Frequency domain bootstrap methods for random fields[*]

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Abstract: This paper develops a frequency domain bootstrap method for random fields on $\mathbb{Z}^2$. Three frequency domain bootstrap schemes are proposed to bootstrap Fourier coefficients of observations. Then, inverse-transformations are applied to obtain resamples in the spatial domain. As a main result, we establish the invariance principle of the bootstrap samples, from which it follows that the bootstrap samples preserve the correct second-order moment structure for a large class of random fields. The frequency domain bootstrap method is simple to apply and is demonstrated to be effective in various applications including constructing confidence intervals of correlograms for linear random fields, testing for signal presence using scan statistics, and testing for spatial isotropy in Gaussian random fields. Simulation studies are conducted to illustrate the finite sample performance of the proposed method and to compare with the existing spatial block bootstrap and subsampling methods.

Contents

1. Introduction

Following Efron's influential paper ([8]), development of bootstrap resampling procedures has been growing rapidly. Bootstrap resampling constitutes a powerful tool for approximating certain characteristics of a statistic, that cannot be easily calculated by analytical means. In addition, bootstrap methods require no explicit knowledge of the underlying dependence mechanism, or the marginal distribution of the observations. These user-friendly features make bootstrap resampling popular for statistical inference.

In recent decades, various resampling methods for dependent data have been proposed. For time series data, block bootstrap and frequency domain bootstrap are two important classes of bootstrap procedures. For block bootstrap methods, include moving block bootstrap ([25] and [29]), non-overlapping block

bootstrap ([3]), circular block bootstrap ([41]), and stationary block bootstrap ([43]). Despite its simplicity, the accuracy of a block bootstrap estimator critically depends on the block size employed. On the other hand, frequency domain bootstrap methods use the periodogram of the data to derive bootstrap approximations for a class of estimators called ratio statistics, see [5], [10] and [22] for details. [21] proposed the time frequency toggle (TFT) bootstrap for time series, which directly resamples the discrete Fourier transform instead of resampling the periodograms. Unlike periodograms, the bootstrapped discrete Fourier transforms can be transformed back to generate bootstrap resamples of a time series. Thus, TFT bootstrap not only comprises the classical frequency domain bootstrap methods, but is also applicable to statistics that are based on the time domain representation of the observations, including the CUSUM statistic for change-point detection, and the least-squares statistic for unit-root testing. By combining a time domain parametric bootstrap and a frequency domain nonparametric bootstrap, [18] extended the autoregressive aided periodogram bootstrap suggested by [22] and proposed a multiple hybrid bootstrap for linear processes which can generate bootstrap resamples in the time domain. For reviews of resampling methods in time series, see [2], [36], [26], and [38].

Apart from time series, subsampling and resampling methods for spatial data have also become increasingly popular in past decades; see [6] for a brief overview. [16] used a block resampling procedure to bootstrap spatial data. [44] developed a subsampling method for random fields. [42] considered a block bootstrap method for homogeneous strong mixing random fields. [46] used a resampling method to estimate variance for statistics computed from spatial data. [40] proposed subsampling methods for statistical inference in irregularly spaced dependent observations. [28] used spatial subsampling for least squares variogram estimation. [34] and [35] developed the optimal block sizes for spatial subsampling and bootstrap methods. However, they are only applicable to variance estimation. [30] proposed a bootstrap method for Gaussian random fields under fixed domain asymptotics. See [26] for a comprehensive review. Recently, [32] proposed an AR sieve bootstrap for linear random fields. To the best of our knowledge, the development of spatial bootstrap methods focuses mainly on the block bootstrap type methods, and a frequency domain bootstrap method for possibly nonlinear random fields remains absent from the literature.

In this paper, we develop a frequency domain bootstrap method for random fields on $\mathbb{Z}^2$. The basic principle of the proposed method is to bootstrap Fourier coefficients of observations, and then inverse-transform the resampled Fourier coefficients to obtain bootstrap samples in the spatial domain. By resampling the discrete Fourier transforms instead of the periodograms, we can handle situations where the statistics of interest are not expressible by periodograms, such as scan statistics for testing presence of spatial signal, see Section 6. The proposed frequency domain bootstrap method is similar in spirit to the TFT bootstrap of [21] for time series. However, resampling the Fourier coefficients in spatial data is not as straightforward as that in time series due to an additional rotational symmetry of the coefficients. In addition, applications of the spatial frequency domain bootstrap method, such as testing for signal presence and

testing for spatial isotropy, are very different from applications of the time series counterpart, such as change-point detection and testing for unit roots. Moreover, to develop the bootstrap theory in spatial context, we establish an invariance principle for the bootstrap partial sum process indexed by a classical example of Vapnik-Chervonenkis-classes (VC-classes) of set $[0, 1]^2$. The results can be generalized to other VC-classes. The proofs of the asymptotic results require different ideas and techniques compared with that for the time series counterpart in [21].

We propose three resampling schemes for bootstrapping the Fourier coefficients of spatial processes in $\mathbb{Z}^2$. We show that the resulting bootstrap sample correctly captures the second-order moment structure for a large class of random fields. The results are illustrated by applications to constructing confidence intervals of correlograms for linear random fields, testing for the presence of signal, and testing for spatial isotropy in Gaussian random fields. Simulation studies are performed to explore the finite sample performance of the proposed method and to compare with existing spatial block bootstrap and subsampling methods.

This paper is organized as follows. Section 2 provides the problem setting and reviews the spectral theory for spatial processes in $\mathbb{Z}^2$. In Section 3, three resampling schemes for the Fourier coefficients are proposed to develop bootstrap procedures for spatial processes in $\mathbb{Z}^2$. The main results are presented in Section 4, in which we establish the validity of the bootstrap procedures by showing the invariance principles of bootstrap samples under some meta-assumptions on the bootstrapped Fourier coefficients. Section 5 verifies these meta-assumptions for the three resampling schemes. In Section 6, we introduce some practical applications for the proposed bootstrap method, and simulation studies on comparing the proposed method with existing spatial block bootstrap and subsampling methods are given. Technical proofs of the theorems and lemmas are provided in Appendices A and B.

2. Problem setting and spectral theory for spatial processes in $\mathbb{Z}^2$

In this section, we describe the problem settings and preliminary about the spectral theory for spatial processes. First, we introduce some notations. For any vector $\mathbf{a} = (a_1, a_2, \ldots, a_q) \in \mathbb{R}^q$, denote $|\mathbf{a}| = \prod_{i=1}^{q} |a_i|$, $\|\mathbf{a}\| = \max_{i=1,\ldots,q}\{|a_i|\}$ and $\|\mathbf{a}\|_p = (\sum_{i=1}^{q} |a_i|^p)^{1/p}$ for $p \geq 1$. Let $T = \{(t_1, t_2) \in \mathbb{Z}^2, 1 \leq t_k \leq d_k, k = 1, 2\}$ be a spatial rectangular lattice, and denote $\mathbf{d}_T = (d_1, d_2) \in \mathbb{N}^2$. Denote $\mathbf{j} \odot \mathbf{d}_T = (j_1 d_1, j_2 d_2)$ for $\mathbf{j} = (j_1, j_2) \in \mathbb{R}^2$. For any set G, denote the cardinality of G by $|G|$. For random variables $X \in \mathbb{L}^p$, denote the $\mathbb{L}^p$ norm as $\|X\|_p = (\mathbb{E}(|X|^p))^{1/p}$. For any two sequences of real numbers $\{a_n\}$ and $\{b_n\}$, denote by $a_n \asymp b_n$ when $a_n = O(b_n)$ and $b_n = O(a_n)$. For any $x \in \mathbb{R}$, $\lfloor x \rfloor$ is the greatest integer that is less than or equal to x. All vectors are column vectors unless specified otherwise, hence for any $\mathbf{a} = (a_1, a_2, \ldots, a_q) \in \mathbb{R}^q$ and $\mathbf{b} = (b_1, b_2, \ldots, b_q) \in \mathbb{R}^q$, the dot product between vectors $\mathbf{a}$ and $\mathbf{b}$ is defined as the vector multiplication $\mathbf{a}'\mathbf{b} = \mathbf{b}'\mathbf{a} = \sum_{i=1}^{q} a_i b_i$.

2.1. Settings and assumptions

Let $\left\{V(\mathbf{t}) : \mathbf{t} \in \mathbb{Z}^2\right\}$ be a stationary random field on a two-dimensional grid with mean $\mu = \mathbb{E}(V(\mathbf{0}))$. Assume that we have observed $\{V(\mathbf{t}) : \mathbf{t} \in T\}$ on a rectangular spatial region $T = \{(t_1, t_2) \in \mathbb{Z}^2, 1 \le t_k \le d_k, k = 1, 2\}$ with $\mathbf{d}_T = (d_1, d_2) \in \mathbb{N}^2$ and cardinality $|T| = d_1 d_2$. Throughout the paper, $T \to \infty$ denotes both $d_1, \, d_2 \to \infty$. We impose the following assumptions about the increasing domain asymptotic framework and the underlying random fields for establishing the asymptotic results.

Assumption $\mathcal{A}.1$. For all sufficiently large $|T|$, there exist $0 < \xi \le 1/2$ and $c_1, \, c_2 > 0$ such that $d_1 > c_1 |T|^\xi$ and $d_2 > c_2 |T|^\xi$.

Assumption $\mathcal{A}.2$. The random field $\left\{V(\mathbf{t}) : \mathbf{t} \in \mathbb{Z}^2\right\}$ is stationary with absolutely summable auto-covariance function $\gamma(\cdot)$, i.e., $\sum_{\mathbf{j} \in \mathbb{Z}^2} |\gamma(\mathbf{j})| < \infty$, where $\gamma(\mathbf{j}) = \mathrm{Cov}(V(\mathbf{0}), V(\mathbf{j}))$. In this case the spectral density of the random field exists and can be expressed as

$$f(\lambda) = \frac{1}{4\pi^2} \sum_{\mathbf{j} \in \mathbb{Z}^2} e^{-i\lambda' \mathbf{j}} \gamma(\mathbf{j}), \tag{2.1}$$

where $\lambda \in [0, 2\pi]^2$. Moreover, $f(\cdot)$ is continuous and bounded. See, e.g., [11], Section 1.2.2, for details.

Assumption $\mathcal{A}.3$. The spectral density is bounded from below, i.e., $f(\lambda) \ge c$ for some $c > 0$ and all $\lambda \in [0, 2\pi]^2$.

Assumption $\mathcal{A}.4(p)$. For $\mathbf{j} \in \mathbb{Z}^2$, assume that $V(\mathbf{j}) - \mu = G(\varepsilon_{\mathbf{j}-\mathbf{s}} : \mathbf{s} \in \mathbb{Z}^2)$, where $G(\cdot)$ is a measurable function and $\{\varepsilon_{\mathbf{i}}\}_{\mathbf{i} \in \mathbb{Z}^2}$ is an i.i.d. random field. Let $\{\widetilde{\varepsilon}_{\mathbf{i}}\}_{\mathbf{i} \in \mathbb{Z}^2}$ be an i.i.d. copy of $\{\varepsilon_{\mathbf{i}}\}_{\mathbf{i} \in \mathbb{Z}^2}$. Define the coupled version of $V(\mathbf{j})$ as $\widetilde{V}(\mathbf{j}) = G(\varepsilon_{\mathbf{j}-\mathbf{s}}^* : \mathbf{s} \in \mathbb{Z}^2)$, where

$$\varepsilon_{\mathbf{j}-\mathbf{s}}^* = \left\{ \begin{array}{ll} \varepsilon_{\mathbf{j}-\mathbf{s}} & \text{if} \quad \mathbf{j} - \mathbf{s} \ne \mathbf{0}, \\ \widetilde{\varepsilon}_{\mathbf{0}} & \text{if} \quad \mathbf{j} - \mathbf{s} = \mathbf{0}. \end{array} \right.$$

Assume that there exists some $p > 0$ such that $V(\mathbf{j})$ belongs to $\mathbb{L}^p$ and $\Delta_p := \sum_{\mathbf{j} \in \mathbb{Z}^2} \delta_{\mathbf{j},p} := \sum_{\mathbf{j} \in \mathbb{Z}^2} \|V(\mathbf{j}) - \widetilde{V}(\mathbf{j})\|_p < \infty$.

Assumption $\mathcal{A}.4(p)$ is the p-stable condition for random fields defined in [9] in which central limit theorems and invariance principles are established for a wide class of stationary random fields. We will discuss the invariance principles in detail in Section 4. The next assumption is a geometric-moment contraction (GMC) condition:

Assumption $\mathcal{A}.5$. Under the notation of Assumption $\mathcal{A}.4(p)$, define another coupled version of $V(\mathbf{j})$ as $\widetilde{V}^\dagger(\mathbf{j}) = G(\varepsilon_{\mathbf{j}-\mathbf{s}}^\dagger : \mathbf{s} \in \mathbb{Z}^2)$, where

$$\varepsilon_{\mathbf{j}-\mathbf{s}}^\dagger = \left\{ \begin{array}{ll} \varepsilon_{\mathbf{j}-\mathbf{s}} & \text{if} \quad \|\mathbf{s}\| < \|\mathbf{j}\|, \\ \widetilde{\varepsilon}_{\mathbf{j}-\mathbf{s}} & \text{if} \quad \|\mathbf{s}\| \ge \|\mathbf{j}\|. \end{array} \right.$$

Assume that there exist $\alpha > 0, C > 0$ and $0 < \rho = \rho(\alpha) < 1$ such that for all $\mathbf{j} \in \mathbb{Z}^2$, $\mathbb{E}(|V(\mathbf{j}) - \widetilde{V}^{\dagger}(\mathbf{j})|^{\alpha}) \leq C\rho^{\|\mathbf{j}\|}$.

Assumption $\mathcal{A}.5$ is the spatial extension of the geometric-moment contraction condition for time series, see [45]. This condition is fulfilled for short-range dependent linear random fields with finite variance, and a large class of nonlinear random fields such as nonlinearly transformed linear random fields, Volterra fields and nonlinear spatial autoregressive models, see [9] and [7].

2.2. Fourier coefficients of spatial processes in $\mathbb{Z}^2$

Denote the sample mean as $\overline{V_T} = |T|^{-1} \sum_{\mathbf{t} \in T} V(\mathbf{t})$, and the centered observations as $Z(\mathbf{j}) = V(\mathbf{j}) - \overline{V_T}$, $\mathbf{j} \in T$. Let the Fourier coefficients be $x(\mathbf{j}) + iy(\mathbf{j})$, where

$$x(\mathbf{j}) = \frac{1}{\sqrt{|T|}} \sum_{\mathbf{t} \in T} V(\mathbf{t})\cos(-\lambda_{\mathbf{j}}'\mathbf{t}), \quad y(\mathbf{j}) = \frac{1}{\sqrt{|T|}} \sum_{\mathbf{t} \in T} V(\mathbf{t})\sin(-\lambda_{\mathbf{j}}'\mathbf{t}), \quad (2.2)$$

for $\mathbf{j} = (j_1, j_2) \in T$ and $\lambda_{\mathbf{j}} = (\lambda_{j1}, \lambda_{j2}) = \left(\frac{2\pi j_1}{d_1}, \frac{2\pi j_2}{d_2}\right)$. Note that the dependence of Fourier coefficients $x(\mathbf{j})$ and $y(\mathbf{j})$ on T is suppressed for notational simplicity. The basic principle of the proposed bootstrap method is to sample the Fourier coefficients of the observations, and then back-transform them to obtain bootstrap samples in the spatial domain. First, we discuss some structural properties of the Fourier coefficients of spatial processes in $\mathbb{Z}^2$. Note from (2.2) that

$$x(\mathbf{d}_T - \mathbf{j}) = x(\mathbf{j}), \ y(\mathbf{d}_T - \mathbf{j}) = -y(\mathbf{j}). \quad (2.3)$$

By using the symmetry property in (2.3), we now partition T as $T = N \cup \tilde{N} \cup M$ such that the Fourier coefficients defined on $\tilde{N}$ are determined by the Fourier coefficients defined on N. Also, the information about the covariance structure and mean of the random field are contained in N and M respectively. Hence, a spatial process can be reconstructed from the Fourier coefficients defined on N and M. In particular, when d_1, d_2 are both odd, define

$$N = \left\{ (t_1, t_2) : 1 \leq t_1 \leq d_1, \ 1 \leq t_2 \leq \frac{d_2 - 1}{2} \right\}$$

$$\cup \left\{ (t_1, t_2) : 1 \leq t_1 \leq \frac{d_1 - 1}{2}, \ t_2 = d_2 \right\}.$$

When d_1 is odd and d_2 is even (similar for d_1 is even and d_2 is odd), define

$$N = \left\{ (t_1, t_2) : 1 \leq t_1 \leq d_1, \ 1 \leq t_2 \leq \frac{d_2}{2} - 1 \right\}$$

$$\cup \left\{ (t_1, t_2) : 1 \leq t_1 \leq \frac{d_1 - 1}{2}, \ t_2 = \frac{d_2}{2}, d_2 \right\}.$$

When d_1, d_2 are both even, define

$$N = \left\{ (t_1, t_2) : 1 \le t_1 \le d_1, \ 1 \le t_2 \le \frac{d_2}{2} - 1 \right\}$$

$$\cup \left\{ (t_1, t_2) : 1 \le t_1 \le \frac{d_1}{2} - 1, \ t_2 = \frac{d_2}{2}, d_2 \right\} .$$

Then, using the symmetry property in (2.3), subset $\tilde{N}$ of T is defined as

$$\tilde{N} = \left\{ (t_1, t_2) \in T : t_i = \left\{ \begin{array}{ll} d_i - s_i & , \text{ if } s_i < d_i \\ d_i & , \text{ if } s_i = d_i \end{array} \right. \text{ for } i = 1, 2 \,, (s_1, s_2) \in N \right\} .$$

Note that the Fourier coefficients at $\mathbf{j} \in \tilde{N}$ can be completely determined by the Fourier coefficients at $\mathbf{j} \in N$. From (2.2), for all $c \in \mathbb{R}$, the Fourier coefficients of $\{V(\mathbf{t}) - c : \mathbf{t} \in T\}$ at $\mathbf{j} \in N$ are the same. In other words, the Fourier coefficients in N and $\tilde{N}$ are invariant under additive constants and thus contain no information about the mean. In contrast, all of the information about the mean is contained in the Fourier coefficients $x(\mathbf{j})$ for $\mathbf{j} \in M$, where

$$M = \left\{ \begin{array}{ll} \{(d_1, d_2)\} & , \text{ when } d_1, d_2 \text{ are odd}, \\ \{(d_1, d_2), (d_1/2, d_2)\} & , \text{ when only } d_1 \text{ is even}, \\ \{(d_1, d_2), (d_1, d_2/2)\} & , \text{ when only } d_2 \text{ is even}, \\ \{(d_1, d_2), (d_1/2, d_2), (d_1, d_2/2), (d_1/2, d_2/2)\} & , \text{ when } d_1, d_2 \text{ are even}. \end{array} \right.$$
(2.4)

Table 1 shows some examples to illustrate the partitions of set T under different scenarios. Table 2 summarizes the value of the Fourier coefficients in M.

TABLE 1
The set N for extracting Fourier coefficients. In the grid, $\star$ are coefficients in N and $\circ$ are coefficients in M.
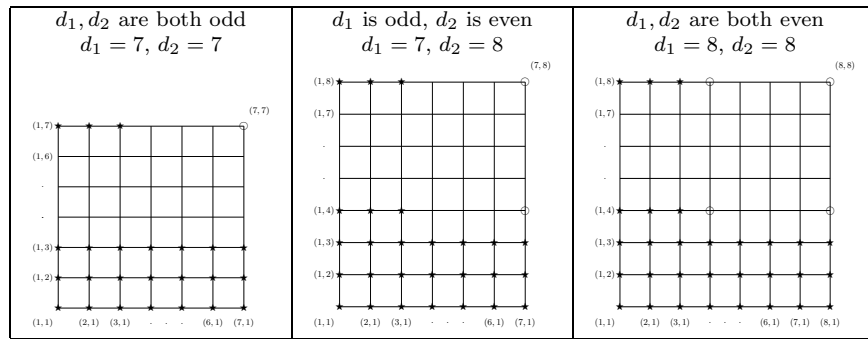
Since in spatial statistics the main concern is the covariance structure of the random field, we focus on bootstrapping the Fourier coefficients in N. The issue of bootstrapping spatial mean is deferred to Section 4.2.3.

TABLE 2
Fourier coefficients in M contain information about the mean.

$\mathbf{j}$	(d_1, d_2)	$(d_1/2, d_2)$	$(d_1, d_2/2)$	$(d_1/2, d_2/2)$
$\mathrm{x}(\mathbf{j})$	$\sqrt{\|T\|}\overline{V}_T$	$\dfrac{\sum_{\mathbf{t}\in T}(-1)^{t_1}V(\mathbf{t})}{\sqrt{\|T\|}}$	$\dfrac{\sum_{\mathbf{t}\in T}(-1)^{t_2}V(\mathbf{t})}{\sqrt{\|T\|}}$	$\dfrac{\sum_{\mathbf{t}\in T}(-1)^{t_1+t_2}V(\mathbf{t})}{\sqrt{\|T\|}}$
$\mathrm{y}(\mathbf{j})$	0	0	0	0

2.3. Kernel spectral density estimation

We consider the kernel spectral density estimator

$$\widehat{f}_T(\lambda) = \frac{\sum_{\mathbf{j}\in\mathbb{Z}^2} K\left(\frac{\lambda_1-\lambda_{j1}}{h_{T1}}, \frac{\lambda_2-\lambda_{j2}}{h_{T2}}\right) I(\mathbf{j})}{4\pi^2 \sum_{\mathbf{j}\in\mathbb{Z}^2} K\left(\frac{\lambda_{j1}}{h_{T1}}, \frac{\lambda_{j2}}{h_{T2}}\right)}, \tag{2.5}$$

for $\lambda = (\lambda_1, \lambda_2) \in [0, 2\pi]^2$, where $\lambda_{\mathbf{j}} = (\lambda_{j1}, \lambda_{j2}) = \left(\frac{2\pi j_1}{d_1}, \frac{2\pi j_2}{d_2}\right)$ for $\mathbf{j} \in \mathbb{Z}^2$, $K(\cdot)$ is the kernel function, $\mathbf{h}_T = (h_{T1}, h_{T2}) \in \mathbb{R}^2$ is the bandwidth satisfying $h_{Tk} \to 0$, $h_{T1} \asymp h_{T2}$ and $h_{Tk}d_k \to \infty$ for $k = 1, 2$, and

$$I(\mathbf{j}) = \begin{cases} 0, & \text{if } \mathbf{j} \in D = \{(c_1 d_1, c_2 d_2) : (c_1, c_2) \in \mathbb{Z}^2\}, \\ x^2(\mathbf{j}) + y^2(\mathbf{j}), & \text{if } \mathbf{j} \in \mathbb{Z}^2 \backslash D, \end{cases}$$

is the periodogram at frequency $\lambda_{\mathbf{j}}$. The periodogram can be set to 0 on D since it only contains information about the mean. We impose the following mild regularity assumptions on the kernel function $K(\cdot)$.

