Weak dependence and GMM estimation of supOU and mixed moving average processes

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Abstract: We consider a mixed moving average (MMA) process X driven by a Lévy basis and prove that it is weakly dependent with rates computable in terms of the moving average kernel and the characteristic quadruple of the Lévy basis. Using this property, we show conditions ensuring that sample mean and autocovariances of X have a limiting normal distribution. We extend these results to stochastic volatility models and then investigate a Generalized Method of Moments estimator for the supOU process and the supOU stochastic volatility model after choosing a suitable distribution for the mean reversion parameter. For these estimators, we analyze the asymptotic behavior in detail.

1. Introduction

Lévy-driven continuous-time moving average processes, i.e. processes $(X_t)_{t\in\mathbb{R}}$ of the form $X_t = \int_{\mathbb{R}} f(t-s)dL_s$ with f a deterministic function and L a Lévy process, are frequently used to model time series, especially, when dealing with data observed at high frequency. Moreover, causal moving averages can be used to model the volatility process when the dynamics of a logarithmic financial asset price are modeled. Popular examples include, for instance CARMA processes [16, 39], the increments of fractionally integrated Lévy processes [38] and non-Gaussian Ornstein-Uhlenbeck type processes [9] where $f(s) = e^{as}1_{[0,\infty)}(s)$ with $a \in \mathbb{R}^-$. By allowing f to depend on a random parameter A and replacing the Lévy process by a Lévy basis one arrives at so-called mixed moving averages (MMA in short) as for instance in [3, 10, 27].

An important example of MMA are the supOU processes studied in [3, 10, 25, 27]. In the univariate case, assume $\int_{|x|>1} \log(|x|)\,\nu(dx) < \infty$ and $\int_{\mathbb{R}^-} -\frac{1}{A}\pi(dA) < \infty$, where ν is a Lévy measure and π is the probability distribution on $\mathbb{R}^-$ of the random parameter A, see Definition 2.1 for details. If Λ is a Lévy basis on

$\mathbb{R}$ with those characteristics, then the process

$$X_t = \int_{\mathbb{R}^-} \int_{-\infty}^t \mathrm{e}^{A(t-s)} \Lambda(dA, ds) \ \forall t \in \mathbb{R},$$

is called a supOU process. Whereas a non-Gaussian Ornstein-Uhlenbeck process necessarily exhibits autocorrelation e^{ah} for $h \in \mathbb{N}$, the supOU process has a flexible dependence structure. For example, its autocorrelations can show a polynomial decay depending on the probability distribution π. Moreover, when a discrete probability distribution π for the random parameter A is considered, we obtain a popular model used, for example, in stochastic volatility models [3], in modeling fractal activity times [36, 37] and in astrophysics [35].

MMA processes can also be used, under suitable conditions, as building blocks for more complex models. We study in this paper the class of MMA stochastic volatility models. An example of the class is the supOU SV model, defined in [10, 11], where the log-price process (of some financial asset) is defined for $t \in \mathbb{R}^+$ as

$$J_t = \int_0^t \sqrt{X_s} dW_s, \ \ J_0 = 0,$$

and $(W_s)_{s \in \mathbb{R}^+}$ is a standard Brownian motion independent of the process $(X_s)_{s \in \mathbb{R}^+}$ which is a non-negative supOU process. Some examples of applications of the supOU SV model can be found in [12, 32, 49].

The aim of this paper is twofold. First, to show that sample moments of an MMA and of the returns of an MMA SV model have a limiting normal distribution. Secondly, to develop a statistical estimation procedure for the MMA and MMA SV model in a semi-parametric framework, where the distribution of the random parameter A is specified in detail, and establishing its asymptotic properties.

For this end, it is of high importance to understand the dependence structure of the class of MMA processes. In [27], it is shown that an MMA process driven by a Lévy basis is mixing. However, in order to prove distributional limit theorems which enable valid asymptotic inference stronger notions of asymptotic independence are needed. Often one applies strong mixing properties (see [21, 46]) to this end. Usually they are established by using a Markovian representation and showing geometric ergodicity of it. In turn this requires often smoothness conditions on the driving random noise and it is well-known that even autoregressive processes of order one are not strongly mixing when the distribution of the noise is not sufficiently regular (see [1]). We want to obtain results for MMA processes in general, which typically have no suitable Markovian representation, and without regularity conditions on the driving Lévy basis apart from moment conditions. As will become obvious later on, the weak dependence concepts introduced by Doukhan and Louhichi [23] and Doukhan and Dedecker [18], respectively called η-weak dependence and θ-weak dependence, are very suitable for our purposes. For an extensive introduction on the weak dependence of causal and non-causal processes we refer the reader to [19]. We

then show the asymptotic normality of the sample mean and the sample autocovariance functions of an MMA process in its non-causal and causal specification. Moreover for the MMA stochastic volatility models, we show the θ-weak dependence of the return process and the distributional limit of its sample moments. In [29, 30, 31], the limiting behavior of integrated and partial sums of supOU processes is analyzed in relation to the growth rate of their moments, called intermittency when the grow rate is fast. This leads to some conclusions regarding their asymptotic finite dimensional distributions and to identify different limiting theorems depending on the short or long memory shown by the supOU process. In our paper, for short memory supOU processes and more general MMA and MMA SV model we can additionally give, exploiting the weak dependence properties, conditions under which functional central limit theorems hold in distribution as well as consider general moments.

