Electronic Journal of Statistics
Vol. 18 (2024) 3931–3974
ISSN: 1935-7524
https://doi.org/10.1214/24-EJS2293

Asymptotic theory for explosive fractional Ornstein-Uhlenbeck processes

Hui Jiang[1], **Yajuan Pan**[1], **Weilin Xiao**[2], **Qingshan Yang**[3], **and Jun Yu**[4]

[1] *School of Mathematics, Nanjing University of Aeronautics and Astronautics, Nanjing, 211106, P. R. China*
e-mail: huijiang@nuaa.edu.cn; panyajuan@nuaa.edu.cn

[2] *School of Management, Zhejiang University, Hangzhou, 310058, P. R. China*
e-mail: wlxiao@zju.edu.cn

[3] *KLAS, Key Laboratory of Big Data Analysis of Jilin Province, School of Mathematics and Statistics, Northeast Normal University, Changchun, 130024, P. R. China*
e-mail: yangqr66@gmail.com

[4] *Faculty of Business Administration, University of Macau, Macao, China*
e-mail: junyu@um.edu.mo

Abstract: This paper proposes estimators for the parameters of an explosive fractional Ornstein-Uhlenbeck process. The asymptotic properties for the diffusion estimators are developed under the in-fill asymptotic scheme, while the asymptotic properties for the drift estimators are developed under the double asymptotic scheme for the full range of the Hurst parameter. The double asymptotic distribution of the estimator of the persistency parameter explicitly depends on the initial condition. Simulation results demonstrate the effectiveness of the proposed estimators, and the asymptotic distributions provide a good approximation in finite samples. An empirical application is presented to demonstrate the model's usefulness and the practical value of the asymptotic theory.

MSC2020 subject classifications: Primary 62N02; secondary 60F10, 60G22.
Keywords and phrases: Explosive process, Hurst parameter, long memory, anti-persistency, double asymptotics, in-fill asymptotics.

Received May 2023.

1. Introduction

In recent years, mildly explosive discrete-time models have been utilized to capture the dynamic behavior of economic and financial time series. This approach has been explored in various studies such as Phillips and Yu [40], Phillips et al. [39], Phillips et al. [37, 38], Harvey et al. [20, 21], Chen et al. [12], Lui et al. [31, 32], and Astill et al. [3].

The mildly explosive model and the asymptotic theory for the least squares estimator were first introduced in the seminal paper by Phillips and Magdalinos [35], where the error terms are assumed to be independent and identically distributed (iid). Phillips and Magdalinos [36] extended the model and asymptotic theory to include weakly dependent errors, while Magdalinos [33] further extended it to strongly dependent errors. Lui et al. [32] expanded the model and asymptotic theory to incorporate anti-persistent errors. In all of these studies, an initial condition was selected to ensure that it becomes negligible in the asymptotic distribution and as a result, the standard Cauchy limiting distribution is obtained.

Wang and Yu [48] demonstrated that mild explosiveness can be achieved from an explosive Ornstein-Uhlenbeck (OU) process under the double asymptotic scheme when the sampling interval approaches zero and the time span becomes infinite. In this scenario, since the randomness is governed by the standard Brownian motion,[1] the error term in the exact discrete-time representation of the model is iid. Wang and Yu [48] obtained the double asymptotic distribution of the least squares estimator of the persistency parameter and showed that it explicitly depends on the initial condition. The reason that the initial condition is given a prominence in the continuous-time setup is because a bigger initial condition than what is typically imposed in the discrete-time literature is allowed in the exact discrete-time representation when the sampling interval shrinks to zero.

In this paper, we extend the OU model of Wang and Yu [48] by replacing the standard Brownian motion with the fractional Brownian motion (fBm), that is, an explosive fractional OU process (fOUp). The exact discrete-time representation of fOUp extends the models considered in Magdalinos [33] and Lui et al. [32] in four aspects. First, our model allows for the full range of the Hurst parameter. Second, we permit a larger initial condition in the exact discrete-time representation of fOUp than that considered in Magdalinos [33]. Third, we estimate and examine the asymptotic properties of all four parameters in the model, not just the persistency parameter. Finally, although the error term in our model shares the same covariance structure as those in Magdalinos [33] and Lui et al. [32], it cannot be expressed as a linear combination of martingale difference sequences. This distinction leads to completely different technical proof procedures.

We adopt the same estimators of the two diffusion parameters, including the Hurst parameter, as those proposed in Wang et al. [47], where a stationary fOUp process is considered. For the drift parameters, including the persistency parameter, we obtain the estimators via least squares, which have analytical expressions and are easy to implement. The asymptotic theory for the diffusion parameters is established under the in-fill asymptotic scheme, while the asymptotic theory for the drift parameters is established under the double asymptotic scheme.

The remainder of the paper is organized as follows. Section 2 introduces the model and compares it with two other models. Section 3 introduces estimators and develops the asymptotic properties of the estimators. Section 4 conducts

[1] In the most general case, Wang and Yu [48] considered the Lévy process instead of the standard Brownian motion.

Monte Carlo studies to check the finite sample performance of the proposed estimators and asymptotic distributions. Section 5 provides an empirical study to illustrate the usefulness of our estimators and the asymptotic theory. Section 6 concludes the paper. All proofs of the theorems are collected in the Appendix. The proofs of Lemmas A.1–A.4 and Proposition A.3, which are useful to prove the theorems, are collected in the supplementary material [28]. Throughout the paper, we use $\stackrel{p}{\to}$, $\stackrel{a.s.}{\to}$, $\stackrel{\mathcal{L}}{\to}$, $\stackrel{d}{=}$, and $\sim$ to denote convergence in probability, convergence almost surely, convergence in distribution, equivalence in distribution, asymptotic equivalence, and asymptotic dominance, respectively. We denote C, C_1, C_2, which may change from line to line, positive constants that depend only on the parameters of fOUp.

2. Model

The fOUp is given by the following stochastic differential equation:

$$dX_t = (\theta X_t + \mu)dt + \sigma dB_t^H, \tag{2.1}$$

where $X_0 = O_p(1)$ is independent of B_t^H, $\sigma \in \mathbb{R}^+$, $\mu \in \mathbb{R}$, $\theta > 0$, and B_t^H is an fBm with the Hurst parameter, $H \in (0, 1)$, with mean zero and the following covariance

$$R(t, s) = \mathbb{E}\big(B_t^H B_s^H\big) = \frac{1}{2}\big(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\big) \ \ \forall\, t, s \geq 0. \tag{2.2}$$

For $t > 0$, Mandelbrot and van Ness [34] presented the following integral representation for B_t^H:

$$B_t^H = \frac{1}{c_H}\left\{\int_{-\infty}^{0}\big[(t-u)^{H-1/2} - (-u)^{H-1/2}\big]dW_u + \int_0^t (t-u)^{H-1/2} dW_u\right\}, \tag{2.3}$$

where W_u is a standard Brownian motion, $c_H = \frac{\Gamma(H+1/2)}{\sqrt{\Gamma(2H+1)\sin(\pi H)}}$, $B_0^H = 0$ and $\Gamma(\cdot)$ denotes the Gamma function.

Obviously, the fBm becomes the standard Brownian motion W_t when $H = 1/2$. Moreover, the fBm is self-similar in the sense that for any $a \in \mathbb{R}$, $B_{at}^H \stackrel{d}{=} |a|^H B_t^H$. Let $L_t^H = B_t^H - B_{t-1}^H$ be the so-called the fractional Gaussian noise (fGn) which is always stationary. The autocovariance function of fGn is

$$\gamma(k) = \mathrm{Cov}\big(L_t^H, L_{t+k}^H\big) = \frac{1}{2}\big[|k+1|^{2H} - 2|k|^{2H} + |k-1|^{2H}\big], \tag{2.4}$$

for $k \geq 0$ and $\gamma(k) = \gamma(-k)$ for $k < 0$.

Applying the Taylor expansion to the right-hand side of (2.4), we can see that if $H \in (0, 1/2) \cup (1/2, 1)$, $\gamma(k) \sim H(2H-1)k^{2H-2}$ for large k. Hence, for $\frac{1}{2} < H < 1$, it has $\gamma(k) > 0$ for all k and $\sum_{k=-\infty}^{\infty} \gamma(k) = \infty$. In this case, fGn has the long memory property and positive (negative) increments are likely to be followed by positive (negative) increments. For $0 < H < \frac{1}{2}$, it can be verified that $\gamma(k) < 0$ for all $k \neq 0$ and $\sum_{k=-\infty}^{\infty} \gamma(k) = 0$. Therefore, the process is anti-persistent.

When $\theta < 0$, X_t is asymptotically stationary and ergodic with the long-run mean $-\mu/\theta$. In this case, the coefficient θ is the speed of adjustment of X_t towards its long-run mean. When $\theta = 0$ and $\mu = 0$, $X_t = B_t^H$ that is non-stationary and null recurrent. When $\theta > 0$, $|\mathbb{E}(X_t|\mathcal{F}_{t-1})| > |X_{t-1}|$, implying X_t is non-stationary and explosive, where $\mathcal{F}_t$ is the sigma-algebra generated by B_s^H with $s \in [0, t]$.

In practice, we often have access to discretely sampled data only. Let $\{X_{i\Delta}\}_{i=1}^{n}$ denote the discretely sampled data, where n is the sample size and Δ is the sampling interval. Let $T (:= n\Delta)$ be the time span. When X_t is annualized and observed monthly (weekly or daily), then $\Delta = 1/12$ ($1/52$ or $1/252$) for assets that are traded five days in a week. The in-fill asymptotics assume $\Delta \to 0$ with T being fixed while the double asymptotics assume $\Delta \to 0, T \to \infty$. In both cases, $n \to \infty$. In model (2.1), there are four parameters, two diffusion parameters, H and σ, and two drift parameters, θ and μ. We would like to estimate these four parameters based on $\{X_{i\Delta}\}_{i=0}^{n}$ generated from model (2.1) with $\theta > 0$, that is, an explosive fOUp.

When $\theta \neq 0$, the strong solution of fOUp is given by

$$X_t = X_0 e^{\theta t} + \frac{\mu}{\theta}\left(e^{\theta t} - 1\right) + \sigma e^{\theta t} \int_0^t e^{-\theta s} dB_s^H, \tag{2.5}$$

where the stochastic integral in (2.5) is interpreted as a Young integral [50].

Therefore, the exact discrete-time representation of model (2.1) is

$$X_{i\Delta} = \beta_\Delta X_{(i-1)\Delta} + \frac{\mu}{\theta}\left(e^{\theta\Delta} - 1\right) + \sigma \epsilon_{i\Delta},\ \beta_\Delta = e^{\theta\Delta}, \tag{2.6}$$

where

$$\epsilon_{i\Delta} = \int_{(i-1)\Delta}^{i\Delta} e^{\theta(i\Delta - s)} dB_s^H = \left(B_{i\Delta}^H - B_{(i-1)\Delta}^H\right) + O_p\left(\Delta^{1+H}\right) = O_p\left(\Delta^H\right).$$

When $\theta > 0$, $\beta_\Delta = e^{\theta\Delta} > 1$ since $\Delta > 0$. If $\Delta \to 0$, $\beta_\Delta \searrow 1$. However, the speed that β_Δ approaches unity depends on whether T is fixed or goes to infinity.

As shown in Wang and Yu [48], under the in-fill asymptotic scheme, model (2.6) with $\theta > 0$ corresponds to a local-to-unity model with the AR(1) parameter larger than unity but approaching to unity as $\Delta \to 0$. It can be seen that with a fixed T and $\Delta \to 0$, we have

$$(1 - \beta_\Delta)n = \left(1 - e^{\theta\Delta}\right)n = \left(-\theta\Delta + o(\Delta)\right)n \to -\theta T,$$

where $-\theta T$ is the scale parameter. Whereas, under the double asymptotic scheme, the exact discrete-time representation of model (2.1) with $\theta > 0$ is an explosive model with the AR(1) coefficient larger than but approaching to unity slower than $1/n$ as $\Delta \to 0$. It can be seen that with $\Delta \to 0, T \to \infty$, we have

$$(1 - \beta_\Delta)n = \left(1 - e^{\theta\Delta}\right)n = \left(-\theta\Delta + o(\Delta)\right)n = -\theta T + o(T) \to -\infty.$$

Using the terminology of Phillips and Magdalinos [35], the model is mildly explosive.

Since $\epsilon_{i\Delta} = O_p(\Delta^H)$, to ensure the error term is $O_p(1)$, dividing both sides of equation (2.6) by Δ^H, we have

$$Y_{i\Delta} = \beta_\Delta Y_{(i-1)\Delta} + \frac{\mu}{\Delta^H \theta}\left(e^{\theta\Delta} - 1\right) + \sigma e_{i\Delta}, \tag{2.7}$$

where $Y_{i\Delta} = X_{i\Delta}/\Delta^H$, $e_{i\Delta} = \epsilon_{i\Delta}/\Delta^H$. Clearly, as $\Delta \to 0$ with a fixed T, $e_{i\Delta} = O_p(1)$ and $Y_0 = X_0/\Delta^H = O_p(n^H/T^H)$.

Magdalinos [33] considered the following AR(1) model,

$$Y_t = \rho_n Y_{t-1} + \sigma u_t, \rho_n = 1 + \frac{c}{n^\alpha}, \alpha \in (0,1), c > 0, Y_0 = o_p\left(n^{\alpha(0.5+d)}\right), \tag{2.8}$$

where $u_t = \sum_{j=0}^\infty c_j v_{t-j}$ with $c_j \sim \gamma j^{d-1}$ for some $d \in (0, 1/2)$ and v_t being a martingale difference sequence and v_t^2 is a uniformly integrable sequence.[2] He showed that the least squares estimator of ρ_n follows the standard Cauchy distribution asymptotically.

His model with $n \to \infty$ is closely linked to model (2.7). To see the connection, in model (2.7), if $\Delta \to 0, T \to \infty$, we have

$$\beta_\Delta \to 1, (1 - \beta_\Delta)n \to -\infty, \mathbb{E}(e_{i\Delta} e_{(i+j)\Delta}) \sim Cj^{2H-2} \text{ for large } j,$$

where the last part is due to Lemma 2.1 of Cheridito et al. [16]. In model (2.8), if $n \to \infty$, we have

$$\rho_n \to 1, (1 - \rho_n)n \to -\infty, \mathbb{E}(u_i u_{i+j}) \sim Cj^{2d-1} \text{ for large } j.$$

If $H = 1/2 + d$, model (2.7) and model (2.8) share the same covariance structure for large j.

Lui et al. [32] considered the following AR(1) model,

$$Y_t = \rho_{n,m} Y_{t-1} + \sigma u_t, \rho_{n,m} = 1 + \frac{cm}{n}, c > 0, Y_0 = o_p\left(n^{0.5+d}\right), \tag{2.9}$$

where $u_t = \sum_{j=0}^\infty c_j v_{t-j}$ with $c_j \sim \gamma j^{d-1}$ for some $d \in (-1/2, 0)$ and v_t being an iid sequence. They showed that the least squares estimator of $\rho_{n,m}$ follows the standard Cauchy distribution asymptotically.[3]

Model (2.9) with $n \to \infty$ followed by $m \to \infty$ is also closely linked to model (2.7). In model (2.9), if $n \to \infty$ followed by $m \to \infty$, we have

$$\rho_{n,m} \to 1, (1 - \rho_{n,m})n \to -\infty, \mathbb{E}(u_i u_{i+j}) \sim Cj^{2d-1} \text{ for large } j.$$

If $H = 1/2 + d$, model (2.8) and model (2.9) also share the same covariance structure for large j.

[2] The exact assumption in Magdalinos [33] is $c_j = L(j)j^{-k}$ for some $k \in (1/2, 1)$ where L is a slowly varying function at infinity in Assumption LP(ii).

[3] In Remark 3.7, Lui et al. [32] argue that the asymptotic theory continues to hold when $d \in (0, 1/2)$.

Table 1
Comparison of model (2.7) and the models considered in Magdalinos (2012) and Lui et al. (2021).

Model (2.7) (with $\Delta \to 0, T \to \infty$)	Model in (2.8) (with $n \to \infty$)	Model in (2.9) (with $n \to \infty, m \to \infty$)
$\beta_\Delta = e^{\theta\Delta} = 1 + \theta\frac{T}{n} + o(\Delta)$	$\rho_n = 1 + \frac{c}{n^\alpha}$	$\rho_{n,m} = 1 + \frac{cm}{n}$
$1 - \beta_\Delta \nearrow 0$, $(1 - \beta_\Delta)n \to -\infty$	$1 - \rho_n \nearrow 0$, $(1 - \rho_n)n \to -\infty$	$1 - \rho_{n,m} \nearrow 0$, $(1 - \rho_{n,m})n \to -\infty$
$Y_0 = O_p(n^H/T^H)$	$Y_0 = o_p(n^{\alpha H})$	$Y_0 \sim o_p(n^H)$
$H \in (0, 1)$	$H \in (1/2, 1)$	$H \in (0, 1/2)$

However, there are three important differences between the two existing models and model (2.7). First, they have different initial conditions. In particular, since $n^{(\alpha-1)H}(\log\Delta)^2 \to 0$ for any $\alpha \in (0, 1)$, the initial condition in model (2.7) is larger than that in (2.8). It turns out the initial condition enters the asymptotic distribution in our model but not in the asymptotic distribution obtained in Magdalinos [33]. Since the finite sample distribution should depend on the initial condition, which is supported by our simulation studies in Section 4, naturally it is expected our asymptotic distribution delivers more accurate finite sample approximations. Second, in (2.8) it is assumed that $d \in (0, 1/2)$ which is equivalent to $H \in (1/2, 1)$, and hence, a long memory error term is assumed. In (2.9) it is assumed that $d \in (-1/2, 0)$ which is equivalent to $H \in (0, 1/2)$, and hence, an anti-persistent error term is assumed. In model (2.7), a full range of $H \in (0, 1)$ is allowed. That is, both long memory error terms and anti-persistent error terms are allowed in our model. While some empirical evidence has been reported to support long memory error terms in the context of the mildly explosive model for equity prices in the literature (see, for example, Lui et al., [31]), some other empirical evidence that supports anti-persistent error terms has also been reported in the literature (see, for example, Gatheral et al., [18], Lui et al., [32], Bennedsen et al., [7], Shi and Yu, [41], Wang et al., [47], Bolk et al., [8]). In practice it is often impossible to have a knowledge about a restricted range of H *ex ante*. Table 1 compares the two existing models with model (2.7). Third, although our model shares the same covariance structure as model (2.7) and model (2.9), unlike the error terms in their models, our $e_{i\Delta}$ in (2.7) cannot be written as a linear combination of a martingale difference sequence. As a result, our proof strategy is remarkably different from those in Magdalinos [33] and Lui et al. [32].

3. Estimators, and asymptotics

3.1. Estimators

Our model is the same as that of Wang et al. [47]. The only difference between the two models is that we assume $\theta > 0$ while Wang et al. [47] assume $\theta < 0$

in fOUp. Following Wang et al. [47], we also consider a two-stage estimation method. Our first stage estimation focuses on estimating the two parameters in the diffusion term following the idea of Wang et al. [47]. In particular, we estimate the Hurst parameter H based on the second-order differences of X_t at two different frequencies:[4]

$$\widehat{H}_\Delta = \frac{1}{2}\log_2\left(\frac{\frac{1}{n-4}\sum_{i=1}^{n-4}(X_{(i+4)\Delta} - 2X_{(i+2)\Delta} + X_{i\Delta})^2}{\frac{1}{n-2}\sum_{i=1}^{n-2}(X_{(i+2)\Delta} - 2X_{(i+1)\Delta} + X_{i\Delta})^2}\right), \tag{3.1}$$

where $\log_2(\cdot)$ is the base-2 logarithm.[5] We estimate the volatility coefficient σ using

$$\widehat{\sigma}_\Delta = \sqrt{\frac{\sum_{i=1}^{n-2}(X_{(i+2)\Delta} - 2X_{(i+1)\Delta} + X_{i\Delta})^2}{n(4 - 2^{2\widehat{H}})\Delta^{2\widehat{H}}}}. \tag{3.2}$$

In the second stage, we consider the estimators of the two drift parameters in (2.1) based on least squares. Let $\alpha_\Delta = \frac{\mu}{\theta}(e^{\theta\Delta} - 1)$. Then, (2.6) can be rewritten as:

$$X_{i\Delta} = \beta_\Delta X_{(i-1)\Delta} + \alpha_\Delta + \sigma\epsilon_{i\Delta}, X_0 = O_p(1).$$

The least squares estimators of α_Δ and β_Δ are

$$\widehat{\beta}_\Delta = \frac{n\sum_{i=1}^n X_{i\Delta}X_{(i-1)\Delta} - \sum_{i=1}^n X_{i\Delta}\sum_{i=1}^n X_{(i-1)\Delta}}{n\sum_{i=1}^n X_{(i-1)\Delta}^2 - (\sum_{i=1}^n X_{(i-1)\Delta})^2}, \tag{3.3}$$

$$\widehat{\alpha}_\Delta = \frac{\sum_{i=1}^n X_{i\Delta}\sum_{i=1}^n X_{(i-1)\Delta}^2 - \sum_{i=1}^n X_{(i-1)\Delta}\sum_{i=1}^n X_{i\Delta}X_{(i-1)\Delta}}{n\sum_{i=1}^n X_{(i-1)\Delta}^2 - (\sum_{i=1}^n X_{(i-1)\Delta})^2}. \tag{3.4}$$

Based on $\widehat{\alpha}_\Delta$ and $\widehat{\beta}_\Delta$, we can propose the least squares estimators of θ and μ as

$$\widehat{\theta}_\Delta = \frac{1}{\Delta}\log\frac{\sum_{i=1}^n X_{i\Delta}X_{(i-1)\Delta} - \frac{1}{n}\sum_{i=1}^n X_{i\Delta}\sum_{i=1}^n X_{(i-1)\Delta}}{\sum_{i=1}^n X_{(i-1)\Delta}^2 - \frac{1}{n}(\sum_{i=1}^n X_{(i-1)\Delta})^2}, \tag{3.5}$$

$$\widehat{\mu}_\Delta = \widehat{\theta}_\Delta\frac{\widehat{\alpha}_\Delta}{\widehat{\beta}_\Delta - 1}. \tag{3.6}$$

Remark 3.1. Wang et al. [47] use the ergodic property of X_t to construct the method-of-moment estimators of θ and μ when $\theta < 0$. With $\theta > 0$, the fOUp is explosive and hence, non-ergodic. Consequently, the estimators for the drift term of Wang et al. [47] are not applicable when $\theta > 0$.