Assumption $\mathcal{K}$.1. The kernel $K(\cdot)$ is a real, positive, even function with $\int_{\mathbb{R}^2} K(\lambda)d\lambda = 1$ and

$$\frac{4\pi^2}{\|\mathbf{h}_T\|\|T\|} \sum_{\mathbf{j}\in\mathbb{Z}^2} K\left(\frac{2\pi j_1}{h_{T1}d_1}, \frac{2\pi j_2}{h_{T2}d_2}\right) = \int_{\mathbb{R}^2} K(\lambda)d\lambda + o(1) = 1 + o(1).$$

Assumption $\mathcal{K}$.2. Assume that $\sup_{\lambda\in[0,2\pi]^2} |K_{\mathbf{h}}(\lambda)| = O\left((h_{T1}h_{T2})^{-1}\right)$, where

$$K_{\mathbf{h}}(\lambda) = \frac{1}{\|\mathbf{h}_T\|} \sum_{\mathbf{j}\in\mathbb{Z}^2} K\left(\frac{\lambda_1 + 2\pi j_1}{h_{T1}}, \frac{\lambda_2 + 2\pi j_2}{h_{T2}}\right). \tag{2.6}$$

Assumption $\mathcal{K}$.3. The inverse Fourier transform of $K(\cdot)$,

$$k(\mathbf{x}) = \int_{\mathbb{R}^2} K(\lambda)\exp(i\mathbf{x}'\lambda)d\lambda, \tag{2.7}$$

is Lipschitz continuous with support $[-1, 1]^2$.

Assumption $\mathcal{K}$.4. The quantity $K_{\mathbf{h}}(\lambda)$ in (2.6) satisfies the following uniformly Lipschitz condition: for some constant $L_K > 0$,

$$\|\mathbf{h}_T\|^{3/2}|K_{\mathbf{h}}(\lambda_{\mathbf{s}}) - K_{\mathbf{h}}(\lambda_{\mathbf{t}})| \le L_K\|\lambda_{\mathbf{s}} - \lambda_{\mathbf{t}}\|,$$

uniformly in $\lambda_{\mathbf{s}} = \left(\frac{2\pi s_1}{d_1}, \frac{2\pi s_2}{d_2}\right)$ and $\lambda_{\mathbf{t}} = \left(\frac{2\pi t_1}{d_1}, \frac{2\pi t_2}{d_2}\right)$.

Remark 2.1. Assumption $\mathcal{K}.2$ is mild, as it holds for any bounded kernels with compact support. Moreover, under Assumption $\mathcal{K}.3$, we have

$$K_{\mathbf{h}}(\lambda) = \frac{1}{4\pi^2} \sum_{\mathbf{j} \in \mathbb{Z}^2} k(j_1 h_{T1}, j_2 h_{T2}) \exp(-i\mathbf{j}'\lambda),$$

and

$$K(\lambda) = \frac{1}{4\pi^2} \int k(\mathbf{x}) \exp(-i\mathbf{x}'\lambda) d\mathbf{x},$$

where $k(\cdot)$ is defined in (2.7). From the above representations it is clear that for large T,

$$K_{\mathbf{h}}(\lambda) \cong \frac{1}{|\mathbf{h}_T|} K\left(\frac{\lambda_1}{h_{T1}}, \frac{\lambda_2}{h_{T2}}\right), \tag{2.8}$$

which is of order $O\left(|\mathbf{h}_T|^{-1}\right)$ for bounded $K(\cdot)$. By (2.8), if the kernel $K(\cdot)$ is uniformly Lipschitz continuous with compact support, then Assumption $\mathcal{K}.4$ holds for a small enough $\mathbf{h}_T = (h_{T1}, h_{T2})$. For infinite support kernels, if $K(\cdot)$ is bounded and continuously differentiable, then Assumption $\mathcal{K}.4$ also holds. Assumptions $\mathcal{K}.1$ to $\mathcal{K}.4$ hold for many commonly used kernels such as uniform kernels $K(\lambda_1, \lambda_2) = \frac{1}{4}\mathbb{1}_{\{|\lambda_1|\leq 1\}}\mathbb{1}_{\{|\lambda_2|\leq 1\}}$, multiplicative 2-$d$ Epanechnikov kernels $K(\lambda_1, \lambda_2) = \frac{9}{16}(1 - \lambda_1^2)(1 - \lambda_2^2)\mathbb{1}_{\{|\lambda_1|\leq 1\}}\mathbb{1}_{\{|\lambda_2|\leq 1\}}$, and Gaussian kernels $K(\lambda_1, \lambda_2) = \frac{1}{2\pi}\exp(\frac{-\lambda_1^2 - \lambda_2^2}{2})$.

3. Frequency domain bootstrap

In this section, we propose three bootstrap schemes in the frequency domain, namely Residual-Based Bootstrap (RB), Wild Bootstrap (WB) and Local Bootstrap (LB), for resampling the Fourier coefficients. Similar bootstrap schemes in time series context are first proposed by [10], [37], and [17] respectively. A bootstrap procedure that produces resamples of spatial processes is then developed in Section 3.2.

3.1. Bootstrap schemes

3.1.1. Residual-based bootstrap (RB)

In RB, we first standardize the Fourier coefficients to obtain a set of residuals, which consists of approximately i.i.d. normal random variables. Hence, i.i.d. resampling methods can be applied to yield a resample of Fourier coefficients.

Step 1: Estimate the spectral density f by $\widehat{f}_T$, which satisfies

$$\sup_{\lambda \in [0,2\pi]^2} \left|\widehat{f}_T(\lambda) - f(\lambda)\right| \xrightarrow{p} 0. \tag{3.1}$$

Step 2: For the Fourier coefficients $x(\mathbf{j})$ and $y(\mathbf{j})$, $\mathbf{j} \in N$, define

$$\widetilde{s}_{\mathbf{j},1} = \frac{x(\mathbf{j})}{\sqrt{2\pi^2 \widehat{f}_T(\lambda_{\mathbf{j}})}}\,, \quad \widetilde{s}_{\mathbf{j},2} = \frac{y(\mathbf{j})}{\sqrt{2\pi^2 \widehat{f}_T(\lambda_{\mathbf{j}})}}\,.$$

For $\mathbf{j} \in N$ and $k = 1, 2$, define the residuals $s_{\mathbf{j},k}$ by standardizing $\widetilde{s}_{\mathbf{j},k}$ as

$$s_{\mathbf{j},k} = \frac{\widetilde{s}_{\mathbf{j},k} - \frac{1}{2|N|} \sum_{\mathbf{l} \in N, k} \widetilde{s}_{\mathbf{l},k}}{\sqrt{\frac{1}{2|N|} \sum_{\mathbf{t} \in N, k} \left(\widetilde{s}_{\mathbf{t},k} - \frac{1}{2|N|} \sum_{\mathbf{l} \in N, k} \widetilde{s}_{\mathbf{l},k} \right)^2}}\,.$$

Note that the residuals $s_{\mathbf{j},k}$ are approximately independent standard normal variables, see Theorem 4.1 of [33].
Step 3: Draw randomly and with replacement from $\mathbb{S} \triangleq \{s_{\mathbf{j},k} : k = 1, 2, \ \mathbf{j} \in N\}$ to obtain an i.i.d. sample $\{s_{\mathbf{j},k}^*, \mathbf{j} \in N, k = 1, 2\}$.
Step 4: Define the bootstrapped Fourier coefficients by

$$x^*(\mathbf{j}) = \sqrt{2\pi^2 \widehat{f}_T(\lambda_{\mathbf{j}})} s_{\mathbf{j},1}^*\,, \quad y^*(\mathbf{j}) = \sqrt{2\pi^2 \widehat{f}_T(\lambda_{\mathbf{j}})} s_{\mathbf{j},2}^*\,,$$

where $\mathbf{j} \in N$.

3.1.2. Wild bootstrap (WB)

Compared to RB, the WB further exploits the asymptotic normality of the Fourier coefficients by generating independent standard normal random variables instead of resampling the residuals.

Step 1: Estimate the spectral density f by $\widehat{f}_T$, which satisfies (3.1).
Step 2: Define the bootstrapped Fourier coefficients by

$$x^*(\mathbf{j}) = \sqrt{2\pi^2 \widehat{f}_T(\lambda_{\mathbf{j}})} G_{\mathbf{j},1}\,, \quad y^*(\mathbf{j}) = \sqrt{2\pi^2 \widehat{f}_T(\lambda_{\mathbf{j}})} G_{\mathbf{j},2}\,,$$

where $\{G_{\mathbf{j},k} : \mathbf{j} \in N, k = 1, 2\}$ are independent standard normal random variables.

For RB and WB, conditions for kernel spectral density estimators satisfying (3.1) can be found in [33] for a large class of random fields.

3.1.3. Local bootstrap (LB)

In contrast to RB and WB, LB does not require any spectral density estimation. Instead, LB makes use of the smoothness of spectral density, which ensures that in a neighborhood of each frequency, the distributions of the Fourier coefficients are nearly identical. Therefore, replicates of the Fourier coefficients can be produced by directly resampling the Fourier coefficients within neighborhoods.

Step 1: Select a symmetric, nonnegative kernel $K(\cdot)$ that satisfies Assumptions $\mathcal{K}.1$ and $\mathcal{K}.4$. Select a bandwidth $\mathbf{h}_T = (h_{T1}, h_{T2}) \in \mathbb{R}^2$ such that $h_{Tk} \to 0$, $h_{T1} \asymp h_{T2}$ and $h_{Tk} d_k \to \infty$ for $k = 1, 2$.
Step 2: Generate i.i.d. jittering variables $\{J_{\mathbf{j},k,T} : \mathbf{j} \in N, k = 1, 2\}$ on $\mathbb{Z}^2$ with

$$p_{\mathbf{s},T} = \mathbb{P}(J_{\mathbf{j},k,T} = \mathbf{s}) = \frac{K\left(\frac{2\pi s_1}{h_{T1} d_1}, \frac{2\pi s_2}{h_{T2} d_2}\right)}{\sum_{\mathbf{j} \in \mathbb{Z}^2} K\left(\frac{2\pi j_1}{h_{T1} d_1}, \frac{2\pi j_2}{h_{T2} d_2}\right)}, \tag{3.2}$$

where $\mathbf{s} = (s_1, s_2) \in \mathbb{Z}^2$. Independent of $\{J_{\mathbf{j},k,T}\}$, generate i.i.d. Bernoulli random variables $\{B_{\mathbf{j},k} : \mathbf{j} \in N, k = 1, 2\}$ with parameter $1/2$.
Step 3: For $\mathbf{j} \in N$, define the *uncentered* bootstrapped Fourier coefficients by resampling within a neighborhood of $\mathbf{j}$.

$$\widetilde{x}^*(\mathbf{j}) = \begin{cases} x(\mathbf{j} + J_{\mathbf{j},1,T}) & \text{if } B_{\mathbf{j},1} = 0, \\ y(\mathbf{j} + J_{\mathbf{j},1,T}) & \text{if } B_{\mathbf{j},1} = 1, \end{cases} \text{ and } \widetilde{y}^*(\mathbf{j}) = \begin{cases} y(\mathbf{j} + J_{\mathbf{j},2,T}) & \text{if } B_{\mathbf{j},2} = 0, \\ x(\mathbf{j} + J_{\mathbf{j},2,T}) & \text{if } B_{\mathbf{j},2} = 1. \end{cases}$$

Step 4: Define the *centered* bootstrapped Fourier coefficients by

$$(x^*(\mathbf{j}), y^*(\mathbf{j})) = (\widetilde{x}^*(\mathbf{j}), \widetilde{y}^*(\mathbf{j})) - \frac{1}{2}\left(\sum_{\mathbf{s} \in \mathbb{Z}^2} p_{\mathbf{s},T}\left(x(\mathbf{j} + \mathbf{s}) + y(\mathbf{j} + \mathbf{s})\right)\right)(1, 1).$$

3.2. Bootstrap procedure for spatial processes

With the three bootstrap schemes for the Fourier coefficients, we develop the bootstrap procedure for resampling spatial processes as follows:

Step 1: Compute the Fourier Coefficients $x(\mathbf{j})$, $y(\mathbf{j})$ for $\mathbf{j} \in T$ using Fast Fourier Transform (FFT).
Step 2: Set $x^*(\mathbf{j}) = y^*(\mathbf{j}) = 0$ for $\mathbf{j} \in M$, where M is defined in (2.4).
Step 3: Obtain a bootstrap sample $\{(x^*(\mathbf{j}), y^*(\mathbf{j})) : \mathbf{j} \in N\}$ using one of the RB, WB or LB schemes.
Step 4: Set bootstrapped Fourier coefficients on $\tilde{N}$ according to the symmetric relations (2.3), i.e. $x^*(\mathbf{d}_T - \mathbf{j}) = x^*(\mathbf{j})$, and $y^*(\mathbf{d}_T - \mathbf{j}) = -y^*(\mathbf{j})$.
Step 5: Use the inverse FFT algorithm to transform the bootstrap Fourier coefficients $\{x^*(\mathbf{j}) + iy^*(\mathbf{j}), \mathbf{j} \in T\}$ back into the spatial domain $\{Z^*(\mathbf{t}) : \mathbf{t} \in T\}$, where

$$Z^*(\mathbf{t}) = \frac{1}{\sqrt{|T|}} \sum_{\mathbf{j} \in T} (x^*(\mathbf{j}) + iy^*(\mathbf{j})) \exp(i\lambda'_{\mathbf{j}} \mathbf{t}).$$

The resulting bootstrap spatial process $\{Z^*(\mathbf{t}) : \mathbf{t} \in T\}$ is real-valued and centered, and can be used for inference on a large class of statistics that are based on partial sums of the centered process $\{Z(\mathbf{t})\}$; see Section 6 for examples. Note that since the Fourier coefficients in N can also be uniquely determined by the Fourier coefficients in $\tilde{N}$, it would be technically the same if we interchange the roles of $\tilde{N}$ and N in the above bootstrap procedure. That is, we can first obtain

a bootstrap sample on $\tilde{N}$ instead of N in Step 3, and then set bootstrapped Fourier coefficients on N instead of $\tilde{N}$ in Step 4 using the rotational symmetry in (2.3).

Remark 3.1. To compare the computational cost of the proposed frequency domain bootstrap methods and the existing spatial block bootstrap method, we first investigate the number of random data required to be simulated for generating one bootstrap spatial resample. The classical block bootstrap with a block size $m_1 \times m_2$ typically requires a simulation of $(\lfloor d_1/m_1 \rfloor + 1)(\lfloor d_2/m_2 \rfloor + 1)$ random data to generate one bootstrap spatial resample, which is with order of $O(|T|/(m_1 m_2))$. On the other hand, the proposed frequency domain bootstrap methods need to simulate $2|N|$ random data for RB and WB and $4|N|$ random data for LB to generate one bootstrap spatial resample, which are of order $O(|T|)$. Hence, the computational cost for generating resamples in the classical block bootstrap method is smaller compared to that of the proposed methods as the block size $m_1 \times m_2$ diverges. However, in practice the computational complexity for evaluating the test statistics under consideration is $O(|T|)$ and hence the computational complexity for conducting bootstrap inference, for example constructing bootstrap confidence intervals, is $O(B|T|)$, where B is the number of bootstrap replications. Thus, the computational cost of both methods are essentially of the same order $O(B|T|)$.

4. Main results

In this section, we first review the invariance principles of the partial sum process of a random field. Then, we present the main results of the paper: the invariance principles of the partial sum process of the bootstrap sample (Theorem 4.3), and the validity of the bootstrap methods (Corollaries 4.4 and 4.5).

4.1. Invariance principles for random fields

To facilitate applications to different situations, we consider the following collection of Borel subsets of $[0,1]^2$ as the index set of the partial sum process:

$$Q_2' = \left\{ [u_1, v_1] \times [u_2, v_2] : u_1 \leq v_1, u_2 \leq v_2, (u_1, u_2), (v_1, v_2) \in [0,1]^2 \right\}.$$

The class Q_2' is a classical example of Vapnik-Chervonenkis-classes (VC-classes), with VC-index equal to 5; see Section 2.6 of [48].

We equip the class Q_2' with the pseudo-metric $\rho(A, B) = \sqrt{L(A \Delta B)}$, where $A, B \in Q_2'$, $L(\cdot)$ is the Lebesgue measure and Δ is the symmetric difference operator. Given $A = [a_{11}, a_{12}] \times [a_{21}, a_{22}] \in Q_2'$ and a rectangular region $E = \left\{ (e_1, e_2) \in \mathbb{Z}^2, b_k \leq e_k \leq c_k, k = 1, 2 \right\}$, define the scaled region as $A \otimes E = ([b_1 + (c_1 - b_1)a_{11}, b_1 + (c_1 - b_1)a_{12}] \times [b_2 + (c_2 - b_2)a_{21}, b_2 + (c_2 - b_2)a_{22}]) \bigcap E$. Note that $A \otimes E = g_E(A)$, where $g_E(u_1, u_2) = (\lfloor b_1 + u_1(c_1 - b_1) \rfloor, \lfloor b_2 + u_2(c_2 - b_2) \rfloor) \in E$ is a linear mapping that translates and rescales $[0,1]^2$ to E.

Define the Q_2'-indexed partial sum processes $\{S_T(A) : A \in Q_2'\}$ as $S_T(A) = \sum_{\mathbf{t} \in A \otimes T}(V(\mathbf{t}) - \mu)$. Let $\mathbb{C}$ be the space of continuous real functions on Q_2', equipped with the norm $\|\cdot\|_{Q_2'}$ defined by $\|f\|_{Q_2'} = \sup_{A \in Q_2'} |f(A)|$. A standard Brownian motion indexed by Q_2' is a mean zero Gaussian process $\{\mathbb{W}(A) : A \in Q_2'\}$ with sample paths in $\mathbb{C}$ and $\mathrm{Cov}(\mathbb{W}(A), \mathbb{W}(B)) = L(A \bigcap B)$, where $L(\cdot)$ is the Lebesgue measure. A standard Brownian bridge indexed by Q_2' is a mean zero Gaussian process $\{\mathbb{B}(A) : A \in Q_2'\}$ with sample paths in $\mathbb{C}$ and $\mathrm{Cov}(\mathbb{B}(A), \mathbb{B}(B)) = L(A \bigcap B) - L(A)L(B)$.

The following lemmas by [9] give the invariance principles of the partial sum processes $\{S_T(A) : A \in Q_2'\}$.

Lemma 4.1. *Under Assumptions* A.1 *and* A.4(p) *with some $p > 8$, the Q_2'-indexed partial sum processes of the random field $\{V(\mathbf{t}) : \mathbf{t} \in T\}$ satisfies the invariance principle: when $T \to \infty$,*

$$\left\{\left(4\pi^2 f(\mathbf{0})|T|\right)^{-1/2} S_T(A) : A \in Q_2'\right\} \xrightarrow{d} \{\mathbb{W}(A) : A \in Q_2'\}\,,$$

where f is the spectral density of $\{V(\cdot)\}$ defined in (2.1).

4.2. Invariance principles for bootstrap samples

This section contains the main result: the invariance principles for the partial sum process of the bootstrap sample. This result implies the validity of the bootstrap methods in case the invariance principles are involved in the asymptotic of the underlying test statistics. To facilitate further extensions to bootstrap schemes other than RB, WB and LB, the results are formulated in a general way under some meta-assumptions on the resampling scheme. In Section 5, we verify the meta-assumptions for the RB, WB and LB schemes.

4.2.1. Assumptions on the bootstrapped Fourier coefficients

Denote $\mathcal{L}^*$, $\mathbb{E}^*$, Var^*, Cov^*, and $\mathbb{P}^*$ as the bootstrap distribution, expectation, variance, covariance and probability, conditional on the data, respectively. Moreover, let $\{\cdot|V(\cdot)\}$ denote conditioning on the data.

Assumption $\mathcal{B}$.1. For the bootstrap scheme in the frequency domain, conditional on the data, the resampled Fourier coefficients $\{x^*(\mathbf{j}), y^*(\mathbf{j}), \mathbf{j} \in N\}$ are mutually independent with

$$\mathbb{E}^*(x^*(\mathbf{k})) = \mathbb{E}^*(y^*(\mathbf{k})) = 0\,.$$

Assumption $\mathcal{B}$.2. Uniform convergence of the variances of the bootstrapped Fourier coefficients:

$$\sup_{\mathbf{j} \in N} \left|\mathrm{Var}^*(x^*(\mathbf{j})) - 2\pi^2 f(\lambda_{\mathbf{j}})\right| = o_p(1)\,, \quad \sup_{\mathbf{j} \in N} \left|\mathrm{Var}^*(y^*(\mathbf{j})) - 2\pi^2 f(\lambda_{\mathbf{j}})\right| = o_p(1)\,.$$

Assumption $\mathcal{B}.3$. There exists some $p > 8$ such that the p-th moments of the bootstrapped Fourier coefficients are uniformly bounded:

$$\sup_{\mathbf{j} \in N} \mathbb{E}^* \left(|x^*(\mathbf{j})|^p \right) \leq C_1 + o_p(1) \,, \quad \sup_{\mathbf{j} \in N} \mathbb{E}^* \left(|y^*(\mathbf{j})|^p \right) \leq C_1 + o_p(1) \,.$$

The Mallows distance on the space of all real Borel probability measures with finite variance is given by $d_2(\mathcal{L}_1, \mathcal{L}_2) = \inf \left(\mathbb{E}|X_1 - X_2|^2 \right)^{1/2}$, where the infimum is taking over all random variables X_1 and X_2 with marginal distributions $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively. Convergence in Mallows distance implies convergence in distribution and convergence in the second moment, see [31].

Assumption $\mathcal{B}.4$. The probability distributions of the bootstrapped Fourier coefficients converge uniformly in the Mallows distance to the same limit as the Fourier coefficients do, i.e.,

$$\sup_{\mathbf{j} \in N} d_2 \left(\mathcal{L}^*(x^*(\mathbf{j})), N \left(0, 2\pi^2 f(\lambda_{\mathbf{j}}) \right) \right) = o_p(1) \,,$$

$$\sup_{\mathbf{j} \in N} d_2 \left(\mathcal{L}^*(y^*(\mathbf{j})), N \left(0, 2\pi^2 f(\lambda_{\mathbf{j}}) \right) \right) = o_p(1) \,.$$

4.2.2. Asymptotic results on bootstrap samples

The following lemma asserts that the bootstrap sample $\{Z^*(\cdot)\}$ and the corresponding partial sum process have correct auto-covariance structures.

