Tempered positive Linnik processes and their representations[*]

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Abstract: This paper analyzes various classes of processes associated with the tempered positive Linnik (TPL) distribution. We provide several subordinated representations of TPL Lévy processes and in particular establish a stochastic self-similarity property with respect to negative binomial subordination. In finite activity regimes we show that the explicit compound Poisson representations give raise to innovations following Mittag-Leffler type laws which are apparently new. We characterize two time-inhomogeneous TPL processes, namely the Ornstein-Uhlenbeck (OU) Lévy-driven processes with stationary distribution and the additive process determined by a TPL law. We finally illustrate how the properties studied come together in a multivariate TPL Lévy framework based on a novel negative binomial mixing methodology. Some potential applications are outlined in the contexts of statistical anti-fraud and financial modelling.

1. Introduction

In recent years, a large body of literature has been devoted to the tempering of heavy-tailed laws – in particular, stable laws – which prove to be extremely

useful in applications to finance and physics (for a recent introduction to the topic, see the monograph by Grabchak 2016). Indeed, even if heavy-tailed distributions are well-motivated models in a probabilistic setting, extremely bold tails are not realistic for most real-world applications. This drawback has led to the introduction of models which are morphologically similar to the original distributions even if they display lighter tails. The initial case for adopting models that are similar to a stable distribution and with lighter tails is introduced in physics by Mantegna and Stanley (1994) and Koponen (1995), and subsequently in economics and finance by the seminal papers of Boyarchenko and Levendorskiĭ (2000) and Carr et al. (2002). For recent accounts on tempered distributions, see e.g. Fallahgoul and Loeper (2021) and Grabchak (2019).

From a static distributional standpoint the tempered (or "tilted") version X of a random variable (r.v.) Y by a parameter $\theta > 0$ is obtained through the Laplace transform

$$L_X(s) = \frac{L_Y(\theta + s)}{L_Y(\theta)} \ . \tag{1.1}$$

From this expression it is apparent that the original r.v. and its tempered version have distributions which are practically indistinguishable for real applications when θ is small, even if their tail behavior is radically different in the sense that the former may have infinite expectation, while the latter has all the moments finite.

This methodology is particularly well-adapted to the case in which Y follows an infinitely-divisible distribution, since in that case X is also infinitely-divisible and expression (1.1) only involves a simple manipulation of characteristic exponents. Furthermore, the Lévy measure $\nu_X(dt)$ of X is itself a tilted version of that of Y, meaning that $\nu_X(dt) = e^{-\theta t}\nu_Y(dt)$. The described tempering procedure may therefore be easily embedded in the theory of Lévy processes. The relationship on Lévy measures highlights that a tempered Lévy process is one whose small jumps occurrence is essentially indistinguishable from that of the base process, but whose large jumps are much more rare events. Additionally, tempering retains a natural interpretation in terms of equivalent measure changes in probability spaces. Let a measure P_θ be defined by means of the following martingale density

$$dP_\theta = e^{-\theta Y_t + \phi_Y(\theta)t}dP \ , \tag{1.2}$$

where ϕ_Y is the characteristic (Laplace) exponent of Y, which goes under the name of Esscher transform. Under P_θ, the Lévy process associated to the infinitely-divisible r.v. Y coincides with that associated to X. This result is of great importance in application fields where the analysis of the process dynamics under equivalent transformation of measures is of relevance, e.g. option pricing (see Hubalek and Sgarra 2006).

In the context of tempering of probability laws, Barabesi et al. (2016a) introduce the tempered positive Linnik (TPL) distribution as a tilted version of the positive Linnik (PL) distribution considered in Pakes (1998) and inspired by the classic paper of Linnik (1963). The PL law has received increasing interest since

it constitutes a generalization of the gamma law and recovers the positive stable (PS) law as a limiting case (for details, see Christoph and Schreiber 2001).

Hence, its tempered version is suitable for modelling real data. In addition, the tempering substantially extends the parameter range of the PL law and accordingly give raise to two distinct regimes embedding positive absolutely-continuous distributions, as well as mixtures of positive absolutely-continuous distributions and the Dirac mass at zero. The latter regime may be useful for modelling zero-inflated data. Finally, the tempered positive stable (TPS) law (or "Tweedie distribution"), which is central in many recent statistical applications, see e.g. Barabesi et al. (2016b), Fontaine et al. (2020), Khalin and Postnikov (2020), Ma et al. (2018), can also be recovered as a limiting case of the TPL law. A discrete version of the TPL law is suggested in Barabesi et al. (2018a), while computational issues dealing with the TPL and TPS laws are discussed in Barabesi (2020) and Barabesi and Pratelli (2014, 2015, 2019).

The different regimes exhibited by the TPL and TPS laws have proven to be valid and flexible tools for stochastic modeling of univariate data, especially in the context of international trade. In that context many peculiar features – including various degrees of skewness, heavy tails and rounding – are often observed for the same traded product and need to be dealt with simultaneously, a task which motivated the development of the TPL law and which is also shared by the three-parameter TPS law in less complicated situations. One of the goals of the present work is to extend the flexibility of such laws to the more ambitious task of modeling the joint distribution of multivariate phenomena.

Regarding the theoretical findings on the TPL law, Barabesi et al. (2016a) obtain closed formulas of the probability density function and the conditional probability density function (under the two regimes, respectively) of the TPL random variable in terms of the Mittag-Leffler function and outline the infinite-divisible and self-decomposable character of the corresponding Lévy measures – as well as their representation as a mixture of TPS laws with a gamma mixing density. Kumar et al. (2019b) study the gamma subordinated representation of the tempered Mittag-Leffler subclass of TPL Lévy process, its moment and covariance properties and provide alternative derivations of the associated Lévy densities and supporting equations for the probability density function. Leonenko et al. (2021) explore in detail the large deviation theory for TPL processes. Kumar et al. (2019a) instead analyze Linnik processes – not necessarily increasing – and their generalizations.

In this paper, we focus on a number of stochastic processes naturally arising from the TPL distribution and illustrate their representations and properties. Our findings are structural as well as methodological. First of all we provide a detailed account of the infinite divisibility property and unify and clarify the Lévy-Khintchine structure of a TPL law and prove additional properties of TPL laws, such as geometric infinite divisibility. We then study the subordinated structure of the TPL Lévy process, showing that besides the defining characterization as a gamma-subordinated distribution, such laws enjoy numerous representations in terms of a negative binomial subordination. This is in turn connected to the stochastic self-similarity property – as introduced by

Kozubowski et al. (2006) – of the PL subordinator with respect to the negative binomial subordinator. We further make clear the role of the tail parameter γ in determining two distinct regimes for the processes associated to the TPL distribution. We find that whenever $\gamma \in (0, 1]$, the TPL law is absolutely continuous and an infinite activity process occurs. In contrast, when $\gamma < 0$, the TPL has a mixed absolutely-continuous and point mass expression. We then find that the corresponding Lévy process is a compound Poisson process which we show to feature increments of "logarithmic" Mittag-Leffler type, which is seemingly a new distribution. In this context a connection with the potential measure of tempered positive stable subordinator is outlined. The absolutely-continuous case instead corresponds to a self-decomposable family distributions to which using classic theory (Barndorff-Nielsen 1997, Sato 1991) we are able to associate an Ornstein-Uhlenbeck (OU) Lévy-driven process with TPL stationary distribution and an additive TPL process. We characterize such processes. In particular, the Lévy driving noise of the OU process with TPL stationary distribution is a compound Poisson process with tempered Mittag-Leffler distribution, a probability law which as far as the authors are aware, has also not been considered before.

The study culminates in the introduction of a novel multivariate TPL law/ Lévy process which is constructed making use of the subordinated representation and the stochastic-self similarity property provided in the prior sections. Such a construction can be easily generalized to different subordinand processes and it encompasses as special cases some well-known multivariate distributions of common use for statistical applications.

The paper is organized as follows. In Section 2 – after reviewing some basic known properties – we discuss the infinite divisibility and the Lévy-Khintchine representation, as well as the self-decomposability and geometric infinite divisibility, of the TPL law. Section 3 is devoted to TPL Lévy subordinators and their various representations. In Section 4, we consider the OU Lévy-driven processes with stationary TPL distribution and the additive process generated by a TPL law. In Section 5, we propose the natural multivariate version of the TPL law, and the connected Lévy process. In Section 6, applications for statistical anti-fraud are considered, and we conclude in Section 7.

2. The tempered positive Linnik distribution

If X represents a positive random variable (r.v.) on a probability space $(\Omega, \mathcal{F}, P)$, we denote its Laplace transform $L_X(s) = \mathrm{E}[e^{-sX}]$, for all values of $s \in \mathbb{C}$ for which such expectation exists.

The PL family of laws $\mathrm{PL}(\gamma, \lambda, \delta)$ was introduced by Pakes (1998), on the basis of the original suggestion of Linnik (1963). A PL r.v. V is characterized by the Laplace transform

$$L_V(s) = \left(\frac{1}{1 + \lambda s^\gamma} \right)^\delta, \quad \mathrm{Re}(s) > 0, \tag{2.1}$$

$(\gamma, \lambda, \delta) \in (0,1] \times \mathbb{R}_+ \times \mathbb{R}_+$. For details on Linnik-type laws see e.g. Christoph and Schreiber (2001), or more recently Korolev et al. (2020) and references therein.

Barabesi et al. (2016a) propose a new family of distributions which is a tempered version of the PL family. By slightly modifying the parametrization thereby proposed, the TPL r.v. X is defined as a member of the four-parameter family $\mathrm{TPL}(\gamma, \lambda, \delta, \theta)$ with Laplace transform given by

$$L_X(s) = \left(\frac{1}{1 + \mathrm{sgn}(\gamma)\lambda((\theta + s)^\gamma - \theta^\gamma)} \right)^\delta, \quad \mathrm{Re}(s) > 0, \qquad (2.2)$$

with parameter space for $(\gamma, \lambda, \delta, \theta)$ given by

$$S = \{(-\infty, 1] \setminus \{0\} \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+\} \cup \{(0, 1] \times \mathbb{R}_+ \times \mathbb{R}_+ \times \{0\}\}. \qquad (2.3)$$

The terminology is motivated by the fact that the genesis of this distribution for $\gamma \in (0,1)$ is that of tempering PL random variables in a way analogous to the classic tempering of stable laws. If V is $\mathrm{PL}(\gamma, \lambda, \delta)$ and $\theta > 0$ then

$$L_X(s) = \frac{L_V(\theta + s)}{L_V(\theta)} = \left(\frac{1}{1 + \lambda'((\theta + s)^\gamma - \theta^\gamma)} \right)^\delta \qquad (2.4)$$

with $\lambda' = \lambda/(1 + \lambda\theta^\gamma)$. See Subsection 2.2 further on for more on this analogy.

The TPL family encompasses the PL law for $\theta = 0$, the Mittag-Leffler law proposed by Pillai (1990) for $\delta = 1$ and $\theta = 0$, and the gamma law for $\gamma = 1$, or – alternatively – for $\gamma \in (0,1]$ and $\lambda\theta^\gamma = 1$. Furthermore, a $\mathrm{TPS}(\gamma, \lambda, \theta)$ can be obtained as a limit in distribution of a $\mathrm{TPL}(\gamma, \delta\lambda, \delta, \theta)$ as $\delta \to \infty$. In addition from (2.2) we see that the TPL family is closed under convolution. More precisely let $(X_k)_{k \in \mathbb{N}}$ be a sequence of independent r.v.s with $\mathrm{TPL}(\gamma, \lambda, \delta_k, \theta)$ distribution; then $\sum_{k=1}^n X_k$ has $\mathrm{TPL}(\gamma, \lambda, \sum_{k=1}^n \delta_k, \theta)$ distribution.

