

Adaptive randomization in network data*

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Abstract: Network data have become increasingly prevalent in recent research. For instance, network models have been proposed to improve the quality of estimation and hypothesis testing when comparing the effects of different types of treatment. In this paper, we focus on efficiently estimating the average treatment effect using an adaptive randomization procedure in networks. We are working on models of causal frameworks in which the outcome of a subject's treatment is influenced not only by their own characteristics but also by the characteristics of their neighbors. The experiment's designer does not observe these characteristics. We also consider situations where only the existing subjects' network is revealed when assigning treatments to current subjects. Balancing the connections in the network among different treatment groups can lessen the impact of unknown characteristics. In network data, it is often difficult to obtain theoretical properties because the numbers of nodes and connections increase simultaneously. Under mild assumptions, our proposed procedure is closely related to a time-varying inhomogeneous Markov chain. We utilize the technique of Lyapunov functions to control the moments of the imbalance measure sequence in our proposed approach. As a result, our theoretical analysis demonstrates that our procedure is capable of decreasing the variability of the observed outcome in the model. The advantages of the proposed procedures are also demonstrated by extensive simulations and experiments on real network data.

Keywords and phrases: Adaptive randomization design, Lamperti process, time-varying Markov process, Lyapunov function, network-correlated outcomes.

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1. Introduction

Evaluation of the effects of different types of treatment is gaining significant attention in social media development, online advertising and clinical testing. The outcome for each subject may depend not only on the treatment allocation, but also the subjects' covariates and the connections between subjects. Random treatment assignment methods often generate unbalanced prognostic factors. In the situation where the covariates are the observed categorical or numerical variables in fixed dimensions, sequential treatment assignment is introduced in [37] to address the issue of unbalancedness. In [47], the author generalizes the idea of sequential design by proposing a marginal urn model. Adaptive randomization methods are studied in [13, 14], and show promising performance in categorical covariate balance with theoretical guarantees. Pairwise sequential randomization is investigated in [39] to reduce the Mahalanobis distance of continuous variables.

In the past decade, the presence of networks in social media, clinical tests and biological experiments has received attention from statisticians [46, 6, 7, 5]. In causal inference studies, the behavior of one individual may be correlated with the behaviors of other individuals, namely peer effects or social interaction [28, 1, 9]. In online social media networks, the behavior of a given user may be similar to his or her friends, as they might share correlated factors. Hence, in causal inference and clinical studies, we assume that if two subjects are connected in the network, then their hidden covariates affect each other's outcomes. To be more precise, we consider *network-correlated outcomes*, where the network informs the correlations among potential outcomes because the potential outcomes of subject i depend on both its own covariates and those of its neighbors in the network [29, 4]. Furthermore, we assume the potential outcome of a certain subject is not affected by the assignment of treatments to other subjects [8]. That is, there is no interference between subjects [2]. In addition, we consider another realistic assumption, which is similar to that proposed in [47]: we assume subjects appear singly and must be treated immediately. In other words, when we decide the treatment assigned to the current subject, only the connections between this subject and the previous subjects are observed; we only observe the sub-adjacency matrix for those subjects observed in the current stage. Rerandomization is proposed in [31] and generalized to network data by [4]; however, their approach requires the whole network to be revealed before deciding the treatment of the first subject. To resolve this issue, here, we generalize adaptive design methods [13, 14] to decide treatment allocation sequentially. The utilization of adaptive design methods in network models is a novel contribution presented in this paper. Prior to the publication of this paper, little attention had been given to this particular approach in the context of network models. However, it is worth noting that during the publication process of this paper, some related works have emerged in the field, such as [24, 45], which have also explored the application of adaptive design methods in similar contexts. Moreover, the performance analysis of the existing adaptive randomization method cannot be applied to the model considered in this paper.

Assuming the observations are network-correlated and sequentially obtained, this paper focuses on improving the estimation of treatment effects by reducing the *imbalance measurement*. We still aim to reduce the effect of prognostic factors by the pairwise sequential randomization method proposed in [39]. Under the assumption of network-correlated outcomes, and supposing the network is observed sequentially, we first derive the formula for the variance of treatment effects under certain statistical assumptions, then we show that our approach reduces the imbalance measurement empirically and theoretically under some reasonable assumptions on the network. Despite the popularity of the model in [29, 4], no previous work has analytically evaluated the variance of the estimator in this model with mathematical verification. To the best of our knowledge, this paper is the first work to provide a theoretical verification for the performance of randomization procedures on models assuming network-correlated outcomes.

In the literature, it is assumed that covariates are identically and independently distributed (i.i.d.), and the number of covariates is fixed, hence turning the imbalance measurement of the adaptive randomization procedure into a Markov process. It is shown in [13] that the Markov process is recurrent when the covariates are categorical variables. To formulate a theoretical analysis of the proposed procedure of this paper, we assume the observed network follows the Erdős-Rényi random graph model. The analysis does not follow from previous work on adaptive design, in the following sense. As we observe a network with extra nodes, the number of possible neighbors of each individual increases simultaneously. Moreover, because the Erdős-Rényi random graph is a probabilistic model for undirected graphs, the entries of the adjacency matrix are not independent. To overcome these difficulties, we analyze this stochastic process as a Lamperti problem [18] and further derive the upper bound of the expectation of imbalance measurement by computing certain Lyapunov functions [30]. We will explain the connection between our analysis method with Lyapunov functions in Section 3. In our model, as more and more subjects join the experiments, the dimension of states changes over time progresses. Thus, this process can be approximated as a time-varying Markov process. The generalization from fixed dimension to increasing dimension is a novel extension in Markov models.

This article is organized as follows. We introduce the network-correlated outcome model and our proposed procedure in Section 2. Theoretical properties under the Erdős-Rényi random graph model are presented in Section 3. In Section 4, we discuss the theoretical properties that arise when we replace the random graph model with a Gaussian orthogonal ensemble. Experiments on simulated and real network data are presented in Section 5. We conclude in Section 6, where possible future works are also discussed. Proofs of the main theorems and auxiliary lemmas appear in Appendix A.

Here, we briefly introduce the notation used in this paper. X^n is the set of vectors with entries belonging to X , where X can be any subset of real numbers. Similarly, $X^{m \times n}$ is the set of $m \times n$ matrices with entries belonging to X . For $A \in S^{m \times n}$, $A_{i*} \in X^n$ is the i -th row of matrix A . For vector a , $\|a\|$ represents the ℓ^2 -norm of vector a . $a_{i:j} = (a_i, a_{i+1}, \dots, a_j)$ for $i < j$. Similarly, for matrix A ,

$A_{i:j,k:l}$ is the submatrix formed by rows $i, i+1, \dots, j$ and columns $k, k+1, \dots, l$. In particular, we write $A^{(i)} = A_{1:i,1:i}$ as the upper-left submatrix.

2. Model assumptions

We assume the treatment outcome of a subject depends on three things.

1. The treatment assignment of the subject.
2. The neighbors of the subject in the network. If the i -th and the j -th subjects are connected, then a covariate Z_j will be added to the i -th outcome.
3. A random noise.

We focus on two treatment groups (treatment 0 and treatment 1) assigned to a finite population of n subjects. Let $T \in \{0, 1\}^n$ be the treatment assignment vector. T_i records the assignment of the i -th subject, that is, $T_i = 0$ for treatment 0 and $T_i = 1$ for treatment 1. The relationship between nodes is recorded by an undirected network, or equivalently, a symmetric binary adjacency matrix $A \in \{0, 1\}^{n \times n}$. We assume self loops always exist, i.e., $A_{ii} = 1$ for $i \in [n]$. We recall that A_{i*} is the i -th row of adjacency matrix A . Given the treatment assignment T_i , the observed outcome of the i -th subject follows the distribution

$$X_i = \mu_0(1 - T_i) + \mu_1 T_i + A_{i*}Z + \varepsilon_i \quad (2.1)$$

where

$$Z \text{ and } \varepsilon_i \text{ has zero mean, } \text{Cov}(Z) = \sigma_Z^2 I_n, \text{ and } \text{Cov}((\varepsilon_1, \dots, \varepsilon_n)^\top) = \sigma_\varepsilon^2 I_n.$$

We assume ε_i are i.i.d. for $i \in [n]$. The observation is the summation of three parts.

1. $\mu_0(1 - T_i) + \mu_1 T_i$ is the treatment effect, where μ_0 and μ_1 are the effect sizes of the corresponding treatments. We note that the outcome has the expectation $\mathbb{E}[X_i] = \mu_0$ if $T_i = 0$, otherwise its expectation is μ_1 .
2. The outcome of the i -th observation is also affected by its unknown covariate Z_i and the covariates of its neighbors in the network. To be precise, let N_i be the set of neighbors of i , and recall that $A_{ii} = 1$, then $A_{i*}Z = Z_i + \sum_{j:j \in N_i} Z_j$. We assume the covariates Z have zero mean, so the outcome can be positively or negatively influenced by the covariates.
3. ε_i is random noise in each observation. They have zero mean, the same variance, and they are uncorrelated.

Following previous studies, to ensure the treatment groups are unbiased, we restrict $T_{2m-1} + T_{2m} = 1$ for positive integer m . That is, (T_{2m-1}, T_{2m}) is either $(0, 1)$ or $(1, 0)$. For notational convenience, we assume the total number of subjects n is even. As a result,

$$\sum_{i=1}^n T_i = \sum_{i=1}^n 1 - T_i = \frac{n}{2}.$$

Hence we have an estimator of $\mu_0 - \mu_1$, defined as

$$\begin{aligned} W &:= \frac{2}{n} \sum_{i=1}^n (1 - T_i) X_i - T_i X_i \\ &= \mu_0 - \mu_1 + \frac{2}{n} \sum_{i=1}^n (1 - T_i) (A_{i*} Z + \varepsilon_i) - T_i (A_{i*} Z + \varepsilon_i) \\ &= \mu_0 - \mu_1 + \frac{2}{n} (\mathbf{1}_n - 2T)^\top (AZ + \varepsilon). \end{aligned}$$

For a fixed adjacency matrix A and an allocation vector T , it is not difficult to check that the estimator is unbiased, as Z and ε have zero means:

$$\begin{aligned} \mathbb{E}[W] &= \mu_0 - \mu_1 + \frac{2}{n} (\mathbf{1}_n - 2T)^\top \mathbb{E}[AZ + \varepsilon] \\ &= \mu_0 - \mu_1 + \frac{2}{n} (\mathbf{1}_n - 2T)^\top A (\mathbb{E}[Z] + \mathbb{E}[\varepsilon]) \\ &= \mu_0 - \mu_1. \end{aligned}$$

