

Inequalities for the false discovery rate (FDR) under dependence*

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Abstract: Inequalities are key tools to prove FDR control of a multiple test. The present paper studies upper and lower bounds for the FDR under various dependence structures of p -values, namely independence, reverse martingale dependence and positive regression dependence on the subset (PRDS) of true null hypotheses. The inequalities are based on exact finite sample formulas which are also of interest for independent uniformly distributed p -values under the null. As applications the asymptotic worst case FDR of step up and step down tests coming from an non-decreasing rejection curve is established. In addition, new step up tests are established and necessary conditions for the FDR control are discussed. The reverse martingale models yield sharper FDR results than the PRDS models. Already in certain multivariate normal dependence models the familywise error rate of the Benjamini Hochberg step up test can be different from the desired level α . The second part of the paper is devoted to adaptive step up tests under dependence. The well-known Storey estimator is modified so that the corresponding step up test has finite sample control for various block wise dependent p -values. These results may be applied to dependent genome data. Within each chromosome the p -values may be reverse martingale dependent while the chromosomes are independent.

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1. Introduction

High dimensional testing problems given by n hypotheses and corresponding ordered p -values $p_{1:n} \leq \dots \leq p_{n:n}$ of the p -value vector (p_1, \dots, p_n) are frequently judged by multiple tests, like step up and step down tests. These tests rely on the component wise comparison of the ordered p -values with a family of critical values $(\alpha_{i:n})_{i \leq n}$, see [1–3, 7–11, 17, 23–25] for instance. The overall control of the error probability of first kind is often too restrictive and leads to very conservative multiple tests. Therefore, Benjamini and Hochberg [1] promoted the false discovery rate (FDR) as error measure to control. The FDR is the expected ratio of the number of falsely rejected null hypotheses among the total number of rejections.

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Starting with the famous choice of critical values $\alpha_{i:n} = \frac{i}{n}\alpha$ by Benjamini and Hochberg [1], quite a lot of authors studied finite sample or asymptotic FDR control (by some given level $0 < \alpha < 1$) under various assumptions. Roughly speaking the finite sample research can be derived in two categories. When the critical values are deterministic, then different sufficient conditions and dependence concepts for the p -values were established in order to ensure FDR control at level α , i.e. $FDR \leq \alpha$, see Benjamini and Hochberg [1], Benjamini and Yekutieli [2], Blanchard and Roquain [5] and Finner et al. [9] among others. In case of data dependent critical values $\hat{\alpha}_{i:n}$, which lead to adaptive multiple tests, typically the i.i.d. structure of the p -values of true null hypotheses is assumed to achieve FDR control, see Storey et al. [24] and Sarkar [21] for instance. They include an estimation of the number of true null hypotheses in the critical values in order to exhaust the predetermined FDR level better. Another branch is the asymptotic FDR control, where milder assumptions like weak dependency may be considered.

In this paper we will again revisit FDR inequalities for step up and step down tests. The results depend on three dependence structures for the p -values, namely the most restrictive basic independence (BI) model, the reverse martingale model and the positive regression dependence on a subset (PRDS) model, respectively, which are introduced in Section 2 beside the basic notation. Martingale arguments were used in Chapter 3 of the dissertation of Scheer [22] for the comparison of the FDR and the expected number of false rejections. Reverse martingale models naturally show up for instance for measurements under restrictions or in multivariate extreme value theory, see Example 3.3 which include a Marshall/Olkin type dependence structure. Section 3 gives some construction methods of reverse martingales and it is pointed out that the FDR of the classical Benjamini Hochberg step up test can exactly be calculated for the reverse martingale structure, whereas already strict inequalities hold under multivariate normal PRDS models, see Example 3.1. Section 4 discusses FDR inequalities for all these models which include inequalities for the FDR and inequalities for the critical values of FDR controlling step up tests. New necessary and sufficient conditions for finite sample FDR control at level α are derived. In particular, critical values considered earlier by Finner et al. [9] and Gavrilov et al. [11] are discussed, see also Section 5.1. The inequalities can be used to modify the critical values of Gavrilov et al. [11] for step up tests, confer Example 5.1 for improved new tests. Theorem 5.1 establishes an exact asymptotic formula for the worst case FDR of step up tests which come from an increasing rejection curve. Observe that our inequalities allow to treat the difficult case when the expected portion of true null hypotheses becomes maximal. That result can be compared with the asymptotic optimal rejection curve (AORC) of Finner et al. [9] which compares concave rejection curves. For concave rejection curves, it is remarkable that the asymptotic worst case FDR value here is the same for the corresponding step up and step down tests, see Section 5.2.

Section 6 deals with adaptive SU tests under dependence which has often been neglected in the past. The adaptive step up tests rely on conservative estimators \hat{n}_0 of the expected number of true null hypotheses. Mostly the basic

independence model is assumed in the literature when the FDR of the adaptive test is shown to be controlled. We will point out that finite sample FDR control of adaptive step up tests is a difficult affair and can not be expected in general under dependence. Recall that the well-known Storey multiple test does not work under positive regression dependence on the subset of true null hypotheses, see Example 6.1 for instance. We will give a simple condition which ensures asymptotic FDR control under different dependence structures, see Theorem 6.1 and 6.2. For fixed sequence of estimators these conditions may also be regarded as conditions for the possible dependence structures. Furthermore, finite sample control can be obtained for various adaptive step up tests under the reverse martingale model. Also necessary conditions for finite sample control will be presented. It is shown that under additional conditions some modified Storey estimators work for dependent but block wise independent p -values, see Theorem 6.3. Under the general assumptions these results are sharp and can not be improved, see Example 6.1. However, when all p -values are independent then the new blockwise test is conservative.

Section 7 contains exact technical FDR formulas which are used in our proofs. Some of them are of separate interest. Many statements of this paper are applications of our central Lemma 7.1. Furthermore, all proofs are outlined in Section 7.

2. Basic notation and dependence models

Throughout, we investigate models with different dependence structures. All of these models are based on the following basic model with random number of true null hypotheses. Let (Ω, \mathcal{A}, P) be a probability space and let

$$(\epsilon_i, U_i, \xi_i)_{i \leq n} : \Omega \longrightarrow (\{0, 1\} \times [0, 1]^2)^n \quad (2.1)$$

be a multivariate random variable where $\epsilon_i = 0$ codes the occurrence of a p -value ξ_i of a false null hypothesis, for short false p -value, and $\epsilon_i = 1$ the occurrence of a p -value U_i of a true null hypothesis, for short true p -value, whose marginal distribution is the uniform distribution on $(0, 1)$. Then the model of the p -values is given by

$$p_i = \epsilon_i U_i + (1 - \epsilon_i) \xi_i, \quad 1 \leq i \leq n, \quad \text{and} \quad N := \sum_{i=1}^n \epsilon_i, \quad (2.2)$$

where N is the random number of true p -values. This model includes the well studied mixture model of Efron et al. [6], where $(U_i)_i$, $(\xi_i)_i$, and $(\epsilon_i)_i$ are i.i.d. and jointly independent. Observe that here N is naturally random. Throughout, true or false null hypotheses are identified with their p -values and for short the corresponding p -values are called “true” or “false”, respectively. Since our multiple tests only rely on p -values this identification may be justified. Moreover, we define $p = (p_1, \dots, p_n)$ and $\epsilon = (\epsilon_1, \dots, \epsilon_n)$. Below, further assumptions about the dependence structure of the vector (2.1) are introduced. To avoid trivial cases let $E(N)$ be always positive and let us assume that our observations

are the order statistics of the p -values, which are introduced as

$$p_{1:n} \leq p_{2:n} \leq \cdots \leq p_{n:n}. \quad (2.3)$$

Moreover, let $\hat{F}_n(t) = \frac{1}{n} \sum_{i=1}^n 1\{p_i \leq t\}$, $0 \leq t \leq 1$, be the empirical cumulative distribution function of the p -values.

Let $\mathcal{B}([0, 1]^n)$ denote the Borel sets of $[0, 1]^n$. A set $C \in \mathcal{B}([0, 1]^n)$ is said to be decreasing iff, $c' \in C$ and $c \leq c'$ component-by-component imply $c \in C$.

Definition 2.1 (Dependence structures).

(a) Let $(\epsilon_i, \xi_i)_{i \leq n}$, U_1, \dots, U_n be independent. Then we call this submodel for the p -values (2.2) to be the **basic independence (BI) model**. Note that $(\epsilon_i, \xi_i)_{i \leq n}$ is considered as one random variable whereas U_1, \dots, U_n are considered as individual random variables in terms of independence.

(b) Let

$$t \mapsto P(p \in C | p_i \leq t, \epsilon = \bar{\epsilon}) \quad (2.4)$$

be non-increasing for every decreasing set $C \in \mathcal{B}([0, 1]^n)$, $\bar{\epsilon} \in \{0, 1\}^n$ and all i with $\epsilon_i = 1$. Then p -value model (2.2) is called the **PRDS** model (positive regression dependent on the subset of true null hypotheses).

(c) Conditioned under ϵ let

$$\frac{1\{p_i \leq t\}}{t}, \quad 0 < t \leq 1, \text{ for all } i \text{ with } \epsilon_i = 1 \quad (2.5)$$

be a reverse martingale with respect to the reverse filtration $\mathcal{F}_t = \sigma(1\{p_j \leq s\}, \epsilon_j) : 1 \leq j \leq n, s \geq t$. Then p -value model (2.2) is called **reverse martingale** model.

Remark 2.1. (a) The assumptions for the PRDS model in Definition 2.1 (b) are a little bit weaker than the usual PRDS assumptions, see Finner et al. [9] for instance. In the literature it is sometimes called weak PRDS. Nevertheless, we will call it PRDS model for brevity.

(b) The BI model is a submodel of the PRDS and reverse martingale model. Furthermore, the intersection of the PRDS and reverse martingale models is at least greater than the BI model. To see this regard k independent disjoint blocks of (U_1, \dots, U_n) with maximal dependence in each block given here by the same uniformly distributed random variable.

We will see that the reverse martingale models yield sharper FDR results than the PRDS concept. A comparison of these models is included in Section 3.

In the past literature usually conditional versions of the BI and PRDS model with deterministic $\epsilon_1, \dots, \epsilon_n$ have been considered, see Benjamini and Hochberg [1], Benjamini and Yekutieli [2], Blanchard and Roquain [5], Finner et al. [9] and Finner and Roters [10] for instance. In all models defined above these conditional versions are included as special case.

In this paper we mainly focus on step up tests (SU tests), which we briefly recall. Suppose that

$$0 < \alpha_{1:n} \leq \alpha_{2:n} \leq \cdots \leq \alpha_{n:n} < 1 \quad (2.6)$$

denote possibly data dependent critical values and set $\alpha_{0:n} = \alpha_{1:n}$ for convenience. The corresponding SU test is based on the number of rejections

$$R = \max\{i : p_{i:n} \leq \alpha_{i:n}\} \tag{2.7}$$

and rejects the null hypotheses corresponding to the set of p -values $\{p_i : p_i \leq \alpha_{R:n}\}$. When the condition in (2.7) is empty no hypothesis is rejected and $R = 0$ holds. Then equivalently all null hypotheses with p -values

$$p_{i:n}, \quad i \leq R \tag{2.8}$$

are rejected. Let

$$V = \#\{i : p_i \leq \alpha_{R:n}, \epsilon_i = 1\} \tag{2.9}$$

be the unobservable number of falsely rejected true null hypotheses. The judgment of multiple tests is often done via the control of the celebrated “false discovery rate” (FDR) which is given by $FDR = E(\frac{V}{R})$ (with $\frac{0}{0} = 0$). More generally we introduce the conditional false discovery rate

$$FDR(n_0) := E\left(\frac{V}{R} \mid N = n_0\right) \tag{2.10}$$

as conditional expectation given $N = n_0$. The conditional quantity (2.10) is a special case since constant N are included.

Benjamini and Hochberg [1] promoted the FDR as new error criterion competing against the well known familywise error rate (FWER) and provided a multiple test procedure controlling the FDR under certain assumptions. The so-called Benjamini Hochberg test (BH test) is the linear SU test with fixed critical values

$$\alpha_{i:n} = \frac{i}{n}\alpha, \quad 1 \leq i \leq n. \tag{2.11}$$

In our setting and notation it was shown by Benjamini and Yekutieli [2] and Finner and Roters [10] that

$$FDR(n_0) = \frac{n_0}{n}\alpha \tag{2.12}$$

holds for the conditional expectation in the BI model. Benjamini and Hochberg [1] previously showed that “ \leq ” holds in (2.12) for the BI model. Moreover, Benjamini and Yekutieli [2] proved that “ \leq ” holds in (2.12) for the PRDS model. In the same work, they also introduced a new SU test based on more conservative critical values

$$\alpha_{i:n} = \frac{i}{n \sum_{j=1}^n \frac{1}{j}}\alpha. \tag{2.13}$$

Benjamini and Yekutieli [2] pointed out that “ \leq ” again holds in (2.12) for this SU test under the basic model with arbitrary dependence structure of

$(\epsilon_i, U_i, \xi_i)_{i \leq n}$. Blanchard and Roquain [5] showed that the critical values (2.13) may also be replaced by

$$\alpha_{i:n} := \frac{\alpha}{n} \int_0^i x d\nu(x), \quad (2.14)$$

where ν is an arbitrary probability measure on $(0, \infty)$. For $\nu(\{i\}) = (i \sum_{j=1}^n \frac{1}{j})^{-1}$, $i = 1, \dots, n$, the critical values correspond to (2.13). Adaptive versions based on (2.13) and (2.14) are presented in Theorem 6.2 under arbitrary dependence.

We will particularly focus on critical values coming from a continuous non-decreasing function

$$f : [0, 1] \rightarrow [0, \infty) \text{ with } f(0) = 0 \text{ and } f(x_0) = 1 \text{ and } f(1) > 1 \quad (2.15)$$

for some $x_0 < 1$. We refer to f as rejection curve. Moreover, let f^{-1} denote the right continuous inverse of f and let the deterministic critical values be generated via

$$\alpha_{i:n} = f^{-1}\left(\frac{i}{n}\right), \quad 1 \leq i \leq n. \quad (2.16)$$

We refer to f^{-1} as critical value curve. Note that the BH test is based on the Simes line $f(t) = t/\alpha$, $t \in [0, 1]$, see Finner et al. [9] for instance.