The fact that in (2.2) the parameter γ is allowed to be negative has many implications and is one of the central aspects of this paper. To begin with, the two parameters subsets $(-\infty, 0)$ and $(0, 1]$ for γ determine distinct regimes in the Lebesgue decomposition of the law of a TPL variable.

Denote with $E_{a,b}^c(z)$, $z \in \mathbb{C}$, the Prabhakar (1971) three-parameter Mittag-Leffler function

$$E_{a,b}^c(z) = \sum_{k=0}^\infty \frac{(c)_k z^k}{k!\Gamma(ak + b)}, \qquad \mathrm{Re}(a) > 0, \quad \mathrm{Re}(b) > 0, \quad c \in \mathbb{C}, \qquad (2.5)$$

where $(c)_k = c(c+1)\ldots(c+k-1)$ is the Pochhammer symbol. The classic one and two-parameter Mittag-Leffler functions E_a and $E_{a,b}$ coincide with $E_{a,1}^1$ and $E_{a,b}^1$ respectively. In Barabesi et al. (2016a) it is shown that for $\gamma \in (0,1)$ a TPL random variable X has probability density function (p.d.f.) f_X given by

$$f_X(x; \gamma, \lambda, \delta, \theta) = \frac{e^{-\theta x} x^{\gamma\delta - 1}}{\lambda^\delta} E_{\gamma, \gamma\delta}^\delta \left(\frac{\lambda\theta^\gamma - 1}{\lambda} x^\gamma \right) \mathbb{1}_{\{x > 0\}}. \qquad (2.6)$$

In the case $\theta = 0$, which forces $\gamma \in (0,1]$, this collapses to the p.d.f. of a PL law, known since Linnik (1963). However, the authors observe that when instead $\gamma \in (-\infty, 0)$ the distribution is not absolutely continuous. Nevertheless, the conditional p.d.f. on the event $\{X > 0\}$ is available. For more details on PL and TPL families of laws see Barabesi et al. (2016a) and Barabesi et al. (2016b).

2.1. Infinite divisibility and Lévy-Khintchine representation

We recall that a positive random variable X is said to be infinitely-divisible if for all $n = 1, 2 \ldots$, there exist n i.i.d. random variables $X_{k,n}$, $k = 1, \ldots, n$, such that $X =^d X_{1,n} + \ldots + X_{n,n}$. Infinite divisibility of a positive r.v. X is equivalent to require that the logarithm of L_X is a Bernstein function (Schilling et al. 2012, Lemma 5.8). This means that there exists a positive measure ν supported on $\mathbb{R}_+$ such that $\int_0^\infty (1 \wedge x)\nu(dx) < \infty$ and constants $a, b \geq 0$ such that

$$\phi_X(s) := -\log(L_X(s)) = a + bs + \int_{(0,\infty)} (1 - e^{-st})\nu(dt). \tag{2.7}$$

Equation (2.7) above is called the Lévy-Khintchine decomposition of X and (a, b, ν) is referred to as the triplet of Lévy characteristics with Lévy measure ν, and the Bernstein function ϕ_X as the characteristic (Laplace) exponent (see e.g. Sato 1999).

If f and g are Bernstein functions so is $f \circ g$ (Schilling et al. 2012, Corollary 3.8, iii). Therefore, if Y and Z are independent positive infinitely-divisible r.v.s then $\phi_Z \circ \phi_Y$ is the characteristic exponent of some positive infinitely-divisible random variable X. Moreover, if (a, b, μ) and (α, β, ρ) are the Lévy characteristics triplets respectively of Y and Z the Lévy triplet of X is given by $(\phi_Z(a), b\beta, \eta)$, where for all Borel sets B not containing zero

$$\eta(B) = \int_{(0,\infty)} \mu_t^Y(B)\rho(dt) + b\mu(B), \tag{2.8}$$

where μ_t^Y is law of Y_t (Schilling et al. 2012, Theorem 5.27). Equation (2.8) has the statistical interpretation of X being a mixture of Y over the mixing density Z, and the dynamic interpretation of a subordination of increasing Lévy processes (Sato 1999, Chapter 30).

A real valued, real-supported infinitely-divisible random variable X with characteristic function $\Psi_X : \mathbb{R} \to \mathbb{C}$ is characterized instead by its Lévy exponent $\psi_X(z) = \log \Psi_X(z)$. It is a classic result that there exists a characteristic triplet (c, σ, ν) where $c \in \mathbb{R}$, $\sigma \geq 0$, and ν is a measure on $\mathbb{R}$ satisfying $\nu(\{0\}) = 0$, $\int_{\mathbb{R}} (x^2 \wedge 1)dx < \infty$, such that ψ_X has the following Lévy-Khintchine representation

$$\psi_X(z) = izc - z^2 \sigma/2 + \int_{\mathbb{R}} \left(e^{izx} - 1 - izx \mathbb{1}_{|x|<1} \right) \nu(dx). \tag{2.9}$$

The function $\psi_X(z)$ possesses a number of analytical properties which equivalently characterize the notion of Lévy exponent.

We recall that for $(\lambda, \delta) \in \mathbb{R}^2_+$ the characteristic exponent ϕ_Z and Lévy density u_Z of a gamma $G(\lambda, \delta)$ r.v. Z, whose p.d.f. is given by

$$f_Z(x; \lambda, \delta) = \frac{x^{\delta-1}}{\Gamma(\delta)\lambda^\delta} e^{-x/\lambda} \mathbb{1}_{\{x>0\}}, \tag{2.10}$$

are respectively

$$\phi_Z(s) = \delta \log(1 + \lambda s), \qquad \mathrm{Re}(s) > -1/\lambda, \tag{2.11}$$

$$u_Z(x) = \delta \frac{e^{-x/\lambda}}{x} \mathbb{1}_{\{x>0\}}. \tag{2.12}$$

The characteristic exponent and Lévy density of a $\mathrm{TPS}(\gamma, \lambda, \theta)$ r.v. Y for $(\gamma, \lambda, \theta) \in \pi^\delta(S)$, where π^δ is the projection on the $\delta = 0$ subspace of $\mathbb{R}^4$, is:

$$\phi_Y(s) = \mathrm{sgn}(\gamma)\lambda((\theta + s)^\gamma - \theta^\gamma), \qquad \mathrm{Re}(s) > 0, \tag{2.13}$$

$$u_Y(x) = \frac{|\gamma|\lambda}{\Gamma(1-\gamma)} \frac{e^{-\theta x}}{x^{\gamma+1}} \mathbb{1}_{\{x>0\}} \tag{2.14}$$

and for $\gamma \in (0, 1)$, the r.v. Y has a density that admits a series representation

$$f_Y(x; \gamma, \lambda, \theta) = \frac{e^{-\theta x + \lambda \theta^\gamma}}{x} \sum_{k=1}^{\infty} \frac{1}{k!\Gamma(-k\gamma)} \left(-\frac{x^\gamma}{\lambda} \right)^{-k} \mathbb{1}_{\{x>0\}}, \tag{2.15}$$

which can be obtained by exponentially tilting with parameter $\theta > 0$ the expression given in Sato (1999, p.88) of the density of the PS (γ, λ) law.

The following result has been established in Barabesi et al. (2016a) by considering limits of the probability measures as t tends to zero. For $\gamma \in (0, 1)$ a proof using (2.8) is offered in Kumar et al. (2019b). In the proposition below, we summarize these results and extend them to the case $\gamma < 0$.

Proposition 2.1. *Let Y and Z be independent r.v.s distributed respectively according to a $\mathrm{TPS}(\gamma, \lambda, \theta)$ and a $G(1, \delta)$ law. Then the r.v. X whose characteristic exponent ϕ_X is given by*

$$\phi_X(s) = \phi_Z(\phi_Y(s)) \tag{2.16}$$

has a $\mathrm{TPL}(\gamma, \lambda, \delta, \theta)$ distribution. As a consequence any TPL r.v. X is infinitely-divisible with triplet $(0, 0, \nu)$, where ν is an absolutely-continuous measure. Furthermore if

$$(\gamma, \lambda, \delta, \theta) \in \{\gamma < 0\} \cup \{0 < \gamma < 1, \lambda \theta^\gamma < 1\} \subset S \tag{2.17}$$

then ν has density u_X given by

$$u_X(x) = |\gamma|\delta \frac{e^{-\theta x}}{x} \left(E_{|\gamma|}\left(c_{\gamma, \lambda, \theta}\, x^{|\gamma|} \right) - \mathbb{1}_{\{\mathrm{sgn}(\gamma)=-1\}} \right) \mathbb{1}_{\{x>0\}} \tag{2.18}$$

with

$$c_{\gamma, \lambda, \theta} := \left(\frac{\lambda \theta^\gamma - \mathrm{sgn}(\gamma)}{\lambda} \right)^{\mathrm{sgn}(\gamma)}. \tag{2.19}$$

Proof. Equation (2.16) is straightforward from (2.2), (2.11) and (2.13), and X is infinitely-divisible because both Z and Y are.

Let us now assume $\gamma \in (0,1)$. We can apply (2.8) with μ_t^Y being the absolutely-continuous measures given by the densities $f_Y(x; \gamma, t\lambda, \theta)$ in (2.15) and the Lévy triplet $(0, 0, \delta \frac{e^{-t}}{t} \mathbb{1}_{\{t>0\}} dt)$ characterizing Z. Whenever $\lambda\theta^\gamma < 1$ this gives for $x > 0$ a uniformly-integrable series, which we can integrate term by term to get the following Lévy density for X

$$
\begin{aligned}
u_X(x) &= \delta \frac{e^{-\theta x}}{x} \sum_{k=1}^\infty \frac{1}{k!\Gamma(-k\gamma)} \left(-\frac{x^\gamma}{\lambda}\right)^{-k} \int_0^\infty t^{k-1} e^{(\lambda\theta^\gamma-1)t} dt \\
&= \delta \frac{e^{-\theta x}}{x} \sum_{k=1}^\infty \frac{1}{k\Gamma(-k\gamma)} \left(\frac{\lambda\theta^\gamma-1}{\lambda} x^\gamma\right)^{-k} \\
&= -\gamma\delta \frac{e^{-\theta x}}{x} \sum_{k=1}^\infty \frac{1}{\Gamma(1-k\gamma)} \left(c_{\gamma,\lambda,\theta}\, x^\gamma\right)^{-k} \\
&= \gamma\delta \frac{e^{-\theta x}}{x} E_\gamma\left(c_{\gamma,\lambda,\theta}\, x^\gamma\right)
\end{aligned}
\tag{2.20}
$$

after having applied Haubold et al. (2011), Equation (9.2), in the third line.