Lemma 4.2. *Let $A_T \in Q_2'$ be a set which depends on T, and Assumptions $\mathcal{A}.1$ and $\mathcal{B}.1$ hold.*

(a) *For any $A_T, B \in Q_2'$, we have $\mathbb{E}^* \left(\frac{1}{\sqrt{|A_T \otimes T|}} \sum_{\mathbf{l} \in B \otimes (A_T \otimes T)} Z^*(\mathbf{l}) \right) = 0 \,.$*

(b) *If Assumptions $\mathcal{A}.2$ and $\mathcal{B}.2$ also hold, then for $A_T, B, C \in Q_2'$,*

$$\mathrm{Cov}^* \left(\frac{1}{\sqrt{|A_T \otimes T|}} \sum_{\mathbf{l}_1 \in B \otimes (A_T \otimes T)} Z^*(\mathbf{l}_1), \frac{1}{\sqrt{|A_T \otimes T|}} \sum_{\mathbf{l}_2 \in C \otimes (A_T \otimes T)} Z^*(\mathbf{l}_2) \right)$$

$$\xrightarrow{p} \left\{ \begin{array}{ll} 4\pi^2 f(\mathbf{0}) L(B \bigcap C) \,, & \text{if } |A_T \otimes T|/|T| \to 0 \,, \\ 4\pi^2 f(\mathbf{0})[L(B \bigcap C) - L(B)L(C)] \,, & \text{if } A_T = [0,1]^2 \,. \end{array} \right.$$

(c) *If Assumptions $\mathcal{A}.2$ and $\mathcal{B}.2$ also hold, then for any fixed $\mathbf{l}_1, \mathbf{l}_2 \in T$,*

$$\begin{aligned} \mathrm{Cov}^*(Z^*(\mathbf{l}_1), Z^*(\mathbf{l}_2)) &= \mathrm{Cov}(V(\mathbf{l}_1), V(\mathbf{l}_2)) + o_p(1) \\ &= \mathrm{Cov}(Z(\mathbf{l}_1), Z(\mathbf{l}_2)) + o_p(1) \,. \end{aligned}$$

For any rectangular region T, denote the Q_2'-indexed partial sum processes of the bootstrap sample $\{S_T^*(B) : B \in Q_2'\}$ as $S_T^*(B) = \sum_{\mathbf{t} \in B \otimes T} Z^*(\mathbf{t})$. Note that if $A_T = [0,1]^2$, then $A_T \otimes T = T$. The following theorem establishes the invariance principles for the Q_2'- indexed partial sum processes of the bootstrap samples.

Theorem 4.3. *Suppose that Assumptions A.1, A.2, B.1-B.3 hold.*

(a) *If $A_T \in Q_2'$ and $|A_T \otimes T|/|T| \to 0$, then it holds in probability*

$$\left\{ (4\pi^2 f(\mathbf{0})|A_T \otimes T|)^{-1/2} S^*_{A_T \otimes T}(B) : B \in Q_2'|V(\cdot) \right\} \xrightarrow{d} \left\{ \mathbb{W}(B) : B \in Q_2' \right\}.$$

(b) *If $A_T = [0,1]^2$ and Assumption B.4 holds, then it holds in probability*

$$\left\{ (4\pi^2 f(\mathbf{0})|T|)^{-1/2} S^*_T(B) : B \in Q_2'|V(\cdot) \right\} \xrightarrow{d} \left\{ \mathbb{B}(B) : B \in Q_2' \right\}.$$

The following corollary states that the bootstrap samples preserve the second-order dependence structure of the random field asymptotically. In particular, if the underlying random field is Gaussian, then the proposed bootstrap procedure produces asymptotically valid approximation of the centered random field $\{V(\mathbf{t}) - \mu\}$.

Corollary 4.4. *If Assumptions A.1, A.2, B.1-B.3 hold, then for any subset $\{\mathbf{l}_1, ..., \mathbf{l}_p\}$ of positive integers in $\mathbb{Z}^2$, we have*

$$(Z^*(\mathbf{l}_1), ..., Z^*(\mathbf{l}_p)|V(\cdot)) \xrightarrow{d} N(0, \Sigma),$$

in probability, where $\Sigma = (\sigma_{i,j})_{i,j=1,...,p}$ with $\sigma_{i,j} = \mathrm{Cov}(V(\mathbf{l}_i), V(\mathbf{l}_j))$.

4.2.3. Bootstrapping the mean

The bootstrap sample $\{Z^*(\mathbf{t}) : \mathbf{t} \in T\}$ obtained from Section 3.2 is real-valued and centered. In order to obtain a non-centered bootstrap process, we may employ a separate bootstrap procedure independent of $\{Z^*(\cdot)\}$ to acquire a bootstrapped mean μ_T^*. For details on bootstrapping the mean μ_T^*, see [16], [42], [46], [26], and [35]. Then, the non-centered bootstrap process $V^*(\cdot) = Z^*(\cdot) + \mu_T^*$ gives a bootstrap approximation of $V(\cdot)$. Note that $\{Z^*(\cdot)\}$ contains information about the covariance structure of the spatial process and μ_T^* contains information about the mean level. The following corollary shows that the non-centered bootstrap sample $V^*(\cdot)$ has the same asymptotic behavior as the original spatial process in terms of the partial sum process.

Corollary 4.5. *Suppose that Assumptions A.1, A.2, B.1-B.4 hold. Let μ_T^* be any bootstrap version of the mean $\mu = \mathbb{E}(V(\mathbf{0}))$ (taken independently from $\{Z^*(\cdot)\}$) such that for all $z \in \mathbb{R}$,*

$$\mathbb{P}^* \left(\sqrt{|T|}(\mu_T^* - \mu) \leq z \right) \xrightarrow{p} \Phi \left(\frac{z}{\sqrt{4\pi^2 f(\mathbf{0})}} \right),$$

where $\Phi(\cdot)$ denotes the standard normal distribution function. Then, it holds in probability that

$$\left\{ (4\pi^2 f(\mathbf{0})|T|)^{-1/2} \sum_{\mathbf{t} \in B \otimes T} (V^*(\mathbf{t}) - \mu) : B \in Q_2' \Big| V(\cdot) \right\} \xrightarrow{d} \left\{ \mathbb{W}(B) : B \in Q_2' \right\},$$

where $V^*(\mathbf{t}) = Z^*(\mathbf{t}) + \mu_T^*$.

Possible generalizations of asymptotic results in Section 4 to other VC-classes with VC-index equal to V as index sets can be established with $p > 2(V - 1)$ moment conditions, see Theorem 2(i) of [9] for the invariance principles indexed by VC-classes. For example, the class $Q_2 = \left\{ [0, v_1] \times [0, v_2] : (v_1, v_2) \in [0, 1]^2 \right\}$ is a classical example of VC-classes, with VC-index equal to 3, and the above asymptotic results hold when the moment condition with $p > 4$ in Assumptions $\mathcal{A}.4(p)$ and $\mathcal{B}.3$ hold.

Remark 4.1. From Theorem 4.3 and Corollary 4.5, the proposed frequency domain bootstrap method can mimic the second-order dependence structure of the random field asymptotically. Hence, the proposed method is applicable to statistics of interest which depend asymptotically on second order dependence structure. In Section 6, we discuss applications of the proposed bootstrap method to confidence intervals construction of correlograms for linear random fields, testing for signal presence in random fields, and spatial isotropy test for Gaussian random fields. The validities of the proposed bootstrap method for the above applications are also theoretically investigated.

5. Validity of meta-assumptions for the resampling schemes RB, WB, and LB

In this section, we prove the validity of the bootstrap schemes RB, WB, and LB under some conditions on the spatial processes. We also give conditions under which the bootstrap schemes remain valid when the bootstrap methods are applied to an estimated field $\{\widehat{V}(\mathbf{t}) : \mathbf{t} \in T\}$ rather than the observed field $\{V(\mathbf{t}) : \mathbf{t} \in T\}$.

Theorem 5.1. *Let Assumptions $\mathcal{A}.1$-$\mathcal{A}.3$, $\mathcal{A}.4(p)$ with some $p > 8$ and $\mathcal{K}.1$ hold.*

(a) *Suppose that Assumptions $\mathcal{K}.3$ and $\mathcal{A}.5$ with $\mathbb{E}\left(|V(\mathbf{0})|^{16}\right) < \infty$ hold, and the bandwidth satisfies $|\mathbf{h}_T| \to 0$, $(|\mathbf{h}_T||T|^\eta)^{-1} = O(1)$ for some $0 < \eta < 1/2$. Then, the kernel spectral density estimator (2.5) satisfies (3.1), and RB satisfies Assumptions $\mathcal{B}.1$ to $\mathcal{B}.4$.*

(b) *Suppose that Assumptions $\mathcal{K}.3$ and $\mathcal{A}.5$ with $\mathbb{E}\left(|V(\mathbf{0})|^\mu\right) < \infty$ for some $4 < \mu \leq 8$ hold, and the bandwidth satisfies $|\mathbf{h}_T| \to 0$, $(|\mathbf{h}_T||T|^\eta)^{-1} = O(1)$ for some $0 < \eta < \frac{(\mu-4)}{\mu}$. Then, the kernel spectral density estimator (2.5) satisfies (3.1), and WB satisfies Assumptions $\mathcal{B}.1$ to $\mathcal{B}.4$.*

(c) *Suppose that Assumptions $\mathcal{K}.4$ and $\mathcal{A}.5$ with $\mathbb{E}\left(|V(\mathbf{0})|^{16}\right) < \infty$ hold, the bandwidth satisfies $|\mathbf{h}_T| + (|\mathbf{h}_T|^4|T|)^{-1} \to 0$, and the kernel $K(\cdot)$ fulfills $\frac{1}{|T||\mathbf{h}_T|} \sum_{\mathbf{j} \in \mathbb{Z}^2} K^2 \left(\frac{2\pi j_1}{h_{T_1} d_1}, \frac{2\pi j_2}{h_{T_2} d_2} \right) = O(1)$. Then, LB satisfies Assumptions $\mathcal{B}.1$ to $\mathcal{B}.4$.*

For illustration, we give some examples where Theorem 5.1 is applicable.

Example 5.1 (Linear Random Fields)**.** Let $\{\varepsilon_{\mathbf{t}}\}_{\mathbf{t} \in \mathbb{Z}^2}$ be an i.i.d. random field with $\mathbb{E}(\varepsilon_{\mathbf{0}}) = 0$, $\mathbb{E}(|\varepsilon_{\mathbf{0}}|^p) < \infty$ with $p \geq 16$. The linear random field $\{V(\mathbf{t}) :$

$\mathbf{t} \in \mathbb{Z}^2\}$, i.e., $V(\mathbf{j}) = \sum_{\mathbf{s} \in \mathbb{Z}^2} a_{\mathbf{s}} \varepsilon_{\mathbf{j}-\mathbf{s}}$ with $|a_{\mathbf{j}}| \leq C\rho^{\|\mathbf{j}\|}$ for some $\rho \in (0,1)$ and $C > 0$, satisfies Assumptions $\mathcal{A}.4(p)$ with $p > 8$ and $\mathcal{A}.5$ with $\mathbb{E}\left(|V(\mathbf{0})|^{16}\right) < \infty$ since $\sum_{\mathbf{j} \in \mathbb{Z}^2} \delta_{\mathbf{j},p} = \sum_{\mathbf{j} \in \mathbb{Z}^2} \|V(\mathbf{j}) - \widetilde{V}(\mathbf{j})\|_p = \sum_{\mathbf{j} \in \mathbb{Z}^2} |a_{\mathbf{j}}| \|\varepsilon_{\mathbf{0}} - \widetilde{\varepsilon}_{\mathbf{0}}\|_p < \infty$ and $\delta_{\mathbf{j},p} = O(\rho^{\|\mathbf{j}\|})$ respectively.

Example 5.2 (Volterra Fields)**.** Volterra Fields is a class of nonlinear random fields which plays an important role in the nonlinear system theory. Let $\{\varepsilon_{\mathbf{t}}\}_{\mathbf{t} \in \mathbb{Z}^2}$ be an i.i.d. random field with $\mathbb{E}(|\varepsilon_{\mathbf{0}}|^p) < \infty$ for some $p \geq 32$. Consider the second order Volterra process $\left\{V(\mathbf{t}) : \mathbf{t} \in \mathbb{Z}^2\right\}$,

$$V(\mathbf{j}) = \sum_{\mathbf{s}_1, \mathbf{s}_2 \in \mathbb{Z}^2} a_{\mathbf{s}_1, \mathbf{s}_2} \varepsilon_{\mathbf{j}-\mathbf{s}_1} \varepsilon_{\mathbf{j}-\mathbf{s}_2} ,$$

where $\{a_{\mathbf{s}_1, \mathbf{s}_2}\}$ are real coefficients with $a_{\mathbf{s}_1, \mathbf{s}_2} = 0$ if $\mathbf{s}_1 = \mathbf{s}_2$. Then, by Rosenthal inequality, there exists a constant $C_p > 0$ such that

$$\delta_{\mathbf{j},p} = \|V(\mathbf{j}) - \widetilde{V}(\mathbf{j})\|_p \leq C_p \left(A_{\mathbf{j}}^{1/2} \|\varepsilon_{\mathbf{0}}\|_2 \|\varepsilon_{\mathbf{0}}\|_p + B_{\mathbf{j}}^{1/p} \|\varepsilon_{\mathbf{0}}\|_p^2 \right) ,$$

where $A_{\mathbf{j}} = \sum_{\mathbf{s}_1, \mathbf{s}_2 \in \mathbb{Z}^2} (a_{\mathbf{s}_1, \mathbf{j}}^2 + a_{\mathbf{j}, \mathbf{s}_2}^2)$ and $B_{\mathbf{j}} = \sum_{\mathbf{s}_1, \mathbf{s}_2 \in \mathbb{Z}^2} (|a_{\mathbf{s}_1, \mathbf{j}}|^p + |a_{\mathbf{j}, \mathbf{s}_2}|^p)$. Thus, if $a_{\mathbf{s}_1, \mathbf{s}_2} = O(\rho^{\max\{\|\mathbf{s}_1\|, \|\mathbf{s}_2\|\}})$ for some $\rho \in (0,1)$, then $\delta_{\mathbf{j},p} = O(\rho^{\|\mathbf{j}\|})$, and Assumptions $\mathcal{A}.4(p)$ with $p > 8$ and $\mathcal{A}.5$ with $\mathbb{E}\left(|V(\mathbf{0})|^{16}\right) < \infty$ hold.

Example 5.3 (Nonlinear Spatial Autoregressive Models)**.** Let $\mathcal{N} \subset \mathbb{Z}^2$ be a finite set and $\mathbf{0} \notin \mathcal{N}$. Consider the spatial process in the form of nonlinear autoregressive scheme

$$V(\mathbf{t}) = G(\{V(\mathbf{t} - \mathbf{s})\}_{\mathbf{s} \in \mathcal{N}}; \varepsilon_{\mathbf{t}}) ,$$

where the function G is such that there exists nonnegative numbers $u_{\mathbf{s}}, \mathbf{s} \in \mathcal{N}$, with $\sum_{\mathbf{s} \in \mathcal{N}} u_{\mathbf{s}} < 1$ and the following holds: for all $\{v(-\mathbf{s})\}_{\mathbf{s} \in \mathcal{N}}$ and $\{v'(-\mathbf{s})\}_{\mathbf{s} \in \mathcal{N}}$,

$$|G(\{v(-\mathbf{s})\}_{\mathbf{s} \in \mathcal{N}}; \varepsilon_{\mathbf{t}}) - G(\{v'(-\mathbf{s})\}_{\mathbf{s} \in \mathcal{N}}; \varepsilon_{\mathbf{t}})| \leq \sum_{\mathbf{s} \in \mathcal{N}} u_{\mathbf{s}} |v(-\mathbf{s}) - v'(-\mathbf{s})| .$$

Also assume that there exists $\{v(-\mathbf{s})\}_{\mathbf{s} \in \mathcal{N}}$ such that $G(\{v(-\mathbf{s})\}_{\mathbf{s} \in \mathcal{N}}; \varepsilon_{\mathbf{0}}) \in \mathbb{L}^p$ for $p \geq 16$. Then following the argument in Section 5 of [45] or Example 2 of [7], we have $\delta_{\mathbf{j},p} = O(\rho^{\|\mathbf{j}\|})$ for some $\rho \in (0,1)$ and hence Assumptions $\mathcal{A}.4(p)$ with $p > 8$ and $\mathcal{A}.5$ with $\mathbb{E}\left(|V(\mathbf{0})|^{16}\right) < \infty$ hold.

In many applications, the bootstrap methods are not applied directly to stationary spatial data $\{V(\mathbf{t})\}$, but to an estimate $\{\widehat{V}(\mathbf{t})\}$ from spatial data $\{Y(\mathbf{t})\}$, see, for example, the testing for signal presence using scan statistics in Section 6.2. The following corollary gives conditions for the validity of the bootstrap schemes in this situation.

Corollary 5.2. *Under the setting of Theorem 5.1, suppose that for the spatial data* $\{Y(\mathbf{t})\}$, *we have an estimator* $\{\widehat{V}(\mathbf{t})\}$ *of* $\{V(\mathbf{t})\}$ *such that the average squared error*

$$\frac{1}{|T|} \sum_{\mathbf{t} \in T} \left(V(\mathbf{t}) - \widehat{V}(\mathbf{t}) \right)^2 = o_p \left(\alpha_T^{-1} \right) ,$$

as $\alpha_T \to \infty$. Furthermore, assume that for RB and WB, the kernel of the spectral density estimator (2.5) satisfies Assumptions $\mathcal{K}.1$ and $\mathcal{K}.2$; and the kernel for LB satisfies Assumptions $\mathcal{K}.1$ and $\mathcal{K}.2$. Let p be the order of uniformly bounded moments of the bootstrapped Fourier coefficients stated in Assumption $\mathcal{B}.3$ which is satisfied under the setting of Theorem 5.1 given $\{V(\mathbf{t}) : \mathbf{t} \in T\}$. Then given $\{Y(\mathbf{t})\}$, we have the followings hold:

(a) *If $\alpha_T = O\big(|T|^{\frac{p-2}{p}} + |\mathbf{h}_T|^{-1}\big)$ with bandwidth $\mathbf{h}_T$ in (2.5), then RB satisfies Assumptions $\mathcal{B}.1$ to $\mathcal{B}.4$.*

(b) *If $\alpha_T = O\big(|\mathbf{h}_T|^{-1}\big)$ with bandwidth $\mathbf{h}_T$ in (2.5), then WB satisfies Assumptions $\mathcal{B}.1$ to $\mathcal{B}.4$.*

(c) *If $\alpha_T = O\left(\big(|T|^{p-2}/|\mathbf{h}_T|^2\big)^{\frac{1}{p}}\right)$ with bandwidth $\mathbf{h}_T$ in (3.2), then LB satifies Assumptions $\mathcal{B}.1$ to $\mathcal{B}.4$.*

Remark 5.1. For random fields exhibiting complex non-linear trends, the proposed bootstrap procedure requires some modifications. Specifically, assume that the underlying random field $\{Y(\mathbf{t})\}$ can be modeled by $Y(\mathbf{t}) = \mu(\mathbf{t}) + V(\mathbf{t})$ for $\mathbf{t} \in T$, where $\mu(\mathbf{t})$ is a non-linear trend, and $V(\mathbf{t})$ is a zero-mean random field which satisfies the conditions stated in the Section 2. From Corollary 5.2, the proposed bootstrap method remains valid for the field $\{\widehat{V}(\mathbf{t})\}$ estimated from the spatial data $\{Y(\mathbf{t})\}$ under some conditions on the decay rate α_T of the average squared error. Hence, to apply the proposed bootstrap method, we can proceed as follows. First, we apply local smoothing or kernel methods to estimate the trend $\hat{\mu}(\mathbf{t})$, and then an estimated field $\{\widehat{V}(\mathbf{t})\}$ can be obtained by $\widehat{V}(\mathbf{t}) = Y(\mathbf{t}) - \hat{\mu}(\mathbf{t})$. The proposed bootstrap method can be applied on $\{\widehat{V}(\mathbf{t})\}$ to get a centered bootstrapped sample $\{Z^*(\mathbf{t})\}$, and then employ a separate bootstrap procedure independent of $\{Z^*(\mathbf{t})\}$ to acquire a bootstrapped mean μ_T^*, and get a non-centered bootstrap field $V^*(\mathbf{t}) = Z^*(\mathbf{t}) + \mu_T^*$ as illustrated in Section 4.2.3. Finally, a bootstrapped sample $Y^*(\mathbf{t}) = \hat{\mu}(\mathbf{t}) + V^*(\mathbf{t})$ can be obtained.

Remark 5.2. To implement the proposed frequency domain bootstrap methods, we need to specify one tuning parameter, the bandwidth $\mathbf{h}_T = (h_{T1}, h_{T2}) \in \mathbb{R}^2$. From Theorem 5.1, the bandwidths have to satisfy some decay rate conditions. To be precise, we require $|\mathbf{h}_T| = O(|T|^{-\eta})$ for some $0 < \eta < 1/2$ for RB and WB, and $|\mathbf{h}_T| \to 0$ and $(|\mathbf{h}_T|^4|T|)^{-1} \to 0$ for LB. For example, $|\mathbf{h}_T| = O(|T|^{-1/5})$ works for all three methods. To provide a more precise guideline for the choice of $\mathbf{h}_T$, in Section 6.1, we first conduct a sensitivity analysis for a wide range of bandwidths, and then select the one with the best coverage of the confidence intervals. As the bandwidths in RB and WB are for the kernel spectral density estimation, we can also employ the adaptive bandwidth selection proposed in [39] or [19]. Although no theoretically supported optimal bandwidth selection method is available for LB, the bandwidth obtained from the adaptive bandwidth selections for RB and WB can be shown to satisfy the required conditions for LB asymptotically. In Section 6, we apply the same range of bandwidths in RB, WB and LB, and similar results occured in all three methods. It indicates that the same suitable bandwidths for RB and WB may

also be appropriate for LB. For more discussions in bandwidth selection in local bootstrap in time series context, see [37].

6. Applications and simulation studies

In this section, we demonstrate applications of the proposed bootstrap procedures to constructing confidence intervals for correlograms for linear random fields, testing for signal presense using scan statistics, and testing for spatial isotropy of Gaussian random fields. We also perform numerical studies to compare the proposed bootstrap methods with existing methods including the spatial block bootstrap and spatial subsampling methods. Unless specified otherwise, in all of the simulation experiments, we consider random fields $\{V(\cdot)\}$ on a 50×50 region T, i.e., $d_1 = 50$, $d_2 = 50$ and $|T| = 2500$. In addition, the number of bootstrap samples is set as 1000. Moreover, the Gaussian kernel $K(\lambda) = \phi(\lambda)$ is employed for the kernel spectral density estimation in RB and WB, and the smoothing function in LB, where $\phi(\cdot)$ is the bivariate standard normal density function. For the spatial block bootstrap, we employ the overlapping block bootstrap ([26]). For the spatial subsampling method, we use the overlapping subblocks subsampling ([15]) with the suggested subblock size $d_1^{1/2} \times d_2^{1/2} = 7 \times 7$.