Finner et al. [9] introduced the Asymptotic Optimal Rejection Curve (AORC) which is constructed to have FDR control by α in an asymptotic Dirac uniform (DU) setting given by $\xi_i = 0$, $i = 1, \dots, n$. The AORC is given by

$$f_\alpha(t) = \frac{t}{t(1-\alpha) + \alpha}, \quad t \in [0, 1], \quad (2.17)$$

but since $f(1) = \alpha_{n:n} = 1$, the above assumptions for rejection curves for SU tests are not fulfilled. There are several modifications of the AORC and corresponding SU tests to overcome this problem. For further details we refer to Finner et al. [9].

3. Examples of reverse martingale models and a comparison with PRDS

At the beginning it is shown that there exist positive dependent multivariate normal models which are PRDS without the martingale property. To prove this we will consider the following example.

Example 3.1. Let X_1 and Y be i.i.d. standard normal random variables with distribution function Φ .

(a) Consider the PRDS model

$$(X_1, X_2) = \left(X_1, \frac{1}{\sqrt{2}}X_1 + \frac{1}{\sqrt{2}}Y \right)$$

and related p -values $(p_1, p_2) = (\Phi(X_1), \Phi(X_2))$. Then the familywise error rate of the BH step up test with critical values (2.11) at level $\alpha = 0.5$ and $n = n_0 = 2$ is $FWER = FDR(2) = \frac{7}{16}$, cf. (2.12), i.e. less than $\alpha = 0.5$.

(b) For the negative dependence model

$$(X_1, X_2) = \left(X_1, -\frac{1}{\sqrt{2}}X_1 + \frac{1}{\sqrt{2}}Y \right)$$

the familywise error rate of the BH step up test is $FWER = FDR(2) = \frac{9}{16}$ and hence greater than $\alpha = 0.5$.

The proof is given in Section 7. Note that Gavrilov et al. [11], p. 625, already derived Monte Carlo results showing that the FWER of the BH step up test may be strictly below α under PRDS. In contrast to PRDS the reverse martingale models allow sharper FDR results, see Section 4 and Lemma 7.1. The next remark summarizes this.

Remark 3.1. Proposition 4.1 always implies the formula $FDR(n_0) = \frac{n_0}{n}\alpha$, see (2.12), for the BH step up test under the reverse martingale model, whereas only “ \leq ” holds under PRDS. Since an strict inequality “ $<$ ” shows up in the PRDS Example 3.1 (a), that normal dependence model is no reverse martingale.

In conclusion we see that reverse martingale models allow sharper FDR formulas as under PRDS. However, we do not know whether every reverse martingale model is PRDS.

The reverse martingale structure yields a rich class of p -values. The next example shows how to construct new reverse martingale models from known ones. In particular, reverse martingale measures on product spaces of $[0, 1]$ are preserved under a lot of operations.

Example 3.2. Let $\epsilon, I \subset \{i : \epsilon_i = 1\}$ and $J \subset \{i : \epsilon_i = 0\}$ be fixed with $|I| > 0$. Define p -values via the canonical projections $p_i : [0, 1]^{|I|+|J|} \rightarrow [0, 1]$. Then

$$\mathcal{M}_I(I \cup J) = \left\{ \begin{array}{l} P \text{ distribution on } [0, 1]^{|I|+|J|} : \\ P \text{ represents a reverse martingale model with true} \\ p\text{-values } p_i, i \in I \text{ and false p-values } p_i, i \in J \end{array} \right\}$$

is the set of reverse martingale measures.

(a) $\mathcal{M}_I(I \cup J)$ is closed under mixtures including convex combinations.

(b) (Independent coupling of reverse martingale regimes) Suppose that there are partitions $\{i : \epsilon_i = 1\} = \sum_{i=1}^r I_i$ and $\{i : \epsilon_i = 0\} = \sum_{i=1}^r J_i$ with $I_i \neq \emptyset$ for all i but J_i is allowed to be empty. Whenever $P_i \in \mathcal{M}_{I_i}(I_i \cup J_i)$ holds for all $i \leq r$, then the product measure $\otimes_{i=1}^r P_i$ belongs to $\mathcal{M}_{\{i:\epsilon_i=1\}}(1, \dots, n)$.

(c) (Optional switching of reverse martingales) It is well known that two independent reverse martingale models given by P_1 and $P_2 \in \mathcal{M}_{\{i:\epsilon_i=1\}}(1, \dots, n)$ may be combined as follows. Let $\tau : [0, 1]^n \rightarrow (0, 1]$ denote a reverse stopping time w.r.t. model P_1 . Whenever $1_{[0,\tau]}(p_i) = 0$ holds, that p -value for the index i comes from the P_1 model. In case $1_{[0,\tau]}(p_i) = 1$, consider the p -value \tilde{p}_i of P_2 and take the renormalized value $\tilde{p}_i\tau$ as new p -value for these coordinates.

(d) Let $p_1, \dots, p_n : \Omega \rightarrow [0, 1]$ be any random variables such that

$$t \mapsto \frac{\frac{1}{n} \sum_{i=1}^n 1_{[0,t]}(p_i)}{nt} \tag{3.1}$$

is a reverse martingale. Suppose that $i \mapsto \sigma(i)$ denotes a uniformly distributed permutation of $\{1, \dots, n\}$ jointly independent of the p_i 's. Then the family of p -values $(p_{\sigma(1)}, \dots, p_{\sigma(n)})$ has the reverse martingale property (2.5) for each component.

Part (d) of that example is easy to prove and grew out of a discussion with Julia Benditkis which is kindly acknowledged.

The subsequent example gives an explicit example which may have the following practical meaning. The statistician can only observe a concentration X_i above a joint random ground level Y . It also occurs in multivariate extreme value theory and risk analysis when the Z_i have a joint risk component Y .

Example 3.3 (Marshall and Olkin type dependence, see Marshall and Olkin [18]). Let X_1, \dots, X_n, Y denote continuous, independent, real random variables, where X_1, \dots, X_n are i.i.d.. Consider $Z_i := \max(X_i, Y)$ for $1 \leq i \leq n$. The transformed true p -values $p_i := H(Z_i)$, $i = 1, \dots, n$ given by $H(t) = P(Z_1 \leq t)$ have the reverse martingale property, see Section 7 for a proof. It is easy to verify, that the present model is also PRDS since it is based on a comonotone transformation of the i.i.d. model X_1, \dots, X_n, Y . Notice that in case $n = 2$ the negative variables $(-Z_1, -Z_2)$ correspond to the bivariate Marshall and Olkin [18] model.

Reverse martingale models can also be obtained allowing some dependence between “true and false” p -values.

Example 3.4 (Dependence between the null and alternatives). The following models may be used as ingredients for Example 3.2 (b). Consider a distribution P_1 on $[0, 1]^{|I_1|+|J_1|}$, where

- (i) the marginal distribution of the “trues” $(p_i)_{i \in I_1}$ belongs to $\mathcal{M}_{I_1}(I_1)$.
- (ii) Suppose $\min(p_i : i \in I_1) \geq \max(p_i : i \in J_1)$ holds P_1 almost everywhere.

Then $P_1 \in \mathcal{M}_{I_1}(I_1 \cup J_1)$ holds. The proof follows the same line as in Example 3.3.

4. Inequalities for the FDR

In this section new inequalities are derived which are used in the proceeding chapters. We start with arbitrary non-decreasing deterministic critical values $0 < \alpha_{1:n} \leq \dots \leq \alpha_{n:n} < 1$ and the following question.

- What can be said about the FDR of the corresponding SU test given by a fixed model (2.2)?

The next inequalities rely on more technical results given in Section 7, in particular in Lemma 7.1.

Proposition 4.1. (a) Assume the reverse martingale model (including the BI model) and consider the SU test with arbitrary deterministic critical values (2.6). Then we have

$$\frac{E(N)}{n} \left(\min_{i \leq n} \frac{n\alpha_{i:n}}{i} \right) \leq FDR \leq \frac{E(N)}{n} \left(\max_{i \leq n} \frac{n\alpha_{i:n}}{i} \right). \quad (4.1)$$

(b) Suppose that $P(R = j) > 0$ holds.

(i) The inequality $\frac{n\alpha_{j:n}}{j} < \max_{i \leq n}(\frac{n\alpha_{i:n}}{i})$ implies the strict inequality

$$FDR < \frac{E(N)}{n} \left(\max_{i \leq n} \frac{n\alpha_{i:n}}{i} \right). \tag{4.2}$$

(ii) Conversely, $\frac{n\alpha_{j:n}}{j} > \min_{i \leq n}(\frac{n\alpha_{i:n}}{i})$ implies

$$\frac{E(N)}{n} \left(\min_{i \leq n} \frac{n\alpha_{i:n}}{i} \right) < FDR. \tag{4.3}$$

(c) Under the PRDS model we still obtain

$$FDR \leq \frac{E(N)}{n} \left(\max_{i \leq n} \frac{n\alpha_{i:n}}{i} \right). \tag{4.4}$$

Observe that Example 3.1 gives a counterexample that the lower bound in (4.1) does not hold under PRDS.

With different methods Guo and Rao [13] already showed that the upper bound in (4.1) holds under the PRDS property. Moreover, Sarkar [20] derived several inequalities and exact expressions for the FDR for so-called generalized step-up-down tests. These inequalities are then used as key tools to prove FDR control of an step-up-down test basically with Benjamini Hochberg critical values (2.11) under the PRDS assumption and a further step-down test under multivariate total positivity of order 2 (MTP₂).

Under regularity assumptions the inequalities are asymptotically sharp. We refer to Section 7 and Lemma 7.4.

For deterministic critical values let us discuss the assumption

$$j \mapsto \frac{\alpha_{j:n}}{j} \text{ is non-decreasing.} \tag{4.5}$$

It is easy to verify that (4.5) holds for the critical values (2.16) which come from a concave rejection curve. Under (4.5) Benjamini and Yekutieli [2] showed that Dirac uniform (DU) configurations (i.e. $\xi_i = 0$) are least favorable parameter configurations for the FDR in the BI model for fixed $N = n_0$. Let us assume that the critical values with (4.5) lead to overall finite sample FDR control for the BI model, the PRDS model or the martingale model, respectively. Then the subsequent results investigate necessary conditions for the critical values $\alpha_{i:n}$ itself and the following question can be treated.

- What can be said about the critical values $\alpha_{i:n}$ when the FDR is controlled by $FDR \leq \alpha$ for all distributions given by a specified class of submodels for fixed n ?

Lemma 4.1. *Suppose that the SU test with deterministic critical values (2.6), satisfying (4.5), always controls the FDR at level α (i.e. $FDR \leq \alpha$) under all distributions of the BI model.*

(a) A necessary condition is then $\alpha_{j:n} \leq \frac{j\alpha}{n+1-j}$ for all $1 \leq j \leq n$.

(b) Suppose that in addition to (4.5) we have $\frac{\alpha_{k:n}}{k} < \frac{\alpha_{k+1:n}}{k+1}$ for one $k < n$. Then we have $\alpha_{j:n} < \frac{j\alpha}{n+1-j}$ for all $j \leq k$.

Corollary 4.1. Consider the assumptions of Lemma 4.1.

(a) The inequality $\alpha_{1:n} \leq \frac{\alpha}{n}$ always holds.

(b) If $\alpha_{1:n} = \frac{\alpha}{n}$ then the SU test is already a BH test at level α . Otherwise, $\alpha_{1:n} < \frac{\alpha}{n}$ holds.

(c) If $\alpha_{1:n} = \frac{\beta}{n}$ for some $\beta \leq \alpha$, then always $FDR \geq \frac{\beta E(N)}{n}$ follows.

Remark 4.1. (a) Consider the Dirac uniform configuration $DU(n_0)$ with $N = n_0$ and $\xi_i = 0$, $i = 1, \dots, n$, for the reverse martingale model. Under (4.5) the lower bound in (4.1) can then be improved following the lines of Proposition 4.1 for $DU(n_0)$ by

$$n_0 \frac{\alpha_{n+1-n_0:n}}{n+1-n_0} \leq FDR_{DU(n_0)}. \quad (4.6)$$

(b) The statements of Lemma 4.1 and Corollary 4.1 naturally hold if $FDR \leq \alpha$ holds for all scenarios described by the PRDS or reverse martingale model, since the BI model is a submodel of both models.

The next example demonstrates an application of our inequalities.

Example 4.1 (About necessary conditions for the BI model).

(a) SU tests with critical values

$$\alpha_{j:n} = \frac{j\alpha}{n+b-ja} \quad \text{and non-negative } a \text{ and } b \quad (4.7)$$

are frequently discussed in the literature. The requirement $\alpha_{n:n} < 1$ for all n implies $0 \leq a \leq 1 - \alpha$. A necessary condition for $FDR \leq \alpha$ is then by Lemma 4.1 the additional condition $a \leq b$. If $a > 0$ is positive then $a < b$ is necessary.

(b) Consider some fixed integer $1 \leq k < n$ and the adjusted critical values

$$\alpha_{j:n} = \frac{j\alpha}{n-j(1-\alpha)}, \quad j \leq k < n, \quad (4.8)$$

of the Asymptotic Optimal Rejection Curve (AORC) (2.17) of Finner et al. [9] which are first only specified for $j \leq k$. There are several possibilities for the choice of $\alpha_{j:n}$, $k < j \leq n$, for the extension of (4.8) such that (4.5) remains true and $\alpha_{n:n} < 1$ holds, see (5.2) below and confer also Finner et al. [9] and Gontscharuk [12]. It is well-known by Finner et al. [9] that the SU tests with adjusted critical values (4.8) do not have finite sample FDR control but asymptotic FDR control. Since $a = 1 - \alpha$ and $b = 0$ we directly observe by (a) that finite sample FDR control can not hold. Even the first critical value $\alpha_{1:n} = \frac{\alpha}{n-(1-\alpha)} \neq \frac{\alpha}{n}$ is too large to allow FDR control.

5. Applications under independence

5.1. FDR control

Our inequalities include a device for the choice of adequate parameters a, b for the critical values (4.7). Below, we restrict ourselves to the FDR adjustment

under the BI model. Some technical inequalities presented in Section 7 also work under dependence.

Proposition 5.1. *Consider SU tests with critical values (4.7) for $0 < a < b$ with fixed value b and an adjustment of a . Let \mathcal{P}_{BI} stand for all distributions of the BI model and let $FDR_{(b,a)}$ be linked to (4.7). There exists a unique parameter $a_1 \in (0, b)$ with*

$$\sup_{P \in \mathcal{P}_{BI}} FDR_{(b,a_1)} = \alpha. \tag{5.1}$$

The worst case $FDR_{(b,a)}$ is strictly smaller (larger) than α for $a < a_1$ ($a > a_1$).

