On the First Two Vassiliev Invariants

Simon Willerton

CONTENTS

2000 AMS Subject Classification: Primary 57M27

Keywords: Vassiliev invariants, knots, fish

The values that the first two Vassiliev invariants take on prime knots with up to fourteen crossings are considered. This leads to interesting fish-like graphs. Several results about the values taken on torus knots are proved.

'First the fish must be caught.'
That is easy: a baby, I think, could have caught it.

— The Red Queen, *Through the Looking Glass.*

1. INTRODUCTION

The two simplest nontrivial Vassiliev knot invariants (see [Vassiliev 92, Birman and Lin 93]) are of type two and type three. These invariants have been studied from various angles: for instance, combinatorial formulæ for evaluating them have been derived, and simple bounds in terms of crossing number have been obtained (see e.g., [Polyak and Viro 94, Lannes 93, Willerton 97]). In this work, the invariants are examined from the novel perspective of the actual values that they take on knots of small crossing number. For instance, one can ask how accurate the known bounds are, as in Section 2. When looking at this question I plotted the values of these invariants which revealed the interesting "fish" plots in Section 3: these pictures form the focus of this paper. Various questions arising from these graphs can be answered for torus knots (see Section 4). Section 5 presents some problems and further questions.

2. ON v_2 AND v_3.

The space of additive invariants of type three is two-dimensional. By "the first two Vassiliev invariants," we mean the elements of a basis $\{v_2, v_3\}$ of this space. The invariants v_2 and v_3 can be defined canonically in the following fashion. The space of additive invariants of type three splits into the direct sum of type three invariants which do not distinguish mirror image knots and the type three invariants which differ by a factor of minus one on

Crossing number	3	4	5	6	7	8	9	10	11	12
Maximum $\lvert v_2\rvert$	1	1	3	2	6	5	10	9	15	14
Bound on $\lvert v_2\rvert$	1.5	2	5	7.5	11.5	14	18	22.5	27.5	33
Maximum $\lvert v_3\rvert$	1	0	5	1	14	10	30	25	55	49
Bound on $\lvert v_3\rvert$	1.5	6	15	30	57.5	84	126	180	247.5	330

TABLE 1. Comparing actual maxima and minima of $|v_2|$ and $|v_3|$ with the bounds of Section 2.

mirror image knots. Pick the vector in each of these one-dimensional spaces that takes the value one on the positive trefoil. The one which is invariant under taking mirror images is of type two and will be denoted v_2; the other will be denoted v_3.

The invariant v_2 has appeared in various guises previously in knot theory: it is the coefficient of z^2 in the Conway polynomial, and its reduction modulo two is the Arf invariant. Both v_2 and v_3 can be obtained from the Jones polynomial in the following fashion. If $J(q)$ is the Jones polynomial of a knot K, and $J^{(n)}(q)$ denotes the n^{th} derivative with respective to q, then

$$v_2(K) = -\tfrac{1}{6}J^{(2)}(1), \quad v_3(K) = -\tfrac{1}{36}\left(J^{(3)}(1) + 3J^{(2)}(1)\right).$$

Combinatorial formulæ for v_2 and v_3 can be given in terms of Gauß diagram formulæ—the reader is referred to [Polyak and Viro 94, Willerton 97]. From the combinatorial formulæ, it is straightforward to obtain simple bounds for v_2 and v_3 in terms of the crossing number, c, of the knot, K: namely,

$$|v_2(K)| \leq \tfrac{1}{4}c(c-1), \quad |v_3(K)| \leq \tfrac{1}{4}c(c-1)(c-2).$$

The first of these bounds was obtained by Lin and Wang [Lin and Wang 96] and led Bar-Natan [Bar-Natan 95] to prove that any type n invariant is bounded by a degree n polynomial in the crossing number—this also follows from Stanford's algorithm [Stanford 97] for calculating Vassiliev invariants. The bound for v_3 was obtained in [Willerton 97] by utilizing Domergue and Donato's integration [Domergue and Donato 96] of a type three weight system.

It is natural to ask how sharp these bounds are, and it is this question that motivated this work. Stanford has calculated Vassiliev invariants up to order six for the prime knots up to ten crossings; the programs and data files of these calculations are available as [Stanford 92]. Thistlethwaite calculated various polynomials for knots up to 15 crossings, which are available in the `knotscape` program [Hoste and Thistlethwaite 99]. Using these data, one can compare the bounds on $|v_2|$ and $|v_3|$ given above, with the actual maximum attained for each crossing number—this comparison is made in Table 1. It is seen, that in this range of crossing numbers, the bounds are not particularly tight.