Remark 3.2. The proposed least squares estimators of θ and μ ignore the dependence structure in the error term and are independent of the two diffusion parameters. Later we will examine the efficiency loss in the least squares estimators relative to the maximum likelihood estimators (MLE) that take account of the dependence structure in the error term.

---

[4]If H is known to be less than 3/4, a more efficient estimator of H may be obtained from first-order differences.

[5]We thank the reference for the estimator of H, by multiplying the numerator by $\frac{1}{n-4}$ and multiplying the denominator by $\frac{1}{n-2}$.

3.2. Asymptotic properties

In this subsection, we develop the in-fill asymptotic theory for $\widehat{H}_\Delta$ and $\widehat{\sigma}_\Delta$ and the double asymptotic theory for $\widehat{\mu}_\Delta$ and $\widehat{\theta}_\Delta$. For $\widehat{H}_\Delta$ and $\widehat{\sigma}_\Delta$, Theorem 4.1 of Wang et al. [47] is directly applicable to fOUp with $\theta > 0$. Hence, we state it here with slightly re-phrasing but without proof.

Theorem 3.1. *Let $\widehat{H}_\Delta$ and $\widehat{\sigma}_\Delta$ be the estimators defined in (3.1) and (3.2) for model (2.1) with $\theta > 0$. For any $H \in (0,1)$, when $\Delta \to 0$ with a fixed $T > 0$,*
(a) $\widehat{H}_\Delta \xrightarrow{a.s.} H$ and

$$\sqrt{n}(\widehat{H}_\Delta - H) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \frac{\Sigma_{11} + \Sigma_{22} - 2\Sigma_{12}}{(2\log 2)^2}\right); \tag{3.7}$$

(b) $\widehat{\sigma}_\Delta \xrightarrow{a.s.} \sigma$ and

$$\frac{\sqrt{n}}{\log(\Delta)}(\widehat{\sigma}_\Delta - \sigma) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \frac{\Sigma_{11} + \Sigma_{22} - 2\Sigma_{12}}{(2\log 2)^2}\sigma^2\right), \tag{3.8}$$

where

$$\Sigma_{11} = 2 + 2^{2-4H}\sum_{j=1}^{\infty}(\rho_{j+2} + 4\rho_{j+1} + 6\rho_j + 4\rho_{|j-1|} + \rho_{|j-2|})^2,$$
$$\Sigma_{12} = 2^{1-2H}\left(4(\rho_1 + 1)^2 + 2\sum_{j=0}^{\infty}(\rho_{j+2} + 2\rho_{j+1} + \rho_j)^2\right),$$
$$\Sigma_{22} = 2 + 4\sum_{j=1}^{\infty}\rho_j^2,$$

with

$$\rho_j = \frac{-|j+2|^{2H} + 4|j+1|^{2H} - 6|j|^{2H} + 4|j-1|^{2H} - |j-2|^{2H}}{2(4 - 2^{2H})}. \tag{3.9}$$

Remark 3.3. Thanks to the Local Asymptotic Normal property of the likelihoods for the fGn (see, [9]) and the Lipschitz condition of the drift part for the fOUp, we can obtain the distribution for $\widehat{H}_\Delta$ and $\widehat{\sigma}_\Delta$ in the explosive fOUp based on the idea of [9]. First, using Lemma 7.2 of [47] and the Lipschitz condition of the drift part for the fOUp, we can see that

$$\frac{1}{\sqrt{n}}\begin{pmatrix} \sum_{i=0}^{n-4}\left[\left(\frac{X^H_{(i+4)\Delta} - 2X^H_{(i+2)\Delta} + X^H_{i\Delta}}{\Delta^H}\right)^2 - 2^{2H}\left(4 - 2^{2H}\right)\right] \\ \sum_{i=0}^{n-2}\left[\left(\frac{X^H_{(i+2)\Delta} - 2X^H_{(i+1)\Delta} + X^H_{i\Delta}}{\Delta^H}\right)^2 - \left(4 - 2^{2H}\right)\right] \end{pmatrix}$$
$$\xrightarrow{\mathcal{L}} \mathcal{N}\left(\begin{pmatrix}0\\0\end{pmatrix}, A_0 + \sum_{j=1}^{\infty}\left(A_j + A_j^\top\right)\right), \tag{3.10}$$

where $^\top$ denotes the transpose of the inverse matrix

$$A_j = 2\big(4-2^{2H}\big)^2 \begin{pmatrix} 2^{4H}\rho_{j,*}^2 & (\rho_{j+2}+2\rho_{j+1}+\rho_j)^2 \\ (\rho_j+2\rho_{|j-1|}+\rho_{|j-2|})^2 & \rho_j^2 \end{pmatrix},$$
$$\rho_{j,*} = 2^{-2H}(\rho_{j+2}+4\rho_{j+1}+6\rho_j+4\rho_{|j-1|}+\rho_{|j-2|}),$$

and ρ_j is defined by (3.9).

Second, for the joint distribution of $\widehat{H}_\Delta$ and $\widehat{\sigma}_\Delta$, let us denote

$$f(u,v) = \begin{pmatrix} h(u,v) \\ s(u,v) \end{pmatrix}$$

where $h(u,v)=\frac{1}{2}\log_2(\frac{v}{u})$, $s(u,v)=\sqrt{\frac{u}{\Delta^{2h(u,v)}(4-2^{2h(u,v)})}}$.

Then, direct computations lead to

$$\frac{\partial}{\partial u}f(u,v) = \begin{pmatrix} \frac{\partial}{\partial u}h(u,v) \\ -s(u,v)\log\Delta\frac{\partial}{\partial u}h(u,v)+g(u,v)\frac{\partial}{\partial u}h(u,v)+\frac{1}{2s(u,v)w(u,v)} \end{pmatrix},$$
$$\frac{\partial}{\partial v}f(u,v) = \begin{pmatrix} \frac{\partial}{\partial v}h(u,v) \\ -s(u,v)\log\Delta\frac{\partial}{\partial v}h(u,v)+g(u,v)\frac{\partial}{\partial v}h(u,v) \end{pmatrix},$$

where $w(u,v)=\Delta^{2h(u,v)}(4-2^{2h(u,v)})$, $g(u,v)=\frac{s(u,v)2^{2h(u,v)}\log 2}{(4-2^{2h}(u,v))}$.

Third, let $\vartheta=(H,\sigma)^\top$ and $\widehat{\vartheta}_\Delta=(\widehat{H}_\Delta,\widehat{\sigma}_\Delta)^\top$. Choose $u^\star=\sigma^2\Delta^{2H}(4-2^{2H})$ and $v^\star=\sigma^2\Delta^{2H}2^{2H}(4-2^{2H})$ such that

$$f\big(u^\star,v^\star\big) = \begin{pmatrix} H \\ \sigma \end{pmatrix}.$$

Moreover, let $\varphi_n(\vartheta)$ be $\varphi_n(H,\sigma)$ defined in Theorem 3.1 of [9]. Let $v=\frac{1}{n-4}\sum_{i=1}^{n-4}(X_{(i+4)\Delta}-2X_{(i+2)\Delta}+X_{i\Delta})^2$, $u=\frac{1}{n-2}\sum_{i=1}^{n-2}(X_{(i+2)\Delta}-2X_{(i+1)\Delta}+X_{i\Delta})^2$. Then, using (3.10), the delta method and a standard calculation, we can see that

$$\begin{aligned}&\varphi_n^{-1}(\vartheta)(\widehat{\vartheta}-\vartheta)\\ &=\varphi_n^{-1}(\vartheta)\big(f(u,v)-f\big(u^\star,v^\star\big)\big)\\ &\xrightarrow{\mathcal{L}}\mathcal{N}\left(\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}a_\vartheta & b_\vartheta\\ c_\vartheta & d_\vartheta\end{pmatrix}\left[A_0+\sum_{j=1}^{\infty}(A_j+A_j^\top)\right]\begin{pmatrix}a_\vartheta & b_\vartheta\\ c_\vartheta & d_\vartheta\end{pmatrix}^\top\right),\end{aligned}\tag{3.11}$$

where

$$a_\vartheta=\frac{1}{\omega}\left[\left(\bar{\gamma}\sigma-\bar{\alpha}\frac{\sigma 2^{2H}\log 2}{4-2^{2H}}\right)\cdot\left(-\frac{1}{2\log 2\sigma^2(4-2^{2H})}\right)-\bar{\alpha}\frac{1}{2\sigma(4-2^{2H})}\right],$$
$$b_\vartheta=\frac{1}{\omega}\left(\bar{\gamma}\sigma-\bar{\alpha}\frac{\sigma 2^{2H}\log 2}{4-2^{2H}}\right)\cdot\left(\frac{1}{2\log 2\sigma^2 2^{2H}(4-2^{2H})}\right),$$

$$c_\vartheta = \frac{1}{\omega}\left[\left(-\gamma\sigma + \alpha\frac{\sigma 2^{2H}\log 2}{4-2^{2H}}\right)\cdot\left(-\frac{1}{2\log 2\sigma^2(4-2^{2H})}\right) + \alpha\frac{1}{2\sigma(4-2^{2H})}\right],$$
$$d_\vartheta = \frac{1}{\omega}\left(-\gamma\sigma + \alpha\frac{\sigma 2^{2H}\log 2}{4-2^{2H}}\right)\cdot\left(\frac{1}{2\log 2\sigma^2 2^{2H}(4-2^{2H})}\right),$$

and α, $\bar{\alpha}$, γ and $\bar{\gamma}$ are defined by (5) of [9].

In the following, we shall state the main results concerning the strong consistency and the asymptotic distributions of $\widehat{\theta}_\Delta$ and $\widehat{\mu}_\Delta$. First, we give the strong consistency of $\widehat{\theta}_\Delta$ and $\widehat{\mu}_\Delta$, as well as the asymptotic theory for $\widehat{\theta}_\Delta$ and $\widehat{\mu}_\Delta$.

Theorem 3.2. *Let $\Delta \to 0$ and $\frac{\log\Delta}{T} \to 0$. If either (i) $H = 1/2$, or (ii) $H \in (1/2, 1)$ and $T^{2H}\Delta \to 0$, or (iii) $H \in (0, 1/2)$ and $T^{2H+1}\Delta \to 0$, then we have $\widehat{\theta}_\Delta \xrightarrow{a.s.} \theta$ and*

$$\frac{e^{\theta T}}{2\theta}(\widehat{\theta}_\Delta - \theta) \xrightarrow{\mathcal{L}} \frac{\sigma\frac{\sqrt{H\Gamma(2H)}}{\theta^H}\nu}{X_0 + \frac{\mu}{\theta} + \sigma\frac{\sqrt{H\Gamma(2H)}}{\theta^H}\omega}, \tag{3.12}$$

where ν and ω are two independent standard normal variables.[6]

Theorem 3.3. *Let $\Delta \to 0$ and $\frac{(\log\Delta)^3}{T^2} \to 0$. If either (i) $H = 1/2$, or (ii) $H \in (1/2, 1)$ and $T^{2H}\Delta \to 0$, or (iii) $H \in (0, 1/2)$ and $T^{2H+1}\Delta \to 0$, then we have $\widehat{\mu}_\Delta \xrightarrow{a.s.} \mu$.*

Second, based on Theorem 3.2, we can develop the following joint distribution for $\widehat{\theta}_\Delta$ and $\widehat{\mu}_\Delta$ in the explosive fOUp.

Theorem 3.4. *Let $\Delta \to 0$ and $\frac{\log\Delta}{T} \to 0$. If either (i) $H = 1/2$, or (ii) $H \in (1/2, 1)$ and $T^{2H}\Delta \to 0$, or (iii) $H \in (0, 1/2)$ and $T^{2H+1}\Delta \to 0$, then we have*

$$\left(\frac{e^{\theta T}}{2\theta}(\widehat{\theta}_\Delta - \theta), T^{1-H}(\widehat{\mu}_\Delta - \mu)\right) \xrightarrow{\mathcal{L}} \left(\frac{\sigma\frac{\sqrt{H\Gamma(2H)}}{\theta^H}\nu}{X_0 + \frac{\mu}{\theta} + \sigma\frac{\sqrt{H\Gamma(2H)}}{\theta^H}\omega}, \sigma\eta\right),$$

where ν, ω and η are independent standard normal variables.

Remark 3.4. In the case of $\mu = 0$, the least squares estimator of θ under the continuous observations on $[0, T]$ is $\widetilde{\theta}_T = \frac{\int_0^T X_t dX_t}{\int_0^T X_t^2 dt}$ [22]). The consistency and asymptotic distribution properties of $\widetilde{\theta}_T$ have been studied thoroughly in both the ergodic ($\theta < 0$, [13, 15, 22, 24, 49]) and explosive ($\theta > 0$, [49]) cases. On the other hand, the discrete version of $\widetilde{\theta}_T$ is to replace dX_t with $(X_{i\Delta} - X_{(i-1)\Delta})$, and $\int_0^T X_t^2 dt$ by $\Delta\sum_{i=1}^n X_{(i-1)\Delta}^2$, i.e. $\widetilde{\theta}_\Delta^* = \frac{n\sum_{i=1}^n X_{(i-1)\Delta}(X_{i\Delta} - X_{(i-1)\Delta})}{\Delta\sum_{i=1}^n X_{(i-1)\Delta}^2}$. If the fOUP X is ergodic ($\theta < 0$), the asymptotic properties of $\widetilde{\theta}_\Delta^*$ in the sense of consistency and asymptotic distribution have been obtained [4, 10, 19, 23, 47].

[6] It is easy to see that $\xrightarrow{\mathcal{L}}$ may be replaced with $\xrightarrow{p}$ in this theorem and subsequent theorem.

If X is explosive ($\theta > 0$), one may expect the estimator $\widetilde{\theta}^*_\Delta$ keeps the asymptotic distribution of $\widetilde{\theta}_T$, i.e. $e^{\theta T}$-rate of convergence in distribution as shown in Xiao and Yu [49]. However, the answer is negative [6, 17, 25, 30, 42]:

$$e^{\theta T}\big(\widetilde{\theta}^*_\Delta - \theta\big) \to \infty, \quad \sqrt{T}\big(\widetilde{\theta}^*_\Delta - \theta\big) \text{ is tight.}$$

To fix this problem, for explosive OU model driven by the standard Brownian motion, Shimizu [43], Wang and Yu [48], Jiang et al. [27] introduce the least squares estimator and consider the associated consistency, asymptotic distribution properties and deviation properties. In our paper, we extend the OU model of Shimizu [43], Wang and Yu [48] by replacing the standard Brownian motion ($H = 1/2$) with the fractional Brownian motion ($H \in (0, 1)$). Moreover, the parameter μ is also assumed to be unknown in our paper. For more details on this topic, one can refer to [1, 5, 11, 14, 26, 29, 44, 45, 46] and the references therein.

Remark 3.5. According to Theorem 3.2, the same asymptotic law holds for the least squares estimator of θ regardless of H in the explosive fOUp. That is, the rate of convergence is $e^{\theta T}$ and, if $X_0 = \mu = 0$, the limit distribution is a standard Cauchy. However, from the technical proofs in the Appendix and Online Supplement, it can be seen that we need to deal with the cases of $H \in (1/2, 1)$, $H = 1/2$ and $H \in (0, 1/2)$ separately. The result in Theorem 3.2 is in sharp contrast with that of the method-of-moments estimator of θ for the stationary fOUp. Theorem 4.4 in Wang et al. [47] shows that the asymptotic law for the method-of-moments estimator of θ changes as H passes $3/4$. In particular, when $H \in (0, 3/4)$, the rate of convergence is $\sqrt{T}$ and the limit distribution is normal; when $H = 3/4$, the rate of convergence is $\sqrt{T}/\log T$ and the limit distribution is different normal; when $H \in (3/4, 1)$, the rate of convergence is T^{2-2H} and the limit distribution is the Rosenblatt random variable.

Remark 3.6. From Theorem 3.4, we can see that the asymptotic law of $\widehat{\mu}_\Delta$ is normal, where the rate of convergence is T^{1-H}. Theorem 5 in Tanaka et al. [44] states that the MLE of μ (denoted by $\widehat{\mu}_{MLE}$) based on a continuous-time record is

$$T^{1-H}(\widehat{\mu}_{MLE} - \mu) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \sigma^2 \frac{2H\Gamma(3-2H)\Gamma(H+1/2)}{\Gamma(3/2-H)}\right).$$

Comparing the above asymptotic theory with Theorem 3.4, we can see that the rate of convergence of the least squares estimator of μ based on the discrete-sampled data is identical to that of the MLE of μ based on a continuous-time record. However, the least squares estimator of μ is less efficient than the MLE of μ since the variance of MLE is smaller when $H \in (0, 1/2) \cup (1/2, 1)$ (i.e., $\frac{2H\Gamma(3-2H)\Gamma(H+1/2)}{\Gamma(3/2-H)} < 1$). This efficiency loss is expected as the least squares estimator ignores the dependence in the error term. When $H = 1/2$, the two variances are the same (i.e., $\frac{2H\Gamma(3-2H)\Gamma(H+1/2)}{\Gamma(3/2-H)} = 1$). This is also expected because, when $H = 1/2$, the error term becomes iid.

Remark 3.7. As stated by [2], when the diffusion parameter is time varying, (2.1) can be written as

$$dX_t = (\theta X_t + \mu)dt + \sigma_t dB_t^H, \tag{3.13}$$

where Hurst parameter $H \in (0,1)$, the volatility σ_t is a stochastic process with β_σ-Hölder continuous trajectories, where $\beta_\sigma > 1-H$. Under this condition on σ_t, the stochastic integral $\int_0^t \sigma_s dB_s^H$ is well defined as a Young integral. It is obvious that the drift function $\theta X_t + \mu$ is Lipschitz continuous. Then X_t in (3.13) has a unique solution under some boundedness conditions for the drift function. In this situation, we can also estimate the Hurst parameter, H, by (3.1). However, the asymptotic properties are complicated and left for future work. Let us also mention that the time varying σ_t has no effect on the estimator of μ from (3.6).

Remark 3.8. From Theorem 3.2, we can see that the limiting distribution of $\widehat{\theta}_\Delta - \theta$ depends explicitly on the initial condition X_0 (as well as μ/θ). This dependence is the same as that in Wang and Yu [48]. The reason is that when $\Delta \to 0$, the initial condition in model (2.7) is larger than those assumed in Magdalinos [33] and in Lui et al. [32]. If $X_0 = -\frac{\mu}{\theta}$ in the fOUp, then the limiting distribution of $\frac{e^{\theta T}}{2\theta}(\widehat{\theta}_\Delta - \theta)$ is a standard Cauchy distribution, which is the same as that obtained in Magdalinos [33] and in Lui et al. [32].

Remark 3.9. From Theorem 3.4, if $\Delta \to 0$ and $\frac{\log \Delta}{T} \to 0$, under either (i) $H = 1/2$, or (ii) $H \in (1/2, 1)$ and $T^{2H}\Delta \to 0$, or (iii) $H \in (0, 1/2)$ and $T^{2H+1}\Delta \to 0$, we can easily get

$$\frac{e^{\theta T}}{2\theta\Delta}(\widehat{\beta}_\Delta - \beta_\Delta) \xrightarrow{\mathcal{L}} \frac{\sigma\frac{\sqrt{H\Gamma(2H)}}{\theta^H}\nu}{X_0 + \frac{\mu}{\theta} + \sigma\frac{\sqrt{H\Gamma(2H)}}{\theta^H}\omega}, \tag{3.14}$$

$$\frac{T^{1-H}}{\Delta}(\widehat{\alpha}_\Delta - \alpha_\Delta) \xrightarrow{\mathcal{L}} \sigma\eta, \tag{3.15}$$

where ν, ω and η are defined by Theorem 3.4. If $H = 1/2$, the asymptotic theory given in (3.14) and (3.15) becomes that given in Theorem 3.3 (a)–(b) in Wang and Yu [48].

Remark 3.10. When $H < 3/4$, based on first-order differences, we can provide a more efficient estimator of H as

$$\widetilde{H}_\Delta = \frac{1}{2}\log_2\left(\frac{\sum_{i=1}^{n-2}(X_{(i+2)\Delta} - X_{i\Delta})^2}{\sum_{i=1}^{n-1}(X_{(i+1)\Delta} - X_{i\Delta})^2}\right). \tag{3.16}$$

Using similar arguments as Theorem 4.1 (a) in Wang et al. [47], we can obtain $\widetilde{H}_\Delta \xrightarrow{a.s} H$ for $H \in (0,1)$. Moreover, for $0 < H < 3/4$, when $\Delta \to 0$ with a fixed T, we can get

$$\sqrt{n}(\widetilde{H}_\Delta - H) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \frac{\Omega_{11} - 2^{1+2H}\Omega_{12} + 2^{4H}\Omega_{22}}{2^{4H+2}\log^2(2)}\right), \tag{3.17}$$

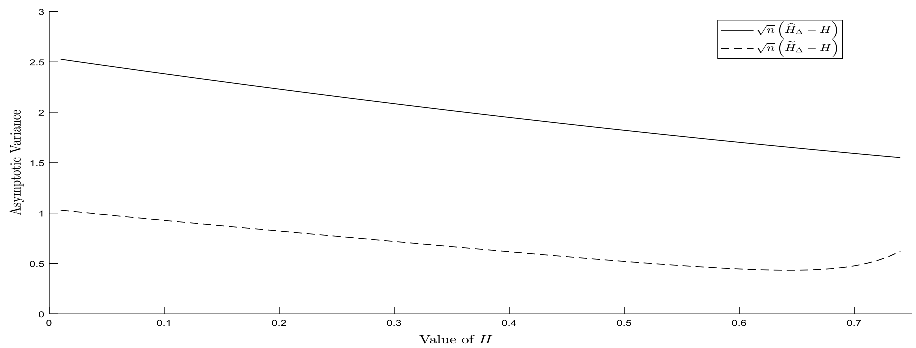

FIG 1. *Asymptotic variance of* $\sqrt{n}(\widehat{H}_\Delta - H)$ *and* $\sqrt{n}(\widetilde{H}_\Delta - H)$ *as functions of* $H \in (0, 3/4)$.

where

$$\Omega_{11} = 2^{4H+1} + \sum_{j=1}^{\infty} 2^{4H+2}\widetilde{\rho}_{j,*}^2, \quad \Omega_{22} = 2 + \sum_{j=1}^{\infty} 4\widetilde{\rho}_j^2,$$

$$\Omega_{12} = \Omega_{21} = 2^{4H-1} + \sum_{j=1}^{\infty}\left[2(\widetilde{\rho}_{j+1} + \widetilde{\rho}_j)^2 + 2(\widetilde{\rho}_{|j-1|} + \widetilde{\rho}_j)^2\right],$$

with

$$\widetilde{\rho}_{j,*} = \frac{1}{2^{2H+1}}\left[|j-2|^{2H} + (j+2)^{2H} - 2j^{2H}\right],$$
$$\widetilde{\rho}_j = \frac{1}{2}\left[|j-1|^{2H} + (j+1)^{2H} - 2j^{2H}\right].$$

When $H = 1/2$, a standard calculation shows that

$$\Omega_{11} - 2^{1+2H}\Omega_{12} + 2^{4H}\Omega_{22} = 4.$$

Consequently, for $H = 1/2$, when $\Delta \to 0$ with a fixed T, we can obtain

$$\sqrt{n}(\widetilde{H}_\Delta - H) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \frac{1}{4\log^2(2)}\right). \tag{3.18}$$

Comparing Corollary 4.2 in Wang et al. [47] with (3.18), we can see that $\widetilde{H}_\Delta$ is more efficient than $\widehat{H}_\Delta$ for $H = 1/2$. Indeed, this conclusion holds true for $0 < H < 3/4$. Figure 1 compares the asymptotic variance of $\sqrt{n}(\widehat{H}_\Delta - H)$ and that of $\sqrt{n}(\widetilde{H}_\Delta - H)$ for $0 < H < 3/4$. When $0 < H < 3/4$, it is more efficient to estimate H via the first-order differences than via the second-order differences. However, when $H > 3/4$, the central limit theorem of the first-order differences does not hold. Whereas, we always have the central limit theorem for the second-order differences.

