On Hyperbolic Plateaus of the Hénon Map

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We propose a rigorous computational method to prove the uniform hyperbolicity of discrete dynamical systems. Applying the method to the real Hénon family, we prove the existence of many regions of hyperbolic parameters in the parameter plane of the family.

1. INTRODUCTION

Consider the problem of determining the set of parameter values for which the real Hénon map

$$H_{a,b} : \mathbb{R}^2 \to \mathbb{R}^2 : (x, y) \mapsto (a - x^2 + by, x) \quad (a, b \in \mathbb{R})$$

is uniformly hyperbolic. If a dynamical system is uniformly hyperbolic, then generally speaking, we can apply the so-called hyperbolic theory of dynamical systems and obtain many results on the behavior of the system. Despite its importance, however, proving hyperbolicity is a difficult problem even for such simple polynomial maps as the Hénon maps.

The first mathematical result about the hyperbolicity of the Hénon map was obtained in [Devaney and Nitecki 79]. The authors showed that for any fixed b, if a is sufficiently large then the nonwandering set of $H_{a,b}$ is uniformly hyperbolic and conjugate to the full horseshoe map, that is, the shift map of the space of bi-infinite sequences of two symbols.

Later, Davis, MacKay, and Sannami [Davis et al. 91] conjectured that besides the uniformly hyperbolic full horseshoe region, there exist some parameter regions in which the nonwandering set of the Hénon map is uniformly hyperbolic and conjugate to a subshift of finite type. For some parameter intervals of the area-preserving Hénon family $H_{a,-1}$, they identified the Markov partition by describing the configuration of stable and unstable manifolds (see also [Sterling et al. 99, Hagiwara and Shudo 04]). Although the mechanism of hyperbolicity at these parameter values is clear by their observations, no mathematical proof of uniform hyperbolicity has been obtained so far.

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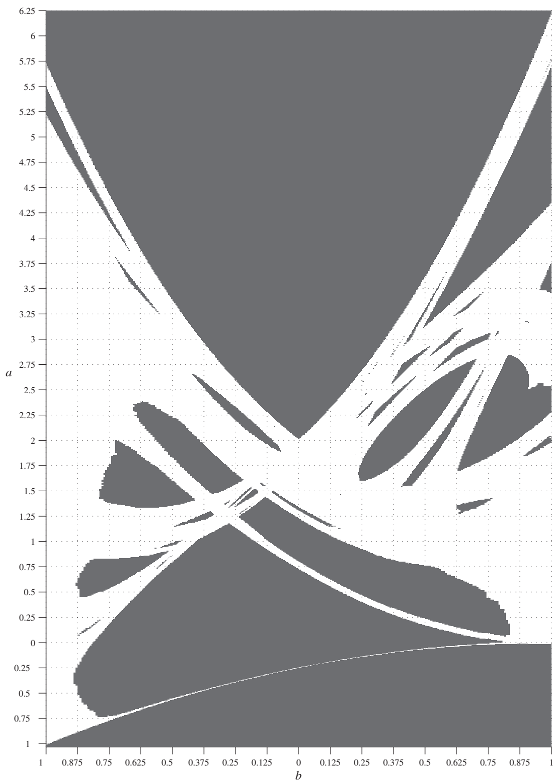

FIGURE 1. Uniformly hyperbolic plateaus.

The purpose of this paper is to propose a general method for proving uniform hyperbolicity of discrete dynamical systems. Applying the method to the Hénon map, we obtain a computer-assisted proof of the hyperbolicity of the Hénon map on many parameter regions including the intervals conjectured by Davis et al.

Our results on the real Hénon map are summarized in the following theorems. We denote by $\mathcal{R}(H_{a,b})$ the chain-recurrent set of $H_{a,b}$.

Theorem 1.1. *There exists a set* $P \subset \mathbb{R}^2$, *which is the union of* 8943 *closed rectangles, such that if* $(a, b) \in P$, *then* $\mathcal{R}(H_{a,b})$ *is uniformly hyperbolic. The set* P *is illustrated in Figure 1 (shaded regions), and the complete list of the rectangles in* P *is given as supplemental material to the paper.*

The hyperbolicity of the chain-recurrent set implies the $\mathcal{R}$-stability. Therefore, on each connected component of P, no bifurcation occurs in $\mathcal{R}(H_{a,b})$, and hence numerical invariants such as the topological entropy and the number of periodic points are constant on it. For this reason, we call it a "plateau."