By looking at the raw data, one can see that, in this range, for odd crossing number $(2b+1)$, the maximum is achieved precisely by the $(2, 2b+1)$-torus knot, and that this dominates the v_2 and v_3 of the $(2b+2)$-crossing knots as well. Letting $T(p,q)$ be the knot type of the (p,q)-torus knot, Alvarez and Labastida [Alvarez and Labastida 96] (see also Section 4 below) give explicitly for crossing number $c = 2b+1$,

$$v_2\left(T\left(2,c\right)\right) = (c^2-1)/8, \quad v_3\left(T\left(2,c\right)\right) = c(c^2-1)/24.$$

One could conjecture that these give bounds on v_2 and v_3. After an earlier version of this paper, Polyak and Viro [Polyak and Viro 01] showed that for a knot with c crossings, $v_2 \leq c^2/8$.

3. PLOTS FOR KNOTS WITH UP TO 14 CROSSINGS

Having stared at Stanford's raw data long enough to start noticing patterns, I was led to plot v_2 against v_3 for knots of each crossing number up to crossing number 14. These plots are contained in Figure 1 and Figure 2. The symmetry in the v_2-axis is expected, as this is just the effect of taking the mirror image of the knots. However, the "fish" shape of these plots is not expected! This shape suggests some bound of the form

$$\text{cubic in } v_2(K) \leq (v_3(K))^2 \leq \text{another cubic in } v_2(K).$$

Such bounds, independent of crossing number do, in fact, exist for torus knots, as will be seen below. However, this cannot be the case in general (unless the bounds depend on the crossing number); we give two reasons.

First, consider the sequence of Whitehead doubles of the unknot, $\{Wh(i)\}_{i\in\mathbb{Z}}$ (see Figure 3). Table 2 gives the value of v_2 and v_3 on these for a range of i. It follows from the theorem of Dean [Dean 94] and Trapp [Trapp 94] on twist sequences that a type n invariant evaluated on the Whitehead doubles is a polynomial in i of degree at most[1] n. A glance at Table 2 shows that $v_2\left(Wh\left(i\right)\right) = i$

[1] In this case, Lin observed that it must be of degree at most $n-1$.