If instead $\gamma < 0$ we observe that we can rewrite

$$
\phi_Y(s) = \lambda\theta^\gamma \left(1 - \left(\frac{1}{1+s/\theta}\right)^{-\gamma}\right) = \lambda\theta^\gamma(1 - e^{-\phi_Z(s)}),
\tag{2.21}
$$

where Z has law $G(-\gamma, 1/\theta)$. Therefore Y is in distribution a compound Poisson process with Lévy density $\lambda\theta^\gamma f_Z$ where f_Z is the p.d.f. of Z given in (2.10), and the measures μ_t^Y are the laws of Y with intensity $\lambda\theta^\gamma$ and i.i.d. excursions Z. It is well-known (e.g. Sato 1999) that μ_t^Y have the Lebesgue decomposition, for any Borel set B

$$
\begin{aligned}
\mu_t^Y(B) &= e^{-\lambda\theta^\gamma t}\delta_0(B) + \sum_{k=1}^\infty \frac{e^{-\lambda\theta^\gamma t}}{k!}(\lambda\theta^\gamma t)^k \int_B f_Z^{*k}(x) dx \\
&= e^{-\lambda\theta^\gamma t}\delta_0(B) + \sum_{k=1}^\infty \frac{e^{-\lambda\theta^\gamma t}}{k!}(t\lambda)^k \int_B \frac{x^{-\gamma k-1}}{\Gamma(-\gamma k)} e^{-\theta x} dx
\end{aligned}
\tag{2.22}
$$

with the gamma convolution $f_Z^{*k}(x) = x^{-\gamma k-1} e^{-\theta x} \theta^{-k\gamma} / \Gamma(-\gamma k)$ and where δ_0 is the Dirac distribution concentrated in 0. According to (2.8) we have the uniformly integrable series

$$
u_X(x) = \delta \frac{e^{-\theta x}}{x} \sum_{k=1}^\infty \frac{x^{-\gamma k}}{k!\Gamma(-\gamma k)} \lambda^k \int_0^\infty t^{k-1} e^{-(\lambda\theta^\gamma+1)t} dt
$$

$$= \delta \frac{e^{-\theta x}}{x} \sum_{k=1}^{\infty} \frac{1}{k\Gamma(-\gamma k)} \left(\frac{\lambda}{\lambda \theta^\gamma + 1} x^{-\gamma} \right)^k$$

$$= -\gamma \delta \frac{e^{-\theta x}}{x} \left(E_{-\gamma} \left(c_{\gamma,\lambda,\theta} \, x^{-\gamma} \right) - 1 \right). \tag{2.23}$$

Combining (2.20) and (2.23) yields (2.18). $\qquad\qquad\square$

The case $\theta = 0$ recovers the Lévy measure of the PL distribution as given in e.g. Barndorff-Nielsen (2000).

Equation (2.18) serves as a starting point for the analysis of the processes based on a TPL a law. Furthermore, it provides the structure of the cumulants of a TPL distribution. The following result is new.

Proposition 2.2. *Under the restrictions on the parameters and in the notation of Proposition 2.1 and assuming additionally $\theta \neq 0$, for $n \in \mathbb{N}$ let κ_n^+ and κ_n^- be the cumulants of the TPL distribution respectively in the regimes $\gamma \in (0,1)$ and $\gamma \in (-\infty, 0)$. We have*

$$\kappa_n^\pm = \frac{|\gamma|\delta}{\theta^n} g_{n-1}^\pm \left(\frac{c_{\gamma,\lambda,\theta}}{\theta^{|\gamma|}} \right) \tag{2.24}$$

where $g_n^\pm(x)$ satisfy the recursion

$$g_n^\pm(x) = x|\gamma| \frac{d}{dx} g_{n-1}^\pm(x) + n g_{n-1}^\pm(x) \tag{2.25}$$

with, for $|x| < 1$,

$$g_0^+(x) = \frac{1}{1-x}, \qquad g_0^-(x) = \frac{x}{1-x}. \tag{2.26}$$

Proof. By differentiating the characteristic function, the cumulants of a positive infinitely-divisible distribution can be seen to be given by the n-th moment integral (modulo adding the linear characteristic when $n = 1$) of the Lévy measure. In our case, recalling (2.8), we have when $\gamma \in (0,1)$

$$\kappa_n^+ = \int_0^\infty x^n u_X(x) dx = \gamma \delta \int_0^\infty e^{-\theta x} \sum_{k=0}^\infty \frac{1}{\Gamma(k\gamma+1)} c_{\gamma,\lambda,\theta}^k x^{\gamma k + n - 1} dx$$

$$= \gamma \delta \sum_{k=0}^\infty \frac{1}{\Gamma(k\gamma+1)} c_{\gamma,\lambda,\theta}^k \int_0^\infty x^{\gamma k + n - 1} e^{-\theta x} dx$$

$$= \frac{\gamma \delta}{\theta^n} \sum_{k=0}^\infty \frac{\Gamma(k\gamma+n)}{\Gamma(k\gamma+1)} \left(\frac{c_{\gamma,\lambda,\theta}}{\theta^\gamma} \right)^k = \frac{\gamma \delta}{\theta^n} \sum_{k=0}^\infty \left(\frac{c_{\gamma,\lambda,\theta}}{\theta^\gamma} \right)^k (k\gamma + 1)_{n-1} \tag{2.27}$$

which is a convergent series. Here $(a)_n = a(a+1)\dots(a+n-1)$, $a \in \mathbb{R}$, is the Pochhammer symbol (rising factorial). The generating function of $g_n^+(x) = \sum_{k=0}^\infty (k\gamma+1)_n x^k$ of the sequence $a_k = (\gamma k + 1)_n$ when n is fixed can be treated as follows

$$\sum_{k=0}^\infty (k\gamma+1)_n x^k = \sum_{k=0}^\infty k\gamma(k\gamma+1)_{n-1} x^k + n \sum_{k=0}^\infty (k\gamma+1)_{n-1} x^k$$

$$= \gamma x \frac{d}{dx} \left(\sum_{k=0}^{\infty} (k\gamma + 1)_{n-1} x^k \right) + n \sum_{k=0}^{\infty} (k\gamma + 1)_{n-1} x^k \quad (2.28)$$

Since $g_0^+(x) = 1/(1-x)$, (2.25) follows in the positive γ regime.

If instead $\gamma < 0$ we have

$$\kappa_n^- = \int_0^{\infty} x^n u_X(x) dx = -\gamma\delta \int_0^{\infty} e^{-\theta x} \sum_{k=1}^{\infty} \frac{1}{\Gamma(-k\gamma + 1)} \, c_{\gamma,\lambda,\theta}^k x^{-\gamma k + n - 1} dx$$

$$= -\gamma\delta \sum_{k=1}^{\infty} \frac{1}{\Gamma(k\gamma + 1)} \, c_{\gamma,\lambda,\theta}^k \int_0^{\infty} x^{-\gamma k + n - 1} e^{-\theta x} dx$$

$$= -\frac{\gamma\delta}{\theta^n} \sum_{k=1}^{\infty} \frac{\Gamma(-k\gamma + n)}{\Gamma(-k\gamma + 1)} \, (c_{\gamma,\lambda,\theta} \, \theta^\gamma)^k = \frac{(-\gamma)\delta}{\theta^n} \sum_{k=1}^{\infty} (c_{\gamma,\lambda,\theta} \, \theta^\gamma)^k \, (-k\gamma + 1)_{n-1}$$

$$(2.29)$$

and the series again converges for all the admissible parameters values. Setting $g_n^-(x) = \sum_{k=1}^{\infty} (-k\gamma + 1)_n x^k$ applying (2.28) and observing $g_0^-(x) = x/(1-x)$ completes the proof. $\qquad\square$

From (2.24) we find the mean and variance of a TPL r.v. X to be

$$\mathrm{E}[X] = |\gamma|\delta\lambda\theta^{\gamma-1}, \qquad \mathrm{Var}[X] = \frac{\mathrm{E}[X]}{\delta} \left(\frac{1-\gamma}{\theta} + \mathrm{E}[X] \right) \qquad (2.30)$$

which correspond to those calculated in Barabesi et al. (2016a) (albeit in a different parametrization).

2.2. *Self-decomposability and geometric infinite divisibility*

A property stronger than infinite divisibility and highly consequential for the study of stochastic processes associated to probability distributions is the so called self-decomposability. A random variable X is said to be self-decomposable if for all $\alpha \in (0,1)$ there exists a r.v. X_α independent from X such that $X =^d \alpha X + X_\alpha$. A self-decomposable distribution is known to be infinitely-divisible, and absolutely continuous with an absolutely continuous Lévy density (Steutel and Van Harn, 2004). Several stochastic processes can be canonically constructed starting from a self-decomposable law, something that we shall exploit in Section 4 once the self-decomposable nature of a TPL r.v. is established.

Another property stronger than infinite divisibility is geometric infinite divisibility introduced by Klebanov et al. (1985). A random variable X is said to be geometrically infinitely-divisible (g.i.d.) if for any $p \in (0,1)$, there exists a geometric random variable G_p with probability mass function (p.m.f.)

$$P(G_p = k) = p^{k-1}(1-p), \qquad k = 1, 2, \ldots, \qquad (2.31)$$

and i.i.d. r.v.s $Z_{n,p}$, $n = 1, 2, \ldots$, such that

$$X =^d \sum_{n=1}^{G_p} Z_{n,p}. \tag{2.32}$$

For example a Mittag-Leffler random variable is g.i.d., as shown in Lin (1998) Remark 2. For other properties of the g.i.d. random variables see Klebanov et al. (1985), Kalashnikov (1997) and Kozubowski and Rachev (1999).

Proposition 2.3. *A* $\mathrm{TPL}(\gamma, \lambda, \delta, \theta)$ *r.v. with* $\gamma \in (0, 1]$ *is self-decomposable. A* $\mathrm{TPL}(\gamma, \lambda, 1, \theta)$ *random variable is g.i.d. for all admissible values of* γ.

Proof. Regarding self-decomposability, according to Steutel and Van Harn (2004) Proposition V.2.14, it is sufficient to check the case $\theta = 0$, i.e. to show self-decomposability of a PL law. This is well-known (see e.g. Christoph and Schreiber 2001, Section 1.2).

As observed by Klebanov et al. (1985), Theorem 2, a distribution is g.i.d. if and only if its characteristic function $\psi_X(z)$ is such that $1 - 1/\psi_X(z)$ is a characteristic Fourier exponent $\phi_R(-iz) = -\log(L_R(-iz))$, $z \in \mathbb{C}$, of an infinitely-divisible r.v. R. In particular, when X has $\mathrm{TPL}(\gamma, \lambda, 1, \theta)$ distribution we have

$$1 - \frac{1}{L_X(-iz)} = \mathrm{sgn}(\gamma)\lambda((\theta - iz)^\gamma - \theta^\gamma) \tag{2.33}$$

which is the Fourier characteristic exponent of a $\mathrm{TPS}(\gamma, \lambda, \theta)$ law. $\qquad \square$

If $\gamma < 0$ then a TPL r.v. is not self-decomposable since it is not absolutely continuous. The proposition above together with (2.4) clarifies the interpretation of TPL laws as "geometric" analogues of TPS laws, or their "geometric versions", in the terminology of Sandhya and Pillai (1999).

In analogy with stable laws one may wish to investigate the stability condition for $p \in (0, 1)$ and some $\alpha > 0$

$$X =^d p^{1/\alpha} \sum_{n=1}^{G_p} Z_{n,p} \tag{2.34}$$

(see Kalashnikov 1997) where X has $Z_{1,p}$ distribution. A r.v. satisfying (2.34) for all p is said to be geometrically strictly stable (g.s.s., Klebanov et al. 1985, Definition 2). Applying Lin (1998) Remark 2 shows that $\mathrm{PL}(\gamma, \lambda, 1)$ (i.e. Mittag-Leffler) r.v.s satisfy (2.34). However, a $\mathrm{TPL}(\gamma, \lambda, 1, \theta)$ r.v. with $\theta > 0$ does not. Indeed (2.34) is very close to characterizing Linnik distributions, as it can be shown that the only symmetric (resp. positive) g.s.s. laws are Linnik (resp. Mittag-Leffler) (Lin 1994, 1998).

3. TPL Lévy subordinators and their representation

Since the TPL distribution class is infinitely-divisible, by general theory, for any given $\mathrm{TPL}(\gamma, \lambda, \delta, \theta)$ law on $(\Omega, \mathcal{F}, P)$ there exists a unique in law increasing

Lévy process (Lévy subordinator) $X = (X_t)_{t \geq 0}$ supported on some filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ such that X_1 has one such law. Furthermore

$$L_{X_t}(s) = \mathrm{E}[e^{-sX_t}] = e^{-t\phi_X(s)} \tag{3.1}$$

and therefore the r.v.s X_t have $\mathrm{TPL}(\gamma, \lambda, t\delta, \theta)$ distribution.