Note that Theorem 1.1 does not claim that a parameter value not in P is a nonhyperbolic parameter. It guarantees only that P is a subset of the uniformly hyperbolic parameter values. We can refine Theorem 1.1 by performing more computations, which yields a set P' of uniformly hyperbolic parameters such that $P \subset P'$.

Since the area-preserving Hénon family is of particular importance, we performed another computation restricted to this one-parameter family and obtained the following.

Theorem 1.2. *If* a *is in one of the following closed intervals,*

$$
\begin{array}{ll}
[4.5383300781250,\ 4.5385742187500], & [4.5388183593750,\ 4.5429687500000], \\
[4.5623779296875,\ 4.5931396484375], & [4.6188964843750,\ 4.6457519531250], \\
[4.6694335937500,\ 4.6881103515625], & [4.7681884765625,\ 4.7993164062500], \\
[4.8530273437500,\ 4.8603515625000], & [4.9665527343750,\ 4.9692382812500], \\
[5.1469726562500,\ 5.1496582031250], & [5.1904296875000,\ 5.5366210937500], \\
[5.5659179687500,\ 5.6077880859375], & [5.6342773437500,\ 5.6768798828125], \\
[5.6821289062500,\ 5.6857910156250], & [5.6859130859375,\ 5.6860351562500], \\
[5.6916503906250,\ 5.6951904296875], & [5.6999511718750,\ \infty),
\end{array}
$$

then $\mathcal{R}(H_{a,-1})$ *is uniformly hyperbolic.*

We remark that the three intervals considered to be hyperbolic parameter values by Davis et al. appear in Theorem 1.2. Thus we can say that Theorem 1.2 justifies their observations.

It is interesting to compare Figure 1 with the bifurcation diagrams of the Hénon map numerically obtained in [Hamouly and Mira 81], and [Sannami 89, Sannami 94]. The boundary of P shown in Figure 1 is very close to the bifurcation curves given in these papers.

Recently, Cao, Luzzatto, and Rios [Cao et al. 05] showed that the Hénon map has a tangency and hence is nonhyperbolic if the parameter is on the boundary of the full horseshoe plateau (see also [Bedford and Smillie 04a, Bedford and Smillie 04b]). This fact and Theorem 1.2 suggests that $H_{a,-1}$ should have a tangency when a is close to 5.699951171875. In fact, we can prove the following result using the rigorous computational method developed in [Arai and Mischaikow 06].

Proposition 1.3. *There exists* $a \in [5.6993102, 5.6993113]$ *such that* $H_{a,-1}$ *has a homoclinic tangency with respect to the saddle fixed point in the third quadrant.*

Consequently, Theorem 1.2 and Proposition 1.3 yield the following.

Corollary 1.4. *When we decrease* $a \in \mathbb{R}$ *of the area-preserving Hénon family* $H_{a,-1}$, *the first tangency occurs in the interval* $[5.6993102, 5.699951171875)$.

We remark that Hruska [Hruska 06a, Hruska 06b] also constructed a rigorous numerical method for proving hyperbolicity of complex Hénon maps. The main difference between our method and Hruska's is that our method does not prove hyperbolicity directly. Instead, we prove quasihyperbolicity, which is equivalent to uniform hyperbolicity under the assumption of chain-recurrence. This rephrasing enables us to avoid the computationally expensive procedure of constructing a metric adapted to the hyperbolic splitting. Another peculiar feature of our algorithm is that it is based on the subdivision algorithm (see [Dellnitz and Junge 02]) and hence is effective for inductive search of hyperbolic parameters.

Finally, we remark that the method developed in this paper can also be applied to higher-dimensional dynamical systems. In fact, by applying the method to the complex Hénon map, we obtain a proof of Conjecture 1.1 in [Bedford and Smillie 06] (see [Arai 07]).

The structure of the rest of the paper is as follows. The notion of quasihyperbolicity will be introduced in Section 2, and then an algorithm for proving quasihyperbolicity will be proposed in Section 3. In the last section, Section 4, we apply the method to the Hénon family and obtain Theorems 1.1 and 1.2.

2. HYPERBOLICITY AND QUASIHYPERBOLICITY

First we recall the definition of hyperbolicity. Let f be a diffeomorphism on a manifold M and Λ a compact invariant set of f. We denote by $T\Lambda$ the restriction of the tangent bundle TM to Λ.