6.1. Confidence intervals construction of correlograms for linear random fields

One major application of frequency domain bootstrap in linear time series is on ratio statistics such as sample autocorrelation functions; see [5] and [21]. Analogously, for spatial statistics, the proposed frequency domain bootstrap method can be applied to construct confidence intervals of the spatial correlograms for linear random fields. Consider a random field $\{V(\mathbf{t}) : \mathbf{t} \in \mathbb{Z}^2\}$ with covariance function $C(\mathbf{h})$. The correlogram at lag $\mathbf{t}$ is defined as the ratio statistic $\rho(\mathbf{t}) = C(\mathbf{t})/C(\mathbf{0})$. A natural estimator of the correlogram is given by $\hat{\rho}(\mathbf{t}) = \hat{C}(\mathbf{t})/\hat{C}(\mathbf{0})$, where $\hat{C}(\mathbf{t}) = \frac{1}{|T(\mathbf{t})|} \times \sum_{\mathbf{s} \in T(\mathbf{t})} \left[V(\mathbf{s}) - \overline{V_T} \right] \left[V(\mathbf{s} + \mathbf{t}) - \overline{V_T} \right]$, with $\overline{V_T} = \frac{1}{|T|} \sum_{s \in T} V(\mathbf{s})$ and $T(\mathbf{t}) = \{\mathbf{s} : \mathbf{s}, \mathbf{s} + \mathbf{t} \in T\}$, is the method-of-moments estimator of the covariogram at lag $\mathbf{t}$. To apply the proposed bootstrap methods, generate B bootstrap samples from either RB, WB or LB. For the i-th bootstrap sample, we compute the correlogram estimator $\hat{\rho}^*_{(i)}(\mathbf{t})$. Then, confidence intervals can be constructed from the sample quantiles of the correlogram estimates of the resamples.

To illustrate the construction of bootstrap confidence intervals for correlograms, consider real-valued mean-zero Gaussian random fields on $\mathbb{Z}^2$ with a Gaussian covariance function

$$C(\mathbf{h}) = \sigma^2 \exp\left(-\left(\frac{\|\mathbf{h}\|}{\phi} \right)^2 \right) + \eta \mathbb{1}_{\{\mathbf{h}=0\}} \,,$$
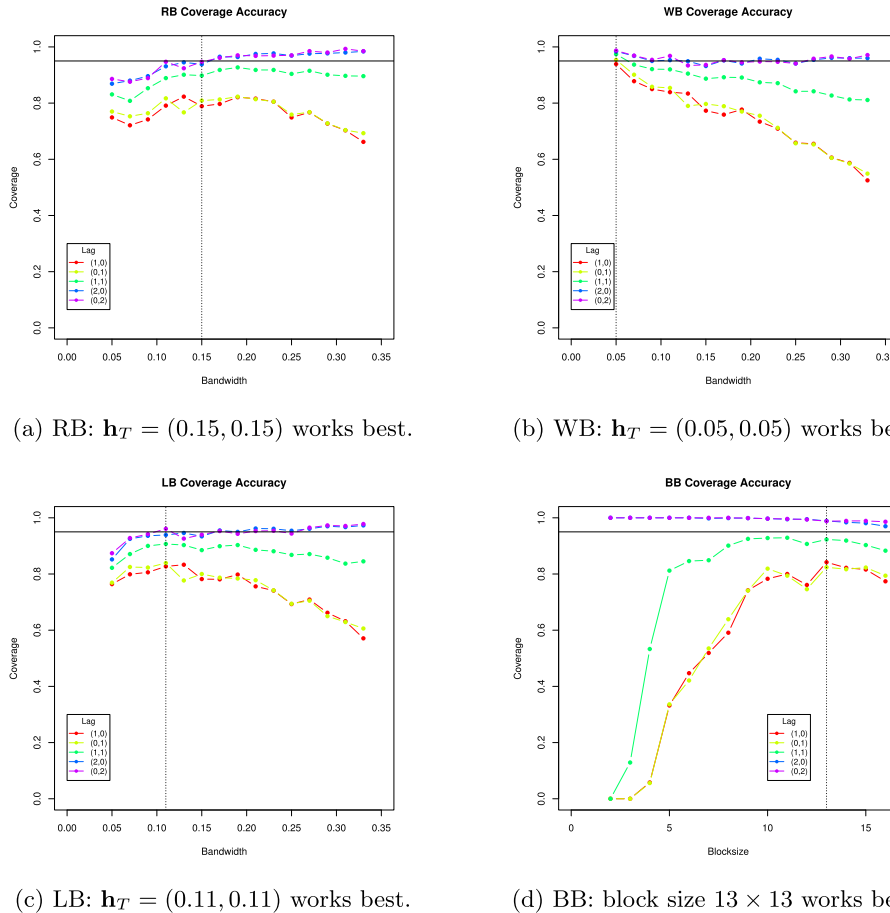
(a) RB: $\mathbf{h}_T = (0.15, 0.15)$ works best.

(b) WB: $\mathbf{h}_T = (0.05, 0.05)$ works best.

(c) LB: $\mathbf{h}_T = (0.11, 0.11)$ works best.

(d) BB: block size 13×13 works best.

FIG 1. *Coverage accuracy of confidence intervals for correlograms under different bandwidths/block sizes.*

where σ^2 is the partial sill parameter, ϕ is the range parameter and η is the nugget effect. First, we consider the model $(\eta, \sigma^2, \phi)=(0, 1, 1)$ to investigate the choice of the bandwidth parameter $\mathbf{h}_T$ for RB, WB and LB schemes, and the choice of block size for the Block Bootstrap (BB). For each resample of data, we compute the correlogram estimates $\hat{\rho}(\mathbf{t})$ at a range of lags: $\mathbf{t} = (1, 0)$, $(0, 1)$, $(1, 1)$, $(2, 0)$ and $(0, 2)$. Then, for each lag, a 95% confidence interval is constructed from the sample quantiles of the correlogram estimates of the resamples. The above procedure is repeated 1000 times to investigate the coverage accuracy of the confidence intervals. The results are summarized in Figure 1. It can be seen that the bandwidths $\mathbf{h}_T = (0.05, 0.05)$, $(0.11, 0.11)$, or $(0.15, 0.15)$ give good performance for all of the proposed RB, WB, and LB schemes. For BB, block sizes 4×4, 7×7, and 13×13 are recommended.

Next, we consider the models $(\eta,\ \sigma^2,\ \phi)$=(0, 1, 0.5), (0, 1, 1), (1, 1, 0.5), and (1, 1, 1) to explore the effect of various decay rates of spatial dependency, and the presence of the nugget effect. For each model, a 95% confidence interval is constructed for each lag based on the sample quantiles of the bootstrapped correlogram estimates. Again, 1000 replications are performed to investigate the coverage accuracy of the confidence intervals. The results are summarized in Tables 3 and 4. It can be seen that the coverage accuracy of the proposed bootstrap methods is much closer to the nominal level of 95% than that of the block bootstrap method. On the other hand, the coverage accuracy of the block bootstrap method is not stable in the sense that it ranges from 50% to nearly 100% coverage under various models.

TABLE 3
The coverage accuracy of the C.I.s for the correlogram by RB, WB, LB, and BB.

	Lag	(1,0)	(0,1)	(1,1)	(2,0)	(0,2)
		$(\eta,\ \sigma^2,\ \phi)$= (0,1,0.5)				
	RB	0.856	0.852	0.885	0.851	0.862
$\mathbf{h}_T = (0.05, 0.05)$	WB	0.962	0.971	0.978	0.980	0.970
	LB	0.852	0.856	0.867	0.862	0.863
	RB	0.895	0.891	0.899	0.898	0.908
$\mathbf{h}_T = (0.11, 0.11)$	WB	0.935	0.932	0.937	0.934	0.943
	LB	0.925	0.934	0.938	0.944	0.953
	RB	0.936	0.937	0.934	0.936	0.942
$\mathbf{h}_T = (0.15, 0.15)$	WB	0.947	0.945	0.941	0.945	0.948
	LB	0.942	0.940	0.934	0.937	0.945
Block size = 4×4	BB	1	0.998	0.999	1	1
Block size = 7×7	BB	0.998	0.998	0.997	0.998	0.999
Block size = 13×13	BB	0.979	0.973	0.981	0.985	0.983
		$(\eta,\ \sigma^2,\ \phi)$= (0,1,1)				
	RB	0.763	0.763	0.862	0.871	0.878
$\mathbf{h}_T = (0.05, 0.05)$	WB	0.941	0.950	0.969	0.978	0.974
	LB	0.777	0.764	0.853	0.855	0.856
	RB	0.782	0.802	0.873	0.924	0.929
$\mathbf{h}_T = (0.11, 0.11)$	WB	0.821	0.842	0.909	0.943	0.959
	LB	0.821	0.841	0.910	0.933	0.948
	RB	0.809	0.826	0.915	0.939	0.940
$\mathbf{h}_T = (0.15, 0.15)$	WB	0.803	0.821	0.903	0.933	0.942
	LB	0.807	0.823	0.907	0.931	0.939
Block size = 4×4	BB	0.063	0.061	0.553	1	1
Block size = 7×7	BB	0.544	0.558	0.877	1	1
Block size = 13×13	BB	0.816	0.818	0.933	0.981	0.991

6.2. Testing for signal presence in random fields

In this subsection, we consider the problem of detecting a deterministic signal against a noisy background. This problem has received considerable attention and has profound applications in epidemiology, astronomy, and biosurveillance. The standard statistical tool to address this problem is the spatial scan statistic; see [24], [13], [14], [12], and [4]. Consider the observations $\{Y(\mathbf{t}) : \mathbf{t} \in T\}$ given by

$$Y(\mathbf{t}) = \begin{cases} V(\mathbf{t}) & \text{if} \quad \mathbf{t} \in T \backslash I_T\,, \\ V(\mathbf{t}) + s & \text{if} \qquad \mathbf{t} \in I_T\,, \end{cases} \tag{6.1}$$

TABLE 4
The coverage accuracy of the C.I.s for the correlogram by RB, WB, LB, and BB.

	Lag	(1,0)	(0,1)	(1,1)	(2,0)	(0,2)
$(\eta, \sigma^2, \phi)= (1,1,0.5)$						
	RB	0.907	0.901	0.893	0.887	0.892
$\mathbf{h}_T = (0.05, 0.05)$	WB	0.973	0.982	0.976	0.972	0.977
	LB	0.905	0.905	0.893	0.865	0.875
	RB	0.912	0.916	0.918	0.912	0.924
$\mathbf{h}_T = (0.11, 0.11)$	WB	0.957	0.958	0.960	0.947	0.952
	LB	0.938	0.944	0.946	0.945	0.947
	RB	0.934	0.942	0.939	0.948	0.938
$\mathbf{h}_T = (0.15, 0.15)$	WB	0.942	0.951	0.949	0.954	0.948
	LB	0.941	0.944	0.946	0.948	0.939
Block size $= 4 \times 4$	BB	1	1	1	1	1
Block size $= 7 \times 7$	BB	0.997	1	1	1	1
Block size $= 13 \times 13$	BB	0.987	0.987	0.985	0.995	0.990
$(\eta, \sigma^2, \phi)= (1,1,1)$						
	RB	0.841	0.844	0.851	0.895	0.891
$\mathbf{h}_T = (0.05, 0.05)$	WB	0.966	0.974	0.980	0.975	0.971
	LB	0.855	0.859	0.861	0.876	0.874
	RB	0.892	0.885	0.912	0.932	0.928
$\mathbf{h}_T = (0.11, 0.11)$	WB	0.928	0.925	0.943	0.967	0.965
	LB	0.915	0.914	0.935	0.955	0.946
	RB	0.935	0.941	0.937	0.933	0.956
$\mathbf{h}_T = (0.15, 0.15)$	WB	0.935	0.937	0.935	0.946	0.966
	LB	0.932	0.935	0.937	0.943	0.962
Block size $= 4 \times 4$	BB	0.888	0.893	0.970	1	1
Block size $= 7 \times 7$	BB	0.981	0.979	0.984	1	1
Block size $= 13 \times 13$	BB	0.954	0.949	0.967	0.995	0.994

where $T \in \mathbb{Z}^2$ is a rectangular region, $\{V(\mathbf{t}) : \mathbf{t} \in T\}$ is a zero-mean process, and $I_T \subset T$ is the location of a deterministic signal with magnitude s. We assume that I_T is sufficiently large in the sense that

$$g_T(\mathcal{C}_{I_T}) \subseteq I_T ,\qquad(6.2)$$

where $\mathcal{C}_{I_T} \subset [0,1]^2$ is a circle with radius $r_{I_T} > 0$, and g_T is the linear mapping defined in Section 4.1. Let $\mathcal{Z}_T = \{g_T(A) : A \in Q_2'\}$ be the collection of all possible rectangular subsets of T, and $\mu_A = \frac{1}{|A|} \sum_{\mathbf{t}\in A} \mathbb{E}(Y(\mathbf{t}))$. To determine whether a signal exists, we consider the hypotheses

$$H_0 : \ \mu_A = \mu_0, \ \ \forall A \in \mathcal{Z}_T \ \ \text{versus} \ \ H_1 : \ \mu_A \neq \mu_0, \ \ \exists A \in \mathcal{Z}_T ,$$

where $\mu_0 \in \mathbb{R}$. To test between H_0 and H_1, the scan statistics

$$SS_T = \max_{A\in\mathcal{Z}_T} \sqrt{|A|} \left| \overline{Y_A} - \overline{Y_T} \right| ,$$

where $\overline{Y_A} = \frac{1}{|A|} \sum_{\mathbf{t}\in A} Y(\mathbf{t})$ and $\overline{Y_T} = \frac{1}{|T|} \sum_{\mathbf{t}\in T} Y(\mathbf{t})$, is a natural candidate; see Theorem 1 of [4] in time series context, and [23] and [49] in spatial context. In particular, H_0 should be rejected for a large SS_T.

To determine the critical value of the test statistics SS_T, the bootstrap methods can be applied to the locally demeaned spatial data $\widehat{V}(\mathbf{t}) = Y(\mathbf{t}) - \overline{Y_{B_\mathbf{t}}}$, where $\overline{Y_{B_\mathbf{t}}} = \frac{1}{|B_\mathbf{t}|} \sum_{\mathbf{t}\in B_\mathbf{t}} Y(\mathbf{t})$ is the local mean in the neighborhood $B_\mathbf{t} = [t_1 - w_T, t_1 + w_T] \times [t_2 - w_T, t_2 + w_T] \bigcap T$ of $\mathbf{t} = (t_1, t_2)$ with window width

$w_T > 0$. Denote $\{Z^*(\mathbf{t}), \mathbf{t} \in T\}$ as the bootstrap sample of $\{\widehat{V}(\mathbf{t}), \mathbf{t} \in T\}$. The following theorem asserts that, by using the locally demeaned data, the frequency domain bootstrap methods asymptotically yield the null distribution even under the presence of a signal.

Theorem 6.1. *Consider $\{Y(\mathbf{t}) : \mathbf{t} \in T\}$ defined in (6.1), α_T defined in Corollary 5.2, and the bootstrapped test statistic*

$$SS_T^* = \max_{A \in \mathcal{Z}_T} \sqrt{|A|} \left| \overline{V_A^*} - \overline{V_T^*} \right| ,$$

where $\overline{V_A^*} = \frac{1}{|A|} \sum_{\mathbf{t} \in A} V^*(\mathbf{t})$, $V^*(\mathbf{t}) = Z^*(\mathbf{t}) + \overline{Y_T}$ and $\overline{V_T^*} = \frac{1}{|T|} \sum_{\mathbf{t} \in T} V^*(\mathbf{t}) = \overline{Y_T}$. If $w_T^{-1} = o_p(\alpha_T^{-1})$, then

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P}_0(SS_T \leq x) - \mathbb{P}^*(SS_T^* \leq x) \right| \to 0 ,$$

where $\mathbb{P}_0$ is the probability measure under H_0 with $s = 0$, and $\mathbb{P}^*$ is the conditional probability measure given $\{Y(\mathbf{t}) : \mathbf{t} \in T\}$ using any bootstrap method satisfying Assumptions B.1 to B.4.

The following theorem proves the consistency of the bootstrap test.

Theorem 6.2. *Let $\{Y(\mathbf{t}) : \mathbf{t} \in T\}$ be defined in (6.1) with a signal $s \neq 0$ at I_T satisfying (6.2) for some circle $\mathcal{C}_{I_T} \subset [0,1]^2$. Assume that there exists some $A \in \mathcal{Z}_T$ such that $\mathcal{C}_{I_T} \subset A$. Then, we have*

$$\mathbb{P}_{1,s}(SS_T \geq c^*) \to 1 ,$$

where $\mathbb{P}_{1,s}$ is the probability measure under H_1 with signal magnitude s, and c^* is the critical value determined by any bootstrap method satisfying Assumptions B.1 to B.4, i.e., $\mathbb{P}^*(SS_T^* \geq c^*) = \alpha$ with significance level α.

The following simulation experiments evaluate the finite sample performance of the bootstrap methods on testing for the presence of spatial signal. We generate real-valued zero-mean non-Gaussian random fields $V(\cdot)$ using point-wise transformation of homogeneous Gaussian random fields. First, we generate real-valued mean-zero Gaussian random fields $X(\cdot)$ on a 50×50 region T using the Gaussian covariance function $C(\mathbf{h}) = \sigma^2 \exp\left(-\left(\frac{\|\mathbf{h}\|}{\phi} \right)^2 \right) + \eta \mathbb{1}_{\{\mathbf{h}=0\}}$ with $(\eta, \sigma^2, \phi) = (1, 1, 1)$. Then, for each $\mathbf{t} \in T$ we transform $X(\mathbf{t})$ to a non-Gaussian $V(\mathbf{t})$ by $V(\mathbf{t}) = F_R^{-1}(F_N(X(\mathbf{t}))$, where F_N is the cumulative distribution function of the standard normal distribution and F_R^{-1} is the inverse distribution function of a centered distribution R. In our simulation, Student's $t(20)$ distribution is used. The signal location I_T is taken as a 8×8 square grid at the center of T. Let $\mathcal{Z}_T = Q_2'$ be the collection of all possible rectangular regions of T. For simplicity, let $\mathcal{Z}_T$ contain all rectangular regions $A \subset T$ with a fixed size of 10×10.

We investigate the size and power of the bootstrap test for signal detection under different magnitudes: $s = -3, -2, -1, 0, 1, 2,$ and 3. The window width

$w_T = 10$ is used to compute the locally demeaned data. We compute the scan statistic for each bootstrap sample and compute the critical values from the quantiles of the bootstrapped scan statistics. Table 5 reports the rejection rate of the test using block bootstrap, RB, WB, and LB under various values of s. Different block sizes 4×4, 7×7, and 13×13 are used for block bootstrap, and a wide range of bandwidths, $\mathbf{h}_T = (0.05, 0.05), (0.1, 0.1), (0.15, 0.15), (0.2, 0.2)$, and $(0.25, 0.25)$ are used for RB, WB, and LB for evaluating the effect of bandwidth selections on the performance. It can be seen that the performance of RB, WB, and LB is superior to that of the block bootstrap, and robust to the choice of the bandwidth $\mathbf{h}_T$. One possible reason for the good performance of frequency domain bootstrap methods is that the frequency domain bootstrap samples have a constant mean of zero; see Lemma 4.2. On the other hand, even though the locally demeaned data are used, the block bootstrap samples may still occasionally contain regions which largely deviate from zero, which affects the performance of the test. From Table 5, the effect of this phenomenon magnifies with an increase in block size.

TABLE 5

The rejection rate of the bootstrap test for the presence of spatial signal using BB, RB, WB, and LB under different block sizes, bandwidths and signal magnitudes s on correlated $t(20)$ random fields.

		Magnitude s						
	Method	-3	-2	-1	0	1	2	3
Block size $= 4 \times 4$	BB	0.979	0.737	0.074	0.006	0.071	0.735	0.979
Block size $= 7 \times 7$	BB	0.051	0.057	0.019	0.003	0.017	0.072	0.059
Block size $= 13 \times 13$	BB	0.001	0	0.006	0.016	0.009	0.004	0
	RB	1	0.959	0.259	0.055	0.233	0.961	1
$\mathbf{h}_T = (0.05, 0.05)$	WB	1	0.910	0.225	0.059	0.209	0.918	1
	LB	1	0.946	0.218	0.045	0.205	0.952	1
	RB	1	0.971	0.249	0.054	0.264	0.970	1
$\mathbf{h}_T = (0.1, 0.1)$	WB	1	0.969	0.262	0.058	0.275	0.971	1
	LB	1	0.963	0.242	0.048	0.249	0.968	1
	RB	1	0.967	0.252	0.053	0.264	0.971	1
$\mathbf{h}_T = (0.15, 0.15)$	WB	1	0.970	0.267	0.055	0.278	0.973	1
	LB	1	0.965	0.258	0.053	0.258	0.971	1
	RB	1	0.973	0.260	0.058	0.282	0.962	1
$\mathbf{h}_T = (0.2, 0.2)$	WB	1	0.974	0.265	0.061	0.292	0.962	1
	LB	1	0.974	0.259	0.056	0.283	0.958	1
	RB	1	0.967	0.308	0.063	0.294	0.977	1
$\mathbf{h}_T = (0.25, 0.25)$	WB	1	0.969	0.310	0.062	0.301	0.977	1
	LB	1	0.969	0.302	0.061	0.295	0.974	1

6.3. Spatial isotropy test for Gaussian random fields

In this subsection, we study the application of the proposed bootstrap methods to test for spatial isotropy of Gaussian random fields, i.e., the covariance between two sites depends on their distance but not direction. Since the asymptotic distributions of spatial covariances and variograms depend on the fourth order structure, Gaussian assumption is needed for the random fields in this application. Since it is difficult to exhaust all possible distances and directions, [15] considered the null hypothesis of isotropy as $H_0 : 2\gamma(\mathbf{t}_i) = 2\gamma(\mathbf{t}_j), \quad \forall \mathbf{t}_i, \mathbf{t}_j \in$

Λ, $\mathbf{t}_i \neq \mathbf{t}_j$, and $\|\mathbf{t}_i\| = \|\mathbf{t}_j\|$, where $\|\mathbf{t}\| = \sqrt{\mathbf{t}'\mathbf{t}}$, $\Lambda = \{\mathbf{t}_1, \ldots, \mathbf{t}_m\}$ is a pre-specified set of sites, and $2\gamma(\mathbf{t}) = E(V(\mathbf{0}) - V(\mathbf{t}))^2$ is the variogram at lag $\mathbf{t}$. Let $\mathbf{G} = (2\gamma(\mathbf{t}_1), \ldots, 2\gamma(\mathbf{t}_m))'$ be a vector of variograms in Λ. Observe that, under H_0, there exists a full row rank matrix $\mathbf{A}$ such that $\mathbf{A}\mathbf{G} = 0$. For example, if $\Lambda = \{(1,0),(0,1)\}$, then $\mathbf{G} = (2\gamma(1,0), 2\gamma(0,1))'$, and we may set $\mathbf{A} = [1 \ -1]$. Based on this observation, [15] derive the test statistic

$$TS_T = |T(\mathbf{t})| \times (\mathbf{A}\hat{\mathbf{G}}_T)'(\mathbf{A}\hat{\Sigma}_R\mathbf{A}')^{-1}(\mathbf{A}\hat{\mathbf{G}}_T) \,, \tag{6.3}$$

where $\hat{\mathbf{G}}_T = (2\hat{\gamma}(\mathbf{t}_1), \ldots, 2\hat{\gamma}(\mathbf{t}_m))'$ is the sample variogram vector that estimates $\mathbf{G}$,

$$2\hat{\gamma}(\mathbf{t}) = \frac{1}{|T(\mathbf{t})|} \times \sum_{\mathbf{s} \in T(\mathbf{t})} [V(\mathbf{s}) - V(\mathbf{s} + \mathbf{t})]^2 \,, \tag{6.4}$$

is the estimator of the variogram at lag $\mathbf{t}$, $T(\mathbf{t}) = \{\mathbf{s} : \mathbf{s}, \mathbf{s} + \mathbf{t} \in T\}$, and $\hat{\Sigma}_R$ is a consistent estimator of Σ_R, the covariance matrix of the sample variograms $\hat{\mathbf{G}}_T$. Under H_0 and some regularity conditions, Theorem 1 of [15] states that $TS_T \xrightarrow{d} \chi_d^2$ as $T \to \infty$, where d is the row rank of $\mathbf{A}$. However, the convergence of the test statistic appears to be slow. Therefore, [15] consider a subsampling method to determine the p-value of the test. In the following, we consider using the proposed frequency domain bootstrap method to determine the p-value of the test.