Sharper inequalities for the range of the parameter a_1 of (5.1) are included in Proposition 7.1 which may be of computational interest in practice. However, the exact value a_1 should be calculated by numerical calculations, see Lemma 7.3 (a) of the proof section. For the step up test with critical values (4.7), $b = 1$, $\alpha = 0.05$ and $n = 10$ we have $a_1 \approx 0.92$ for instance which is not far away from the upper bound $b = 1$.

In the next step we establish another FDR adjustment as in (5.1) of critical values which may have some advantage in practice. The new proposal relies on the following observation. Typically the largest coefficients of (2.6) are responsible for a worst case FDR value with $FDR > \alpha$, cf. Finner et al. [7]. For these reasons we propose to bound the largest critical values as follows.

Proposition 5.2. *Fix $\epsilon > 0$ which is typically small. Consider SU tests with deterministic critical values (2.6) satisfying (4.5) and $\alpha_{1:n} < \frac{\alpha}{n}$. Introduce for fixed $1 \leq k \leq n$ the new coefficients*

$$\alpha_{j:n}^{(k)} := \min \left(\alpha_{j:n}, \frac{j}{k} \alpha_{k:n} \right), \quad j = 1, \dots, n. \tag{5.2}$$

If $\sup_{P \in \mathcal{P}_{BI}} FDR((\alpha_{j:n})_j) > \alpha + \epsilon$ holds for the FDR of the corresponding SU test, then there exists some $1 \leq k_0 < n$ with

$$\sup_{P \in \mathcal{P}_{BI}} FDR((\alpha_{j:n}^{(k)})_j) \leq \sup_{P \in \mathcal{P}_{BI}} FDR((\alpha_{j:n}^{(k_0)})_j) \leq \alpha + \epsilon \tag{5.3}$$

for all $k \leq k_0$ and “ $>$ ” for all $k > k_0$.

The modification (5.2) of the critical values has also been considered by Finner et al. [9] Example 3.2 for the special case of critical values coming from the AORC. Moreover, for this type of modification Finner et al. [7] propose to increase the parameter b in a further step in order to decrease the FDR below α . In contrast to earlier work Proposition 5.2 works for general critical values with (4.5). In principal, Proposition 5.2 also works for $\epsilon = 0$. For practical purposes a choice of a very small value $\epsilon > 0$ can be recommended, see Figure 1 in Example 5.1.

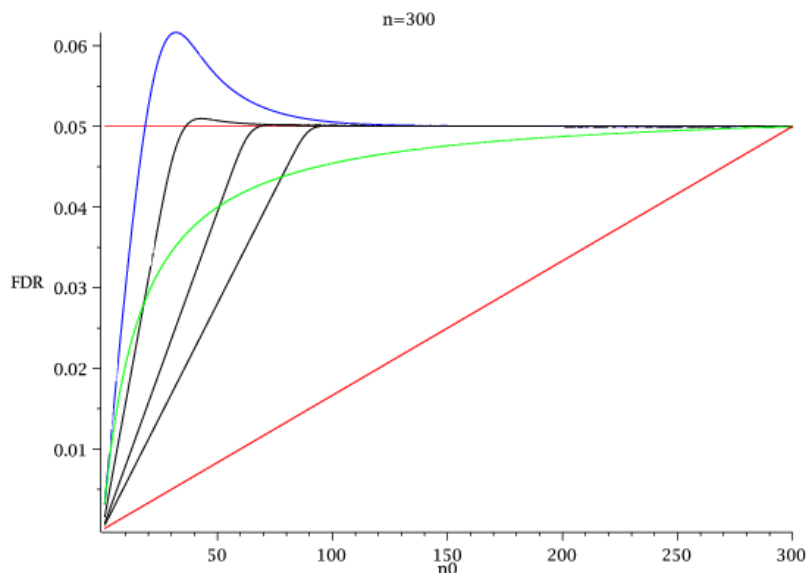


FIG 1. *SU FDR* of the Gavrilov et al. [11] critical values (blue curve), critical values (5.2) for $k = 283, 250, 223$ (black curves), *BH SU FDR* (lower red line), and lower bound (4.6) (green curve).

Example 5.1 (Under the BI model).

(a) Let us consider the step down critical values

$$\alpha_{j:n} = \frac{j\alpha}{n+1-j(1-\alpha)}, \quad j \leq n, \quad (5.4)$$

of Gavrilov et al. [11]. It is well-known that the corresponding SD test, see Section 5.2 for the notation, yields finite sample FDR control, whereas the corresponding SU test has no finite sample FDR control. On the other hand the necessary conditions for finite sample FDR control of Lemma 4.1 (a) are fulfilled. In this case our results do not exclude this procedure but we get a meaningful lower bound based on (4.6) for the worst case of $FDR(n_0)$ and a hint how the critical values (5.4) can be modified.

(b) Figure 1 shows the FDR of the SU test for $\alpha = 0.05$ and $n = 300$ for the least favorable DU configurations for different values of $N = n_0$ with $n_0 = 1, \dots, n$, given by the critical values (5.4). The lower bound in (4.1) is based on

$$\min_{i \leq n} \frac{n\alpha_{i:n}}{i} = \frac{\alpha}{1 + \frac{\alpha}{n}} \rightarrow \alpha. \quad (5.5)$$

Thus, this lower bound is close to the FDR for fixed $N \approx n$ for this example. Moreover, Figure 1 shows the FDR_{DU} plot for different choices $k = 300, 283, 250, 223$ given by (5.2) based on the critical values (5.4) with graphs decreasing in k .

TABLE 1
 Worst case FDR for different choices of k in (5.2) for the critical values (5.4)

k	300	283	250	223	2
$\sup_{P \in \mathcal{P}_{BI}} \text{FDR}$	0.06165	0.05098	0.05020	0.05009	0.050006
$\operatorname{argmax}_{n_0} \text{FDR}(n_0)$	32	43	74	100	300

The straight line represents the FDR of the BH test and the green curve is the lower bound (4.6). Numerical results yield the value $k_0 = 283$ for $\epsilon = 10^{-3}$ and $k_0 = 223$ for $\epsilon = 10^{-4}$ given by (5.3), see Table 1. Here, $k = 1$ leads to a BH test and is the only k with $FDR \leq \alpha$.

(c) In practice the statistician can accept the enlarged FDR value $\alpha + \epsilon$ or he can reduce the critical values (5.4) by a minor reduction of α .

The results given in Figure 1 are quite promising. A minor modification of the critical values (5.4) exploits the FDR of the BH test. The value of $\text{FDR}(n_0)$ is quite good for large n_0 , where the power of the multiple test is really needed.

5.2. Asymptotic worst FDR case

Our technique applies to the worst case FDR asymptotics for SU tests given by rejection curves f .

Theorem 5.1. *Let \mathcal{P}_n be the set of all possible distributions of the BI model for fixed n . Consider a non-decreasing continuous rejection curve $f : [0, x_0] \rightarrow [0, 1]$ for some $0 < x_0 < 1$ with $f(0) = 0$, $f(x_0) = 1$. Assume also that f is left-sided differentiable on $(0, x_0)$ and let $f(x) \geq (1 + \epsilon)x$ for all x and some $\epsilon > 0$. For the sequence of SU tests based on the critical values (2.16) the asymptotic worst SU case FDR is*

$$\beta := \limsup_{n \rightarrow \infty} \sup_{P_n \in \mathcal{P}_n} \text{FDR}_{P_n} = \sup \left\{ \frac{x}{1-x} \frac{1-f(x)}{f(x)} : 0 \leq x \leq x_0 \right\}. \tag{5.6}$$

Moreover, $0 < \beta < 1$ holds.

Remark 5.1. (a) Note that the AORC curve f_α (2.17) yields

$$\frac{x}{1-x} \frac{1-f_\alpha(x)}{f_\alpha(x)} = \alpha, \quad x \in (0, 1), \tag{5.7}$$

which again supports the optimality of f_α .

(b) Suppose that $\beta \leq \alpha < 1$ holds for some rejection curve f treated in Theorem 5.1. The proof of Theorem 5.1 implies $f \geq f_\alpha$ on $[0, x_0]$ with the upper bound $f^{-1}(\frac{i}{n}) \leq f_\alpha^{-1}(\frac{i}{n})$ for the critical values of the SU tests.

(c) The question about the limiting FDR for $\frac{R_n}{n} \rightarrow 0$ was addressed in Remark 5.1 of Finner et al. [9]. Our approach contributes to this open problem.

For concave rejection curves we briefly point out that the asymptotic bound β of Theorem 5.1 is the same for step down (SD) tests. Consider again the

critical values (2.16). The step down critical index is given by the number of SD rejections

$$R_{SD} := \max\{i : p_{j:n} \leq \alpha_{j:n} \text{ for all } 1 \leq j \leq i\}. \quad (5.8)$$

The modification of (2.7) and (2.8) for SD tests requires that all null hypotheses with p -values

$$p_i \leq \alpha_{R_{SD}:n} \quad (5.9)$$

are rejected. When the condition in (5.8) is empty no hypothesis is rejected and $R_{SD} = 0$ holds. Similarly to (2.9) put $V_{SD} = \#\{\text{true } p_i : p_i \leq \alpha_{R_{SD}:n}\}$ to be the number of false positive rejections.

Theorem 5.2. *Under the assumptions of Theorem 5.1, let us additionally assume that f is a concave rejection curve. Then we obtain the same asymptotic upper bound*

$$\limsup_{n \rightarrow \infty} \sup_{P_n \in \mathcal{P}_n} FDR_{P_n, SD} = \beta \quad (5.10)$$

for the sequence of SD tests generated by (2.16) as for the corresponding sequence of SU tests.

Remark 5.2. Consider a sequence of SD tests with critical values given by (2.16) based on a concave rejection curve (2.15) and which has finite sample FDR control by α for all $n \in \mathbb{N}$ in the BI model. Then the same holds asymptotically for the corresponding sequence of SU tests. This technique does not apply for the sequence of SU tests based on the Gavrilov et al. [11] critical values (5.4), since the critical values are not generated by one rejection curve, but by the sequence of rejection curves $(1 + \frac{1}{n})f_\alpha$ based on the AORC.

6. Applications to adaptive control under dependence

In contrast to the preceding sections now data driven critical values are considered in order to exhaust the FDR level of given SU tests. Much effort was done in order to establish adaptive SU tests which are based on the linear SU test of Benjamini and Hochberg [1]. These tests are typically based on conservatively biased estimators \hat{n}_0 of N in order to exploit the FDR level better. The approach is motivated by the substitution of α by $\alpha' = \frac{n}{\hat{n}_0}\alpha$ which leads to the heuristic $FDR \approx \alpha E(\frac{N}{\hat{n}_0}) \approx \alpha$ for consistent \hat{n}_0 and to data dependent BH type critical values

$$\hat{\alpha}_{i:n} = \frac{i}{\hat{n}_0} \alpha. \quad (6.1)$$

We refer to the well-known and frequently applied so called Storey type estimators given by the empirical distribution function \hat{F}_n of the p -values

$$\hat{n}_0(\lambda) = n \frac{1 - \hat{F}_n(\lambda) + \kappa_n}{1 - \lambda}, \quad \kappa_n > 0, \quad (6.2)$$

where λ is often chosen to be close to 0.5, see Storey et al. [24] and Storey [23] for the choice of $\kappa_n = \frac{1}{n}$. There are several estimators and conditions for FDR

control in the literature, for example see Benjamini et al. [3], Sarkar [21] and Zeisel et al. [25].

The finite sample FDR control of the adaptive SU test of Storey based on the critical values (6.1) and estimator (6.2) with $\kappa_n = \frac{1}{n}$ seems to be restricted to the BI model. Even for the reverse martingale model, which allows that some p -values coincide, further assumptions are required, see Example 6.1 below for instance.

The aim of this section is twofold for the reverse martingale model.

- In Section 6.1 sufficient conditions for estimators of N are introduced which ensure asymptotic FDR control.
- In Section 6.3 modified Storey SU tests are introduced which have finite sample FDR control for various block wise dependence models.

Moreover, we propose an adaptive multiple test for arbitrary dependent data and we also give a sufficient condition for the estimator and dependence structure, respectively, which again ensures asymptotic FDR control, compare with (2.13) and (2.14).

The estimators and multiple tests are based on various assumptions. Let $0 < \lambda < 1$ and divide $[0, 1]$ into two areas, the rejection area $[0, \lambda]$ and the estimation area $[\lambda, 1]$. Let us specify different assumptions.

(A1) The unknown value N is estimated by an estimator

$$\hat{n}_0 = g((\hat{F}_n(t))_{\lambda \leq t \leq 1}) > 0 \quad \text{almost surely} \tag{6.3}$$

via the empirical cumulative distribution function \hat{F}_n on the **estimation area** $[\lambda, 1]$ and a measurable function g .

(A2) The unknown value N is estimated by

$$\hat{n}_0 = g((\hat{F}_n(t))_{0 \leq t \leq 1}) > 0 \quad \text{almost surely.} \tag{6.4}$$

(A3) The multiple test is applied to the **rejection area** $[0, \lambda]$ with data dependent critical values

$$\hat{\alpha}_{i:n} = \left(\frac{i}{\hat{n}_0} \alpha \right) \wedge \lambda, \quad 1 \leq i \leq n. \tag{6.5}$$

(A4) The multiple test is applied to the **rejection area** $[0, 1]$ with the following data dependent critical values

$$\hat{\alpha}_{i:n} = \frac{\alpha}{n} \int_0^{in/\hat{n}_0} x d\nu(x), \quad 1 \leq i \leq n, \tag{6.6}$$

where ν is an arbitrary probability measure on $(0, \infty)$.

Taking the minimum in (6.5) goes back to Storey et al. [24] and ensures that one does not reject p -values greater than λ , therefore the name rejection area. Statisticians often do not like to reject a hypothesis when the p -value is too high. The estimated critical values (6.6) are based on the deterministic family of critical values (2.14) of Blanchard and Roquain [5] which also include the critical values (2.13) of Benjamini and Yekutieli [2].

6.1. Asymptotic results

The central Lemma 7.1 now establishes sufficient conditions for asymptotic FDR control of adaptive SU tests under different dependence structures.