We can also compute the variance of W :

$$\text{var}[W] = \frac{4}{n^2} \text{var}[(\mathbf{1}_n - 2T)^\top (AZ + \varepsilon)] = \frac{4}{n^2} \|A(\mathbf{1}_n - 2T)\|^2 \sigma_Z^2 + \frac{4}{n} \sigma_\varepsilon^2,$$

where $\|\cdot\|$ denotes the ℓ^2 -norm throughout this paper. We note that W is an unbiased estimation and the term $4\sigma_\varepsilon^2/n$ converges to 0 as $n \rightarrow \infty$, so the best strategy in this experiment is to reduce the term $\|A(\mathbf{1}_n - 2T)\|^2$ by assigning an appropriate treatment to each pair of subjects. As the variance of estimator W decreases, the hypothesis testing on the effectiveness of the treatment becomes more powerful. We assume each pair of subjects joins the experiment sequentially, and we need to decide their treatment assignment soon after they join. In pairwise sequential randomization [39, 27], we assign different treatments to each pair of subjects simultaneously. In the m -th stage, we determine the treatment assignments to the $(2m - 1)$ -th and the $2m$ -th subjects, which may depend on two factors. First, after the first $2m$ subjects join the experiment, we only observe the connection between these subjects, while all other connections are concealed. In other words, we observe the (upper-left) sub-adjacency matrix $A^{(2m)} := (A_{ij})_{1 \leq i, j \leq 2m}$. Second, when we determine the assignment to the $(2m - 1)$ -th and $2m$ -th subjects, we have the record of the assignments to the first $(2m - 2)$ subjects, although we cannot update them. Therefore, given the submatrix $A^{(2m)}$ and T_1, \dots, T_{2m-2} , we need to determine T_{2m-1} and T_{2m} to reduce the *imbalance measurement*, defined as

$$I_{2m} = \|A^{(2m)}(\mathbf{1}_{2m} - 2T_{1:2m})\|, \quad (2.2)$$

where $\mathbf{1}_{2m} \in \mathbb{R}^{2m}$ with all entries equal to 1 and $T_{1:2m}$ consists of the first $2m$ entries of T . To reduce the imbalance measurement, we propose the following procedure:

1. The first two subjects are randomly assigned to different treatments.
2. Suppose $2m - 2$ patients have been assigned to treatments, we define the imbalance measurement when $(T_{2m-1}, T_{2m}) = (0, 1)$

$$I_{2m}^{(0,1)} = \|A^{(2m)}(\mathbf{1}_{2m} - 2(T_{1:(2m-2)}^\top, 0, 1)^\top)\|,$$

and in the same manner, when $(T_{2m-1}, T_{2m}) = (1, 0)$, we have

$$I_{2m}^{(1,0)} = \|A^{(2m)}(\mathbf{1}_{2m} - 2(T_{1:(2m-2)}^\top, 1, 0)^\top)\|.$$

3. We decide (T_{2m-1}, T_{2m}) according to the following probabilities,

$$\mathbb{P}((T_{2m-1}, T_{2m}) = (0, 1)) = \begin{cases} b, & \text{if } I_m^{(0,1)} < I_m^{(1,0)}; \\ 1 - b, & \text{if } I_m^{(0,1)} > I_m^{(1,0)}; \\ 0.5, & \text{otherwise.} \end{cases}$$

Here $b \in (1/2, 1]$ is a fixed biasing probability.

4. We repeat steps 2 and 3 until $2m \geq n - 1$. If $2m = n - 1$, we arbitrarily assign a treatment to subject n .

The general idea of this procedure can be summarized as follows. In each stage, we consider two possible assignments to (T_{2m-1}, T_{2m}) and compute which assignment minimizes the imbalance measurement. In pairwise sequential randomization, the assignments are either $(T_{2m-1}, T_{2m}) = (0, 1)$, or $(1, 0)$. We use the assignment that results in the smallest imbalance measurement with the biasing probability $b \in (1/2, 1]$. It is clear that letting $b = 1$ would reduce the expected imbalance measurement as far as possible, but we allow randomness in the procedure for several practical reasons. We further discuss this biasing probability in Remark 1. Notably, the proposed procedure does not require any information on subjects joining the experiment in the future. To be more specific, the choice of treatment for subjects $2m - 1$ and $2m$ only depends on their connection with previous subjects and the current imbalance measurement. The procedure can be applied to the case when n is odd, as long as we assign a random treatment to the last subject. If b is a constant greater than $1/2$, the adaptive procedure can significantly reduce the imbalance measurement under mild assumptions on the network.

Remark 1 (Biased coin design). *Suppose we let $b = 1$ in our proposed procedure, then each pair of assignments in the procedure reduces the imbalance measurement as far as possible, and treatment allocation is completely determined by the network. However, deterministic treatment assignment is not desirable from the standpoint of (un)predictability and the principle of randomness [20]. The authors of [20] mention “treatments are prepacked in accordance with a suitable randomization schedule, . . . so that no one involved in the conduct of the trial is aware of the specific treatment allocated to any particular subject”. Thus, an appropriate allocation probability $\in (1/2, 1)$ should be selected. The concept of biased coin design was first proposed in [10] to ensure an equal distribution of*

different treatments, and the author recommended a value of $b = 2/3$. For the purpose of balancing prognostic factors between treatment groups, the authors of [12] suggest an allocation probability between 0.70 and 0.95 according to the sample size. In [41], the authors simulate the effects of allocation probability. In this paper, we assume that b can be any constant greater than 0.5 and no more than 1.

Remark 2 (Binary Integer Programming). *Suppose the whole network is observed, the goal of reducing the imbalance measurement $I = \|A(\mathbf{1} - 2T)\|$ with unbiased treatment groups is equivalent to the following optimization problem:*

$$\min_{x \in \{-1,1\}, \mathbf{1}^\top x = 0} \|Ax\| = \min_{x \in \{-1,1\}, \mathbf{1}^\top x = 0} x^\top Hx,$$

where $H = A^\top A = A^2$. It is not difficult to observe that H_{ij} counts the number of common neighbors of node i and j in the adjacency matrix A . The constrained $\mathbf{1}^\top x = 0$ can be converted to a penalty function:

$$\min_{x \in \{-1,1\}} x^\top Hx + \lambda(\mathbf{1}^\top x)^2 = \min_{x \in \{-1,1\}} x^\top (H + \lambda \mathbf{1}\mathbf{1}^\top)x.$$

This formulation is summarized as an unconstrained binary programming problem (UBQP) in [17]. The authors of that survey also mention that the UBQP is an NP-hard problem, whose proof is provided in [34], except for some special cases with very strong assumptions on H [36, 3, 33]. H in these special cases is restricted to be an adjacency matrix with certain regularization conditions, so their results cannot apply to our case $H = A^2$. In the general case, heuristic methods such as the continuous approach [35, 32], tabu search algorithms [25, 43], and semi-definite relaxation [42] have been proposed for finding inexact but high-quality solutions. However, it is worth noting that the setting we consider here is very different from a UBQP problem. We have to determine x_i when only the upper-left $i \times i$ submatrix of A is observed.

3. Theoretical properties of the proposed design

In this section, we study the asymptotic property of the imbalance measurement quantity of (2.2) under the following stochastic assumption on the symmetric adjacency matrix A . We assume for some $p \in (0, 1)$,

$$A - I \sim G(n, p), \tag{3.1}$$

where $G(n, p)$ represents the Erdős-Rényi random graph model. In other words, on the diagonal of A , we have determinant entries $A_{ii} = 1$ for $i \in [n]$, and

$$A_{ij} = A_{ji} \sim \text{Bernoulli}(p) \text{ independently for } 1 \leq i < j \leq n.$$

In the graph sense, the Erdős-Rényi random graph model indicates that an edge between distinct nodes exists with probability p [11]. Under this assumption on

A , we aim to analyze the asymptotic behavior of the imbalance measurement $I_{2m} = \|A^{(2m)}(\mathbf{1}_{2m} - 2T_{1:2m})\|$ defined in (2.2). Let us also define the state after the m -th iteration of the procedure:

$$S_{2m} = A_{2m}(\mathbf{1}_{2m} - 2T_{1:2m})$$

so that $I_{2m} = \|S_{2m}\|$. For convenience of notation, we let

$$I_{2m+1} = I_{2m} \quad \text{for } m \in \mathbb{N}, \quad (3.2)$$

so the imbalance measurement I_i can be defined for all positive integers i . Suppose A were not symmetric, i.e., a_{ij} and a_{ji} were i.i.d., then $\{S_i\}_{i \in \mathbb{N}}$ would be a *time-varying Markov chain*, where the randomness comes from entrywise Bernoulli distribution and random assignments in step 3 of the procedure. In the symmetric case, we still approximately have the following Markov property:

$$\mathbb{P}(S_i = x | S_1, \dots, S_{i-1}) \approx \mathbb{P}(S_i = x | S_{i-1})$$

We will show that the imbalance measurement I_n is significantly reduced compared with random design if we apply our proposed procedure.

A random design indicates that we assign $(T_{2m-1}, T_{2m}) = (0, 1)$ or $(1, 0)$ with probability $1/2$. In other words, we implement step 3 of our proposed design with $b = 1/2$. We denote the resulting assignments by the vector T_{random} , then for fixed $p \in (0, 1)$, we have the following theorem about random assignment.

Theorem 1. *Suppose the $n \times n$ network follows the Erdős-Rényi random graph model in (3.1) with Bernoulli parameter p , then using random assignment, the imbalance measurement satisfies the following limit*

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}[\|A(1 - 2T_{\text{random}})\|^2]}{n^2} = p(1 - p). \quad (3.3)$$

The next theorem shows that our proposed design can significantly reduce the imbalance measurement.

Theorem 2. *Suppose the $n \times n$ network follows the Erdős-Rényi random graph model in (3.1) with Bernoulli parameter p , then using our proposed design, the imbalance measurement I_n satisfies the following upper bound:*

$$\limsup_{n \rightarrow \infty} \frac{\mathbb{E}[I_n^4]}{n^4} \leq p^2(1 - p)^2 - \frac{1}{8}(2b - 1)(2 - \sqrt{2}(2b - 1))^{3/2} p^{5/2} (1 - p)^{5/2}. \quad (3.4)$$

Remark 3. *Theorem 2 provides the upper bound of the fourth moments of the imbalance measurement. Because $\mathbb{E}[I_n^4] \geq \mathbb{E}[I_n^2]^2$, we immediately obtain an upper bound of the second moment $\mathbb{E}[I_n^2]$. For fixed $p \in (0, 1)$ and $b \in (1/2, 1]$,*

$$\limsup_{n \rightarrow \infty} \frac{\mathbb{E}[I_n^2]}{n^2} < p(1 - p).$$

Hence the proposed procedure provides a strictly smaller imbalance measurement than random design in expectation. Suppose $b = 1/2$, i.e., $2b - 1 = 0$, then the proposed method is identical to random design. As a result, the second term of (3.4) vanishes. Meanwhile, suppose the network is very sparse, that is, p is very small, then the reduction of imbalance measurement by the proposed design is not very great, because $p^{5/2}$ is much smaller than p^2 .

