On the Computation of Clebsch–Gordan Coefficients and the Dilation Effect

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CONTENTS

We investigate the problem of computing tensor product multiplicities for complex semisimple Lie algebras. Even though computing these numbers is $\#P$-hard in general, we show that when the rank of the Lie algebra is assumed fixed, then there is a polynomial-time algorithm, based on counting lattice points in polytopes. In fact, for Lie algebras of type A, there is an algorithm, based on the ellipsoid algorithm, to decide when the coefficients are nonzero in polynomial time for arbitrary rank. Our experiments show that the lattice point algorithm is superior in practice to the standard techniques for computing multiplicities when the weights have large entries but small rank. Using an implementation of this algorithm, we provide experimental evidence for two conjectured generalizations of the saturation property of Littlewood–Richardson coefficients. One of these conjectures seems to be valid for types B, C, and D.

1. INTRODUCTION

Given highest weights λ, μ, and ν for a finite-dimensional complex semisimple Lie algebra, we denote by $C_{\lambda\mu}^{\nu}$ the multiplicity of the irreducible representation V_{ν} in the tensor product of V_{λ} and V_{μ}; that is, we write

$$V_{\lambda} \otimes V_{\mu} = \bigoplus_{\nu} C_{\lambda\mu}^{\nu} V_{\nu}. \qquad (1\text{--}1)$$

In general, the numbers $C_{\lambda\mu}^{\nu}$ are called *Clebsch–Gordan coefficients*. In the specific case of type-A Lie algebras, the values $C_{\lambda\mu}^{\nu}$ defined in equation (1–1) are called *Littlewood–Richardson coefficients*. When we are specifically discussing the type-A case, we will adhere to convention and write $c_{\lambda\mu}^{\nu}$ for $C_{\lambda\mu}^{\nu}$.

The concrete computation of Clebsch–Gordan coefficients (sometimes known as the *Clebsch–Gordan problem* [Fulton and Harris 91]) has attracted considerable attention from not only representation theorists, but also from physicists, who employ them in the study of quantum mechanics, e.g., [Belinfante and Kolman 89, Cohen and de Graaf 96, Wybourne 90]. The importance

of these coefficients is also evident from their widespread appearance in other fields of mathematics besides representation theory. For example, the Littlewood–Richardson coefficients appear in combinatorics via symmetric functions and in enumerative algebraic geometry via Schubert varieties and Grassmannians; see, for instance, [Fulton 97, Stanley 77]. More recently, Clebsch–Gordan coefficients are also playing an important role in the study of P vs. NP; see [Mulmuley and Sohoni 03].

Very recently, Narayanan has proved that the computation of Clebsch–Gordan coefficients is in general a $\#P$-complete problem [Narayanan 05]. Nonetheless, one can ask for an algorithm that behaves well when some parameter is fixed. Stembridge has raised the challenge of crafting algorithms based on geometric ideas such as Littelmann's paths [Littelmann 98] or Kashiwara's crystal bases [Kashiwara 90] (see comment on page 29, section 7, of [Stembridge 01b]). As we show below, there is such an algorithm, based on the polyhedral geometry of the Clebsch–Gordan coefficients.

The first encoding of Littlewood–Richardson numbers as integral points in polytopes appeared in the Ph.D. thesis of S. Johnson [Johnson 79]. In 1988, Berenstein and Zelevinsky presented another combinatorial interpretation of the Littlewood–Richardson coefficients as the number of lattice points in members of a certain family of polytopes [Berenstein and Zelevinsky 88a]. Around the same time, they also presented a more general family of polytopes, which we call *BZ-polytopes*, that they conjectured would enumerate Clebsch–Gordan coefficients for any finite-dimensional complex semisimple Lie algebra [Berenstein and Zelevinsky 88b]. In 1999, Knutson and Tao reformulated the type-A versions of these polytopes, which they called *hive polytopes*, and used them to prove the saturation theorem [Knutson and Tao 99]. Other mathematicians have also looked at these polyhedra [Pak and Vallejo 05, Kirillov 01]. In this paper, we will use the presentation by Knutson and Tao in studying the type-A case. Finally, in 2001, the polyhedral picture of Clebsch–Gordan coefficients was completed when Berenstein and Zelevinsky proved their 1988 conjecture [Berenstein and Zelevinsky 01]. Here are our contributions:

1. We combine the lattice point enumeration algorithm of Barvinok [Barvinok 94] with the results of Berenstein and Zelevinsky [Berenstein and Zelevinsky 01] on the polyhedral realization of Clebsch–Gordan coefficients to produce a new algorithm for computing these coefficients. Our main theoretical result is the following.

Theorem 1.1. (Proved in Section 2.) *Given a fixed finite-dimensional complex semisimple Lie algebra* $\mathfrak{g}$, *one can compute a Clebsch–Gordan coefficient* $C^\nu_{\lambda\mu}$ *of* $\mathfrak{g}$ *in time polynomial in the input size of the defining weights.*

Moreover, in the type-A case, deciding whether $c^\nu_{\lambda\mu} \neq 0$ *can be done in polynomial time even when the rank is not fixed.*

2. We implemented this algorithm for types A_r, B_r, C_r, and D_r (the so-called classical Lie algebras) using the software packages LattE [De Loera et al. 03] and Maple 9 [Maplesoft]. In many instances, our implementation performs faster than standard methods, such as those implemented in the software LiE [van Leeuwen 94]. Our software is freely available at http://math.ucdavis.edu/~tmcal.

3. Using our software, we explored general properties satisfied by the Clebsch–Gordan coefficients for the classical Lie algebras under the operation of *stretching of multiplicities* in the sense of [King et al. 04]. Our computer experiments provided evidence for the following proposition.

Proposition 1.2. (Proved in Section 4.2.) *The minimum quasiperiod of a stretched Clebsch–Gordan coefficient for a classical Lie algebra is at most* 2.

On the basis of abundant experimental evidence, we also propose two conjectured generalizations of the saturation theorem of Knutson and Tao [Knutson and Tao 99]. One of them, which applies to all of the classical root systems, is an extension of earlier work by King et al. [King et al. 04].

The paper is organized as follows. In Section 2, after a review of some background material, we prove Theorem 1.1. Section 3 explains our experiments comparing our software with LiE. In Section 4, motivated by our computational results, we prove Proposition 1.2 and present two conjectures: Conjectures 4.5 and 4.7. Each of these conjectures, if true, would generalize the saturation theorem of Knutson and Tao.