Theorem 6.1. *Let \mathcal{P}_n be the set of all possible distributions of the reverse martingale models for fixed n and let $(P_n)_n$ be a sequence of distributions with $P_n \in \mathcal{P}_n$. Moreover, let $\hat{n}_{0,n}$ be a sequence of estimators for N_n which fulfill (A1). If*

$$P_n \left(\frac{\hat{n}_{0,n}}{N_n} \leq 1 - \delta \right) \rightarrow 0 \quad (6.7)$$

holds for all $\delta > 0$, where $\frac{x}{0} := \infty$ for $x > 0$, then

$$\limsup_{n \rightarrow \infty} FDR_{P_n} \leq \alpha \quad (6.8)$$

holds for the sequence of adaptive SU tests given by (A3).

Finner and Gontscharuk [8] and Gontscharuk [12] already used condition (6.7) to show asymptotic FWER control of a specific sequence of adaptive Bonferroni tests and adaptive SD tests, respectively. Under mild regularity assumptions, Liang and Nettleton [17] showed that the FDR of the adaptive SU test of Storey with altered estimator $\hat{n}_0(\lambda) = n \frac{1 - \hat{F}_n(\lambda)}{1 - \lambda}$ and critical values $\hat{\alpha}_{i:n} = \frac{i}{\hat{n}_0(\lambda)} \alpha$ is asymptotically controlled at level α for every arbitrary and data dependent selection of the tuning parameter λ out of a candidate set $\{0 = \lambda_0 < \dots < \lambda_m < 1\}$. This result may also be proved by application of Theorem 6.1, but note that Theorem 6.1 works for a much broader class of estimators and also under the reverse martingale model. The conservative consistency (6.7) is a very weak condition. Under mild regularity assumptions the crucial assumption (6.7) is also necessary for (6.8) for Storey type estimators (6.2).

Proposition 6.1. *Let $(P_n)_n$ be a sequence of reverse martingale models with either*

- (i) $\frac{N_n}{n} \rightarrow 1$ or
- (ii) $\xi_i \leq \lambda$ for all i and $0 < \eta \leq \frac{N_n}{n}$ for some η and all n .

Consider the FDR_{P_n} of the sequence of SU tests based on (A3) and (6.2) so that (6.8) holds. Let $\kappa_n \rightarrow 0$ and suppose that $P_n(\{\hat{\alpha}_{R:n} = \lambda\}) \rightarrow 0$ holds as $n \rightarrow \infty$. Then the ratio $\frac{\hat{n}_0}{N_n} \rightarrow 1$ converges to one in P_n -probability as $n \rightarrow \infty$.

Remark 6.1 (About asymptotic FDR of Storey type SU tests). Consider the reverse martingale model. As long as enough variability of the variables $(\epsilon_i 1\{p_i \leq \lambda\})_{i \leq n}$ is present condition (6.7) can be verified. Let \hat{n}_0 be the estimator (6.2) for some positive sequence κ_n . Let

$$\hat{F}_{0,n}(\lambda) = \frac{1}{N_n} \sum_{i=1}^n \epsilon_i 1\{p_i \leq \lambda\} \quad (6.9)$$

be the cumulative distribution function of true p -values. Then a sufficient condition to ensure (6.7) is (6.10), where the conditional variances

$$\text{Var}_{P_n} \left(\hat{F}_{0,n}(\lambda) \middle| (\epsilon_i)_{i \leq n} \right) = E_{P_n} \left(\left(\hat{F}_{0,n}(\lambda) - \lambda \right)^2 \middle| (\epsilon_i)_{i \leq n} \right) \rightarrow 0 \quad (6.10)$$

tends to zero in probability as $n \rightarrow \infty$. The corresponding SU tests then have asymptotic FDR control.

At this point, Theorem 6.1 can be extended to treat arbitrary p -values. Like Benjamini and Yekutieli [2] and Blanchard and Roquain [5], who considered non data dependent SU tests for arbitrary dependence structures. Therefore we have to consider more conservative test procedures. The adaptive SU test (A4) is based on the critical values (2.14) of Blanchard and Roquain [5] and yields asymptotic FDR control if (6.7) is satisfied.

Theorem 6.2. *Let $\tilde{\mathcal{P}}_n$ be the set of all possible distributions of the p -value model (2.2) for fixed n , where each U_1, \dots, U_n is distributed according to the uniform distribution on $(0, 1)$. No further distributional assumption and no dependence structure is assumed. Again, let $(P_n)_n$ be a sequence of distributions with $P_n \in \tilde{\mathcal{P}}_n$ and $\hat{n}_{0,n}$ be a sequence of estimators for N_n which fulfill (A2). If (6.7) is fulfilled, then (6.8) holds for the sequence of adaptive SU tests given by (A4).*

6.2. Finite sample results

Let us now come back to finite sample FDR control. We will give a condition for FDR control for the reverse martingale model and the next very useful Lemma offers an exact formula for the FDR of our adaptive tests.

Lemma 6.1. *Let $V(\lambda) := \#\{p_i \leq \lambda, p_i \text{ true}\}$ denote the number of true null hypotheses with p -values $p_i \leq \lambda$. Under the reverse martingale model, the adaptive SU test with critical values (A3) and estimator (A1) fulfills*

$$E \left(\frac{V}{R} \right) = \frac{\alpha}{\lambda} E \left(V(\lambda) \min \left\{ \frac{1}{\hat{n}_0}, \frac{\lambda}{n \hat{F}_n(\lambda) \alpha} \right\} \right) \leq \frac{\alpha}{\lambda} E \left(\frac{V(\lambda)}{\hat{n}_0} \right). \quad (6.11)$$

This result generalizes Lemma 3.1 of Heesen and Janssen [15], where the BI model is treated only. For the control of the FDR by α one merely has to show

$$E \left(\frac{V(\lambda)}{\hat{n}_0} \right) \leq \lambda. \quad (6.12)$$

For example the benchmark result of Liang and Nettleton [17, Theorem 7] works for estimators \hat{n}'_0 with $\hat{n}'_0 \geq \hat{n}_0(\lambda)$ almost surely with $\hat{n}_0(\lambda)$ defined in (6.2) with $\kappa_n = \frac{1}{n}$ and for some $\lambda \in [0, 1)$. In comparison to that, Lemma 6.1 works for the class of estimators (A1) and also for the reverse martingale model. Some interesting estimators which do not satisfy $\hat{n}'_0 \geq \hat{n}_0(\lambda)$ are given Heesen and Janssen [15].

The following negative result explains first that the use of adaptive SU tests is limited under dependence and further results are needed for finite sample FDR control.

Proposition 6.2. *Consider an adaptive SU test based on (A1), (A3) and $\alpha < \lambda$. Assume that the estimator $p_i \mapsto \hat{n}_0$ is non-decreasing for each coordinate i . If we have $FDR \leq \alpha$ for all reverse martingale models then $\hat{n}_0 \geq n$ holds and the adaptive critical values $\hat{\alpha}_{i:n} \leq \frac{i}{n}\alpha$ are dominated by the BH critical values.*

The adaptive step up test of Storey does not yield FDR control in the reverse martingale and PRDS model. For instance Blanchard and Roquain [4, Theorem 17] proved that in case of $n = n_0 \geq 2$ and $p_1 = \dots = p_n = U$, for a uniform distributed U on $(0, 1)$,

$$E\left(\frac{V}{R}\right) = \min(\alpha n(1 - \lambda), \lambda)$$

holds for the Storey estimator (6.2) with $\kappa_n = \frac{1}{n}$. This result corresponds to Lemma 6.1. Note that condition (6.7) is violated since $\frac{1}{n_0} = 1 - \lambda$ holds on $\{U \leq \lambda\}$ for the Storey estimator (6.2).

6.3. Case of a block model

Finally the following modified adaptive SU test is considered when mild additional dependence assumptions are present. Suppose that the p -values can be divided by

$$\{p_1, \dots, p_n\} = \bigcup_{i=1}^k G_i \tag{6.13}$$

in k disjoint blocks or groups G_i . Suppose the reverse martingale condition for (p_1, \dots, p_n) . Assume in addition that for each group the subset $\tilde{G}_i \subset G_i$ corresponds to uniformly distributed p -values given by true null hypotheses. Below let the groups \tilde{G}_i , $1 \leq i \leq k$, be conditionally independent given the signs ϵ whereas within group \tilde{G}_i a reverse martingale dependence structure is allowed.

Remark 6.2. In practice the model may have the following meaning for genome data. \tilde{G}_i may stand for independent portions of true p -values which may come from different chromosomes. The p -values of \tilde{G}_i may be reverse martingale dependent, for instance some of them may be equal.

Consider the maximal size m of the groups

$$m := \max_{i \leq k} |G_i| \quad \text{and} \quad n = mk - r \tag{6.14}$$

with a remainder $r \geq 0$. Furthermore, let us assume that the number of true p -values N is almost surely lower bounded by N_{min} . For the tuning parameters $0 < \lambda < 1$ and $\kappa \geq 1$ the modified Storey estimator

$$\hat{n}_0(\kappa) := n \frac{1 - \hat{F}_n(\lambda) + \frac{\kappa}{n}}{1 - \lambda} \tag{6.15}$$

with $\kappa_n = \frac{\kappa}{n}$ is introduced. Again (6.15) can be improved by the factor $(1 - \lambda^k)$, i.e. also $(1 - \lambda^k)\hat{n}_0(\kappa)$ will work. We show that the step up test with estimated critical values

$$\hat{\alpha}_{i:n} = \left(\frac{i}{\hat{n}_0(\kappa)} \alpha \right) \wedge \lambda \tag{6.16}$$

yields FDR control, $FDR \leq \alpha$, under the present block wise dependence model for all $\kappa \geq m + r + (n - N_{min})$. If the groups are balanced, i.e. $|G_1| = \dots = |G_k|$ holds, then r vanishes and the best fit is expected. Of course $\hat{n}_0(\kappa) > n$ may happen for large r in the unbalanced case and there would then be no advantage in comparison with the BH test when the p -values are all independent.

Theorem 6.3. *Consider the reverse martingale model for the p -values. Assume that the p -values can be divided in $k \geq 2$ disjoint groups, see (6.13) and (6.14) above. Moreover, assume that $N \geq N_{min}$ holds almost surely for a lower bound $N_{min} > 0$. Let conditionally on the signs ϵ the groups $\tilde{G}_1, \dots, \tilde{G}_k$ of the true p -values be independent. If $\kappa \geq m + r + (n - N_{min})$ holds, then the modified adaptive SU test with critical values (6.16) and estimator (6.15) has finite sample FDR control, i.e. $FDR \leq \alpha$, and (6.12) holds.*

Gontscharuk [12] considered a similar block model which leads to dependent p -values and an adaptive Bonferroni type procedure with asymptotic FWER control. Theorem 6.3 works for finite n .

If the group structure is known and balanced with $|G_1| = \dots = |G_k| = m$, then Guo and Sarkar [14] propose an adaptive multiple test with FDR control under PRDS within each group. The ingredients are based on the Storey type estimator (6.2), where λ depends on the number of blocks and $\kappa = \frac{m}{n}$. However, every rejected p -values has to be less than or equal to $\frac{k}{\hat{n}_0(\lambda)}\alpha$ in comparison to $\frac{n}{\hat{n}_0(\lambda)}\alpha$ for the adaptive SU test considered in Theorem 6.3.

As mentioned above the estimator (6.15) may produce very conservative SU tests for independent p -values. However, Theorem 6.3 is designed for block dependent p -values and we will see by the inspection of the FWER that this procedure can not always be improved. The necessary calculations for Example 6.1 are included in the proof of Theorem 6.3.

Example 6.1. Consider k blocks of size m with $N = n = mk$ true p -values. Suppose that for each block all p -values coincide with the same uniformly distributed random variable whereas the blocks are independent. The model has both the PRDS and the reverse martingale property. When $\alpha \leq \frac{\lambda}{k(1-\lambda)}$ holds then the choice $\kappa = m$ for the procedures (6.15) and (6.16) yields

$$FDR = FWER = \alpha(1 - \lambda^k). \tag{6.17}$$

Here the modified estimator $(1 - \lambda^k)\hat{n}_0(\kappa)$ attains the FWER bound α and we obtain a sharp result for the block model.

We conducted a small Monte-Carlo simulation with 100.000 repetitions to explore the situation beyond the case of Example 6.1 for the adaptive SU test with critical values (6.16) and estimator (6.15) with $\alpha = 0.05$ at size $n = 100$. All false p -values are set to 0. Consider a setting of $k = 5$ groups with $m = 20$ p -values and

16 equal true p -values in each group. The choices of $\kappa = 1, 16, 20$ lead to $FDR \approx 0.0886, 0.0476, 0.0438$. Furthermore, the group frequencies 25, 25, 20, 15, 15 with 20, 20, 16, 12, 12 equal true p -values within these groups and the choices of $\kappa = 1, 12, 20, 25$ lead to $FDR \approx 0.0921, 0.0567, 0.0446, 0.0385$. Since the number of true p -values within each group is unknown, the simulation and Example 6.1 indicate that here $\kappa \approx m$ is an appropriate tuning parameter for the adaptive SU test. Moreover, a simulation with equal group frequencies under the global intersection hypothesis shows that κ slightly smaller than m already yields $FDR > \alpha$. For $k = 10$ groups of equal size with $m = 100$ and only true p -values in each group, the choice of $\kappa = 97$ yields $FDR \approx 0.051$.

7. Technical results and proofs

Lemma 7.1. (a) Let $0 < \hat{\alpha}_{1:n} \leq \dots \leq \hat{\alpha}_{n:n} \leq \lambda < 1$ be data dependent critical values

$$\hat{\alpha}_{i:n} = g_i((\hat{F}_n(t))_{t \geq \lambda}), \quad i = 1, \dots, n, \quad (7.1)$$

given by measurable functions g_i and introduce $\hat{\alpha}_{0:n} = \hat{\alpha}_{1:n}$. Moreover define $\gamma(i) := n\hat{\alpha}_{i:n}$. Then

$$E\left(\frac{V}{\gamma(R)}\right) = \frac{E(N)}{n} \quad (7.2)$$

holds for the corresponding adaptive SU tests under the reverse martingale model (including the BI model).