Proof overview of Theorem 2 We will use the notation:

$$\tilde{S}_m := S_{2m} = A^{(2m)}\tilde{T}_{2m}, \quad \tilde{I}_m := I_{2m} = \|A^{(2m)}\tilde{T}_{2m}\|.$$

Our goal is to show that the expected value of \tilde{I}_m has a nontrivial upper bound. However, the reduction achieved by applying our proposed method actually depends on the lower bound of certain moments of \tilde{I}_m . More details are described below:

1. It is easy to find a trivial upper bound for $\mathbb{E}[\tilde{I}_m]$, which is approximately $2n\sqrt{p(1-p)}$ in Theorem 1.
2. Let Y_m be the network connection data from the new units. Since Y_m is random, the expected value of $\cos(\tilde{S}_m, Y_m)$ cannot be too large. We can choose the sign of Y_m in a biased manner according to which sign can reduce the norm of \tilde{S}_{m+1} , so the reduction depends on $|\tilde{S}_m^\top Y_m|$. This reduction cannot be too large because \tilde{I}_m has an upper bound, and $\mathbb{E}[\cos(\tilde{S}_m, Y_m)]$ is not too large. Using this fact, we can show that $\mathbb{E}[\tilde{I}_m^2]$ has a lower bound, approximately $4 - c_b m^2 p(1-p)$, with a constant c_b depending on b only.
3. $\|\tilde{S}_m\| = \tilde{I}_m$ has a lower bound, as does $|\tilde{S}_m^\top Y_m|$. Now we can find a lower bound for the imbalance reduction for each step. Cumulatively, we find the upper bound for the imbalance measurement.

In the proof, we utilized the concept of a Lyapunov function, as introduced in [30], to bound $\|S_m\|$. We achieved this by studying the conditional expected increment

$$\mathbb{E}[\|\tilde{S}_{m+1}\|^p - \|\tilde{S}_m\|^p \mid S_m]$$

By bounding this conditional expectation, we can obtain the bound for $\|S_m\|^p$ by adding the conditional expected difference. It is important to note that $\|\cdot\|^p$ serves as a Lyapunov function in the definition provided in [30].

4. Discussion on the Gaussian case

In previous sections, we assumed the network followed the Erdős-Rényi random graph model. As discussed in Remark 3, Theorem 2 has not shown that the reduction of imbalance measurement is asymptotically smaller than the imbalance measurement itself if $p \rightarrow 0$. To discuss whether the reduction is rate optimal,

we investigate a weighted adjacency matrix with Gaussian entries. Specifically, in Wigner matrix A , we have determinant entries $A_{ii} = 1$ for $i \in [n]$, and

$$A_{ij} = A_{ji} \sim \mathcal{N}(0, \sigma^2) \text{ independently for } 1 \leq i < j \leq n.$$

In other words, we consider the Gaussian orthogonal ensemble (GOE) instead of the Erdős-Rényi random graph model. This assumption corresponds to the following scenario. We observe a weighted network in which the weights can be either positive or negative. Under assumption (2.1), the observation X_i is still well defined. In this case, the unknown covariate Z_j can affect the i -th observation X_i positively or negatively, depending on the weight A_{ij} . Under this assumption, the proposed procedure in Section 2 is still valid. If we adopt the definition of imbalance measurement I_n , we have the following asymptotic upper bound.

Theorem 3. *Suppose the $n \times n$ weighted network follows the GOE with variance σ^2 where σ depends on n . Assuming $\sigma = O(1)$ and $n\sigma^2 \rightarrow \infty$, then using our proposed design, the imbalance measurement I_n satisfies the following upper bound:*

$$\limsup_{n \rightarrow \infty} \frac{\mathbb{E}[I_n^4]}{n^4 \sigma^4} \leq 1 - \frac{1}{4}(2b-1)\sqrt{2/\pi}(4 - \sqrt{2/\pi}(2b-1))^{3/2}.$$

In the Erdős-Rényi random graph model, the entrywise variance of the adjacency matrix is $p(1-p)$. This quantity is comparable to σ^2 in the GOE. When $\sigma \rightarrow 0$, the reduction of the imbalance measurement is still significantly large. This is a stronger result than that in the Erdős-Rényi random graph model. An essential technical reason is the lower bound of $\mathbb{E}[|x^\top Y|]$ for fixed subject vector x and centered random vector $Y \in \mathbb{R}^m$. If we only assume Y is a sub-Gaussian vector, then for general x , we obtain the best lower bound by Khinchin-Kahane inequality, see Lemma 1. If we further assume $Y_i \sim \mathcal{N}(0, \sigma^2)$ independently, then $|x^\top Y|$ is a folded normal random variable and $\mathbb{E}[|x^\top Y|] = \sigma\sqrt{n/\pi}$ for all subject vectors x . It is still an open problem whether the term $p^{5/2}$ can be improved to p^2 . An empirical comparison of these two cases can be found in the next section.

5. Experiments

In this section, we empirically study the behavior of imbalance measurement in (2.2). The experiments demonstrate that our proposed algorithm improves the estimation of treatment effects for both simulated data and real network data.

5.1. Experiments on simulated network data

The plots in Figure 1 show the result of the Erdős-Rényi random graph model in (3.1). We fix $p = 0.2$ and simulate different sizes of random graphs. We

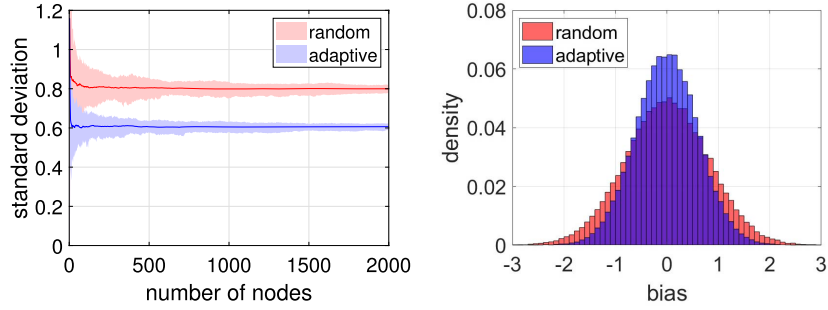


FIG 1. Left: the standard deviation of W in Section 2 for different n . Right: the histogram of W when $\mu_0 = \mu_1$ and $\sigma_\varepsilon = 1$.

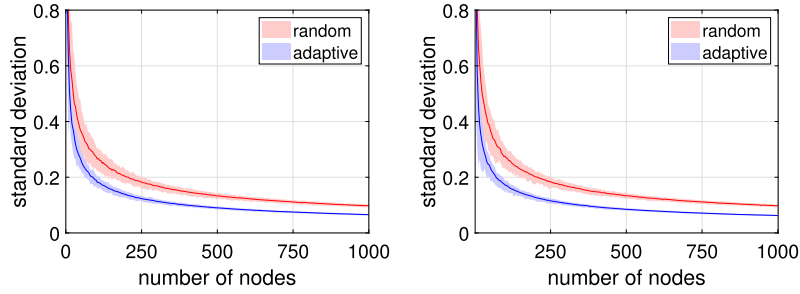


FIG 2. Left: the result when the graph is sparse. Right: the simulation on the Gaussian orthogonal ensemble.

consider random assignment and our proposed adaptive design algorithm with $b = 0.95$. On the left plot, the shaded region is the 95% confidence interval for 100 iterations. All other plots with shaded regions in this section have confidence intervals with the same confidence coefficient. The plot of random assignment shows that the imbalance measurement concentrates around 0.8. This coincides with the theoretical limit $\sqrt{4p(1-p)} = 0.8$ suggested by Theorem 1. Applying the proposed algorithm, the imbalance measurement decreases to approximately 0.6. The right plot shows the bias of estimation of $\mu_0 - \mu_1$ when $n = 100$. This experiment is repeated 20000 times.

The left plot in Figure 2 considers the sparse Erdős-Rényi random graph model. For an $n \times n$ network, we consider the density regime $\frac{\log n}{n}$. In particular, we generate random networks $G(n, \frac{\log n}{5n})$. The shaded region is the interquartile range over 100 iterations. The imbalance measurement of random design monotonically decreases because its maximum expectation is $\sqrt{4p(1-p)}$, which converges to 0 as the network becomes more sparse. The right plot in Figure 2 considers the GOE instead. The entrywise variance remains the same. In other words, for the network with n nodes, $p_n = \frac{\log n}{n}$, the corresponding variance of the GOE is $\sigma_n^2 = p_n(1-p_n)$. The results of this simulation show that the imbalance measures have very similar asymptotic behavior for both models.

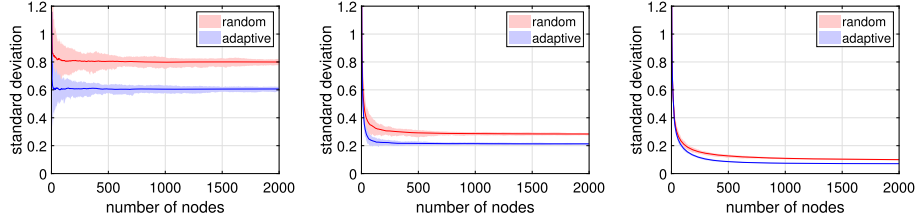


FIG 3. Comparison of the Erdős-Rényi random graph model with different density.

The plots in Figure 3 compare the performance of our proposed method on the Erdős-Rényi random graph model. In the plots, we let $p = 0.2, 0.02,$ and 0.002 and plot the imbalance measurement on different sizes of random graphs. The result of this experiment is identical to that shown in the left plot in Figure 1, but in different densities.

In Figure 4, we consider the GOE with different levels of variance. For the left, center, and right plots, we let $\sigma^2 = p(1-p)$ for $p = 0.2, 0.02, 0.002$ respectively, so that the entrywise variance is the same as that in the Erdős-Rényi random graph model. As we can see in the plots, the empirical performances are very similar for these two models.

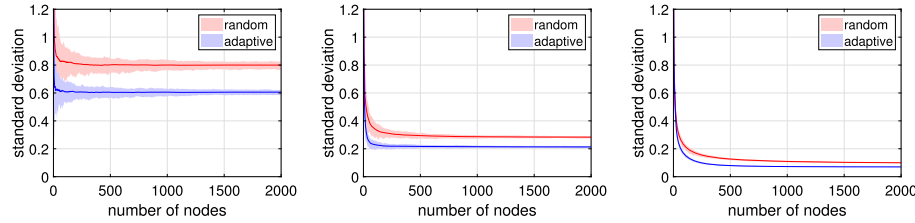


FIG 4. Comparison of Gaussian orthogonal ensemble with different variance.

Figure 5 considers the stochastic block model. The subjects are randomly divided into two groups. In our setting, if two subjects belong to the same group, then the probability of connection between them is p_1 , and the between-group probability is p_2 . The plots consider $p_1 = 0.3, 0.03, 0.003$ and $p_2 = 0.1, 0.01, 0.001$ from left to right respectively. The overall density is the same as the previous experiment on the Erdős-Rényi random graph in Figure 3, and the empirical performances of the two experiments are very similar. However, on the left plot, we can observe that the confidence intervals are wider than those in Figure 3 and Figure 4.