2. CLEBSCH–GORDAN COEFFICIENTS: POLYHEDRAL ALGORITHMS

As stated in the introduction, we are interested in the problem of efficiently computing $C^\nu_{\lambda\mu}$ in the tensor product expansion $V_\lambda \otimes V_\mu = \bigoplus_\nu C^\nu_{\lambda\mu} V_\nu$. It appears that the most common method used to compute Clebsch–Gordan coefficients is based on Klimyk's formula (see Lemma
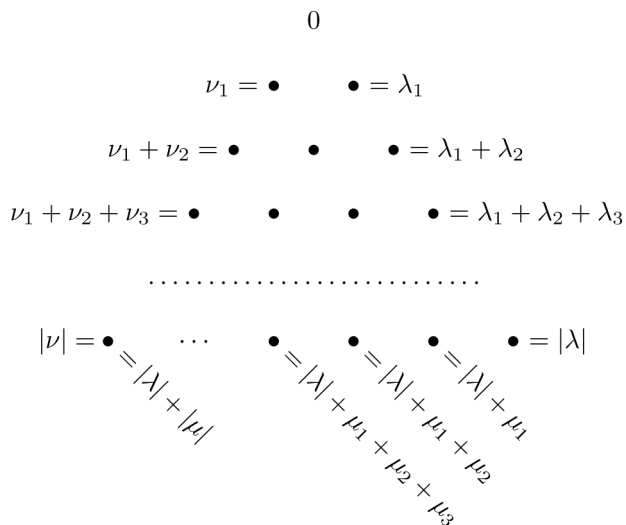
2.1 below). For example, it is used in the package LiE [van Leeuwen 94] and the Maple packages `coxeter/weyl` [Stembridge 01a].

Lemma 2.1. [Humphreys 72, Exercise 24.9] *Fix a complex semisimple Lie algebra $\mathfrak{g}$, and let $\mathfrak{W}$ be the associated Weyl group. For each weight β of $\mathfrak{g}$, let $\operatorname{sgn}(\beta)$ denote the parity of the minimum length of an element $w \in \mathfrak{W}$ such that $w(\beta)$ is a highest weight, and let $\{\beta\}$ denote that highest weight. Let δ be one-half the sum of the positive simple roots of $\mathfrak{g}$. Finally, for each highest weight λ of $\mathfrak{g}$, let $K_{\lambda\beta}$ be the multiplicity of β in V_λ.*

Then, given highest weights λ and μ of $\mathfrak{g}$, we have that

$$V_\lambda \otimes V_\mu = \bigoplus_\varepsilon K_{\lambda\varepsilon} \operatorname{sgn}(\varepsilon + \mu + \delta) V_{\{\varepsilon+\mu+\delta\}-\delta},$$

where the sum is over weights ε of $\mathfrak{g}$ with trivial stabilizer subgroup in $\mathfrak{W}$.

Implementations of Klimyk's algorithm begin by computing the weight spaces appearing with nonzero multiplicity in the representation V_λ. Then, for each such weight ε with trivial stabilizer, one computes the Weyl group orbit of $\varepsilon + \mu + \delta$. One then finds the dominant member of the orbit and notes the number ℓ of reflections needed to reach it. Finally, one adds $(-1)^\ell K_{\lambda\varepsilon}$ to the multiplicity of $V_{\{\varepsilon+\mu+\delta\}-\delta}$.

Observe that as we perform the Klimyk algorithm, we compute the coefficient of each V_ν in $V_\lambda \otimes V_\mu$ "in parallel." In other words, we do not know the value of any particular Clebsch–Gordan coefficient until we have carried out the entire computation and produced the complete decomposition $V_\lambda \otimes V_\mu = \bigoplus_\nu C^\nu_{\lambda\mu} V_\nu$. Since the number of terms in this decomposition grows exponentially as the sizes of λ and μ grow, these sizes need to be small in practice. This is the main disadvantage of Klimyk's algorithm from the point of view of computational complexity. One can then ask for an algorithm that behaves well as the sizes of the input weights increase, at least if some other parameter is fixed.

As discussed in the introduction, it was shown in [Berenstein and Zelevinsky 88a] that the Littlewood–Richardson coefficients are equal to the number of integral lattice points in members of a particular family of polytopes. In 1999, Knutson and Tao [Knutson and Tao 99] used these polytopes to prove the saturation theorem. More precisely, Knutson and Tao applied a lattice-preserving linear map to the polytopes of Berenstein and Zelevinsky, producing what they call

hive polytopes. These polytopes exist in the polyhedral cone of *hive patterns*, which we now define.

Definition 2.2. Fix $r \in \mathbb{Z}_{\geq 0}$ and let $\mathcal{H} = \{(i, j, k) \in \mathbb{Z}^3_{\geq 0} : i + j + k = r\}$. A *hive pattern* is a map

$$h \colon \mathcal{H} \to \mathbb{R}_{\geq 0}, \qquad (i, j, k) \mapsto h_{ijk},$$

satisfying the *rhombus inequalities*:

- $h_{i,j-1,k+1} + h_{i-1,j+1,k} \leq h_{ijk} + h_{i-1,j,k+1}$,

- $h_{ijk} + h_{i-1,j-1,k+2} \leq h_{i,j-1,k+1} + h_{i-1,j,k+1}$,

- $h_{i+1,j-1,k} + h_{i-1,j,k+1} \leq h_{ijk} + h_{i,j-1,k+1}$,

for $(i, j, k) \in \mathcal{H}$, $i, j \geq 1$.

We usually think of a hive pattern of size r as a triangular array of real numbers:

$$h_{0,0,r}$$
$$h_{1,0,r-1} \quad h_{0,1,r-1}$$
$$h_{2,0,r-2} \quad h_{1,1,r-2} \quad h_{0,2,r-2}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$h_{r,0,0} \quad h_{r-1,1,0} \quad \cdots \quad h_{1,r-1,0} \quad h_{0,r,0}$$

In this representation, the rhombus inequalities state that for each "little rhombus" of entries (which comes in one of three orientations), the sum of the entries on the long diagonal does not exceed the sum of the entries on the short diagonal. That is, in each of the three cases

$$
\begin{array}{ccccccc}
 & & & c & & & \\
c & \; b & & & & a & \; c \\
 & & a & \; b & & & \\
a & \; d & & & & d & \; b \\
 & & d & & & &
\end{array}
$$

we have that $a + b \geq c + d$. Here is an example of a hive pattern with $r = 4$:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & 8 & & 5 & & & \\
 & & 13 & & 12 & & 8 & & \\
 & 18 & & 17 & & 15 & & 11 & \\
20 & & 20 & & 18 & & 16 & & 12
\end{array}
$$

$$0$$

$$\nu_1 = \bullet \qquad \bullet = \lambda_1$$

$$\nu_1 + \nu_2 = \bullet \qquad \bullet \qquad \bullet = \lambda_1 + \lambda_2$$

$$\nu_1 + \nu_2 + \nu_3 = \bullet \qquad \bullet \qquad \bullet \qquad \bullet = \lambda_1 + \lambda_2 + \lambda_3$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$|\nu| = \bullet \quad \cdots \quad \bullet \quad = \bullet \quad = \bullet \quad = \bullet \quad = |\lambda|$$

FIGURE 1. The boundary entries of hive patterns in $H_{\lambda\mu}^{\nu}$.

Recall that a partition of length r is a sequence λ of integers $\lambda_1 \geq \cdots \geq \lambda_r \geq 0$. We write $|\lambda|$ for $\sum_{i=1}^{r} \lambda_i$, the *size* of the partition λ.

Definition 2.3. Given partitions $\lambda, \mu, \nu \in \mathbb{Z}_{\geq 0}^r$, the *hive polytope* $H_{\lambda\mu}^{\nu}$ is the set of hive patterns with boundary entries fixed as in Figure 1.