Definition 2.1. We say that f is *uniformly hyperbolic* on Λ, or Λ is a *uniformly hyperbolic invariant set* of f, if $T\Lambda$ splits into a direct sum $T\Lambda = E^s \oplus E^u$ of two Tf-invariant subbundles and there are constants $c > 0$ and $0 < \lambda < 1$ such that

$$\|Tf^n|_{E^s}\| < c\lambda^n \quad \text{and} \quad \|Tf^{-n}|_{E^u}\| < c\lambda^n$$

hold for all $n \geq 0$. Here $\|\cdot\|$ denotes a metric on M.

We note that this definition involves many ingredients: constants c and λ, a splitting of $T\Lambda$, and a metric on M. If we try to prove hyperbolicity according to this definition, we must control these objects at the same time, and the algorithm would be rather complicated. Although we can omit the constant c by choosing a suitable metric on M, constructing such a metric is also a difficult problem in general. The situation is the same even if we use the standard "cone fields" argument.

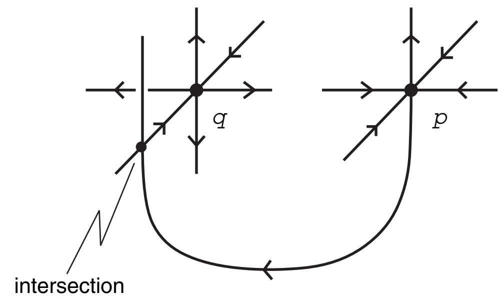

FIGURE 2. $\Lambda := \{p\} \cup \{q\} \cup (W^u(p) \cap W^s(q))$ is quasihyperbolic but is not uniformly hyperbolic.

To avoid this computational difficulty, we introduce the notion of quasihyperbolicity. Recall that the differential of f induces a dynamical system $Tf : TM \to TM$. By restricting it to the invariant set $T\Lambda$, we obtain $Tf : T\Lambda \to T\Lambda$. An orbit of Tf is called a trivial orbit if it is contained in the zero section of the bundle $T\Lambda$.

Definition 2.2. We say that f is *quasihyperbolic* on Λ if $Tf : T\Lambda \to T\Lambda$ has no nontrivial bounded orbit.

This definition is much simpler than that of uniform hyperbolicity and is a purely topological condition for Tf. It is easy to see that hyperbolicity implies quasihyperbolicity. The converse is not true in general, although the hyperbolicity of periodic points and the nonexistence of a tangency follows from quasihyperbolicity.

However, when $f|_\Lambda$ is chain-recurrent, these two notions coincide.

Theorem 2.3. [Churchill et al. 77, Sacker and Sell 74] *Assume that $f|_\Lambda$ is chain-recurrent, that is, $\mathcal{R}(f|_\Lambda) = \Lambda$. Then f is uniformly hyperbolic on Λ if and only if f is quasihyperbolic on it.*

Remark 2.4. The assumption of chain-recurrence is essential for uniform hyperbolicity. For example, consider two hyperbolic saddle fixed points p and q in $\mathbb{R}^3$, with 1- and 2-dimensional unstable direction respectively. Assume that the unstable manifold $W^u(p)$ of p intersects the stable manifold $W^s(q)$ of q in a way that the sum of the tangent spaces of these two 1-dimensional manifolds span a 2-dimensional subspace of $\mathbb{R}^3$ (see Figure 2). Let $\Lambda := \{p\} \cup \{q\} \cup (W^u(p) \cap W^s(q))$. Then Λ is quasihyperbolic, but clearly not uniformly hyperbolic because it contains fixed points with different unstable dimensions and a connecting orbit between them.

Next, we rephrase the definition of quasihyperbolicity in terms of isolating neighborhoods. Recall that a compact set N is an isolating neighborhood (see [Mischaikow and Mrozek 02]) with respect to f if the maximal invariant set

$$\mathrm{Inv}(f, N) := \{x \in N \mid f^n(x) \in N \text{ for all } n \in \mathbb{Z}\}$$

is contained in $\mathrm{int}\, N$, the interior of N. An invariant set S of f is said to be isolated if there is an isolating neighborhood N such that $\mathrm{Inv}(f, N) = S$.