5.2. Experiments on real network data

We implement our proposed algorithm on 11 real undirected network datasets from SNAP [19]. For each network, we randomly sample a subnetwork with

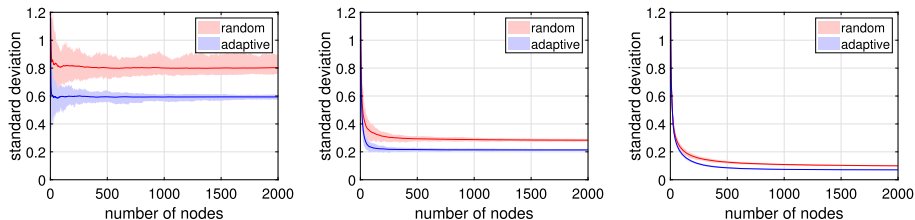


FIG 5. Comparison of stochastic block models with different density.

10000 nodes, then we apply both adaptive design and random design to the subnetwork. We compare the imbalance measurement in (2.2) in each dataset. Because theoretical analysis shows that the network density plays an important role in imbalance measurement, the densities are recorded in the last column. In our model, we always assume the existence of self loops. When we compute the density of the subnetwork, we only consider the connections between different nodes. For example, the density of the network `com-youtube.ungraph` is approximately 0.00076. In other words, the average degree of the graph is approximately 8.6, including self loops.

TABLE 1
Comparison of adaptive random design applied to real network data from SNAP.

Dataset	Adaptive	Random	Reduction	Density
Email-Enron	212.4147	346.6583	39%	11.7178×10^{-4}
<code>com-youtube.ungraph</code>	172.6210	269.6257	36%	7.6078×10^{-4}
HR_edges	143.2271	212.1933	33%	3.4652×10^{-4}
HU_edges	117.8813	174.4821	32%	1.9725×10^{-4}
RO_edges	106.7895	157.6959	32%	1.4708×10^{-4}
CA-GrQc	99.2774	117.0982	15%	0.3776×10^{-4}
CA-HepPh	100.3494	113.8508	12%	0.3766×10^{-4}
CA-AstroPh	100.2098	112.5256	11%	0.2304×10^{-4}
CA-CondMat	97.8877	107.7311	9%	0.1720×10^{-4}
CA-HepTh	99.1060	104.9667	6%	0.1200×10^{-4}

The edges in these networks might have different meanings. We now explain how our model and algorithm are applied on the network `com-youtube.ungraph`. The nodes of this network represent users on YouTube. Two nodes are connected in the network if they are friends on YouTube. Under the assumption of a network-correlated outcome, friends share common unknown factors that affect observations. To reduce the effect of a factor, we should propose treatment allocation such that friends sharing the corresponding factor are divided into two treatment groups. Suppose we apply our proposed adaptive design on this network with $b = 0.85$ (see Remark 1 for the choice of b), then the imbalance measurement is reduced by 36%.

We also implement the proposed adaptive design algorithm in the other real networks. The result of the experiment is presented in Table 1. We observe that the percentage of imbalance measurement reduction depends on the density of the network. For instance, the network of `CA-HepTh` has a low density

of 0.1200×10^{-4} , hence, our method can only reduce the imbalance measurement by 6%. According to Remark 3, there is no evidence that our proposed method can reduce the imbalance measurement significantly if the network is very sparse.

In Figure 6, we implement both random and adaptive designs on different sizes of the network `com-youtube.ungraph`. To repeat the experiments, we keep the random graphs generated in the previous experiment. In this real dataset, to obtain the confidence interval, we repeatedly sample subgraphs from the network and apply the proposed algorithm on each subgraph. The empirical results again show that our proposed method significantly reduces the imbalance measurement and improves the accuracy of estimation of treatment effects.

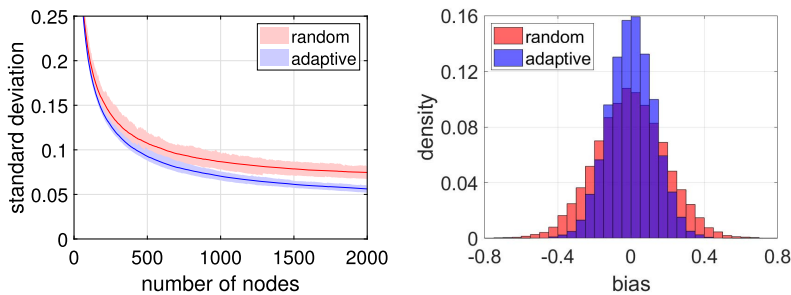


FIG 6. These plots repeat the experiments in Figure 1 on the real dataset from YouTube.

6. Conclusion

In this paper, we consider the problem of estimating treatment effects under the assumption that the outcomes are network-correlated. We propose an adaptive randomization procedure to reduce the variance of the estimation. The algorithm assigns different treatments to each pair of subjects sequentially. The biased coin design enforces the assignments, with the result that a smaller imbalance measurement will be chosen with higher probability. For theoretical analysis, we assume the network is generated by the Erdős-Rényi random graph model. As the number of subjects increases, the states of the Markov process have different dimensions as time progresses. We provide a novel mathematical proof that our adaptive randomization algorithm significantly reduces the imbalance measurement. Our empirical results also show that this proposed algorithm reduces the variance of the unbiased estimator in both simulated and real data.

One potential extension of our method is to balance both the network correlation and observed covariates simultaneously. Previous works [13, 14] have mainly focused on balancing the covariates alone. By incorporating both network information and observed covariates, we can assign treatment to subjects sequentially in a more effective manner.

The new procedure can still be generalized in several ways. To guarantee a balanced treatment allocation, we consider pairwise sequential design, which

determines treatments to two subjects simultaneously. Conventional adaptive design [13] is still applicable in network data. The empirical results can be expected to be similar to the proposed method, but the theoretical analysis will be different, and this is an interesting topic for future work. We have demonstrated our theoretical results for networks generated by the Erdős-Rényi random graph model. However, this assumption can be extended to other types of random networks as long as the connections are independent. This is due to the fact that we rely solely on the fact that random vectors are almost perpendicular to other vectors in high-dimensional Euclidean space.

In the methods described so far, uniform weights are assigned to subjects. If different weights were allowed for different subjects, it might be possible to further reduce the variance of estimation. If a subject has a high degree in the network, its outcome is affected by many other subjects. As a result, the outcome of this node has high variance. If we can reduce the weight of such nodes, the performance of the algorithm will be further improved.

If we assume there is interference between subjects, i.e., the outcome of a certain subject might be affected by the treatment of its neighbors, then the analysis in this paper is no longer strictly applicable. However, the tools introduced in this paper could still powerfully reduce the variance under such an interference assumption. As long as we can define the variance after each step sequentially, then we assign the desired assignment to the current subject with a probability greater than 0.5. We believe this procedure at least performs better than random assignment. We leave these as future research topics.

Last but not least, it is possible that the theoretical analysis in Theorem 2 can be further improved. As mentioned in Section 3, the result in Theorem 3 allows $\sigma \rightarrow 0$. If $p \rightarrow 0$ (sparse) in Theorem 2, the current analysis does not show that the proposed design still achieves significant improvement. This could be a very interesting problem for further research. In general, it has proven difficult for researchers to obtain theoretical results on the designs of network data, due to the complexity of the problem and the lack of technical tools. In this paper, we introduce the technique of Lyapunov functions. This technique could provide a feasible way of studying the properties of general designs in network data.

Appendix A: Proofs

A.1. Proof of Theorem 1

In this proof, we briefly denote $T := T_{\text{random}}$. We have

$$\|A(1 - 2T)\|^2 = \sum_{i=1}^n (A_{i*}(1 - 2T))^2$$

and observe that the distributions of $A_{i*}(1 - 2T_{\text{random}})$ are identical for all $i \in [n]$. Without loss of generality, we consider $i = 1$. By the definition of random design, $\mathbb{E}[1 - 2T] = 0$. Furthermore, A and T are independent in the

random design, so $\mathbb{E}[A_{1*}(1 - 2T)] = 0$. Hence we have

$$\mathbb{E}[(A_{1*}(1 - 2T))^2] = \text{var}[A_{1*}(1 - 2T)].$$

We recall that $(T_{2m-1}, T_{2m}) = (0, 1)$ or $(1, 0)$ with equiprobability. By independence, we have

$$\text{var}[A_{1*}(1 - 2T)] = \sum_{m=1}^{n/2} \text{var}[A_{1,2m-1}(1 - 2T_{2m-1}) + A_{1,2m}(1 - 2T_{2m})]$$

For $m \in [[2, n/2]]$, the distributions of $A_{1,2m-1}(1 - 2T_{2m-1}) + A_{1,2m}(1 - 2T_{2m})$ are identical. When $m = 1$, we are in the special case that $A_{11} = 1$. Hence, it suffices to consider the cases when $m = 1$ and $m = 2$. When $m = 1$, we have

$$\text{var}[A_{11}(1 - 2T_1) + A_{12}(1 - 2T_2)] = \mathbb{E}[(1 - A_{12})^2] = 1 - p.$$

When $m = 2$, we have

$$\text{var}[A_{13}(1 - 2T_3) + A_{14}(1 - 2T_4)] = \mathbb{E}[(A_{13} - A_{14})^2] = 2p(1 - p).$$

Hence $\mathbb{E}[(A_{1*}(1 - 2T))^2] = np(1 - p) + (1 - 2p)(1 - p)$, and $\mathbb{E}[\|A(1 - 2T)\|^2] = n^2p(1 - p) + n(1 - 2p)(1 - p)$. Taking the limit, we have

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}[\|A(1 - 2T_{\text{random}})\|^2]}{n^2} = p(1 - p)$$

as desired.