Knutson and Tao do not require in their definition that $\lambda, \mu, \nu \geq 0$. However, in considering Littlewood–Richardson coefficients, it suffices to restrict ourselves to the case in which λ, μ, and ν are partitions; in particular, their coordinates are nonnegative. This has the consequence that hive polytopes lie in the nonnegative orthant, which will be convenient in Section 4.1.

From the perspective of computational complexity, it is important to note that for fixed r, the input size of a hive polytope $H_{\lambda\mu}^{\nu}$ grows linearly with the input sizes of the weights λ, μ, and ν. As we have indicated several times, our interest in hive polytopes arises from the following result:

Lemma 2.4. [Berenstein and Zelevinsky 88a, Knutson et al. 04] *The Littlewood–Richardson coefficient $c_{\lambda\mu}^{\nu}$ equals the number of integer lattice points in the hive polytope $H_{\lambda\mu}^{\nu}$.*

Unfortunately, the description of the BZ-polytopes for the other classical Lie algebras is more involved than that of the hive polytopes above. Therefore, we refer the reader to Theorems 2.5 and 2.6 of [Berenstein and Zelevinsky 01], which give their descrip-

tion as systems of linear equalities and inequalities in terms of the root systems B_r, C_r, and D_r. The reader may also view the contents of our Maple notebooks, available at http://math.ucdavis.edu/~tmcal, for completely explicit descriptions of the necessary inequalities. The specific properties of the BZ-polytopes that we need to prove our theorem are (1) for fixed rank r, the dimensions of the BZ-polytopes are bounded above by a constant, (2) the input size of a BZ-polytope grows linearly with the input sizes of the weights λ, μ, and ν, and (3) we have the following result describing the relationship between BZ-polytopes and Clebsch–Gordan coefficients:

Lemma 2.5. [Berenstein and Zelevinsky 01, Theorem 2.4] *Fix a finite-dimensional complex semisimple Lie algebra $\mathfrak{g}$ and a triple of highest weights (λ, μ, ν) for $\mathfrak{g}$. Then the Clebsch–Gordan coefficient $C_{\lambda\mu}^{\nu}$ equals the number of integer lattice points in the corresponding BZ-polytope.*

The final necessary ingredient is Barvinok's algorithm for counting lattice points in polytopes in polynomial time for fixed dimension. Several detailed descriptions of the algorithm in Lemma 2.6 are now available in the literature; see, for example, [De Loera et al. 04] and references therein.

Lemma 2.6. [Barvinok 94] *Fix $d \in \mathbb{Z}_{\geq 0}$. Then, given a system of equalities and inequalities defining a rational convex polytope $P \subset \mathbb{R}^d$, we can compute $\#(P \cap \mathbb{Z}^d)$ in time polynomial in the input size of the polytope.*

Having stated these prior results, we are now ready to prove our main theorem.

Proof of Theorem 1.1: First, if we fix the rank of the Lie algebra, then we fix an upper bound on the dimension of the hive or BZ-polytope. Moreover, the input sizes of these polytopes grow linearly with the input sizes of the weights. Thus, by Barvinok's theorem (Lemma 2.6 stated above), their lattice points can be computed in time polynomial in the input sizes of the weights. Therefore, the first part of the theorem follows by Lemmas 2.4 and 2.5.

For the second part of the theorem, regarding type A, the hive polytopes provide a very fast method for determining whether $c_{\lambda\mu}^{\nu} \neq 0$. According to the saturation theorem (see Section 4), $c_{\lambda\mu}^{\nu} \neq 0$ if and only if the corresponding hive polytope is nonempty. Hence, it suffices to check whether the system of inequalities defining the hive

polytope is (real) feasible, which can be done in polynomial time for *arbitrary* dimension as a corollary of the polynomiality of linear programming via Khachian's ellipsoid algorithm; see [Schrijver 86].   □

An analogue of Theorem 1.1 also applies to Kostka numbers $K_{\lambda\beta}$, which are another significant family of invariants in the representation theory of type-A Lie algebras.

Proposition 2.7. *For fixed rank, the Kostka number $K_{\lambda\beta}$ can be computed in polynomial time in the size of the highest weight λ and the weight β. For arbitrary rank, one can decide in polynomial time whether $K_{\lambda\beta} \neq 0$.*

The polynomiality of computing Kostka numbers for fixed rank follows because these numbers can be expressed as the number of lattice points in Gelfand–Tsetlin polytopes [Gelfand and Tsetlin 50]. The polynomiality of determining whether $K_{\lambda\beta} \neq 0$ for arbitrary rank follows from the well-known criterion that $K_{\lambda\beta} \neq 0$ if and only if λ dominates β, which may be checked in polynomial time.

It is also worth noticing that Proposition 2.7 follows directly from Theorem 1.1. This is because each Kostka number $K_{\lambda\beta}$ is a Littlewood–Richardson coefficient for some choice of highest weights. For example, as observed in [King et al. 04], if $\lambda, \beta \in \mathbb{Z}^r$, then $K_{\lambda\beta} = c_{\mu\lambda}^{\nu}$, where

$$\begin{cases} \nu_i = \beta_i + \beta_{i+1} + \cdots, \\ \mu_i = \beta_{i+1} + \beta_{i+2} + \cdots, \end{cases} \quad \text{for } i = 1, 2, \ldots, r.$$

For those familiar with the enumeration of semistandard Young tableaux and Littlewood–Richardson tableaux by Kostka numbers and Littlewood–Richardson coefficients respectively (see, e.g., [Stanley 99]), the bijection establishing this relation is straightforward: given a semistandard Young tableau Y with shape λ and content β, construct a Littlewood–Richardson tableau L with shape ν/μ and content λ by filling the boxes as follows. Start with a skew Young diagram D with shape ν/μ. For $1 \leq i, j \leq r$, place a number of j's in the ith row of D equal to the number of i's in the jth row of Y, ordering the entries in each row so that they are weakly increasing. Let L be the tableau produced by filling the boxes of D in this fashion. (See Figure 2 for an example.)

It is not hard to see that under this map, the column-strictness condition on Y is equivalent to the lattice permutation condition on L. It follows that the map just described is a bijection between semistandard Young
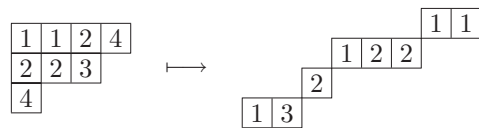
FIGURE 2.   Corresponding semistandard Young and Littlewood–Richardson tableaux.

tableaux with shape λ and content β and Littlewood–Richardson tableaux with shape ν/μ and content λ. Thus, computing Kostka numbers reduces to computing Littlewood–Richardson coefficients.

3.   USING THE ALGORITHM IN PRACTICE

Using the explicit definitions for the hive and BZ-polytopes as the sets of solutions to systems of linear inequalities and equalities, we wrote a Maple notebook that given a triple of highest weights produces the corresponding hive or BZ-polytope in a LattE-readable format. The notebook is available from http://math.ucdavis.edu/~tmcal.