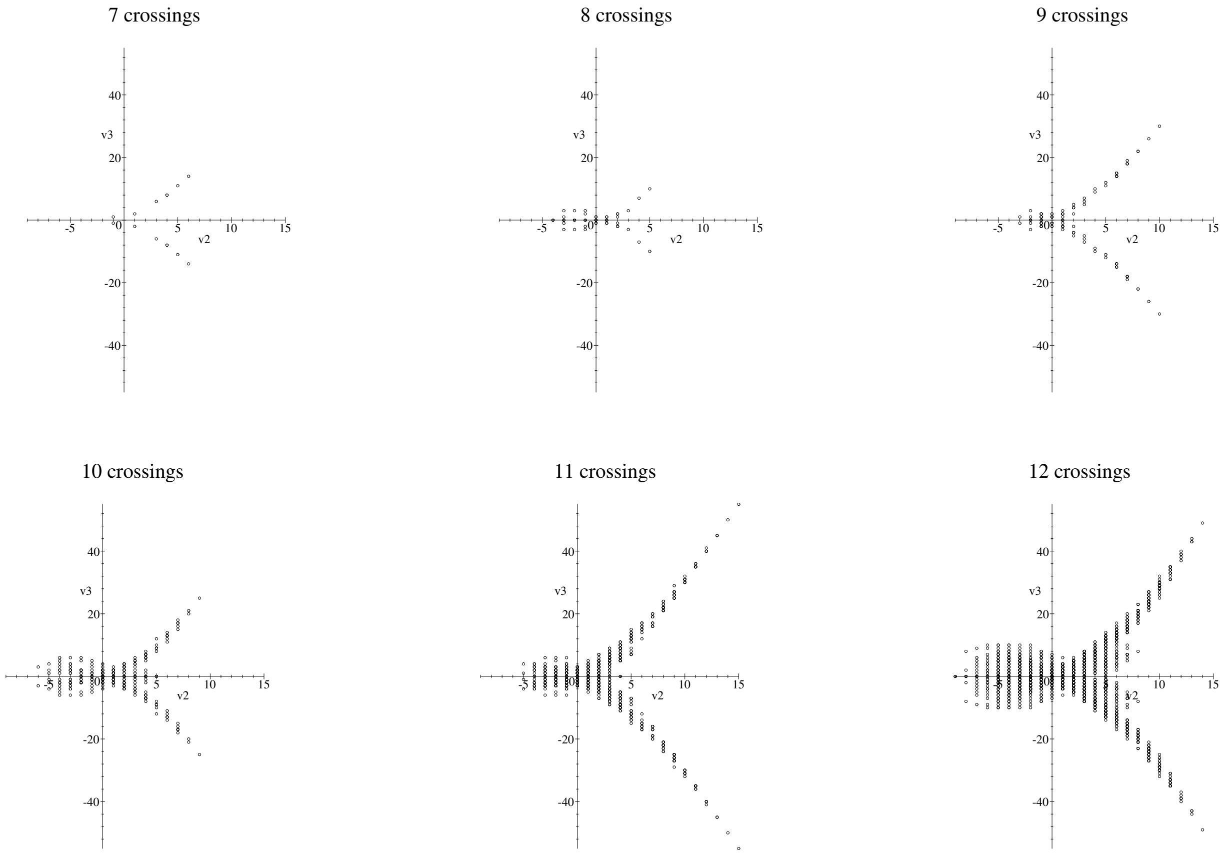

FIGURE 1. Plots by crossing number of v_2 and v_3 for the prime knots up to 12 crossings.

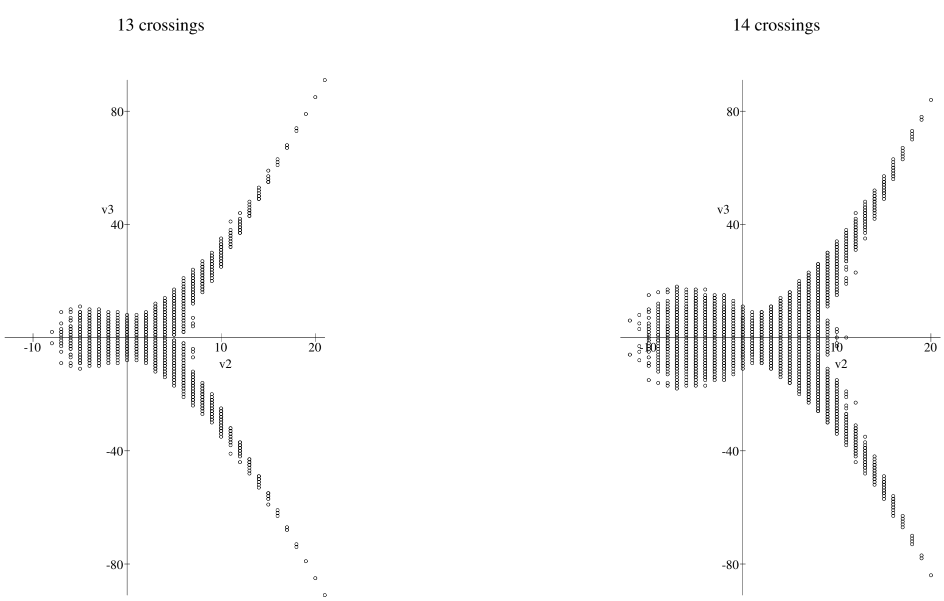

FIGURE 2. Plots by crossing number of v_2 and v_3 for the prime knots with 13 and 14 crossings.

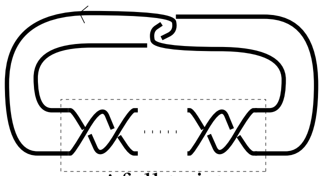

FIGURE 3. The ith twisted Whitehead double of the unknot, $Wh(i)$. For i negative, i full twists means $-i$ negative twists.

and $v_3\left(Wh\left(i\right)\right) = \frac{1}{2}i(i+1)$. Thus, there is a sequence of knots (all except the unknot having unknotting number equal to one) that maps into the (v_2, v_3)-plane as a nice quadratic. This contradicts any bounds of the above form.

Second, for any $(a,b) \in \mathbb{Z}^2$ one can obtain a prime (alternating) knot with (v_2, v_3) equal to (a,b) in the following manner: connect some suitably many positive and negative trefoil knots (with $(v_2, v_3) = (1, \pm 1)$) and figure eight knots (with $(v_2, v_3) = (-1, 0)$), to obtain a composite knot with $(v_2, v_3) = (a,b)$; then Stanford [Stanford 96] gives a method for constructing a prime knot with the same v_2 and v_3.

There does appear to be a qualitative difference between the pictures for odd and even crossing numbers in Figures 1 and 2. The even crossing number ones seem to be more concentrated in the 'body' of the 'fish' and the odd ones more in the 'tail'.

Note that for each odd crossing number, c, there is the $(2,c)$-torus knot and the Whitehead double $Wh((c-1)/2)$ with a (v_2, v_3) of $\left((c-1)/2, \left(c^2-1\right)/8\right)$; and for even crossing number, c, there is the Whitehead double $Wh(1-c/2)$ with a (v_2, v_3) of $(1-c/2, (c-2)\,c/8)$.