Following the simulation study in [15], we employ a mean-zero Gaussian random field on $\mathbb{Z}^2$ with a spherical covariance function

$$C(\mathbf{h}) = \begin{cases} \sigma^2\left(1 - \frac{3r}{2\phi} + \frac{r^3}{2\phi^3}\right) + \eta\mathbb{1}_{\{\mathbf{h}=0\}} & \text{if } 0 \leq r \leq \phi \,, \\ \eta\mathbb{1}_{\{\mathbf{h}=0\}} & \text{otherwise} \,, \end{cases} \tag{6.5}$$

where σ^2 is the partial sill parameter, ϕ is the range parameter, η is the nugget effect, and $r = \sqrt{\mathbf{h}'\mathbf{B}\mathbf{h}}$ is related to a geometric anisotropy transformation. Specifically, given an anisotropy angle ψ_A and anisotropy ratio ψ_R, define the rotation matrix R and shrinking matrix T as

$$R = \begin{bmatrix} \cos(\psi_A) & \sin(\psi_A) \\ -\sin(\psi_A) & \cos(\psi_A) \end{bmatrix} \text{ and } T = \begin{bmatrix} 1 & 0 \\ 0 & \psi_R \end{bmatrix} \,,$$

then $\mathbf{B} = R'T'TR$ is a 2×2 positive definite matrix representing a geometric anisotropy transformation. A random field with spherical covariance function (6.5) is in general anisotropy except that it is isotropy when $\psi_R \neq 1$. In addition, if $\psi_A = 0$, then the main anisotropic axes are aligned with the (x, y) axes. See, for example, Section 5.1 of [47] for details. For the Gaussian process, it can be shown that the covariance function (6.5) satisfies the absolute integrability condition, which is sufficient for Theorem 1 of [15] to hold.

We consider model parameters $(\eta, \sigma^2, \phi, \psi_A, \psi_R) = (2, 3, 4, 0, \psi_R)$ for different anisotropy ratio ψ_R. Also, set $\Lambda = \{(1,0),(0,1)\}$, $\mathbf{G} = (2\gamma(1,0), 2\gamma(0,1))'$,

$\mathbf{A} = [1 \quad -1]$ and $\hat{\mathbf{G}}_T = (2\hat{\gamma}(1,0), 2\hat{\gamma}(0,1))'$. Thus, the test statistics (6.3) becomes

$$TS_T = |T(\mathbf{t})| \times (2\hat{\gamma}(1,0) - 2\hat{\gamma}(0,1))^2 \left(\mathbf{A}\hat{\Sigma}_R\mathbf{A}'\right)^{-1} , \qquad (6.6)$$

where $\hat{\Sigma}_R$ may be estimated by subsampling or by the proposed bootstrap methods. However, since $(\mathbf{A}\hat{\Sigma}_R\mathbf{A}')^{-1}$ is only a normalizing factor in (6.6), we may focus on subsampling and bootstrapping the test statistic

$$TS_T = |T(\mathbf{t})| \times (2\hat{\gamma}(1,0) - 2\hat{\gamma}(0,1))^2 .$$

Next, we briefly outline the subsampling and bootstrap methods. For the spatial subsampling, the region T is divided into k_T small overlapping subblocks, known as subsampling windows, which are congruent to T in both configuration and orientation. Denote T^i_{sub}, as the i-th subblock. For each of the k_T subblocks, compute the statistic

$$TS^i_{T,sub} = |T^i_{sub}| \times \left(2\hat{\gamma}^i_{sub}(1,0) - 2\hat{\gamma}^i_{sub}(0,1)\right)^2 ,$$

where $\hat{\gamma}^i_{sub}(\mathbf{t})$ is defined similarly as in (6.4), but with $T(\mathbf{t})$ replaced by $T^i_{sub}(\mathbf{t}) = \{\mathbf{s} : \mathbf{s}, \mathbf{s}+\mathbf{t} \in T^i_{sub}\}$. Using the $TS^i_{T,sub}$s, the p-value for the test can be calculated by $\sum_{i=1}^{k_T} \mathbb{1}_{\{TS^i_{T,sub} \geq TS_T\}}/k_T$, and the null hypothesis is rejected if the p-value is smaller than the significance level α.

For the proposed bootstrap methods, B bootstrap samples are generated from either RB, WB or LB. For the i-th bootstrap sample, we compute the variogram $\hat{\gamma}^i_{boot}(1,0)$ using (6.4). Next, define the variogram difference $VD_i = 2\hat{\gamma}^i_{boot}(1,0) - 2\hat{\gamma}^i_{boot}(0,1)$, and the bootstrapped test statistic

$$TS^i_{T,boot} = |T| \times \left(VD_i - \overline{VD}\right)^2 ,$$

where $\overline{VD} = \sum_{i=1}^{B} VD_i/B$. Note that centering of VD_is is required since VD_i has a non-zero mean under the alternative. Similar to the test for signal presence, this centering procedure allows the bootstrapped test statistic to converge to the null distribution even under the alternative hypothesis; see Theorem 6.3 below. Finally, the p-value of the test can be calculated by $\sum_{i=1}^{B} \mathbb{1}_{\{TS^i_{T,boot} \geq TS_T\}}/B$, and the null hypothesis is rejected if the p-value is smaller than the significance level α.

The following theorem states that the bootstrapped test statistic converges to the same limit as that of TS_T, and hence the proposed bootstrap method is valid.

Theorem 6.3. *For a stationary Gaussian random field $\{V(\mathbf{t}) : \mathbf{t} \in T\}$, under the assumptions of Theorem 1 in [15], we have*

$$\sup_{x \in \mathbb{R}} \left|\mathbb{P}_0(TS_T \leq x) - \mathbb{P}^*(TS^i_{T,boot} \leq x)\right| \to 0 ,$$

where $\mathbb{P}_0$ denotes the probability measure under H_0, and $\mathbb{P}^*$ denotes the conditional probability given $\{V(\mathbf{t}) : \mathbf{t} \in T\}$ using any bootstrap method satisfying Assumptions B.1 to B.4.

Table 6 summarizes the rejection rates of the bootstrap test by spatial subsampling, RB, WB and LB under different values of anisotropy ratio ψ_R. It can be seen that the performance of the proposed methods is superior to that of spatial subsampling in both size and power.

TABLE 6
The rejection rate of the isotropy tests under various anisotropy ratio ψ_R.

ψ_R	Subsampling	RB	WB	LB
1	0.025	0.052	0.049	0.054
1.1	0.110	0.197	0.167	0.173
1.2	0.426	0.580	0.561	0.523
1.3	0.770	0.872	0.848	0.862
1.4	0.946	0.969	0.962	0.967
1.5	0.997	1	1	1

7. Conclusion

This paper develops a frequency domain bootstrap method for random fields on $\mathbb{Z}^2$. Three bootstrap schemes for resampling the Fourier coefficients are proposed. Inverse-transformations are then applied to obtain resamples in the spatial domain. The resulting bootstrap resamples capture the correct second-order moment structure for a large class of random fields. Moreover, invariance principles of the partial sum process indexed by a classical example of Vapnik-Chervonenkis classes of Borel subsets of $[0, 1]^2$ are established. The results can be easily generalized to other Vapnik-Chervonenkis classes. The frequency domain bootstrap method is simple to apply and is demonstrated to be effective in various applications including constructing confidence intervals of correlograms for linear random fields, testing for signal presence using scan statistics, and testing for spatial isotropy in Gaussian random fields.

Simulation studies are conducted to illustrate the finite sample performance of the proposed method and to compare with the existing spatial block bootstrap and subsampling methods. For small or moderate sample sizes, since the effective number of blocks for block bootstrap and subsampling methods would be small when the block size is chosen to be large, severe bias would be induced, and the finite sample performances are sensitive to the selection of block size. However, this problem cannot be resolved by choosing a smaller block size as the dependency structure of the underlying spatial fields cannot be preserved when the block size is too small. On the other hand, although the bandwidth selection may also affect the performance of the proposed frequency domain bootstrap for small or moderate sample sizes, its effect is relatively small compared to that of block bootstrap as shown in the simulation studies in Section 6. However, as shown in Section 4, the proposed frequency domain bootstrap can only mimic the second order moment structure of the underlying spatial fields, and hence may not be appropriate for general statistics which involve higher moment structures, or may require some form of transformations of the data. On the other hand, in general the block bootstrap methods can be applied directly without any transformations beforehand.

Appendix A: Proofs and technical details

A.1. Proofs of Section 4

We start with a short lemma needed to prove Lemma 4.2. Its proof is deferred to Appendix B.

Lemma A.1. *Under Assumption A.2, the following representation holds,*

$$\frac{8\pi^2}{|T|} \sum_{\mathbf{j} \in N} f(\lambda_{\mathbf{j}}) \cos(\lambda'_{\mathbf{j}} \mathbf{h}) = \sum_{\mathbf{l} \in \mathbb{Z}^2} \gamma(\mathbf{h} + \mathbf{l} \odot \mathbf{d}_T) - \frac{4\pi^2}{|T|} f(\mathbf{0})$$

$$- \frac{4\pi^2}{|T|} f(\pi, 2\pi) \exp(i(\pi, 2\pi)' \mathbf{h}) \mathbb{1}_{\{d_1 \text{ even}\}}$$

$$- \frac{4\pi^2}{|T|} f(2\pi, \pi) \exp(i(2\pi, \pi)' \mathbf{h}) \mathbb{1}_{\{d_2 \text{ even}\}}$$

$$- \frac{4\pi^2}{|T|} f(\pi, \pi) \exp(i(\pi, \pi)' \mathbf{h}) \mathbb{1}_{\{d_1, d_2 \text{ even}\}},$$

where $\lambda_{\mathbf{j}} = \left(\frac{2\pi j_1}{d_1}, \frac{2\pi j_2}{d_2} \right)$, $\mathbf{j} = (j_1, j_2) \in T$.

We are now ready to prove Lemma 4.2. Note that $|T| = d_1 d_2$, for $A_T = [a_{11T}, a_{12T}] \times [a_{21T}, a_{22T}] \in Q'_2$, $|A_T \otimes T| = L(A_T) d_1 d_2 = (a_{12T} - a_{11T})(a_{22T} - a_{21T}) d_1 d_2$, and $|A_T \otimes T| \to \infty$ as $|T| \to \infty$.

Proof of Lemma 4.2. Assertion (a) follows immediately from the fact that the bootstrapped Fourier coefficients are conditionally centered. For assertion (b), by Lemma A.3 in [20], it holds that (uniformly in u) for $\lambda_k = \frac{2\pi k}{T}$,

$$\sum_{l=1}^{\lfloor mu \rfloor} \cos(\lambda_k l) = O\left(\min\left(\frac{T}{k}, m \right) \right) \quad \text{and} \quad \sum_{l=1}^{\lfloor mu \rfloor} \sin(\lambda_k l) = O\left(\min\left(\frac{T}{k}, m \right) \right). \tag{A.1}$$

We will show that for all $\lambda_{\mathbf{k}} = \left(\frac{2\pi k_1}{d_1}, \frac{2\pi k_2}{d_2} \right)$, $\mathbf{k} = (k_1, k_2) \in \mathbb{Z}^2$, and sets A_T, B, C, where $A_T = [a_{11T}, a_{12T}] \times [a_{21T}, a_{22T}] \in Q'_2$, $B = [b_{11}, b_{12}] \times [b_{21}, b_{22}] \in Q'_2$ and $C = [c_{11}, c_{12}] \times [c_{21}, c_{22}] \in Q'_2$, we have

$$\sum_{\mathbf{l} \in B \otimes (A_T \otimes T)} \cos(\lambda'_{\mathbf{k}} \mathbf{l})$$

$$= O\left(\min\left(\frac{d_1}{k_1}, (a_{12T} - a_{11T}) d_1 \right) \right) O\left(\min\left(\frac{d_2}{k_2}, (a_{22T} - a_{21T}) d_2 \right) \right),$$

$$\sum_{\mathbf{l} \in B \otimes (A_T \otimes T)} \sin(\lambda'_{\mathbf{k}} \mathbf{l})$$

$$= O\left(\min\left(\frac{d_1}{k_1}, (a_{12T} - a_{11T}) d_1 \right) \right) O\left(\min\left(\frac{d_2}{k_2}, (a_{22T} - a_{21T}) d_2 \right) \right).$$

Note that

$$\sum_{\mathbf{l}\in B\otimes(A_T\otimes T)}\cos(\lambda'_{\mathbf{k}}\mathbf{l})$$

$$= \sum_{(l_1,l_2)\in B\otimes(A_T\otimes T)}\left[\cos\left(\frac{2\pi k_1}{d_1}l_1\right)\cos\left(\frac{2\pi k_2}{d_2}l_2\right)\right.$$

$$\left. - \sin\left(\frac{2\pi k_1}{d_1}l_1\right)\sin\left(\frac{2\pi k_2}{d_2}l_2\right)\right]$$

$$= O\left(\min\left(\frac{d_1}{k_1},(a_{12T}-a_{11T})d_1\right)\right)O\left(\min\left(\frac{d_2}{k_2},(a_{22T}-a_{21T})d_2\right)\right),$$

where the last equality follows from (A.1) and that $B\otimes(A_T\otimes T)$ is a rectangular region. Similar arguments apply to the term $\sum_{\mathbf{l}\in B\otimes(A_T\otimes T)}\sin(\lambda'_{\mathbf{k}}\mathbf{l})$.

Then, we have

$$\sum_{\mathbf{l_1}\in B\otimes(A_T\otimes T)}\sum_{\mathbf{l_2}\in C\otimes(A_T\otimes T)}\sum_{\mathbf{k}\in N}\cos(\lambda'_{\mathbf{k}}\mathbf{l_1})\cos(\lambda'_{\mathbf{k}}\mathbf{l_2}) \tag{A.2}$$

$$= \sum_{\mathbf{k}\in N}\left(\sum_{\mathbf{l_1}\in B\otimes(A_T\otimes T)}\cos(\lambda'_{\mathbf{k}}\mathbf{l_1})\right)\left(\sum_{\mathbf{l_2}\in C\otimes(A_T\otimes T)}\cos(\lambda'_{\mathbf{k}}\mathbf{l_2})\right)$$

$$= \sum_{\mathbf{k}\in N}\left[O\left(\min\left(\frac{d_1}{k_1},(a_{12T}-a_{11T})d_1\right)\right)O\left(\min\left(\frac{d_2}{k_2},(a_{22T}-a_{21T})d_2\right)\right)\right]^2$$

$$= O\left((a_{12T}-a_{11T})d_1^2\right)O\left((a_{22T}-a_{21T})d_2^2\right) = O\left(|A_T\otimes T||T|\right).$$

The same equation holds true if we replace cosine by sine.

By Assumptions $\mathcal{B}.1$ and $\mathcal{B}.2$, and the definition of $Z^*(\mathbf{l})$, it holds that

$$\mathrm{Cov}^*\left(\frac{1}{\sqrt{|A_T\otimes T|}}\sum_{\mathbf{l_1}\in B\otimes(A_T\otimes T)}Z^*(\mathbf{l_1}),\frac{1}{\sqrt{|A_T\otimes T|}}\sum_{\mathbf{l_2}\in C\otimes(A_T\otimes T)}Z^*(\mathbf{l_2})\right)$$

$$= \frac{1}{|A_T\otimes T|}\sum_{\mathbf{l_1}\in B\otimes(A_T\otimes T)}\sum_{\mathbf{l_2}\in C\otimes(A_T\otimes T)}\mathrm{Cov}^*(Z^*(\mathbf{l_1}),Z^*(\mathbf{l_2}))$$

$$= \frac{1}{|A_T\otimes T|}\frac{4}{|T|}\times$$

$$\sum_{\mathbf{l_1}\in B\otimes(A_T\otimes T)}\sum_{\mathbf{l_2}\in C\otimes(A_T\otimes T)}\sum_{\mathbf{k}\in N}\mathrm{Cov}^*(x^*(\mathbf{k})\cos(\lambda'_{\mathbf{k}}\mathbf{l_1})-y^*(\mathbf{k})\sin(\lambda'_{\mathbf{k}}\mathbf{l_1}),$$

$$x^*(\mathbf{k})\cos(\lambda'_{\mathbf{k}}\mathbf{l_2})-y^*(\mathbf{k})\sin(\lambda'_{\mathbf{k}}\mathbf{l_2}))$$

$$= \frac{4}{|A_T\otimes T||T|}\times$$

$$\sum_{\mathbf{l_1}\in B\otimes(A_T\otimes T)}\sum_{\mathbf{l_2}\in C\otimes(A_T\otimes T)}\sum_{\mathbf{k}\in N}[\mathrm{Var}^*(x^*(\mathbf{k}))\cos(\lambda'_{\mathbf{k}}\mathbf{l_1})\cos(\lambda'_{\mathbf{k}}\mathbf{l_2})$$

$$+\mathrm{Var}^*(y^*(\mathbf{k}))\sin(\lambda'_{\mathbf{k}}\mathbf{l_1})\sin(\lambda'_{\mathbf{k}}\mathbf{l_2})]$$

$$= \frac{4}{|A_T \otimes T||T|} \sum_{\mathbf{l_1} \in B \otimes (A_T \otimes T)} \sum_{\mathbf{l_2} \in C \otimes (A_T \otimes T)} \sum_{\mathbf{k} \in N} \{[2\pi^2 f(\lambda_{\mathbf{k}}) \cos(\lambda'_{\mathbf{k}}(\mathbf{l_1} - \mathbf{l_2}))]$$

$$+ o_p(1) \cos(\lambda'_{\mathbf{k}}(\mathbf{l_1} - \mathbf{l_2}))\}$$

$$= \frac{8\pi^2}{|A_T \otimes T||T|} \sum_{\mathbf{l_1} \in B \otimes (A_T \otimes T)} \sum_{\mathbf{l_2} \in C \otimes (A_T \otimes T)} \sum_{\mathbf{k} \in N} f(\lambda_{\mathbf{k}}) \cos(\lambda'_{\mathbf{k}}(\mathbf{l_1} - \mathbf{l_2}))$$

$$+ o_p(1) .$$

Using Lemma A.1, we can handle the above sum by decomposing the inmost sum into the 5 terms. For the first term, by the absolute summability of $\gamma(\cdot)$, we have

$$\frac{1}{|A_T \otimes T|} \sum_{\mathbf{l_1} \in B \otimes (A_T \otimes T)} \sum_{\mathbf{l_2} \in C \otimes (A_T \otimes T)} \gamma(\mathbf{l_1} - \mathbf{l_2})$$

$$= \frac{1}{|A_T \otimes T|} \left\{ \sum_{\mathbf{l_1}, \mathbf{l_2} \in (B \otimes (A_T \otimes T)) \bigcap (C \otimes (A_T \otimes T))} \gamma(\mathbf{l_1} - \mathbf{l_2}) \right.$$

$$+ \sum_{\mathbf{l_1} \in B \otimes (A_T \otimes T)} \sum_{\mathbf{l_2} \in C \otimes (A_T \otimes T) \backslash (B \otimes (A_T \otimes T))} \gamma(\mathbf{l_1} - \mathbf{l_2})$$

$$\left. + \sum_{\mathbf{l_1} \in B \otimes (A_T \otimes T) \backslash (C \otimes (A_T \otimes T))} \sum_{\mathbf{l_2} \in (B \otimes (A_T \otimes T)) \bigcap (C \otimes (A_T \otimes T))} \gamma(\mathbf{l_1} - \mathbf{l_2}) \right\}$$

$$= \frac{1}{|A_T \otimes T|} \sum_{\mathbf{l_1}, \mathbf{l_2} \in (B \otimes (A_T \otimes T)) \bigcap (C \otimes (A_T \otimes T))} \gamma(\mathbf{l_1} - \mathbf{l_2}) + o_p(1)$$

$$= 4\pi^2 f(\mathbf{0}) \frac{|(B \otimes (A_T \otimes T)) \bigcap (C \otimes (A_T \otimes T))|}{|A_T \otimes T|} + o_p(1)$$

$$= 4\pi^2 f(\mathbf{0}) L(B \bigcap C) + o_p(1) .$$

Also, by the absolute summability of $\gamma(\cdot)$, we have

$$\frac{1}{|A_T \otimes T|} \sum_{\mathbf{l_1} \in B \otimes (A_T \otimes T)} \sum_{\mathbf{l_2} \in C \otimes (A_T \otimes T)} \sum_{\mathbf{j} \neq \mathbf{0}} |\gamma(\mathbf{l_1} - \mathbf{l_2} + \mathbf{j} \odot \mathbf{d}_T)|$$

$$\leq \frac{1}{|A_T \otimes T|} \sum_{\mathbf{l_1} \in A_T \otimes T} \sum_{\mathbf{l_2} \in A_T \otimes T} \sum_{\mathbf{j} \neq \mathbf{0}} |\gamma(\mathbf{l_1} - \mathbf{l_2} + \mathbf{j} \odot \mathbf{d}_T)|$$

$$\leq \sum_{\substack{\mathbf{h} \in \{(h_1, h_2): \\ |h_1| \leq (a_{12T} - a_{11T})d_1, \\ |h_2| \leq (a_{22T} - a_{21T})d_2\}}} \left[\left(\frac{(a_{12T} - a_{11T})d_1 - |h_1|}{(a_{12T} - a_{11T})d_1} \right) \left(\frac{(a_{22T} - a_{21T})d_2 - |h_2|}{(a_{22T} - a_{21T})d_2} \right) \right.$$

$$\left. \times \sum_{\mathbf{j} \neq \mathbf{0}} |\gamma(\mathbf{h} + \mathbf{j} \odot \mathbf{d}_T)| \right]$$

$$\leq \frac{2\sum\limits_{\mathbf{j}\in\mathbb{Z}^2}|\gamma(\mathbf{j})|}{\sqrt{|A_T\otimes T|}} + \sum_{\substack{\mathbf{h}\in\{(t_1,t_2): \\ |t_1|\leq(a_{12T}-a_{11T})d_1-\sqrt{(a_{12T}-a_{11T})d_1}, \\ |t_2|\leq(a_{22T}-a_{21T})d_2-\sqrt{(a_{22T}-a_{21T})d_2}\}}} \sum_{\mathbf{j}\neq\mathbf{0}} |\gamma(\mathbf{h}+\mathbf{j}\odot\mathbf{d}_T)|$$

$$\leq o(1) + 2 \sum_{\mathbf{k}\in\left\{(t_1,t_2):\ |t_1|\geq\sqrt{(a_{12T}-a_{11T})d_1},\ |t_2|\geq\sqrt{(a_{22T}-a_{21T})d_2}\right\}} |\gamma(\mathbf{k})| = o(1)\,.$$

For the sum of the last three terms of Lemma A.1, we have

$$\frac{4\pi^2}{|T||A_T\otimes T|}\sum_{\mathbf{l_1}\in B\otimes(A_T\otimes T)}\sum_{\mathbf{l_2}\in C\otimes(A_T\otimes T)}\exp(i(\pi,2\pi)'(\mathbf{l_1}-\mathbf{l_2}))$$

$$=\frac{4\pi^2}{|T||A_T\otimes T|}\sum_{\mathbf{l_1}\in B\otimes(A_T\otimes T)}\exp(i(\pi,2\pi)'\mathbf{l_1})\sum_{\mathbf{l_2}\in C\otimes(A_T\otimes T)}\exp(-i(\pi,2\pi)'\mathbf{l_2})$$

$$=O\left(\frac{1}{|T||A_T\otimes T|}\right)\,.$$

Similar arguments can be applied on the remaining two terms. Finally, the sum of the second term is

$$\frac{4\pi^2 f(\mathbf{0})}{|T||A_T\otimes T|}(|B\otimes(A_T\otimes T)|)(|C\otimes(A_T\otimes T)|)$$

$$=\begin{cases} o(1) & \text{if}\quad \frac{|A_T\otimes T|}{|T|}\to 0\,, \\ 4\pi^2 f(\mathbf{0})L(B)L(C)+o(1) & \text{if}\quad A_T\otimes T=T\,. \end{cases}$$

Putting everything together, we obtain (b). The proof of (c) is analogous. A simple calculation shows that $\mathrm{Cov}(Z(\mathbf{l_1}),Z(\mathbf{l_2}))=\mathrm{Cov}(V(\mathbf{l_1}),V(\mathbf{l_2}))+o(1)$ by the absolute summability of the auto-covariance function. $\qquad\square$

Proof of Theorem 4.3. To prove the invariance principle of the Q_2'-indexed partial sum process of the bootstrap sample, we have to show the finite-dimensional convergence and also the tightness of the partial sum process. The following lemma shows the convergence of the finite-dimensions distribution of the Q_2'-indexed partial sum process of the bootstrap sample. Its proof is deferred to Appendix B.