All of our computations were done on a Linux PC with a 2-GHz CPU and 4 GB of memory. From our experiments, we conclude that (1) the polyhedral method of computing tensor product multiplicities complements the method employed in LiE. LiE is effective for slightly larger ranks (up to $r = 10$, say), but the sizes of the weights must be kept small. This is because LiE uses the Klimyk formula to generate the entire direct sum decomposition of the tensor product, after which it dispenses with all but the single desired term. However, computing all of the terms in the direct sum decomposition is not feasible when the sizes of the entries in the weights grow into the 100s. On the other hand, (2) lattice point enumeration is often effective for very large weights (in particular, the algorithm is suitable for investigating the stretching properties of Section 4). However, the rank must be relatively low (roughly $r < 6$) because lattice point enumeration complexity grows exponentially in the dimension of the polytope, and the dimensions of these polytopes grow quadratically with the rank of the Lie algebra. Together, the two algorithms cover a larger range of problems.

We would also like to mention that Charles Cochet [Cochet 05] also uses lattice points in polytopes to compute Clebsch–Gordan coefficients. Using the Steinberg formula (see Equation (4–3), Section 4.2), together with techniques developed in [Baldoni et al. 05], he has written software that, like ours, can compute with large sizes of weight entries. Indeed, in the comparison of running

times reported in [Cochet 05], his software seems to compute Clebsch–Gordan coefficients approximately five to ten times more quickly than ours. It would be interesting to determine whether these computation times differ by a constant factor in general and whether this factor is due to the theoretical complexity of the computations or to implementation issues.

Applying the Steinberg formula consists in computing an alternating sum of vector partition functions over $\mathfrak{W} \times \mathfrak{W}$, the Cartesian square of the Weyl group. Since this is a fixed set for fixed rank, and since evaluating each vector partition function in the sum amounts to enumerating the lattice points in a polytope, Cochet's techniques also yield a polynomial-time algorithm for computing Clebsch–Gordan coefficients in fixed rank. However, because applying the Steinberg formula involves computing an alternating sum, the techniques in [Cochet 05] cannot be used to yield the second theoretical result in Theorem 1.1.

3.1 Experiments for Type A_r

In the tables below, we index Littlewood–Richardson coefficients for type A_r with triples of partitions with $r + 1$ parts. Experiments indicate that lattice point enumeration is very efficient for computing Littlewood–Richardson numbers when $r \leq 4$. First, we computed over 30 instances with randomly generated weights with leading entries larger than 40 with our approach and with L_iE. In all cases our algorithm was faster. After that, we did a "worst case" sampling for Table 1 comparing the computation times of LattE and L_iE. To produce the ith row of that table, we selected uniformly at random 1000 triples of weights (λ, μ, ν) in which the largest parts of λ and μ were bounded above by $10i$ and $|\nu| = |\lambda| + |\mu|$ (this is a necessary condition for $c_{\lambda\mu}^{\nu} \neq 0$). Then we evaluated the corresponding hive polytopes with LattE. The LattE input files are created with our Maple program. The weight triple in the ith row is the one that LattE took the longest time to compute. We then computed the same tensor product multiplicity with L_iE. Table 2 shows the running time needed when one uses LattE to compute weight triples with entries in the thousands or millions.

When $r \geq 5$, the running time under LattE begins to blow up. Still, for $r = 5$, all examples we attempted could be computed in under 30 minutes using LattE, and most could be computed in under 5 minutes. For example, among 54 nonempty hive polytopes chosen uniformly at random among those in which the weights had entries less than 100, all but seven could be computed in under

5 minutes with LattE, and the remaining seven could all be computed in under 30 minutes. None of these Littlewood–Richardson coefficients could be computed with L_iE. At $r = 6$, lattice point enumeration becomes less effective, with examples typically taking several hours or more to evaluate.

3.2 Experiments for Types B_r, C_r, and D_r

To compute Clebsch–Gordan coefficients in types B_r, C_r, and D_r, we used the BZ-polytopes. In the tables that follow, all weights are given in the basis of fundamental weights for the corresponding Lie algebra.

Our experiments followed the same process we used for A_r: first, for each root system, we computed over 30 instances with randomly generated weights with entries larger than 40 with our approach and with L_iE. In all cases our algorithm was faster. After that, we did a "worst case" sampling to produce Table 3, comparing the computation times of LattE and L_iE. As in Section 3.1, these weight triples were the ones that LattE took the longest to evaluate among thousands of instances generated with the following procedure: first, to produce line i of a table, we set an upper bound U_i for the entries of each weight. Then, we generated 1000 random-weight triples with entry sizes no larger than U_i. Here are the specific values of U_i used in each of the three subtables in Table 3. For type B_r, the bounds U_i were 50, 60, 70, and 10,000, respectively. For type C_r, the bounds U_i were 50, 60, 80, and 10,000, respectively. Finally, for type D_r, the bounds U_i were 20, 30, 40, and 10,000, respectively. For each generated triple of weights, we produced the associated BZ-polytopes (using our Maple notebook) and counted their lattice points with LattE. Table 3 includes those instances that were slowest in LattE. We also recorded in the table the time taken by L_iE for the same instances. One can see that the running time needed by LattE is hardly affected by growth in the size of the input weights, while the time needed by L_iE grows rapidly.

We found that for types B_r and C_r, lattice point enumeration with the BZ-polytopes is very effective when $r \leq 3$. Each of the many thousands of examples we generated could be evaluated by LattE in under 10 seconds (the examples in Table 3 were the worst cases). When $r = 4$, the computation time begins to blow up, with examples typically taking half an hour or more to compute. The polyhedral method is also reasonably efficient for type-D Lie algebras with rank 4, the lowest rank in which they are defined. All of the examples we generated could be evaluated by LattE in under five minutes.

λ, μ, ν	$c^{\nu}_{\lambda\mu}$	LattE runtime	LiE runtime
(9,7,3,0,0), (9,9,3,2,0), (10,9,9,8,6)	2	0m 00.74s	0m00.01s
(18,11,9,4,2), (20,17,9,4,0), (26,25,19,16,8)	453	0m 03.86s	0m00.12s
(30,24,17,10,2), (27,23,13,8,2), (47,36,33,29,11)	5231	0m 05.21s	0m02.71s
(38,27,14,4,2), (35,26,16,11,2), (58,49,29,26,13)	16784	0m 06.33s	0m25.31s
(47,44,25,12,10), (40,34,25,15,8), (77,68,55,31,29)	5449	0m 04.35s	1m55.83s
(60,35,19,12,10), (60,54,27,25,3), (96,83,61,42,23)	13637	0m 04.32s	23m32.10s
(64,30,27,17,9), (55,48,32,12,4), (84,75,66,49,24)	49307	0m 04.63s	45m52.61s
(73,58,41,21,4), (77,61,46,27,1), (124,117,71,52,45)	557744	0m 07.02s	> 24 hours

TABLE 1. A "worst case" comparison of running times between LattE and LiE for A_4.

λ, μ, ν	$c^{\nu}_{\lambda\mu}$	LattE runtime
(935,639,283,75,48) (921,683,386,136,21) (1529,1142,743,488,225)	1303088213330	0m 07.84s
(6797,5843,4136,2770,707) (6071,5175,4035,1169,135) (10527,9398,8040,5803,3070)	459072901240524338	0m 09.63s
(859647,444276,283294,33686,24714) (482907,437967,280801,79229,26997) (1120207,699019,624861,351784,157647)	1171112000038700713912948714 75	0m 08.15s

TABLE 2. Computing large weights with LattE for case A_4.