(b) Let $\hat{\rho}(i) = \hat{\rho}(i, (\hat{F}_n(t))_{0 \leq t \leq 1}) > 0$, $i = 0, \dots, n$, be non-decreasing in i and let $\gamma(i) := \alpha \hat{\rho}(i)$. Moreover, assume that ν is a probability measure on $(0, \infty)$ and define the data dependent critical values via

$$\hat{\alpha}_{i:n} = \frac{\alpha}{n} \int_0^{\hat{\rho}(i)} x d\nu(x). \quad (7.3)$$

Then “ \leq ” holds in (7.2) for the corresponding adaptive SU test with critical values (7.3) for arbitrary dependent variables $(\epsilon_i, U_i, \xi_i)_{i \leq n}$.

Remark 7.1. (a) Lemma 7.1 (a) also applies to deterministic critical values $0 < \alpha_{1:n} \leq \dots \leq \alpha_{n:n} < 1$ if we put $\lambda = \alpha_{n:n}$. In this case, the reverse martingale assumption (2.5) can be weakened in order to prove (7.2). It is only necessary to assume that (2.5) is a reverse martingale w.r.t. the discrete parameter set $I := \{\alpha_{1:n}, \dots, \alpha_{n:n}, 1\}$ and $t \in I$.

(b) Storey et al. [24] already used martingale arguments which have been outlined by Scheer [22].

(c) In case of deterministic critical values $0 < \alpha_{1:n} < \dots < \alpha_{n:n} < 1$ we obtain the inequality “ \leq ” in (7.2) under the PRDS model. The proof follows straightforward classical lines, see Heesen [16] and Meskaldji et al. [19].

PROOF OF LEMMA 7.1. (a) Observe that the SU test can be represented by the reverse stopping time

$$\tau = \sup\{\hat{\alpha}_{i:n}, i = 1, \dots, n : p_{i:n} \leq \hat{\alpha}_{i:n}\} \vee \hat{\alpha}_{1:n},$$

which is adapted to the reverse Filtration $(\mathcal{F}_t)_{0 < t \leq 1}$ and where $\sup \emptyset := 0$. Then every p -value $p_i \leq \tau$ is rejected. For $V(t) := \#\{p_i \leq t, p_i \text{ true}\}$ and $R(t) = n\hat{F}_n(t)$ for $0 \leq t \leq 1$ we have $\frac{V}{\gamma(R)} = \frac{V(\tau)}{\gamma(R(\tau))}$. Conditioned under \mathcal{F}_λ the critical values $\hat{\alpha}_{1:n}, \dots, \hat{\alpha}_{n:n}$ are fixed and therefore, τ is a discrete stopping time w.r.t. the reverse martingale (2.5) for the periode $\hat{\alpha}_{1:n} \leq t \leq \lambda$. Furthermore, observe that $\hat{\alpha}_{R(\tau):n} = \tau$ holds if $R > 0$ since $R(\tau) = R$. Thus by (2.5) and the discrete version of the optional stopping theorem

$$\begin{aligned} E\left(\frac{V}{\gamma(R)} \middle| \mathcal{F}_\lambda\right) &= E\left(\frac{V(\tau)}{\gamma(R(\tau))} \middle| \mathcal{F}_\lambda\right) = E\left(\frac{V(\tau)}{n\hat{\alpha}_{R(\tau):n}} \mathbf{1}\{R(\tau) > 0\} \middle| \mathcal{F}_\lambda\right) \\ &= \frac{1}{n} E\left(\frac{V(\tau)}{\tau} \mathbf{1}\{R(\tau) > 0\} \middle| \mathcal{F}_\lambda\right) = \frac{1}{n} E\left(\frac{V(\tau)}{\tau} \middle| \mathcal{F}_\lambda\right) \\ &= \frac{1}{n} E\left(\frac{V(\lambda)}{\lambda} \middle| \mathcal{F}_\lambda\right) \end{aligned}$$

holds and integration yields

$$E\left(\frac{V}{\gamma(R)}\right) = \frac{1}{n} E\left(\frac{V(\lambda)}{\lambda}\right) = \frac{1}{n} E\left(\frac{V(1)}{1}\right) = \frac{E(N)}{n}.$$

(b) Applying the technique of the proof of Lemma 3.2 of Blanchard and Roquain [4] yields

$$E\left(\frac{V}{\gamma(R)} \middle| \epsilon\right) = \sum_{i: \epsilon_i=1} E\left(\frac{\mathbf{1}\{p_i \leq \hat{\alpha}_{R:n}\}}{\alpha \hat{\rho}(R)} \middle| \epsilon\right) \leq \frac{N}{n}.$$

There, the technique is formulated for deterministic critical values, but observe that it also works for data dependent critical values. \square

When the proof was finished we came across the early paper of Meskaldji et al. [19] which covers the special non-adaptive case of our technical Lemma 7.1 (b). For deterministic critical values their proof also follows the lines of Blanchard and Roquain [5].

7.1. Proofs of Section 3

PROOF OF EXAMPLE 3.1. Below the proof of part (a) is sketched. The calculations for (b) are similar. Recall from Finner et al. [9], p. 604, the FDR formula for $n = n_0 = 2$

$$\begin{aligned} FWER &= FDR(2) \\ &= \alpha \left[P\left(R \geq 1 \middle| p_1 \leq \frac{\alpha}{2}\right) - P\left(R \geq 2 \middle| p_1 \leq \frac{\alpha}{2}\right) + P\left(R \geq 2 \middle| p_1 \leq \alpha\right) \right] \end{aligned} \tag{7.4}$$

since $(X_1, X_2) \stackrel{d}{=} (X_2, X_1)$ holds for the normal sample. For $\alpha = \frac{1}{2}$ we have by conditioning w.r.t. $X_1 = x_1$

$$P\left(R \geq 2, p_1 \leq \frac{1}{2}\right) = P\left(p_1 \leq \frac{1}{2}, p_2 \leq \frac{1}{2}\right) = P(X_1 \leq 0, X_2 \leq 0)$$

$$= \int_0^\infty \Phi(x_1)\phi(x_1)dx_1,$$

where $\phi = \Phi'$. Similarly,

$$P\left(R \geq 2, p_1 \leq \frac{1}{4}\right) = \int_{-\Phi^{-1}(\frac{1}{4})}^\infty \Phi(x_1)\phi(x_1)dx_1. \tag{7.5}$$

Notice that

$$\int_a^\infty \Phi(x)\phi(x)dx = \frac{1}{2}\Phi(x)^2\Big|_a^\infty = \frac{1}{2}[1 - \Phi(a)^2] \tag{7.6}$$

follows. Thus, $P(R \geq 2|p_1 \leq \frac{1}{2}) = \frac{3}{4}$ and $P(R \geq 2|p_1 \leq \frac{1}{4}) = \frac{7}{8}$ holds which implies the result. \square

PROOF OF EXAMPLE 3.3. Part (b) and (c) are obvious. To prove (a) define

$$M_t^{(i)} = \frac{1\{Z_i \leq t\}}{H(t)}, \quad \text{for } t \in \{H > 0\}.$$

In case $n = 1$ it is well known that $M_t^{(1)}$ is a reverse martingale w.r.t. $G_t = \sigma(M_s^{(1)} : s \geq t)$. In case $n > 1$ let us prove that $M_t^{(1)}$ is a reverse martingale w.r.t. $\mathcal{F}_t := \sigma((M_s^{(j)})_{s \geq t}, 1 \leq j \leq n)$. Obviously, $E(M_t^{(1)}|\mathcal{F}_s) = 0 = M_s^{(1)}$ holds if $Z_1 > s$. Otherwise, $Z_1 \leq s$ implies $X_1 \leq s$ and $Y \leq s$. Thus $1\{Z_i \leq \tau\} = 1\{X_i \leq \tau\}$ holds for $i \geq 2$ and all $\tau \geq s$. Let $f_i : [s, \infty) \rightarrow \{0, 1\}^{[s, \infty)}$ be a possible path of $\tau \mapsto 1\{X_i \leq \tau\}$, $\tau \geq s$. If $1\{Z_1 \leq s\} = 1$ holds we have

$$\begin{aligned} & E(M_t^{(1)}|\mathcal{F}_s) \\ &= E(M_t^{(1)}|1\{Z_1 \leq s\} = 1, (1\{X_i \leq \tau\})_{\tau \geq s} = (f_i(\tau))_{\tau \geq s}, i = 2, \dots, n) \\ &= E(M_t^{(1)}|1\{Z_1 \leq s\} = 1) \\ &= E\left(M_t^{(1)}\Big| M_s^{(1)} = \frac{1}{H(s)}\right) = M_s^{(1)}. \end{aligned}$$

Above we used that $1\{Z_1 \leq s\}$ and (X_2, \dots, X_n) are independent. Observe that the time change

$$u \mapsto M_{H^{-1}(u)}^{(i)} = \frac{1\{H(Z_i) \leq u\}}{u}$$

by the inverse distribution function H^{-1} preserves the reverse martingale. \square

7.2. Proofs of Section 4

Lemma 7.2. (a) Assume the PRDS model or reverse martingale model (which include the BI model). Consider the SU test with deterministic critical values (2.6) and suppose that there exist some k and $0 < c < 1$ with $\alpha_{j:n} \leq \frac{jc}{n}$ for all $j \leq k$.

- (i) Then $FDR \leq \frac{cE(N)}{n} + P(R > k)$ holds.
- (ii) Suppose that the assumption holds for $k = n$. If in addition $\alpha_{j:n} < \frac{jc}{n}$ holds for a fixed j with $P(R = j, V > 0) > 0$, then $FDR < \frac{cE(N)}{n}$ follows for the BI model.

(b) Assume the reverse martingale model, consider the SU test with critical values (2.6) and suppose that $\alpha_{j:n} \geq \frac{ic}{n}$ holds for all $j \leq n$.

(i) Then $FDR \geq \frac{cE(N)}{n}$ holds.

(ii) If in addition $\alpha_{j:n} > \frac{ic}{n}$ holds for some j with $P(R = j, V > 0) > 0$, then $FDR > \frac{cE(N)}{n}$ holds at least under the BI model.

Proof. (a) (i) Let $\gamma(i) = n\alpha_{i:n}$ and observe that $\frac{\gamma(j)}{j} \leq c$ holds for all $j \leq k$. Then Lemma 7.1 (a) and Remark 7.1 (c) imply

$$\begin{aligned} FDR &\leq E\left(\frac{V}{\gamma(R)} \frac{\gamma(R)}{R} 1_{\{R \leq k\}}\right) + P(R > k) \\ &\leq \frac{cE(N)}{n} + P(R > k) \end{aligned}$$

for the PRDS and reverse martingale model, respectively.

(ii) By (i) we already know $FDR \leq \frac{cE(N)}{n}$ and in case of equality we have $0 = E\left(\frac{cV}{\gamma(R)} - \frac{V}{R}\right)$. But observe that $\frac{cV}{\gamma(R)} - \frac{V}{R} \geq 0$ holds. Thus,

$$E\left(V\left(\frac{c}{\gamma(R)} - \frac{1}{R}\right)\right) \geq \left(\frac{c}{\gamma(j)} - \frac{1}{j}\right) \cdot P(R = j, V > 0) \tag{7.7}$$

follows by our assumptions, a contradiction.

(b) (i) Observe that $\frac{\gamma(j)}{j} \geq c$ holds for all $j \leq n$. Thus Lemma 7.1 implies

$$FDR = E\left(\frac{V}{\gamma(R)} \frac{\gamma(R)}{R}\right) \geq cE\left(\frac{V}{\gamma(R)}\right) \geq \frac{cE(N)}{n}$$

for the reverse martingale model.

(ii) Observe that $\frac{cV}{\gamma(R)} - \frac{V}{R} \leq 0$ and $\frac{c}{\gamma(j)} - \frac{1}{j} < 0$ hold and the assertion follows in the same way as in (a) (ii). \square

PROOF OF PROPOSITION 4.1. The proposition is a direct application of Lemma 7.1 (a) and Remark 7.1 (c) with deterministic critical values. Again let $\gamma(i) = n\alpha_{i:n}$, $0 \leq i \leq n$. For the reverse martingale and PRDS models we have

$$FDR = E\left(\frac{V}{\gamma(R)} \frac{\gamma(R)}{R}\right) \leq E\left(\frac{V}{\gamma(R)}\right) \max_{i \leq n} \frac{n\alpha_{i:n}}{i} \leq \frac{E(N)}{n} \max_{i \leq n} \frac{n\alpha_{i:n}}{i} \tag{7.8}$$

by Lemma 7.1 (a) and Remark 7.1 (c). Under the assumptions of (b) (i) the first inequality in (7.8) is actually a strict inequality “<”. The other inequalities follow analogously. \square

PROOF OF LEMMA 4.1. (a) In this proof we may always choose the Dirac uniform case $\vec{f} = (p_j : \epsilon_j = 0) = (0, \dots, 0)$ for the false p -values. Let $N = n_0 = n + 1 - j$ be deterministic. Thus by Lemma 7.1 (a)

$$\frac{n_0}{n} = E\left(\frac{V}{\gamma(R)} \mid n_0, \vec{f}\right) = E\left(\frac{V}{R} \frac{R}{\gamma(R)} \mid n_0, \vec{f}\right)$$

$$\leq E\left(\frac{V}{R} \mid n_0, \bar{f}\right) \frac{j}{\gamma(j)}$$

since $R \geq j$ holds when V is positive and $j \mapsto \frac{j}{\gamma(j)} = \frac{j\alpha}{n\alpha_{j:n}}$ is non-increasing.

By our assumption the inequality $E(\frac{V}{R} \mid n_0, \bar{f}) \leq \alpha$ proves the result.

(b) In that case we have $\frac{j}{\gamma(j)} \geq \frac{k}{\gamma(k)} > \frac{k+1}{\gamma(k+1)}$ for $j \leq k$. Obviously, $P(R = k+1 \mid n_0, \bar{f}) > 0$ holds for $N = n_0 = n+1-j$ with $j \leq k$. Thus we have strict inequality in the proof of part (a) and $\frac{n_0}{n} < \frac{j\alpha}{\gamma(j)}$ follows. \square

PROOF OF COROLLARY 4.1. (a) is a special case of Lemma 4.1.

(b) Assume the multiple test is not a BH test. Take the first value k with

$$\frac{\alpha}{n} = \frac{\alpha_{1:n}}{1} = \dots = \frac{\alpha_{k:n}}{k} < \frac{\alpha_{k+1:n}}{k+1}.$$

Then Lemma 4.1 (b) implies $\alpha_{1:n} < \frac{\alpha}{n}$ which contradicts our assumption.

(c) On the other hand $\alpha_{1:n} = \frac{\beta}{n}$ implies $\frac{\alpha_{j:n}}{j} \geq \frac{\beta}{n}$ and $FDR \geq \frac{\beta E(N)}{n}$ by Lemma 7.2 (b). \square

The following technical tools of Lemma 7.3 and Remark 7.2 supplement the inequalities of Section 4.