Remark 3.11. Remark 3.10 suggests a two-step procedure to estimate the Hurst parameter for the fOUp. Thus, we first test the hypothesis $\mathcal{H}_0 : H \geq 3/4$ versus $\mathcal{H}_1 : H < 3/4$ using the estimator $\hat{H}_\Delta$ and the asymptotic distribution of (3.7). Then, if $\mathcal{H}_0$ is not rejected, we use the estimator $\hat{H}_\Delta$ proposed in (3.1). Otherwise, if $\mathcal{H}_0$ is rejected and $\mathcal{H}_1$ is accepted, we can use $\widetilde{H}_\Delta$ proposed in (3.16) for the sake of efficiency.

4. Simulation studies

In this section, we conduct Monte Carlo simulations to evaluate the finite sample performance of the proposed estimator and the derived asymptotic limit theory. Following Wang and Yu [48] and Chen et al. [12], we first examine the sensitivity of the Monte Carlo empirical distribution (MCED) of $\widehat{\theta}_\Delta$ and $\widehat{\beta}_\Delta$ with respect to the initial condition and to μ. We then check the finite sample properties (3.12) and (3.14).

For this purpose, we simulate 10,000 sample paths from model (2.1) with $\theta = 2$, $\sigma = 1$ and $\mu = 0$. However, we allow H to take different values, $0.15, 0.35, 0.55, 0.75$. The first two values imply anti-persistent errors while the last two values imply long-memory errors. We set the sampling interval $\Delta = 1/252, 1/52, 1/12$, the time span $T = 10$, the initial value $X_0 \in \{0, 3.5, 10\}$. For each simulated path, we estimate θ by (3.5) and calculate $\frac{e^{\theta T}}{2\theta}(\widehat{\theta}_\Delta - \theta)$. Moreover, we also estimate β by (3.3) and calculate $\frac{e^{\theta T}}{2\theta\Delta}(\widehat{\beta}_\Delta - \beta)$. We report percentiles at levels $\{1\%, 2.5\%, 10\%, 90\%, 97.5\%, 99\%\}$ in the limit distributions of (3.12) and (3.14). Tables 2–4 report the percentiles of the Cauchy asymptotic distribution, the newly derived asymptotic distributions, and the Monte Carlo empirical distribution when $X_0 = 0, 3.5, 10$, respectively.

When $X_0 = 0$, since $\mu = 0$, the newly derived asymptotic distribution becomes the Cauchy asymptotic distribution. Table 2 only report the percentiles of the Cauchy asymptotic distribution and the finite sample distributions. It is clear that the Monte Carlo empirical distributions are close to the Cauchy asymptotic distribution.

When $X_0 \neq 0$, the newly derived asymptotic distribution is different from the Cauchy asymptotic distribution. From Table 3, when $X_0 = 3.5$, it is clear that the Monte Carlo empirical distributions are sensitive to the change of the initial condition and very far away from the Cauchy asymptotic distribution. For example, the 1 percentile of the Cauchy asymptotic distribution is -31.8205 while the 1 percentiles of the finite sample distribution move around -0.4. In sharp contrast, the 1 percentile of the newly derived asymptotic distribution is -0.4388, suggesting the newly derived asymptotic distribution yields good approximations to the finite sample distributions. From Table 4, when $X_0 = 10$, the finite sample distributions are even further away from the Cauchy asymptotic distribution. Whereas, the newly derived asymptotic distribution yields good approximations to the Monte Carlo empirical distributions.

Next, we investigate the sensitivity of the Monte Carlo empirical distribution of $\widehat{\theta}$ and $\widehat{\beta}$ with respect to the value of μ. For this purpose, we set $\theta = 1$, $\sigma = 0.2$,

TABLE 2

This table reports six percentiles of the Cauchy distribution and the Monte Carlo empirical distribution of $\frac{e^{\theta T}}{2\theta}(\widehat{\theta}_\Delta - \theta)$ *and* $\frac{e^{\theta T}}{2\theta\Delta}(\widehat{\beta}_\Delta - \beta)$ *when* $\mu = 0$ *and* $X_0 = 0$.

		Percentiles	1%	2.5%	10%	90%	97.5%	99%
		Cauchy Asym.	−31.8205	−12.7062	−3.0777	3.0777	12.7062	31.8205
$H = 0.15$	$\Delta = 1/252$	MCED$^\theta$	−39.9078	−13.9255	−3.0534	3.0426	11.6275	28.0315
		MCED$^\beta$	−39.9073	−13.9253	−3.0533	3.0426	11.6273	28.0312
	$\Delta = 1/52$	MCED$^\theta$	−30.3777	−12.4841	−2.9167	3.1683	12.8068	36.0481
		MCED$^\beta$	−30.3702	−12.4810	−2.9160	3.1675	12.8037	36.0392
	$\Delta = 1/12$	MCED$^\theta$	−34.8777	−15.4491	−3.6273	3.6117	15.5121	39.6442
		MCED$^\beta$	−34.7167	−15.3778	−3.6106	3.5951	15.4405	39.4613
$H = 0.35$	$\Delta = 1/252$	MCED$^\theta$	−24.0441	−12.2667	−3.0488	3.0094	12.8997	32.8611
		MCED$^\beta$	−24.0438	−12.2666	−3.0487	3.0093	12.8996	32.8607
	$\Delta = 1/52$	MCED$^\theta$	−28.2827	−11.7502	−2.9669	3.2007	12.2671	29.8610
		MCED$^\beta$	−28.2757	−11.7474	−2.9662	3.1999	12.2641	29.8536
	$\Delta = 1/12$	MCED$^\theta$	−35.1576	−14.5487	−3.5612	3.6065	14.4238	37.9829
		MCED$^\beta$	−34.9954	−14.4816	−3.5448	3.5898	14.3572	37.8076
$H = 0.55$	$\Delta = 1/252$	MCED$^\theta$	−31.6969	−13.8355	−3.0380	2.9457	12.8903	31.5700
		MCED$^\beta$	−31.6966	−13.8353	−3.0379	2.9456	12.8902	31.5696
	$\Delta = 1/52$	MCED$^\theta$	−30.6575	−13.2005	−3.1697	3.2053	12.6823	31.7275
		MCED$^\beta$	−30.6500	−13.1973	−3.1689	3.2045	12.6792	31.7197
	$\Delta = 1/12$	MCED$^\theta$	−33.2604	−13.7933	−3.4648	3.5866	17.0857	40.2489
		MCED$^\beta$	−33.1069	−13.7297	−3.4488	3.5700	17.0069	40.0631
$H = 0.75$	$\Delta = 1/252$	MCED$^\theta$	−29.8430	−12.2791	−2.8160	3.1137	13.7555	35.4082
		MCED$^\beta$	−29.8427	−12.2790	−2.8160	3.1137	13.7553	35.4078
	$\Delta = 1/52$	MCED$^\theta$	−31.0889	−13.5346	−3.1458	3.3624	13.4632	33.0715
		MCED$^\beta$	−31.0812	−13.5313	−3.1450	3.3616	13.4599	33.0634
	$\Delta = 1/12$	MCED$^\theta$	−35.3672	−13.8410	−3.3564	3.7404	14.9353	40.5611
		MCED$^\beta$	−35.2040	−13.7772	−3.3409	3.7231	14.8664	40.3739

TABLE 3

The Cauchy distribution, the new asymptotic distribution and the Monte Carlo empirical distribution of $\frac{e^{\theta T}}{2\theta}(\widehat{\theta}_\Delta - \theta)$ *and* $\frac{e^{\theta T}}{2\theta\Delta}(\widehat{\beta}_\Delta - \beta)$ *when* $\mu = 0$ *and* $X_0 = 3.5$.

		Percentiles	1%	2.5%	10%	90%	97.5%	99%
		Cauchy Asym.	−31.8205	−12.7062	−3.0777	3.0777	12.7062	31.8205
$H = 0.15$		New Asym.	−0.4388	−0.3595	−0.2269	0.2271	0.3593	0.4374
	$\Delta = 1/252$	MCED$^\theta$	−0.4129	−0.3442	−0.2178	0.2089	0.3237	0.3918
		MCED$^\beta$	−0.4129	−0.3442	−0.2178	0.2089	0.3237	0.3918
	$\Delta = 1/52$	MCED$^\theta$	−0.4013	−0.3335	−0.2138	0.2088	0.3272	0.3977
		MCED$^\beta$	−0.4012	−0.3335	−0.2138	0.2088	0.3271	0.3976
	$\Delta = 1/12$	MCED$^\theta$	−0.5114	−0.4090	−0.2602	0.2608	0.4093	0.5026
		MCED$^\beta$	−0.5090	−0.4071	−0.2590	0.2596	0.4074	0.5003
$H = 0.35$		New Asym.	−0.3761	−0.3106	−0.1977	0.1976	0.3100	0.3750
	$\Delta = 1/252$	MCED$^\theta$	−0.3240	−0.2667	−0.1697	0.1645	0.2538	0.3060
		MCED$^\beta$	−0.3240	−0.2667	−0.1697	0.1645	0.2538	0.3060
	$\Delta = 1/52$	MCED$^\theta$	−0.2958	−0.2502	−0.1620	0.1605	0.2454	0.3014
		MCED$^\beta$	−0.2957	−0.2501	−0.1619	0.1604	0.2453	0.3013
	$\Delta = 1/12$	MCED$^\theta$	−0.4283	−0.3529	−0.2254	0.2240	0.3526	0.4463
		MCED$^\beta$	−0.4263	−0.3513	−0.2243	0.2229	0.3510	0.4442
$H = 0.55$		New Asym.	−0.3481	−0.2882	−0.1841	0.1843	0.2881	0.3473
	$\Delta = 1/252$	MCED$^\theta$	−0.2715	−0.2251	−0.1448	0.1370	0.2123	0.2507
		MCED$^\beta$	−0.2715	−0.2251	−0.1448	0.1370	0.2123	0.2507
	$\Delta = 1/52$	MCED$^\theta$	−0.2417	−0.2044	−0.1328	0.1302	0.1993	0.2387
		MCED$^\beta$	−0.2416	−0.2043	−0.1328	0.1301	0.1992	0.2386
	$\Delta = 1/12$	MCED$^\theta$	−0.3958	−0.3259	−0.2091	0.2072	0.3232	0.3983
		MCED$^\beta$	−0.3940	−0.3244	−0.2081	0.2062	0.3217	0.3965
$H = 0.75$		New Asym.	−0.3407	−0.2822	−0.1805	0.1807	0.2822	0.3400
	$\Delta = 1/252$	MCED$^\theta$	−0.2447	−0.2016	−0.1293	0.1204	0.1844	0.2206
		MCED$^\beta$	−0.2447	−0.2016	−0.1293	0.1204	0.1844	0.2206
	$\Delta = 1/52$	MCED$^\theta$	−0.2159	−0.1792	−0.1154	0.1100	0.1649	0.2025
		MCED$^\beta$	−0.2159	−0.1792	−0.1154	0.1100	0.1649	0.2024
	$\Delta = 1/12$	MCED$^\theta$	−0.4022	−0.3268	−0.2068	0.1980	0.3044	0.3595
		MCED$^\beta$	−0.4003	−0.3253	−0.2058	0.1970	0.3030	0.3579

Table 4

The Cauchy distribution, the new asymptotic distribution and the Monte Carlo empirical distribution of $\frac{e^{\theta T}}{2\theta}(\widehat{\theta}_\Delta-\theta)$ *and* $\frac{e^{\theta T}}{2\theta\Delta}(\widehat{\beta}_\Delta-\beta)$ *when* $\mu=0$ *and* $X_0=10$.

		Percentiles	1%	2.5%	10%	90%	97.5%	99%
		Cauchy Asym.	−31.8205	−12.7062	−3.0777	3.0777	12.7062	31.8205
$H=0.15$		New Asym.	−0.1421	−0.1192	−0.0777	0.0777	0.1192	0.1417
	$\Delta=1/252$	MCED^θ	−0.1379	−0.1129	−0.0745	0.0717	0.1088	0.1291
		MCED^β	−0.1378	−0.1129	−0.0745	0.0717	0.1088	0.1291
	$\Delta=1/52$	MCED^θ	−0.1330	−0.1127	−0.0733	0.0717	0.1084	0.1325
		MCED^β	−0.1329	−0.1127	−0.0733	0.0717	0.1084	0.1324
	$\Delta=1/12$	MCED^θ	−0.1642	−0.1383	−0.0896	0.0899	0.1374	0.1621
		MCED^β	−0.1634	−0.1377	−0.0892	0.0895	0.1367	0.1613
$H=0.35$		New Asym.	−0.1243	−0.1043	−0.0681	0.0680	0.1043	0.1240
	$\Delta=1/252$	MCED^θ	−0.1087	−0.0902	−0.0587	0.0565	0.0863	0.1018
		MCED^β	−0.1087	−0.0902	−0.0587	0.0565	0.0863	0.1018
	$\Delta=1/52$	MCED^θ	−0.1002	−0.0853	−0.0560	0.0557	0.0839	0.1010
		MCED^β	−0.1001	−0.0853	−0.0560	0.0557	0.0838	0.1009
	$\Delta=1/12$	MCED^θ	−0.1411	−0.1209	−0.0778	0.0771	0.1192	0.1421
		MCED^β	−0.1405	−0.1203	−0.0774	0.0767	0.1186	0.1415
$H=0.55$		New Asym.	−0.1160	−0.0974	−0.0636	0.0635	0.0974	0.1157
	$\Delta=1/252$	MCED^θ	−0.0907	−0.0768	−0.0498	0.0478	0.0728	0.0863
		MCED^β	−0.0907	−0.0768	−0.0498	0.0478	0.0728	0.0863
	$\Delta=1/52$	MCED^θ	−0.0817	−0.0712	−0.0458	0.0455	0.0690	0.0816
		MCED^β	−0.0817	−0.0712	−0.0458	0.0455	0.0690	0.0816
	$\Delta=1/12$	MCED^θ	−0.1310	−0.1103	−0.0729	0.0712	0.1094	0.1309
		MCED^β	−0.1304	−0.1098	−0.0725	0.0708	0.1089	0.1303
$H=0.75$		New Asym.	−0.1138	−0.0955	−0.0624	0.0623	0.0955	0.1135
	$\Delta=1/252$	MCED^θ	−0.0807	−0.0690	−0.0444	0.0425	0.0648	0.0753
		MCED^β	−0.0807	−0.0690	−0.0443	0.0425	0.0648	0.0753
	$\Delta=1/52$	MCED^θ	−0.0728	−0.0607	−0.0399	0.0386	0.0583	0.0713
		MCED^β	−0.0727	−0.0607	−0.0398	0.0386	0.0583	0.0713
	$\Delta=1/12$	MCED^θ	−0.1303	−0.1089	−0.0702	0.0698	0.1064	0.1265
		MCED^β	−0.1297	−0.1084	−0.0698	0.0695	0.1059	0.1259

$X_0=0$, $\mu=-0.7$. Table 5 reports the percentiles of the Monte Carlo empirical distribution, the Cauchy asymptotic distribution, and the new asymptotic distribution. Compared with Table 2, Table 5 suggests that the Monte Carlo empirical distributions are sensitive to the change of μ and far away to the Cauchy asymptotic distribution. Whereas, the newly derived asymptotic distribution yields good approximations to the Monte Carlo empirical distributions.

Thirdly, we conduct Monte Carlo simulations to evaluate the finite sample performance of the derived asymptotic distributions of $\widehat{H}_\Delta$, $\widehat{\mu}_\Delta$ and $\widehat{\sigma}_\Delta$. In particular, we obtain the Monte Carlo empirical distributions of the following statistics:

$$\Phi_{\widehat{H}_\Delta}=\sqrt{n}(\widehat{H}_\Delta-H),\ \Phi_{\widehat{\sigma}_\Delta}=\frac{\sqrt{n}}{\log(\Delta)}(\widehat{\sigma}_\Delta-\sigma),\ \Phi_{\widehat{\mu}_\Delta}=T^{1-H}(\widehat{\mu}_\Delta-\mu). \quad (4.1)$$

To simulate data, we set $\theta=0.2$, $\sigma=0.2$ and $\mu=-1$ and allow H to take different values in the range of $(0,1)$. For convenience, we choose the sampling interval $\Delta=1/252$ and the time span $T=10$ with 10,000 simulated sample paths from model (2.1). We then report the mean, variance, skewness and kurtosis of the Monte Carlo empirical distributions of $\Phi_{\widehat{H}_\Delta}$, $\Phi_{\widehat{\sigma}_\Delta}$ and $\Phi_{\widehat{\mu}_\Delta}$ and those of the asymptotic standard normal distributions (i.e., $\mathcal{N}(0,1)$) in Table 7. As we can see from Table 7, the derived asymptotic distributions well approximate the Monte Carlo empirical distributions for all three parameters.

TABLE 5
The Cauchy asymptotic distribution, the new asymptotic distribution and the Monte Carlo empirical distribution of $\frac{e^{\theta T}}{2\theta}(\widehat{\theta}_\Delta - \theta)$ *and* $\frac{e^{\theta T}}{2\theta\Delta}(\widehat{\beta}_\Delta - \beta)$ *when* $\mu = -0.7$ *and* $X_0 = 0$.

		Percentiles	1%	2.5%	10%	90%	97.5%	99%
		Cauchy Asym.	−31.8205	−12.7062	−3.0777	3.0777	12.7062	31.8205
$H = 0.15$		New Asym.	−0.4981	−0.4052	−0.2529	0.2540	0.4065	0.4980
	$\Delta = 1/252$	MCED^θ	−0.4972	−0.4154	−0.2692	0.2730	0.4379	0.5303
		MCED^β	−0.4972	−0.4154	−0.2692	0.2730	0.4379	0.5303
	$\Delta = 1/52$	MCED^θ	−0.4679	−0.3919	−0.2484	0.2575	0.4309	0.5121
		MCED^β	−0.4679	−0.3918	−0.2484	0.2575	0.4309	0.5120
	$\Delta = 1/12$	MCED^θ	−0.5212	−0.4326	−0.2814	0.2945	0.4784	0.5960
		MCED^β	−0.5206	−0.4321	−0.2811	0.2942	0.4778	0.5954
$H = 0.35$		New Asym.	−0.5021	−0.4082	−0.2546	0.2557	0.4094	0.5020
	$\Delta = 1/252$	MCED^θ	−0.4407	−0.3700	−0.2385	0.2486	0.4001	0.4907
		MCED^β	−0.4407	−0.3700	−0.2385	0.2486	0.4001	0.4907
	$\Delta = 1/52$	MCED^θ	−0.4011	−0.3414	−0.2206	0.2322	0.3753	0.4605
		MCED^β	−0.4011	−0.3413	−0.2206	0.2321	0.3752	0.4605
	$\Delta = 1/12$	MCED^θ	−0.5097	−0.4187	−0.2731	0.3011	0.4910	0.6077
		MCED^β	−0.5091	−0.4183	−0.2728	0.3007	0.4904	0.6070
$H = 0.55$		New Asym.	−0.5495	−0.4438	−0.2746	0.2757	0.4451	0.5495
	$\Delta = 1/252$	MCED^θ	−0.4051	−0.3355	−0.2192	0.2414	0.3822	0.4775
		MCED^β	−0.4051	−0.3355	−0.2192	0.2414	0.3822	0.4775
	$\Delta = 1/52$	MCED^θ	−0.3635	−0.3015	−0.2001	0.2102	0.3467	0.4197
		MCED^β	−0.3634	−0.3015	−0.2001	0.2102	0.3467	0.4197
	$\Delta = 1/12$	MCED^θ	−0.5142	−0.4193	−0.2746	0.3147	0.5312	0.6669
		MCED^β	−0.5136	−0.4188	−0.2743	0.3144	0.5306	0.6661
$H = 0.75$		New Asym.	−0.6460	−0.5142	−0.3126	0.3139	0.5155	0.6462
	$\Delta = 1/252$	MCED^θ	−0.3563	−0.2932	−0.1950	0.2209	0.3552	0.4257
		MCED^β	−0.3563	−0.2932	−0.1950	0.2209	0.3552	0.4257
	$\Delta = 1/52$	MCED^θ	−0.3037	−0.2620	−0.1710	0.1842	0.3025	0.3641
		MCED^β	−0.3037	−0.2620	−0.1710	0.1842	0.3025	0.3640
	$\Delta = 1/12$	MCED^θ	−0.4819	−0.4023	−0.2627	0.3191	0.5614	0.7115
		MCED^β	−0.4814	−0.4018	−0.2624	0.3187	0.5608	0.7106

For testing the influence of the sampling interval, we choose the sampling interval $\Delta = 1/52$ and $\Delta = 1/12$. Moreover we set $H \in \{0.1, 0.3, 0.6, 0.8\}$ and other parameters are the same as those in Table 7. Table 6 provides mean, variance, skewness and kurtosis of the Monte Carlo empirical distributions of $\Phi_{\widehat{H}_\Delta}$, $\Phi_{\widehat{\sigma}_\Delta}$ and $\Phi_{\widehat{\mu}_\Delta}$ and those of the asymptotic standard normal distributions. From Table 6, we can see that both $\Phi_{\widehat{H}_\Delta}$ and $\Phi_{\widehat{\sigma}_\Delta}$ is a little different from the standard normal distribution. Hence, we can obtain that the smaller of Δ, more accurate of $\widehat{H}_\Delta$, $\widehat{\sigma}_\Delta$ and $\widehat{\mu}_\Delta$.