	λ, μ, ν	$C^{\mu}_{\lambda\nu}$	LattE runtime	LiE runtime
B_3	(46,42,38), (38,36,42), (41,36,44)	354440672	0m09.58s	1m 45.27s
	(46,42,41), (14,58,17), (50,54,38)	88429965	0m06.38s	3m 16.01s
	(15,60,67), (58,70,52), (57,38,63)	626863031	0m07.14s	6m 01.43s
	(5567,2146,6241), (6932,1819,8227), (3538,4733,3648)	2156768818765690849679	0m 7.07s	n/a
C_3	(25,42,22), (36,38,50), (31,33,48)	87348857	0m 07.48s	0m 17.21s
	(34,56,36), (44,51,49), (37,51,54)	606746767	0m 08.42s	2m 57.27s
	(39,64,58), (65,15,72), (70,41,44)	519379044	0m 07.63s	8m 00.35s
	(5046,5267,7266), (7091,3228,9528), (9655,7698,2728)	1578943284716032240384	0m 07.66s	n/a
D_4	(13,20,10,14), (10,20,13,20), (5,11,15,18)	41336415	2m 46.88s	0m 12.29s
	(12,22,9,30), (28,14,15,26), (10,24,10,26)	322610723	3m 04.31s	7m 03.44s
	(37,16,31,29), (40,18,35,41), (36,27,19,37)	18538329184	4m 29.63s	>60m
	(2883,8198,3874,5423), (1901,9609,889,4288), (5284,9031,2959,5527)	18912932567045743565651493 44	2m 06.42s	n/a

TABLE 3. A "worst case" sample comparison of running times between LattE and LiE.

4. TWO NEW CONJECTURES THAT WOULD GENERALIZE THE SATURATION THEOREM

In 1999, Knutson and Tao used the hive polytopes to prove the *saturation theorem.*

Theorem 4.1. (Saturation.) [Knutson and Tao 99] *Given highest weights λ, μ, and ν for $\mathfrak{sl}_r(\mathbb{C})$, and given an integer $n > 0$, the Littlewood–Richardson coefficient $c^{\nu}_{\lambda\mu}$ satisfies*

$$c^{\nu}_{\lambda\mu} \neq 0 \iff c^{n\nu}_{n\lambda,n\mu} \neq 0.$$

Buch [Buch 00] has written a very readable exposition of Knutson and Tao's proof of this theorem. Several conjectured generalizations of Knutson and Tao's result have appeared in the literature since then. For example, Littlewood–Richardson numbers can be expressed as coefficients of certain polynomials arising in the study of parabolic Kostka polynomials. Kirillov conjectures an extension of the saturation theorem to other coefficients of these polynomials [Kirillov 04]. Another conjecture, by Kapovich and Millson, concerns the degree to which the other classical Lie algebras fail to be saturated

[Kapovich and Millson 04]. This conjecture implies saturation in the type-A case.

We would like to state two additional conjectures that, if true, generalize Theorem 4.1. Our conjectures concern the polyhedral geometry arising in the interpretation provided by Berenstein and Zelevinsky. First, we translate Theorem 4.1 into the language of hive polytopes, where it may be restated as

$$\# \left(H_{\lambda\mu}^{\nu} \cap \mathbb{Z}^d \right) \neq \varnothing \quad \Longleftrightarrow \quad \# \left(H_{n\lambda,n\mu}^{n\nu} \cap \mathbb{Z}^d \right) \neq \varnothing,$$

where $d = \binom{r+2}{2}$. The definition of hive polytopes (see Definition 2.3 above) implies that $H_{n\lambda,n\mu}^{n\nu} = n\, H_{\lambda\mu}^{\nu}$. Since $n\, H_{\lambda\mu}^{\nu}$ clearly contains an integer lattice point for sufficiently large n, it follows that the saturation theorem is equivalent to the statement that every nonempty hive polytope contains an integral lattice point.

4.1 First Conjecture

To show that every hive polytope contains an integral point, Knutson and Tao actually proved that every hive polytope contains an integral *vertex*. Our idea was to take a different approach to prove a generalization of this last result using the theory of triangulations of semigroups. To develop this idea, observe that the boundary equalities and rhombus inequalities that define a hive polytope may be expressed as the set of solutions to a system of matrix equalities and inequalities:

$$H_{\lambda\mu}^{\nu} = \left\{ h \in \mathbb{R}^{(r+1)(r+2)/2} : \begin{array}{l} Bh = b(\lambda,\mu,\nu), \\ Rh \leq 0 \end{array} \right\}, \quad (4\text{--}1)$$

where B and R are integral matrices (depending on r), and $b(\lambda,\mu,\nu)$ is an integral vector depending linearly on λ, μ, and ν. Here we think of a hive pattern h as a column vector of dimension $(r+1)(r+2)/2$. There is some degree of choice in how the boundary equalities and rhombus inequalities are encoded as matrices B and R, respectively, but all such encodings are equivalent for our purposes.

Recall that in our definition of hive polytopes, we required that $\lambda, \mu, \nu \geq 0$. This has the convenient consequence that the hive polytope $H_{\lambda\mu}^{\nu}$ is contained in the nonnegative orthant. Such a polytope defined by a system of equalities and inequalities may be *homogenized* by adding "slack variables" to produce an equivalent polytope defined as the set of nonnegative solutions to a system of linear equations. Following this procedure, we define the *homogenized hive polytope* $\tilde{H}_{\lambda\mu}^{\nu}$ by

$$\tilde{H}_{\lambda\mu}^{\nu} = \left\{ \tilde{h} : \begin{bmatrix} B & 0 \\ R & I \end{bmatrix} \tilde{h} = \begin{bmatrix} b(\lambda,\mu,\nu) \\ 0 \end{bmatrix}, \ \tilde{h} \geq 0 \right\}$$

(where I is an identity matrix). The equivalence between $H_{\lambda\mu}^{\nu}$ and $\tilde{H}_{\lambda\mu}^{\nu}$ is given by the linear map

$$h \mapsto \begin{bmatrix} h \\ -Rh \end{bmatrix}.$$

Note that this linear map preserves vertices and integrality. Therefore, to prove the saturation theorem, it suffices to show that every homogenized hive polytope contains an integral vertex. Proceeding with this idea, we make the following definitions.

Definition 4.2. Fix $r \in \mathbb{Z}$. Define the *homogenized hive matrix* to be

$$M = \begin{bmatrix} B & 0 \\ R & I \end{bmatrix}$$

(where B and R are as in equation (4–1)). Given an integral vector b with dimension equal to the number of rows in M, define the *generalized hive polytope* or *g-hive polytope* H_b by

$$H_b = \left\{ \tilde{h} : M\tilde{h} = b, \ \tilde{h} \geq 0 \right\}. \quad (4\text{--}2)$$

Note that the homogenized hive polytopes are g-hive polytopes that satisfy certain additional conditions on the right-hand-side vector b, such as its final entries being 0.