Note that the linearity of Tf in the fibers of $T\Lambda$ implies that if there is a nontrivial bounded orbit of $Tf : T\Lambda \to T\Lambda$, then its multiplication by a constant is also a nontrivial bounded orbit, and hence any compact neighborhood N of the zero section of $T\Lambda$ contains a nontrivial bounded orbit. Therefore, the definition of quasihyperbolicity is equivalent to saying that the zero section of the tangent bundle $T\Lambda$ is an isolated invariant set with respect to $Tf : T\Lambda \to T\Lambda$.

Furthermore, it suffices to find an isolating neighborhood that contains the zero section.

Proposition 2.5. *Assume that $N \subset T\Lambda$ is an isolating neighborhood with respect to $Tf : T\Lambda \to T\Lambda$ and N contains the image of the zero section of $T\Lambda$. Then Λ is quasihyperbolic.*

Proof: For a subset S of TM and $\delta \geq 0$, we define $\delta S := \{\delta \cdot v \mid v \in S\}$. By linearity of Tf, if S is Tf-invariant, so is δS. Now we assume that N is an isolating neighborhood, that is, $\mathrm{Inv}(Tf, N) \subset \mathrm{int}\, N$. A standard compactness argument shows that there is $\delta > 1$ such that $\delta\, \mathrm{Inv}(Tf, N) \subset N$. Since $\delta\, \mathrm{Inv}(Tf, N)$ is Tf-invariant and contained in N, we have $\delta\, \mathrm{Inv}(Tf, N) \subset \mathrm{Inv}(Tf, N)$, by definition of the maximal invariant set. It follows that if $v \in \mathrm{Inv}(Tf, N)$, we have $\delta^n v \in \mathrm{Inv}(Tf, N)$ for all $n \geq 0$. Since $\mathrm{Inv}(Tf, N)$ is compact and hence bounded, v must be the zero vector. This implies that there is no nontrivial bounded orbit of $Tf : T\Lambda \to T\Lambda$. □

3. ALGORITHMS

In this section, we assume that $M = \mathbb{R}^n$ and consider a family of diffeomorphisms $f_a : \mathbb{R}^n \to \mathbb{R}^n$ that depends on an r-tuple of real parameters $a = (a_1, \ldots, a_r) \in \mathbb{R}^r$. Define $F : \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^n$ and $TF : T\mathbb{R}^n \times \mathbb{R}^r \to T\mathbb{R}^n$ by

$$F(x, a) := f_a(x) \quad \text{and} \quad TF(x, v, a) := Tf_a(x, v),$$

where $x \in \mathbb{R}^n$ and $v \in T_x\mathbb{R}^n$.

We denote by $\mathbb{F}$ the set of floating-point numbers, or the set of numbers our computer can handle. Let $\mathbb{IF}$ be the set of intervals whose endpoints are in $\mathbb{F}$. That is,

$$\mathbb{IF} := \{I = [a, b] \subset \mathbb{R} \mid a, b \in \mathbb{F}\}.$$

Similarly, we define a set of n-dimensional cubes by

$$\mathbb{IF}^n := \{I_1 \times \cdots \times I_n \subset \mathbb{R}^n \mid I_i \in \mathbb{IF}\}.$$

Let $X, F \in \mathbb{IF}^n$ and $A \in \mathbb{IF}^r$. We consider these cubes respectively as subspaces of the manifold M, the tangent space of M, and the parameter space. What we want to compute is the image of these cubes under the maps F and TF, namely $F(X \times A)$ and $TF(X \times V \times A)$. Note that these images are not objects of $\mathbb{IF}^n$ or $\mathbb{IF}^{2n}$ in general. By this fact and the effect of rounding errors, we cannot hope that a computer can exactly compute these images. Instead, we require that our computer be able to enclose these images using elements of $\mathbb{IF}^n$ and $\mathbb{IF}^{2n}$.

Assumption 3.1. *There exists a computational method such that for any $X, V \in \mathbb{IF}^n$ and $A \in \mathbb{IF}^r$, it can compute $Y \in \mathbb{IF}^n$ and $W \in \mathbb{IF}^{2n}$ such that*

$$F(X \times A) \subset \mathrm{int}\, Y$$

and

$$TF(X \times V \times A) \subset \mathrm{int}\, W$$

hold rigorously.

Obviously, if the outer approximations Y and W in Assumption 3.1 are too large, we cannot derive any information about F or TF. As we will mention in the last section, for many classes of dynamical systems including polynomial maps, the rigorous interval arithmetic can be used to satisfy this assumption, and it gives effectively good outer approximations.