Finally, we investigate the effect of the time varying diffusion parameter on estimates of $\widehat{\mu}_\Delta$ proposed by (3.6) and $\widehat{H}_\Delta$ proposed by (3.1). To simulate data, we set $\theta = 0.2$, $\mu = -1$ and $H \in \{0.1, 0.3, 0.6, 0.8\}$. The time varying diffusion parameters used are $\sigma_t = \sqrt{t}$ and $\sigma_t = \sin(t)$. For convenience, we choose the sampling interval $\Delta = 1/252$ and the time span $T = 10$ with 10,000 simulated sample paths from model (2.1). Similarly, we provide the mean, variance, skewness and kurtosis of the Monte Carlo empirical distributions of $\Phi_{\widehat{H}_\Delta}$ and $\Phi_{\widehat{\mu}_\Delta}$ and those of the asymptotic standard normal distributions (i.e., $\mathcal{N}(0,1)$) in Table 8. As we can see from Table 8, the derived asymptotic distributions well approximate the Monte Carlo empirical distributions for time varying diffusion parameters.

5. Empirical studies

To illustrate the usefulness of the proposed model and the derived limit distribution in practice, we consider an empirical study. Our study is motivated from

TABLE 6
Mean, variance, skewness and kurtosis of $\Phi_{\widehat{H}_\Delta}$, $\Phi_{\widehat{\sigma}_\Delta}$, $\Phi_{\widehat{\mu}_\Delta}$ in (4.1) and the standard normal limiting distribution with $\Delta = 1/252$.

Value of H	Statistics	Mean	Variance	Skewness	Kurtosis
	$\mathcal{N}(0,1)$	0	1	0	3
$H = 0.1$	$\Phi_{\widehat{H}_\Delta}$	−0.017311	0.923305	−0.013949	3.024885
	$\Phi_{\widehat{\sigma}_\Delta}$	0.022569	0.888150	−0.091405	3.176769
	$\Phi_{\widehat{\mu}_\Delta}$	0.0315831	1.062469	−0.163131	3.570116
$H = 0.2$	$\Phi_{\widehat{H}_\Delta}$	−0.019234	0.953352	−0.007881	2.991773
	$\Phi_{\widehat{\sigma}_\Delta}$	0.064560	0.927148	−0.087279	3.172227
	$\Phi_{\widehat{\mu}_\Delta}$	0.044777	0.938544	0.108009	3.096121
$H = 0.3$	$\Phi_{\widehat{H}_\Delta}$	−0.020780	0.925791	−0.006025	2.978431
	$\Phi_{\widehat{\sigma}_\Delta}$	0.029536	0.933122	−0.086998	3.168624
	$\Phi_{\widehat{\mu}_\Delta}$	0.033007	0.952455	−0.096756	3.131023
$H = 0.4$	$\Phi_{\widehat{H}_\Delta}$	−0.022140	0.929721	−0.007054	2.979255
	$\Phi_{\widehat{\sigma}_\Delta}$	0.030560	1.085842	−0.089672	3.268379
	$\Phi_{\widehat{\mu}_\Delta}$	−0.015380	0.915075	0.058212	2.962478
$H = 0.5$	$\Phi_{\widehat{H}_\Delta}$	−0.011831	0.934033	−0.029452	3.108228
	$\Phi_{\widehat{\sigma}_\Delta}$	−0.065678	1.022217	−0.035021	3.207613
	$\Phi_{\widehat{\mu}_\Delta}$	−0.140621	0.922738	−0.084704	3.231211
$H = 0.6$	$\Phi_{\widehat{H}_\Delta}$	−0.022225	0.942600	−0.014839	2.991179
	$\Phi_{\widehat{\sigma}_\Delta}$	0.075118	0.911895	−0.053187	3.217130
	$\Phi_{\widehat{\mu}_\Delta}$	−0.029949	1.036335	−0.083410	3.278550
$H = 0.7$	$\Phi_{\widehat{H}_\Delta}$	−0.013036	0.952611	−0.021022	2.991612
	$\Phi_{\widehat{\sigma}_\Delta}$	0.081108	0.942304	−0.051543	3.299545
	$\Phi_{\widehat{\mu}_\Delta}$	−0.018987	1.039631	−0.070227	2.896728
$H = 0.8$	$\Phi_{\widehat{H}_\Delta}$	0.039712	0.967824	−0.027516	2.986774
	$\Phi_{\widehat{\sigma}_\Delta}$	0.061376	0.945947	−0.167950	3.281394
	$\Phi_{\widehat{\mu}_\Delta}$	−0.016915	1.056022	0.040067	2.856668

Phillips et al. [39] where explosiveness is found in the monthly Nasdaq between January 1990 to June 2000 when a pure AR(1) model is fitted. In our study, Model (2.1) is fitted to the monthly price-dividend ratio of Nasdaq between January 1990 to June 2000 with $\Delta = 1/12$, $T = 10.5$, $n = 126$, and $X_0 = 1.7753$ (which is the price-dividend ratio of Nasdaq in December 1989).[7]

We estimate H, σ, θ, μ using (3.1), (3.2), (3.5) and (3.6), respectively. The point estimates and their corresponding 90% confidence intervals based on the derived asymptotic distributions are reported in Table 9. Since the estimated θ is greater than zero, model (2.1) is relevant and the asymptotic theory developed in this paper is applicable. From Table 9, we can see that the 90% confidence interval of θ excludes zero, which implies explosiveness. Moreover, the point estimate of H is much smaller than 0.5, implying anti-persistence in the error term. The evidence of anti-persistence is statistically significant.

[7]The data are obtained from https://www.nasdaq.com/market-activity/index/comp

TABLE 7
Mean, variance, skewness and kurtosis of $\Phi_{\widehat{H}_\Delta}$, $\Phi_{\widehat{\sigma}_\Delta}$, $\Phi_{\widehat{\mu}_\Delta}$ *in (4.1) and the standard normal limiting distribution with* $\Delta = 1/52$ *and* $\Delta = 1/12$.

	Value of H	Statistics	Mean	Variance	Skewness	Kurtosis
		$\mathcal{N}(0,1)$	0	1	0	3
$\Delta = 1/52$	$H = 0.1$	$\Phi_{\widehat{H}_\Delta}$	−0.027001	0.933378	−0.033449	3.073117
		$\Phi_{\widehat{\sigma}_\Delta}$	0.065119	0.903888	−0.066673	3.703790
		$\Phi_{\widehat{\mu}_\Delta}$	−0.062120	0.932149	0.028473	3.004746
	$H = 0.3$	$\Phi_{\widehat{H}_\Delta}$	0.073222	0.932350	−0.030244	3.024067
		$\Phi_{\widehat{\sigma}_\Delta}$	0.089004	0.879408	−0.182250	4.322952
		$\Phi_{\widehat{\mu}_\Delta}$	−0.005251	0.892314	0.023862	2.981330
	$H = 0.6$	$\Phi_{\widehat{H}_\Delta}$	0.060516	1.187440	0.041642	2.998527
		$\Phi_{\widehat{\sigma}_\Delta}$	−0.066142	1.150020	−0.258197	4.639454
		$\Phi_{\widehat{\mu}_\Delta}$	0.000291	0.939629	0.020138	2.999654
	$H = 0.8$	$\Phi_{\widehat{H}_\Delta}$	0.043335	1.271114	0.043377	2.850611
		$\Phi_{\widehat{\sigma}_\Delta}$	−0.089881	1.105603	0.162334	2.936305
		$\Phi_{\widehat{\mu}_\Delta}$	0.000233	0.970088	0.015274	3.013257
$\Delta = 1/12$	$H = 0.1$	$\Phi_{\widehat{H}_\Delta}$	−0.012826	0.928770	−0.112388	2.965402
		$\Phi_{\widehat{\sigma}_\Delta}$	−0.050260	0.940998	−0.231826	4.880821
		$\Phi_{\widehat{\mu}_\Delta}$	−0.016434	0.981458	0.030186	3.018325
	$H = 0.3$	$\Phi_{\widehat{H}_\Delta}$	0.079236	0.922651	−0.092458	2.925360
		$\Phi_{\widehat{\sigma}_\Delta}$	−0.095521	0.940193	−0.121790	5.061145
		$\Phi_{\widehat{\mu}_\Delta}$	−0.031046	0.960453	0.030111	2.940064
	$H = 0.6$	$\Phi_{\widehat{H}_\Delta}$	0.164374	0.990257	−0.082454	3.001363
		$\Phi_{\widehat{\sigma}_\Delta}$	−0.068967	0.977749	−0.543642	5.026407
		$\Phi_{\widehat{\mu}_\Delta}$	−0.002422	0.968370	0.056569	3.075413
	$H = 0.8$	$\Phi_{\widehat{H}_\Delta}$	0.026784	1.106585	0.080360	2.598159
		$\Phi_{\widehat{\sigma}_\Delta}$	0.026791	1.290227	0.872756	7.735156
		$\Phi_{\widehat{\mu}_\Delta}$	0.007610	0.963467	0.232312	3.951518

TABLE 8
Mean, variance, skewness and kurtosis of $\Phi_{\widehat{H}_\Delta}$, $\Phi_{\widehat{\mu}_\Delta}$ *and the standard normal limiting distribution with* $\sigma_t = \sqrt{t}$ *and* $\sigma_t = \sin(t)$.

	Value of H	Statistics	Mean	Variance	Skewness	Kurtosis
		$\mathcal{N}(0,1)$	0	1	0	3
$\sigma_t = \sqrt{t}$	$H = 0.1$	$\Phi_{\widehat{H}_\Delta}$	−0.011914	1.067501	−0.035781	2.953839
		$\Phi_{\widehat{\mu}_\Delta}$	−0.081175	1.098299	0.057418	3.036918
	$H = 0.3$	$\Phi_{\widehat{H}_\Delta}$	−0.014314	1.065272	−0.031240	2.969305
		$\Phi_{\widehat{\mu}_\Delta}$	0.093090	1.084782	−0.033541	3.177708
	$H = 0.6$	$\Phi_{\widehat{H}_\Delta}$	−0.018544	1.081048	−0.029286	3.005216
		$\Phi_{\widehat{\mu}_\Delta}$	0.055843	1.077630	0.063124	2.947798
	$H = 0.8$	$\Phi_{\widehat{H}_\Delta}$	−0.003935	1.108419	−0.034659	2.991984
		$\Phi_{\widehat{\mu}_\Delta}$	−0.001668	1.082284	−0.105158	3.037013
$\sigma_t = \sin(t)$	$H = 0.1$	$\Phi_{\widehat{H}_\Delta}$	0.008673	1.114837	0.031816	3.035688
		$\Phi_{\widehat{\mu}_\Delta}$	0.083700	0.976268	−0.083029	3.196050
	$H = 0.3$	$\Phi_{\widehat{H}_\Delta}$	0.005612	1.109477	−0.014354	3.072120
		$\Phi_{\widehat{\mu}_\Delta}$	0.069933	1.121313	−0.027690	3.066687
	$H = 0.6$	$\Phi_{\widehat{H}_\Delta}$	0.003653	1.106876	−0.010307	3.066465
		$\Phi_{\widehat{\mu}_\Delta}$	0.029138	1.037571	−0.059882	3.174537
	$H = 0.8$	$\Phi_{\widehat{H}_\Delta}$	0.025356	1.056075	−0.017431	3.042485
		$\Phi_{\widehat{\mu}_\Delta}$	0.002881	0.980997	0.111312	3.124255

Table 9
Empirical results for the monthly price-dividend ratio of Nasdaq.

$\widehat{H}$	$\widehat{\sigma}$	$\widehat{\mu}$	$\widehat{\theta}$
0.1206	0.9879	0.8759	0.0521
(−0.1429, 0.3842)	(0.8831, 1.0927)	(0.6293, 1.1225)	(0.0460, 0.0582)

6. Conclusions

In recent years, the fOUp has been used to model the realized volatility in financial time series. Moreover, the discrete-time representation of fOUp has been used to model equity price (Lui et al. [31, 32]). In this paper, we introduce estimators for all four parameters in fOUp. The estimators of two diffusion parameters are the same as those in Wang et al. [47]. The estimators of two drift parameters are based on the least squares method. The asymptotic theory for the diffusion estimators are established under the in-fill asymptotic scheme. The asymptotic theory for the drift estimators are established under the double asymptotic scheme for explosive fOUp with a full range of the Hurst parameter.

Our double asymptotic theory contributes to the literature in two aspects. First, our theory permits explicit consideration of the effects from the initial condition. Monte Carlo evidence suggests that the new asymptotic theory provides a better approximation to the Monte Carlo empirical distribution than the limit theory that is independent of the initial condition. Second, our theory works for the full range value of $H \in (0,1)$. Our asymptotic distribution for θ is the same whether $H < 1/2$ or $H > 1/2$.

Our simulation studies show that the Monte Carlo empirical distribution of θ is indeed very sensitive to the change of the initial condition and that our asymptotic distribution can well approximate the Monte Carlo empirical distribution not only for θ but also for other parameters in the model.

Appendix A: Proofs of Theorems 3.2–3.4

In order to give the proofs to Theorems 3.2–3.4, we first establish some crucial useful lemmas and propositions.

A.1. Technical lemmas

Let $\alpha_H = H(2H-1)$. The following four lemmas relate to the explicit fractional calculus, which will play an important role in our analysis. They may have independent interests. The proofs of Lemma A.3 and A.4 are postponed to the supplementary material file [28] for the readability of the paper.

Lemma A.1. *Assume $H \in (1/2, 1)$. Let $\Delta \to 0$ and $T \to \infty$. Then we have*

$$\alpha_H \sum_{i=1}^{n} \sum_{j=1}^{n} e^{-2\theta T} e^{2\theta i\Delta} e^{2\theta j\Delta} \int_{(i-1)\Delta}^{i\Delta} \int_{(j-1)\Delta}^{j\Delta} e^{-\theta t} e^{-\theta s} |t-s|^{2H-2} dtds \to \frac{H\Gamma(2H)}{\theta^{2H}}. \tag{A.1}$$

Proof. A standard calculation shows

$$
\begin{aligned}
&\alpha_H \sum_{i,j=1}^{n} \sum_{j \neq i} e^{-2\theta T} e^{2\theta i\Delta} e^{2\theta j\Delta} \int_{(i-1)\Delta}^{i\Delta} \int_{(j-1)\Delta}^{j\Delta} e^{-\theta t} e^{-\theta s} |t-s|^{2H-2} dtds \\
&= 2\alpha_H \sum_{i=1}^{n} \sum_{j=i+1}^{n} e^{-2\theta T} e^{\theta(i+j+2)\Delta} \int_0^{\Delta} \int_0^{\Delta} e^{-\theta t} e^{-\theta s} |t-s+j\Delta - i\Delta|^{2H-2} dtds \\
&\sim 2\alpha_H \sum_{i=1}^{n} \sum_{j=i+1}^{n} e^{-2\theta T} e^{\theta i\Delta} e^{\theta j\Delta} e^{2\theta\Delta} \int_0^{\Delta} \int_0^{\Delta} |t-s+j\Delta - i\Delta|^{2H-2} dsdt \\
&= \sum_{i=1}^{n} \sum_{j=i+2}^{n+1} e^{-2\theta T} e^{\theta i\Delta} e^{\theta j\Delta} e^{\theta\Delta} (j\Delta - i\Delta)^{2H} \\
&\quad - 2 \sum_{i=1}^{n} \sum_{j=i+1}^{n} e^{-2\theta T} e^{\theta i\Delta} e^{\theta j\Delta} e^{2\theta\Delta} (j\Delta - i\Delta)^{2H} \\
&\quad + \sum_{i=1}^{n} \sum_{j=i+1}^{n-1} e^{-2\theta T} e^{\theta i\Delta} e^{\theta j\Delta} e^{3\theta\Delta} (j\Delta - i\Delta)^{2H} \\
&= \sum_{i=1}^{n} \sum_{j=i+1}^{n} e^{-2\theta T} e^{\theta i\Delta} e^{\theta j\Delta} e^{\theta\Delta} \left(1 + e^{2\theta\Delta} - 2e^{\theta\Delta}\right) (j\Delta - i\Delta)^{2H} \\
&\quad - \sum_{i=1}^{n} e^{-2\theta T} e^{2\theta i\Delta} e^{2\theta\Delta} \Delta^{2H} + \sum_{i=1}^{n} e^{-\theta T} e^{\theta i\Delta} e^{2\theta\Delta} (n\Delta + \Delta - i\Delta)^{2H} \\
&\quad - \sum_{i=1}^{n} e^{-\theta T} e^{\theta i\Delta} e^{3\theta\Delta} (n\Delta - i\Delta)^{2H} \\
&= \sum_{i=1}^{n} \sum_{j=i+1}^{n} e^{-2\theta T} e^{\theta i\Delta} e^{\theta j\Delta} e^{\theta\Delta} \left(1 + e^{2\theta\Delta} - 2e^{\theta\Delta}\right) (j\Delta - i\Delta)^{2H} \\
&\quad - \frac{e^{4\theta\Delta}(1 - e^{-2\theta T})}{e^{2\theta\Delta} - 1} \Delta^{2H} + \sum_{i=1}^{n} e^{-\theta i\Delta} e^{3\theta\Delta} (i\Delta)^{2H} - \sum_{i=1}^{n-1} e^{-\theta i\Delta} e^{3\theta\Delta} (i\Delta)^{2H} \\
&= \sum_{i=1}^{n} \sum_{j=i+1}^{n} e^{-2\theta T} e^{\theta i\Delta} e^{\theta j\Delta} e^{\theta\Delta} \left(1 + e^{2\theta\Delta} - 2e^{\theta\Delta}\right) (j\Delta - i\Delta)^{2H} \\
&\quad - \frac{e^{4\theta\Delta}(1 - e^{-2\theta T})}{e^{2\theta\Delta} - 1} \Delta^{2H} + e^{-\theta T} e^{3\theta\Delta} T^{2H} \\
&\sim \theta^2 \sum_{i=1}^{n} \sum_{j=i+1}^{n} e^{-2\theta T} e^{\theta i\Delta} e^{\theta j\Delta} (j\Delta - i\Delta)^{2H} \Delta^2 - \frac{1 - e^{-2\theta T}}{2\theta} \Delta^{2H-1} + e^{-\theta T} T^{2H} \\
&\sim \theta^2 \int_0^T \int_s^T e^{-2\theta T} e^{\theta t} e^{\theta s} (t-s)^{2H} dtds + o(1),
\end{aligned}
$$

where for the first '$\sim$', $e^{-\theta t} e^{-\theta s}$ is approximated as 1, which is because $e^{-2\theta\Delta} \leq e^{-\theta t} e^{-\theta s} \leq 1$ for $t, s \in [0, \Delta]$.

Furthermore, we can obtain

$$
\begin{aligned}
&\theta^2 \int_0^T \int_s^T e^{-2\theta T} e^{\theta t} e^{\theta s} (t-s)^{2H} dtds \\
&= \theta^2 \int_0^T \int_0^{T-s} e^{-2\theta T} e^{\theta t} e^{2\theta s} t^{2H} dtds \\
&= \theta^2 \int_0^T \int_0^{T-t} e^{-2\theta T} e^{\theta t} e^{2\theta s} t^{2H} dsdt \\
&= \frac{\theta}{2} \int_0^T e^{-2\theta T} \big(e^{2\theta T} e^{-2\theta t} - 1\big) e^{\theta t} t^{2H} dt \\
&= \frac{\theta}{2} \int_0^T e^{-\theta t} t^{2H} dt - \frac{\theta}{2} e^{-2\theta T} \int_0^T e^{\theta t} t^{2H} dt \\
&\sim \frac{\theta}{2} \int_0^T e^{-\theta t} t^{2H} dt - \frac{1}{2} e^{-\theta T} T^{2H} \to \frac{H\Gamma(2H)}{\theta^{2H}}, \text{ as } T \to \infty \text{ and } \Delta \to 0.
\end{aligned}
$$

When $j = i$, it is easy to see that

$$
\begin{aligned}
&\alpha_H \sum_{i=1}^n e^{-2\theta T} e^{4\theta i\Delta} \int_{(i-1)\Delta}^{i\Delta} \int_{(i-1)\Delta}^{i\Delta} e^{-\theta t} e^{-\theta s} |t-s|^{2H-2} dtds \\
&= \alpha_H \sum_{i=1}^n e^{-2\theta T} e^{2\theta i\Delta} e^{2\theta \Delta} \int_0^\Delta \int_0^\Delta e^{-\theta t} e^{-\theta s} |t-s|^{2H-2} dtds \\
&\sim \sum_{i=1}^n e^{-2\theta T} e^{2\theta i\Delta} e^{2\theta\Delta} \Delta^{2H} = O\big(\Delta^{2H-1}\big).
\end{aligned}
$$

Hence, the proof of this lemma is completed. □

Lemma A.2. *Assume $H \in (0, 1/2)$. Let $\Delta \to 0$ and $T \to \infty$. Then we have*

$$
\begin{aligned}
&\sum_{i=1}^n \sum_{j=1}^n e^{-2\theta T} e^{2\theta(i+j)\Delta} \Bigg(\theta \int_{(j-1)\Delta}^{j\Delta} \int_{(i-1)\Delta}^{i\Delta} e^{-\theta t} e^{-\theta s} \frac{\partial R(t,s)}{\partial t} dtds \\
&\qquad + \int_{(i-1)\Delta}^{i\Delta} e^{-\theta t} \left(e^{-\theta j\Delta} \frac{\partial R(t, j\Delta)}{\partial t} - e^{-\theta(j-1)\Delta} \frac{\partial R(t,(j-1)\Delta)}{\partial t} \right) dt \Bigg) \\
&\to \frac{H\Gamma(2H)}{\theta^{2H}},
\end{aligned}
\tag{A.2}
$$

where $R(t, s)$ is defined by (2.2).