We now state some very basic facts about vertices of polyhedra expressed in the form $\{x : Ax = b, \ x \geq 0\}$. Let a finite collection of integral points $\{a_1, \ldots, a_d\} \subset \mathbb{Z}^m$ be given, and let A be the matrix with columns $a_1, \ldots, a_d$. Define $\operatorname{cone} A$ to be the cone in $\mathbb{R}^m$ generated by the point set $\{a_1, \ldots, a_d\}$:

$$\operatorname{cone} A = \{x_1 a_1 + \cdots + x_d a_d : x_1, \ldots, x_d \geq 0\}.$$

Then, for each vector $b \in \mathbb{Z}^m$, we have a polytope

$$P_b = \{x : Ax = b, \ x \geq 0\} \subset \mathbb{R}^d,$$

and $P_b \neq \varnothing$ if and only if $b \in \operatorname{cone} A$. In other words, there is a correspondence between nonempty polytopes P_b, $b \in \mathbb{Z}^m$, and the elements of the semigroup of integral lattice points contained in the cone generated by the columns of A. The crucial property for our purposes is the following.

Lemma 4.3. *If $b \in (\operatorname{cone} A') \cap \mathbb{Z}^m$ for some $m \times m$ submatrix A' of A with $\det A' = \pm 1$, then P_b has an integral vertex.*

Proof: Suppose that $b \in (\text{cone } A') \cap \mathbb{Z}^m$ for some $m \times m$ submatrix A' of A with $\det A' = \pm 1$. Let the columns of A' be $a_{i_1}, \ldots, a_{i_m}$, and let $J = \{i_1, \ldots, i_m\}$ be the indices of these columns. Then there is a vector $x = (x_1, \ldots, x_d)^T \in \mathbb{R}^d_{\geq 0}$ such that $Ax = b$ and $x_i = 0$ for each $i \notin J$. Letting $x' = (x_{i_1}, \ldots, x_{i_m})^T$ and using Cramer's rule to solve for x' in $A'x' = b$, we find that x is an integral vector. Thus, x is an integral lattice point in the polytope P_b.

To see that x is in fact a vertex of P_b, recall that the codimension (with respect to the ambient space) of the face containing a solution to the system $Ax = b$, $x \geq 0$, of linear equalities and inequalities is the number of linearly independent equalities or inequalities satisfied with *equality*. Observe that x is a solution to the system of d equalities

$$\begin{cases} Ax = b, \\ x_i = 0, \quad i \notin J. \end{cases}$$

We claim that this is a linearly independent system of equalities. Suppose otherwise; then the zero row vector is a nontrivial linear combination of the rows of A and the row vectors $(0, \ldots, 0, 1, 0, \ldots, 0)$, where the 1 is in the ith position, $i \notin J$. But this implies that the zero row vector is a nontrivial linear combination of the rows of A', which is impossible because $\det A' \neq 0$.

Thus, having shown that x satisfies the d linearly independent equalities above, we have shown that x lies in a codimension-d face of P_b; that is, x is a vertex. $\square$

We say that $a_{i_1}, \ldots, a_{i_m}$ is a *unimodular* subset if the submatrix A' of A with columns $a_{i_1}, \ldots, a_{i_m}$ satisfies $\det A' = \pm 1$. We say that the matrix A has a *unimodular cover* (respectively *unimodular triangulation*) if the point set $\{a_1, \ldots, a_d\}$ has a unimodular cover (respectively unimodular triangulation).

Corollary 4.4. *If A has a unimodular cover, then P_b has an integral vertex for every integral $b \in \text{cone}(A)$.*

Our conjecture is that this corollary applies to the homogenized hive matrix. More precisely, we conjecture the following.

Conjecture 4.5. *The homogenized hive matrix has a unimodular triangulation. Consequently, every g-hive polytope has an integral vertex.*

Since the hive polytopes are special cases of the g-hive polytopes, Conjecture 4.5 generalizes the saturation theorem.

Theorem 4.6. *Conjecture 4.5 is true for $r \leq 6$.*

To compute the unimodular triangulations that provide a proof of Theorem 4.6 we used the software topcom [Rambau 02]. It may be worth noting that the triangulations used to prove Theorem 4.6 were all placing triangulations.

4.2 Second Conjecture

For our second conjecture, we looked at general properties satisfied by Clebsch–Gordan coefficients for semisimple Lie algebras of types B_r, C_r, and D_r under the operation of *stretching of multiplicities*. By *stretching of multiplicities*, we refer to the function $e \colon \mathbb{Z}_{>0} \to Z_{\geq 0}$ defined by $e(n) = C^{n\nu}_{n\lambda, n\mu}$.

The BZ-polytopes are defined as the set of solutions to a system of linear equalities and inequalities $Ax \leq b$, $Cx = d$, where b and d are linear functions of λ, μ, and ν with rational coefficients [Berenstein and Zelevinsky 01]. It follows that given any highest weights λ, μ, and ν of a semisimple Lie algebra, $e(n)$ is a quasipolynomial in n. Indeed, $e(n)$ is, in polyhedral language, the *Ehrhart quasipolynomial* of the corresponding BZ-polytope. We recall the basic theory of Ehrhart quasipolynomials. Its origins can be traced to the work of Ehrhart [Ehrhart 77] in the 1960s (see [Stanley 97, Chapter 4] for an excellent introduction).

Given a convex polytope $P \subset \mathbb{R}^k$, let $nP = \{x : (1/n)x \in P\}$ for $n = 1, 2, \ldots$. If P is a d-dimensional rational polytope, then the counting function $i_P(n) = \#(nP \cap \mathbb{Z}^k)$ is a quasipolynomial function of degree d; that is, there are polynomials $f_1(n), \ldots, f_N(n)$ such that each $(f_j(n))$ is either the zero polynomial or is of degree d, and

$$i_P(n) = \begin{cases} f_1(n) & \text{if } n \equiv 1 \mod N, \\ \cdots \\ f_N(n) & \text{if } n \equiv N \mod N. \end{cases}$$

If $i_P(n)$ can be expressed in terms of N polynomials in this fashion, we say that N is a *quasiperiod* of $i_P(n)$. We do not assume that N is the minimum such number.

If we put $P = H^\nu_{\lambda\mu}$, then the Ehrhart quasipolynomial of P is just the stretched Littlewood–Richardson coefficient $c^{n\nu}_{n\lambda, n\mu}$. The Ehrhart quasipolynomials of hive polytopes have been studied by several authors. Since lattice point enumeration can compute with large weights, it is possible to produce the Ehrhart quasipolynomials for the stretched Clebsch–Gordan coefficients in the other types. See Tables 4–6 for some examples out of the many hundreds generated.