Let $K \subset \mathbb{R}^n$ be a compact set that contains Λ and $L \subset T\mathbb{R}^n$, the product of K and $[-1, 1]^n$. We assume that K is decomposed into a finite union of elements of $\mathbb{IF}^n$, namely

$$K = \bigcup_{i=1}^{k} K_i, \quad \text{where} \quad K_i \in \mathbb{IF}^n.$$

We also decompose the fiber $[-1, 1]^n \subset T_x\mathbb{R}^n$ into a finite union of elements of $\mathbb{IF}^n$. By making products of cubes contained in the decompositions of K and $[-1,1]$, we obtain a decomposition of L such as

$$L = \bigcup_{j=1}^{\ell} L_j, \quad \text{where} \quad L_j \in \mathbb{IF}^{2n}.$$

By Assumption 3.1, we can compute $Y_i \in \mathbb{IF}^n$ and $W_j \in \mathbb{IF}^{2n}$ such that

$$F(K_i \times A) \subset \operatorname{int} Y_i \quad \text{and} \quad TF(L_j \times A) \subset \operatorname{int} W_j$$

for any $1 \le i \le k$ and $1 \le j \le \ell$.

From this information about Y_i and W_j, we then construct directed graphs $\mathcal{G}(F, K, A)$ and $\mathcal{G}(TF, L, A)$ as follows:

- $\mathcal{G}(F, K, A)$ has k vertices $\{v_1, v_2, \ldots, v_k\}$.
- There exists an edge from v_p to v_q if and only if $Y_p \cap K_q \neq \varnothing$.

And similarly,

- $\mathcal{G}(TF, L, A)$ has ℓ vertices $\{w_1, w_2, \ldots, w_\ell\}$.
- There exists an edge from w_p to w_q if and only if $W_p \cap N_q \neq \varnothing$.

The most important property of $\mathcal{G}(F, K, A)$ is that if there exists $x \in K_p$ that is mapped into K_q by f_a for some $a \in A$, then there must be an edge of $\mathcal{G}(F, K, A)$ from v_p to v_q. This property also holds for $\mathcal{G}(TF, L, A)$.

We use these directed graphs to enclose the chain-recurrent set of f_a and the maximal invariant set of N. For this purpose, we define the following notions.

Definition 3.2. Let G be a directed graph. The vertices of $\operatorname{Inv} G$, the *invariant set* of G, is defined by

$$\{v \in G \mid \exists \text{ a bi-infinitely long path through } v\}.$$

The vertices of $\operatorname{Scc} G$, the set of *strongly connected components* of G, is

$$\{v \in G \mid \exists \text{ a path from } v \text{ to itself}\}.$$

The edges of these graphs are defined to be the restriction of those of G.

Note that by definition, $\operatorname{Scc} G$ is a subgraph of $\operatorname{Inv} G$.

For subgraphs G of $\mathcal{G}(F, K, A)$ and G' of $\mathcal{G}(TF, L, A)$, we define their *geometric representations* $|G| \subset \mathbb{R}^n$ and $|G'| \subset \mathbb{R}^{2n}$ by

$$|G| := \bigcup_{v_i \in G} K_i$$

and

$$|G'| := \bigcup_{w_j \in G'} L_j.$$

Obviously, $|\mathcal{G}(F, K, A)| = K$ and $|\mathcal{G}(TF, L, A)| = L$.

Proposition 3.3. *For any $a \in A$,*

$$\operatorname{Inv}(f_a, K) \subset |\operatorname{Inv} \mathcal{G}(F, K, A)|$$

and

$$\operatorname{Inv}(Tf_a, L) \subset |\operatorname{Inv} \mathcal{G}(TF, L, A)|.$$

Furthermore, if $\mathcal{R}(f_a) \subset \operatorname{int} K$ holds for all $a \in A$, then we have

$$\mathcal{R}(f_a) \subset |\operatorname{Scc} \mathcal{G}(F, K, A)|$$

for all $a \in A$.

Proof: The claims for maximal invariant sets follow from the construction of $\mathcal{G}(F, K, A)$ and $\mathcal{G}(TF, L, A)$. We prove only $\mathcal{R}(f_a) \subset |\operatorname{Scc} \mathcal{G}(F, K, A)|$. Since $F(K_i \times \{a\}) \subset \operatorname{int} Y_i$ holds for all i and the number of cubes in K is finite, we can choose $\varepsilon > 0$ such that for any i and $x \in K_i$, if y is a point with $d(f_a(x), y) < \varepsilon$, then y must be contained in Y_i. Here d denotes a fixed metric of $\mathbb{R}^n$.