Also for up to 12 crossings the amphicheiral knots—that is those equivalent to their mirror image, and hence with $v_3 = 0$—all have an even crossing number, but this is not true in general, as the 15 crossing knot 15_{224980} is amphicheiral.

i	-3	-2	-1	0	1	2	3	4
$Wh(i)$	8_1	6_1	4_1	0_1	3_1	5_2	7_2	9_2
$v_2\left(Wh(i)\right)$	-3	-2	-1	0	1	2	3	4
$v_3\left(Wh(i)\right)$	3	1	0	0	1	3	6	10

TABLE 2. The values of v_2 and v_3 on the twisted Whitehead doubles of the unknot. The knot notation, e.g. 3_1, refers to Alexander-Briggs notation (see [Burde and Zieschang 85]).

4. TORUS KNOTS

The purpose of this section is to show that the torus knots map into the (v_2, v_3)-plane in a nice manner. In particular, they satisfy cubic bounds of the form described above, implying that they lie on the tails of the fish; further, torus knots of the same unknotting number, or crossing number, lie on nice curves in the (v_2, v_3)-plane. The results of this section are summarized diagrammatically in Figure 4.

For p and q coprime, let $T(p,q)$ be the knot type of the (p,q)-torus knot. Then $T(p,q)$ is the unknot if and only if p or q is ± 1, and for $T(p,q)$ nontrivial, $T(p,q)$ is the same knot as $T(p',q')$ if and only if (p',q') equals one of the following: (p,q), (q,p), $(-p,-q)$, or $(-q,-p)$. Further, $T(p,-q)$ is the mirror image of $T(p,q)$. See [Burde and Zieschang 85].

The key to this section is the following pair of formulæ of Alvarez and Labastida [Alvarez and Labastida 96]:

$$v_2(T(p,q)) = \tfrac{1}{24}(p^2-1)(q^2-1),$$
$$v_3(T(p,q)) = \tfrac{1}{144}pq(p^2-1)(q^2-1).$$

Note that these have the required properties under the symmetries of p and q mentioned above, and that these are integer valued on torus knots (i.e., when p and q are coprime). Also, $T \mapsto (v_2(T), v_3(T))$ is injective for torus knots, i.e., torus knots are determined by their (v_2, v_3).

4.1 Cubic Bounds

With the above formulæ of Alvarez and Labastida it is straightforward to prove bounds for torus knots of the form suggested in the last section.

Proposition 4.1. *If T is a torus knot then*

$$\tfrac{2}{3}v_2(T)^3 + \tfrac{1}{3}v_2(T)^2 \le v_3(T)^2 \le \tfrac{8}{9}v_2(T)^3 + \tfrac{1}{9}v_2(T)^2.$$

Further, the righthand bound is tight in the sense that there exist torus knots with arbitrarily large v_2 and v_3 such that equality holds.

Proof: Suppose that T is a (p,q)-torus knot. Then

$$\begin{aligned} v_3(T)^2 - \tfrac{2}{3}v_2(T)^3 &= \left(\tfrac{1}{144}pq(p^2-1)(q^2-1)\right)^2 \\ &\quad - \tfrac{2}{3}\left(\tfrac{1}{24}(p^2-1)(q^2-1)\right)^3 \\ &= \tfrac{1}{12^4}(p^2-1)^2(q^2-1)^2[p^2+q^2-1] \\ &\ge \tfrac{1}{12^3}(p^2-1)^2(q^2-1)^2 \\ &\qquad\qquad \text{as } p^2+q^2 \ge 13 \\ &= \tfrac{1}{3}v_2(T)^2. \end{aligned}$$