Proof. Firstly, we can see that

$$
\begin{aligned}
&\theta \int_{(j-1)\Delta}^{j\Delta} \int_{(i-1)\Delta}^{i\Delta} e^{-\theta t} e^{-\theta s} \frac{\partial R(t,s)}{\partial t} dtds \\
&+ \int_{(i-1)\Delta}^{i\Delta} e^{-\theta t} \left(e^{-\theta j\Delta} \frac{\partial R(t, j\Delta)}{\partial t} - e^{-\theta(i-1)\Delta} \frac{\partial R(t,(j-1)\Delta)}{\partial t} \right) dt
\end{aligned}
$$

$$
\begin{aligned}
&= \theta \int_0^{\Delta} \int_0^{\Delta} e^{-\theta(t+(i-1)\Delta)} e^{-\theta(s+(j-1)\Delta)} \frac{\partial R(t+(i-1)\Delta, s+(j-1)\Delta)}{\partial t} dtds \\
&\quad + \int_0^{\Delta} e^{-\theta(t+(i-1)\Delta)} e^{-\theta j\Delta} \frac{\partial R(t+(i-1)\Delta, j\Delta)}{\partial t} dt \\
&\quad - \int_0^{\Delta} e^{-\theta(t+(i-1)\Delta)} e^{-\theta(j-1)\Delta} \frac{\partial R(t+(i-1)\Delta, (j-1)\Delta)}{\partial t} dt \\
&= \theta e^{-\theta(i+j-2)\Delta} \int_0^{\Delta} \int_0^{\Delta} e^{-\theta(t+s)} \frac{\partial \frac{1}{2}(|t+(i-1)\Delta|^{2H} - |t-s+i\Delta-j\Delta|^{2H})}{\partial t} dtds \\
&\quad + e^{-\theta i\Delta} e^{-\theta j\Delta} e^{\theta\Delta} \int_0^{\Delta} e^{-\theta t} \frac{\partial \frac{1}{2}(|t+(i-1)\Delta|^{2H} - |t+i\Delta-j\Delta-\Delta|^{2H})}{\partial t} dt \\
&\quad - e^{-\theta i\Delta} e^{-\theta j\Delta} e^{2\theta\Delta} \int_0^{\Delta} e^{-\theta t} \frac{\partial \frac{1}{2}(|t+(i-1)\Delta|^{2H} - |t+i\Delta-j\Delta|^{2H})}{\partial t} dt \\
&= e^{-\theta i\Delta} e^{-\theta j\Delta} e^{2\theta\Delta} \int_0^{\Delta} e^{-\theta t} \frac{\partial \frac{1}{2}|t+i\Delta-j\Delta|^{2H}}{\partial t} dt \\
&\quad - e^{-\theta i\Delta} e^{-\theta j\Delta} e^{\theta\Delta} \int_0^{\Delta} e^{-\theta t} \frac{\partial \frac{1}{2}|t+i\Delta-j\Delta-\Delta|^{2H}}{\partial t} dt \\
&\quad - \theta e^{-\theta i\Delta} e^{-\theta j\Delta} e^{2\theta\Delta} \int_0^{\Delta} \int_0^{\Delta} e^{-\theta t} e^{-\theta s} \frac{\partial \frac{1}{2}|t-s+i\Delta-j\Delta|^{2H}}{\partial t} dtds. \qquad \text{(A.3)}
\end{aligned}
$$

Using (A.3), we can obtain

$$
\begin{aligned}
&\sum_{i=1}^{n} \sum_{j=1}^{n} e^{-2\theta T} e^{2\theta(i+j)\Delta} \Bigg(\theta \int_{(j-1)\Delta}^{j\Delta} \int_{(i-1)\Delta}^{i\Delta} e^{-\theta t} e^{-\theta s} \frac{\partial R(t,s)}{\partial t} dtds \\
&+ \int_{(i-1)\Delta}^{i\Delta} e^{-\theta t} \left(e^{-\theta j\Delta} \frac{\partial R(t, j\Delta)}{\partial t} - e^{-\theta(j-1)\Delta} \frac{\partial R(t,(j-1)\Delta)}{\partial t} \right) dt \Bigg) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} e^{-2\theta T} e^{\theta i\Delta} e^{\theta j\Delta} e^{2\theta\Delta} \int_0^{\Delta} e^{-\theta t} \frac{\partial \frac{1}{2}|t+i\Delta-j\Delta|^{2H}}{\partial t} dt \\
&\quad - \sum_{i=1}^{n} \sum_{j=1}^{n} e^{-2\theta T} e^{\theta i\Delta} e^{\theta j\Delta} e^{\theta\Delta} \int_0^{\Delta} e^{-\theta t} \frac{\partial \frac{1}{2}|t+i\Delta-j\Delta-\Delta|^{2H}}{\partial t} dt \\
&\quad - \theta \sum_{i=1}^{n} \sum_{j=1}^{n} e^{-2\theta T} e^{\theta i\Delta} e^{\theta j\Delta} e^{2\theta\Delta} \int_0^{\Delta} \int_0^{\Delta} e^{-\theta t} e^{-\theta s} \frac{\partial \frac{1}{2}|t-s+i\Delta-j\Delta|^{2H}}{\partial t} dtds \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} e^{-2\theta T} e^{\theta i\Delta} e^{\theta j\Delta} e^{2\theta\Delta} \int_0^{\Delta} e^{-\theta t} \frac{\partial \frac{1}{2}|t+i\Delta-j\Delta|^{2H}}{\partial t} dt \\
&\quad - \sum_{i=1}^{n} \sum_{j=2}^{n+1} e^{-2\theta T} e^{\theta i\Delta} e^{\theta j\Delta} \int_0^{\Delta} e^{-\theta t} \frac{\partial \frac{1}{2}|t+i\Delta-j\Delta|^{2H}}{\partial t} dt \\
&\quad - \theta \sum_{i=1}^{n} \sum_{j=1}^{n} e^{-2\theta T} e^{\theta i\Delta} e^{\theta j\Delta} e^{2\theta\Delta} \int_0^{\Delta} \int_0^{\Delta} e^{-\theta t} e^{-\theta s} \frac{\partial \frac{1}{2}|t-s+i\Delta-j\Delta|^{2H}}{\partial t} dtds
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=1}^{n}\sum_{j=1}^{n} e^{-2\theta T} e^{\theta i\Delta} e^{\theta j\Delta}\left(e^{2\theta\Delta}-1\right)\int_0^{\Delta} e^{-\theta t}\frac{\partial \frac{1}{2}|t+i\Delta-j\Delta|^{2H}}{\partial t}dt\\
&\quad - \sum_{i=1}^{n} e^{-\theta T} e^{\theta i\Delta} e^{\theta\Delta}\int_0^{\Delta} e^{-\theta t}\frac{\partial \frac{1}{2}|t+i\Delta-n\Delta-\Delta|^{2H}}{\partial t}dt\\
&\quad + \sum_{i=1}^{n} e^{-2\theta T} e^{\theta i\Delta} e^{\theta\Delta}\int_0^{\Delta} e^{-\theta t}\frac{\partial \frac{1}{2}|t+i\Delta-\Delta|^{2H}}{\partial t}dt\\
&\quad - \theta\sum_{i=1}^{n}\sum_{j=1}^{n} e^{-2\theta T} e^{\theta i\Delta} e^{\theta j\Delta} e^{2\theta\Delta}\int_0^{\Delta}\int_0^{\Delta} e^{-\theta t}e^{-\theta s}\frac{\partial \frac{1}{2}|t-s+i\Delta-j\Delta|^{2H}}{\partial t}dtds\\
&= -\sum_{i=1}^{n} e^{-\theta i\Delta} e^{2\theta\Delta}\int_0^{\Delta} e^{-\theta t}\frac{\partial \frac{1}{2}|-t+i\Delta|^{2H}}{\partial t}dt\\
&\quad + \sum_{i=0}^{n-1} e^{-2\theta T} e^{\theta i\Delta} e^{2\theta\Delta}\int_0^{\Delta} e^{-\theta t}\frac{\partial \frac{1}{2}|t+i\Delta|^{2H}}{\partial t}dt\\
&\quad + \sum_{i=1}^{n}\sum_{j=1}^{n} e^{-2\theta T} e^{\theta i\Delta} e^{\theta j\Delta}\left(e^{2\theta\Delta}-1\right)\int_0^{\Delta} e^{-\theta t}\frac{\partial \frac{1}{2}|t+i\Delta-j\Delta|^{2H}}{\partial t}dt\\
&\quad - \theta\sum_{i=1}^{n}\sum_{j=1}^{n} e^{-2\theta T} e^{\theta i\Delta} e^{\theta j\Delta} e^{2\theta\Delta}\int_0^{\Delta}\int_0^{\Delta} e^{-\theta t}e^{-\theta s}\frac{\partial \frac{1}{2}|t-s+i\Delta-j\Delta|^{2H}}{\partial t}dtds\\
&=: J_{1n}+J_{2n}+J_{3n}, \qquad \text{(A.4)}
\end{aligned}
$$

where

$$
\begin{aligned}
J_{1n} &= -\sum_{i=1}^{n} e^{-\theta i\Delta} e^{2\theta\Delta}\int_0^{\Delta} e^{-\theta t}\frac{\partial \frac{1}{2}|-t+i\Delta|^{2H}}{\partial t}dt\\
&\quad + \sum_{i=0}^{n-1} e^{-2\theta T} e^{\theta i\Delta} e^{2\theta\Delta}\int_0^{\Delta} e^{-\theta t}\frac{\partial \frac{1}{2}|t+i\Delta|^{2H}}{\partial t}dt,\\
J_{2n} &= \sum_{i=1}^{n}\sum_{j=1}^{n} e^{-2\theta T} e^{\theta i\Delta} e^{\theta j\Delta}\left(e^{2\theta\Delta}-1\right)\int_0^{\Delta} e^{-\theta t}\frac{\partial \frac{1}{2}|t+i\Delta-j\Delta|^{2H}}{\partial t}dt,\\
J_{3n} &= -\theta\sum_{i=1}^{n}\sum_{j=1}^{n} e^{-2\theta T} e^{\theta(i+j+2)\Delta}\int_0^{\Delta}\int_0^{\Delta} e^{-\theta(t+s)}\frac{\partial \frac{1}{2}|t-s+i\Delta-j\Delta|^{2H}}{\partial t}dtds.
\end{aligned}
$$

First, for J_{1n}, we have

$$
\begin{aligned}
&-\sum_{i=1}^{n} e^{-\theta i\Delta} e^{2\theta\Delta}\int_0^{\Delta} e^{-\theta t}\frac{\partial \frac{1}{2}|-t+i\Delta|^{2H}}{\partial t}dt\\
&\quad + \sum_{i=0}^{n-1} e^{-2\theta T} e^{\theta i\Delta} e^{2\theta\Delta}\int_0^{\Delta} e^{-\theta t}\frac{\partial \frac{1}{2}|t+i\Delta|^{2H}}{\partial t}dt
\end{aligned}
$$

$$\begin{aligned}
&\sim \frac{1}{2}\sum_{i=1}^{n} e^{-\theta i\Delta}e^{2\theta\Delta}\big[(i\Delta)^{2H}-(i\Delta-\Delta)^{2H}\big] \\
&\quad+\frac{1}{2}\sum_{i=0}^{n-1} e^{-2\theta T}e^{\theta i\Delta}e^{2\theta\Delta}\big[(i\Delta+\Delta)^{2H}-(i\Delta)^{2H}\big] \\
&=\frac{1}{2}\sum_{i=1}^{n} e^{-\theta i\Delta}e^{2\theta\Delta}(i\Delta)^{2H}-\frac{1}{2}\sum_{i=1}^{n-1} e^{-\theta i\Delta}e^{\theta\Delta}(i\Delta)^{2H} \\
&\quad-\frac{1}{2}\sum_{i=1}^{n} e^{-2\theta T}e^{\theta i\Delta}e^{\theta\Delta}(i\Delta)^{2H}+\frac{1}{2}\sum_{i=1}^{n} e^{-2\theta T}e^{\theta i\Delta}e^{2\theta\Delta}(i\Delta)^{2H} \\
&=\frac{1}{2}\sum_{i=1}^{n} e^{-\theta i\Delta}e^{\theta\Delta}\big(e^{\theta\Delta}-1\big)(i\Delta)^{2H}+\frac{1}{2}e^{-\theta T}e^{\theta\Delta}T^{2H} \\
&\quad-\frac{1}{2}\sum_{i=1}^{n} e^{-2\theta T}e^{\theta i\Delta}e^{\theta\Delta}\big(e^{\theta\Delta}-1\big)(i\Delta)^{2H}+\frac{1}{2}e^{-\theta T}e^{2\theta\Delta}T^{2H} \\
&\sim \frac{\theta}{2}\int_0^T e^{-\theta t}t^{2H}dt-\frac{\theta}{2}e^{-2\theta T}\int_0^T e^{\theta t}t^{2H}dt+o(1) \\
&=\frac{\theta}{2}\int_0^T e^{-\theta t}t^{2H}dt+o(1)\to\frac{H\Gamma(2H)}{\theta^{2H}}. \qquad \text{(A.5)}
\end{aligned}$$

Second, for J_{2n}, we can see that

$$\begin{aligned}
&\sum_{i=1}^{n}\sum_{j=1}^{n} e^{-2\theta T}e^{\theta i\Delta}e^{\theta j\Delta}\big(e^{2\theta\Delta}-1\big)\int_0^\Delta e^{-\theta t}\frac{\partial\frac{1}{2}|t+i\Delta-j\Delta|^{2H}}{\partial t}dt \\
&=H\sum_{i=1}^{n}\sum_{j=1}^{i-1} e^{-2\theta T}e^{\theta i\Delta}e^{\theta j\Delta}\big(e^{2\theta\Delta}-1\big)\int_0^\Delta e^{-\theta t}(t+i\Delta-j\Delta)^{2H-1}dt \\
&\quad+H\sum_{i=1}^{n} e^{-2\theta T}e^{2\theta i\Delta}\big(e^{2\theta\Delta}-1\big)\int_0^\Delta e^{-\theta t}t^{2H-1}dt \\
&\quad-H\sum_{i=1}^{n}\sum_{j=i+1}^{n} e^{-2\theta T}e^{\theta i\Delta}e^{\theta j\Delta}\big(e^{2\theta\Delta}-1\big)\int_0^\Delta e^{-\theta t}(-t+j\Delta-i\Delta)^{2H-1}dt \\
&\sim H\sum_{i=1}^{n}\sum_{j=1}^{i-1} e^{-2\theta T}e^{\theta i\Delta}e^{\theta j\Delta}\big(e^{2\theta\Delta}-1\big)\int_0^\Delta (t+i\Delta-j\Delta)^{2H-1}dt \\
&\quad+H\sum_{i=1}^{n} e^{-2\theta T}e^{2\theta i\Delta}\big(e^{2\theta\Delta}-1\big)\int_0^\Delta t^{2H-1}dt \\
&\quad-H\sum_{i=1}^{n}\sum_{j=i+1}^{n} e^{-2\theta T}e^{\theta i\Delta}e^{\theta j\Delta}\big(e^{2\theta\Delta}-1\big)\int_0^\Delta (-t+j\Delta-i\Delta)^{2H-1}dt \\
&=\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{i-1} e^{-2\theta T}e^{\theta i\Delta}e^{\theta j\Delta}\big(e^{2\theta\Delta}-1\big)\big[(i\Delta-j\Delta+\Delta)^{2H}-(i\Delta-j\Delta)^{2H}\big]
\end{aligned}$$

$$+\frac{1}{2}e^{2\theta\Delta}\big(1-e^{-2\theta T}\big)\Delta^{2H}$$
$$-\frac{1}{2}\sum_{i=1}^{n}\sum_{j=i+1}^{n}e^{-2\theta T}e^{\theta i\Delta}e^{\theta j\Delta}\big(e^{2\theta\Delta}-1\big)\big[(j\Delta-i\Delta)^{2H}-(j\Delta-i\Delta-\Delta)^{2H}\big]$$
$$=\frac{1}{2}\sum_{i=1}^{n}\sum_{j=i+1}^{n}e^{-2\theta T}e^{\theta i\Delta}e^{\theta j\Delta}\big(e^{2\theta\Delta}-1\big)\big[(j\Delta-i\Delta+\Delta)^{2H}-2(j\Delta-i\Delta)^{2H}$$
$$+(j\Delta-i\Delta-\Delta)^{2H}\big]+\frac{1}{2}e^{2\theta\Delta}\big(1-e^{-2\theta T}\big)\Delta^{2H}$$
$$=\frac{1}{2}\sum_{i=1}^{n}\sum_{j=i+1}^{n}e^{-2\theta T}e^{\theta i\Delta}e^{\theta j\Delta}e^{-\theta\Delta}\big(e^{2\theta\Delta}-1\big)\big(1+e^{2\theta\Delta}-2e^{\theta\Delta}\big)(j\Delta-i\Delta)^{2H}$$
$$+\frac{1}{2}\sum_{i=1}^{n}e^{-\theta i\Delta}e^{\theta\Delta}\big(e^{2\theta\Delta}-1\big)(i\Delta)^{2H}-\frac{1}{2}\sum_{i=1}^{n-1}e^{-\theta i\Delta}e^{\theta\Delta}\big(e^{2\theta\Delta}-1\big)(i\Delta)^{2H}$$
$$\sim\theta^{3}\sum_{i=1}^{n}\sum_{j=i+1}^{n}e^{-2\theta T}e^{\theta i\Delta}e^{\theta j\Delta}e^{-\theta\Delta}(j\Delta-i\Delta)^{2H}\Delta^{3}+\frac{1}{2}e^{-\theta T}e^{\theta\Delta}\big(e^{2\theta\Delta}-1\big)T^{2H}$$
$$=o(1). \tag{A.6}$$

Third, for J_{3n}, it follows that

$$-\theta\sum_{i=1}^{n}\sum_{j=1}^{n}e^{-2\theta T}e^{\theta i\Delta}e^{\theta j\Delta}e^{2\theta\Delta}\int_0^{\Delta}\int_0^{\Delta}e^{-2\theta\Delta}\frac{\partial\frac{1}{2}|t-s+i\Delta-j\Delta|^{2H}}{\partial t}dtds$$
$$\leq-\theta\sum_{i=1}^{n}\sum_{j=1}^{n}e^{-2\theta T}e^{\theta i\Delta}e^{\theta j\Delta}e^{2\theta\Delta}\int_0^{\Delta}\int_0^{\Delta}e^{-\theta t}e^{-\theta s}\frac{\partial\frac{1}{2}|t-s+i\Delta-j\Delta|^{2H}}{\partial t}dtds$$
$$\leq-\theta\sum_{i=1}^{n}\sum_{j=1}^{n}e^{-2\theta T}e^{\theta i\Delta}e^{\theta j\Delta}e^{2\theta\Delta}\int_0^{\Delta}\int_0^{\Delta}\frac{\partial\frac{1}{2}|t-s+i\Delta-j\Delta|^{2H}}{\partial t}dtds$$

here, the first term equals to

$$-\theta H\sum_{i=1}^{n}\sum_{j=1}^{i-1}e^{-2\theta T}e^{\theta i\Delta}e^{\theta j\Delta}e^{2\theta\Delta}\int_0^{\Delta}\int_0^{\Delta}e^{-\theta t}e^{-\theta s}(t-s+i\Delta-j\Delta)^{2H-1}dtds$$
$$+\theta H\sum_{i=1}^{n}\sum_{j=i+1}^{n}e^{-2\theta T}e^{\theta i\Delta}e^{\theta j\Delta}e^{2\theta\Delta}\int_0^{\Delta}\int_0^{\Delta}e^{-\theta t}e^{-\theta s}(-t+s+j\Delta-i\Delta)^{2H-1}dtds$$
$$-\theta\sum_{i=1}^{n}e^{-2\theta T}e^{2\theta i\Delta}e^{2\theta\Delta}\int_0^{\Delta}\int_0^{\Delta}e^{-\theta t}e^{-\theta s}\frac{\partial\frac{1}{2}|t-s|^{2H}}{\partial t}dtds$$
$$\sim-\theta H\sum_{i=1}^{n}\sum_{j=1}^{i-1}e^{-2\theta T}e^{\theta i\Delta}e^{\theta j\Delta}e^{2\theta\Delta}\int_0^{\Delta}\int_0^{\Delta}(t-s+i\Delta-j\Delta)^{2H-1}dtds$$
$$+\theta H\sum_{i=1}^{n}\sum_{j=i+1}^{n}e^{-2\theta T}e^{\theta i\Delta}e^{\theta j\Delta}e^{2\theta\Delta}\int_0^{\Delta}\int_0^{\Delta}(-t+s+j\Delta-i\Delta)^{2H-1}dtds$$

$$
\begin{aligned}
&= -\frac{\theta}{2}\sum_{i=1}^{n}\sum_{j=1}^{i-1} e^{-2\theta T}e^{\theta i\Delta}e^{\theta j\Delta}e^{2\theta\Delta}\big[(i\Delta - j\Delta + \Delta)^{2H+1} - 2(i\Delta - j\Delta)^{2H+1} \\
&\qquad\qquad + (i\Delta - j\Delta - \Delta)^{2H+1}\big] \\
&\quad + \frac{\theta}{2}\sum_{i=1}^{n}\sum_{j=i+1}^{n} e^{-2\theta T}e^{\theta i\Delta}e^{\theta j\Delta}e^{2\theta\Delta}\big[(j\Delta - i\Delta + \Delta)^{2H+1} - 2(j\Delta - i\Delta)^{2H+1} \\
&\qquad\qquad + (j\Delta - i\Delta - \Delta)^{2H+1}\big] \\
&= 0. \qquad\qquad (A.7)
\end{aligned}
$$

The last term also equals to 0. By Sandwich Theorem, we have $J_3 \sim 0$. Together with (A.4)–(A.6), we complete the proof of this lemma. □