This implies that if such y is contained in K_j, there must be an edge from v_i to v_j. Let $x \in \mathcal{R}(f_a)$. From the assumption, there exists p such that $x \in K_p$. Since $\mathcal{R}(f_a) \subset \operatorname{int} K$, we can assume that there is an ε-chain from x to itself that is contained in K by choosing ε smaller if necessary. It follows that there must be a path of $\mathcal{G}(F, K, A)$ from v_p to itself and therefore $x \in |\operatorname{Scc} \mathcal{G}(F, K, A)|$. This proves the claim. □

For the computation of $\operatorname{Inv} G$, the algorithm of [Szymczak 97] can be used. There is also an algorithm for computing $\operatorname{Scc} G$ that is standard in algorithmic graph theory (see [Sedgewick 83], for example).

Now we can describe the algorithm to prove quasihyperbolicity. It involves the subdivision algorithm [Dellnitz and Junge 02]. Roughly speaking, this means that if we fail to prove quasihyperbolicity, then we subdivide all of the cubes in K and L to obtain a better approximation of the invariant set, and repeat the whole step until we succeed with the proof.

In the following, we first develop an algorithm for a fixed set of parameter values. That is, we fix the set A and try to check whether $\mathcal{R}(f_a)$ is quasihyperbolic for all $a \in A$. Note that we do not exclude the case in which A contains only one parameter value, namely $A = \{a\}$, where a is an r-tuple of floating-point numbers.

Algorithm 3.4. (For proving quasihyperbolicity for all $a \in A$.)

1. Find K such that $\mathcal{R}(f_a) \subset \operatorname{int} K$ holds for all $a \in A$ and let $L := K \times [-1,1]^n$.
2. Compute $\operatorname{Scc} \mathcal{G}(F, K, A)$ and replace K with $|\operatorname{Scc} \mathcal{G}(F, K, A)|$.
3. Replace L with $L \cap (K \times [-1,1]^n)$.
4. Compute $\operatorname{Inv} \mathcal{G}(TF, L, A)$.
5. If $|\operatorname{Inv} \mathcal{G}(TF, K, A)| \subset K \times \operatorname{int}[-1,1]^n$, then stop.
6. Otherwise, replace L with $|\operatorname{Inv} \mathcal{G}(TF, L, A)|$ and refine the decomposition of K and L by bisecting all cubes in them. Then go to step 2.

Theorem 3.5. *If Algorithm 3.4 stops, then f_a is quasihyperbolic on $\mathcal{R}(f_a)$ for every $a \in A$.*

Proof: Assume that Algorithm 3.4 stops and choose $a \in A$. Let

$$N_a = L \cap (\mathcal{R}(f_a) \times [-1,1]^n).$$

Then N_a contains the zero-section of $T\mathcal{R}(f_a)$. By Proposition 2.5, it suffices to show that N_a is an isolating neighborhood with respect to $Tf_a : T\mathcal{R}(f_a) \to T\mathcal{R}(f_a)$. Since the algorithm stops, then

$$\begin{aligned} \operatorname{Inv}(Tf_a, N_a) &\subset \operatorname{Inv}(Tf_a, L) \\ &\subset |\operatorname{Inv} \mathcal{G}(TF, L)| \\ &\subset K \times \operatorname{int}[-1,1]^n. \end{aligned}$$

Then it follows from $\operatorname{Inv}(Tf_a, N_a) \subset N_a \subset \mathcal{R}(f_a) \times [-1,1]^n$ that

$$\operatorname{Inv}(Tf_a, N_a) \subset \mathcal{R}(f_a) \times \operatorname{int}[-1,1]^n.$$

But $\mathcal{R}(f_a) \times \operatorname{int}[-1,1]^n$ is the interior of N_a with respect to $T\mathcal{R}(f_a)$, and this proves $\operatorname{Inv}(Tf_a, N_a) \subset \operatorname{int} N_a$. □

In other words, if A contains a nonquasihyperbolic parameter value, then Algorithm 3.4 never stops. Therefore, if we want to apply the method for a large family of diffeomorphisms, the algorithm should involve an automatic selection of parameter values.