The Laplace exponent of the Lévy subordinator is by definition the Laplace exponent of its unit time marginal. Henceforth, when we refer to a Lévy process using a distribution, we mean the Lévy process having such a marginal distribution at time 1. The Lévy measure of a process is the Lévy measure of its time-1 marginal. Unless otherwise stated, when we write equality of processes we mean equality of the finite-dimensional distributions.

The TPL process X enjoys a plethora of different representations. The main one is provided directly by Proposition 2.1 and is given by Lévy subordination. Since the characteristic exponent of X is the composition of the characteristic exponents of a gamma law and a TPS law then for all t

$$L_{X_t}(s) = e^{-t\phi_Z(\phi_Y(s))} \tag{3.2}$$

and therefore by a familiar conditioning argument

$$X_t =^d Y_{Z_t}. \tag{3.3}$$

In other words X can be represented as a $\mathrm{TPS}(\gamma, \lambda, \theta)$ process $Y = (Y_t)_{t \geq 0}$ (a tempered stable subordinator) subordinated to an independent $G(1, \delta)$ subordinator $Z = (Z_t)_{t \geq 0}$. We indicate subordination of a process Y to a subordinator Z with $X = Y_Z$.

Remark 3.1. Associating differently the scale parameter, a fully equivalent representation in distribution for the TPL process is of the form $X = Y'_{Z'}$ where Y' is a $\mathrm{TPS}(\gamma, 1, \theta)$ process and Z' a $G(\lambda, \delta)$ independent subordinator.

From (2.13) we have that $\int_0^\infty \nu^Y(dx) = \infty$ or $\int_0^\infty \nu^Y(dx) < \infty$ depending on whether $\gamma < 0$ or $\gamma \in (0, 1]$. By (2.14) in the former case the process Y is of finite activity, that is, Y is a compound Poisson process (CPP). Furthermore, as already observed:

$$\phi_Y(s) = \lambda\theta^\gamma(1 - e^{-\phi_Z(s)}) \tag{3.4}$$

where Z is a $G(-\gamma, 1/\theta)$ r.v.. Thus Y is a CPP that can be written explicitly as

$$Y_t = \sum_{n=0}^{N_t} Z_n, \tag{3.5}$$

where Z_n, $n \geq 0$ are i.i.d. with same distribution as Z and $(N_t)_{t \geq 0}$ is a Poisson process of rate $\lambda\theta^\gamma$ independent of the Z_ns.

The case $\gamma < 0$ is of particular interest for data modelling (see e.g. Barabesi et al. 2016b). In such a case Y is of finite activity and the representation above holds we shall write Y_-, and Y_+ when instead $\gamma \in (0, 1]$. Correspondingly we define X_- and X_+. Although these Lévy process have different path properties representation (3.3) holds in both cases.

3.1. *Compound Poisson representation and the logarithmic Mittag-Leffler distribution*

Since Y_- is a driftless CPP process, then equation (2.8) and Fubini's Theorem imply that X_- must be a CPP too. In the following we explicitly identify its structure.

In applications the following class of functions is of interest

$$p(x; a, b, c, \alpha, \beta) = x^{a-1} E_{\alpha,\beta}(cx^b), \qquad x, a, b > 0, \ c \in \mathbb{R}, \ \mathrm{Re}(\alpha), \mathrm{Re}(\beta) > 0, \tag{3.6}$$

which are often seen to appear in connection with the solution of fractional differential problems. We can express the Laplace transform $L(\cdot; x, s)$, $\mathrm{Re}(s) > 0$, of (3.6) in terms of the Fox-Wright function (e.g. Wright 1935) as follows (Mathai and Haubold 2008, equation 2.2.22)

$$L(p(x; a, b, c, \alpha, \beta); x, s) = \int_0^\infty e^{-sx} x^{a-1} E_{\alpha,\beta}(cx^b) dx = \sum_{k=0}^\infty \frac{\Gamma(bk+a)}{\Gamma(\alpha k + \beta)} \frac{c^k}{s^{ak+b}}$$
$$= \frac{1}{s^b} \, {}_2\Psi_1 \left[\begin{array}{cc} (1,1) & (a,b) \\ (\beta, \alpha) & \end{array} ; \frac{c}{s^a} \right], \qquad |c| < |s^a|. \tag{3.7}$$

This latter expression is not always the transform of a probability density function. For example in

$$L(p(x; a, b, -1, a, b); x, s) = \frac{s^{a-b}}{s^a + 1}, \tag{3.8}$$

a function which is pivotal to fractional calculus and its applications (e.g. Haubold et al. 2011), we have that as $s \to 0$, then $L(p(x; a, b, -1, a, b), x, s) \to 1$ if and only if $a = b$, in which case the associated distribution is the Mittag-Leffler distribution.

However we can exponentially temper (3.6) by $\theta > 0$ obtaining, with slight notation abuse,

$$p(x; a, b, c, \alpha, \beta, \theta) = e^{-\theta x} x^{a-1} E_{\alpha,\beta}(cx^b), \quad x, a, b, \theta > 0, \ c \in \mathbb{R}, \ \mathrm{Re}(\alpha), \ \mathrm{Re}(\beta) > 0, \tag{3.9}$$

which in turn after applying the shifting rule determines the Laplace transform

$$L(p(x; a, b, c, \alpha, \beta, \theta); x, s) =$$
$$\frac{1}{(s+\theta)^b} \, {}_2\Psi_1 \left[\begin{array}{cc} (1,1) & (a,b) \\ (\beta, \alpha) & \end{array} ; \frac{c}{(s+\theta)^a} \right], \quad |c| < |(s+\theta)^a|. \tag{3.10}$$

As $s \to 0$ the limit of the expression of the above is always finite, so after appropriate normalization, $p(x; a, b, c, \alpha, \beta, \theta)$ determines a probability distribution.

In particular if we let $a = b = \alpha$, $\beta = a + 1$, $|c| < \theta^a$ we have:

$$L(p(x; a, a, c, a, a + 1, \theta), x, s) = \frac{1}{ac} \sum_{k=0}^{\infty} \frac{1}{k+1} \left(\frac{c}{(\theta + s)^a} \right)^{k+1}$$

$$= -\frac{1}{ac} \log \left(1 - c(s + \theta)^{-a} \right). \qquad (3.11)$$

Therefore, defining the normalizing constant

$$n(a, c, \theta) = -\frac{ac}{\log \left(1 - c\theta^{-a} \right)}, \qquad (3.12)$$

we conclude that $n(a, c, \theta)p(x; a, a, c, a, a + 1, \theta)$ is a p.d.f. of known Laplace transform. Calculating this product explicitly we can introduce the following probability distribution.

Definition 3.1. A logarithmic Mittag-Leffler (LML) distribution $\mathrm{LML}(a, c, \theta)$ $a, \theta > 0$, $|c| < \theta^a$, is defined such that an LML r.v. W has p.d.f.

$$f_W(x; a, c, \theta) = -\frac{ac}{\log \left(1 - c\theta^{-a} \right)} e^{-\theta x} x^{a-1} E_{a,a+1}(cx^a) \mathbb{1}_{\{x>0\}} \qquad (3.13)$$

and Laplace transform

$$L_W(s) = \frac{\log \left(1 - c(s + \theta)^{-a} \right)}{\log \left(1 - c\theta^{-a} \right)}, \qquad \mathrm{Re}(s) > 0. \qquad (3.14)$$

The terminology is motivated by the similitude of the Laplace transform of this distribution with that of the discrete logarithmic probability law. This distribution is not a generalization of the Mittag-Leffler law as it does not admit it as a particular case, nor does it admit the degenerate case $\theta = 0$. Hence it is structurally different from a TPL law. As it turns out, the LML law arises naturally in the CPP structure of X_-.

Proposition 3.2. *The process X_- admits the* CPP *representation*

$$X_{-t} = \sum_{n=0}^{N_t} J_n \qquad (3.15)$$

where $(N_t)_{t \geq 0}$ is a Poisson process of rate $\delta \log(1 + \lambda\theta^\gamma)$ and $(J_n)_{n \geq 0}$, is an i.i.d. sequence of random variables having $\mathrm{LML}(-\gamma, c_{\gamma,\lambda,\theta}, \theta)$ distribution, where $c_{\gamma,\lambda,\theta}$ is given by (2.19).

Proof. Because of (2.8) X_- is driftless, being the gamma subordination of the driftless CPP in (3.5). Using Haubold et al. (2011) Theorem 5.1 in the negative determination of (2.18), we have the equivalent expression for the Lévy density of X_-

$$u_{X_-}(x) = -\gamma\delta c_{\gamma,\lambda,\theta} e^{-\theta x} x^{-\gamma-1} E_{-\gamma,-\gamma+1}(c_{\gamma,\lambda,\theta} x^{-\gamma}) \mathbb{1}_{\{x>0\}}. \qquad (3.16)$$

But $c_{\gamma,\lambda,\theta}\theta^\gamma < 1$ so that from (3.16) and (3.13) we have

$$u_{X_-}(x) = \delta \log(1 + \lambda\theta^\gamma) f_W(x; -\gamma, c_{\gamma,\lambda,\theta}, \theta)\mathbb{1}_{\{x>0\}} \qquad (3.17)$$

which proves the proposition. □

Another type of tilted Mittag-Leffler distribution following the construction outlined in this section will appear in Section 4.

3.2. Stochastic self-similarity and negative binomial subordination

The TPL processes subordinated structure is extremely rich and, for reasons which we shall shortly explore, mostly revolves around the negative binomial subordinator. We introduce a lattice-valued version of the Lévy subordinator $B = (B_t)_{t\geq0}$ with unit time distribution in the family of laws $\mathrm{NB}(p,\kappa,\alpha,\mu)$ given by the Laplace transform

$$L_B(s) = \left(\frac{p}{1 - (1-p)e^{-\alpha s}}\right)^\kappa e^{-\mu s}, \qquad p \in (0,1), \kappa, \alpha > 0, \mu \in \mathbb{R}, \mathrm{Re}(s) > 0. \tag{3.18}$$

The above law is a scale-location modification of the negative binomial law, and it thus give raise to an infinitely-divisible distribution. Taking the logarithm of L_B and considering the corresponding characteristic exponent we see that B is such that B_t has $\mathrm{NB}(p, \kappa t, \alpha, \mu)$ distribution and it can be represented as a CPP with drift as follows

$$B_t = \sum_{n=0}^{N_t} J_n + \mu t \tag{3.19}$$

with the J_n being i.i.d distributed r.v.s with lattice-valued logarithmic probability mass function

$$P(J_n = \alpha k) = \frac{(1-p)^k}{-k\log p}, \qquad k = 1, 2, \ldots \tag{3.20}$$

and $N = (N_t)_{t\geq0}$ is an independent Poisson process of intensity $-1/\kappa \log p$. For these and other properties of the negative binomial subordinator see Kozubowski and Podgórski (2009).

Negative binomial processes appear naturally in connection to the concept of stochastic self-similarity introduced in Kozubowski et al. (2006), Definition 4.1. Let $X = (X_t)_{t\geq0}$ be any stochastic process and assume that there exists a family of processes $T^c = \{(T_t^c)_{t\geq0}, c > 1\}$ almost surely increasing and diverging as $t \to \infty$ such that

$$X_{T_t^c} =^d c^H X_t \tag{3.21}$$

for some $H > 0$. Then X is said to be stochastically self-similar of index H with respect to T^c.