Lemma A.3. *Assume $H \in (1/2, 1)$. Let $\Delta \to 0$ and $T \to \infty$. Then we have the following results*

$$
\begin{aligned}
&(i)\ \alpha_H \sum_{i=1}^{n}\sum_{j=1}^{n} e^{-\theta T}e^{2\theta i\Delta}\int_{(i-1)\Delta}^{i\Delta}\int_{(j-1)\Delta}^{j\Delta} e^{-\theta t}e^{-\theta s}|t-s|^{2H-2}dtds \\
&\quad \sim \theta T^{2H}\Delta + Te^{-\theta T}\Delta^{2H-1} + o(1), \qquad (A.8) \\
&(ii)\ \alpha_H \sum_{i=1}^{n}\sum_{j=1}^{n} e^{\theta(i+j)\Delta}\int_{(i-1)\Delta}^{i\Delta}\int_{(j-1)\Delta}^{j\Delta} e^{-\theta(t+s)}|t-s|^{2H-2}dtds \\
&\quad \sim T^{2H} + o\big(T^{2H}\big), \qquad (A.9) \\
&(iii)\ \alpha_H \sum_{i=1}^{n}\sum_{j=1}^{n} e^{\theta i\Delta}\int_{(i-1)\Delta}^{i\Delta}\int_{(j-1)\Delta}^{j\Delta} e^{-\theta(t+s)}|t-s|^{2H-2}dtds \\
&\quad \sim \frac{\theta}{2}T^{2H}\Delta + \theta H T^{2H-1} + O(1), \qquad (A.10) \\
&(iv)\ \alpha_H \sum_{i=1}^{n}\sum_{j=1}^{n} e^{-\theta T}e^{\theta(i+2j)\Delta}\int_{(i-1)\Delta}^{i\Delta}\int_{(j-1)\Delta}^{j\Delta} e^{-\theta(t+s)}|t-s|^{2H-2}dtds \\
&\quad \sim \frac{\theta}{2}T^{2H}\Delta + \theta H T^{2H-1} + O(1). \qquad (A.11)
\end{aligned}
$$

Lemma A.4. *Assume $H \in (0, 1/2)$. Let $\Delta \to 0$ and $T \to \infty$. Then we have the following statements*

$$
\begin{aligned}
&(i)\ \sum_{i=1}^{n}\sum_{j=1}^{n} e^{-\theta T}e^{2\theta i\Delta}\bigg(\theta\int_{(j-1)\Delta}^{j\Delta}\int_{(i-1)\Delta}^{i\Delta} e^{-\theta t}e^{-\theta s}\frac{\partial R(t,s)}{\partial t}dtds \\
&\qquad + \int_{(i-1)\Delta}^{i\Delta} e^{-\theta t}\bigg(e^{-\theta j\Delta}\frac{\partial R(t,j\Delta)}{\partial t} - e^{-\theta(j-1)\Delta}\frac{\partial R(t,(j-1)\Delta)}{\partial t}\bigg)dt\bigg) \\
&\quad \sim -\frac{\theta^2}{2H+1}T^{2H+1}\Delta + \frac{\theta}{2}T^{2H}\Delta + o(1), \qquad (A.12) \\
&(ii) \sum_{i=1}^{n}\sum_{j=1}^{n} e^{\theta(i+j)\Delta}\bigg(\theta\int_{(j-1)\Delta}^{j\Delta}\int_{(i-1)\Delta}^{i\Delta} e^{-\theta t}e^{-\theta s}\frac{\partial R(t,s)}{\partial t}dtds
\end{aligned}
$$

$$+\int_{(i-1)\Delta}^{i\Delta} e^{-\theta t}\left(e^{-\theta j\Delta}\frac{\partial R(t,j\Delta)}{\partial t} - e^{-\theta(j-1)\Delta}\frac{\partial R(t,(j-1)\Delta)}{\partial t}\right)dt\Bigg)$$

$$\sim T^{2H} + o\big(T^{2H}\big), \tag{A.13}$$

$$(iii)\ \sum_{i=1}^{n}\sum_{j=1}^{n} e^{\theta i\Delta}\Bigg(\theta\int_{(j-1)\Delta}^{j\Delta}\int_{(i-1)\Delta}^{i\Delta} e^{-\theta t}e^{-\theta s}\frac{\partial R(t,s)}{\partial t}dtds$$

$$+\int_{(i-1)\Delta}^{i\Delta} e^{-\theta t}\left(e^{-\theta j\Delta}\frac{\partial R(t,j\Delta)}{\partial t} - e^{-\theta(j-1)\Delta}\frac{\partial R(t,(j-1)\Delta)}{\partial t}\right)dt\Bigg)$$

$$\sim -\frac{\theta}{2(2H+1)}T^{2H+1}\Delta + O(1), \tag{A.14}$$

$$(iv)\ \sum_{i=1}^{n}\sum_{j=1}^{n} e^{-\theta T}e^{\theta(i+2j)\Delta}\Bigg(\theta\int_{(j-1)\Delta}^{j\Delta}\int_{(i-1)\Delta}^{i\Delta} e^{-\theta t}e^{-\theta s}\frac{\partial R(t,s)}{\partial t}dtds$$

$$+\int_{(i-1)\Delta}^{i\Delta} e^{-\theta t}\left(e^{-\theta j\Delta}\frac{\partial R(t,j\Delta)}{\partial t} - e^{-\theta(j-1)\Delta}\frac{\partial R(t,(j-1)\Delta)}{\partial t}\right)dt\Bigg)$$

$$= O\big(T^{2H+1}\Delta\big) + O(1). \tag{A.15}$$

A.2. Useful lemmas, propositions and their proofs

Now, we rewrite $\widehat{\theta}_\Delta$ and $\widehat{\mu}_\Delta$ as

$$\widehat{\theta}_\Delta = \theta + \frac{1}{\Delta}\log\left(1 + e^{-\theta\Delta}\frac{U_T}{V_T}\right), \tag{A.16}$$

$$\widehat{\mu}_\Delta = \mu + (\widehat{\theta}_\Delta - \theta)\left(\frac{\mu}{\theta} + \frac{M_T}{N_T}\right) + \theta\frac{M_T}{N_T}, \tag{A.17}$$

where

$$U_T = \sigma\sum_{i=1}^{n}\epsilon_{i\Delta}X_{(i-1)\Delta} - \sigma\frac{1}{n}\sum_{i=1}^{n}\epsilon_{i\Delta}\sum_{i=1}^{n}X_{(i-1)\Delta}, \tag{A.18}$$

$$V_T = \sum_{i=1}^{n}X_{(i-1)\Delta}^2 - \frac{1}{n}\left(\sum_{i=1}^{n}X_{(i-1)\Delta}\right)^2, \tag{A.19}$$

$$M_T = \sigma\sum_{i=1}^{n}\epsilon_{i\Delta}\sum_{i=1}^{n}X_{(i-1)\Delta}^2 + \sigma\frac{\mu}{\theta}\sum_{i=1}^{n}\epsilon_{i\Delta}\sum_{i=1}^{n}X_{(i-1)\Delta}$$

$$-\sigma\frac{\mu}{\theta}n\sum_{i=1}^{n}\epsilon_{i\Delta}X_{(i-1)\Delta} - \sigma\sum_{i=1}^{n}X_{(i-1)\Delta}\sum_{i=1}^{n}\epsilon_{i\Delta}X_{(i-1)\Delta}, \tag{A.20}$$

$$N_T = n\big(e^{\theta\Delta}-1\big)\sum_{i=1}^{n}X_{(i-1)\Delta}^2 - \big(e^{\theta\Delta}-1\big)\left(\sum_{i=1}^{n}X_{(i-1)\Delta}\right)^2$$

$$-\sigma\sum_{i=1}^{n}\epsilon_{i\Delta}\sum_{i=1}^{n}X_{(i-1)\Delta} + \sigma n\sum_{i=1}^{n}\epsilon_{i\Delta}X_{(i-1)\Delta}. \tag{A.21}$$

Moreover, let

$$\widetilde{U}_T = \sigma \sum_{i=1}^{n} e^{-\theta(n-i)\Delta} \epsilon_{i\Delta} = \sigma \sum_{i=1}^{n} e^{-\theta T} e^{2\theta i\Delta} \int_{(i-1)\Delta}^{i\Delta} e^{-\theta s} dB_s^H, \tag{A.22}$$

$$\widetilde{X}_T = e^{-\theta T} X_T = X_0 + \frac{\mu}{\theta}\big(1 - e^{-\theta T}\big) + \sigma \int_0^T e^{-\theta s} dB_s^H, \tag{A.23}$$

$$\Xi_T = T^{-H} \sum_{i=1}^{n} \epsilon_{i\Delta}. \tag{A.24}$$

Then, in Proposition A.1, we present further expansions for the terms U_T, V_T, M_T and N_T, which play crucial roles in our analysis.

Propostion A.1. *Let U_T, V_T, M_T, N_T, $\widetilde{U}_T$, $\widetilde{X}_T$ and Ξ_T be defined by (A.18)–(A.24), respectively. Then, we have*

$$e^{-\theta(T+\Delta)} U_T = e^{-2\theta\Delta} \widetilde{U}_T \widetilde{X}_T + R_{1n}, \tag{A.25}$$

$$\big(1 - e^{-2\theta\Delta}\big) e^{-2\theta T} V_T = e^{-2\theta\Delta} \widetilde{X}^2_{(n-1)\Delta} - R_{2n}, \tag{A.26}$$

$$T^{-H}\big(1 - e^{-2\theta\Delta}\big) e^{-2\theta T} M_T = \sigma e^{-2\theta\Delta} \Xi_T \widetilde{X}^2_{(n-1)\Delta} - R_{3n}, \tag{A.27}$$

$$\big(n\big(e^{\theta\Delta} - 1\big)\big)^{-1}\big(1 - e^{-2\theta\Delta}\big) e^{-2\theta T} N_T = e^{-2\theta\Delta} \widetilde{X}^2_{(n-1)\Delta} - R_{4n}, \tag{A.28}$$

where the remainder terms R_{1n}, R_{2n}, R_{3n} and R_{4n} are defined as

$$\begin{aligned} R_{1n} &= \sigma e^{-2\theta\Delta} \sum_{i=1}^{n} e^{-\theta(T-i\Delta)} \epsilon_{i\Delta} (\widetilde{X}_{(i-1)\Delta} - \widetilde{X}_T) \\ &\quad - \sigma \frac{e^{-\theta(T+\Delta)}}{n} \sum_{i=1}^{n} \epsilon_{i\Delta} \sum_{i=1}^{n} X_{(i-1)\Delta}, \end{aligned} \tag{A.29}$$

$$\begin{aligned} R_{2n} &= e^{-2\theta\Delta} \sum_{i=2}^{n} e^{-2\theta(T-i\Delta+\Delta)} \big(\widetilde{X}^2_{(i-1)\Delta} - \widetilde{X}^2_{(i-2)\Delta}\big) + e^{-2\theta\Delta} e^{-2\theta T} \widetilde{X}_0^2 \\ &\quad + \frac{(1 - e^{-2\theta\Delta}) e^{-2\theta T}}{n} \left(\sum_{i=1}^{n} X_{(i-1)\Delta} \right)^2, \end{aligned} \tag{A.30}$$

$$\begin{aligned} R_{3n} &= \sigma \Xi_T \left[e^{-2\theta\Delta} \sum_{i=2}^{n} e^{-2\theta(T-i\Delta+\Delta)} \big(\widetilde{X}^2_{(i-1)\Delta} - \widetilde{X}^2_{(i-2)\Delta}\big) + e^{-2\theta\Delta} e^{-2\theta T} \widetilde{X}_0^2 \right] \\ &\quad - \sigma \frac{\mu}{\theta} \big(1 - e^{-2\theta\Delta}\big) e^{-2\theta T} \Xi_T \sum_{i=1}^{n} X_{(i-1)\Delta} + e^{-2\theta\Delta} e^{-2\theta T} \widetilde{X}_0^2 \\ &\quad + \sigma \frac{\mu}{\theta} n T^{-H} \big(1 - e^{-2\theta\Delta}\big) e^{-2\theta T} \sum_{i=1}^{n} \epsilon_{i\Delta} X_{(i-1)\Delta} \\ &\quad + \sigma T^{-H} \big(1 - e^{-2\theta\Delta}\big) e^{-2\theta T} \sum_{i=1}^{n} X_{(i-1)\Delta} \sum_{i=1}^{n} \epsilon_{i\Delta} X_{(i-1)\Delta}, \end{aligned} \tag{A.31}$$

$$R_{4n} = e^{-2\theta\Delta}\sum_{i=2}^{n} e^{-2\theta(T-i\Delta+\Delta)}\big(\widetilde{X}^2_{(i-1)\Delta} - \widetilde{X}^2_{(i-2)\Delta}\big) + e^{-2\theta\Delta}e^{-2\theta T}\widetilde{X}^2_0$$
$$+ \frac{1}{n}\big(1-e^{-2\theta\Delta}\big)e^{-2\theta T}\left(\sum_{i=1}^{n} X_{(i-1)\Delta}\right)^2$$
$$+ \sigma\frac{(1-e^{-2\theta\Delta})e^{-2\theta T}}{n(e^{\theta\Delta}-1)}\sum_{i=1}^{n}\epsilon_{i\Delta}\sum_{i=1}^{n} X_{(i-1)\Delta}$$
$$- \sigma\frac{(1-e^{-2\theta\Delta})e^{-2\theta T}}{e^{\theta\Delta}-1}\sum_{i=1}^{n}\epsilon_{i\Delta}X_{(i-1)\Delta}. \tag{A.32}$$

Proof. Firstly, we can write

$$\sigma e^{-\theta(T+\Delta)}\sum_{i=1}^{n}\epsilon_{i\Delta}X_{(i-1)\Delta}$$
$$= e^{-2\theta\Delta}\widetilde{U}_T\widetilde{X}_T + \sigma e^{-2\theta\Delta}\sum_{i=1}^{n} e^{-\theta(T-i\Delta)}\epsilon_{i\Delta}(\widetilde{X}_{(i-1)\Delta} - \widetilde{X}_T), \tag{A.33}$$

where $\widetilde{U}_T$ and $\widetilde{X}_T$ are defined by (A.22) and (A.23), respectively. Then, together with (A.18) and (A.33), we can easily obtain (A.25). We can see that

$$\big(1-e^{-2\theta\Delta}\big)e^{-2\theta T}\sum_{i=1}^{n} X^2_{(i-1)\Delta}$$
$$= e^{-2\theta\Delta}\widetilde{X}^2_{(n-1)\Delta} - e^{-2\theta\Delta}\sum_{i=2}^{n} e^{-2\theta(n-i+1)\Delta}\big(\widetilde{X}^2_{(i-1)\Delta} - \widetilde{X}^2_{(i-2)\Delta}\big). \tag{A.34}$$

Together with (A.19), we have (A.26). Furthermore, combining (A.20), (A.21) and (A.26), we can obtain (A.27) and (A.28) easily. □

The following four lemmas characterize the relationship between $\widetilde{U}_T, \widetilde{X}_T$ and Ξ_T, whose proofs are postponed to the supplementary material file [28] for the readability of this paper.

Lemma A.5. *Let $\Delta \to 0$, $T \to \infty$. If (i) $H = 1/2$, (ii) $H \in (1/2, 1)$ and $T^{2H}\Delta \to 0$, or (iii) $H \in (0, 1/2)$ and $T^{2H+1}\Delta \to 0$, then we have $\mathbb{E}(\widetilde{U}_T\widetilde{X}_T) \to 0$.*

Proof. **Case 1:** $H \in (0, 1/2)$. By Lemma A.4 and $T^{2H+1}\Delta \to 0$, we can get

$$\mathbb{E}(\widetilde{U}_T\widetilde{X}_T)$$
$$= \sigma^2\sum_{i=1}^{n}\sum_{j=1}^{n} e^{-\theta T}e^{2\theta i\Delta}\mathbb{E}\left(\int_{(i-1)\Delta}^{i\Delta} e^{-\theta s}dB^H_s\int_{(j-1)\Delta}^{j\Delta} e^{-\theta s}dB^H_s\right)$$
$$= \sigma^2\sum_{i=1}^{n}\sum_{j=1}^{n} e^{-\theta T}e^{2\theta i\Delta}\left(\theta\int_{(j-1)\Delta}^{j\Delta}\int_{(i-1)\Delta}^{i\Delta} e^{-\theta t}e^{-\theta s}\frac{\partial R(t,s)}{\partial t}dtds\right.$$

$$+\int_{(i-1)\Delta}^{i\Delta} e^{-\theta t}\left(e^{-\theta j\Delta}\frac{\partial R(t,j\Delta)}{\partial t}-e^{-\theta(j-1)\Delta}\frac{\partial R(t,(j-1)\Delta)}{\partial t}\right)dt\Bigg)$$

$$\to 0. \tag{A.35}$$

Case 2: $H=1/2$. By (A.22) and (A.23), we have

$$\begin{aligned}\mathbb{E}(\widetilde{U}_T\widetilde{X}_T) &= \mathbb{E}\left(\sigma^2\sum_{i=1}^n e^{-\theta T}e^{2\theta i\Delta}\left(\int_{(i-1)\Delta}^{i\Delta} e^{-\theta s}dB_s\right)^2\right)\\ &= \sigma^2\sum_{i=1}^n e^{-\theta T}e^{2\theta i\Delta}\int_{(i-1)\Delta}^{i\Delta} e^{-2\theta s}ds\\ &= \frac{\sigma^2}{2\theta}\sum_{i=1}^n e^{-\theta T}\left(e^{2\theta\Delta}-1\right)\to 0,\end{aligned} \tag{A.36}$$

as $T\to\infty$ and $\Delta\to 0$.

Case 3: $H\in(1/2,1)$. From (A.22), (A.23), Lemma A.3 and $T^{2H}\Delta\to 0$, we can write the following result immediately,

$$\begin{aligned}&\mathbb{E}(\widetilde{U}_T\widetilde{X}_T)\\ &= \sigma^2\sum_{i=1}^n\sum_{j=1}^n e^{-\theta T}e^{2\theta i\Delta}\mathbb{E}\left(\int_{(i-1)\Delta}^{i\Delta} e^{-\theta s}dB_s^H\int_{(j-1)\Delta}^{j\Delta} e^{-\theta s}dB_s^H\right)\\ &= \sigma^2\alpha_H\sum_{i=1}^n\sum_{j=1}^n e^{-\theta T}e^{2\theta i\Delta}\int_{(i-1)\Delta}^{i\Delta}\int_{(j-1)\Delta}^{j\Delta} e^{-\theta(s+r)}|s-r|^{2H-2}dsdr\\ &\to 0.\end{aligned} \tag{A.37}$$

Using (A.35)–(A.37), we complete the proof of this lemma. □

Lemma A.6. *Let $\Delta\to 0$ and $T\to\infty$. If (i) $H=1/2$, (ii) $H\in(1/2,1)$ and $T^H\Delta\to 0$, or (iii) $H\in(0,1/2)$ and $T^{H+1}\Delta\to 0$, then we have $\mathbb{E}(\Xi_T\widetilde{X}_T)\to 0$.*

Proof. **Case 1:** $H\in(0,1/2)$. By Lemma A.4 and $T^{H+1}\Delta\to 0$, it holds that

$$\begin{aligned}&\mathbb{E}(\Xi_T\widetilde{X}_T)\\ &= \sigma T^{-H}\sum_{i=1}^n\sum_{j=1}^n\mathbb{E}\left(e^{\theta i\Delta}\int_{(i-1)\Delta}^{i\Delta} e^{-\theta s}dB_s^H\int_{(j-1)\Delta}^{j\Delta} e^{-\theta s}dB_s^H\right)\\ &= \sigma T^{-H}\sum_{i=1}^n\sum_{j=1}^n e^{\theta i\Delta}\Bigg(\theta\int_{(j-1)\Delta}^{j\Delta}\int_{(i-1)\Delta}^{i\Delta} e^{-\theta t}e^{-\theta s}\frac{\partial R(t,s)}{\partial t}dtds\\ &\qquad+\int_{(i-1)\Delta}^{i\Delta} e^{-\theta t}\left(e^{-\theta j\Delta}\frac{\partial R(t,j\Delta)}{\partial t}-e^{-\theta(j-1)\Delta}\frac{\partial R(t,(j-1)\Delta)}{\partial t}\right)dt\Bigg)\\ &\to 0,\end{aligned} \tag{A.38}$$

Case 2: $H = 1/2$. Straightforward calculations lead to

$$\begin{aligned}\mathbb{E}(\Xi_T \widetilde{X}_T) &= \sigma T^{-1/2} \sum_{i=1}^{n} e^{\theta i\Delta} \mathbb{E}\bigg(\int_{(i-1)\Delta}^{i\Delta} e^{-\theta s} dB_s\bigg)^2 \\ &= \sigma T^{-1/2} \sum_{i=1}^{n} e^{\theta i\Delta} \int_{(i-1)\Delta}^{i\Delta} e^{-2\theta s} ds \\ &= \sigma \frac{T^{-1/2}}{2\theta} \frac{(e^{2\theta\Delta}-1)(1-e^{-\theta T})}{e^{\theta\Delta}-1} \to 0,\end{aligned} \tag{A.39}$$

as $T \to \infty$ and $\Delta \to 0$.