Lemma A.2. *For* $A_T\in Q_2'$, *let* $S^*_{A_T\otimes T}(B)=\sum_{\mathbf{l}\in B\otimes(A_T\otimes T)}Z^*(\mathbf{l})$, *where* $B\in Q_2'$.

(a) *If Assumptions A.2, B.1 to B.3 are fulfilled and* $\frac{|A_T\otimes T|}{|T|}\to 0$ *for* $B_1, B_2, \dots, B_p\in Q_2'$, *we have*

$$\frac{1}{\sqrt{|A_T\otimes T|}}(S^*_{A_T\otimes T}(B_1),...,S^*_{A_T\otimes T}(B_p))\xrightarrow{d} N(0,\Sigma)\,,$$

in probability, where $\Sigma=(c_{i,j})_{i,j=1,...,p}$ *with* $c_{i,j}=4\pi^2 f(\mathbf{0})L(B_i\cap B_j)$.

(b) *If Assumptions A.2, B.1 and B.4 are fulfilled, for $B_1, B_2, \ldots, B_p \in Q'_2$, we have*

$$\frac{1}{\sqrt{|T|}}(S_T^*(B_1), ..., S_T^*(B_p)) \xrightarrow{d} N(0, \Sigma),$$

in probability, where $\Sigma = (c_{i,j})_{i,j=1,...,p}$ with $c_{i,j} = 4\pi^2 f(\mathbf{0})[L(B_i \bigcap B_j) - L(B_i)L(B_j)]$.

The following lemma gives the critical step towards tightness of the Q'_2-indexed partial sum process of the bootstrap sample. Its proof is deferred to Appendix B.

Lemma A.3. *Under Assumptions A.2, B.1 to B.3, for any $\epsilon > 0$ and $A_T \in Q'_2$, we have*

$$\lim_{\delta \to 0} \limsup_{T \to \infty} \mathbb{P} \left(\sup_{\substack{B,C \in Q'_2 \\ \rho(B,C) < \delta}} \left| \frac{S_{A_T \otimes T}^*(B)}{\sqrt{|A_T \otimes T|}} - \frac{S_{A_T \otimes T}^*(C)}{\sqrt{|A_T \otimes T|}} \right| > \epsilon \right) = 0. \qquad (A.3)$$

Theorem 13.5 of [1] gives a characterization of weak convergence via convergence of the finite-dimensions distributions as well as tightness. Lemmas A.2 and A.3 show that these conditions are fulfilled. It completes the proof of Theorem 4.3. □

Proof of Corollary 4.4. The proof is analogous to the proof of Lemma A.2(a) and is thus omitted. □

Proof of Corollary 4.5. The corollary is an immediate consequence of Theorem 4.3, thus the proof is omitted. □

A.2. Proofs of Section 5

Proof of Theorem 5.1. In the following we only prove the assertions for $x^*(\cdot)$, while the assertions for $y^*(\cdot)$ follow because $x^*(\mathbf{j}) \stackrel{d}{=} y^*(\mathbf{j})$ (conditionally given $V(\cdot)$). Since Assumption B.1 directly follows from the definition of the bootstrap schemes, we will show that Assumptions B.2 to B.4 are also valid under the assumptions stated in the theorem.

(a) **Residual-Based Bootstrap (RB):** It follows directly from Theorem 5.1 and Corollary 4.2 in [33] that we have $\sup_{\lambda \in [0,2\pi]^2} \left| \widehat{f}_T(\lambda) - f(\lambda) \right| \xrightarrow{p} 0$, and the empirical distribution function of the Fourier coefficients converges uniformly to the standard normal distribution function $\Phi(\cdot)$, i.e., $\sup_{z \in \mathbb{R}} \left| \frac{1}{2|N|} \sum_{\mathbf{j} \in N} \left(\mathbb{1}_{\left\{ x(\mathbf{j}) \le z \sqrt{2\pi^2 f(\lambda_{\mathbf{j}})} \right\}} + \mathbb{1}_{\left\{ y(\mathbf{j}) \le z \sqrt{2\pi^2 f(\lambda_{\mathbf{j}})} \right\}} \right) - \Phi(z) \right| \xrightarrow{p} 0$. Also, by Theorem 4.3 in [33], we also have the following four conditions

on the sums of the periodograms and Fourier coefficients:

$$\text{(i)} \qquad \frac{1}{2|N|} \sum_{\mathbf{j} \in N} \frac{x(\mathbf{j}) + y(\mathbf{j})}{\sqrt{f(\lambda_{\mathbf{j}})}} \xrightarrow{p} 0 \,,$$

$$\text{(ii)} \qquad \frac{1}{4\pi^2 |N|} \sum_{\mathbf{j} \in N} \frac{I(\mathbf{j})}{f(\lambda_{\mathbf{j}})} \xrightarrow{p} 1 \,,$$

$$\text{(iii)} \quad \frac{1}{|N|} \sum_{\mathbf{j} \in N} \frac{I^2(\mathbf{j})}{f^2(\lambda_{\mathbf{j}})} \leq C_1 + o_p(1) \,,$$

$$\text{(iv)} \quad \frac{1}{|N|} \sum_{\mathbf{j} \in N} \frac{I^q(\mathbf{j})}{f^q(\lambda_{\mathbf{j}})} \leq C_2 + o_p(1) \,,$$

for some constant $C_1, C_2 \geq 0$ and $q = 4 + \epsilon$ with some $\epsilon \in (0, 1)$. Next, since we have for $k = 1, 2$,

$$\frac{1}{2|N|} \sum_{\mathbf{j} \in N, k} \left(\widetilde{s}_{\mathbf{j},k} - \frac{1}{2|N|} \sum_{\mathbf{l} \in N, k} \widetilde{s}_{\mathbf{l},k} \right)^2$$

$$= \frac{1}{2|N|} \sum_{\mathbf{j} \in N} \frac{I(\mathbf{j})}{2\pi^2 \widehat{f}_T(\lambda_{\mathbf{j}})} - \left(\frac{1}{2|N|} \sum_{\mathbf{j} \in N} \frac{x(\mathbf{j}) + y(\mathbf{j})}{\sqrt{2\pi^2 \widehat{f}_T(\lambda_{\mathbf{j}})}} \right)^2$$

$$= \frac{1}{2|N|} \sum_{\mathbf{j} \in N} \frac{I(\mathbf{j})}{2\pi^2 f(\lambda_{\mathbf{j}})} - \left(\frac{1}{2|N|} \sum_{\mathbf{j} \in N} \frac{x(\mathbf{j}) + y(\mathbf{j})}{\sqrt{2\pi^2 f(\lambda_{\mathbf{j}})}} \right)^2$$

$$+ O(1) \left(\sup_{\mathbf{l}} \left| \frac{f(\lambda_{\mathbf{l}}) - \widehat{f}_T(\lambda_{\mathbf{l}})}{2\pi^2 \widehat{f}_T(\lambda_{\mathbf{l}})} \right| \right) \left(\frac{1}{|N|} \sum_{\mathbf{j} \in N} \frac{I(\mathbf{j})}{f(\lambda_{\mathbf{j}})} \right) \xrightarrow{p} 1 \,.$$

Also, we have for $k = 1, 2$,

$$\frac{1}{2|N|} \sum_{\mathbf{j} \in N, k} |\widetilde{s}_{\mathbf{j},k}|^p \leq \frac{1}{2|N|} \sum_{\mathbf{j} \in N} \frac{I(\mathbf{j})^q}{(2\pi^2 f(\lambda_{\mathbf{j}}))^q} \left(1 + \sup_{\mathbf{l}} \left| \frac{f(\lambda_{\mathbf{l}}) - \widehat{f}_T(\lambda_{\mathbf{l}})}{2\pi^2 \widehat{f}_T(\lambda_{\mathbf{l}})} \right|^q \right)$$

$$\leq C + o_p(1) \,.$$

Hence, Assumption $\mathcal{B}$.3 holds with $p = 2q > 8$ since

$$\sup_{\mathbf{j} \in N} \mathbb{E}^* (|x^*(\mathbf{j})|)^p = \sup_{\mathbf{j} \in N} \mathbb{E}^* \left(\sqrt{2\pi^2 \widehat{f}_T(\lambda_{\mathbf{j}})} |s_{\mathbf{j},k}^*| \right)^p$$

$$= \sup_{\mathbf{j} \in N} \left(2\pi^2 \widehat{f}_T(\lambda_{\mathbf{j}}) \right)^{\frac{p}{2}} \mathbb{E}^* (|s_{\mathbf{j},k}^*|^p)$$

$$= \frac{C_p}{2|N|} \frac{\sum\limits_{\mathbf{j}\in N,k} \left| \widetilde{s}_{\mathbf{j},k} - \frac{1}{2|N|} \sum\limits_{\mathbf{t}\in N,k} \widetilde{s}_{\mathbf{t},k} \right|^p}{\left[\frac{1}{2|N|} \sum\limits_{\mathbf{l}\in N,k} \left(\widetilde{s}_{\mathbf{l},k} - \frac{1}{2|N|} \sum\limits_{\mathbf{t}\in N,k} \widetilde{s}_{\mathbf{t},k} \right)^2 \right]^{\frac{p}{2}}}$$

$$\leq C + o_p(1),$$

Lastly, for Assumption *B.4*, let $\{U_N(\mathbf{j}, k) : \mathbf{j} \in N, k = 1, 2\}$ be i.i.d. taking values in N with equal probability. Denote $\widetilde{s}^*_{\mathbf{j},k} = \widetilde{s}_{U_N(\mathbf{j},k)}$ (i.i.d.), $\widetilde{\widetilde{s}}^*_{\mathbf{j},k} = \sqrt{\widehat{f}_T(\lambda_{U_N(\mathbf{j},k)})/f(\lambda_{U_N(\mathbf{j},k)})} \widetilde{s}_{U_N(\mathbf{j},k)}$ (i.i.d.) and $s^*_{\mathbf{j},k} = s_{U_N(\mathbf{j},k)}$ (i.i.d.), furthermore $x^*(\mathbf{j}) \stackrel{d}{=} \sqrt{2\pi^2 \widehat{f}_T(\lambda_{\mathbf{j}})} s^*_{\mathbf{j},1}$, $y^*(\mathbf{j}) \stackrel{d}{=} \sqrt{2\pi^2 \widehat{f}_T(\lambda_{\mathbf{j}})} s^*_{\mathbf{j},2}$, $\mathbf{j} \in N$ and $k = 1, 2$. Since $s^*_{\mathbf{j},k}$ are from the standardized residues, we have $\mathbb{E}^*(s^*_{\mathbf{1},k})^2 = 1$,

$$\mathbb{E}^*(s^*_{\mathbf{1},k} - \widetilde{s}^*_{\mathbf{1},k})^2$$

$$\asymp \frac{\left| \frac{1}{2|N|} \sum_{\mathbf{j}\in N,k} \left(\widetilde{s}_{\mathbf{j},k} - \frac{1}{2|N|} \sum_{\mathbf{l}\in N,k} \widetilde{s}_{\mathbf{l},k} \right)^2 - 1 \right|}{\frac{1}{2|N|} \sum_{\mathbf{j}\in N,k} \left(\widetilde{s}_{\mathbf{j},k} - \frac{1}{2|N|} \sum_{\mathbf{l}\in N,k} \widetilde{s}_{\mathbf{l},k} \right)^2} \frac{1}{2|N|} \sum_{\mathbf{j}\in N,k} \widetilde{s}^2_{\mathbf{j},k}$$

$$+ \frac{\left(\frac{1}{2|N|} \sum_{\mathbf{j}\in N,k} \widetilde{s}_{\mathbf{j},k} \right)^2}{\frac{1}{2|N|} \sum_{\mathbf{j}\in N,k} \left(\widetilde{s}_{\mathbf{j},k} - \frac{1}{2|N|} \sum_{\mathbf{l}\in N,k} \widetilde{s}_{\mathbf{l},k} \right)^2} = o_p(1),$$

and

$$\mathbb{E}^*(\widetilde{s}^*_{\mathbf{1},k} - \widetilde{\widetilde{s}}^*_{\mathbf{1},k})^2 \asymp \sup_{\mathbf{l}\in N} \frac{\left| f(\lambda_{\mathbf{l}}) - \widehat{f}_T(\lambda_{\mathbf{l}}) \right|}{f(\lambda_{\mathbf{l}})} \frac{1}{2|N|} \sum_{\mathbf{j}\in N,k} \widetilde{s}^2_{\mathbf{j},k} = o_p(1).$$

From this, it follows that

$$\sup_{\mathbf{j}\in N} d_2^2(\mathcal{L}^*(x^*(\mathbf{j})), N(0, 2\pi^2 f(\lambda_{\mathbf{j}})))$$

$$\leq \sup_{\mathbf{j}\in N} d_2^2 \left(\mathcal{L}^*(x^*(\mathbf{j})), \mathcal{L}^* \left(\sqrt{2\pi^2 f(\lambda_{\mathbf{j}})} s^*_{\mathbf{j},k} \right) \right)$$

$$+ \sup_{\mathbf{j}\in N} d_2^2 \left(\mathcal{L}^* \left(\sqrt{2\pi^2 f(\lambda_{\mathbf{j}})} s^*_{\mathbf{j},k} \right), N(0, 2\pi^2 f(\lambda_{\mathbf{j}})) \right)$$

$$\leq 2\pi^2 \sup_{\mathbf{j}} |f(\lambda_{\mathbf{j}}) - \widehat{f}_T(\lambda_{\mathbf{j}})| \mathbb{E}^*(s^*_{\mathbf{1},k})^2 + 2\pi^2 \sup_{\mathbf{j}} |f(\lambda_{\mathbf{j}})| d_2^2(\mathcal{L}^*(s^*_{\mathbf{1},k}), N(0, 1))$$

$$\asymp o_p(1) + d_2^2(\mathcal{L}^*(s^*_{\mathbf{1},k}), \mathcal{L}^*(\widetilde{s}^*_{\mathbf{1},k}))$$

$$+ d_2^2 \left(\mathcal{L}^*(\widetilde{s}^*_{\mathbf{1},k}), \mathcal{L}^* \left(\widetilde{\widetilde{s}}^*_{\mathbf{1},k} \right) \right) + d_2^2 \left(\mathcal{L}^* \left(\widetilde{\widetilde{s}}^*_{\mathbf{1},k} \right), N(0, 1) \right)$$

$$\asymp o_p(1) + \mathbb{E}^*(s^*_{\mathbf{1},k} - \widetilde{s}^*_{\mathbf{1},k})^2 + \mathbb{E}^* \left(\widetilde{s}^*_{\mathbf{1},k} - \widetilde{\widetilde{s}}^*_{\mathbf{1},k} \right)^2 + d_2^2 \left(\mathcal{L}^* \left(\widetilde{\widetilde{s}}^*_{\mathbf{1},k} \right), N(0, 1) \right)$$

$$= o_p(1) \,,$$

where $\mathcal{L}^* \left(\widetilde{s}_{1,k}^* \right)$ is given by the (empirical) distribution (conditionally on $\{V(\cdot)\}$) which gives the correct convergence of the first and second moment, which together gives the convergence in the Mallows distance.

(b) **Wild Bootstrap (WB):**

By Theorem 5.1 in [33], we have the following condition:

$$\sup_{\lambda \in [0,2\pi]^2} \left| \widehat{f}_T(\lambda) - f(\lambda) \right| \xrightarrow{p} 0 \,.$$

Hence, Assumption $\mathcal{B}.2$ holds since

$$\sup_{\mathbf{j} \in N} |\mathrm{Var}^*(x^*(\mathbf{j})) - 2\pi^2 f(\lambda_{\mathbf{j}})| = \sup_{\mathbf{j} \in N} |2\pi^2 \widehat{f}_T(\lambda_j) - 2\pi^2 f(\lambda_{\mathbf{j}})| \xrightarrow{p} 0 \,.$$

Also, Assumption $\mathcal{B}.3$ holds since

$$\sup_{\mathbf{j} \in N} \mathbb{E}^*(|x^*(\mathbf{j})|)^p = C_p \sup_{\mathbf{j} \in N} \widehat{f}_T(\lambda_{\mathbf{j}})^p \leq C_p \sup_{\mathbf{j} \in N} f(\lambda_{\mathbf{j}})^p + o_p(1) \leq C + o_p(1) \,.$$

For Assumption $\mathcal{B}.4$, let $X \sim N(0,1)$ such that $\sqrt{2\pi^2 \widehat{f}_T(\lambda_j)} X \sim x^*(j)$. Then,

$$\sup_{\mathbf{j} \in N} d_2^2 \big(\mathcal{L}^*(x^*(\mathbf{j})), N(0, 2\pi^2 f(\lambda_{\mathbf{j}})) \big) \leq 2\pi^2 \sup_{\mathbf{j} \in N} |\widehat{f}_T(\lambda_{\mathbf{j}}) - f(\lambda_{\mathbf{j}})| = o_p(1) \,.$$

(c) **Local Bootstrap (LB):**

By Corollary 4.2 in [33], the empirical distribution function of the Fourier coefficients converges uniformly to the standard normal distribution function $\Phi(\cdot)$, i.e.,

$$\sup_{\mathbf{s} \in \mathbb{Z}^2} \mathbb{P} \bigg(\sup_{z \in \mathbb{R}} \bigg| \frac{1}{2} \sum_{\mathbf{j} \in N} p_{\mathbf{j}-\mathbf{s},T} \Big(\mathbb{1}_{\left\{ x(\mathbf{j}) \leq z \sqrt{2\pi^2 f(\lambda_{\mathbf{j}})} \right\}} + \mathbb{1}_{\left\{ y(\mathbf{j}) \leq z \sqrt{2\pi^2 f(\lambda_{\mathbf{j}})} \right\}} \Big)$$
$$- \Phi(z) \bigg| \geq \epsilon \bigg) \xrightarrow{p} 0 \,.$$

Also, by by Theorem 4.4 in [33], we also have the following conditions on the sums of the periodograms and Fourier coefficients:

$$\text{(i)} \quad \sup_{\mathbf{j} \in N} \bigg| \sum_{\mathbf{s} \in \mathbb{Z}^2} p_{\mathbf{s},T}(x(\mathbf{j}+\mathbf{s}) + y(\mathbf{j}+\mathbf{s})) \bigg| = o_p(1) \,,$$

$$\text{(ii)} \quad \sup_{\mathbf{j} \in N} \bigg| \sum_{\mathbf{s} \in \mathbb{Z}^2} p_{\mathbf{s},T} I(\mathbf{j}+\mathbf{s}) - 4\pi^2 f(\lambda_{\mathbf{j}}) \bigg| = o_p(1) \,,$$

$$\text{(iii)} \quad \sup_{\mathbf{j} \in N} \sum_{\mathbf{s} \in \mathbb{Z}^2} p_{\mathbf{s},T} I^2(\mathbf{j}+\mathbf{s}) \leq C_1 + o_p(1) \,,$$

$$\text{(iv)} \quad \sup_{\mathbf{j} \in N} \sum_{\mathbf{s} \in \mathbb{Z}^2} p_{\mathbf{s},T} I^q(\mathbf{j}+\mathbf{s}) \leq C_2 + o_p(1) \,,$$

for some constant $C_1, C_2 \geq 0$ and $q = 4 + \epsilon$ with some $\epsilon \in (0,1)$. Hence, Assumption $\mathcal{B}.2$ holds since

$$\sup_{\mathbf{j} \in N} |\mathrm{Var}^*(x^*(\mathbf{j})) - 2\pi^2 f(\lambda_{\mathbf{j}})|$$

$$\asymp \sup_{\mathbf{j} \in N} \left| \sum_{\mathbf{s} \in \mathbb{Z}^2} p_{\mathbf{s},T} I(\mathbf{s} + \mathbf{j}) - 4\pi^2 f(\lambda_{\mathbf{j}}) \right| + \sup_{\mathbf{j} \in N} \left(\sum_{\mathbf{s} \in \mathbb{Z}^2} p_{\mathbf{s},T} (x(\mathbf{j} + \mathbf{s}) + y(\mathbf{j} + \mathbf{s})) \right)$$

$$= o_p(1) \,.$$

Furthermore, by Assumption $\mathcal{K}.1$, Assumption $\mathcal{B}.3$ holds with $p = 2q > 8$, since

$$\sup_{\mathbf{j} \in N} \mathbb{E}^*(|x^*(\mathbf{j})|^p)$$

$$\asymp \sup_{\mathbf{j} \in N} \frac{1}{2} \sum_{\mathbf{s} \in \mathbb{Z}^2} p_{\mathbf{s},T} (|x(\mathbf{j} + \mathbf{s})|^p + |y(\mathbf{j} + \mathbf{s})|^p)$$

$$+ \sup_{\mathbf{j} \in N} \left| \frac{1}{2} \sum_{\mathbf{s} \in \mathbb{Z}^2} p_{\mathbf{s},T} (x(\mathbf{j} + \mathbf{s}) + y(\mathbf{j} + \mathbf{s})) \right|^p$$

$$\leq \sup_{\mathbf{j} \in N} \sum_{\mathbf{s} \in \mathbb{Z}^2} p_{\mathbf{s},T} I^q(\mathbf{j} + \mathbf{s}) + o_p(1) \leq C + o_p(1) \,.$$