Stochastic self-similarity is intimately related to geometric infinite-divisibility and in particular to the stability property. Based on this relationship we establish a general invariance property of the PL processes which extends Kozubowski

et al. (2006), Proposition 4.2, and which in particular implies that Linnik processes are stochastically self-similar with respect to families of negative binomial processes (see also Barndorff-Nielsen et al. 2001, Example 2.2).

Proposition 3.3. *Let $H > 0$ and $X = (X_t)_{t \geq 0}$ be a $\mathrm{PL}(\gamma, \lambda, \delta)$ Lévy process. Denote with $B^c = \{(B_t^c)_{t \geq 0}, c > 1\}$ the family of $\mathrm{NB}(c^{-H}, \kappa, 1/\delta, \kappa/\delta)$ subordinators independent of X. Then $Y = X_{B^c}$ is a $\mathrm{PL}(\gamma, \lambda c^H, \kappa)$ process for all $c > 1$. In particular, X is stochastically self-similar of index H with respect to B^c.*

Proof. The first claim is equivalent to $\phi_{B^c}(\phi_X(s)) = \phi_Y(sc^H)$ for all $\mathrm{Re}(s) > 0$ and $c > 1$. Composing the exponents and using (2.1) and (3.18) we obtain

$$
\begin{aligned}
\phi_{B^c}(\phi_X(s)) &= -\kappa \log \left(\frac{c^{-H} e^{-\phi_X(s)/\delta}}{1 - (1 - c^{-H}) e^{-\phi_X(s)/\delta}} \right) \\
&= -\kappa \log \left(\frac{c^{-H}(1 + \lambda s^\gamma)^{-1}}{1 - (1 - c^{-H})(1 + \lambda s^\gamma)^{-1}} \right) \\
&= \kappa \log \left(1 - c^H + c^H(1 + \lambda s^\gamma) \right) \\
&= \kappa \log \left(1 + c^H \lambda s^\gamma \right).
\end{aligned} \tag{3.22}
$$

Stochastic self-similarity follows setting $\kappa = \delta$.   □

Using the subordinated structure of the TPL Lévy process, the distributional invariance part of the proposition above can be extended to X_+, although stochastic self-similarity does not hold because PL and TPL processes scale differently.

Corollary 3.4. *Let X be a $\mathrm{TPL}(\gamma, \lambda, \delta, \theta)$ Lévy process and $B^p = \{(B_t^p)_{t \geq 0}, p \in (0, 1)\}$ be a family of $\mathrm{NB}(p, \kappa, 1/\delta, \kappa/\delta)$ subordinators independent of X. Then $X^p = X_{B^p}$ is a $\mathrm{TPL}(\gamma, \lambda p^{-1}, \kappa, \theta)$ Lévy process for all $p \in (0, 1)$.*

Proof. Using Remark 3.1 we recall $X = Y_Z$ where Y is a $\mathrm{TPS}(\gamma, 1, \theta)$ subordinator and Z is a gamma $G(\lambda, \delta)$ process. Applying Proposition 3.3 with $\gamma = H = 1$, $c = p^{-1}$ we have that $Z^p := Z_{B^p}$ is a $G(\lambda p^{-1}, \kappa)$ gamma process. Using independence we have the equalities

$$
X_t^p =^d (Y_Z)_{B_t^p} =^d Y_{Z_t^p} \tag{3.23}
$$

and the conclusion follows using again Remark 3.1.   □

Unit scale negative binomial subordinators provide an additional representation for non degenerate ($\theta > 0$) TPL Lévy processes X_+ and X_- as subordinated gamma processes seemingly unrelated to the results above.

Proposition 3.5. *For $\gamma \in (-\infty, 0) \cup (0, 1)$, $\theta > 0$, let Z be a gamma $G(\theta, |\gamma|)$ Lévy process and B_+^p and B_-^p to be two negative binomial processes independent of Z respectively of unit marginals $\mathrm{NB}(p, \delta, 1, \delta)$ and $\mathrm{NB}(p, \delta, 1, 0)$. Then if $\gamma \in (0, 1)$*

$$
X_+^p := Z_{B_+^p} \tag{3.24}
$$

is a TPL $(\gamma, \theta^\gamma p^{-1}, \delta, \theta^{-1})$ *Lévy process. If* $\gamma < 0$

$$X_-^p := Z_{B_-^p} \tag{3.25}$$

is a TPL$(\gamma, \theta^\gamma(p^{-1} - 1), \delta, \theta^{-1})$ *Lévy process.*

Proof. For X_+^p we have

$$
\begin{aligned}
\phi_{B_+^p}(\phi_Z(s)) &= -\delta \log \left(\frac{pe^{-\phi_Z(s)}}{1 - (1-p)e^{-\phi_Z(s)}} \right) = -\delta \log \left(\frac{p(1 + \theta s)^{-\gamma}}{1 - (1-p)(1 + \theta s)^{-\gamma}} \right) \\
&= \delta \log \left(1 - \frac{1}{p} + \frac{1}{p}(1 + \theta s)^\gamma \right) = \delta \log \left(1 + \frac{\theta^\gamma}{p}((\theta^{-1} + s)^\gamma - \theta^{-\gamma}) \right)
\end{aligned}
\tag{3.26}
$$

whereas for X_-^p

$$
\begin{aligned}
\phi_{B_-^p}(\phi_Z(s)) &= -\delta \log \left(\frac{p}{1 - (1-p)e^{-\phi_Z(s)}} \right) = \delta \log \left(1 + \frac{1-p}{p}\left(1 - (1 + \theta s)^\gamma \right) \right) \\
&= \delta \log \left(1 - \theta^\gamma \frac{1-p}{p}\left((\theta^{-1} + s)^\gamma - \theta^{-\gamma} \right) \right).
\end{aligned}
\tag{3.27}
$$

$\square$

Notice that the negative binomial representation of X_- is obtained applying a driftless CPP to the infinite activity process Z which correctly determines finite activity. In contrast, the one for X_+ features a CPP with drift which maintains the infinite activity of the subordinand process Y_+.

3.3. A connection with potential theory

There exists an interesting connection between gamma-subordinated Lévy processes and the potential measure. Following Sato (1999), Chapter 6, define for any Borel set $B \subset \mathbb{R}$ the q-potential measure of a process $X = (X_t)_{t \geq 0}$ with probability laws μ_t^X as

$$V^q(B) = \int_0^\infty e^{-qu} \mu_u^X(B) du. \tag{3.28}$$

Now, by (2.8) for $q > 0$ the laws $\mu_t^{Y,q}$ of $Y^q := X_{Z^q}$, where Z^q is a $G(1/q, 1)$ gamma process independent of X, write as

$$\mu_t^{Y,q}(B) = \frac{q^t}{\Gamma(t)} \int_0^\infty \mu_u^X(B) u^{t-1} e^{-qu} du. \tag{3.29}$$

Clearly the law of Y_1^q coincides with qV^q, $q > 0$. Therefore, the knowledge of the unit time law of Y^q completely determines the q-potential measure of X. But according to Proposition 2.1, TPL processes are TPS processes subject to

gamma-subordination, and moreover their probability laws are known by (2.6). According to the discussion above, this means that the whole q-potential structure, $q > 0$, of a $\text{TPS}(\gamma, \lambda, \theta)$ law can be made explicit. Simply setting $\lambda = 1/q$ and $\delta = 1$ in (2.6) then shows the following:

Proposition 3.6. *Let Y be a $\text{TPS}(\gamma, \lambda, \theta)$ subordinator. The $q > 0$ potential measures of Y are absolutely continuous, and the potential densities $v^q(x)$ are given by*

$$v^q(x) = e^{-\theta x} x^{\gamma-1} E_{\gamma,\gamma}\left((\theta^\gamma - q)x^\gamma\right). \tag{3.30}$$

The 0-potential measure (the potential measure *tout court*) of tempered stable subordinators has been calculated using contour integration methods by Kumar and Verma (2020).

4. Inhomogeneous TPL processes

We discuss two non-homogeneous (non-Lévy) Markovian TPL processes: the Lévy-driven OU process with TPL stationary distribution and the self-similar process with independent increments (Sato process) with unit time TPL marginal. The existence of these processes essentially stem from the self-decomposability property of the TPL distribution whenever $\gamma \in (0, 1]$.

4.1. The OU process with stationary TPL distribution

A Lévy-driven OU process is the solution $X = (X_t)_{t \geq 0}$ on $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ of the stochastic differential equation (SDE)

$$X_t = X_0 - \alpha \int_0^t X_u du + \int_0^t dZ_u^\alpha \tag{4.1}$$

given by

$$X_t = e^{-\alpha t} X_0 + \int_0^t e^{-\alpha(t-u)} dZ_u^\alpha \tag{4.2}$$

for some adapted Lévy process $Z^\alpha = (Z_t^\alpha)_{t \geq 0}$, $\alpha > 0$. The theory of OU Lévy-driven SDEs and their applications is fully detailed in Barndorff-Nielsen (1997), Barndorff-Nielsen et al. (2002) and Barndorff-Nielsen and Shephard (2001) using prior results of Jurek and Vervaat (1983) and Wolfe (1982).

The law of X is clearly determined by that of Z^α. Conversely, under some conditions, for any self-decomposable distribution D there exists a Lévy process Z^α of known Lévy triplet, such that (4.1) admits a stationary solution X (i.e. a solution with time-independent distribution) with law D (see the aforementioned references).

In order to study the process Z^α determining a TPL stationary solution X to (4.1), we begin by introducing the tempered Mittag-Leffler (TML) distribution. A TML distribution is obtained by exponentially tempering with $\theta > 0$ the survival function of Pillai (1990) Mittag-Leffler distribution. We illustrate such a family in the following result.

Proposition 4.1. *A* TML *r.v. U with* $\mathrm{TML}(a, c, \theta)$ *distribution where* $(a, c, \theta) \in (0, 1] \times \mathbb{R} \times \mathbb{R}_+$, *is a positive r.v. defined by the cumulative distribution function (c.d.f.)*

$$F_U(x; a, c, \theta) = \left(1 - e^{-\theta x} E_a(-cx^a)\right) \mathbb{1}_{\{x \geq 0\}}, \tag{4.3}$$

with p.d.f.

$$f_U(x; a, c, \theta) = e^{-\theta x}(\theta E_a(-cx^a) + c\, x^{a-1} E_{a,a}(-c\, x^a)) \mathbb{1}_{\{x \geq 0\}} \tag{4.4}$$

and Laplace transform

$$L_U(s) = \frac{\theta(s+\theta)^{a-1} + c}{(s+\theta)^a + c}, \qquad \mathrm{Re}(s) > 0. \tag{4.5}$$

Furthermore, U is infinitely-divisible.

Proof. Using the properties of the Mittag-Leffler function, that F_U is a positively-supported c.d.f. is clear. By differentiating in x we have

$$f_U(x; a, c, \theta) = e^{-\theta x}\left(\theta E_a\left(-cx^a\right) - x^{a-1}\sum_{k=0}^{\infty} a(k+1)(-c)^{k+1}\frac{x^{ak}}{\Gamma(ak+a+1)}\right)$$

$$= e^{-\theta x}\left(\theta E_a(-cx^a) + c\, x^{a-1}\sum_{k=0}^{\infty}\frac{(-c\, x^a)^k}{\Gamma(ak+a)}\right) \tag{4.6}$$

which yields (4.4). Using (3.10) with the appropriate parameters on both terms in (4.4) (see also Haubold et al. (2011), equation (7.1)) we have

$$L_U(s) = \theta\frac{(s+\theta)^{a-1}}{(s+\theta)^a + c} + \frac{c}{(s+\theta)^a + c} \tag{4.7}$$

and (4.5) follows. To show that the TML distribution is infinitely-divisible is necessary and sufficient to show that the logarithmic derivative of $-L_U$ is a completely monotone function (e.g. Gorenflo et al. 2020, Chapter 9). But for $s > 0$

$$-\frac{d}{ds}\log(L_U(s)) = (s+\theta)^{a-1}\left(\frac{\theta(1-a)}{c(s+\theta) + \theta(s+\theta)^a} + \frac{a}{c + (s+\theta)^a}\right) \tag{4.8}$$

which is a product of positive linear combinations of completely monotone functions, and hence is itself completely monotone (Schilling et al. 2012, Corollary 1.6). □

The TML distribution dictates the activity of the CPP process Z^α when X is a stationary solution to (4.1). The next proposition closely mirrors Proposition 3.2.