Case 3: $H \in (1/2, 1)$. By Lemma A.3 and $T^H \Delta \to 0$, it holds

$$\begin{aligned}&\mathbb{E}(\Xi_T \widetilde{X}_T) \\ &= \sigma T^{-H} \sum_{i=1}^{n} \sum_{j=1}^{n} \mathbb{E}\bigg(e^{\theta i\Delta} \int_{(i-1)\Delta}^{i\Delta} e^{-\theta s} dB_s^H \int_{(j-1)\Delta}^{j\Delta} e^{-\theta s} dB_s^H\bigg) \\ &= \sigma \alpha_H T^{-H} \sum_{i=1}^{n} \sum_{j=1}^{n} e^{\theta i\Delta} \int_{(i-1)\Delta}^{i\Delta} \int_{(j-1)\Delta}^{j\Delta} e^{-\theta(s+r)} |s-r|^{2H-2} ds dr \\ &\to 0,\end{aligned}$$

which together with (A.38) and (A.39) completes the proof of this lemma. ☐

Lemma A.7. *Let $\Delta \to 0$ and $T \to \infty$. If (i) $H = 1/2$, (ii) $H \in (1/2, 1)$ and $T^H \Delta \to 0$, or (iii) $H \in (0, 1/2)$ and $T^{H+1}\Delta \to 0$, then we have $\mathbb{E}(\Xi_T \widetilde{U}_T) \to 0$.*

Proof. **Case 1:** $H \in (0, 1/2)$. By Lemma A.4 and $T^{H+1}\Delta \to 0$, it holds that

$$\begin{aligned}&\mathbb{E}\Bigg(\Xi_T \sum_{j=1}^{n} e^{-\theta(n-j)\Delta} \epsilon_{j\Delta}\Bigg) \\ &= T^{-H} \sum_{i=1}^{n} \sum_{j=1}^{n} e^{-\theta T} e^{\theta(i+2j)\Delta} \mathbb{E}\bigg(\int_{(i-1)\Delta}^{i\Delta} e^{-\theta s} dB_s^H \int_{(j-1)\Delta}^{j\Delta} e^{-\theta s} dB_s^H\bigg) \\ &= T^{-H} \sum_{i=1}^{n} \sum_{j=1}^{n} e^{-\theta T} e^{\theta(i+2j)\Delta} \bigg(\theta \int_{(j-1)\Delta}^{j\Delta} \int_{(i-1)\Delta}^{i\Delta} e^{-\theta t} e^{-\theta s} \frac{\partial R(t,s)}{\partial t} dt ds \\ &\qquad + \int_{(i-1)\Delta}^{i\Delta} e^{-\theta t} \bigg(e^{-\theta j\Delta} \frac{\partial R(t, j\Delta)}{\partial t} - e^{-\theta(j-1)\Delta} \frac{\partial R(t,(j-1)\Delta)}{\partial t}\bigg) dt\bigg) \\ &\to 0.\end{aligned} \tag{A.40}$$

Case 2: $H = 1/2$. It follows that

$$\mathbb{E}(\Xi_T \widetilde{U}_T) = \sigma \mathbb{E}\Bigg(T^{-1/2} \sum_{i=1}^{n} \epsilon_{i\Delta} \sum_{j=1}^{n} e^{-\theta(n-j)\Delta} \epsilon_{j\Delta}\Bigg)$$

$$= \sigma T^{-1/2} \sum_{i=1}^{n} e^{-\theta T} e^{3\theta i\Delta} \mathbb{E}\bigg(\int_{(i-1)\Delta}^{i\Delta} e^{-\theta s} dW_s \bigg)^2$$
$$= \sigma \frac{e^{\theta\Delta}}{2\theta} T^{-1/2} \frac{e^{2\theta\Delta}-1}{e^{\theta\Delta}-1} \big(1-e^{-\theta T}\big)$$
$$= O\big(T^{-1/2}\big). \tag{A.41}$$

Case 3: $H \in (1/2, 1)$. By Lemma A.3 and $T^{H+1}\Delta \to 0$, we have

$$\mathbb{E}\Bigg(\Xi_T \sum_{j=1}^{n} e^{-\theta(n-j)\Delta} \epsilon_{j\Delta} \Bigg)$$
$$= T^{-H} \sum_{i=1}^{n} \sum_{j=1}^{n} e^{-\theta T} e^{\theta(i+2j)\Delta} \mathbb{E}\bigg(\int_{(i-1)\Delta}^{i\Delta} e^{-\theta s} dB_s^H \int_{(j-1)\Delta}^{j\Delta} e^{-\theta s} dB_s^H \bigg)$$
$$= \alpha_H T^{-H} \sum_{i=1}^{n} \sum_{j=1}^{n} e^{-\theta T} e^{\theta(i+2j)\Delta} \int_{(i-1)\Delta}^{i\Delta} \int_{(j-1)\Delta}^{j\Delta} e^{-\theta(s+r)} |s-r|^{2H-2} ds dr$$
$$\to 0. \tag{A.42}$$

Together with (A.40), (A.41) and (A.42), we complete the proof of this lemma. □

Lemma A.8. *Let $\Delta \to 0$ and $T \to \infty$. Then for any $m < 1$, we have*

$$T^m \frac{1}{n} e^{-\theta T} \sum_{i=1}^{n} X_{(i-1)\Delta} \overset{a.s.}{\to} 0.$$

Proof. First, using (2.6), we can write

$$X_{(i-1)\Delta} = X_0 e^{\theta(i-1)\Delta} + \frac{\mu}{\theta}\big(e^{\theta(i-1)\Delta} - 1\big) + \sigma e^{\theta(i-1)\Delta} \int_0^{(i-1)\Delta} e^{-\theta s} dB_s^H. \tag{A.43}$$

It is obvious that

$$\sum_{i=1}^{n} X_{(i-1)\Delta} - \Bigg[\bigg(X_0 + \frac{\mu}{\theta} \bigg) \frac{1-e^{\theta T}}{1-e^{\theta\Delta}} - \frac{\mu}{\theta} n \Bigg] \sim \mathcal{N}\big(0, \sigma^2_{\sum_{i=1}^n X_{(i-1)\Delta}}\big), \tag{A.44}$$

where $\sigma^2_{\sum_{i=1}^n X_{(i-1)\Delta}} = \mathbb{E}(\sum_{i=1}^n \sigma e^{\theta(i-1)\Delta} \int_0^{(i-1)\Delta} e^{-\theta s} dB_s^H)^2$.

Case 1: $H = 1/2$. We can deduce that, as $n \to \infty$

$$\sigma^2_{\sum_{i=1}^n X_{(i-1)\Delta}}$$
$$= \sigma^2 \sum_{i=1}^{n} e^{2\theta(i-1)\Delta} \int_0^{(i-1)\Delta} e^{-2\theta s} ds$$
$$+ 2\sigma^2 \sum_{i=1}^{n} \sum_{j=i+1}^{n} e^{\theta(i-1)\Delta} e^{\theta(j-1)\Delta} \int_0^{(i-1)\Delta} e^{-2\theta s} ds$$

$$
\begin{aligned}
&= \frac{\sigma^2}{2\theta}\sum_{i=1}^{n} e^{2\theta(i-1)\Delta}\left(1-e^{-2\theta(i-1)\Delta}\right)\\
&\quad + \frac{\sigma^2}{\theta}\sum_{i=1}^{n}\sum_{j=i+1}^{n} e^{\theta(i-1)\Delta}e^{\theta(j-1)\Delta}\left(1-e^{-2\theta(i-1)\Delta}\right)\\
&= \frac{\sigma^2}{2\theta}\left(\frac{1-e^{2\theta T}}{1-e^{2\theta\Delta}}-n\right)+\frac{\sigma^2}{\theta(e^{\theta\Delta}-1)}\left[e^{\theta T}\left(\frac{1-e^{\theta T}}{1-e^{\theta\Delta}}-\frac{1-e^{-\theta T}}{1-e^{-\theta\Delta}}\right)\right.\\
&\qquad\left. -\frac{(1-e^{2\theta T})e^{\theta\Delta}}{1-e^{2\theta\Delta}}+ne^{\theta\Delta}\right]\\
&\le Ce^{2\theta T}\Delta^{-2}. \qquad \text{(A.45)}
\end{aligned}
$$

Case 2: $H \in (0,1/2)\cup(1/2,1)$. Using the Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
&\sigma^2_{\sum_{i=1}^n X_{(i-1)\Delta}}\\
&= \sigma^2\sum_{i=1}^{n}\sum_{j=1}^{n} e^{\theta(i-1)\Delta}e^{\theta(j-1)\Delta}\mathbb{E}\left(\int_0^{(i-1)\Delta} e^{-\theta s}dB_s^H\int_0^{(j-1)\Delta} e^{-\theta s}dB_s^H\right)\\
&\le \sigma^2\sum_{i=1}^{n}\sum_{j=1}^{n} e^{\theta(i-1)\Delta}e^{\theta(j-1)\Delta}\mathbb{E}\left|\int_0^{(i-1)\Delta} e^{-\theta s}dB_s^H\int_0^{(j-1)\Delta} e^{-\theta s}dB_s^H\right|\\
&\le \sigma^2\sum_{i=1}^{n}\sum_{j=1}^{n} e^{\theta(i-1)\Delta}e^{\theta(j-1)\Delta}\left(\mathbb{E}\left(\int_0^{(i-1)\Delta} e^{-\theta s}dB_s^H\right)^2\right)^{1/2}\\
&\qquad\cdot\left(\mathbb{E}\left(\int_0^{(j-1)\Delta} e^{-\theta s}dB_s^H\right)^2\right)^{1/2}\\
&\le \sigma^2\sum_{i=1}^{n}\sum_{j=1}^{n} e^{\theta(i-1)\Delta}e^{\theta(j-1)\Delta}\left(\mathbb{E}\left(\int_0^{\infty} e^{-\theta s}dB_s^H\right)^2\right)\\
&= \sigma^2\frac{H\Gamma(2H)}{\theta^{2H}}\left(\frac{1-e^{\theta T}}{1-e^{\theta\Delta}}\right)^2 \le Ce^{2\theta T}\Delta^{-2}. \qquad \text{(A.46)}
\end{aligned}
$$

Using (A.45) and (A.46), for $H \in (0,1)$, we have

$$
\sigma^2_{\sum_{i=1}^n X_{(i-1)\Delta}} \le Ce^{2\theta T}\Delta^{-2}. \qquad \text{(A.47)}
$$

Note that

$$
\begin{aligned}
&T^m\frac{1}{n}e^{-\theta T}\sum_{i=1}^{n} X_{(i-1)\Delta}\\
&= T^m\frac{1}{n}e^{-\theta T}\sigma_{\sum_{i=1}^n X_{(i-1)\Delta}}\frac{1}{\sigma_{\sum_{i=1}^n X_{(i-1)\Delta}}}\\
&\qquad\cdot\left\{\sum_{i=1}^{n} X_{(i-1)\Delta}-\left[\left(X_0+\frac{\mu}{\theta}\right)\frac{1-e^{\theta T}}{1-e^{\theta\Delta}}-\frac{\mu}{\theta}n\right]\right\}
\end{aligned}
$$

$$+ T^m \frac{1}{n} e^{-\theta T} \left[\left(X_0 + \frac{\mu}{\theta} \right) \frac{1 - e^{\theta T}}{1 - e^{\theta \Delta}} - \frac{\mu}{\theta} n \right] \tag{A.48}$$

and by (A.47), we can get

$$T^m \frac{1}{n} e^{-\theta T} \sigma_{\sum_{i=1}^n X_{(i-1)\Delta}} \le C T^{m-1},$$
$$T^m \frac{1}{n} e^{-\theta T} \left[\left(X_0 + \frac{\mu}{\theta} \right) \frac{1 - e^{\theta T}}{1 - e^{\theta \Delta}} - \frac{\mu}{\theta} n \right] \le C T^{m-1} - C T^m e^{-\theta T}.$$

Therefore, we have

$$\mathbb{P}\left(\left| T^m \frac{1}{n} e^{-\theta T} \sum_{i=1}^n X_{(i-1)\Delta} \right| \ge \epsilon \right) \le 2 \exp\{ -C \epsilon^2 T^{2-2m} \},$$

which implies that

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\left| T^m \frac{1}{n} e^{-\theta T} \sum_{i=1}^n X_{(i-1)\Delta} \right| \ge \epsilon \right) < \infty.$$

By the Borel-Cantelli lemma, we complete the proof of this lemma. □

Propostion A.2. *Let $\Delta \to 0$ and $T \to \infty$. Then, we have*

$$\widetilde{U}_T \xrightarrow{\mathcal{L}} \sigma \frac{\sqrt{H\Gamma(2H)}}{\theta^H} \nu, \quad \widetilde{X}_T \xrightarrow{a.s.} \widetilde{X}_\infty, \quad \Xi_T \xrightarrow{\mathcal{L}} \eta. \tag{A.49}$$

Moreover, if (i) $H = 1/2$, (ii) $H \in (1/2, 1)$ and $T^{2H}\Delta \to 0$, or (iii) $H \in (0, 1/2)$ and $T^{2H+1}\Delta \to 0$,, then we have

$$(\widetilde{U}_T, \widetilde{X}_T, \Xi_T) \xrightarrow{\mathcal{L}} \left(\sigma \frac{\sqrt{H\Gamma(2H)}}{\theta^H} \nu, X_0 + \frac{\mu}{\theta} + \sigma \frac{\sqrt{H\Gamma(2H)}}{\theta^H} \omega, \eta \right), \tag{A.50}$$

where ν, ω and η are independent standard normal variables, and $\widetilde{X}_\infty = X_0 + \frac{\mu}{\theta} + \sigma \int_0^\infty e^{-\theta s} dB_s^H$.

Proof. (i). We first consider the limiting distribution for $\widetilde{U}_T$. Since $\widetilde{U}_T$ is a Gaussian process, for any n and $\Delta > 0$, we have $\widetilde{U}_T \stackrel{d}{=} \sigma_{\widetilde{U}_T} \mathcal{N}(0, 1)$, where $\sigma_{\widetilde{U}_T}$ denotes the standard deviation of $\widetilde{U}_T$. Thus, it is sufficient to show as $\Delta \to 0,\ T \to \infty$

$$\sigma^2_{\widetilde{U}_T} = \sigma^2 \sum_{i=1}^n \sum_{j=1}^n e^{-2\theta T} e^{2\theta(i+j)\Delta} \mathbb{E}\left(\int_{(i-1)\Delta}^{i\Delta} e^{-\theta s} dB_s^H \int_{(j-1)\Delta}^{j\Delta} e^{-\theta s} dB_s^H \right)$$
$$\to \sigma \frac{\sqrt{H\Gamma(2H)}}{\theta^H}.$$

Case 1: $H \in (0, 1/2)$. Using Lemma A.2, as $\Delta \to 0,\ T \to \infty$, we get

$$\begin{aligned}\sigma^2_{\widetilde{U}_T} &= \sigma^2 \sum_{i=1}^{n} \sum_{j=1}^{n} e^{-2\theta T} e^{2\theta(i+j)\Delta} \mathbb{E}\left(\int_{(i-1)\Delta}^{i\Delta} e^{-\theta s} dB_s^H\right)\left(\int_{(j-1)\Delta}^{j\Delta} e^{-\theta s} dB_s^H\right) \\ &= \sigma^2 \sum_{i=1}^{n} \sum_{j=1}^{n} e^{-2\theta T} e^{2\theta(i+j)\Delta} \Bigg(\theta \int_{(j-1)\Delta}^{j\Delta} \int_{(i-1)\Delta}^{i\Delta} e^{-\theta t} e^{-\theta s} \frac{\partial R(t,s)}{\partial t} dt ds \\ &\quad + \int_{(i-1)\Delta}^{i\Delta} e^{-\theta t}\left(e^{-\theta j\Delta} \frac{\partial R(t, j\Delta)}{\partial t} - e^{-\theta(j-1)\Delta} \frac{\partial R(t,(j-1)\Delta)}{\partial t}\right) dt\Bigg) \\ &\to \sigma^2 \frac{H\Gamma(2H)}{\theta^{2H}}.\end{aligned}$$

The first equation is from (8) in Chen and Li [15], which is a modification of (2.3) in Hu et al. [22]; see also Hu et al. [24].

Case 2: $H = 1/2$. As $\Delta \to 0,\ T \to \infty$,

$$\sigma^2_{\widetilde{U}_T} = \sigma^2 \sum_{i=1}^{n} e^{-2\theta T} e^{4\theta i\Delta} \int_{(i-1)\Delta}^{i\Delta} e^{-2\theta s} ds = \frac{\sigma^2}{2\theta} e^{2\theta\Delta}\left(1 - e^{2\theta T}\right) \to \frac{\sigma^2}{2\theta}.$$

Case 3: $H \in (1/2, 1)$. Using Lemma A.1, we can get as $\Delta \to 0,\ T \to \infty$,

$$\begin{aligned}\sigma^2_{\widetilde{U}_T} &= \sigma^2 \alpha_H \sum_{i=1}^{n} \sum_{j=1}^{n} e^{-2\theta T} e^{2\theta i\Delta} e^{2\theta j\Delta} \int_{(i-1)\Delta}^{i\Delta} \int_{(j-1)\Delta}^{j\Delta} e^{-\theta t} e^{-\theta s} |t-s|^{2H-2} dt ds \\ &\to \sigma^2 \frac{H\Gamma(2H)}{\theta^{2H}}.\end{aligned}$$

Therefore, we have $\widetilde{U}_T \xrightarrow{\mathcal{L}} \sigma \frac{\sqrt{H\Gamma(2H)}}{\theta^H} \nu$, as $\Delta \to 0,\ T \to \infty$.

(ii). Now, we consider the limiting distribution for $\widetilde{X}_T$. Recall that

$$\widetilde{X}_\infty = X_0 + \frac{\mu}{\theta} + \sigma \int_0^\infty e^{-\theta s} dB_s^H.$$

Then, using (A.23), we have

$$\begin{aligned}\mathbb{E}|\widetilde{X}_T - \widetilde{X}_\infty| &= \mathbb{E}\left|-\frac{\mu}{\theta} e^{-\theta T} - \sigma \int_T^\infty e^{-\theta s} dB_s^H\right| \\ &\le \frac{|\mu|}{\theta} e^{-\theta T} + \sigma\left(\mathbb{E}\left(\int_T^\infty e^{-\theta s} dB_s^H\right)^2\right)^{1/2}.\end{aligned}$$

Case 1: $H \in (0, 1/2)$. Straightforward calculations lead to

$$\begin{aligned}&\mathbb{E}|\widetilde{X}_T - \widetilde{X}_\infty| \\ &\le \frac{|\mu|}{\theta} e^{-\theta T} + \sigma\left(\theta \int_T^\infty \int_T^\infty e^{-\theta(t+s)} \frac{\partial R(t,s)}{\partial t} dt ds + e^{-\theta T} \int_T^\infty e^{-\theta t} \frac{\partial R(t,s)}{\partial t} dt\right)^{1/2}\end{aligned}$$

$$= \frac{|\mu|}{\theta}e^{-\theta T} + \sigma\big(He^{-2\theta T}\int_0^\infty e^{-\theta u}u^{2H-1}du\big)^{1/2}$$
$$= \frac{|\mu|}{\theta}e^{-\theta T} + \sigma\left(\frac{H\Gamma(2H)}{\theta^{2H}}e^{-2\theta T}\right)^{1/2} \leq Ce^{-\theta T}.$$

Case 2: $H = 1/2$. Using the Itô isometry formula, we obtain

$$\mathbb{E}|\widetilde{X}_T - \widetilde{X}_\infty| \leq \frac{|\mu|}{\theta}e^{-\theta T} + \sigma\frac{1}{\sqrt{2\theta}}e^{-\theta T} \leq Ce^{-\theta T}.$$

Case 3: $H \in (1/2, 1)$. It follows that

$$\begin{aligned}\mathbb{E}|\widetilde{X}_T - \widetilde{X}_\infty| &\leq \frac{|\mu|}{\theta}e^{-\theta T} + \sigma\left(\alpha_H\int_T^\infty\int_T^\infty e^{-\theta(s+r)}|s-r|^{2H-2}dsdr\right)^{1/2}\\ &= \frac{|\mu|}{\theta}e^{-\theta T} + \sigma\left(2\alpha_H\int_T^\infty\int_0^{r-T} e^{-2\theta r}e^{\theta u}u^{2H-2}dudr\right)^{1/2}\\ &= \frac{|\mu|}{\theta}e^{-\theta T} + \sigma\left(2\alpha_H e^{-2\theta T}\int_0^\infty\int_0^{v} e^{-2\theta v}e^{\theta u}u^{2H-2}dudv\right)^{1/2}\\ &= \frac{|\mu|}{\theta}e^{-\theta T} + \sigma\left(\frac{H\Gamma(2H)}{\theta^{2H}}e^{-2\theta T}\right)^{1/2} \leq Ce^{-\theta T}.\end{aligned}$$

Hence, for $H \in (0,1)$, it holds that $\mathbb{E}|\widetilde{X}_T - \widetilde{X}_\infty| \leq Ce^{-\theta T}$, and then we have $\mathbb{P}(|\widetilde{X}_T - \widetilde{X}_\infty| \geq \epsilon) \leq C\epsilon^{-1}e^{-\theta T}$. Consequently, by the Borel-Cantelli lemma, we have $\widetilde{X}_T \overset{a.s.}{\to} \widetilde{X}_\infty$.

(iii). In this part, we prove $\Xi_T = T^{-H}\sum_{i=1}^n \epsilon_{i\Delta} \overset{\mathcal{L}}{\to} \mathcal{N}(0,1)$. In fact, using the definition of $\epsilon_{i\Delta}$, we can easily obtain $\sum_{i=1}^n \epsilon_{i\Delta} \overset{d}{=} \sigma_{\sum_{i=1}^n \epsilon_{i\Delta}}\mathcal{N}(0,1)$. Here $\sigma_{\sum_{i=1}^n \epsilon_{i\Delta}}$ denotes the standard deviation of $\sum_{i=1}^n \epsilon_{i\Delta}$, which will be calculated as follows.

Case 1: $H \in (0, 1/2)$. By (A.13) in Lemma A.4, we have

$$\begin{aligned}\sigma^2_{\sum_{i=1}^n \epsilon_{i\Delta}} &= \mathbb{E}\left(\sum_{i=1}^n e^{\theta i\Delta}\int_{(i-1)\Delta}^{i\Delta} e^{-\theta s}dB_s^H\right)^2\\ &= \sum_{i=1}^n\sum_{j=1}^n e^{\theta(i+j)\Delta}\left(\theta\int_{(j-1)\Delta}^{j\Delta}\int_{(i-1)\Delta}^{i\Delta} e^{-\theta t}e^{-\theta s}\frac{\partial R(t,s)}{\partial t}dtds\right.\\ &\quad \left.+\int_{(i-1)\Delta}^{i\Delta} e^{-\theta t}\left(e^{-\theta j\Delta}\frac{\partial R(t,j\Delta)}{\partial t} - e^{-\theta(j-1)\Delta}\frac{\partial R(t,(j-1)\Delta)}{\partial t}\right)dt\right)\\ &\sim T^{2H} + o\big(T^{2H}\big).\end{aligned}$$

Case 2: $H = 1/2$. Using the Itô isometry formula, we can see that

$$\sigma^2_{\sum_{i=1}^n \epsilon_{i\Delta}} = \sum_{i=1}^n e^{2\theta i\Delta}\mathbb{E}\left(\int_{(i-1)\Delta}^{i\Delta} e^{-\theta s}dW_s\right)^2 = \frac{1}{2\theta}\sum_{i=1}^n\big(e^{2\theta\Delta} - 1\big) \sim T.$$

Case 3: $H \in (1/2, 1)$. Using (A.9) in Lemma A.3, we have

$$\sigma^2_{\sum_{i=1}^n \epsilon_{i\Delta}} = \mathbb{E}\left(\sum_{i=1}^n e^{\theta i\Delta} \int_{(i-1)\Delta}^{i\Delta} e^{-\theta s} dB_s^H\right)^2$$
$$= \alpha_H \sum_{i=1}^n \sum_{j=1}^n e^{\theta(i+j)\Delta} \int_{(i-1)\Delta}^{i\Delta} \int_{(j-1)\Delta}^{j\Delta} e^{-\theta(s+r)} |s-r|^{2H-2} ds dr \sim T^{2H}.$$

Therefore, we have $T^{-H}\sigma_{\sum_{i=1}^n \epsilon_{i\Delta}} \to 1$, which implies $\Xi_T \xrightarrow{\mathcal{L}} \mathcal{N}(0,1)$.