A.2. Proof of Theorem 2

Proof. For $i \in [2m]$ and $j \in m$, let us define

$$Y_{ij} = A_{i,2j} - A_{i,2j-1}$$

If $i \neq 2j - 1$ and $i \neq 2j$, i.e., neither $A_{i,2j}$ nor $A_{i,2j-1}$ is on the diagonal, we have

$$Y_{ij} = \begin{cases} -1, & \text{with probability } p(1 - p); \\ 0, & \text{with probability } p^2 + (1 - p)^2; \\ 1, & \text{with probability } p(1 - p) \end{cases} \quad (\text{A.1})$$

We recall that $A^{(2m)}$ is the $2m \times 2m$ submatrix of A , $\tilde{T}_{2m} = \mathbf{1}_{2m} - 2T_{1:2m}$, and define $Y_m = Y_{1:2m, m+1} \in \mathbb{R}^{2m}$. In this section, we use the notations

$$\tilde{S}_m := S_{2m} = A^{(2m)}\tilde{T}_{2m}, \quad \tilde{I}_m := I_{2m} = \|A^{(2m)}\tilde{T}_{2m}\|.$$

Now, as we define $I_{2m+1} = I_{2m} = \tilde{I}_m$ in (3.2), it suffices to show

$$\limsup_{n \rightarrow \infty} \frac{\mathbb{E}[\tilde{I}_n^4]}{16n^4} \leq p^2(1 - p)^2 - \frac{1}{8}(2b - 1)(2 - \sqrt{2}(2b - 1))^{3/2}p^{5/2}(1 - p)^{5/2}, \quad (\text{A.2})$$

which is equivalent to (3.4). As the entries of Y_m follow the distribution of (A.1) independently, we have

$$\mathbb{E}[\|Y_m\|^2] = 2m \operatorname{var}[Y_{1,m+1}] = 2m\mathbb{E}[Y_{1,m+1}^2] = 4mp(1-p).$$

We have $\mathbb{E}[\|Y_m\|^2] = 4mp(1-p)$. We also define

$$Z_{2m+1} = \sum_{i=1}^{2m} A_{2m+1,i} \tilde{T}_i \quad \text{and} \quad Z_{2m+2} = \sum_{i=1}^{2m} A_{2m+2,i} \tilde{T}_i.$$

By definition, we have $Z_{2m+2} - Z_{2m+1} = \tilde{T}_{2m}^\top Y_m$. As $(T_{2i-1}, T_{2i}) = (1, 0)$ or $(0, 1)$, we can write

$$Z_{2m+1} = \sum_{j=1}^m (A_{2m+1,2j} - A_{2m+1,2j-1}) \tilde{T}_{2i} = \sum_{j=1}^m Y_{2m+1,j} \tilde{T}_{2i}.$$

By symmetry of $Y_{ij} \tilde{T}_{2i}$, we have that Z_{2m+1} shares the same distribution as $\sum_{j=1}^m Y_{2m+1,j}$. It is also clear that Z_{2m+2} shares that same distribution. Hence we have $\mathbb{E}[Z_{2m+1}] = \mathbb{E}[Z_{2m+2}] = 0$ and

$$\begin{aligned} \mathbb{E}[Z_{2m+1}^2] &= \mathbb{E}[Z_{2m+2}^2] = \mathbb{E}\left[\left(\sum_{j=1}^m Y_{2m+1,j}\right)^2\right] \\ &= \sum_{j=1}^m \operatorname{var}[Y_{2m+1,j}] = m\mathbb{E}[Y_{2m+1,1}^2] = 2mp(1-p). \end{aligned}$$

In the $m+1$ step of our proposed procedure, we observe two new columns and new rows of the adjacency matrix, which will change the imbalance measurement. The square of the imbalance measurement in the $m+1$ step will be either

$$U_m = \|\tilde{S}_m + Y_m\|^2 + (Z_{2m+1} - 1 + A_{2m+1,2m+2})^2 + (Z_{2m+2} + 1 - A_{2m+1,2m+2})^2$$

or

$$V_m = \|\tilde{S}_m - Y_m\|^2 + (Z_{2m+1} + 1 - A_{2m+1,2m+2})^2 + (Z_{2m+2} - 1 + A_{2m+1,2m+2})^2.$$

Step 3 of the procedure indicates that our new design will pick the smaller one of the above two with probability $b > 1/2$, and choose the larger one otherwise. By the symmetry of the distributions of Y_m , one can observe that these two terms have the same expectation, and by direct calculation, we obtain

$$\begin{aligned} \mathbb{E}[(Z_{2m+1} - 1 + A_{2m+1,2m+2})^2] &= \mathbb{E}[(Z_{2m+2} + 1 - A_{2m+1,2m+2})^2] \\ &= 2mp(1-p) + 1 - p. \end{aligned}$$

1. Upper bound of $\mathbb{E}[\tilde{I}_n]$ We have the conditional expectation of \tilde{I}_{m+1}^2 ,

$$\begin{aligned}
& \mathbb{E}[\tilde{I}_{m+1}^2 | \tilde{S}_m] \\
& \leq \mathbb{E}[\|\tilde{S}_m + Y_m\|^2 + (Z_{2m+1} - 1 + A_{2m+1,2m+2})^2 \\
& \quad + (Z_{2m+2} + 1 - A_{2m+1,2m+2})^2 | \tilde{S}_m] \\
& = \mathbb{E}[\|\tilde{S}_m\|^2 + 2\tilde{S}_m^\top Y_m + \|Y_m\|^2 + (Z_{2m+2} + 1 - A_{2m+1,2m+2})^2 \\
& \quad + (Z_{2m+1} - 1 + A_{2m+1,2m+2})^2 | \tilde{S}_m] \\
& = \|\tilde{S}_m\|^2 + 4mp(1-p) + 2mp(1-p) + 1-p + 2mp(1-p) + 1-p \\
& = \|\tilde{S}_m\|^2 + 8mp(1-p) + 2(1-p).
\end{aligned}$$

Therefore, $\mathbb{E}[\tilde{I}_{m+1}^2 - \tilde{I}_m^2 | \tilde{I}_m^2] = 8mp(1-p) + 2(1-p)$. Hence

$$\mathbb{E}[\tilde{I}_{m+1}^2 - \tilde{I}_m^2] = \mathbb{E}[\mathbb{E}[\tilde{I}_{m+1}^2 - \tilde{I}_m^2 | \tilde{I}_m^2]] \leq 8mp(1-p) + 2(1-p).$$

In the first stage, the imbalance measurement $\mathbb{E}[\tilde{I}_1^2] = (1 - A_{12})^2 + (A_{21} - 1)^2 = 2(1-p)$. Thus

$$\begin{aligned}
\mathbb{E}[\tilde{I}_n^2] & \leq \sum_{m=0}^{n-1} 8mp(1-p) + 2(1-p) = 4n(n-1)p(1-p) + 2n(1-p) \\
& \leq \left(2n\sqrt{p(1-p)} + \sqrt{\frac{1-p}{4p}}\right)^2.
\end{aligned}$$

By Jensen's inequality,

$$\mathbb{E}[\tilde{I}_n] \leq \sqrt{\mathbb{E}[\tilde{I}_n^2]} \leq 2n\sqrt{p(1-p)} + \sqrt{\frac{1-p}{4p}}.$$

2. Lower bound of $\mathbb{E}[\tilde{I}_n^2]$ With the upper bound of the first moment, we can derive the lower bound of the second moment.

$$\begin{aligned}
\mathbb{E}[\tilde{I}_{m+1}^2 | \tilde{S}_m] & = \mathbb{E}[B \min(U_m, V_m) + (1-B) \max(U_m, V_m) | \tilde{S}_m] \\
& = \mathbb{E}[U_m | \tilde{S}_m] - (2b-1) \mathbb{E}[|U_m - V_m| | \tilde{S}_m] \\
& = \|\tilde{S}_m\|^2 + 8mp(1-p) + 2(1-p) \\
& \quad - (2b-1) \mathbb{E}[|2\tilde{S}_m^\top Y_m + 2(1 - A_{2m+1,2m+2})(Z_{2m+1} - Z_{2m+2})| | \tilde{S}_m]
\end{aligned}$$

Now we aim to find an upper bound of $\mathbb{E}[|2\tilde{S}_m^\top Y_m + 2(1 - A_{2m+1,2m+2})(Z_{2m+1} - Z_{2m+2})| | \tilde{S}_m]$. By Jensen's inequality,

$$\begin{aligned}
& (\mathbb{E}[|2\tilde{S}_m^\top Y_m + 2(1 - A_{2m+1,2m+2})(Z_{2m+1} - Z_{2m+2})| | \tilde{S}_m])^2 \\
& \leq 4\mathbb{E}[(\tilde{S}_m^\top Y_m + (1 - A_{2m+1,2m+2})(Z_{2m+1} - Z_{2m+2}))^2 | \tilde{S}_m] \\
& = 4\mathbb{E}[(\tilde{S}_m^\top Y_m)^2 | \tilde{S}_m] + 8\mathbb{E}[(1 - A_{2m+1,2m+2})(Z_{2m+1} - Z_{2m+2})\tilde{S}_m^\top Y_m | \tilde{S}_m] \\
& \quad + 4\mathbb{E}[(1 - A_{2m+1,2m+2})(Z_{2m+1} - Z_{2m+2})^2 | \tilde{S}_m]
\end{aligned}$$

Now we will find the condition expectation of these three terms. As the entries of Y_m are i.i.d. with distribution (A.1) we have

$$\mathbb{E}[(\tilde{S}_m^\top Y_m)^2 | \tilde{S}_m] = \|\tilde{S}_m\|^2 \mathbb{E}[Y_{21}^2] = 2\|\tilde{S}_m\|^2 p(1-p).$$

For the second term, by the definition of Y_m , Z_{2m+1} and Z_{2m+2} , we have $Z_{2m+2} - Z_{2m+1} = Y_m^\top \tilde{T}_{2m}$. Hence

$$\begin{aligned} & \mathbb{E}[(1 - A_{2m+1,2m+2})(Z_{2m+1} - Z_{2m+2})\tilde{S}_m^\top Y_m | \tilde{S}_m] \\ &= (1-p)\mathbb{E}[-Y_m^\top \tilde{T}_{2m} \tilde{S}_m^\top Y_m | \tilde{S}_m]. \end{aligned}$$

As the distributions of the $(2i-1)$ -th and $2i$ -th rows are identical, we have $\mathbb{P}(\tilde{T}_i = -1 | \tilde{S}_m) = \mathbb{P}(\tilde{T}_i = 1 | \tilde{S}_m) = 0.5$. Hence for all i and m , T_i and \tilde{S}_m are independent. T_i and \tilde{S}_m only depend on the submatrix $A^{(2m)}$, so they are independent of Y_m . Thus,

$$\mathbb{E}[Y_m^\top \tilde{T}_{2m} \tilde{S}_m^\top Y_m | \tilde{S}_m] = 0,$$

which implies the second term vanishes. Using $Z_{2m+2} - Z_{2m+1} = \tilde{T}_{2m}^\top Y_m$ again, we have

$$\begin{aligned} \mathbb{E}[(1 - A_{2m+1,2m+2})(Z_{2m+1} - Z_{2m+2})^2 | \tilde{S}_m] &= (1-p)\mathbb{E}[(\tilde{T}_{2m}^\top Y_m)^2] \\ &= (1-p)\|\tilde{T}_{2m}\|^2 \mathbb{E}[Y_{21}^2] \\ &= 4mp(1-p)^2. \end{aligned}$$

Therefore, using the fact that $\sqrt{x+y} \leq \sqrt{x} + \sqrt{y}$ for $x, y \geq 0$, we have

$$\begin{aligned} & \mathbb{E}[|2\tilde{S}_m^\top Y_m + 2(1 - A_{2m+1,2m+2})(Z_{2m+1} - Z_{2m+2})| | \tilde{S}_m] \\ & \leq 2\sqrt{2\|\tilde{S}_m\|^2 p(1-p) + 2mp(1-p)^2} \\ & \leq 2\|\tilde{S}_m\| \sqrt{2p(1-p)} + 4\sqrt{mp}(1-p). \end{aligned}$$