λ, μ, ν	$C^{n\mu}_{n\lambda,n\nu}$	LattE runtime
(0, 15, 5) (12, 15, 3) (6, 15, 6)	$\begin{cases} \frac{68339}{64}n^5 + \frac{407513}{384}n^4 + \frac{13405}{32}n^3 + \frac{9499}{96}n^2 + \frac{107}{8}n + 1, \ n \text{ even} \\ \frac{68339}{64}n^5 + \frac{407513}{384}n^4 + \frac{13405}{32}n^3 + \frac{16355}{192}n^2 + \frac{659}{64}n + \frac{75}{128}, \ n \text{ odd} \end{cases}$	7m08.57s
(4, 8, 11) (3, 15, 10) (10, 1, 3)	$\begin{cases} \frac{13}{4}n^2 + 3n + 1, \ n \text{ even} \\ \frac{13}{4}n^2 + 3n + 3/4, \ n \text{ odd} \end{cases}$	0m01.64s
(8, 1, 3) (11, 13, 3) (8, 6, 14)	$\begin{cases} \frac{121}{576}n^6 + \frac{1129}{640}n^5 + \frac{6809}{1152}n^4 + \frac{163}{16}n^3 + \frac{2771}{288}n^2 + \frac{191}{40}n + 1, \ n \text{ even} \\ \frac{121}{576}n^6 + \frac{1129}{640}n^5 + \frac{6809}{1152}n^4 + \frac{1933}{192}n^3 + \frac{659}{72}n^2 + \frac{8003}{1920}n + \frac{93}{128}, \ n \text{ odd} \end{cases}$	0m10.26s
(8, 9, 14) (8, 4, 5) (1, 5, 15)	$\begin{cases} \frac{4117}{192}n^6 + \frac{50369}{640}n^5 + \frac{14829}{128}n^4 + \frac{703}{8}n^3 + \frac{3541}{96}n^2 + \frac{341}{40}n + 1, \ n \text{ even} \\ \frac{4117}{192}n^6 + \frac{50369}{640}n^5 + \frac{14829}{128}n^4 + \frac{5599}{64}n^3 + \frac{3451}{96}n^2 + \frac{5001}{640}n + \frac{97}{128}, \ n \text{ odd} \end{cases}$	0m13.29s
(10, 5, 6) (5, 4, 10) (0, 7, 12)	$\begin{cases} \frac{669989}{960}n^5 + \frac{286355}{384}n^4 + \frac{10803}{32}n^3 + \frac{7993}{96}n^2 + \frac{1427}{120}n + 1, \ n \text{ even} \\ \frac{669989}{960}n^5 + \frac{286355}{384}n^4 + \frac{10803}{32}n^3 + \frac{15509}{192}n^2 + \frac{10081}{960}n + \frac{65}{128}, \ n \text{ odd} \end{cases}$	2m52.39s

TABLE 4. Ehrhart quasipolynomials for BZ-polytopes of type B_3.

λ, μ, ν	$C^{n\mu}_{n\lambda,n\nu}$	LattE runtime
(1,13,6) (14,15,5) (5,11,7)	$\begin{cases} \frac{5937739}{5760}n^6 + \frac{87023}{40}n^5 + \frac{936097}{576}n^4 + \frac{27961}{48}n^3 + \frac{85397}{720}n^2 + \frac{883}{60}n + 1, \ n \text{ even} \\ \frac{5937739}{5760}n^6 + \frac{87023}{40}n^5 + \frac{936097}{576}n^4 + \frac{27961}{48}n^3 + \frac{657931}{5760}n^2 + \frac{3097}{240}n + 3/4, \ n \text{ odd} \end{cases}$	21m20.59s
(4,15,14) (12,12,10) (4,9,8)	$\begin{cases} \frac{22199219}{2880}n^6 + \frac{8154617}{960}n^5 + \frac{4500665}{1152}n^4 + \frac{31297}{32}n^3 + \frac{226903}{1440}n^2 + \frac{2021}{120}n + 1, \ n \text{ even} \\ \frac{22199219}{2880}n^6 + \frac{8154617}{960}n^5 + \frac{4500665}{1152}n^4 + \frac{31297}{32}n^3 + \frac{217363}{1440}n^2 + \frac{13513}{960}n + \frac{85}{128}, \ n \text{ odd} \end{cases}$	17m05.74s
(9,0,8) (8,12,9) (7,7,3)	$\frac{1}{30}n^5 + \frac{3}{8}n^4 + \frac{19}{12}n^3 + \frac{25}{8}n^2 + \frac{173}{60}n + 1$	0m00.61s
(10,2,7) (8,10,1) (7,5,5)	$\begin{cases} \frac{596153}{1152}n^6 + \frac{53425}{48}n^5 + \frac{502621}{576}n^4 + \frac{5577}{16}n^3 + \frac{11941}{144}n^2 + \frac{149}{12}n + 1, \ n \text{ even} \\ \frac{596153}{1152}n^6 + \frac{53425}{48}n^5 + \frac{502621}{576}n^4 + \frac{5577}{16}n^3 + \frac{94097}{1152}n^2 + \frac{131}{12}n + \frac{23}{32}, \ n \text{ odd} \end{cases}$	19m24.55s
(10,10,15) (11,3,15) (10,7,15)	$\begin{cases} \frac{6084163}{320}n^6 + \frac{507527}{30}n^5 + \frac{1185853}{192}n^4 + \frac{59995}{48}n^3 + \frac{43039}{240}n^2 + \frac{357}{20}n + 1, \ n \text{ even} \\ \frac{6084163}{320}n^6 + \frac{507527}{30}n^5 + \frac{1185853}{192}n^4 + \frac{59995}{48}n^3 + \frac{144751}{960}n^2 + \frac{883}{80}n + \frac{25}{64}, \ n \text{ odd} \end{cases}$	16m05.08s

TABLE 5. Ehrhart quasipolynomials for BZ-polytopes of type C_3.

The reader will observe that each of the quasipolynomials in Tables 4–6 have quasiperiod 2. We now show that this property holds in general for each of the classical Lie algebras. Derksen and Weyman [Derksen and Weyman 02] have already shown that the stretched Littlewood–Richardson coefficients are polynomials, so it remains only to consider the root systems B_r, C_r, and D_r.

Proof of Proposition 1.2: Let $\mathfrak{g}$ be a classical Lie algebra of type B_r, C_r, or D_r. Since we already know that $C^{n\nu}_{n\lambda,n\mu}$ is a quasipolynomial in n, it will suffice to show that, for all sufficiently large n, $C^{n\nu}_{n\lambda,n\mu}$ has quasiperiod 2. Once this is established, we can interpolate to show that the claim holds for all values of n.

Let Λ be the weight lattice of $\mathfrak{g}$ and let $N_{\mathfrak{g}}(b)$ denote the number of ways to write a vector $b \in \Lambda$ as an integral linear combination of the positive roots of $\mathfrak{g}$. The function $N_{\mathfrak{g}}$ is a *vector partition function*; that is, its support is contained in a union of polyhedral cones in $\Lambda \otimes \mathbb{R}$, called *chambers*, such that the restriction of $N_{\mathfrak{g}}(b)$ to the lattice points in any chamber is a multivariate quasipolynomial function of the coordinates of b [Sturmfels 95].