(iv). Finally, we show (A.50). In fact, it holds $(\widetilde{U}_T, \widetilde{X}_T - X_0, \Xi_T)^\top \overset{d}{=} \mathcal{N}(\overrightarrow{a}^*, B^*)$, where $^\top$ denotes the vector transposition. Here, $\overrightarrow{a}^*$ and B^* have the following representation $\overrightarrow{a}^* = (\mathbb{E}\widetilde{U}_T, \mathbb{E}(\widetilde{X}_T - X_0), \mathbb{E}\Xi_T)^\top$ and

$$B^* = \begin{pmatrix} \sigma^2_{\widetilde{U}_T} & Cov(\widetilde{U}_T, \widetilde{X}_T) & Cov(\widetilde{U}_T, \Xi_T) \\ Cov(\widetilde{X}_T, \widetilde{U}_T) & \sigma^2_{\widetilde{X}_T} & Cov(\widetilde{X}_T, \Xi_T) \\ Cov(\Xi_T, \widetilde{U}_T) & Cov(\Xi_T, \widetilde{X}_T) & \sigma^2_{\Xi_T} \end{pmatrix}.$$

Notice that $\mathbb{E}\Xi_T = 0$, $\mathbb{E}\widetilde{U}_T = 0$ and $\mathbb{E}(\widetilde{X}_T - X_0) \to \frac{\mu}{\theta}$. Then B^* can written as

$$B^* = \begin{pmatrix} \sigma^2_{\widetilde{U}_T} & \mathbb{E}(\widetilde{U}_T \widetilde{X}_T) & \mathbb{E}(\widetilde{U}_T \Xi_T) \\ \mathbb{E}(\widetilde{X}_T \widetilde{U}_T) & \sigma^2_{\widetilde{X}_T} & \mathbb{E}(\widetilde{X}_T \Xi_T) \\ \mathbb{E}(\Xi_T \widetilde{U}_T) & \mathbb{E}(\Xi_T \widetilde{X}_T) & \sigma^2_{\Xi_T} \end{pmatrix}.$$

By (A.49) and Lemmas A.5–A.7, we can obtain (A.50). The proof of this proposition is completed. □

Finally, in Proposition A.3, we can see the remainders R_{1n}, R_{2n}, R_{3n} and R_{4n} are negligible in the asymptotic analysis.

Propostion A.3. *Let R_{1n}, R_{2n}, R_{3n} and R_{4n} be defined by (A.29)–(A.32), respectively. Then, as $\Delta \to 0$ and $\frac{\log \Delta}{T} \to 0$, we have $R_{1n} \xrightarrow{a.s.} 0$, $R_{2n} \xrightarrow{a.s.} 0$, $R_{3n} \xrightarrow{\mathcal{L}} 0$, $R_{4n} \xrightarrow{a.s.} 0$.*

In particular, as $\Delta \to 0$ and $\frac{(\log \Delta)^3}{T^2} \to 0$, we get $T^{H-1}R_{3n} \xrightarrow{a.s.} 0$.

For the sake of the length and readability of this paper, we delegate it to the supplementary material [28].

A.3. Proofs of Theorems 3.2–3.4

Proof of Theorem 3.2. (i). We first prove the strong consistency of $\widehat{\theta}_\Delta$. By Proposition A.1, we have

$$\frac{e^{-\theta\Delta}}{\Delta}\frac{U_T}{V_T} = \frac{e^{-\theta\Delta}\Delta^{-1}(1-e^{-2\theta\Delta})e^{-2\theta T}U_T}{(1-e^{-2\theta\Delta})e^{-2\theta T}V_T}$$

$$= \frac{\Delta^{-1}(1-e^{-2\theta\Delta})e^{-2\theta\Delta}e^{-\theta T}\widetilde{U}_T\widetilde{X}_T + \Delta^{-1}(1-e^{-2\theta\Delta})e^{-\theta T}R_{1n}}{e^{-2\theta\Delta}\widetilde{X}^2_{(n-1)\Delta} - R_{2n}}.$$

Using the fact $\mathbb{E}|e^{-\theta T}\widetilde{U}_T\widetilde{X}_T| \leq e^{-\theta T}(\mathbb{E}\widetilde{U}_T^2)^{1/2}(\mathbb{E}\widetilde{X}_T^2)^{1/2} \leq Ce^{-\theta T}$, and the Borel-Cantelli lemma, we obtain that

$$e^{-\theta T}\widetilde{U}_T\widetilde{X}_T \xrightarrow{a.s.} 0. \tag{A.51}$$

Moreover, Proposition A.3 gives that

$$e^{-\theta T}R_{1n} \xrightarrow{a.s.} 0, \quad R_{2n} \xrightarrow{a.s.} 0, \quad e^{-2\theta\Delta}\widetilde{X}^2_{(n-1)\Delta} \xrightarrow{a.s.} \widetilde{X}^2_\infty. \tag{A.52}$$

From (A.51) and (A.52), it follows that $\frac{e^{-\theta\Delta}}{\Delta}\frac{U_T}{V_T} \xrightarrow{a.s.} 0$, which together with (A.16) implies $\widehat{\theta}_\Delta - \theta \xrightarrow{a.s.} 0$.

(ii). Now, we turn to prove the asymptotic distribution of $\widehat{\theta}_\Delta$, that is, (3.12). By (A.16), Proposition A.1, Proposition A.3, we can get

$$\begin{aligned}
\frac{\Delta e^{\theta T}}{1-e^{-2\theta\Delta}}(\widehat{\theta}_\Delta - \theta) &= \frac{e^{\theta T}}{1-e^{-2\theta\Delta}}\log\left(1 + e^{-\theta\Delta}\frac{U_T}{V_T}\right) \\
&= \frac{e^{-\theta(n+1)\Delta}U_T}{(1-e^{-2\theta\Delta})e^{-2\theta T}V_T}\big(1+o(1)\big) \\
&= \frac{e^{-2\theta\Delta}\widetilde{U}_T\widetilde{X}_T + R_{1n}}{e^{-2\theta\Delta}\widetilde{X}^2_{(n-1)\Delta} - R_{2n}}\big(1+o(1)\big) = \frac{\widetilde{U}_T\widetilde{X}_T}{\widetilde{X}^2_{(n-1)\Delta}}\big(1+o_p(1)\big),
\end{aligned} \tag{A.53}$$

where U_T and V_T are defined by (A.18) and (A.19), respectively. Consequently, from Propositions A.2, it follows that

$$\frac{e^{\theta T}}{2\theta}(\widehat{\theta}_\Delta - \theta) \xrightarrow{\mathcal{L}} \frac{\sigma\frac{\sqrt{H\Gamma(2H)}}{\theta^H}\nu}{X_0 + \frac{\mu}{\theta} + \sigma\frac{\sqrt{H\Gamma(2H)}}{\theta^H}\omega},$$

which completes the proof of this theorem. □

Proof of Theorem 3.3. Note that $\widehat{\theta}_\Delta \xrightarrow{a.s.} \theta$ and

$$\widehat{\mu}_\Delta = \mu + (\widehat{\theta}_\Delta - \theta)\left(\frac{\mu}{\theta} + \frac{M_T}{N_T}\right) + \theta\cdot\frac{M_T}{N_T}. \tag{A.54}$$

To verify the strong consistency of $\widehat{\mu}_\Delta$, it is sufficient to show $\frac{M_T}{N_T} \xrightarrow{a.s.} 0$, as $\Delta \to 0$ and $T \to \infty$.

In fact, using (A.28), Proposition A.2 and Proposition A.3, we have

$$\big(n(e^{\theta\Delta}-1)\big)^{-1}\big(1-e^{-2\theta\Delta}\big)e^{-2\theta T}N_T \xrightarrow{a.s.} \widetilde{X}^2_\infty, \quad \Delta \to 0, T \to \infty. \tag{A.55}$$

Moreover, using (iii) in the proof of Proposition A.2, we obtain $T^{H-1}\Xi_T \sim \mathcal{N}(0, T^{-2} + o(T^{-2}))$. Then, Borel-Cantelli lemma gives that $T^{H-1}\Xi_T \xrightarrow{a.s.} 0$, as $\Delta \to 0, T \to \infty$. Therefore, under the condition $\Delta \to 0$ and $\frac{T^2}{(\log n)^3} \to \infty$, by (A.27), Proposition A.2 and Proposition A.3, we obtain

$$\big(n(e^{\theta\Delta} - 1)\big)^{-1}(1 - e^{-2\theta\Delta})e^{-2\theta T} M_T \overset{a.s.}{\to} 0,$$

which together with (A.55) implies $\frac{M_T}{N_T} \overset{a.s.}{\to} 0$. □

Proof of Theorem 3.4. Using (A.53) and (A.54), we can write

$$\begin{aligned}
\frac{e^{\theta T}}{2\theta}(\widehat{\theta}_\Delta - \theta) &= \frac{\widetilde{U}_T \widetilde{X}_T}{\widetilde{X}^2_{(n-1)\Delta}} + o_p(1), \\
T^{1-H}(\widehat{\mu}_\Delta - \mu) &= \frac{e^{\theta T}}{2\theta}(\widehat{\theta}_\Delta - \theta)\left(2\mu e^{-\theta T} T^{1-H} + 2e^{-\theta T}\frac{T^{-H} M_T}{(\theta n \Delta)^{-1} N_T}\right) \\
&\quad + \frac{T^{-H} M_T}{(\theta n \Delta)^{-1} N_T}.
\end{aligned} \tag{A.56}$$

Together with Proposition A.1 and Proposition A.3, we can obtain

$$T^{1-H}(\widehat{\mu}_\Delta - \mu) = \sigma \Xi_T + o_p(1). \tag{A.57}$$

From (A.56) and (A.57), we have

$$\left(\frac{e^{\theta T}}{2\theta}(\widehat{\theta}_\Delta - \theta), T^{1-H}(\widehat{\mu}_\Delta - \mu)\right) = \left(\frac{\widetilde{U}_T \widetilde{X}_T}{\widetilde{X}^2_{(n-1)\Delta}}, \sigma \Xi_T\right) + o_p(1).$$

Using Proposition A.2, we can complete the proof of this theorem. □

Acknowledgments

We would like to express our greatest gratitude to AE and the anonymous reviewer for the careful reading and insightful comments, which surely lead to an improved presentation of this paper.

Funding

Hui Jiang is supported by the National Natural Science Foundation of China (No. 12471146) and Natural Science Foundation of Jiangsu Province of China (No. 20231435). Yajuan Pan is supported by Funding for Outstanding Doctoral Dissertation in NUAA (No. BCXJ23-12). Weilin Xiao is supported by the National Natural Science Foundation of China (No. 71871202), Humanities and Social Sciences of Ministry of Education Planning Fund of China (No. 23YJA630102), the Yangtze River Delta Technology Innovation Community Joint Tackling Plan (No. 2022CSJGG0800) and the National Social Science Fund of China (No. 22AZD039). Qingshan Yang is supported by the National Natural Science Foundation of China (No. 11401090, 11971097, 11971098).

Supplementary Material

Supplement to "Asymptotic Theory for Explosive Fractional Ornstein-Uhlenbeck Processes"
(doi: 10.1214/24-EJS2293SUPP; .pdf). The supplement contains the proofs of Lemmas A.1–A.4 and Proposition A.3.

References

[1] Altmeyer, R., Cialenco, I. and Paseman, G. (2023). Parameter estimation for semilinear SPDEs from local measurements. *Bernoulli*, **29(3)** 2035–2061. MR4580906

[2] Arvanitis, S. and Magdalinos, T. (2018). Mildly explosive autoregression under stationary conditional heteroskedasticity. *Journal of Time Series Analysis*, **39(6)** 892–908. MR3867129

[3] Astill, S., Harvey, D. I., Leybourne, S. J., Sollis, R. and Taylor, A.M.R. (2018). Real-time monitoring for explosive financial bubbles. *Journal of Time Series Analysis*, **39(6)** 863–891. MR3867128

[4] Azmoodeh, E. and Viitasaari, L. (2015). Parameter estimation based on discrete observations of fractional Ornstein-Uhlenbeck process of the second kind. *Statistical Inference for Stochastic Processes*, **18** 205–227. MR3395605

[5] Barboza, L. A. and Viens, F. G. (2017). Parameter estimation of Gaussian stationary processes using the generalized method of moments. *Electronic Journal of Statistics*, **11(1)** 401–439. MR3611508

[6] Belfadli, R., Es-sebaiy, K. and Ouknine, Y. (2011). Parameter estimation for fractional Ornstein-Uhlenbeck processes: Non-ergodic case. *Frontiers in Science and Engineering*, **1** 1–16.

[7] Bennedsen, M., Lunde, A. and Pakkanen, M. S. (2022). Decoupling the short-and long-term behavior of stochastic volatility. *Journal of Financial Econometrics*, **20(5)** 961–1006.

[8] Bolko, A. E., Christensen, K., Pakkanen, M. S. and Veliyev, B. (2023). A GMM approach to estimate the roughness of stochastic volatility. *Journal of Econometrics*, **235(2)** 745–778. MR4602890

[9] Brouste, A. and Fukasawa, M. (2018). Local asymptotic normality property for fractional Gaussian noise under high-frequency observations. *Annals of Statistics*, **46(5)** 2045–2061. MR3845010

[10] Brouste, A. and Iacus, S. M. (2013). Parameter estimation for the discretely observed fractional Ornstein-Uhlenbeck process and the Yuima R package. *Computational Statistics*, **28** 1529–1547. MR3120827

[11] Cialencl, I. and Kim, H. J. (2022). Parameter estimation for discretely sampled stochastic heat equation driven by space-only noise. *Stochastic Processes and their Applications*, **143** 1–30. MR4332773

[12] Chen, Y., Phillips, P. C. B. and Yu, J. (2017). Inference in continuous systems with mildly explosive regressors. *Journal of Econometrics*, **201(2)** 400–416. MR3717572

[13] Chen, Y., Hu, Y. Z. and Wang, Z. (2017). Parameter estimation of complex fractional Ornstein-Uhlenbeck processes with fractional noise. *ALEA*, **14(1)** 613–629. MR3680192

[14] Cheng, Y. Y., Hu, Y. Z. and Long, H. W. (2020). Generalized moment estimators for α-stable Ornstein-Uhlenbeck motions from discrete observations. *Statistical Inference for Stochastic Processes*, **23** 53–81. MR4072252

[15] Chen, Y. and Li, Y. (2021). Berry-Esséen bound for the parameter estimation of fractional Ornstein-Uhlenbeck processes with the Hurst parameter $H \in (0, 1/2)$. *Communications in Statistics-Theory and Methods*, **50(12)** 2996–3013. MR4275508

[16] Cheridito, P., Kawaguchi, H. and Maejima, M. (2003). Fractional Ornstein-Uhlenbeck processes. *Electronic Journal of Probability*, **8(3)** 1–14. MR1961165

[17] El Onsy, B., Es-Sebaiy, K. and Ndiaye, D. (2018). Parameter estimation for discretely observed non-ergodic fractional Ornstein-Uhlenbeck processes of the second kind. *Brazilian Journal of Probability and Statistics*, **32(3)** 545–558. MR3812381

[18] Gatheral, J., Jaisson, T. and Rosenbaum, M. (2018). Volatility is rough. *Quantitative Finance*, **18(6)** 933–949. MR3805308

[19] Haress, E. M. and Hu, Y. Z. (2021). Estimation of all parameters in the fractional Ornstein-Uhlenbeck model under discrete observations. *Statistical Inference for Stochastic Processes*, **24** 327–351. MR4265053

[20] Harvey, D. I., Leybourne, S. J. and Sollis, R. (2017). Improving the accuracy of asset price bubble start and end date estimators. *Journal of Empirical Finance*, **40** 121–138.

[21] Harvey, D. I., Leybourne, S. J., Sollis, R. and Taylor, A. M. R. (2016). Tests for explosive financial bubbles in the presence of non-stationary volatility. *Journal of Empirical Finance*, **38** 548–574

[22] Hu, Y. Z. and Nualart, D. (2010). Parameter estimation for fractional Ornstein-Uhlenbeck processes. *Statistics and Probability Letters*, **80** 1030–1038. MR2638974

[23] Hu, Y. Z. and Song, J. (2013). Parameter estimation for fractional Ornstein-Uhlenbeck processes with discrete observations. In: Viens F., Feng J., Hu Y., Nualart E. (eds.) *Malliavin Calculus and Stochastic Analysis*, Springer Proceedings in Mathematics and Statistics, vol. 34. Springer, Boston, 427–442. MR3070455

[24] Hu, Y. Z., Nualart, D. and Zhou, H. J. (2019). Parameter estimation for fractional Ornstein-Uhlenbeck processes of general Hurst parameter. *Statistical Inference for Stochastic Processes*, **22** 111–142. MR3918739

[25] Jacod, J. (2006). Parametric inference for discretely observed non-ergodic diffusions. *Bernoulli*, **12(3)** 383–401. MR2232724

[26] Jiang, H., Li, S. M. and Wang, W. G. (2024). Moderate deviations for parameter estimation in the fractional Ornstein-Uhlenbeck processes with periodic mean. *Acta Mathematica Sinica, English Series*, **40** 1308–1324. MR4746061

[27] Jiang, H., Liu, H. and Zhou, Y. Z. (2020). Asymptotic properties for

the parameter estimation in Ornstein-Uhlenbeck process with discrete observations. *Electronic Journal of Statistics*, **14(2)** 3192–3229. MR4146360

[28] Jiang, H., Pan, Y. J., Xiao, W. L., Yang, Q. S. and Yu, J. (2023). Supplement to "Asymptotic theory for explosive fractional Ornstein-Uhlenbeck processes."

[29] Kleptsyna, M. L. and Le Breton, A. (2002). Statistical analysis of the fractional Ornstein-Uhlenbeck type process. *Statistical Inference for Stochastic Processes*, **5(3)** 229–248. MR1943832

[30] Kubilius, K., Mishura, Y., Ralchenko, K. and Seleznjev, O. (2015). Consistency of the drift parameter estimator for the discretized fractional Ornstein-Uhlenbeck process with Hurst index $H \in (0, \frac{1}{2})$. *Electronic Journal of Statistics*, **9(2)** 1799–1825. MR3391120

[31] Lui, Y. L., Phillips, P. C. B. and Yu, J. (2024). Robust testing for explosive behavior with strongly dependent errors. *Journal of Econometrics*, **238(2)** 105626. MR4674010

[32] Lui, Y. L., Xiao, W. L. and Yu, J. (2021). Mildly explosive autoregression with anti-persistent errors. *Oxford Bulletin of Economics and Statistics*, **83(2)** 518–539.

[33] Magdalinos, T. (2012). Mildly explosive autoregression under weak and strong dependence. *Journal of Econometrics*, **169(2)** 179–187. MR2947876

[34] Mandelbrot, B. B. and van Ness, J. W. (1968). Fractional Brownian motions, fractional noises and applications. *SIAM Review*, **10(4)** 422–437. MR0242239

[35] Phillips, P. C. B. and Magdalinos, T. (2007a). Limit theory for moderate deviations from a unit root. *Journal of Econometrics*, **136(1)** 115–130. MR2328587

[36] Phillips, P. C. B. and Magdalinos, T. (2007b). Limit theory for moderate deviations from a unit root under weak dependence. In: Phillips, Garry D. A. and Tzavalis, Elias (eds.) *The Refinement of Econometric Estimation and Test Procedures: Finite Sample and Asymptotic Analysis*. Cambridge University Press, pp. 123–162.

[37] Phillips, P. C. B., Shi, S. P. and Yu, J. (2015a). Testing for multiple bubbles: historical episodes of exuberance and collapse in the S&P 500. *International Economic Review*, **56(4)** 1043–1078.

[38] Phillips, P. C. B., Shi, S. P. and Yu, J. (2015b). Testing for multiple bubbles: limit theory of real time detector. *International Economic Review*, **56(4)** 1079–1134. MR3424282

[39] Phillips, P. C. B., Wu, Y. R. and Yu, J. (2011). Explosive behavior in the 1990s Nasdaq: When did exuberance escalate asset values? *International Economic Review*, **52(1)** 201–226.

[40] Phillips, P. C. B. and Yu, J. (2011). Dating the timeline of financial bubbles during the subprime crisis. *Quantitative Economics*, **2(3)** 455–491. MR2947140

[41] Shi, S. P. and Yu, J. (2023). Volatility puzzle: long memory or antipersistency. *Management Science*, **69(7)** 3861–3883.

[42] Shimizu, Y. (2009). Notes on drift estimation for certain non-recurrent

diffusion processes from sampled data. *Statistics and Probability Letters*, **79(20)**, 2200–2207. MR2572052

[43] Shimizu, Y. (2012). Local asymptotic mixed normality for discretely observed non-recurrent Ornstein-Uhlenbeck processes. *Annals of the Institute of Statistical Mathematics*, **64**, 193–211. MR2878923

[44] Tanaka, K., Xiao, W. L. and Yu, J. (2020). Maximum likelihood estimation for the fractional Vasicek model. *Econometrics*, **8(3)** 32. MR4343184

[45] Tran N. K. and Ngo, H. L. (2022). LAMN property for multivariate inhomogeneous diffusions with discrete observations. *Electronic Journal of Statistics*, **16(2)** 4275–4331. MR4474575

[46] Tudor, C. A. and Viens, F. G. (2007). Statistical aspects of the fractional stochastic calculus. *Annals of Statistics*, **35(3)** 1183–1212. MR2341703

[47] Wang, X. H., Xiao, W. L. and Yu, J. (2023). Modeling and forecasting realized volatility with the fractional Ornstein-Uhlenbeck process. *Journal of Econometrics*, **232(2)** 389–415 MR4539492

[48] Wang, X. H. and Yu, J. (2016). Double asymptotics for explosive continuous time models. *Journal of Econometrics*, **193(1)** 35–53. MR3500175

[49] Xiao, W. L. and Yu, J. (2019). Asymptotic theory for estimating drift parameters in the fractional Vasicek model. *Econometric Theory*, **35** 198–231. MR3904176

[50] Young, L. C. (1936). An inequality of the Hölder type, connected with Stieltjes integration. *Acta Mathematica*, **67** 251–282. MR1555421