As $\|\tilde{S}_m\| = \tilde{I}_m$, we have

$$\begin{aligned} & \mathbb{E}[\tilde{I}_{m+1}^2 - \tilde{I}_m^2] \\ &= \mathbb{E}[\mathbb{E}[\tilde{I}_{m+1}^2 - \tilde{I}_m^2 | \tilde{S}_m]] \\ & \geq 8mp(1-p) + 2(1-p) - 2(2b-1)(\mathbb{E}[\tilde{I}_m] \sqrt{2p(1-p)} + 2\sqrt{2mp}(1-p)) \\ &= 8mp(1-p) + 2(1-p) - 4\sqrt{2}(2b-1)mp(1-p) - \sqrt{2}(2b-1)p(1-p) \\ & \quad - 4\sqrt{mp}(2b-1)(1-p) \\ &= (8 - 4\sqrt{2}(2b-1))mp(1-p) - (4\sqrt{mp}(2b-1) + 2 - \sqrt{2}(2b-1)p)(1-p). \end{aligned}$$

Recalling that $\mathbb{E}[\tilde{I}_1^2] = 2(1-p)$, and using $\sum_{m=1}^{n-1} \sqrt{m} \geq \frac{2}{3}(n-1)^{3/2}$, we have

$$\mathbb{E}[\tilde{I}_n^2] = 2(1-p) + \sum_{m=1}^{n-1} (8 - 4\sqrt{2}(2b-1))mp(1-p)$$

$$\begin{aligned}
& - (4\sqrt{mp}(2b-1) + 2 - \sqrt{2}(2b-1)p)(1-p) \\
& \geq (4 - 2\sqrt{2}(2b-1))n(n-1)p(1-p) - \frac{8}{3}(2b-1)\sqrt{p}(1-p)(n-1)^{3/2} \\
& \quad - (2 - \sqrt{2}(2b-1)p)(n-1)(1-p).
\end{aligned}$$

3. Lower bound of $\mathbb{E}[\tilde{I}_n^3]$ Combining with Jensen's inequality, we have

$$\mathbb{E}[\tilde{I}_n^3] \geq \mathbb{E}[\tilde{I}_n^2]^{3/2} \geq (4 - 2\sqrt{2}(2b-1))n^2p(1-p) + O(n^{3/2})\sqrt{p}(1-p)^{3/2}.$$

Since $(x+y)^{3/2} \geq x^{3/2} + y^{3/2}$ for $x, y \geq 0$, we have

$$\mathbb{E}[\tilde{I}_n^3] \geq (4 - 2\sqrt{2}(2b-1))^{3/2}n^3p^{3/2}(1-p)^{3/2} + O(n^{9/4}p^{3/4}(1-p)^{3/2}).$$

4. Upper bound of $\mathbb{E}[\tilde{I}_n^4]$ Now we are ready to establish the upper bound of the fourth moment of \tilde{I}_n . We have

$$\begin{aligned}
\mathbb{E}[\tilde{I}_{m+1}^4] &= \mathbb{E}[B \max(U_m^2, V_m^2) + (1-B) \min(U_m^2, V_m^2)] \\
&= \mathbb{E}[U_m^2] - (2b-1)\mathbb{E}[|U_m^2 - V_m^2|].
\end{aligned} \tag{A.3}$$

where the first term

$$\begin{aligned}
\mathbb{E}[U_m^2] &= \mathbb{E}[(\|\tilde{S}_m + Y_m\|^2 + (Z_{2m+1} - 1 + A_{2m+1,2m+2})^2 \\
&\quad + (Z_{2m+2} + 1 - A_{2m+1,2m+2})^2)].
\end{aligned}$$

As $\mathbb{E}[\tilde{S}_m] = 0$ and \tilde{S}_m is independent of Y_m , Z_{2m+1} and Z_{2m+2} , all of the cross terms containing $S_m^\top Y_m$ have expectation 0. Using $\mathbb{E}[\tilde{S}_m^\top Y_m] = 0$, it is easy to check

$$\mathbb{E}[(\tilde{S}_m^\top Y_m)^2] = \text{var}(\tilde{S}_m^\top Y_m) = 2p(1-p)\mathbb{E}[\|\tilde{S}_m\|^2].$$

By independence again, we have

$$\mathbb{E}[2\|\tilde{S}_m\|^2\|Y_m\|^2] = 2\mathbb{E}[\|\tilde{S}_m\|^2]\mathbb{E}[\|Y_m\|^2] = 8mp(1-p)\mathbb{E}[\|\tilde{S}_m\|^2],$$

and recalling that $\mathbb{E}[(Z_{2m+1}-1+A_{2m+1,2m+2})^2] = \mathbb{E}[(Z_{2m+2}+1-A_{2m+1,2m+2})^2] = 2mp(1-p) + 1-p$, we have

$$\begin{aligned}
\mathbb{E}[2\|\tilde{S}_m\|^2(Z_{2m+1}-1+A_{2m+1,2m+2})^2] &= \mathbb{E}[\|\tilde{S}_m\|^2(Z_{2m+2}+A_{2m+2,2m+1}-1)^2] \\
&= 2(2mp+1)(1-p)\mathbb{E}[\|\tilde{S}_m\|^2].
\end{aligned}$$

The other terms do not contain \tilde{S}_m . We first compute the fourth moments:

$$\begin{aligned}
\mathbb{E}[\|Y_m\|^4] &= \mathbb{E}\left[\left(\sum_{i=1}^{2m} Y_{i,m+1}\right)^4\right] = \sum_{i=1}^{2m} \mathbb{E}[Y_{i,m+1}^4] + \sum_{1 \leq i < j \leq m} \mathbb{E}[Y_{i,m+1}^2 Y_{j,m+1}^2] \\
&= 4mp(1-p) + 2m(2m-1)p^2(1-p)^2,
\end{aligned}$$

and we recall that Z_{2m+1} has the same distribution as $\sum_{i=1}^m Y_{2m+1,i}$

$$\begin{aligned}\mathbb{E}[Z_{2m+1}^4] &= \sum_{i=1}^m \mathbb{E}[Y_{2m+1,i}^4] + \binom{4}{2} \sum_{1 \leq i < j \leq m} \mathbb{E}[Y_{2m+1,i}^2 Y_{2m+1,j}^2] \\ &= 2mp(1-p) + 12m(m-1)p^2(1-p)^2.\end{aligned}$$

We have $\mathbb{E}[Z_{2m+2}^4] = 2mp(1-p) + 12m(m-1)p^2(1-p)^2$ in the same manner. By symmetry of Z_{2m+1} , we have $\mathbb{E}[Z_{2m+1}] = \mathbb{E}[Z_{2m+1}^3] = 0$, so

$$\begin{aligned}\mathbb{E}[(Z_{2m+1} - 1 + A_{2m+1,2m+2})^4] &= \mathbb{E}[Z_{2m+1}^4 + 6Z_{2m+1}^2(A_{2m+1,2m+2} - 1)^2 + (A_{2m+1,2m+2} - 1)^4] \\ &= 2mp(1-p) + 12m(m-1)p^2(1-p)^2 + 6(2mp(1-p))(1-p) + 1-p \\ &\leq (14mp+1)(1-p) + 12m^2p^2(1-p)^2.\end{aligned}$$

Applying these bounds, we have

$$\begin{aligned}\mathbb{E}[U_m^2] &= \mathbb{E}[(\|\tilde{S}_m + Y_m\|^2 + (Z_{2m+1} - 1 + A_{2m+1,2m+2})^2 \\ &\quad + (Z_{2m+2} + 1 - A_{2m+1,2m+2})^2)^2] \\ &= \mathbb{E}[\|\tilde{S}_m\|^4 + (2\tilde{S}_m^\top Y_m)^2 + \|Y_m\|^4 + (Z_{2m+1} - 1 + A_{2m+1,2m+2})^4 \\ &\quad + (Z_{2m+2} + 1 - A_{2m+1,2m+2})^4 \\ &\quad + 2\|\tilde{S}_m\|^2\|Y_m\|^2 + 2\|\tilde{S}_m\|^2(Z_{2m+1} - 1 + A_{2m+1,2m+2})^2 \\ &\quad + 2\|\tilde{S}_m\|^2(Z_{2m+2} + 1 - A_{2m+1,2m+2})^2 \\ &\quad + 2\|Y_m\|^2(Z_{2m+1} - 1 + A_{2m+1,2m+2})^2 + 2\|Y_m\|^2(Z_{2m+2} + 1 - A_{2m+1,2m+2})^2 \\ &\quad + 2(Z_{2m+1} - 1 + A_{2m+1,2m+2})^2(Z_{2m+2} + 1 - A_{2m+1,2m+2})^2] \\ &= \mathbb{E}[\|\tilde{S}_m\|^4] + 4p(1-p)\mathbb{E}[\|\tilde{S}_m\|^2] + 4mp(1-p) + 2m(2m-1)p^2(1-p)^2 \\ &\quad + 2(2mp(1-p) + 12m(m-1)p^2(1-p)^2) + 2\mathbb{E}[\|\tilde{S}_m\|^2](4mp(1-p)) \\ &\quad + 4\mathbb{E}[\|\tilde{S}_m\|^2](2mp(1-p) + 1-p) + 4(4mp(1-p))(2mp(1-p) + 1-p) \\ &\quad + 4(2mp(1-p) + 1-p)^2 \\ &= \mathbb{E}[\|\tilde{S}_m\|^4] + 16mp(1-p)\mathbb{E}[\|\tilde{S}_m\|^2] + O((m^2p(1-p) + \mathbb{E}[\|\tilde{S}_m\|^2])) \\ &= \mathbb{E}[\|\tilde{S}_m\|^4] + 16mp(1-p)\mathbb{E}[\|\tilde{S}_m\|^2] + O(m^2p(1-p)).\end{aligned}$$

We denote as ‘‘high-order terms’’ those whose expected values have an order of at most $O((m^2p(1-p) + \mathbb{E}[\|\tilde{S}_m\|^2])p(1-p))$. Now let us consider the terms in $U_m^2 - V_m^2$ with absolute value. The only term that does not belong to the high-order terms is $4(2b-1)\mathbb{E}[\|\tilde{S}_m\|^2|\tilde{S}_m^\top Y_m|]$. In other words,

$$|\mathbb{E}[U_m^2 - V_m^2] - 4(2b-1)\mathbb{E}[\|\tilde{S}_m\|^2|\tilde{S}_m^\top Y_m|]| = O(m^2p(1-p)).$$