Next, for Assumption $\mathcal{B}.4$, note first that f is uniformly continuous (since it is continuous by Assumption $\mathcal{A}.2$ and periodic on $[0, 2\pi]^2$). Hence,

$$\sup_{\mathbf{j} \in N} \sup_{\mathbf{s} \in \{(t_1, t_2) : -h_{T1} d_1 \leq t_1 \leq h_{T1} d_1, -h_{T2} d_2 \leq t_2 \leq h_{T2} d_2\}} |f(\lambda_{\mathbf{j}+\mathbf{s}}) - f(\lambda_{\mathbf{j}})| = o(1) \,.$$

Denote $\widetilde{x}^*(\mathbf{j}) = \frac{\widetilde{x}^*(\mathbf{j})}{\sqrt{2\pi^2 f(\lambda_{\mathbf{j}+J_{\mathbf{j},1,T}})}}$, where $J_{\mathbf{j},1,T}$ is defined in LB. Then, we get

$$\sup_{\mathbf{j} \in N} d_2^2 \big(\mathcal{L}^*(x^*(\mathbf{j})), N(0, 2\pi^2 f(\lambda_{\mathbf{j}})) \big)$$

$$\asymp \sup_{\mathbf{j} \in N} d_2^2 \big(\mathcal{L}^*(x^*(\mathbf{j})), \mathcal{L}^*(\widetilde{x}^*(\mathbf{j})) \big)$$

$$+ \pi \sup_{\mathbf{l} \in N} |f(\lambda_{\mathbf{l}})| \sup_{\mathbf{j} \in N} d_2^2 \left(\mathcal{L}^* \left(\frac{\widetilde{x}^*(\mathbf{j})}{\sqrt{2\pi^2 f(\lambda_{\mathbf{j}})}} \right), N(0,1) \right)$$

$$\asymp \sup_{\mathbf{j} \in N} \left[\sum_{\mathbf{s} \in \mathbb{Z}^2} p_{\mathbf{s},T} (x(\mathbf{j} + \mathbf{s}) + y(\mathbf{j} + \mathbf{s})) \right]^2$$

$$+ \sup_{\mathbf{j} \in N} d_2^2 \left(\mathcal{L}^* \left(\frac{\widetilde{x}^*(\mathbf{j})}{\sqrt{2\pi^2 f(\lambda_{\mathbf{j}})}} \right), \mathcal{L}^*(\widetilde{\widetilde{x}}^*(\mathbf{j})) \right)$$

$$+ \sup_{\mathbf{j} \in N} d_2^2 \big(\mathcal{L}^*(\widetilde{\widetilde{x}}^*(\mathbf{j})), N(0,1) \big)$$

$$= o_p(1) \,.$$

The last line follows from the uniform convergence of the empirical distribution function of the Fourier coefficients. Note that convergence in Mallows distance is equivalent to convergence in distribution in addition to convergence of the first two moments. In this case, the convergences in all three cases hold uniformly in **j**. ☐

Proof of Corollary 5.2. It follows directly from Theorem 6.1 in [33] and the above proof of Theorem 5.1. ☐

A.3. Proofs of Section 6

Proof of Theorem 6.1. By the conditions stated in the theorem, it is easy to see that $\{\widehat{V}(\mathbf{t})\}$ satisfies the condition in Corollary 5.2. Since $\mathcal{Z}_T \subset Q_2'$, the results follows from Theorem 4.3(a). ☐

Proof of Theorem 6.2. It is easy to see that the scan statistics diverges to infinity under the conditions stated in the theorem. ☐

Proof of Theorem 6.3. To simplify the notation used, we illustrate the proof by a simpler example using $\Lambda = \{(1,0),(0,1)\}$, then $\mathbf{G} = (2\gamma(1,0), 2\gamma(0,1))'$. The general results can be proved similarly.

Denote the vectors $\hat{\mathbf{G}}_T = (2\hat{\gamma}(1,0), 2\hat{\gamma}(0,1))'$ and $\mathbf{G} = (2\gamma(1,0), 2\gamma(0,1))'$ where $\gamma(\mathbf{j}) = \text{Var}[Z(\mathbf{0}) - Z(\mathbf{j})]$. Respectively, for the bootstrapped version, denote $\hat{\mathbf{G}}_{T,boot} = (2\hat{\gamma}_{boot}(1,0), 2\hat{\gamma}_{boot}(0,1))'$ and $\mathbf{G}_{boot} = \mathbb{E}^*(\hat{\mathbf{G}}_{T,boot})$. From Theorem 1 of [15], we have $\sqrt{|T|} \times \left(\hat{\mathbf{G}}_T - \mathbf{G} \right) \xrightarrow{d} N(\mathbf{0}, \mathbf{\Sigma}_R)$, where $\mathbf{\Sigma}_R = \lim_{T \to \infty} |T| \times \text{Cov}\left(\hat{\mathbf{G}}_T, \hat{\mathbf{G}}_T \right)$. In particular, its $(1,2)$th element is

$$\sum_{\mathbf{s} \in \mathbb{Z}} \text{Cov}\left([Z(0,0) - Z(1,0)]^2, [Z(\mathbf{s}) - Z(\mathbf{s} + (0,1))]^2 \right).$$

The other elements are defined similarly. To prove Theorem 6.3, it suffices to show that

$$\sqrt{|T|} \times \left(\hat{\mathbf{G}}_{T,boot} - \mathbf{G}_{boot} \right) \xrightarrow{d} N(\mathbf{0}, \mathbf{\Sigma}_R).$$

First, we show that for all $\mathbf{s}_1, \mathbf{s}_2 \in \mathbb{Z}^2$,

$$\lim_{T \to \infty} \sum_{\mathbf{s} \in T} \left| \text{Cov}^*\left([Z^*(\mathbf{0}) - Z^*(\mathbf{s}_1)]^2, [Z^*(\mathbf{s}) - Z^*(\mathbf{s} + \mathbf{s}_2)]^2 \right) \right| < \infty,$$

and

$$\lim_{T \to \infty} \sum_{\mathbf{s} \in T} \text{Cov}^*\left([Z^*(\mathbf{0}) - Z^*(\mathbf{s}_1)]^2, [Z^*(\mathbf{s}) - Z^*(\mathbf{s} + \mathbf{s}_2)]^2 \right) = \mathbf{\Sigma}_R.$$

Note that for fixed $\mathbf{s}, \mathbf{s}_1, \mathbf{s}_2 \in \mathbb{Z}^2$, we have

$$Z^*(\mathbf{0}) - Z^*(\mathbf{s}_1) = \frac{2}{\sqrt{|T|}} \sum_{\mathbf{j} \in N} \left[x^*(\mathbf{j})(1 - \cos(\lambda_{\mathbf{j}}'\mathbf{s}_1)) + y^*(\mathbf{j}) \sin(\lambda_{\mathbf{j}}'\mathbf{s}_1) \right]$$

$$= \quad \frac{2}{\sqrt{|T|}} \sum_{\mathbf{j}\in N} \left(x^*(\mathbf{j})A_{\mathbf{j}} + y^*(\mathbf{j})B_{\mathbf{j}}\right),$$

and

$$Z^*(\mathbf{s}) - Z^*(\mathbf{s} + \mathbf{s}_2)$$

$$= \quad \frac{2}{\sqrt{|T|}} \sum_{\mathbf{j}\in N} \left\{ x^*(\mathbf{j})[\cos(\lambda_{\mathbf{j}}'\mathbf{s}) - \cos(\lambda_{\mathbf{j}}'(\mathbf{s} + \mathbf{s}_2))] \right.$$

$$\left. -y^*(\mathbf{j})[\sin(\lambda_{\mathbf{j}}'\mathbf{s}) - \sin(\lambda_{\mathbf{j}}'(\mathbf{s} + \mathbf{s}_2))] \right\}$$

$$= \quad \frac{2}{\sqrt{|T|}} \sum_{\mathbf{j}\in N} \left(x^*(\mathbf{j})C_{\mathbf{j}} + y^*(\mathbf{j})D_{\mathbf{j}}\right),$$

where $A_{\mathbf{j}}$, $B_{\mathbf{j}}$, $C_{\mathbf{j}}$ and $D_{\mathbf{j}}$ are defined as $A_{\mathbf{j}} = 1 - \cos(\lambda_{\mathbf{j}}'\mathbf{s}_1)$, $B_{\mathbf{j}} = \sin(\lambda_{\mathbf{j}}'\mathbf{s}_1)$, $C_{\mathbf{j}} = \cos(\lambda_{\mathbf{j}}'\mathbf{s}) - \cos(\lambda_{\mathbf{j}}'(\mathbf{s} + \mathbf{s}_2))$ and $D_{\mathbf{j}} = \sin(\lambda_{\mathbf{j}}'(\mathbf{s} + \mathbf{s}_2)) - \sin(\lambda_{\mathbf{j}}'\mathbf{s})$.
Consider

$$\mathrm{Cov}^* \left([Z^*(\mathbf{0}) - Z^*(\mathbf{s}_1)]^2, [Z^*(\mathbf{s}) - Z^*(\mathbf{s} + \mathbf{s}_2)]^2 \right)$$

$$= \mathrm{Cov}^* \left(\frac{4}{|T|} \left(\sum_{\mathbf{j}\in N}\{x^*(\mathbf{j})A_{\mathbf{j}} + y^*(\mathbf{j})B_{\mathbf{j}}\} \right)^2, \frac{4}{|T|} \left(\sum_{\mathbf{j}\in N}\{x^*(\mathbf{j})C_{\mathbf{j}} + y^*(\mathbf{j})D_{\mathbf{j}}\} \right)^2 \right)$$

$$= \frac{16}{|T|^2} \sum_{\mathbf{i}\in N}\sum_{\mathbf{j}\in N}\sum_{\mathbf{k}\in N}\sum_{\mathbf{l}\in N} \left[A_{\mathbf{i}}A_{\mathbf{j}}C_{\mathbf{k}}C_{\mathbf{l}}\mathrm{Cov}^*(x^*(\mathbf{i})x^*(\mathbf{j}), x^*(\mathbf{k})x^*(\mathbf{l})) \right.$$

$$+ A_{\mathbf{i}}B_{\mathbf{j}}C_{\mathbf{k}}D_{\mathbf{l}}\mathrm{Cov}^*(x^*(\mathbf{i})y^*(\mathbf{j}), x^*(\mathbf{k})y^*(\mathbf{l}))$$

$$\left. + B_{\mathbf{i}}B_{\mathbf{j}}D_{\mathbf{k}}D_{\mathbf{l}}\mathrm{Cov}^*(y^*(\mathbf{i})y^*(\mathbf{j}), y^*(\mathbf{k})y^*(\mathbf{l})) \right]$$

$$= \frac{16}{|T|^2} \left[\sum_{\mathbf{i}\in N} A_{\mathbf{i}}^2 C_{\mathbf{i}}^2 \mathrm{Var}^*(x^*(\mathbf{i})^2) + \sum_{\mathbf{i}\in N} B_{\mathbf{i}}^2 D_{\mathbf{i}}^2 \mathrm{Var}^*(y^*(\mathbf{i})^2) \right.$$

$$+ 2 \sum_{\mathbf{i}\in N, \mathbf{j}\in N, \mathbf{i}\neq\mathbf{j}} A_{\mathbf{i}}A_{\mathbf{j}}C_{\mathbf{i}}C_{\mathbf{j}}\mathbb{E}^*(x^*(\mathbf{i})^2)\mathbb{E}^*(x^*(\mathbf{j})^2)$$

$$+ 2 \sum_{\mathbf{i}\in N, \mathbf{j}\in N, \mathbf{i}\neq\mathbf{j}} B_{\mathbf{i}}B_{\mathbf{j}}D_{\mathbf{i}}D_{\mathbf{j}}\mathbb{E}^*(y^*(\mathbf{i})^2)\mathbb{E}^*(y^*(\mathbf{j})^2)$$

$$\left. + 4 \sum_{\mathbf{i}\in N, \mathbf{j}\in N} A_{\mathbf{i}}B_{\mathbf{j}}C_{\mathbf{i}}D_{\mathbf{j}}\mathbb{E}^*(x^*(\mathbf{i})^2)\mathbb{E}^*(y^*(\mathbf{j})^2) \right]$$

$$= \frac{16}{|T|^2} \left[\sum_{\mathbf{i}\in N} A_{\mathbf{i}}^2 C_{\mathbf{i}}^2 \mathrm{Var}^*(x^*(\mathbf{i})^2) + \sum_{\mathbf{i}\in N} B_{\mathbf{i}}^2 D_{\mathbf{i}}^2 \mathrm{Var}^*(y^*(\mathbf{i})^2) \right]$$

$$+ \frac{32}{|T|^2} \left[\sum_{\mathbf{i}\in N, \mathbf{j}\in N, \mathbf{i}\neq\mathbf{j}} A_{\mathbf{i}}A_{\mathbf{j}}C_{\mathbf{i}}C_{\mathbf{j}}\mathbb{E}^*(x^*(\mathbf{i})^2)\mathbb{E}^*(x^*(\mathbf{j})^2) \right.$$

$$\left. + \sum_{\mathbf{i}\in N, \mathbf{j}\in N, \mathbf{i}\neq\mathbf{j}} B_{\mathbf{i}}B_{\mathbf{j}}D_{\mathbf{i}}D_{\mathbf{j}}\mathbb{E}^*(y^*(\mathbf{i})^2)\mathbb{E}^*(y^*(\mathbf{j})^2) \right.$$

$$+ 2 \sum_{\mathbf{i} \in N, \mathbf{j} \in N} A_{\mathbf{i}} B_{\mathbf{j}} C_{\mathbf{i}} D_{\mathbf{j}} \mathbb{E}^*(x^*(\mathbf{i})^2) \mathbb{E}^*(y^*(\mathbf{j})^2) \Big] .$$

By adding and subtracting terms and completing the square, we have

$$\frac{32}{|T|^2} \Big[\sum_{\mathbf{i} \in N} A_{\mathbf{i}} C_{\mathbf{i}} \mathbb{E}^*(x^*(\mathbf{i})^2) + \sum_{\mathbf{j} \in N} B_{\mathbf{j}} D_{\mathbf{j}} \mathbb{E}^*(y^*(\mathbf{j})^2) \Big]^2$$

$$- \frac{16}{|T|^2} \Big[\sum_{\mathbf{i} \in N} A_{\mathbf{i}}^2 C_{\mathbf{i}}^2 K_x^*(\mathbf{i}) + \sum_{\mathbf{j} \in N} B_{\mathbf{j}}^2 D_{\mathbf{j}}^2 K_y^*(\mathbf{j}) \Big]$$

$$= J_1 - J_2 ,$$

where $K_x^*(\mathbf{i}) = \mathbb{E}^*(x^*(\mathbf{i})^4) - 3\mathbb{E}^*(x^*(\mathbf{i})^2)^2$ and $K_y^*(\mathbf{j}) = \mathbb{E}^*(y^*(\mathbf{j})^4) - 3\mathbb{E}^*(y^*(\mathbf{j})^2)^2$, respectively.

For the first term J_1,

$$J_1 = \frac{32}{|T|^2} \Big[\sum_{\mathbf{i} \in N} A_{\mathbf{i}} C_{\mathbf{i}} \mathbb{E}^*(x^*(\mathbf{i})^2) + \sum_{\mathbf{j} \in N} B_{\mathbf{j}} D_{\mathbf{j}} \mathbb{E}^*(y^*(\mathbf{j})^2) \Big]^2$$

$$= \frac{32}{|T|^2} \Big[\mathrm{Cov}^* \Big(\sum_{\mathbf{i} \in N} [A_{\mathbf{i}} x^*(\mathbf{i}) + B_{\mathbf{i}} y^*(\mathbf{i})], \sum_{\mathbf{j} \in N} [C_{\mathbf{j}} x^*(\mathbf{j}) + D_{\mathbf{j}} y^*(\mathbf{j})] \Big) \Big]^2$$

$$= 2 \Big[\mathrm{Cov}^* \left(Z^*(\mathbf{0}) - Z^*(\mathbf{s}_1), Z^*(\mathbf{s}) - Z^*(\mathbf{s} + \mathbf{s}_2) \right) \Big]^2 .$$

For the second term J_2, since the underlying field is Gaussian, the Fourier coefficients $x(\mathbf{i})$ and $y(\mathbf{i})$ are Gaussian. By the Gaussianity of $x(\mathbf{i})$ and $y(\mathbf{i})$, asymptotic independence and strong law of large numbers, we have $K_x^*(\mathbf{i}) = o(1)$ and $K_y^*(\mathbf{j}) = o(1)$. Hence, we have $J_2 = o(\frac{1}{|T|})$. Therefore,

$$\sum_{\mathbf{s} \in T} \Big| \mathrm{Cov}^* \left([Z^*(\mathbf{0}) - Z^*(\mathbf{s}_1)]^2, [Z^*(\mathbf{s}) - Z^*(\mathbf{s} + \mathbf{s}_2)]^2 \right) \Big|$$

$$= \sum_{\mathbf{s} \in T} 2 \left[\mathrm{Cov}^* \left(Z^*(\mathbf{0}) - Z^*(\mathbf{s}_1), Z^*(\mathbf{s}) - Z^*(\mathbf{s} + \mathbf{s}_2) \right) \right]^2 + o(1) .$$

Furthermore, we have

$$\sum_{\mathbf{s} \in T} 2 \left[\mathrm{Cov}^* \left(Z^*(\mathbf{0}) - Z^*(\mathbf{s}_1), Z^*(\mathbf{s}) - Z^*(\mathbf{s} + \mathbf{s}_2) \right) \right]^2$$

$$= \sum_{\mathbf{s} \in T} \frac{32}{|T|^2} \Big[\sum_{\mathbf{i} \in N} \Big[(1 - \cos(\lambda_{\mathbf{i}}'\mathbf{s}_1))(\cos(\lambda_{\mathbf{i}}'\mathbf{s}) - \cos(\lambda_{\mathbf{i}}'(\mathbf{s} + \mathbf{s}_2)))$$

$$+ \sin(\lambda_{\mathbf{j}}'\mathbf{s}_1)(\sin(\lambda_{\mathbf{j}}'(\mathbf{s} + \mathbf{s}_2)) - \sin(\lambda_{\mathbf{j}}'\mathbf{s})) \Big] \mathbb{E}^*(x^*(\mathbf{i})^2) \Big]^2$$

$$= \sum_{\mathbf{s} \in T} \frac{32}{|T|^2} \left[\sum_{\mathbf{i} \in N} \left[(\cos(\lambda_{\mathbf{i}}' \mathbf{s}) - \cos(\lambda_{\mathbf{i}}'(\mathbf{s} - \mathbf{s}_1)) \right. \right.$$

$$\left. \left. + \cos(\lambda_{\mathbf{i}}'(\mathbf{s} + \mathbf{s}_2 - \mathbf{s}_1)) - \cos(\lambda_{\mathbf{i}}'(\mathbf{s} + \mathbf{s}_2)) \right] 2\pi^2 f(\lambda_{\mathbf{i}}) \right]^2 + o_p(1)$$

$$= \sum_{\mathbf{s} \in T} 2 \left[\mathrm{Cov}\left(Z(\mathbf{0}) - Z(\mathbf{s}_1), Z(\mathbf{s}) - Z(\mathbf{s} + \mathbf{s}_2) \right) \right]^2 + o_p(1)$$

$$= \sum_{\mathbf{s} \in T} \mathrm{Cov}\left([Z(\mathbf{0}) - Z(\mathbf{s}_1)]^2, [Z(\mathbf{s}) - Z(\mathbf{s} + \mathbf{s}_2)]^2 \right) + o_p(1) \xrightarrow{p} \mathbf{\Sigma}_R.$$

Hence, we have

$$\lim_{T \to \infty} \sum_{\mathbf{s} \in T} \mathrm{Cov}^*\left([Z^*(\mathbf{0}) - Z^*(\mathbf{s}_1)]^2, [Z^*(\mathbf{s}) - Z^*(\mathbf{s} + \mathbf{s}_2)]^2 \right) = \mathbf{\Sigma}_R.$$

Then, we have

$$\sqrt{|T|} \frac{1}{|T|} \sum_{\mathbf{s} \in T} [Z^*(\mathbf{s}) - Z^*(\mathbf{s} + \mathbf{t})]^2$$

$$= \frac{1}{\sqrt{|T|}} \sum_{\mathbf{s} \in T} \left[\frac{1}{\sqrt{|T|}} \sum_{\mathbf{j} \in T} (x^*(\mathbf{j}) + iy^*(\mathbf{j}))[\exp(i\lambda_{\mathbf{j}}' \mathbf{s}) - \exp(i\lambda_{\mathbf{j}}'(\mathbf{s} + \mathbf{t}))] \right]^2$$

$$= \frac{1}{|T|^{3/2}} \sum_{\mathbf{s} \in T} \left[\sum_{\mathbf{k} \in T} (x^*(\mathbf{k}) + iy^*(\mathbf{k}))[\exp(i\lambda_{\mathbf{k}}' \mathbf{s}) - \exp(i\lambda_{\mathbf{k}}'(\mathbf{s} + \mathbf{t}))] \right] \times$$

$$\left[\sum_{\mathbf{l} \in T} (x^*(\mathbf{l}) - iy^*(\mathbf{l}))[\exp(-i\lambda_{\mathbf{l}}' \mathbf{s}) - \exp(-i\lambda_{\mathbf{l}}'(\mathbf{s} + \mathbf{t}))] \right]$$

$$= \frac{1}{|T|^{3/2}} \left[\left(\sum_{\mathbf{k} \in T} (x^*(\mathbf{k}) + iy^*(\mathbf{k}))[1 - \exp(i\lambda_{\mathbf{k}}' \mathbf{t})] \right) \times \right.$$

$$\left. \left(\sum_{\mathbf{l} \in T} (x^*(\mathbf{l}) - iy^*(\mathbf{l}))[1 - \exp(-i\lambda_{\mathbf{l}}' \mathbf{t})] \right) \times \left(\sum_{\mathbf{s} \in T} \exp(i\lambda_{\mathbf{k}}' \mathbf{s}) \exp(-i\lambda_{\mathbf{l}}' \mathbf{s}) \right) \right]$$

$$= \frac{1}{|T|^{3/2}} \left[\left(\sum_{\mathbf{k} \in T} (x^*(\mathbf{k}) + iy^*(\mathbf{k}))[1 - \exp(i\lambda_{\mathbf{k}}' \mathbf{t})] \right) \right.$$

$$\left. \times \left(\sum_{\mathbf{l} \in T} (x^*(\mathbf{l}) - iy^*(\mathbf{l}))[1 - \exp(-i\lambda_{\mathbf{l}}' \mathbf{t})] \right) \times |T| \mathbb{1}_{\{\mathbf{k} = \mathbf{l}\}} \right],$$

by grouping the terms with $\mathbf{k} = \mathbf{l}$ in the summation, we have

$$\frac{1}{\sqrt{|T|}} \left[\sum_{\mathbf{k} \in T} (x^*(\mathbf{k})^2 + y^*(\mathbf{k})^2)[2 - \exp(i\lambda_{\mathbf{k}}' \mathbf{t}) - \exp(-i\lambda_{\mathbf{k}}' \mathbf{t})] \right]$$

$$= \frac{4}{\sqrt{|T|}} \left[\sum_{\mathbf{k} \in N} (x^*(\mathbf{k})^2 + y^*(\mathbf{k})^2)[1 - \cos(\lambda'_{\mathbf{k}} \mathbf{t})] \right].$$