Proposition 4.2. *Let X have* $\mathrm{TPL}(\gamma, \lambda, \delta, \theta)$ *distribution with $\gamma \in (0, 1]$, and $\lambda\theta^\gamma < 1$. Then X is the law of the stationary solution to (4.1) with*

$$Z_t^\alpha = \sum_{n=0}^{N_t^\alpha} U_n \tag{4.9}$$

where $N^\alpha = (N_t^\alpha)_{t \geq 0}$ is a Poisson process of intensity $\alpha \delta \gamma$ while $(U_n)_{n \geq 0}$ is an i.i.d. sequence of r.v.s independent of N^α all with common distribution $\mathrm{TML}(\gamma, -c_{\gamma,\lambda,\theta}, \theta)$ and $c_{\gamma,\lambda,\theta}$ is given by (2.19). Moreover

$$\phi_\alpha(s) := \phi_{Z^\alpha}(s) = \alpha \delta \gamma \frac{\lambda s (\theta + s)^{\gamma - 1}}{1 + \lambda((\theta + s)^\gamma - \theta^\gamma)}. \tag{4.10}$$

Proof. According to Proposition 2.3 whenever $\gamma \in (0,1]$, the r.v. X is self-decomposable. According to e.g. Barndorff-Nielsen (1997) Theorem 2.3, it holds $\phi_\alpha(s) = \alpha s \phi_X'(s)$ so long as this latter expressions is continuous at 0. Such a calculation produces (4.10) and the continuity is easily checked. Moreover, it is easy to show that

$$\phi_\alpha(s) = \alpha \delta \gamma \left(1 - \frac{\theta(\theta + s)^{\gamma - 1} - c_{\gamma,\lambda,\theta}}{(\theta + s)^\gamma - c_{\gamma,\lambda,\theta}} \right) \tag{4.11}$$

and from the second term inside the parentheses we recognize the Laplace transform (4.5) with the required parameters. This characterizes the law of Z^α as that of the CPP in (4.9). □

The CPP structure of the Lévy driving noise is typical for a large class of self-decomposable distributions D. It is known (Steutel and Van Harn 2004, Theorem V.6.12) that $u_D(x)$ must be such that $k(x) := x u_D(x)$ is non-increasing. On the other hand, as observed in e.g. Barndorff-Nielsen and Shephard (2001), equations (16)–(17), we have

$$\int_x^\infty u_{Z^\alpha}(u) du = \alpha x u_D(x) = \alpha k(x). \tag{4.12}$$

Assume now that $k(x)$ is differentiable and finite in zero. From (4.12) it follows

$$u_{Z^\alpha}(x) = -\alpha k'(x). \tag{4.13}$$

and setting $x = 0$ in (4.12) shows that Z^α is a CPP and the p.d.f. of the increments equals $-k'(x)/\alpha k(0+)$. This argument does provide an alternative proof of Proposition (4.2): from (2.18) specifying

$$k(x) = \gamma \delta e^{-\theta x} E_\gamma \left(c_{\gamma,\lambda,\theta} \, x^\gamma \right) \mathbb{1}_{\{x > 0\}}, \tag{4.14}$$

differentiating and substituting in (4.13) recovers the Lévy density of the CPP in (4.9).

In the case $\gamma = 1$ or $\lambda \theta^\gamma = 1$ we fall back to two instances of the popular gamma OU Lévy-driven model discussed in Barndorff-Nielsen and Shephard (2001), Barndorff-Nielsen et al. (2001) with respectively $G(\lambda, \delta)$ and $G(1/\theta, \delta\gamma)$ stationary solution. Accordingly, in such a case the TML increments reduce to exponential variables of parameter λ (resp. $1/\theta$). Furthermore, we have the notable particular case $\theta = 0$ in which the stationary OU solution with PL distribution has Mittag-Leffler driving noise. The PL stationary OU Lévy-driven
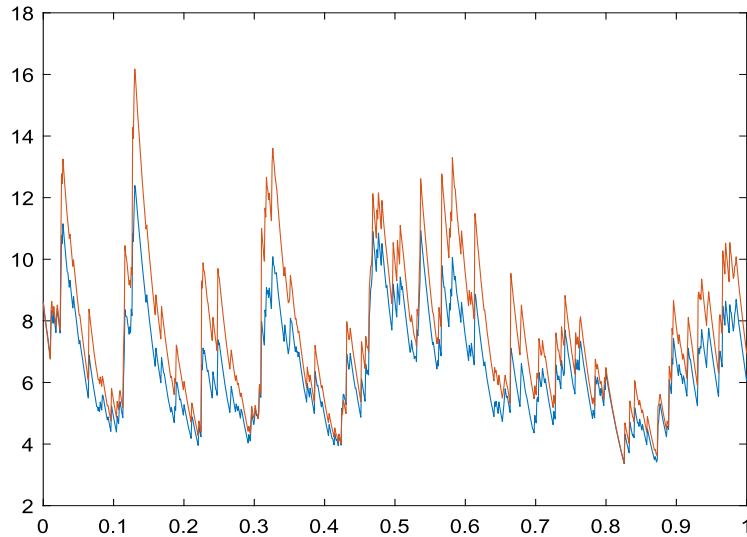
FIG 1. *In blue a trajectory of the OU gamma stationary model and in red a trajectory of its TPL extension from the same random drawing. The parameters are $\alpha = 25$, $\delta = 20$, $\lambda = \theta = 0.5$, $\gamma = 0.7$.*

process can be thus seen as the natural modification of a gamma stationary OU process upon introduction of the γ tail parameter.

The OU representation of a stationary TPL process is very well-suited for numerical schemes of Euler type, where the innovation U_i can be treated by inverse-CDF sampling using equation (4.3). We exemplify this in Figure 1 where we simulate both the stationary gamma OU Lévy-driven model and its TPL counterpart with same random variate drawings. The former is attained from the latter by using same parameters but changing γ to 1. In the TPL model γ and θ govern the tail of the TML jumps: the smaller such parameters the biggest the incidence of large upward jumps in the OU process, a feature which is particularly appealing for modeling financial returns volatility.

Another classic application of Lévy driven SDEs is the explicit construction of a stationary process with (quasi) long-range dependence, which can be attained using a superposition of the SDEs (4.1) as explained in Barndorff-Nielsen (1997), Theorem 4.1.

4.2. Self-similar TPL processes with independent increments

As shown by Sato (1991), for all $H > 0$ to any self-decomposable distribution D we can associate a self-similar process with Hurst exponent H with independent increments (s.s.i.i.) having D as unit time marginal. Unless D is a stable distribution, such process will not be the same one as the Lévy process with unit time law D. As it turns out, when D is TPL all the marginals of the TPL

s.s.i.i. process remain TPL and we thus have an explicit representation for its law and Lévy measure.

Proposition 4.3. *Let $H > 0$ and X be a $\mathrm{TPL}(\gamma, \lambda, \delta, \theta)$ r.v. with $\gamma \in (0,1)$. There exists a stochastically-continuous s.s.i.i. process $X^H = (X_t^H)_{t \geq 0}$ of Hurst index H with independent increments such that X_1^H has the same distribution as X, and whose triplet of the integrated semimartingale characteristics is $(0, 0, U^H(dt, dx))$ with $U^H(dt, dx)$ having density*

$$u_X^H(t, x) = \gamma \delta \frac{e^{-\theta t^{-H} x}}{x} E_\gamma \left(c_{\gamma, \lambda, \theta} (t^{-H} x)^\gamma \right). \tag{4.15}$$

In particular X_t^H has $\mathrm{TPL}(\gamma, \delta, \lambda t^{H\gamma}, \theta t^{-H})$ distribution. Furthermore we have the subordinated representation

$$X^H = Y_Z^H \tag{4.16}$$

where Y^H is the s.s.i.i. process associated to a $\mathrm{TPS}(\gamma, \lambda, \theta)$ law which is such that Y_t^H has $\mathrm{TPS}(\gamma, \lambda t^{H\gamma}, \theta t^{-H})$ distribution, and Z is a $G(1, \delta)$ independent gamma Lévy process.

Proof. The existence of X^H for a given unit time self-decomposable marginal X, and its characterization in terms of the integrated semimartingale characteristic triplet is provided in Sato (1991). In particular the integrated Lévy measure of X^H is absolutely continuous and its density is given by

$$u_X^H(t, x) = t^{-H} u_X(t^{-H} x). \tag{4.17}$$

Remembering (2.18) and the density (4.15) follows.

To prove the second statement consider the s.s.i.i. process with independent increments Y_t^H and combine the substitution in (4.17) with the Lévy density (2.13) which determines the integrated Lévy density $u_Y^H(t, x)$ of Y_t^H as

$$u_Y^H(t, x) = \frac{\gamma \lambda t^{-H}}{\Gamma(1 - \gamma)} \frac{e^{-\theta t^{-H} x}}{(x t^{-H})^{\gamma + 1}} \mathbb{1}_{\{x > 0\}} dx = \frac{\gamma \lambda t^{\gamma H}}{\Gamma(1 - \gamma)} \frac{e^{-\theta t^{-H} x}}{x^{\gamma + 1}} \mathbb{1}_{\{x > 0\}} dx \quad (4.18)$$

showing that Y_t^H has $\mathrm{TPS}(\gamma, \lambda t^{H\gamma}, \theta t^{-H})$ law. Now using (4.17) in (2.16) implies

$$\phi_{X^H}(s) = \phi_X(s t^H) = \phi_Z(\phi_Y(s t^H)) = \phi_Z(\phi_{Y^H}(s)) \tag{4.19}$$

which terminates the proof. $\qquad\square$

We observe that as $t \to \infty$ tempering tends to zero and X_t^H approaches a large scale PL variable.

Self-similarity is a property which is often observed in financial returns time series. Using additive processes in place of Lévy ones in finance has also benefits for valuation of derivative securities. It is recognized that normalized cumulants of risk-neutral distributions implicit in option prices do not decrease with time to expiration of contracts, or at least not as rapidly as the linear rate of decay predicted by Lévy process, a behaviour which is corrected by removing the assumption of returns stationarity.

5. Multivariate TPL processes

Stochastic self-similarity with respect to the negative binomial subordinator of the gamma process can be exploited for generating multivariate TPL Lévy processes in a natural way, which we illustrate in the following. Multivariate g.i.d. laws are studied in Mittnik and Rachev (1991): recently, multivariate Mittag-Leffler distributions have been explored in Albrecher et al. (2021) and Khokhlov et al. (2020).