By Lemma 1, we have

$$\mathbb{E}[\|\tilde{S}_m\|^2|\tilde{S}_m^\top Y_m|] = \mathbb{E}[\mathbb{E}[\|\tilde{S}_m\|^2|\tilde{S}_m^\top Y_m||\tilde{S}_m]] \geq \mathbb{E}[2p(1-p)\|\tilde{S}_m\|^3]$$

$$= 2p(1-p)\mathbb{E}[\tilde{I}_m^3].$$

Applying this inequality to (A.3), we have

$$\begin{aligned} \mathbb{E}[\tilde{I}_{m+1}^4] &\leq \mathbb{E}[\tilde{I}_m^4] + 16mp(1-p)\mathbb{E}[\tilde{I}_m^2] - 8(2b-1)p(1-p)\mathbb{E}[\tilde{I}_m^3] \\ &\quad + O((m^2p(1-p) + \mathbb{E}[\tilde{I}_m^2])p(1-p)) \\ &\leq \mathbb{E}[\tilde{I}_m^4] + 16mp(1-p)(4m^2p(1-p) + 2m(1-p)) + O(m^2p(1-p)) \\ &\quad - 8(2b-1)p(1-p)[(4-2\sqrt{2}(2b-1))^{3/2}m^3p^{3/2}(1-p)^{3/2} \\ &\quad + O(m^{9/4}p^{3/4}(1-p)^{3/2})] \\ &= \mathbb{E}[\tilde{I}_m^4] + 64m^3p^2(1-p)^2 - 8(2b-1)(4-2\sqrt{2}(2b-1))^{3/2} \\ &\quad \times m^3p^{5/2}(1-p)^{5/2} + O(m^{9/4}p(1-p)). \end{aligned}$$

In the first stage, the imbalance measurement $\mathbb{E}[\tilde{I}_1^4] = (1-A_{12})^4 + (A_{21}-1)^4 = 2(1-p)$. Thus

$$\begin{aligned} \mathbb{E}[\tilde{I}_n^4] &= 2(1-p) + \sum_{m=1}^{n-1} \mathbb{E}[\tilde{I}_{m+1}^4 - \tilde{I}_m^4] \\ &= 2(1-p) + \sum_{m=1}^{n-1} 64m^3p^2(1-p)^2 - 8(2b-1)(4-2\sqrt{2}(2b-1))^{3/2} \\ &\quad \times m^3p^{5/2}(1-p)^{5/2} + O(m^{9/4}p(1-p)) \\ &= (16p^2(1-p)^2 - 2(2b-1)(2-\sqrt{2}(2b-1))^{3/2}p^{5/2}(1-p)^{5/2})n^4 \\ &\quad + O(n^{13/4}p(1-p)). \end{aligned}$$

Therefore,

$$\limsup_{n \rightarrow \infty} \frac{\mathbb{E}[\tilde{I}_n^4]}{n^4} \leq 16p^2(1-p)^2 - 2(2b-1)(2-\sqrt{2}(2b-1))^{3/2}p^{5/2}(1-p)^{5/2},$$

which proves (A.2). \square

A.3. Proof of Theorem 3

The outline of the proof is very similar to that of Theorem 2. We adopt the definitions of \tilde{I}_m , \tilde{S}_m , Y_{ij} , Y_m , Z_{2m+1} and Z_{2m+2} , U_m and V_m . As we replace the Erdős-Rényi random graph model by the GOE, we have $A_{ij} \sim (0, \sigma^2)$ for $1 \leq i \leq j \leq n$ and $A_{ij} = A_{ji}$ if $1 \leq i < j \leq n$. Then $Y_{ij} \sim \mathcal{N}(0, 2\sigma^2)$, $Z_{2m+1}, Z_{2m+2} \sim \mathcal{N}(0, 2m\sigma^2)$. Hence

$$\|Y_{ij}\|^2 \sim 2\sigma^2\chi_{2m}^2 \quad \text{and} \quad Z_{2m+1}^2, Z_{2m+2}^2 \sim 2m\sigma^2\chi_1^2.$$

We need to replace the moments of these variables in the proof of Theorem 2.

1. Upper bound of $\mathbb{E}[\tilde{I}_n]$ The conditional expectation of \tilde{I}_{m+1}^2 is bounded by

$$\begin{aligned}
& \mathbb{E}[\tilde{I}_{m+1}^2 | \tilde{S}_m] \\
& \leq \mathbb{E}[\|\tilde{S}_m + Y_m\|^2 + (Z_{2m+1} - 1 + A_{2m+1,2m+2})^2 \\
& \quad + (Z_{2m+2} + 1 - A_{2m+1,2m+2})^2 | \tilde{S}_m] \\
& = \mathbb{E}[\|\tilde{S}_m\|^2 + 2\tilde{S}_m^\top Y_m + \|Y_m\|^2 + (Z_{2m+2} + 1 - A_{2m+1,2m+2})^2 \\
& \quad + (Z_{2m+2} + 1 - A_{2m+1,2m+2})^2 | \tilde{S}_m] \\
& = \|\tilde{S}_m\|^2 + 4m\sigma^2 + (2m+1)\sigma^2 + (2m+1)\sigma^2 + 2 \\
& = \|\tilde{S}_m\|^2 + (8m+2)\sigma^2 + 2.
\end{aligned}$$

Therefore, $\mathbb{E}[\tilde{I}_{m+1}^2 - \tilde{I}_m^2 | \tilde{I}_m^2] \leq (8m+2)\sigma^2 + 2$. Hence

$$\mathbb{E}[\tilde{I}_{m+1}^2 - \tilde{I}_m^2] = \mathbb{E}[\mathbb{E}[\tilde{I}_{m+1}^2 - \tilde{I}_m^2 | \tilde{I}_m^2]] \leq (8m+2)\sigma^2 + 2.$$

In the first stage, $\mathbb{E}[\tilde{I}_1^2] \leq 2 + 2\sigma^2$. Thus,

$$\mathbb{E}[\tilde{I}_n^2] \leq 2 + 2\sigma^2 + \sum_{m=1}^{n-1} (8m+2)\sigma^2 + 2 \leq 4n^2\sigma^2 + 2n \leq \left(2n\sigma + \frac{1}{2\sigma}\right)^2.$$

By Jensen's inequality,

$$\mathbb{E}[\tilde{I}_n] \leq \sqrt{\mathbb{E}[\tilde{I}_n^2]} \leq 2n\sigma + \frac{1}{2\sigma}.$$

Lower bound of $\mathbb{E}[\tilde{I}_n^2]$ As we did in the proof of Theorem 2, to find the lower bound of $\mathbb{E}[\tilde{I}_{m+1}^2 - \tilde{I}_m^2]$, we need to find the upper bound for

$$\begin{aligned}
& (\mathbb{E}[|2\tilde{S}_m^\top Y_m + 2(1 - A_{2m+1,2m+2})(Z_{2m+1} - Z_{2m+2})| | \tilde{S}_m])^2 \\
& \leq \mathbb{E}[4(\tilde{S}_m^\top Y_m)^2 + 8(1 - A_{2m+1,2m+2})(Z_{2m+1} - Z_{2m+2})\tilde{S}_m^\top Y_m \\
& \quad + 4(1 - A_{2m+1,2m+2})(Z_{2m+1} - Z_{2m+2})^2 | \tilde{S}_m].
\end{aligned}$$

The second term has expectation 0 for the same reason as in the proof of Theorem 2. Since $Y_m \sim \mathcal{N}(0, 2\sigma^2 \mathbf{I}_m)$, we have $\mathbb{E}[(\tilde{S}_m^\top Y_m)^2 | \tilde{S}_m] = 2\sigma^2 \|\tilde{S}_m\|^2$. We recall that $Z_{2m+1}, Z_{2m+2} \sim \mathcal{N}(0, 2m\sigma^2)$, so

$$\mathbb{E}[(1 - A_{2m+1,2m+2})(Z_{2m+1} - Z_{2m+2})^2 | \tilde{S}_m] = 4m\sigma^2.$$

Applying $Z_{2m+1} - Z_{2m+2} = \mathbf{1}_{2m}^\top Y_m$, we have

$$\begin{aligned}
& \mathbb{E}[|2\tilde{S}_m^\top Y_m + 2(1 - A_{2m+1,2m+2})(Z_{2m+1} - Z_{2m+2})| | \tilde{S}_m] \\
& = 2\mathbb{E}[(\tilde{S}_m + (1 - A_{2m+1,2m+2})\mathbf{1}_{2m})^\top Y_m | \tilde{S}_m] \\
& = 2\sqrt{2/\pi} \mathbb{E}[\|\tilde{S}_m + (1 - A_{2m+1,2m+2})\mathbf{1}_{2m}\| | \tilde{S}_m]
\end{aligned}$$

$$\geq 2\sqrt{2/\pi}(\|\tilde{S}_m\| + \sqrt{2m(1+\sigma^2)})\sigma,$$

where in the last equality, we use if $X \sim \mathcal{N}(0, \sigma^2)$, then $\mathbb{E}[|X|] = \sigma\sqrt{2/\pi}$. Using the same definition of U_m and V_m from the proof of Theorem 2, we have

$$\begin{aligned} & \mathbb{E}[\tilde{I}_{m+1}^2 | \tilde{S}_m] \\ &= \mathbb{E}[B \min(U_m, V_m) + (1-B) \max(U_m, V_m) | \tilde{S}_m] \\ &= \mathbb{E}[U_m | \tilde{S}_m] - (2b-1)\mathbb{E}[|U_m - V_m| | \tilde{S}_m] \\ &= \|\tilde{S}_m\|^2 + 8m\sigma^2 + 2 \\ &\quad - (2b-1)\mathbb{E}[|2\tilde{S}_m^\top Y_m + 2(1-A_{2m+1,2m+2})(Z_{2m+1} - Z_{2m+2})| | \tilde{S}_m] \\ &\geq \|\tilde{S}_m\|^2 + 8m\sigma^2 + 2 - 2(2b-1)\sqrt{2/\pi}(\|\tilde{S}_m\| + \sqrt{2m(1+\sigma^2)})\sigma \\ &\geq \|\tilde{S}_m\|^2 + 8m\sigma^2 + 2 - 2(2b-1)\sqrt{2/\pi}(2m\sigma^2 + \sqrt{2m(1+\sigma^2)}\sigma + 0.5) \\ &= \|\tilde{S}_m\|^2 + (8 - 4(2b-1)\sqrt{2/\pi})m\sigma^2 + O(\sqrt{m}\sigma). \end{aligned}$$

Using $\sum_{m=1}^{n-1} \sqrt{m} \geq \frac{2}{3}(n-1)^{3/2}$, we have

$$\begin{aligned} \mathbb{E}[\tilde{I}_n^2] &= \mathbb{E}[\tilde{I}_1^2] + \sum_{i=1}^{n-1} \mathbb{E}[\mathbb{E}[\tilde{I}_{i+1}^2 - \tilde{I}_i^2 | \tilde{S}_i]] \\ &\geq 2 + 2\sigma^2 + \sum_{m=1}^{n-1} (8 - 4(2b-1)\sqrt{2/\pi})m\sigma^2 + O(\sqrt{m}\sigma) \\ &\geq (4 - 2\sqrt{2/\pi}(2b-1))n^2\sigma^2 + O(n^{3/2}\sigma). \end{aligned}$$