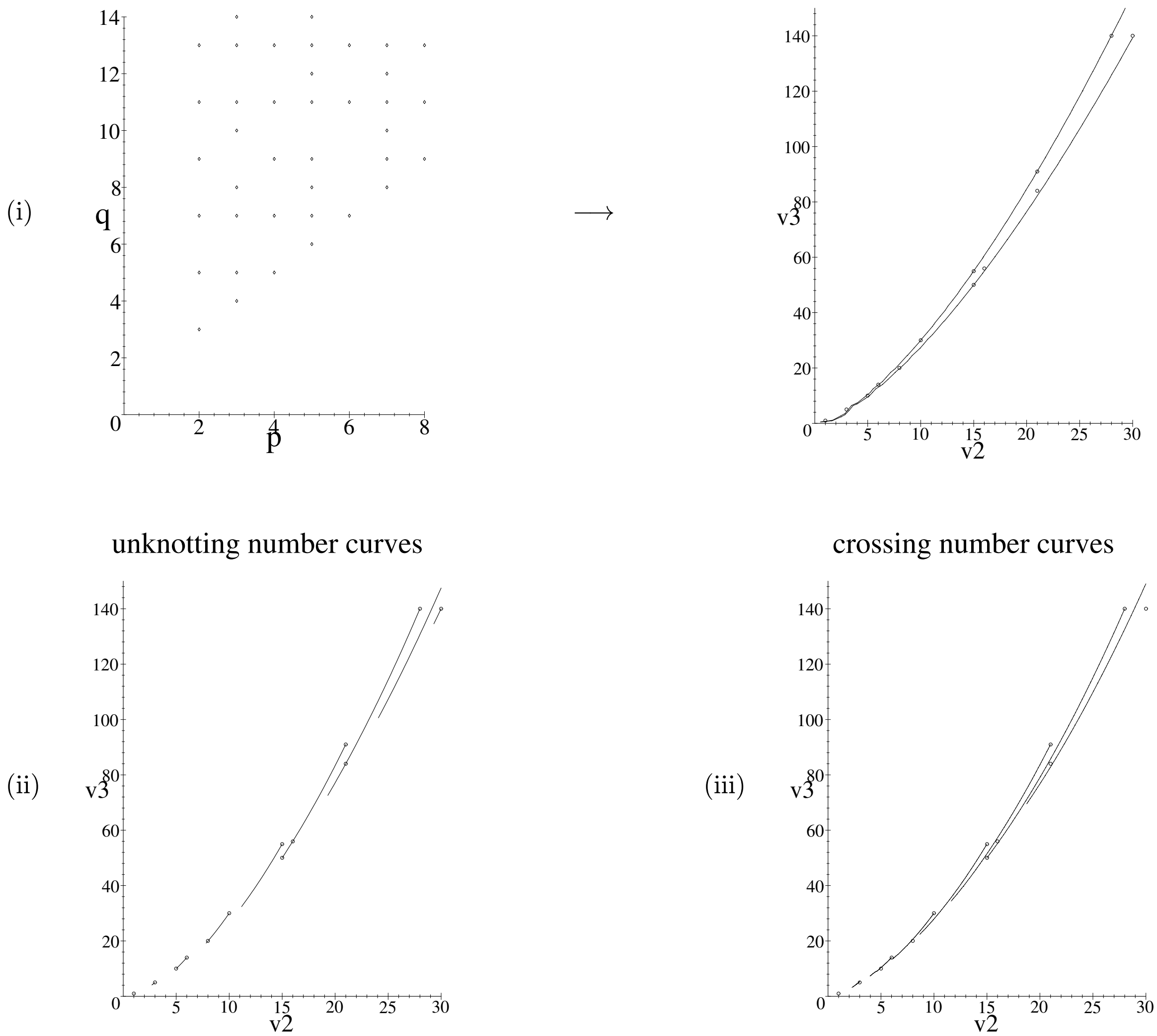

FIGURE 4. Torus knots in the (v_2, v_3)-plane: (i) mapping torus knots from the (p, q)-plane into the region of the (v_2, v_3)-plane given by Propositions 4.1 and 4.2; (ii) torus unknotting number curves for $u = 1, \ldots, 9$ (see Section 4.2); (iii) torus crossing number curves for $c = 3, 5, \ldots, 17$ (see Section 4.3).

Hence, the first inequality holds (with equality only in the case of trefoil knots).

For the second inequality,

$$\begin{aligned}\tfrac{8}{9}v_2(T)^3 - v_3(T)^2 &= \tfrac{8}{9}\left(\tfrac{1}{24}(p^2-1)(q^2-1)\right)^3 \\ &\quad - \left(\tfrac{1}{144}pq(p^2-1)(q^2-1)\right)^2 \\ &= \tfrac{1}{4.27}\left[\tfrac{1}{24}(p^2-1)(q^2-1)\right]^2 \\ &\quad \times \left\{4(p^2-1)(q^2-1) - 3p^2q^2\right\} \\ &= \tfrac{1}{4.27}v_2(T)^2\left\{(p^2-4)(q^2-4) - 12\right\} \\ &\geq \tfrac{1}{4.27}v_2(T)^2\{-12\} = -\tfrac{1}{9}v_2(T)^2.\end{aligned}$$

Note that equality occurs precisely when T is a $(2, q)$-torus knot. □

Although the lefthand bound has the correct asymptotic behaviour, a different form of cubic is required for a tight bound.

Proposition 4.2. *For a torus knot T,*

$$\tfrac{2}{3}v_2(T)^3 + \tfrac{1}{3}v_2(T)v_3(T) \leq v_3(T)^2,$$

and this bound is tight in the sense of the previous proposition.