Let $d \in \mathbb{N}$ and $X = (X_t)_{t \geq 0}$ with $X_t = (X_t^1, \ldots, X_t^d)$ be some independent multivariate TPL Lévy process i.e. X is such that for all $i = 1, \ldots, d$, X^i is a $\text{TPL}(\gamma_i, \lambda_i, 1, \theta_i)$ Lévy process independent from X^j, $j = 1, \ldots, d$ whenever $i \neq j$. Let B^p be a negative binomial process $\text{NB}(p, \delta, 1, \delta)$ independent of X. According to Corollary 3.4 the subordinate multivariate Lévy process $X^p = (X_t^p)_{t \geq 0}$ with

$$X_t^p = (X_t^{p,1}, \ldots, X_t^{p,d}) := (X_{B_t^p}^1, \ldots, X_{B_t^p}^d) \tag{5.1}$$

is such that $X^{p,i} = (X^{p,i})_{t \geq 0}$ has $\text{TPL}(\gamma_i, \lambda_i p^{-1}, \delta, \theta_i)$ unit time law. Therefore X^p is a multivariate Lévy process with correlated TPL marginals, conditionally independent given B^p, and the success probability p plays the role of a dependence parameter with the degenerate case $p = 1$ amounting to the independent case (B^p being pure drift in such a case). According to the general properties of TPL laws illustrated in Section 2, depending on whether $\gamma_i \in (0, 1]$ or $\gamma_i < 0$ the marginal processes can be either infinite activity with nonintegrable Lévy marginal measure and absolutely-continuous law, or CPPs, whose law has a point mass in zero.

A useful alternative representation of X^p can also be provided. By virtue of Remark 3.1 for all $i = 1, \ldots, d$, we can interpret the marginal processes X^i as $X^i = Y_{Z^i}^i$, for two independent multivariate Lévy processes Y^i and Z^i, where Y^i is a $\text{TPS}(\gamma_i, 1, \theta_i)$ process and Z^i is a gamma $G(\lambda_i, 1)$ process independent of Y^i and therefore

$$X_t =^d \left(Y_{Z_t^1}^1, \ldots, Y_{Z_t^d}^d\right). \tag{5.2}$$

Choosing further Y^j, Y^i and Z^i, Z^j to be independent whenever $i \neq j$, we can introduce two independent multivariate Lévy processes $Y = (Y_t^1, \ldots, Y_t^d)$ and $Z = (Z_t^1, \ldots, Z_t^d)$ with independent marginals and (5.2) has the interpretation of a multivariate subordination of Y to Z, as detailed in Barndorff-Nielsen et al. (2001). We shall denote multivariate subordination in the same way as the standard one, and therefore (5.2) implicates $X = Y_Z$. Furthermore, by Proposition 3.3 it holds

$$X_t^p =^d \left(\left(Y_{Z^1}^1\right)_{B_t^p}, \ldots, \left(Y_{Z^d}^d\right)_{B_t^p}\right) =^d \left(Y_{Z_t^{p,1}}^1, \ldots, Y_{Z_t^{p,d}}^d\right), \tag{5.3}$$

where $Z^{p,i} = Z_{B^p}^i$ are $G(\lambda_i p^{-1}, \delta)$ processes, making $Z^p = (Z_t^p)_{t \geq 0}$ given by

$$Z_t^p = (Z_t^{p,1}, \ldots, Z_t^{p,d}) \tag{5.4}$$

into a multivariate gamma subordinator with dependent marginals. Therefore, X^p enjoys the multivariate subordinated representation

$$X^p = Y_{Z^p}. \tag{5.5}$$

In order to investigate the Lévy structure of X^p we hence first compute the Lévy density of Z^p, which is of independent interest. Notice that unlike X^p, Z^p is a multivariate process attained by ordinary subordination. We have the following lemma

Lemma 5.1. *The process Z^p is a multivariate Lévy subordinator with zero drift and Lévy measure*

$$\zeta^p(dt_1 \dots dt_d) = \delta \prod_{i=1}^d \frac{e^{-t_i p/\lambda_i} - e^{-t_i/\lambda_i}}{t_i} \mathbb{1}_{\{t_i > 0\}} dt_1 \dots dt_d + \delta \mu_Z(dt_1 \dots dt_d) \tag{5.6}$$

where the Lévy measure μ_Z of Z is given by

$$\mu_Z(A) = \sum_{i=1}^d \int_{\pi_i(A)} \frac{e^{-t_i/\lambda_i}}{t_i} \mathbb{1}_{\{t_i > 0\}} dt_i \tag{5.7}$$

for each Borel set A in $\mathbb{R}^d$, with π_i the projection onto the i-th coordinate axis.

Proof. By using the p.m.f. (3.20) one can show that the Lévy measure r^p of B^p is (e.g. Kozubowski and Podgórski 2009)

$$r^p = \delta \sum_{k=1}^\infty \frac{(1-p)^k}{k} \delta_k, \tag{5.8}$$

where δ_k is the Dirac measure concentrated in k. That the multivariate Lévy density of Z is given by (5.7) follows from the fact that by independence the Laplace exponent is additive i.e. $\phi_Z(z_1, \dots, z_d) = \sum_{i=1}^d \phi_{Z^i}(z_i)$, so that we can use the Lévy-Khintchine representation component-wise. Moreover the multivariate independent gamma law μ_t^Z writes as

$$\mu_t^Z = \prod_{i=1}^d f_{Z^i}(t_i; \lambda_i, t) dt_1 \dots dt_d = \prod_{i=1}^d \frac{t_i^{t-1}}{\Gamma(t)\lambda_i^t} e^{-t_i/\lambda_i} \mathbb{1}_{\{t_i > 0\}} dt_1 \dots dt_d. \tag{5.9}$$

We can then apply the multivariate version of the (ordinary) subordination integral (2.8) (Sato 1999, Theorem 30.1) to the process Z with triplet $(0, 0, u_Z)$ and probability law μ_t^Z, and the process B^p with triplet $(0, \delta, r^p)$. We thus have, applying monotone convergence, interchanging the series with the integral sign, and performing the integration in du with respect to the Dirac measure

$$\zeta^p(dt_1 \dots dt_d) = \delta \sum_{k=1}^\infty \frac{(1-p)^k}{k} \int_0^\infty \left(\prod_{i=1}^d \frac{t_i^{u-1}}{\Gamma(u)\lambda_i^u} e^{-t_i/\lambda_i} \mathbb{1}_{\{t_i > 0\}} dt_1 \dots dt_d \right) \delta_k(du)$$

$$+ \delta \mu_Z(dt_1 \cdots dt_d)$$

$$= \delta \prod_{i=1}^{d} \frac{e^{-t_i/\lambda_i}}{t_i} \mathbb{1}_{\{t_i>0\}} \sum_{k=1}^{\infty} \left(\frac{t_i(1-p)}{\lambda_i} \right)^k \frac{1}{k!} dt_1 \ldots dt_d + \delta \mu_Z(dt_1 \ldots dt_d)$$

$$= \delta \prod_{i=1}^{d} \frac{e^{-t_i/\lambda_i}}{t_i} \mathbb{1}_{\{t_i>0\}} \left(e^{t_i(1-p)/\lambda_i} - 1 \right) dt_1 \ldots dt_d + \delta \mu_Z(dt_1 \ldots dt_d) \quad (5.10)$$

which proves (5.6). $\qquad\square$

Together with the foregoing discussion, Lemma 5.1 allows the identification of the Lévy structure of X^p.

Theorem 5.2. *The process X^p is a multidimensional Lévy subordinator with multivariate characteristic exponent, for $\mathrm{Re}(s_i) > 0$, $i = 1, \ldots, d$ given by*

$$\phi_{X^p}(s_1, \ldots, s_d) = \delta \log \left(1 - \frac{1}{p} + \frac{1}{p} \prod_{i=1}^{d} (1 + \mathrm{sgn}(\gamma_i)\lambda_i((\theta_i + s_i)^{\gamma_i} - \theta_i^{\gamma_i})) \right).$$

$$(5.11)$$

Furthermore X^p has zero drift, and Lévy density

$$u_X^p(x_1, \ldots, x_d) =$$

$$\delta \sum_{A \subseteq \{1,\ldots,d\}} (-1)^{|A|} \prod_{i \in A} |\gamma_i| \frac{e^{-\theta_i x_i}}{x_i} \left(E_{|\gamma_i|} \left(c_{\gamma_i, \lambda_i, \theta_i} x_i^{|\gamma_i|} \right) - \mathbb{1}_{\{\mathrm{sgn}(\gamma_i)=-1\}} \right)$$

$$\times \prod_{i \in A^c} |\gamma_i| \frac{e^{-\theta_i x_i}}{x_i} \left(E_{|\gamma_i|} \left(c_{\gamma_i, p\lambda_i, \theta_i} x_i^{|\gamma_i|} \right) - \mathbb{1}_{\{\mathrm{sgn}(\gamma_i)=-1\}} \right)$$

$$+ \delta \sum_{i=1}^{d} |\gamma_i| \frac{e^{-\theta_i x_i}}{x_i} \left(E_{|\gamma_i|} \left(c_{\gamma_i, \lambda_i, \theta_i} x_i^{|\gamma_i|} \right) - \mathbb{1}_{\{\mathrm{sgn}(\gamma_i)=-1\}} \right) \quad (5.12)$$

whenever $x_i > 0$ for all i, and zero otherwise, with $(\gamma_i, \lambda_i, \delta_i, \theta_i) \in S$ for all $i = 1, \ldots, d$, and where the constants $c_{\gamma_i, \lambda_i, \theta_i}$ and $c_{\gamma_i, p\lambda_i, \theta_i}$ are given by (2.19).

Proof. By definition of X^p and independence

$$\phi_{X^p}(s_1, \ldots, s_d) = \phi_{B^p}(\phi_X(s_1, \ldots, s_d)) = \phi_{B^p}(\phi_{X_1}(s_1) \ldots \phi_{X_d}(s_d)) \quad (5.13)$$

(5.11) is then clear from (2.2) and (3.18).

We indicate with $\mu_{t_1,\ldots,t_d}^Y$ the probability law of the random vector $(Y_{t_1}^1, \ldots Y_{t_d}^d)$ and by ζ^p the Lévy measure of Z^p given in Lemma 5.1. By virtue of (5.5) we can apply Barndorff-Nielsen et al. (2001) Theorem 3.3, and we see that X^p has Lévy triplet $(0, 0, \eta^p)$ with

$$\eta^p(B) = \int_{\mathbb{R}_+^d} \mu_{t_1,\ldots,t_d}^Y(B)\zeta^p(dt_1 \ldots dt_d) \quad (5.14)$$

for all Borel sets $B \subseteq \mathbb{R}^d_+$ not containing zero. Now by independence we have the product measure

$$\mu^Y_{t_1,\dots,t_d} = \prod_{i=1}^{d} \mu^{Y^i}_{t_i} \tag{5.15}$$

where $\mu^{Y^i}_t$ indicates the law of Y^i_t. Substituting (5.6) and (5.15) in (5.14) and using Proposition 2.1 term by term in (5.7), we obtain the density for $x_i > 0$,

$$u^p_X(x_1,\dots,x_d) = \delta \int_{\mathbb{R}^d_+} \prod_{i=1}^{d} f_{Y^i}(x_i;\gamma_i,\theta_i,t_i) \frac{e^{-t_i p/\lambda_i} - e^{-t_i/\lambda_i}}{t_i} dt_1 \dots dt_d$$
$$+ \delta \sum_{i=1}^{d} |\gamma_i| \frac{e^{-\theta_i x_i}}{x_i} \left(E_{|\gamma_i|} \left(c_{\gamma_i,\lambda_i,\theta_i} x_i^{|\gamma_i|} \right) \mathbb{1}_{\{\mathrm{sgn}(\gamma_i)=-1\}} \right) \tag{5.16}$$

where $f_{Y^i}(x_i;\gamma_i,\theta_i,t_i)$ is given by (2.15) if $\gamma_i \in (0,1)$, or the absolutely continuous part of (2.22) if $\gamma_i < 0$ (the point mass at zero is irrelevant, it being excluded from the Bochner integral). Now observe the additive expansion

$$\prod_{i=1}^{d} \frac{e^{-t_i p/\lambda_i} - e^{-t_i/\lambda_i}}{t_i} = \sum_{A \subseteq \{1,\dots,d\}} (-1)^{|A|} \exp\left(-\sum_{i \in A} \frac{t_i}{\lambda_i} - p \sum_{i \in A^c} \frac{t_i}{\lambda_i} \right) \prod_{i=1}^{d} t_i^{-1} \tag{5.17}$$

where $|A|$ denotes the cardinality of the subset A. In view of (5.17) we can rewrite the first term in (5.16) as

$$\int_{\mathbb{R}^d_+} \prod_{i=1}^{d} f_{Y^i}(x_i;\gamma_i,\theta_i,t_i) \frac{e^{-t_i p/\lambda_i} - e^{-t_i/\lambda_i}}{t_i} dt_1 \dots dt_d$$
$$= \sum_{A \subseteq \{1,\dots,d\}} (-1)^{|A|} \left(\prod_{i \in A} \int_{\mathbb{R}_+} f_{Y^i}(x_i;\gamma_i,\theta_i,t_i) \frac{e^{-t_i/\lambda_i}}{t_i} dt_i \right.$$
$$\left. \prod_{i \in A^c} \int_{\mathbb{R}_+} f_{Y^i}(x_i;\gamma_i,\theta_i,t_i) \frac{e^{-p t_i/\lambda_i}}{t_i} dt_i \right) \tag{5.18}$$

with the product term corresponding to the empty set being 1. Replicating the integrations in Proposition 2.1 we finally arrive at (5.12). $\qquad\square$