Lemma 7.3. Consider SU tests based on the critical values (4.7) with $0 < a < b$ with fixed value b and an adjustment of a .

(a) Under the reverse martingale model and the Dirac uniform configuration $DU(n_0)$ the FDR is given by

$$g_{a,b}(n_0) := FDR_{DU}(n_0) = \frac{\alpha n_0}{n+b} + \frac{a E_{DU}(V \mid n_0)}{n+b}, \quad (7.9)$$

with “ \leq ” under PRDS. Then

$$\sup_{P \in \mathcal{P}_{BI}} FDR_{(b,a)} = \max_{1 \leq n_0 \leq n} g_{a,b}(n_0) \quad (7.10)$$

holds.

(b) The following inequality holds for the expected number of false rejections $E_{DU}(V \mid n_0)$ under $DU(n_0)$

$$n_0 \alpha_{n+1-n_0:n} \leq E_{DU}(V \mid n_0).$$

Note that this statement holds without any dependence assumption on U_1, \dots, U_n .

Proof. Part (a) follows from (7.2) which reads under DU with $n_1 = n - n_0$ as

$$\frac{n_0}{n} = \frac{1}{n\alpha} E\left(\frac{V(n+b) - (n_1+V)aV}{n_1+V}\right).$$

Equation (7.10) holds since the DU configuration is least favorable.

(b) Without restrictions we assume that the true p -values are given by p_1, \dots, p_{n_0} . Then we obtain

$$\begin{aligned} E_{DU}(V|n_0) &= \sum_{i=1}^{n_0} E_{DU}(1\{p_i \leq \alpha_{R:n}\}|n_0) \\ &\geq \sum_{i=1}^{n_0} E_{DU}(1\{p_i \leq \alpha_{n+1-n_0:n}\}|n_0) = n_0\alpha_{n+1-n_0:n} \end{aligned}$$

since $R \geq n + 1 - n_0$ holds on $\{p_i \leq \alpha_{R:n}\}$ for the DU configuration for all true p_i . \square

By different methods Scheer [22] obtained (7.9) for $a = 1 - \alpha$.

Remark 7.2. The expected number of false rejections $h(n_0, \alpha) = E_{DU}(V^{BH,\alpha}|n_0)$ is easy to compute by the following recursion $h(1, \alpha) = \alpha$ and

$$h(n_0, \alpha) = \frac{n_0\alpha}{n} [h(n_0 - 1, \alpha) + n - n_0 + 1] \tag{7.11}$$

for the BH SU test which equals by induction

$$h(n_0, \alpha) = \frac{n_0!}{n^{n_0-1}} \alpha^{n_0} + \sum_{j=1}^{n_0-1} \frac{n_0!}{j!} \left(\frac{\alpha}{n}\right)^{n_0-j} (n - j). \tag{7.12}$$

Formula (7.12) is due to Finner and Roters [10, p. 991] which is now a direct consequence of our equalities, see (7.14).

PROOF OF THE RECURSION (7.11) OF REMARK 7.2. Consider a true p -value, say p_1 , of $p = (p_1, \dots, p_n)$. Put $p^{(1)} = (0, p_2, \dots, p_n)$ and let $R(p), R(p^{(1)})$ be the number of rejections. Simple calculations show that $R(p) = R(p^{(1)})$ holds on the set $\{p_1 \leq \alpha_{R(p):n}\}$. Observe, when p_1 is rejected, then p_1 could also be zero. Moreover, we have $\{p_1 \leq \alpha_{R(p):n}\} = \{p_1 \leq \alpha_{R(p^{(1)}):n}\}$ in any case for SU tests which implies

$$E(1\{p_1 \leq \alpha_{R(p):n}\}|n_0) = E(\alpha_{R(p^{(1)}):n}|n_0) = \frac{\alpha}{n} E(R(p^{(1)})|n_0) \tag{7.13}$$

by Fubini's theorem and $E(V|n_0) = \alpha \frac{n_0}{n} E(R(p^{(1)}|n_0)$. Under $DU(n_0)$ we obtain

$$E_{DU}(R(p^{(1)}|n_0) = E_{DU}(V(p^{(1)}) + n - n_0|n_0) = E_{DU}(V(p) + n - n_0 + 1|n_0 - 1) \tag{7.14}$$

and the recursion. Formula (7.12) follows by induction. \square

Remark 7.3. Under the BI model the proof of statement (7.2) can be simplified as follows. Consider deterministic critical values. Using the proof of Remark 7.2 above we may again conclude by Fubini's Theorem

$$E\left(\frac{1\{p_1 \leq \alpha_{R:n}\}}{\gamma(R)} \Big| \epsilon_1 = 1\right) = E\left(\frac{1\{p_1 \leq \alpha_{R(p^{(1)}):n}\}}{\gamma(R(p^{(1)}))} \Big| \epsilon_1 = 1\right)$$

$$= E \left(\frac{\alpha_{R(p^{(1)}) : n}}{\gamma(R(p^{(1)}))} \mid \epsilon_1 = 1 \right) = \frac{1}{n}.$$

The method of proof can be used to prove the discussion about least favorable “false p-values” given by Benjamini and Yekutieli [2] Theorem 5.3.

7.3. Proofs of Section 5.1

PROOF OF PROPOSITION 5.1. By (7.10) we may concentrate on the DU case for the BI model. Note also that $g_{a,b}(n_0)$ is strictly increasing in a since the critical values are ordered. On the other hand expression (7.10) converges to $\frac{\alpha n}{n+b} = \alpha' < \alpha$ for $a \searrow 0$. Thus, the solution a_1 of (5.1) is unique and it is easy to see that $0 < a_1 < b$ holds. \square

Lemma 7.3 leads to a new upper bound for the crucial parameter a_1 established in Proposition 5.1.

Proposition 7.1. *Consider the assumptions of Proposition 5.1. Introduce a_0 as the unique positive solution of*

$$\alpha = \max_{1 \leq n_0 \leq n} \left(\frac{\alpha n_0}{n+b} + \frac{ah(n_0, \alpha')}{n+b} \right), \quad (7.15)$$

where $\alpha' := \frac{\alpha n}{n+b}$. Then the crucial parameter a_0 satisfies $0 < a_1 < a_0$.

Proof. Observe first, that the coefficients $\alpha_{i:n} \geq \frac{\alpha' i}{n}$ dominate the BH critical values for the choice of α' . Thus we have $E_{DU}(V | n_0) \geq h(n_0, \alpha')$ and

$$g_{a,b}(n_0) \geq \frac{\alpha n_0}{n+b} + a \frac{h(n_0, \alpha')}{n+b}.$$

Hence, it is easy to see that the solution a_1 of (5.1) satisfies $0 < a_1 < a_0$. \square

PROOF OF PROPOSITION 5.2. The requirements (4.5) remain true for the new coefficients (5.2) and we may restrict ourselves to the worst DU case. Here we have

$$FDR_{DU}(n_0) = E \left(\frac{V}{n - n_0 + V} \right).$$

We see that $FDR_{DU}(n_0)$ is ordered by “ \leq ” when the critical values and thus the V ’s are ordered since $x \mapsto \frac{x}{n-n_0-x}$ increases. Observe that $k = 1$ yield a BH test with $FDR < \alpha$ and the proof is finished. \square

7.4. Proofs of Section 5.2

In regular cases the inequalities (4.1) are asymptotically sharp. For this purpose assume that the critical values are generated by a function h via

$$\alpha_{i:n} = h \left(\frac{i}{n} \right), \quad 1 \leq i \leq n. \quad (7.16)$$

At this stage we require that $h : [0, 1] \rightarrow [0, 1]$ is a non-decreasing function with $h(0) = 0$, $h(1) < 1$ and $0 < c_1 = \inf_{0 < x < 1} \frac{h(x)}{x}$, $\sup_{0 < x < 1} \frac{h(x)}{x} = c_2 < 1$. Then under regularity assumptions these bounds (4.1) are asymptotically sharp in the sense that for $c \in \{c_1, c_2\}$ there exist sequences of BI distributions P_n so that

$$\lim_{n \rightarrow \infty} FDR_{P_n} = \lim_{n \rightarrow \infty} \frac{E(N_n)}{n} c. \tag{7.17}$$

Lemma 7.4. *Consider the SU test with critical values (7.16) given by a continuous function h . Suppose that $\frac{R_n}{n} \rightarrow K$ converges in probability for some $0 < K < 1$ and suppose that $0 < \lim_{n \rightarrow \infty} \frac{E(N_n)}{n} < 1$ exists. Then $\lim_{n \rightarrow \infty} FDR_n = \frac{h(K)}{K} \lim_{n \rightarrow \infty} \frac{E(N_n)}{n}$.*

Suppose that $c_1 = \frac{h(x_1)}{x_1}$ as well as $c_2 = \frac{h(x_2)}{x_2}$ are attained for some $x_1, x_2 \in (0, 1)$. Furthermore, assume that there exist sequences of distributions with $\frac{R_n}{n} \rightarrow x_1, x_2$, in probability, respectively, and $0 < \lim_{n \rightarrow \infty} \frac{E(N_n)}{n} < 1$. Then Lemma 7.4 can be applied in order to get sharp bounds in (4.1).

PROOF OF LEMMA 7.4. Let $A(K)$ be an open neighborhood of K with $\frac{h(x)}{x} \geq d > 0$ on $A(K)$. Then

$$\lim_{n \rightarrow \infty} FDR_n = \lim_{n \rightarrow \infty} E \left(\frac{V_n}{R_n} 1 \left\{ \frac{R_n}{n} \in A(K) \right\} \right)$$

holds. Introduce

$$0 \leq Z_n := \frac{V_n}{\gamma(R_n)} 1 \left\{ \frac{R_n}{n} \in A(K) \right\} = \frac{V_n}{nh(\frac{R_n}{n})} 1 \left\{ \frac{R_n}{n} \in A(K) \right\} \leq \frac{1}{d}$$

which is a tight sequence of random variables. On the other hand the bounded sequence of random variables

$$W_n := \frac{nh(\frac{R_n}{n})}{R_n} \rightarrow \frac{h(K)}{K}$$

converges in probability. Turning to distributional convergent subsequences of Z_n we finally obtain

$$\lim_{n \rightarrow \infty} E \left(\frac{V_n}{R_n} \right) = \lim_{n \rightarrow \infty} E(Z_n W_n) = \frac{h(K)}{K} \lim_{n \rightarrow \infty} \frac{E(N_n)}{n}$$

since $\lim_{n \rightarrow \infty} E(Z_n) = \lim_{n \rightarrow \infty} \frac{E(N_n)}{n}$ holds by Lemma 7.1. □

PROOF OF THEOREM 5.1. The present proof is given by several steps. Below the following elementary geometric property of the AORC is used. Consider for $0 < \alpha < 1$ the function $A(x, y) := \frac{x}{1-x} \frac{1-y}{y}$ for $(x, y) \in (0, 1)^2$ in the plain. Then

1. $A(x, y) = \alpha$ iff (x, y) belongs to the graph of f_α , i.e. $f_\alpha(x) = y$.
2. $A(x, y) > \alpha$ iff (x, y) is below the graph of f_α , i.e. $f_\alpha(x) > y$.
3. $A(x, y) < \alpha$ iff (x, y) lies above the graph of f_α , i.e. $f_\alpha(x) < y$.

(I) We claim $0 < \beta < 1$. We first show $\beta > 0$. Therefore, choose $n_0 = \lfloor \frac{n}{2} \rfloor$ and a Dirac uniform $DU(n, n_0)$ configuration. Then $\frac{R_n}{n} \geq \frac{1}{2}$ holds. Hence every p -value $p_i \leq f^{-1}(\frac{R_n}{n})$ will be rejected and in particular every p -value $p_i \leq f^{-1}(\frac{1}{2})$. Thus

$$\beta \geq \liminf_{n \rightarrow \infty} FDR_{DU(n, n_0)} \geq \liminf_{n \rightarrow \infty} E \left(\frac{V_n}{n} \right) \geq \frac{1}{2} f^{-1} \left(\frac{1}{2} \right) > 0$$

for this sequence of Dirac uniform models. Let us next show $\beta < 1$. Therefore, let n_0 be arbitrary and observe that f lies above the Benjamini Hochberg rejection curve $f_{BH}(x) = (1 + \epsilon)x$ of the BH test. By Lemma 7.2 (a) (i) we always have

$$FDR_{P_n} \leq \frac{1}{1 + \epsilon}.$$

(II) The statement (5.6) of the Theorem is first proved for concave rejection curves f . Therefore, observe that under the distribution $P_n^{N_n}$ of N_n

$$FDR_{P_n} = \int FDR_{P_n}(n_0) P_n^{N_n}(dn_0) \leq \sup_{n_0 \leq n} FDR_{P_n}(n_0) \leq \sup_{n_0 \leq n} FDR_{DU(n, n_0)}$$

for all $P_n \in \mathcal{P}_n$, since the Dirac uniform configuration is least favorable for the FDR for SU tests with critical values fulfilling (4.5). Hence we get

$$\beta = \limsup_{n \rightarrow \infty} \sup_{n_0 \leq n} FDR_{DU(n, n_0)} \tag{7.18}$$

since $DU(n, n_0)$ belongs to \mathcal{P}_n and thus there exists a subsequence $(n, n_0) = (n_n, n_{0,n})_n$, again denoted by n , with

$$FDR_{DU(n, n_0)} \rightarrow \beta \quad \text{and} \quad \frac{n_0}{n} \rightarrow 1 - y \quad \text{for } n \rightarrow \infty \tag{7.19}$$

and some $1 - y \in [0, 1]$. Now we determine the limit of $FDR_{DU(n, n_0)}$ for every sequence $\frac{n_0}{n} \rightarrow 1 - y \in [0, 1]$.

1. For $\frac{n_0}{n} \rightarrow 0$ observe that

$$FDR_{DU(n, n_0)} = E \left(\frac{V_n}{n - n_0 + V_n} \right) \leq \frac{n_0}{n} \xrightarrow{n \rightarrow \infty} 0$$

and by a similar argument it follows that the limit of $FDR_{DU(n, n_0)}$ is continuous in $1 - y$ at 0.