Proof: Using the notation of the previous proof,

$$\begin{aligned} v_3(T)^2 - \tfrac{2}{3}v_2(T)^3 - \tfrac{1}{3}v_2(T)v_3(T) \\ &= \tfrac{1}{36.24^2}(p^2-1)^2(q^2-1)^2\left((p-q)^2-1\right) \\ &\geq 0, \end{aligned}$$

with equality if and only if T is a $(p,p+1)$ torus knot. □

Given that half the torus knots (those with positive v_3) can be thought of as lying in the region $q > p > 0$ in the (p,q)-plane, these bounds are not surprising. Graphically, this can be seen in Figure 4.

4.2 Torus Knots and Unknotting Number

By Kronheimer and Mrowka's [Kronheimer and Mrowka 93] positive solution to the Milnor conjecture, the following formula is known for the unknotting number, u, of torus knots:

$$u(T(p,q)) = \tfrac{1}{2}\left(|p|-1\right)\left(|q|-1\right).$$

As a consequence, the following easily verifiable relationship is obtained:

Proposition 4.3. *For a torus knot T,*

$$v_2(T)^2 + \tfrac{1}{6}u(T)(u(T)-1)v_2(T) = u(T)|v_3(T)|,$$

and given $v_2(T)$ and $v_3(T)$, then $u(T)$ is the smaller of the two roots.

So for a fixed unknotting number, the torus knots lie on a quadratic in the (v_2,v_3)-plane (c.f., the Whitehead knots in Section 3). This is pictured in Figure 4. The segments of curves shown were chosen by the following proposition.

Proposition 4.4. *For a torus knot T,*

$$\begin{aligned} \tfrac{1}{2}u(T)(u(T)+1) &\geq v_2(T) \\ &\geq \tfrac{1}{6}u(T)\left(u(T)+\sqrt{8u(T)+1}+2\right), \end{aligned}$$

and both bounds are tight.

Proof: If T is a (p,q)-torus knot, then a minimal amount of manipulation gives

$$\begin{aligned} \tfrac{1}{2}u(T)(u(T)+1) - v_2(T) \\ &= \tfrac{1}{12}(|p|-1)(|q|-1)(|p|-2)(|q|-2) \\ &\geq 0, \end{aligned}$$

with equality if and only if T is a $(2,q)$-torus knot.

For the righthand bound, first let a and b be distinct positive integers, then $(a-b)^2 \geq 1$, so $(a+b)^2 \geq 4ab+1$, and thus $a+b \geq \sqrt{4ab+1}$, with equality precisely when a and b differ by one.

Now for T, a (p,q)-torus knot,

$$\begin{aligned} v_2(T) - \tfrac{1}{6}u(T)\left(u(T)+\sqrt{8u(T)+1}+2\right) \\ &= \tfrac{1}{12}(|p|-1)(|q|-1) \\ &\quad\times\left\{|p|+|q|-2-\sqrt{4(|p|-1)(|q|-1)+1}\right\} \\ &\geq 0, \end{aligned}$$

by setting $a = |p|-1,\ b = |q|-1$ in the above paragraph. Note that equality occurs precisely when T is a $(p,p+1)$-torus knot. □

If we weaken the righthand bound to $v_2 \geq \frac{1}{6}u(T)(u(T)+5)$ and invert the inequalities, we have the following corollary.

Corollary 4.5. *For a torus knot T,*

$$\sqrt{1+8v_2(T)} - 1 \leq 2u(T) \leq \sqrt{24v_2(T)+25} - 5,$$

and the lefthand bound is tight (in the sense of Proposition 4.1).

4.3 Torus Knots and Crossing Number

In the work of Murasugi [Murasugi 91], a similar formula can be found for the crossing number, c, of torus knots:

$$c(T(p,q)) = |q|(|p|-1), \qquad \text{when } |p| < |q|.$$

This leads to the following relation:

Proposition 4.6. *If T is a torus knot, and $\rho(T) = \left|\frac{6v_3(T)}{v_2(T)}\right|$, then*

$$\begin{aligned} 24v_2(T)(c(T)-\rho(T))^2 \\ &= c(T)\left((c(T)-\rho(T))^2-1\right)(2\rho(T)-c(T)), \end{aligned}$$

and

$$\begin{aligned} c(T) = \rho(T) - \tfrac{1}{2}\Big(&\sqrt{(\rho(T)-1)^2-24v_2(T)} \\ &+\sqrt{(\rho(T)+1)^2-24v_2(T)}\Big). \end{aligned}$$

Proof: This is easily verified; note that if T is a (p,q)-torus knot, then $\rho(T) = |pq|$ and $c(T)-\rho(T) = |q|$. □

This isn't as nice a relationship as with the unknotting number: for a fixed crossing number, the relationship is a not particularly nice quartic between v_2 and v_3. However, the crossing number curves can still be graphed, as in Figure 4—the length of arc segments plotted there being determined by the following proposition.