The Lévy measure of X^p thus decomposes in an independent multivariate TPL measure plus a combinatorial expression of one-dimensional TPL Lévy measures depending on p accounting for the dependence across the marginals, which increasingly gains weight as p decreases from one to zero. In the case $\gamma_i = 1$ for all i we notice from (5.11) that X^p_t follows the multivariate gamma law discussed in Gaver (1970) and generalizing Kibble (1941), which is widely popular for applications.

6. Statistical anti-fraud applications

One major motivation for our interest in the univariate TPS and TPL laws is their ability to model international trade data, with particular reference to imports into (and exports from) the Member States of the European Union (EU). Due to the combination of economic activities and normative constraints, the empirical distribution of traded quantities and traded values in imports and exports is often markedly skewed with heavy tails, featuring a large number of rounding errors in small-scale transactions due to data registration problems, and structural zeros arising because of confidentiality issues related to national regulations within the EU. While such features are not easy to be combined into a single statistical model, Barabesi et al. (2016b) and Barabesi et al. (2016a) show that in the univariate case both the TPS and TPL distributions do provide reliable models for the monthly aggregates of import quantities of several products of interest.

The potential number of fraud-sensitive products of interest for law-enforcement agents is typically very large. It spans over numerous chapters of the Combined Nomenclature (CN) of Eurostat (2021), which classifies the traded products by their material, function and degree of processing with more than 10,000 sub-headings. Many different categories of goods were seized in the last years by customs operations; see, e.g., the official EU press releases available at the database http://europa.eu/rapid of the European Commission. The surveyed products may be classified according to the main fraud area to which they were associated by the press releases. Two important examples of these categories are counterfeiting (including products such as alcohol, textiles, footwear, electrical and electronic apparatus, etc.) and mis-declaration of product or origin. Barabesi et al. (2016a) demonstrate that several major products of interest within the cited areas exhibit empirical distributions with very different shapes, in terms of skewness, tail heaviness and rounding. For instance, in the case of a traded product within the CN chapter of Beverages, Spirits and Vinegar for which both skewness and kurtosis are impressively high due to the presence of a small number of transactions involving more than 5,000 liters, they show that the TPL law with $\gamma \in (0, 1]$ can provide a very sensible fit with a substantial improvement over the simpler Tweedie distribution, especially in terms of accuracy in the right-hand tail of the distribution. By assuming a TPL law with $\gamma \in (-\infty, 0]$, a similar good fit is instead obtained to the traded quantities of a product belonging to the CN chapter of Fish and Crustaceans, Molluscs and other Aquatic Invertebrates, for which both a non-negligible mass of null values and a rather heavy tail are present.

These examples emphasize the importance of developing statistical models that are both flexible enough to capture such different shapes in the same data set and sound enough to allow for almost automatic implementation without substantial tuning intervention, given the large number of goods of potential interest. The univariate TPS and TPL models are very natural candidates for this task, as they share both desired features. Instead, Barabesi et al. (2016b) show that a naive mixture approach enlarging classical models for skewed distribu-

tions with a spike at the origin, in order to allow for the existence of null values, does not yield the same satisfactory properties. In fact the fit of these mixtures often exhibits numerical instability and less-than-optimal results, even if the models take advantage of the a priori information that rounding occurs exactly at zero. The results obtained in Section 5 further extend the flexible approach provided by the TPL law (and by the TPS one in less extreme situations) to multivariate stochastic modeling. A simple, but prominent, example is shown in Figure 2 below, where a bivariate TPL process is adopted to develop a model which has a very good qualitative fit to both traded quantities and traded values for a relevant fraud-sensitive product. Another active research path with potential impact on anti-fraud analysis is the adoption of (bivariate) TPL laws as the components of a mixture distribution that could model heterogeneous trade data. Underlying unknown heterogeneity may hamper the validity of standard diagnostic tools by boosting the proportion of false alarms and a model-based clustering approach can help to alleviate it (Cerioli and Perrotta 2014, Torti et al. 2019). A further improvement will then be obtained by replacing elliptical distributional assumptions in such mixtures with the much more realistic data-generating process provided by the TPL law.

The main operational target of the research line on international trade data sketched above is the construction of sound statistical methods for the detection of customs frauds, such as the under-valuation of import duties, and the investigation of other trade-related infringements, such as money laundering and circumvention of regulatory measures. In this framework flexible statistical models that can accurately describe the distribution of traded quantities and values for a very large number of products is of paramount importance for several reasons. Firstly, such models could provide direct support to policy makers, e.g. in the form of tools for monitoring the effect of policy measures and for providing factual background for the official communications on trade policy. Another goal, which is perhaps even more prominent from a statistical standpoint, is their use in model-based assessments of the performance of methods used for finding relevant signals of potential fraud.

Most of the fraud detection tools adopted in the context of international trade look for anomalies in the data. Therefore, they typically make use of outlier detection methods for multivariate and regression data, such as those described in Cerioli (2010) and Perrotta et al. (2020a), as well as of robust clustering techniques, see e.g. Cerioli and Perrotta (2014). All of these techniques assume that the available data have been generated by an appropriate contamination model, which in the context of international trade typically involves at least two variables, in view of the basic economic relationship that yields the value of an individual import (export) transaction as the product of the traded amount and the unit price. Any parameter of the distribution that models the "genuine" part of the data must then be estimated in a robust way, in order to avoid the well-known masking and swamping effects due to the anomalies themselves. Relying on the theory of robust high-breakdown estimation, that typically assumes elliptical symmetry of the uncontaminated data-generation process, it is very difficult to derive analytical results for such methods when skewed distributions

should be used for realistic modeling of economic processes. The available methods need thus to be compared, evaluated and eventually tuned on a large number of data sets artificially generated with known statistical properties, which must reflect the distributions observed in real-world trade data. Cerioli and Perrotta (2014) show a first attempt in this direction under a rather specialized ad-hoc model. The class of multivariate TPL processes described in Section 5 provides instead a very natural and general reference model, extending the framework suggested by Barabesi et al. (2016a) to the simultaneous description of (at least) traded quantities and traded values. Reliable inferential results for anti-fraud diagnostics computed on trade data could then be obtained by simulation from this class of processes following a model-based Monte Carlo scheme, in the spirit e.g. of Besag and Diggle (1977).

A similar requirement holds for an alternative approach to fraud detection which has recently attracted considerable attention also in international trade and which rests on the development of powerful and accurate conformance tests of Benford's law Barabesi et al. (2018b), Cerioli et al. (2019), Barabesi et al. (2021). This approach aims at unveiling serial fraudsters and has proven to be especially effective for the analysis of individual customs declarations, instead of monthly aggregates of them. A variety of statistical procedures are compared by Cerioli et al. (2019) and in Section 7.2 of Barabesi et al. (2021) through a bootstrap algorithm that generates pseudo-observations mimicking a national database of one calendar-year customs declarations, after appropriate anonymization that makes it impossible to infer the features of individual operators. The class of multivariate TPL processes can again provide a suitable reference framework for such comparisons when a model-based approach replicating international trade conditions is deemed desirable.

Figure 2 displays one sample of 5,000 observations from a bivariate TPL process simulated using representation (5.3). The parameters of the marginal processes are $\gamma_1 = \gamma_2 = -2.2$, $\lambda_1 = \lambda_2 = 10$, $\theta_1 = \theta_2 = 0.5$, while $\delta = 1$ and $p = 0.01$ is the success probability relating $X^{p,1}$ and $X^{p,2}$. The visual similarity between the simulated scatter and the scatter shown in Section 4, p. 10, of Perrotta et al. (2020b) for a homogeneous sample from a fraud-sensitive commodity is striking and confirms the potential of multivariate TPL processes for describing the joint distribution of relevant variables arising in international trade. Therefore, we argue that suitably tuned versions of X^p could lead to reliable simulation inference for outlier labeling rules and other anti-fraud diagnostics in trade data structures, when the distributional assumption of symmetry for individual uncontaminated observations, typically implied by such methods, is not met. This is an important research goal for anti-fraud applications and international trade analysis, also foreseen in Section 7.2 of Barabesi et al. (2021).

7. Conclusion

In this paper we have detailed the nature of a number of stochastic processes with TPL marginals. The TPL distribution appears naturally in the context
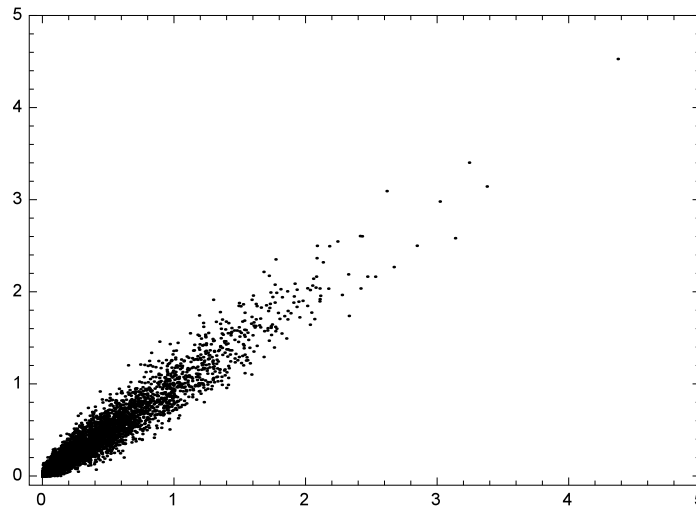
FIG 2. *Simulated sample from a multivariate TPL process with $d = 2$, $\delta = 1$, $p = 0.01$ and marginal parameters given in the text.*

of subordinated Lévy modelling and it can be associated to both finite and infinite activity Lévy processes. For some parameters sets, the TPL distribution naturally determines finite activity Lévy processes whose jumps follow new probability laws of generalized Mittag-Leffler type. It also enjoys a number of desirable properties such as geometric infinite divisibility and stochastic self-similarity with respect to the negative binomial law. In particular, the latter can be fruitfully exploited to construct a multivariate TPL process in which the success probability plays the role of a correlation parameter. It is envisaged that this mixing procedure can be also applied outside a strict TPL context. Furthermore, we have provided initial evidence that the multivariate process we introduced could lead to improvements in the statistical analysis of real data in settings in which the TPL and TPS distributions find application, such as international trade and fraud detection.

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