2. Let us consider $\frac{n_0}{n} \rightarrow 1 - y \in (0, 1)$ with positive y and introduce the straight line $g(t) = y + (1 - y)t$ which runs through the points $(0, y)$ and $(1, 1)$ and has the unique crossing point (x, K) , $0 < K < 1$, with f . Observe that

$$\hat{F}_n(\alpha_{R_n:n}) = f(\alpha_{R_n:n})$$

holds. Now let Z be a weak accumulation point of $\alpha_{R_n:n}$. Since f is continuous and \hat{F}_n converges uniformly to g with probability 1 the equation

$$g(Z) = y + (1 - y)Z = f(Z)$$

follows. There is only one crossing point and thus $Z = x$ is constant for each weak accumulation point. From $f^{-1}(\frac{R_n}{n}) = \alpha_{R_n:n}$ we now deduce

$$\frac{R_n}{n} = f(\alpha_{R_n:n}) \rightarrow f(x) = K \quad a.s. \tag{7.20}$$

at least along subsequences. Similar arguments were used by Scheer [22], Lemma 2.9, in his set up in order to prove that $\alpha_{R:n}$ converges to the crossing point x .

A simple geometric argument for the gradient of g yields

$$\frac{1 - y}{1} = \frac{f(x) - y}{x}$$

and hence

$$y = \frac{f(x) - x}{1 - x}, \quad 1 - y = \frac{1 - f(x)}{1 - x}. \tag{7.21}$$

By Lemma 7.4 and subsequence arguments

$$\lim_{n \rightarrow \infty} FDR_{DU(n, n_0)} = \frac{f^{-1}(K)}{K} (1 - y) = \frac{x}{f(x)} \frac{1 - f(x)}{1 - x} \tag{7.22}$$

holds for all sequences $\frac{n_0}{n} \rightarrow 1 - y \in (0, 1)$.

3. Now consider $\frac{n_0}{n} \rightarrow 1$. Again $\frac{R_n}{n} \rightarrow 0$ in distribution. This follows from (7.20) by the monotonicity of R_n in n_0 since $K \rightarrow 0$ holds for $y \rightarrow 0$. Observe next that for every $x > 0$ we have $z := f(x) > 0$ and hence by (4.5)

$$\frac{n\alpha_{j:n}}{j} \leq \frac{n\alpha_{\lfloor nz \rfloor:n}}{\lfloor nz \rfloor} = \frac{nf^{-1}(\frac{\lfloor nz \rfloor}{n})}{\lfloor nz \rfloor} < 1$$

holds for $j \leq \lfloor nz \rfloor$. By Lemma 7.2 (a) (i)

$$FDR_{DU(n, n_0)} \leq \frac{h(\frac{\lfloor nz \rfloor}{n})}{\frac{\lfloor nz \rfloor}{n}} \frac{n_0}{n} + P(R_n > \lfloor nz \rfloor) \rightarrow \frac{f^{-1}(z)}{z} = \frac{x}{f(x)}$$

holds and hence

$$\limsup_{n \rightarrow \infty} FDR_{DU(n, n_0)} \leq \lim_{x \searrow 0} \frac{x}{f(x)} = \lim_{x \searrow 0} \frac{x}{f(x)} \frac{1 - f(x)}{1 - x}$$

when $\frac{n_0}{n} \rightarrow 1$ since $\lim_{x \searrow 0} \frac{1 - f(x)}{1 - x} = 1$.

Altogether, by subsequence arguments and (7.18) we directly obtain (5.6)
(III) Let now f be the general rejection curve of Theorem 5.1. Introduce $\gamma := \sup\{\frac{x}{1-x} \frac{1-f(x)}{f(x)} : 0 \leq x \leq x_0\}$.

(a) Claim: $\beta \leq \gamma$. The geometric arguments 1.–3. at the beginning of the proof imply $f \geq f_\gamma$ for the AORC with parameter γ . Next f_γ is modified as follows. Let $x \mapsto r(x)$ be the tangent straight line attached at f_γ at the point $(x_0, f_\gamma(x_0))$. Then

$$\tilde{f}(x) = \begin{cases} f_\gamma(x) & x \leq x_0 \\ r(x) & x > x_0 \end{cases} \tag{7.23}$$

defines a concave rejection curve $\tilde{f} \leq f$ with $\tilde{f}(x_1) = 1$ for some $x_0 < x_1 < 1$. By Lemma 7.5, given below, the $FDR(\tilde{f})$ is always an upper bound of the $FDR(f)$. By step (II) the asymptotic worst case FDR of $FDR(\tilde{f})$ equals

$$\sup \left\{ \frac{x}{1-x} \frac{1-\tilde{f}(x)}{\tilde{f}} : 0 \leq x \leq x_1 \right\} \geq \beta. \quad (7.24)$$

It is easy to see that the left hand side is just γ .

(b) The proof will be completed by showing $\beta \geq \gamma$. The inequality follows from the next special construction of mixture models, where now in contrast to part (II) of this proof the DU configuration is no longer least favorable. Introduce for each $0 < \delta < 1$ the straight line $g_\delta(t) = 1 - \delta + t\delta$ and the intersection set $D_\delta := \{x \in (0, x_0) : g_\delta(x) = f(x)\}$. Note that D_δ is a compact set bounded away from 0 and x_0 with $D_\delta \neq \emptyset$ and

$$\bigcup_{0 < \delta < 1} D_\delta = (0, x_0). \quad (7.25)$$

Thus, there exists some maximal element $x_\delta \in D_\delta$ with $x \leq x_\delta$ for all $x \in D_\delta$. In the next step we will introduce for each set D_δ a mixture model with appropriate distribution function

$$F_\delta(t) = \delta t + (1 - \delta)G_\delta(t). \quad (7.26)$$

The non-uniform part G_δ will have the following properties:

- (i) $G_\delta(x_\delta) = 1$, which implies $F_\delta(t) = g_\delta(t)$ for all $t \geq x_\delta$ and $F_\delta(x_\delta) = f(x_\delta)$.
- (ii) $F_\delta(t) < f(t)$ for all $t < x_\delta$.

In order to do so, let h be a straight line through $(x_\delta, f(x_\delta))$ with sufficiently large slope such that $h(t) < f(t)$ for all $0 < t < x_\delta$. Here the left-sided differentiability of f is used. Put now

$$F_\delta(t) = \max(\delta t, h(t)), \quad \text{for } t \leq x_\delta. \quad (7.27)$$

This x_δ is the only cut point of F_δ and f . Consider now a mixture model P_n with distribution function f_δ . Similarly as in (II) we have $\frac{N_n}{n} \rightarrow \delta$ and via the cut point consideration we arrive at

$$FDR_{P_n} \rightarrow \frac{x_\delta}{1-x_\delta} \frac{1-f(x_\delta)}{f(x_\delta)} =: \beta_\delta \quad (7.28)$$

at least along suitable subsequences. A comparison of the line segment $\{(x, g_\delta(x)) : x \in D_\delta\}$ with the AORC f_{β_δ} yields $\frac{x}{1-x} \frac{1-f(x)}{f(x)} \leq \beta_\delta$ for each $x \in D_\delta$ if we take 1.–3. into account. This construction can be done for each set D_δ . Thus, the proof of the inequality $\beta \geq \gamma$ is complete. \square

The next lemma is used in the last proof and may be of separate interest.

Lemma 7.5. *Let $f : [0, 1] \rightarrow \mathbb{R}$ be a non-decreasing rejection curve with $f(0) = 0$, $f(x_0) = 1$ for some $x_0 < 1$ and $f(x) \geq (1 + \epsilon)x$ for some $\epsilon > 0$. Moreover, let $r_0 : [0, 1] \rightarrow \mathbb{R}$ be a concave rejection curve and a lower bound*

of f . Under the BI Model for fixed n and $N = n_0$ we have

$$FDR(f) \leq FDR(r_0) \leq FDR_{DU(n, n_0)}(r_0) \tag{7.29}$$

for the FDR of the SU tests based on f and r_0 via (2.16).

Proof. Let $\alpha_{i:n}$ and $\alpha_{i:n}^{(0)}$ be the critical values of the rejection curves f and r_0 , respectively, defined by (2.16). Define R and R_0 as the number of rejections of the SU test based on $\alpha_{i:n}$ and $\alpha_{i:n}^{(0)}$, respectively. It is easy to see that $\alpha_{i:n} \leq \alpha_{i:n}^{(0)}$ and hence $R \leq R_0$ almost surely holds. Using the technique of Remark 7.3 we obtain

$$\begin{aligned} FDR(f) &= n_0 E \left(\frac{\alpha_{R(p^{(1)}) : n}}{R(p^{(1)})} \mid \epsilon_1 = 1 \right) \leq n_0 E \left(\frac{\alpha_{R(p^{(1)}) : n}^{(0)}}{R(p^{(1)})} \mid \epsilon_1 = 1 \right) \\ &\leq n_0 E \left(\frac{\alpha_{R_0(p^{(1)}) : n}^{(0)}}{R_0(p^{(1)})} \mid \epsilon_1 = 1 \right) \leq n_0 E_{DU(n, n_0)} \left(\frac{\alpha_{R_0(p^{(1)}) : n}^{(0)}}{R_0(p^{(1)})} \mid \epsilon_1 = 1 \right) \\ &= FDR_{DU(n, n_0)}(r_0). \end{aligned}$$

since $i \mapsto \frac{\alpha_{i:n}^{(0)}}{i}$ is non-decreasing and the $DU(n, n_0)$ configuration is least favorable for the FDR, cf. Benjamini and Yekutieli [2]. \square

Remark 7.4. Consider the BI Model and let the false p -values be independent and stochastically smaller than the uniform distribution. Let $FDR_{U(0,1)}$ denote the FDR under uniformly distributed false p -values. If we replace r_0 in Lemma 7.5 by a convex rejection curve r_0 which is an upper bound of f , it is easy to see by similar arguments that

$$FDR(f) \geq FDR_{U(0,1)}(r_0) \tag{7.30}$$

holds for the FDR of the SU tests based on f and r_0 , since $i \mapsto \frac{\alpha_{i:n}^{(0)}}{i}$ is non-increasing for the critical values $\alpha_{i:n}^{(0)}$ corresponding to the rejection curve r_0 . According to Benjamini and Yekutieli [2], here the uniform distribution is least favorable for the FDR under stochastically smaller p -values.

If (2.4) is non-increasing, then the p -values are called to be negative regression dependent on the subset of true null hypotheses (NRDS). Under this assumption the lower bound in (4.1) and Lemma 7.2 (b) (i) stay true, see Heesen [16] for instance.

PROOF OF THEOREM 5.2. By Gontscharuk [12, Theorem 3.10] it is well known that

$$E \left(\frac{V_{SD}}{R_{SD}} \mid n_0, \bar{f} \right) \leq E \left(\frac{V_{SU}}{R_{SU}} \mid n_0, \bar{f} \right)$$

holds for tests with critical values (4.5), see also Heesen [16] Lemma 2.29. The inequality also follows from the technique used in Remark 7.3. Hence

$$\limsup_{n \rightarrow \infty} \sup_{P_n \in \mathcal{P}_n} FDR_{P_n, SD} \leq \limsup_{n \rightarrow \infty} \sup_{P_n \in \mathcal{P}_n} FDR_{P_n} =: \beta. \tag{7.31}$$

In the proof of Theorem 5.1 we already showed that β is attained by $\frac{x}{1-x} \frac{1-f(x)}{f(x)}$ for some x . Thus, to show “=” on the left hand side in (7.31) it suffices to show that for all such x there is some sequence of distributions $(P_n)_n$ so that $FDR_{P_n,SD}$ converges to $\frac{x}{1-x} \frac{1-f(x)}{f(x)}$. Therefore, let us now consider sequences of $DU(n, n_0)$ configurations with $\frac{n_0}{n} \rightarrow 1 - y \in (0, 1)$. Along the lines of the proof of Theorem 5.1 we have

$$\frac{R_{SD,n}}{n} \rightarrow K \quad a.s. \text{ with } 0 < K < 1. \quad (7.32)$$

Observe that $0 < K$ holds since $\frac{R_{SD,n}}{n} \geq \frac{n-n_0}{n} \rightarrow y > 0$. Moreover, $K < 1$ holds since $f(x) = 1$ for all $x_0 \leq x$ for some $x_0 < 1$. Now with $x, f(x)$ and y as in (7.20) and (7.21) in the proof of Theorem 5.1 we have

$$\begin{aligned} FDR_{DU(n,n_0),SD} &= E_{DU(n,n_0)} \left(\frac{V_{SD,n}}{R_{SD,n}} \right) = E_{DU(n,n_0)} \left(\frac{\frac{R_{SD,n}}{n} - \frac{n-n_0}{n}}{\frac{R_{SD,n}}{n}} \right) \\ &\xrightarrow{n \rightarrow \infty} \frac{K - y}{K} = \frac{x}{f(x)} \frac{1 - f(x)}{1 - x} \end{aligned}$$

by dominated convergence. Hence we have “=” in (7.31) since the above formula holds for all $1 - y \in (0, 1)$ and the representation (5.6). \square

7.5. Proofs of Section 6

PROOF OF THEOREM 6.1. First observe that $FDR_{P_n} = 0 \leq \alpha$ obviously holds if $N_n = 0$. Thus, without loss of generality let $N_n > 0$ almost surely for all n . Conditioned under $\epsilon = (\epsilon_1, \dots, \epsilon_n)$ we obtain by Lemma 7.1 (a) and (6.5)

$$\frac{N_n}{n} = E \left(\frac{V}{n \hat{\alpha}_{R:n}} \mid \epsilon \right) \geq E \left(\frac{\hat{n}_{0,n}}{n} \frac{V}{R \alpha} \mid \epsilon \right) \quad (7.33)$$

for the reverse martingale model since the conditional case is also included. Thus by integration

$$\begin{aligned} \alpha &\geq E_{P_n} \left(\frac{\hat{n}_{0,n}}{N_n} \frac{V}{R} \right) \geq E_{P_n} \left((1 - \delta) \frac{V}{R} 1_{\left\{ \frac{\hat{n}_{0,n}}{N_n} > 1 - \delta \right\}} \right) \\ &= (1 - \delta) E_{P_n} \left(\frac{V}{R} \right) - (1 - \delta) E_{P_n} \left(\frac{V}{R} 1_{\left\{ \frac{\hat{n}_{0,n}}{N_n} \leq 1 - \delta \right\}} \right) \\ &\geq (1 - \delta) E_{P_n} \left(\frac{V}{R} \right) - (1 - \delta) P_n \left(\frac{\hat{n}_{0,n}}{N_n} \leq 1 - \delta \right) \end{aligned}$$

holds for every $\delta > 0$ and the statement follows by (6.7). \square

PROOF OF PROPOSITION 6.1. Introduce the set $A_n = \{\hat{\alpha}_{R:n} < \lambda\}$. An inspection of the proof of Lemma 6.1 (given below) yields

$$E_{P_n} \left(\frac{V_n}{R_n} 1_{A_n} \right) = \frac{\alpha}{\lambda} E_{P_n} \left(\frac{V_n(\lambda)}{\hat{n}_0} 1_{A_n} \right)$$

restricted on A_n . Define $S_n(\lambda) := \sum_{i=1}^n (1 - \epsilon_i) 1\{p_i \leq \lambda\}$, then

$$\alpha \geq \limsup_{n \rightarrow \infty} \frac{\alpha(1 - \lambda)}{\lambda} E_{P_n} \left(\frac{V_n(\lambda)}{n - V_n(\lambda) - S_n(\lambda) + \kappa_n} \right)$$

follows. First, let us consider case (ii). For each $\delta > 0$ we have

$$1 \geq \frac{(1 - \lambda)}{\lambda} \limsup_{n \rightarrow \infty} E_{P_n} \left(\frac{V_n(\lambda)/N_n}{1 - V_n(\lambda)/N_n + \kappa_n/N_n + \delta} 1_{A_n} \right). \quad (7.34)$$