Proposition 4.7. *For a torus knot T,*

$$\begin{aligned}\tfrac{1}{8}\left(c(T)^2-1\right) &\geq v_2(T)\\ &\geq \tfrac{1}{24}c(T)\left(c(T)+1+2\sqrt{c(T)+1}\right),\end{aligned}$$

and these bounds are tight (in the sense of Proposition 4.1).

Proof: Suppose that T is a (p,q)-torus knot with $q > p > 0$ (this just avoids excessive modulus signs in the calculation), then for the lefthand bound,

$$\begin{aligned}&\tfrac{1}{8}\left(c(T)^2-1\right)-v_2(T)\\ &\quad=\tfrac{1}{24}\left\{3\left([q(p-1)]^2-1\right)-(p^2-1)(q^2-1)\right\}\\ &\quad=\tfrac{1}{24}\left\{2q^2p^2-6q^2p+4q^2+p^2-4\right\}\\ &\quad=\tfrac{1}{24}(p-2)\left\{(2q^2+1)(p-1)+3\right\}\\ &\quad\geq 0,\end{aligned}$$

and equality occurs precisely when T is a $(2,q)$-torus knot.

For the righthand bound,

$$\begin{aligned}&24v_2(T)-c(T)\left(c(T)+1+2\sqrt{c(T)+1}\right)\\ &\quad=(p^2-1)(q^2-1)-q(p-1)\\ &\qquad\times\left(q(p-1)+1+2\sqrt{q(p-1)+1}\right)\\ &\quad=(p-1)\left\{2q^2-q-1-p-2q\sqrt{qp-q+1}\right\},\end{aligned}$$

and claim that this is nonnegative and is zero precisely when $q=p+1$.

To prove the claim, note

$$(q-1)^2 = q(q-1)-q-1 \geq qp-q-1$$

as $q-p-1\geq 0$, and so also

$$(q-1)^2+\tfrac{2(q-1)(q-p-1)}{2q}+\left[\tfrac{q-p-1}{2q}\right]^2 \geq qp-q-1>0.$$

Thus, by taking square roots,

$$(q-1)+\tfrac{q-p-1}{2q}\geq\sqrt{qp-q-1},$$

from which the claim follows on multiplying through by $2q$. □

Weakening the righthand bound to $v_2 \geq \frac{1}{24}c(c+5)$ and inverting gives

Corollary 4.8. *For a torus knot T*

$$\tfrac{1}{24}\left(\sqrt{25+96v_2(T)}-5\right)\geq c(T)\geq 2\sqrt{8v_2(T)+1},$$

and the righthand bound is tight in the previous sense.

5. PROBLEMS AND FURTHER QUESTIONS

Problem 5.1. Does the fish pattern persist in the graphs of knots with higher crossing number?

Problem 5.2. Is there some qualitative distinction between knots with odd and even crossing number which explains the perceived difference in the fish?

Problem 5.3. Is there any relationship with unknotting number? Note that the n-fold connect sum of 8_{14} has unknotting number n and $(v_2, v_3) = (0,0)$. (Stoimenov pointed this out to me.)

Problem 5.4. For a knot K with $(6|v_3(K)|-|v_2(K)|)^2 \geq 24v_2(K)^3$, let $\rho(K) = 6|v_3(k)/v_2(K)|$ and then define the *pseudo-unknotting number*, $\tilde{u}(K)$, and the *pseudo-uncrossing number*, $\tilde{c}(K)$, by

$$\begin{aligned}\tilde{u}(K) &:= \tfrac{1}{2}\left(1+\rho(K)-\sqrt{(1+\rho(K))^2-24v_2(K)}\right);\\ \tilde{c}(K) &:= \rho-\tfrac{1}{2}\left(\sqrt{(1+\rho(K))^2-24v_2(K)}\right.\\ &\qquad\left.+\sqrt{(1-\rho(K))^2-24v_2(K)}\right).\end{aligned}$$

For torus knots, the pseudo-unknotting and pseudo-crossing numbers coincide with the usual unknotting and crossing numbers. Do they have any meaning for other knots? Does the necessary bound for K have any topological interpretation?

As an example, consider the Whitehead knots $Wh(i)$, for $i>0$ these all have unknotting number equal to one. In this case $\tilde{u}(Wh(1))=1$, and $\tilde{u}(Wh(i))\to 2$ as $i\to\infty$.