The Clebsch–Gordan coefficients can be expressed in terms of the vector partition function $N_{\mathfrak{g}}$ with the Steinberg multiplicity formula (see, e.g., [Fulton and Harris 91]), according to which

$$C^{n\nu}_{n\lambda,n\mu} = \sum (-1)^{\sigma(ww')} N_{\mathfrak{g}}\big(w(n\lambda + \rho) + w'(n\mu + \rho) - (n\nu + 2\rho)\big), \quad (4\text{–}3)$$

where $\mathfrak{W}$ is the Weyl group of $\mathfrak{g}$, $\sigma(w)$ is the sign of w in $\mathfrak{W}$, and the sum is over $(w, w') \in \mathfrak{W} \times \mathfrak{W}$.

λ, μ, ν	$C^{n\mu}_{n\lambda, n\nu}$	LattE runtime
(0, 2, 10, 5) (4, 11, 9, 11) (5, 8, 6, 9)	$\begin{cases} \frac{625007}{10080} n^7 + \frac{729157}{2880} n^6 + \frac{77197}{180} n^5 + \frac{449539}{1152} n^4 + \frac{298979}{1440} n^3 + \frac{95189}{1440} n^2 \\ \quad + \frac{10079}{840} n + 1, \ n \text{ even} \\ \frac{625007}{10080} n^7 + \frac{729157}{2880} n^6 + \frac{77197}{180} n^5 + \frac{449539}{1152} n^4 + \frac{298979}{1440} n^3 + \frac{95189}{1440} n^2 \\ \quad + \frac{10079}{840} n + \frac{127}{128}, \ n \text{ odd} \end{cases}$	20m24.79s
(2, 7, 12, 2) (11, 10, 5, 9) (13, 11, 1, 1)	$\begin{cases} \frac{34675903}{80640} n^8 + \frac{3037051}{1680} n^7 + \frac{9121453}{2880} n^6 + \frac{241181}{80} n^5 + \frac{615083}{360} n^4 + \frac{8947}{15} n^3 \\ \quad + \frac{107791}{840} n^2 + \frac{6721}{420} n + 1 \ n \text{ even} \\ \frac{34675903}{80640} n^8 + \frac{3037051}{1680} n^7 + \frac{9121453}{2880} n^6 + \frac{241181}{80} n^5 + \frac{615083}{360} n^4 + \frac{8947}{15} n^3 \\ \quad + \frac{107791}{840} n^2 + \frac{6721}{420} n + \frac{239}{256} \ n \text{ odd} \end{cases}$	123m59.76s
(3, 11, 0, 10) (2, 15, 10, 15) (10, 12, 11, 0)	$\begin{cases} \frac{53609}{60} n^6 + \frac{25631}{15} n^5 + \frac{63779}{48} n^4 + \frac{1627}{3} n^3 \\ \quad + \frac{2497}{20} n^2 + \frac{239}{15} n + 1, \ n \text{ even} \\ \frac{53609}{60} n^6 + \frac{25631}{15} n^5 + \frac{63779}{48} n^4 + \frac{1627}{3} n^3 \\ \quad + \frac{2497}{20} n^2 + \frac{239}{15} n + \frac{15}{16}, \ n \text{ odd} \end{cases}$	2m37.73s
(10, 1, 12, 4) (1, 12, 0, 3) (0, 5, 3, 4)	$5 n^2 + 4 n + 1$	0m01.63s
(12, 2, 5, 13) (15, 6, 10, 11) (2, 0, 12, 13)	$\begin{cases} \frac{455263}{2016} n^7 + \frac{447281}{576} n^6 + \frac{198433}{180} n^5 + \frac{971011}{1152} n^4 + \frac{108787}{288} n^3 \\ \quad + \frac{28969}{288} n^2 + \frac{12631}{840} n + 1, \ n \text{ even} \\ \frac{455263}{2016} n^7 + \frac{447281}{576} n^6 + \frac{198433}{180} n^5 + \frac{971011}{1152} n^4 + \frac{108787}{288} n^3 \\ \quad + \frac{28969}{288} n^2 + \frac{12631}{840} n + \frac{127}{128}, \ n \text{ odd} \end{cases}$	4m25.90s

TABLE 6. Ehrhart quasipolynomials for BZ-polytopes of type D_3.

To prove that $C^{n\nu}_{n\lambda, n\mu}$ is a quasiperiod-2 quasipolynomial of n, it will suffice to show that each term in the sum on the right-hand side of (4–3) is a quasiperiod-2 quasipolynomial of n. The key fact from which this will follow is Corollary 3.6 in [Baldoni et al. 05], which states that, in each of its chambers, $N_{\mathfrak{g}}(b)$ is a multivariate quasipolynomial function of b with quasiperiod 2. Thus, we need only show that for all sufficiently large n, the vectors

$$b_n = w(n\lambda + \rho) + w'(n\mu + \rho) - (n\nu + 2\rho)$$

remain in a single chamber.

To see this, note that as n increases, the direction of b_n approaches that of $b' = w(\lambda) + w'(\mu) - \nu$ along a straight line. Hence, for all n sufficiently large, b_n and b' share a chamber, so that the value of $N_{\mathfrak{g}}(b_n)$ is given by a single quasipolynomial function of the coordinates of b_n. Because we are looking at a particular term in the right side of (4–3), w, w', λ, μ, and ν are all fixed, so $N_{\mathfrak{g}}(b_n)$ reduces to a quasipolynomial of n. This reduction does not increase the quasiperiod, so we have shown that the right-hand side of (4–3) is a sum of quasiperiod-2 quasipolynomials when n is sufficiently large. Thus it follows that $C^{n\nu}_{n\lambda, n\mu}$ is also a quasipolynomial of quasiperiod 2. □

Our experiments also motivate the following "stretching conjecture," which generalizes the saturation theorem.

Conjecture 4.7. (Stretching conjecture.) *Given highest weights* λ, μ, ν *of a Lie algebra of type* A_r, B_r, C_r, *or* D_r, *let*

$$C^{n\nu}_{n\lambda, n\mu} = \begin{cases} f_1(n) & \text{if } n \text{ is odd,} \\ f_2(n) & \text{if } n \text{ is even} \end{cases}$$

be the quasipolynomial representation of the stretched Clebsch–Gordan coefficient $C^{n\nu}_{n\lambda, n\mu}$. *Then the coefficients of* f_1 *and* f_2 *are all nonnegative.*

The type-A_r case of this conjecture was made by King, Tollu, and Toumazet in [King et al. 04]. That Conjecture 4.7 implies the saturation theorem follows from a result of Derksen and Weyman [Derksen and Weyman 02] showing that the Ehrhart quasipolynomials of hive polytopes are in fact just polynomials.

We should remark that the saturation property is known not to hold in the root systems B_r, C_r, and D_r. A simple example in B_2, due to Kapovich, Leeb, and Millson [Kapovich et al. 02], is given by setting $\lambda = \mu = \nu = (1, 0)$ (with respect to the basis of fundamental weights).

In this case we have

$$C_{n\lambda,n\mu}^{n\nu} = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ 1 & \text{if } n \text{ is even.} \end{cases}$$

This example also demonstrates why the stretching conjecture is not contradicted by the failure of the saturation property in the root systems B_r, C_r, or D_r. Since the stretched multiplicities are not necessarily polynomials in these cases, it is possible for them to evaluate to zero for some nonnegative integer while still having all nonnegative coefficients.

ACKNOWLEDGMENTS

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