Observe next that $0 \leq V_n(\lambda)/N_n \leq 1$ is tight and we consider an arbitrary distributional cluster point Z of $V_n(\lambda)/N_n$. The appertaining subsequence is for convenience also denoted by $\{n\}$, i.e. $V_n(\lambda)/N_n \rightarrow Z$ in distribution. Note that $E(Z) = \lambda$ and

$$1 \geq \frac{1 - \lambda}{\lambda} E \left(\frac{Z}{1 - Z + \delta} \right)$$

hold. Now Jensen's inequality implies

$$\frac{\lambda}{1 - \lambda} \geq E \left(\frac{Z}{1 - Z} \right) \geq \frac{\lambda}{1 - \lambda}$$

when $\delta \searrow 0$. Since $x \mapsto \frac{x}{1-x}$ is strictly convex we have $Z = \lambda$ a.e. Since Z was an arbitrary cluster point we conclude $V_n(\lambda)/N_n \rightarrow \lambda$ in P_n -probability. This statement implies the result

$$\frac{\hat{n}_0}{N_n} = \frac{N_n - V_n(\lambda) + n\kappa_n}{(1 - \lambda)N_n} \rightarrow 1.$$

In case (i) the proof is similar. Note that the assumption then implies

$$\frac{(n - N_n) - S_n(\lambda)}{N_n} \rightarrow 0$$

and we may proceed as in (7.34) □

PROOF OF REMARK 6.1. Observe that

$$\begin{aligned} \frac{\hat{n}_0}{N_n} &\geq \frac{N_n - \sum_{i=1}^n \epsilon_i 1\{p_i \leq \lambda\} + \kappa_n}{N_n(1 - \lambda)} \\ &\geq \frac{1 - \frac{1}{N_n} \sum_{i=1}^n \epsilon_i 1\{p_i \leq \lambda\}}{1 - \lambda} \rightarrow 1 \end{aligned}$$

holds, where the right hand side converges by (6.10) in probability w.r.t. the conditional convergence. □

PROOF OF THEOREM 6.2. Conditioned under $\epsilon = (\epsilon_1, \dots, \epsilon_n)$ we directly obtain by Lemma 7.1 (b) and (6.6) that

$$\frac{N_n}{n} \geq E \left(\frac{\hat{n}_{0,n}}{n} \frac{V}{R\alpha} \middle| \epsilon \right) \quad (7.35)$$

holds and the statement follows by the same arguments as in the proof of Theorem 6.1. □

PROOF OF LEMMA 6.1. Conditioned under $(\epsilon_i)_{i \leq n} = \epsilon$, $(p_i : \epsilon_i = 0) = \bar{f}$ and $(n\hat{F}_n(t))_{t \geq \lambda} = (n(t))_{t \geq \lambda}$ we have exactly $V(\lambda)$ true p -values smaller or equal to λ where $V(\lambda)$ is a fixed number. Without restriction we assume $N \geq V(\lambda) > 0$ since everything is obviously fine for the excluded cases. Let us now consider new rescaled p -values q_i , $i = 1, \dots, n(\lambda)$ defined by

$$q_{i:n(\lambda)} := \frac{p_{i:n}}{\lambda}, \quad i = 1, \dots, n(\lambda).$$

When a new p -values q_i corresponds to a true null hypothesis it is again uniformly distributed on $(0, 1)$ and $\frac{1\{q_i \leq t\}}{t}$ is a reverse martingale with respect to the reverse filtration $\mathcal{F}_t^q = \sigma(1\{q_j \leq s\} : 1 \leq j \leq n(\lambda), s \geq t)$ under the above conditional assumption. The exact positions of the $V(\lambda)$ true p -values in $(q_1, \dots, q_{n(\lambda)})$ does not matter for our considerations. We now apply Lemma 7.1 (a) for the SU multiple test with critical values

$$\alpha_{i:n(\lambda)}^{(q)} := \frac{i}{n(\lambda)} \alpha' \quad \text{and} \quad \alpha' := \frac{n(\lambda)}{\lambda \hat{n}_0} \alpha$$

on the q 's. The data dependent level α' only depends by assumption (A1) on the information given by \mathcal{F}_λ . Conditionally under \mathcal{F}_λ we have a regular non data dependent SU procedure on the q 's. Let R_q and V_q denote the number of rejections and false rejections respectively by the above SU test. Observe that

$$E \left(\frac{V_q}{R_q} \mid \epsilon, \bar{f}, (n(t))_{t \geq \lambda} \right) = \frac{V(\lambda)}{n(\lambda)} \min(\alpha', 1) \quad (7.36)$$

holds by Lemma 7.1 (a). Obviously (7.36) is $\frac{V(\lambda)}{n(\lambda)}$ in case $\alpha' \geq 1$.

Now observe that

$$\begin{aligned} R_q &= \max\{i \leq n(\lambda) : q_{i:n(\lambda)} \leq \alpha_{i:n(\lambda)}^{(q)}\} \\ &= \max\left\{i \leq n(\lambda) : \frac{p_{i:n}}{\lambda} \leq \frac{i}{n(\lambda)} \frac{n(\lambda)}{\lambda \hat{n}_0} \alpha\right\} \\ &= \max\left\{i \leq n : p_{i:n} \leq \left(\frac{i}{\hat{n}_0} \alpha\right) \wedge \lambda\right\} = R \end{aligned}$$

and hence $V_q = V$ since both tests, belonging to R and R_q , are rejecting the same hypotheses. Thus by (7.36) we get

$$\begin{aligned} E \left(\frac{V}{R} \right) &= E \left(E \left(\frac{V}{R} \mid \epsilon, \bar{f}, (n(t))_{t \geq \lambda} \right) \right) \\ &= E \left(E \left(\frac{V_q}{R_q} \mid \epsilon, \bar{f}, (n(t))_{t \geq \lambda} \right) \right) \\ &= E \left(\frac{V(\lambda)}{n(\lambda)} \min(\alpha', 1) \right) = \frac{\alpha}{\lambda} E \left(V(\lambda) \min \left\{ \frac{1}{\hat{n}_0}, \frac{\lambda}{n\hat{F}_n(\lambda)\alpha} \right\} \right). \quad \square \end{aligned}$$

PROOF OF PROPOSITION 6.2. Choose $N = n$ and $p_i = U$ for all $i \leq n$. Thus $V(\lambda) = n1\{U \leq \lambda\}$ and $n\hat{F}_n(\lambda) = n1\{U \leq \lambda\}$. The exact FDR formula (6.11)

yields

$$\alpha \geq FDR = \frac{\alpha}{\lambda} n \lambda \min \left(\frac{1}{\hat{n}_0(\vec{1})}, \frac{\lambda}{n\alpha} \right)$$

where $\hat{n}_0(\vec{1})$ stands for the value of the estimator when $\hat{F}_n(\rho) = 1$ for all $\rho \geq \lambda$. Thus $\frac{n}{\hat{n}_0(\vec{1})} \leq 1$ holds. By our assumption we have $\hat{n}_0 \geq \hat{n}_0(\vec{1})$. \square

We need the following “balayage” lemma for the proof of Theorem 6.3.

Lemma 7.6. *Let $f : \{0, 1, \dots, m\} \rightarrow \mathbb{R}$ be a convex function and let $P = \sum_{j=1}^m p_j \epsilon_j$ be a distribution on $\{0, 1, \dots, m\}$, where ϵ_j denotes the Dirac distribution on $\{j\}$. Then $E_P(id) = E_{P'}(id)$ and $E_P(f) \leq E_{P'}(f)$ hold for the distribution*

$$P' = \left(1 - \sum_{j=1}^m \frac{j}{m} p_j \right) \epsilon_0 + \sum_{j=1}^m \frac{j}{m} p_j \epsilon_m.$$

Proof. $E_P(id) = E_{P'}(id)$ obviously holds by the definition of P' . Then by the convexity of f we have

$$\begin{aligned} E(f) &= \sum_{j=0}^m f(j) p_j \leq \sum_{j=0}^m \left(\frac{m-j}{m} f(0) p_j + \frac{j}{m} f(m) p_j \right) \\ &= \left(1 - \sum_{j=1}^m \frac{j}{m} p_j \right) f(0) + \sum_{j=1}^m \frac{j}{m} p_j f(m) = E_{P'}(f). \quad \square \end{aligned}$$

PROOF OF THEOREM 6.3. Throughout the condition (6.12) will be verified. For the portion \tilde{G}_i of true p -values introduce the quantities

$$V_i(\lambda) := \sum_{j: p_j \in \tilde{G}_i} 1\{p_j \leq \lambda\}, \quad N_i := |\tilde{G}_i| \tag{7.37}$$

with $V(\lambda) = \sum_{i=1}^k V_i(\lambda)$. Whenever $\tilde{G}_i \neq \emptyset$ holds select one true p -value $\tilde{p}_i \in \tilde{G}_i$. Observe that the \tilde{p}_i are conditionally independent given ϵ . Under that condition we have for (6.14)

$$\begin{aligned} E \left(\frac{V(\lambda)}{\hat{n}_0(\kappa)} \middle| \epsilon \right) &= (1 - \lambda) E \left(\frac{\sum_{i=1}^k V_i(\lambda)}{n - n\hat{F}_n(\lambda) + \kappa} \middle| \epsilon \right) \\ &\leq (1 - \lambda) E \left(\frac{\sum_{i=1}^k V_i(\lambda)}{N - \sum_{i=1}^k V_i(\lambda) + \kappa} \middle| \epsilon \right) = (\star). \end{aligned}$$

In the next step we are going to condition under $a = \sum_{i=2}^k V_i(\lambda)$. Note that then $V_1(\lambda)$ is a $B(N_1, \lambda)$ distributed random variable. By Lemma 7.6, $V_1(\lambda)$ can be substituted by the worst case random variable $N_1 1\{\tilde{p}_1 \leq \lambda\}$, i.e.

$$E \left(\frac{V_1(\lambda) + a}{N - V_1(\lambda) - a + \kappa} \middle| \epsilon, a \right) \leq E \left(\frac{N_1 1\{\tilde{p}_1 \leq \lambda\} + a}{N - N_1 1\{\tilde{p}_1 \leq \lambda\} - a + \kappa} \middle| \epsilon, a \right).$$

If we proceed k times we arrive at the upper bound

$$\begin{aligned}
 (\star) &\leq (1-\lambda)E\left(\frac{\sum_{i=1}^k N_i 1\{\tilde{p}_i \leq \lambda\}}{N - \sum_{i=1}^k N_i 1\{\tilde{p}_i \leq \lambda\} + \kappa} \middle| \epsilon\right) \\
 &\leq (1-\lambda)E\left(\frac{\sum_{i=1}^k m 1\{\tilde{p}_i \leq \lambda\}}{N - \sum_{i=1}^k m 1\{\tilde{p}_i \leq \lambda\} + \kappa} \middle| \epsilon\right) \\
 &= (1-\lambda)E\left(\frac{\sum_{i=1}^k 1\{\tilde{p}_i \leq \lambda\}}{\frac{N}{m} - \sum_{i=1}^k 1\{\tilde{p}_i \leq \lambda\} + \frac{\kappa}{m}} \middle| \epsilon\right) \\
 &\leq (1-\lambda)E\left(\frac{\sum_{i=1}^k 1\{\tilde{p}_i \leq \lambda\}}{k+1 - \sum_{i=1}^k 1\{\tilde{p}_i \leq \lambda\}} \middle| \epsilon\right) \leq \lambda
 \end{aligned}$$

since $x \mapsto \frac{x}{N-x+\kappa}$ is increasing for $x \in [0, N+\kappa)$ and $m \sum_{i=1}^k 1\{\tilde{p}_i \leq \lambda\} \leq km < km + m + N - N_{min} \leq N + \kappa$ holds. Moreover, $\frac{N+\kappa}{m} \geq \frac{N-N_{min}+m+r+n}{m} = \frac{N-N_{min}+m+m\kappa}{m} \geq k+1$ holds and the last inequality follows by the well known result for Binomial variables

$$E\left(\frac{\sum_{i=1}^k 1\{\tilde{p}_i \leq \lambda\}}{k+1 - \sum_{i=1}^k 1\{\tilde{p}_i \leq \lambda\}} \middle| \epsilon\right) = \frac{\lambda}{1-\lambda}(1-\lambda^k). \quad (7.38) \quad \square$$

PROOF OF EXAMPLE 6.1. The proof for the FWER is mostly included in the proof of Theorem 6.3. Notice first that

$$\frac{\alpha}{\lambda} \frac{V(\lambda)}{\hat{n}_0(\kappa)} = \frac{\alpha(1-\lambda)}{\lambda} \frac{X}{k+1-X} \leq 1 \quad (7.39)$$

always holds under our assumptions, where $X = \sum_{i=1}^k 1\{\tilde{p}_i \leq \lambda\}$ is a binomial variable. Thus, we have equality in (6.11). By an inspection of the proof above we arrive at a sequence of equality with

$$FWER = \frac{\alpha(1-\lambda)}{\lambda} E\left(\frac{X}{k+1-X}\right) \quad (7.40)$$

and (7.38) can be applied. \square

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