6. RELATED WORK

Since an early version of this paper was circulated, Dasbach, Le, and Lin [Dasbach et al. 01] have considered the fish phenomenon from the point of view of the Jones polynomial evaluated at roots of unity for knots with 13 and 14 crossings, and Okuda [Okuda 02] has examined the plots of v_2/c^2 versus v_3/c^3 (where c is the crossing number) for various infinite families of knots, although, at the time of writing, I have not seen this work.

ACKNOWLEDGMENTS

This work formed part of the author's University of Edinburgh PhD Thesis, and was supported by an EPSRC studentship. Thanks to Oliver Dasbach and Alexander Stoimenow for useful comments. The diagrams and numerical experiments were done using Maple.

REFERENCES

[Alvarez and Labastida 96] M. Alvarez and J. M. F. Labastida. "Vassiliev invariants for torus knots." *J. Knot Theory Ramifications* **5** (1996), 779–803.

[Bar-Natan 95] D. Bar-Natan. "Polynomial invariants are polynomial." *Math. Res. Lett.* **2** (1995), 239–246.

[Birman and Lin 93] J. Birman and X.-S. Lin. "Knot polynomials and Vassiliev's invariants." *Invent. Math.* **111** (1993), 225–270.

[Burde and Zieschang 85] G. Burde and H. Zieschang. *Knots.* Number 5 in Studies in Mathematics, de Gruyter, Berlin, 1980.

[Dasbach et al. 01] O. T. Dasbach, T. D. Le, and X.-S. Lin. "Quantum morphing and the Jones polynomial." *Comm. Math. Phys.* **224**:2 (2001), 427–442.

[Dean 94] J. Dean. "Many classical knot invariants are not Vassiliev invariants." *J. Knot Theory Ramifications* **3** (1994), 7–9.

[Domergue and Donato 96] M. Domergue and P. Donato. "Integrating a weight system of order n to an invariant of $(n-1)$-singular knots." *J. Knot Theory Ramifications*, **5** (1996), 23–35.

[Hoste and Thistlethwaite 99] J. Hoste and M. Thistlethwaite. `knotscape`. available at http://www.math.utk.edu/∼morwen, 1999.

[Kronheimer and Mrowka 93] P. Kronheimer and T. Mrowka. "Gauge theory for embedded surfaces. I." *Topology* **32** (1993), 773–826.

[Lannes 93] J. Lannes. "Sur les invariants de Vassiliev de degré inférieur ou égal à 3." *Enseign. Math. (2)* **39** (1993), 295–316.

[Lin and Wang 96] X.-S. Lin and Z. Wang. "Integral geometry of plane curves and knot invariants." *J. Differential Geom.* **44** (1996), 74–95.

[Maple 97] Maple symbolic computation software, Waterloo Maple Inc., www.maplesoft.com, (1981-1997).

[Murasugi 91] K. Murasugi. "On the braid index of alternating links." *Trans. Amer. Math. Soc.* **326**:1 (1991), 237–260.

[Okuda 02] N. Okuda. *On the first two Vassiliev invariants of some sequences of knots.* Master's thesis, Tokyo Inst. Tech., 2002, in Japanese.

[Polyak and Viro 94] M. Polyak and O. Viro. "Gauss diagram formulas for Vassiliev invariants." *International Mathematical Research Notes* **11** (1994), 445–454.

[Polyak and Viro 01] M. Polyak and O. Viro. "On the Casson knot invariant." *J. Knot Theory Ramifications* **10**:5 (2001), 711–738.

[Stanford 92] T. Stanford Computer programs and data tables are available online at: ftp://geom.umn.edu/pub/contrib/vassiliev.tar.Z.

[Stanford 96] T. Stanford. "Braid commutators and Vassiliev invariants." *Pacific J. Math.* **174** (1996), 269–276.

[Stanford 97] T. Stanford. "Computing Vassiliev's invariants." *Topology Appl.* **77** (1997), 261–276. see also data files [Stanford 92].

[Trapp 94] R. Trapp. "Twist sequences and Vassiliev invariants." *J. Knot Theory Ramifications* **3** (1994), 391–405.

[Vassiliev 92] V. A. Vassiliev. *Complements of discriminants of smooth maps: topology and applications*, volume 98 of *Trans. of Math. Mono.* Amer. Math. Soc., Providence, 1992.

[Willerton 97] S. Willerton. *On the Vassiliev invariants for knots and for pure braids.* PhD thesis, University of Edinburgh, 1997.

Simon Willerton, Department of Pure Mathematics, University of Sheffield, Hicks Building, Sheffield S3 7RH, United Kingdom (swillerton@sheffield.ac.uk)

Received April 3, 2001; accepted in revised form February 19, 2002.