We can also use the subdivision algorithm to realize such a procedure. That is, we will inductively decompose A into a finite union of elements of $\mathbb{IF}^r$ and remove cubes in which the hyperbolicity has been proved. We denote by $\mathcal{A}$ the set of cubes in the decomposition of A.

Algorithm 3.6. (Adaptive selection of quasihyperbolic parameters.)

1. Find K such that $\mathcal{R}(f_a) \subset K$ holds for all $a \in A$.
2. Let $\mathcal{A} = \{A_0\}$, where $A_0 = A$ and $K_0 = K$, $L_0 = K_0 \times [-1,1]^n$.
3. Choose a cube $A_i \in \mathcal{A}$ according to the "selection rule."
4. Apply steps 2, 3, and 4 of Algorithm 3.4 with $A = A_i$, $K = K_i$, and $L = L_i$.
5. If $|\operatorname{Inv} \mathcal{G}(TF, L_i)| \subset \operatorname{int} L_i$, then remove A_i from $\mathcal{A}$ and go to step 2.
6. Otherwise, bisect A_i into two cubes A_j, A_k. Remove A_i from $\mathcal{A}$ and add A_j, A_k to $\mathcal{A}$. Put $K_j = K_k = K_i$ and $L_j = L_k = \operatorname{Inv} \mathcal{G}(TF, L_i)$ and then go to step 2.

This algorithm does not stop if there is a nonquasihyperbolic parameter in A. But it follows from Theorem 3.5 that if the cube A_i is removed in the procedure of Algorithm 3.6, then A_i consists of quasihyperbolic parameter values.

We did not specify the "selection rule" that appears in step 2 of Algorithm 3.6. Various rules can be applied, and the effectiveness of a rule depends on the case.

One example of such a rule is to select A_i such that N_i and K_i consist of smaller numbers of cubes. Since the computational cost of the algorithm depends on the number of cubes, this rule implies that our computation will be concentrated on parameter values on which the computation is relatively fast. By applying this rule, we can avoid wasting too much time trying to prove the hyperbolicity for apparently nonhyperbolic parameter values. This rule works sufficiently well for general purposes.

The problem with this rule is that sometimes the computation is focused only on parameters with a small invariant set, for example, in the case that $\mathcal{R}(f_a)$ is a single fixed point. If this is the case, then most of the computation will be done on parameter cubes close to the bifurcation curve of the fixed point. To avoid this, we can use the number of cubes multiplied by the subdivision depth of A_i instead of the number of cubes itself.

Alternatively, we can distribute our computational effort across the whole of the parameter space equally simply by selecting all cubes in $\mathcal{A}$ sequentially.

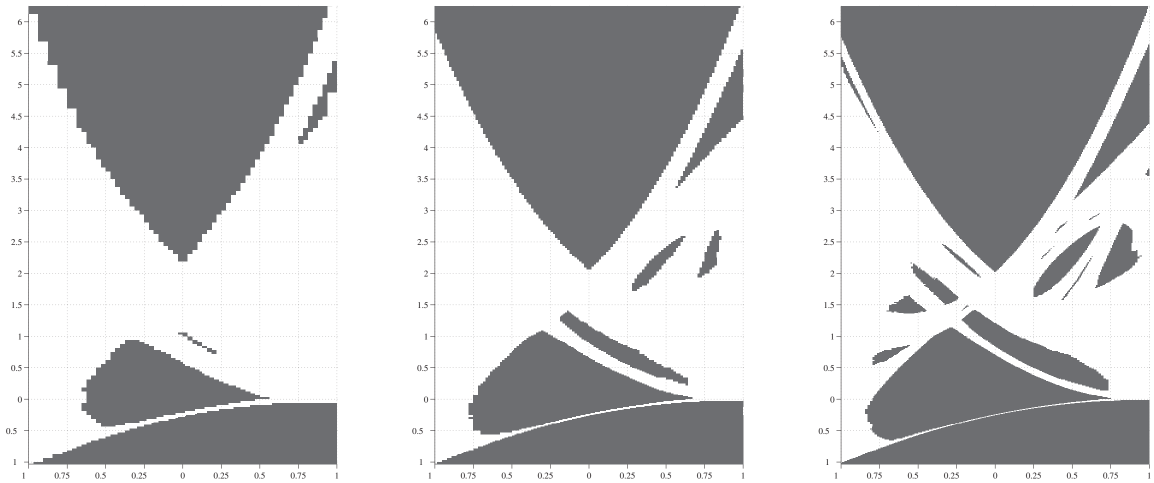

FIGURE 3. Results after computations of 1, 10, and 100 hours.