Since the above sum is sum of independent random variables with finite second moment, by the central limit theorem, it converges to normal distribution with variance $\mathbf{\Sigma}_R$. Hence, the results follows directly by applying Cramer-Wold device. $\qquad \square$

Appendix B: Proofs of lemmas in Appendix A

Proof of Lemma A.1. Because $\cos(\lambda'_{\mathbf{j}} \mathbf{h}) = \frac{1}{2}[\exp(i\lambda'_{\mathbf{j}} \mathbf{h}) + \exp(i\lambda'_{\mathbf{d}_T - \mathbf{j}} \mathbf{h})]$, we obtain

$$\frac{8\pi^2}{|T|} \sum_{\mathbf{j} \in N} f(\lambda_{\mathbf{j}}) \cos(\lambda'_{\mathbf{j}} \mathbf{h})$$

$$= \frac{4\pi^2}{|T|} \sum_{\mathbf{j} \in T} f(\lambda_{\mathbf{j}}) \exp(i\lambda'_{\mathbf{j}} \mathbf{h}) - \frac{4\pi^2}{|T|} f(\mathbf{0})$$

$$- \frac{4\pi^2}{|T|} f(\pi, 2\pi) \exp(i(\pi, 2\pi)' \mathbf{h}) \mathbb{1}_{\{d_1 \text{ even}\}}$$

$$- \frac{4\pi^2}{|T|} f(2\pi, \pi) \exp(i(2\pi, \pi)' \mathbf{h}) \mathbb{1}_{\{d_2 \text{ even}\}}$$

$$- \frac{4\pi^2}{|T|} f(\pi, \pi) \exp(i(\pi, \pi)' \mathbf{h}) \mathbb{1}_{\{d_1, d_2 \text{ even}\}}.$$

Moreover, since $f(\lambda) = \frac{1}{4\pi^2} \sum_{\mathbf{j} \in \mathbb{Z}^2} \gamma(\mathbf{j}) \exp(-i\lambda' \mathbf{j})$, we have

$$\frac{4\pi^2}{|T|} \sum_{\mathbf{j} \in T} f(\lambda_{\mathbf{j}}) \exp(i\lambda'_{\mathbf{j}} \mathbf{h}) = \frac{1}{|T|} \sum_{\mathbf{j} \in T} \left[\sum_{\mathbf{k} \in \mathbb{Z}^2} \gamma(\mathbf{k}) \exp(-i\lambda'_{\mathbf{j}} \mathbf{k}) \right] \exp(i\lambda'_{\mathbf{j}} \mathbf{h})$$

$$= \frac{1}{|T|} \sum_{\mathbf{k} \in \mathbb{Z}^2} \gamma(\mathbf{k}) \sum_{\mathbf{j} \in T} \exp(i\lambda'_{\mathbf{j}} (\mathbf{h} - \mathbf{k}))$$

$$= \sum_{\mathbf{k} \in \mathbb{Z}^2} \gamma(\mathbf{h} + \mathbf{k} \odot \mathbf{d}_T).$$

where the exchange of summations is due to Fubini's Theorem. $\qquad \square$

Proof of Lemma A.2. For the assertion in (a), we use the Cramer-Wold device

and prove a Lyapunov type condition. Let $\alpha_i \in \mathbb{R}$. Consider

$$\sum_{i=1}^{p} \alpha_i S^*_{A_T \otimes T}(\mathbf{u_i})$$

$$= \frac{1}{\sqrt{|A_T \otimes T|}\sqrt{|T|}} \sum_{\mathbf{j} \in N} 2\Big[x^*(\mathbf{j}) \sum_{i=1}^{p} \alpha_i \sum_{\mathbf{l} \in B_i \otimes (A_T \otimes T)} \cos(\lambda'_{\mathbf{j}}\mathbf{l})$$

$$- y^*(\mathbf{j}) \sum_{i=1}^{p} \alpha_i \sum_{\mathbf{l} \in B_i \otimes (A_T \otimes T)} \sin(\lambda'_{\mathbf{j}}\mathbf{l}) \Big]$$

$$= \frac{1}{\sqrt{|A_T \otimes T|}\sqrt{|T|}} \sum_{\mathbf{j} \in N} \big[A^*_{\mathbf{j},N} + B^*_{\mathbf{j},N} \big] \,,$$

where $A^*_{\mathbf{j},N}, B^*_{\mathbf{j},N}$ for all $\mathbf{j} \in N$ are conditionally row-wise independent. The Lyapunov conditon is then (in probability) fulfilled since by Assumptions $\mathcal{B}.2$ and $\mathcal{B}.3$, similar to (A.2), we have

$$\frac{1}{|A_T \otimes T|^2 |T|^2} \sum_{\mathbf{j} \in N} \big[\mathbb{E}^*(A^*_{\mathbf{j},N} - \mathbb{E}(A^*_{\mathbf{j},N}))^4 + \mathbb{E}^*(B^*_{\mathbf{j},N} - \mathbb{E}(B^*_{\mathbf{j},N}))^4 \big]$$

$$\leq (C + o_p(1)) \frac{1}{|A_T \otimes T|^2 |T|^2} \times$$

$$\sum_{\mathbf{j} \in N} \left[\min\left(\frac{d_1}{j_1}, (a_{12T} - a_{11T})d_1 \right) \min\left(\frac{d_2}{j_2}, (a_{22T} - a_{21T})d_2 \right) \right]^4$$

$$\leq (C + o_p(1)) \frac{|A_T \otimes T|^2}{|T|^2} = o_p(1)\,.$$

Together with Lemma A.1, this gives assertion (a). Note that it is essential that $\frac{|A_T \otimes T|}{|T|} \to 0$. In fact, it is easy to see that for $A_T \otimes T = T$, the Feller condition is not fulfilled. Thus, the Lindeberg condition cannot be fulfilled. Therefore, we need a different argument to obtain the asymptotic normality of $A_T \otimes T = T$. We now use the Cramer Wold device and Lemma 3 of [31] which gives an upper bound for the Mallows distance ρ_2 of weighted sums of independent $N(0,1)$ random variables. The assertion then follows, since by the proof of Lemma A.1,

$$\sum_{i=1}^{p} \sum_{j=1}^{p} \alpha_i \alpha_j \frac{8\pi^2}{|T|^2} \sum_{\mathbf{j} \in N} f(\lambda_{\mathbf{j}}) \sum_{\mathbf{l_1} \in B_i \otimes T} \sum_{\mathbf{l_2} \in B_j \otimes T} \cos(\lambda'_{\mathbf{j}}(\mathbf{l_1} - \mathbf{l_2}))$$

$$= \sum_{i=1}^{p} \sum_{j=1}^{p} \alpha_i \alpha_j 4\pi^2 f(\mathbf{0})[L(B_i \bigcap B_j) - L(B_i)L(B_j)] + o_p(1)\,. \quad \square$$

Proof of Lemma A.3. Recall that the pseudo-metric ρ is defined as $\rho(B,C) = \sqrt{L(B\Delta C)}$ for any fixed $B, C \in Q'_2$. We have

$$S^*_{A_T \otimes T}(B) - S^*_{A_T \otimes T}(C) = \sum_{\mathbf{l} \in A_T \otimes T} a_{\mathbf{l}} Z^*(\mathbf{l})\,,$$

where $a_{\mathbf{l}} = \mathbb{1}_{\{\mathbf{l} \in B \otimes (A_T \otimes T)\}} - \mathbb{1}_{\{\mathbf{l} \in C \otimes (A_T \otimes T)\}}$. First, we show that

$$|A_T \otimes T|^{-1/2} \left\| \sum_{\mathbf{l} \in A_T \otimes T} a_{\mathbf{l}} Z^*(\mathbf{l}) \right\|_p \leq D_p \left(\sum_{\mathbf{l} \in A_T \otimes T} a_{\mathbf{l}}^2 \right)^{\frac{1}{2}} = D_p \rho(B, C), \quad \text{(B.1)}$$

where D_p is a constant depends only on p, and the value of p is the same as in the Assumption B.3, which requires $p > 8$. Recall that

$$Z^*(\mathbf{t}) = \frac{2}{\sqrt{|T|}} \sum_{\mathbf{j} \in N} \left[x^*(\mathbf{j}) \cos(\lambda_{\mathbf{j}}' \mathbf{t}) - y^*(\mathbf{j}) \sin(\lambda_{\mathbf{j}}' \mathbf{t}) \right].$$

Hence by Burkholder inequality and $x^*(\mathbf{j}) \overset{d}{=} y^*(\mathbf{j})$, we have

$$\left\| \sum_{\mathbf{l} \in A_T \otimes T} a_{\mathbf{l}} Z^*(\mathbf{l}) \right\|_p$$

$$= \frac{2}{\sqrt{|T|}} \left\| \sum_{\mathbf{l} \in A_T \otimes T} a_{\mathbf{l}} \left(\sum_{\mathbf{j} \in N} (x^*(\mathbf{j}) \cos(\lambda_{\mathbf{j}}' \mathbf{l}) - y^*(\mathbf{j}) \sin(\lambda_{\mathbf{j}}' \mathbf{l})) \right) \right\|_p$$

$$= \frac{2}{\sqrt{|T|}} \left\| \sum_{\mathbf{l} \in A_T \otimes T} \sum_{\mathbf{j} \in N} a_{\mathbf{l}} \left[(x^*(\mathbf{j}) \cos(\lambda_{\mathbf{j}}' \mathbf{l}) - y^*(\mathbf{j}) \sin(\lambda_{\mathbf{j}}' \mathbf{l})) \right] \right\|_p$$

$$\leq \frac{2}{\sqrt{|T|}} \left(2p \sum_{\mathbf{l} \in A_T \otimes T} \sum_{\mathbf{j} \in N} \|a_{\mathbf{l}} x^*(\mathbf{j}) \cos(\lambda_{\mathbf{j}}' \mathbf{l})\|_p^2 + \|a_{\mathbf{l}} y^*(\mathbf{j}) \sin(\lambda_{\mathbf{j}}' \mathbf{l})\|_p^2 \right)^{\frac{1}{2}}$$

$$\leq \frac{2}{\sqrt{|T|}} \left(2p \sum_{\mathbf{l} \in A_T \otimes T} \sum_{\mathbf{j} \in N} a_{\mathbf{l}}^2 \|x^*(\mathbf{j})\|_p^2 \right)^{\frac{1}{2}}$$

$$= \left(2p \sum_{\mathbf{l} \in A_T \otimes T} a_{\mathbf{l}}^2 \right)^{\frac{1}{2}} \left(\frac{4}{|T|} \sum_{\mathbf{j} \in N} \|x^*(\mathbf{j})\|_p^2 \right)^{\frac{1}{2}}.$$

Finally, by Assumption B.3, we have

$$|A_T \otimes T|^{-1/2} \left\| \sum_{\mathbf{l} \in A_T \otimes T} a_{\mathbf{l}} Z^*(\mathbf{l}) \right\|_p$$

$$\leq |A_T \otimes T|^{-1/2} \left(2p \sum_{\mathbf{l} \in A_T \otimes T} a_{\mathbf{l}}^2 \right)^{\frac{1}{2}} \left(\frac{4}{|T|} \sum_{\mathbf{j} \in N} \|x^*(\mathbf{j})\|_p^2 \right)^{\frac{1}{2}}$$

$$= \left(\frac{1}{|A_T \otimes T|} \sum_{\mathbf{l} \in A_T \otimes T} a_{\mathbf{l}}^2 \right)^{\frac{1}{2}} \left(\frac{8p}{|T|} \sum_{\mathbf{j} \in N} \|x^*(\mathbf{j})\|_p^2 \right)^{\frac{1}{2}}$$

$$= D_p \left(\frac{1}{|A_T \otimes T|} \sum_{\mathbf{l} \in A_T \otimes T} a_{\mathbf{l}}^2 \right)^{\frac{1}{2}} = D_p \sqrt{L(B \Delta C)} = D_p \rho(B, C) \,.$$

From Theorem 2.6.4 of [48], with $\mathcal{C} = Q_2'$ and $V(Q_2') = 5$, there exists a positive constant K such that $N(\epsilon, Q_2', \rho) \leq KV(Q_2')(4e)^{V(Q_2')}(1/\epsilon)^{2(V(Q_2')-1)} = 5K(4e)^5(1/\epsilon)^8$, where $N(\epsilon, Q_2', \rho)$ is the smallest number of open balls of radius ϵ with respect to ρ which forms a covering of Q_2'. Since $p > 8$, we have

$$\int_0^1 (N(\epsilon, Q_2', \rho))^{\frac{1}{p}} d\epsilon < \infty \,. \tag{B.2}$$

By (B.1) and (B.2), we can apply Theorem 11.6 in [27] and we have that for each $\epsilon > 0$, there exists a $\delta > 0$ which depends on ϵ and on the value of the entropy integral (B.2) but not on T, such that

$$\mathbb{E} \left(\sup_{\substack{B,C \in Q_2' \\ \rho(B,C) < \delta}} \left| |A_T \otimes T|^{-1/2} S^*_{A_T \otimes T}(B) - |A_T \otimes T|^{-1/2} S^*_{A_T \otimes T}(C) \right| \right) < \epsilon \,.$$

Thus, (A.3) follows and hence the sequence of the bootstrap sample processes

$$\left\{ (4\pi^2 f(\mathbf{0})|A_T \otimes T|)^{-1/2} S^*_{A_T \otimes T}(B) : B \in Q_2'|V(\cdot) \right\} \,,$$

is tight in the space of continuous real functions on Q_2'. $\qquad\qquad \square$

Acknowledgments

References

[1] BILLINGSLEY, P. (2013). *Convergence of Probability Measures*. John Wiley & Sons. MR0233396

[2] BÜHLMANN, P. (2002). Bootstraps for time series. *Statistical Science* **17** 52–72. MR1910074

[3] CARLSTEIN, E. (1986). The use of subseries values for estimating the variance of a general statistic from a stationary sequence. *The Annals of Statistics* **14** 1171–1179. MR0856813

[4] CHAN, H. P. and WALTHER, G. (2013). Detection with the scan and the average likelihood ratio. *Statistica Sinica* **23** 409–428. MR3076173

[5] DAHLHAUS, R. and JANAS, D. (1996). A frequency domain bootstrap for ratio statistics in time series analysis. *The Annals of Statistics* **24** 1934–1963. MR1421155

[6] Davison, A. C. and Hinkley, D. V. (1997). *Bootstrap Methods and Their Application* **1**. Cambridge University Press. MR1478673

[7] Deb, S., Pourahmadi, M. and Wu, W. B. (2017). An asymptotic theory for spectral analysis of random fields. *Electronic Journal of Statistics* **11** 4297–4322. MR3724221

[8] Efron, B. (1979). Bootstrap methods: another look at the jackknife. *The Annals of Statistics* **7** 1–26. MR0515681

[9] El Machkouri, M., Volný, D. and Wu, W. B. (2013). A central limit theorem for stationary random fields. *Stochastic Processes and their Applications* **123** 1–14. MR2988107

[10] Franke, J. and Hardle, W. (1992). On bootstrapping kernel spectral estimates. *The Annals of Statistics* **20** 121–145. MR1150337

[11] Gaetan, C. and Guyon, X. (2009). *Spatial Statistics and Modeling*. Springer-Verlag. MR2569034

[12] Glaz, J. and Balakrishnan, N. (2012). *Scan Statistics and Applications*. Springer Science & Business Media.

[13] Glaz, J., Naus, J. I., Wallenstein, S., Wallenstein, S. and Naus, J. I. (2001). *Scan Statistics*. Springer. MR1869112

[14] Glaz, J., Pozdnyakov, V. and Wallenstein, S. (2009). *Scan Statistics: Methods and Applications*. Springer Science & Business Media. MR2761094

[15] Guan, Y., Sherman, M. and Calvin, J. A. (2004). A nonparametric test for spatial isotropy using subsampling. *Journal of the American Statistical Association* **99** 810–821. MR2090914

[16] Hall, P. (1985). Resampling a coverage pattern. *Stochastic Processes and their Applications* **20** 231–246. MR0808159

[17] Hurvich, C. M. and Zeger, S. L. (1987). Frequency domain bootstrap methods for time series. *Preprint, New York: Department of Statistics and Operations Research, New York University*. MR0909989

[18] Jentsch, C. and Kreiss, J.-P. (2010). The multiple hybrid bootstrap – resampling multivariate linear processes. *Journal of Multivariate Analysis* **101** 2320–2345. MR2719865

[19] Jiang, Z., Ling, N., Lu, Z., Tjøstheim, D. and Zhang, Q. (2020). On bandwidth choice for spatial data density estimation. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)* **82** 817–840. MR4112786

[20] Kirch, C. (2007). Resampling in the frequency domain of time series to determine critical values for change-point tests. *Statistics & Decisions* **25** 237–261. MR2412072

[21] Kirch, C. and Politis, D. N. (2011). TFT-bootstrap: Resampling time series in the frequency domain to obtain replicates in the time domain. *The Annals of Statistics* **39** 1427–1470. MR2850208

[22] Kreiss, J.-P. and Paparoditis, E. (2003). Autoregressive-aided periodogram bootstrap for time series. *The Annals of Statistics* **31** 1923–1955. MR2036395

[23] Kulldorff, M. (1997). A spatial scan statistic. *Communications in Statistics - Theory and Methods* **26** 1481–1496. MR1456844

[24] KULLDORFF, M. (1999). Spatial scan statistics: models, calculations, and applications. In *Scan Statistics and Applications* (J. Glaz and N. Balakrishnan, eds.) 303–322. Springer. MR1697758

[25] KÜNSCH, H. R. (1989). The jackknife and the bootstrap for general stationary observations. *The Annals of Statistics* **17** 1217–1241. MR1015147

[26] LAHIRI, S. N. (2003). *Resampling methods for dependent data.* Springer Series in Statistics, Springer-Verlag. MR2001447

[27] LEDOUX, M. and TALAGRAND, M. (1991). *Probability in Banach Spaces.* Springer, New York. MR1102015

[28] LEE, Y. D. and LAHIRI, S. N. (2002). Least squares variogram fitting by spatial subsampling. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)* **64** 837–854. MR1979390

[29] LIU, R. Y. and SINGH, K. (1992). Moving blocks jackknife and bootstrap capture weak dependence. In *Exploring the Limits of Bootstrap* (R. LePage and L. Billard, eds.) 225–248. John Wiley, New York. MR1197787

[30] LOH, J. M. and STEIN, M. L. (2008). Spatial bootstrap with increasing observations in a fixed domain. *Statistica Sinica* **18** 667–688. MR2411618

[31] MALLOWS, C. L. (1972). A note on asymptotic joint normality. *The Annals of Mathematical Statistics* **43** 508–515. MR0298812

[32] MEYER, M., JENTSCH, C. and KREISS, J.-P. (2017). Baxter's inequality and sieve bootstrap for random fields. *Bernoulli* **23** 2988–3020. MR3654797

[33] NG, W. L. and YAU, C. Y. (2020). Asymptotic Spectral Theory for Spatial Data. *arXiv:2005.13274.*

[34] NORDMAN, D. J. and LAHIRI, S. N. (2004). On optimal spatial subsample size for variance estimation. *The Annals of Statistics* **32** 1981–2027. MR2102500

[35] NORDMAN, D. J., LAHIRI, S. N. and FRIDLEY, B. L. (2007). Optimal block size for variance estimation by a spatial block bootstrap method. *Sankhyā: The Indian Journal of Statistics* **69** 468–493. MR2460005

[36] PAPARODITIS, E. (2002). Frequency domain bootstrap for time series. In *Empirical Process Techniques for Dependent Data* (H. Dehling, T. Mikosch and M. Sørensen, eds.) 365–381. Springer. MR1958790

[37] PAPARODITIS, E. and POLITIS, D. N. (1999). The local bootstrap for periodogram statistics. *Journal of Time Series Analysis* **20** 193–222. MR1701054

[38] POLITIS, D. N. (2003). The impact of bootstrap methods on time series analysis. *Statistical Science* **18** 219–230. MR2026081

[39] POLITIS, D. N. (2003). Adaptive bandwidth choice. *Journal of Nonparametric Statistics* **15** 517–533. MR2017485

[40] POLITIS, D. N., PAPARODITIS, E. and ROMANO, J. P. (1998). Large sample inference for irregularly spaced dependent observations based on subsampling. *Sankhyā: The Indian Journal of Statistics, Series A* **60** 274–292. MR1711685

[41] POLITIS, D. N. and ROMANO, J. P. (1992). A circular block-resampling procedure for stationary data. In *Exploring the Limits of Bootstrap* (R. LePage and L. Billard, eds.) 263–270. John Wiley, New York. MR1197789

[42] POLITIS, D. N. and ROMANO, J. P. (1993). Nonparametric resampling for homogeneous strong mixing random fields. *Journal of Multivariate Analysis* **47** 301–328. MR1247380

[43] POLITIS, D. N. and ROMANO, J. P. (1994). The stationary bootstrap. *Journal of the American Statistical association* **89** 1303–1313. MR1310224

[44] POSSOLO, A. (1991). Subsampling a random field. *IMS Lecture Notes Monograph Series* **20** 286–294. MR1195571

[45] SHAO, X. and WU, W. B. (2007). Asymptotic spectral theory for nonlinear time series. *The Annals of Statistics* **35** 1773–1801. MR2351105

[46] SHERMAN, M. (1996). Variance estimation for statistics computed from spatial lattice data. *Journal of the Royal Statistical Society. Series B (Methodological)* **58** 509–523. MR1394363

[47] SHERMAN, M. (2011). *Spatial Statistics and Spatio-temporal Data: Covariance Functions and Directional Properties.* John Wiley & Sons. MR2815783

[48] VAN DER VAART, A. W. and WELLNER, J. A. (1996). *Weak Convergence and Empirical Processes with Applications to Statistics.* Springer. MR1385671

[49] ZHAO, B. and GLAZ, J. (2017). Scan statistics for detecting a local change in variance for two-dimensional normal data. *Communications in Statistics - Theory and Methods* **46** 5517–5530. MR3610341