Lower bound of $\mathbb{E}[\tilde{I}_n^3]$ By Jensen's inequality and the fact that $(x+y)^{3/2} \geq x^{3/2} + y^{3/2}$, we have

$$\mathbb{E}[\tilde{I}_n^3] \geq \mathbb{E}[\tilde{I}_n^2]^{3/2} \geq (4 - \sqrt{2/\pi}(2b-1))^{3/2} n^3 \sigma^3 + O(n^{9/4} \sigma^{3/2}).$$

Upper bound of $\mathbb{E}[\tilde{I}_n^4]$ We have

$$\begin{aligned} \mathbb{E}[\tilde{I}_{m+1}^4] &= \mathbb{E}[B \max(U_m^2, V_m^2) + (1-B) \min(U_m^2, V_m^2)] \\ &= \mathbb{E}[U_m^2] - (2b-1)\mathbb{E}[|U_m^2 - V_m^2|], \end{aligned}$$

from (A.3). Using the same arguments in the proof of Theorem 2, we have

$$\mathbb{E}[U_m^2] = \mathbb{E}[\|\tilde{S}_m\|^4] + 16m\sigma^2\mathbb{E}[\|\tilde{S}_m\|^2] + O(m^2\sigma^2),$$

and

$$\mathbb{E}[|U_m^2 - V_m^2|] = 4(2b-1)\mathbb{E}[\|\tilde{S}_m\|^2 |\tilde{S}_m^\top Y_m|] + O(m^2\sigma^2).$$

Using the expectation of a folded normal random variable again, we have

$$\mathbb{E}[\|\tilde{S}_m\|^2 |\tilde{S}_m^\top Y_m|] = \mathbb{E}[\|\tilde{S}_m\|^2 |\tilde{S}_m^\top Y_m|]$$

$$\begin{aligned}
&= \mathbb{E}[\mathbb{E}[\|\tilde{S}_m\|^2 | \tilde{S}_m^\top Y_m] | \tilde{S}_m] \\
&= \sqrt{2/\pi} \sigma \mathbb{E}[\tilde{I}_m^3].
\end{aligned}$$

We apply the upper bound of $\mathbb{E}[\tilde{I}_m^2]$ and lower bound of $\mathbb{E}[\tilde{I}_m^3]$, and have

$$\begin{aligned}
\mathbb{E}[\tilde{I}_{m+1}^4] &\leq \mathbb{E}[\tilde{I}_m^4] + 64m^3\sigma^4 - 4(2b-1)\sqrt{2/\pi}(4 - \sqrt{2/\pi}(2b-1))^{3/2}m^3\sigma^4 \\
&\quad + O(m^{9/4}\sigma^{5/2} + m^2\sigma^2).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\mathbb{E}[\tilde{I}_n^4] &= \mathbb{E}[\tilde{I}_1^4] + \sum_{m=1}^{n-1} \mathbb{E}[\tilde{I}_{m+1}^4 - \tilde{I}_m^4] \\
&\leq (16 - 4(2b-1)\sqrt{2/\pi}(4 - \sqrt{2/\pi}(2b-1))^{3/2})m^4\sigma^4 \\
&\quad + O(m^{13/4}\sigma^{5/2} + m^3\sigma^2).
\end{aligned}$$

Assuming $n\sigma^2 \rightarrow \infty$, we have

$$O\left(\frac{m^{13/4}\sigma^{5/2} + m^3\sigma^2}{m^4\sigma^4}\right) = O((n\sigma^2)^{-3/4} + (n\sigma^2)^{-1}) \rightarrow 0.$$

Hence

$$\limsup_{n \rightarrow \infty} \frac{\mathbb{E}[\tilde{I}_n^4]}{m^4\sigma^4} \leq (16 - 4(2b-1)\sqrt{2/\pi}(4 - \sqrt{2/\pi}(2b-1))^{3/2}),$$

as desired.

A.4. Auxiliary lemmas

Lemma 1 (Khinchin-Kahane inequality). *For $i \in [n]$, let $Y_i = -1, 0, 1$ with probability $p_i, 1-2p_i, p_i$ identically and independently distributed for $p_i \in (0, 1/2)$. Then we have*

$$\min_{i \in [n]} 2p_i \leq \inf_{x \in S^{n-1}} \mathbb{E}[|x^\top Y|] \leq \sup_{x \in S^{n-1}} \mathbb{E}[|x^\top Y|] \leq \max_{i \in [n]} \sqrt{2p_i},$$

where $S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$.

Proof.

Upper bound For $\|x\| = 1$, we have

$$\mathbb{E}[|x^\top Y|^2] \leq \mathbb{E}[|x^\top Y|^2] = \sum_{i=1}^n x_i^2 \mathbb{E}[|Y_i|^2] = \|x\|^2 \max_{i \in [n]} \mathbb{E}[Y_i^2] = \max_{i \in [n]} \mathbb{E}[Y_i^2].$$

Hence $\mathbb{E}[|x^\top Y|] \leq \max_{i \in [n]} \sqrt{\mathbb{E}[Y_i^2]} = \max_{i \in [n]} \sqrt{2p_i}$.

Lower bound Let us define $S_+^{n-1} = \{x \in S^{n-1} : \forall i \in [n], x_i \geq 0\}$. By symmetry of Y_i , we only need to consider the infimum for x over S_+^{n-1} to avoid loss of generality. We claim that if $j = \arg \min_{i \in [n]} 2p_i$, then the minimum is achieved at $x = e_j$. We will prove this by induction. The claim is clearly correct when $n = 1$. Now let us consider the case $n + 1$, given the statement is true for n . In other words, it suffices to show that

$$\inf_{x \in S^n} \mathbb{E}[|x^\top Y|] = \inf_{x \in S^{n-1}} \inf_{\theta \in [0, \pi/2]} \mathbb{E}[|x^\top Y \cos \theta + Y_{n+1} \sin \theta|] \geq \min_{i \in [n+1]} 2p_i,$$

given

$$\inf_{x \in S^{n-1}} \mathbb{E}[|x^\top Y|] \geq \min_{i \in [n]} 2p_i.$$

We note that for $x \in S_+^{n-1}$, $\|(x \cos \theta, \sin \theta)\|^2 = \|x\|^2 \cos^2 \theta + \sin^2 \theta = 1$. We have

$$\begin{aligned} & \mathbb{E}[|x^\top Y \cos \theta + Y_{n+1} \sin \theta|] \\ &= \sum_{y=-1}^1 \mathbb{E}[|x^\top Y \cos \theta + Y_{n+1} \sin \theta| | Y_{n+1} = y] \mathbb{P}(Y_{n+1} = y) \\ &= (1 - 2p_{n+1}) \cos \theta \mathbb{E}[|x^\top Y|] + p_{n+1} \mathbb{E}[|x^\top Y \cos \theta + \sin \theta|] \\ & \quad + p_{n+1} \mathbb{E}[|x^\top Y \cos \theta - \sin \theta|]. \end{aligned}$$

By symmetry of $x^\top Y$, we have

$$\mathbb{E}[|x^\top Y \cos \theta - \sin \theta|] = \mathbb{E}[|-x^\top Y \cos \theta - \sin \theta|] = \mathbb{E}[|x^\top Y \cos \theta + \sin \theta|].$$

By Lemma 2, we have

$$\mathbb{E}[|x^\top Y \cos \theta + \sin \theta|] = \mathbb{E}[\max\{|x^\top Y| \cos \theta, \sin \theta\}] \geq \max\{\mathbb{E}[|x^\top Y|] \cos \theta, \sin \theta\}.$$

Therefore,

$$\begin{aligned} & \mathbb{E}[|x^\top Y \cos \theta + Y_{n+1} \sin \theta|] \\ &= (1 - 2p_{n+1}) \mathbb{E}[|x^\top Y|] \cos \theta + 2p_{n+1} \mathbb{E}[|x^\top Y \cos \theta + \sin \theta|] \\ &\geq (1 - 2p_{n+1}) \mathbb{E}[|x^\top Y|] \cos \theta + 2p_{n+1} \max\{\mathbb{E}[|x^\top Y|] \cos \theta, \sin \theta\} \\ &= \max\{\mathbb{E}[|x^\top Y|] \cos \theta, (1 - 2p_{n+1}) \mathbb{E}[|x^\top Y|] \cos \theta + 2p_{n+1} \sin \theta\}. \end{aligned}$$

The last line is a concave function corresponding to the variable $\theta \in [0, \pi/2]$. For every $x \in S_+^{n-1}$, it achieves the minimum when either $\theta = 0$ or $\theta = 1$. Thus for every $x \in S^n$ and $\theta \in [0, \pi/2]$,

$$\begin{aligned} & \mathbb{E}[|x^\top Y \cos \theta + Y_{n+1} \sin \theta|] \\ &\geq \min\{\max\{\mathbb{E}[|x^\top Y|], (1 - 2p_{n+1}) \mathbb{E}[|x^\top Y|]\}, \max\{0, 2p_{n+1}\}\} \\ &= \min\{\mathbb{E}[|x^\top Y|], 2p_{n+1}\}. \end{aligned}$$

By the inductive assumption, $\mathbb{E}[|x^\top Y|] \geq \min_{i \in [n]} 2p_i$, so

$$\inf_{x \in S^n} \mathbb{E}[|x^\top Y|] = \inf_{x \in S^{n-1}} \inf_{\theta \in [0, \pi/2]} \mathbb{E}[|x^\top Y \cos \theta + Y_{n+1} \sin \theta|] \geq \min_{i \in [n+1]} 2p_i,$$

which finishes the proof. \square

Lemma 2. *Suppose Y is a symmetric random variable and $x \geq 0$ is fixed, then*

$$\mathbb{E}[|Y + x|] \geq \max\{\mathbb{E}[|Y|], x\}.$$

Proof. We assume Y is discrete. Other cases simply follow from the arguments below.

$$\begin{aligned} \mathbb{E}[|Y + x|] &= \sum_y |y + x| \mathbb{P}(Y = y) = x \mathbb{P}(Y = 0) + \sum_{y > 0} (y + x + |-y + x|) \mathbb{P}(Y = y) \\ &\geq x \mathbb{P}(Y = 0) + \sum_{y > 0} (y + x + y - x) \mathbb{P}(Y = y) \geq \sum_{y > 0} 2y \mathbb{P}(Y = y) = \mathbb{E}[|Y|]. \end{aligned}$$

Additionally,

$$\begin{aligned} \mathbb{E}[|Y + x|] &= x \mathbb{P}(Y = 0) + \sum_{y > 0} (y + x + |-y + x|) \mathbb{P}(Y = y) \\ &\geq x \mathbb{P}(Y = 0) + \sum_{y > 0} (y + x - x + y) \mathbb{P}(Y = y) = x. \end{aligned}$$

The proof is complete. \square

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