4. APPLICATION TO THE HÉNON MAP

In this section we apply the method developed in Sections 2 and 3 to the chain-recurrent set $\mathcal{R}(H_{a,b})$ of the Hénon family.

In order to apply the algorithm, we must know a priori the size of $\mathcal{R}(H_{a,b})$. Further, to apply Theorem 2.3, we need to check that the dynamics restricted to $\mathcal{R}(H_{a,b})$ are chain-recurrent.

First we recall the numbers defined in [Devaney and Nitecki 79]. Let

$$R(a,b) := \frac{1}{2}\left(1 + |b| + \sqrt{(1+|b|)^2 + 4a}\right),$$
$$S(a,b) := \left\{(x,y) \in \mathbb{R}^2 : |x| \le R(a,b), |y| \le R(a,b)\right\}.$$

Then we can prove the following.

Lemma 4.1. *The chain-recurrent set $\mathcal{R}(H_{a,b})$ is contained in $S(a,b)$, and $H_{a,b}$ restricted to $\mathcal{R}(H_{a,b})$ is chain-recurrent.*

Proof: If $x \notin S(a,b)$, we can choose $\varepsilon_0 > 0$ so small that if $\varepsilon < \varepsilon_0$, then all ε-chains through x must diverge to infinity, and hence x cannot be chain-recurrent (this is a special case of [Bedford and Smillie 91, Corollary 2.7]). The proof for the second claim is the same as that for the compact case (see [Robinson 99], for example), because we can choose a compact neighborhood S' of $S(a,b)$ and $\varepsilon_0 > 0$ such that if $\varepsilon < \varepsilon_0$, then all ε-chains from $x \in \mathcal{R}$ to x must be contained in S'. □

In the case of the Hénon map, Assumption 3.1 can be satisfied with rigorous interval arithmetic on a CPU that satisfies the IEEE754 standard for binary floating-point arithmetic. This is also the case for an arbitrary polynomial map of $\mathbb{R}^n$.

We remark that we need to consider only the case $b \in [-1,1]$, because the inverse of the Hénon map $H_{a,b}$ is again conjugate to the Hénon map $H_{a/b^2,1/b}$, whose Jacobian is $1/b$, and the hyperbolicity of a diffeomorphism is equivalent to that of the inverse map. Further, we can restrict our computation to the case $(a,b) \in [-1,12] \times [-1,1]$, for otherwise it follows from the proof of [Devaney and Nitecki 79] that $\mathcal{R}(H_{a,b})$ is hyperbolic or empty.

Therefore, we start with $A := [-1,12] \times [-1,1]$, $K := [-8,8] \times [-8,8]$, and $L = K \times [-1,1]^2$. Then Lemma 4.1 implies that $\mathcal{R}(H_{a,b}) \subset \operatorname{int} K$ holds for all $(a,b) \in A$. With this initial data, Theorem 1.1 is proven by applying Algorithm 3.6.

To obtain Theorem 1.2, we fix $b = -1$ and start the computation with $A := [4,12]$. The sets K and L are the same as in the computation for Theorem 1.1.

Finally, we mention the computational cost of the method. To achieve Theorem 1.1, we needed 1000 hours of computation using a 2-GHz PowerPC 970 CPU. With the same CPU, 260 hours were used for Theorem 1.2. Figure 3 shows the intermediate results obtained after 1, 10, and 100 hours of computation toward Theorem 1.2. We remark that as these figures suggest, almost all computation time was spent on parameter values close to the bifurcation curves.

All of the source code used in these computations is available at the home page of the author, http://www.math.kyoto-u.ac.jp/~arai/, as well as at http://www.expmath.org/expmath/volumes/16/16.2/Arai/realhenon.tar.gz. The data structure and the subdivision algorithm are implemented in the GAIO package, available at http://math-www.uni-paderborn.de/~agdellnitz/gaio/; see also [Dellnitz and Junge 02]. For the interval arithmetic, we use the package CAPD (http://capd.wsb-nlu.edu.pl/). You can also use the PROFIL/BIAS interval arithmetic package for this purpose (http://www.ti3.tu-harburg.de/knueppel/profil).

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