Later in the paper, we discuss a Generalized Method of Moments (GMM in short) procedure to estimate the parameters of a supOU and supOU SV model. Unfortunately, the classical and efficient maximum-likelihood approach seems not applicable in this case, since the density of the supOU processes is not known in general. However, the supOU process has a known moment structure and GMM estimators can be defined as in [48]. In a semiparametric framework, we consider in detail the case in which the random parameter A is Gamma distributed and the moment functions are known in closed form. For the GMM estimators of the supOU process and the return process of a supOU SV model we show the asymptotic normality of both estimators (whose consistency has been shown in [48]). Finally, via an explicit computation of the third and fourth order cumulants of the supOU and return process, we give the explicit form of the asymptotic covariance matrices of the GMM estimators.

Interestingly, our result can also be seen as a first step in obtaining an estimation theory for the ambit processes (homogeneous and stationary) which include an additional multiplicative random input in the definition of an MMA process, see [4, 5, 8].

The paper is organized as follows. In Section 2, the definition of a Lévy basis and MMA process is given. In Section 3, the weak dependence properties of an MMA process are discussed. In Section 4, the asymptotic distributions of the moments of non-causal and causal MMA processes are shown. In Section 5, the definition of an MMA SV model is given and the θ-weak dependence of the return process is analyzed along with its sample moments asymptotic. In Section 6, the asymptotic normality of the GMM estimators of the supOU process and of the supOU SV model is then proven.

2. Lévy bases and mixed moving average processes

We start with some preliminary results leading to the definition of an MMA process. Throughout, we assume that all random variables and processes are defined on a given complete probability space $(\Omega, \mathcal{A}, \mathbb{P})$ equipped with a filtration when relevant. Let S denote a non-empty topological space, $\mathcal{B}(S)$ the Borel σ-field on S, π some probability measure on $(S, \mathcal{B}(S))$ and $\mathcal{B}_b(S \times \mathbb{R})$ the bounded

Borel sets of $S \times \mathbb{R}$. A Lévy basis, which is also known as an infinitely divisible independently scattered random measure, is defined as follows.

Definition 2.1. *A family* $\Lambda = \{\Lambda(B) : B \in \mathcal{B}_b(S \times \mathbb{R})\}$ *of* $\mathbb{R}^d$*-valued random variables is called an* $\mathbb{R}^d$*-valued Lévy basis on* $S \times \mathbb{R}$ *if:*

- *the distribution of* $\Lambda(B)$ *is infinitely divisible for all* $B \in \mathcal{B}_b(S \times \mathbb{R})$,
- *for arbitrary* $n \in \mathbb{N}$ *and pairwise disjoint sets* $B_1, \ldots, B_n \in \mathcal{B}_b(S \times \mathbb{R})$ *the random variables* $\Lambda(B_1), \ldots, \Lambda(B_n)$ *are independent and*
- *for any pairwise disjoint sets* $B_1, B_2, \ldots \in \mathcal{B}_b(S \times \mathbb{R})$ *with* $\bigcup_{n \in \mathbb{N}} B_n \in \mathcal{B}_b(S \times \mathbb{R})$ *we have, almost surely,* $\Lambda(\bigcup_{n \in \mathbb{N}} B_n) = \sum_{n \in \mathbb{N}} \Lambda(B_n)$.

We restrict ourselves to time-homogeneous and factorisable Lévy bases, i.e. Lévy bases with characteristic function

$$\mathbb{E}[e^{i\langle u, \Lambda(B)\rangle}] = e^{\Phi(u)\Pi(B)} \tag{2.1}$$

for all $u \in \mathbb{R}^d$ and $B \in \mathcal{B}_b(S \times \mathbb{R})$, where $\Pi = \pi \times \lambda$ is the product of a probability measure π on S and the Lebesgue measure λ on $\mathbb{R}$ and

$$\Phi(u) = i\langle \gamma, u \rangle - \frac{1}{2}\langle u, \Sigma u \rangle + \int_{\mathbb{R}^d} e^{i\langle u, x \rangle} - 1 - i\langle u, x \rangle \mathbb{1}_{[0,1]}(\|x\|) \; \nu(dx)$$

is the cumulant transform of an infinitely divisible (i.d. in short) distribution with characteristic triplet (γ, Σ, ν), where $\gamma \in \mathbb{R}^d$, $\Sigma \in \mathbb{S}_d^+$ - i.e. the space of the positive semi-definite matrices - and ν is a Lévy measure - a Borel measure on $\mathbb{R}^d$ with $\nu(0) = 0$ and $\int_{\mathbb{R}^d}(\|x\|^2 \wedge 1)\nu(dx) < \infty$. By L we denote the underlying Lévy process associated with (γ, Σ, ν) and given by

$$L_t = \Lambda(S \times (0, t]) \text{ and } L_{-t} = -\Lambda(S \times (-t, 0)) \text{ for } t \in \mathbb{R}^+.$$

The quadruple $(\gamma, \Sigma, \nu, \pi)$ determines the distribution of the Lévy basis completely and therefore it is called the generating quadruple.

In the following, norms of vectors or matrices are denoted by $\| \cdot \|$. We are going to work especially with the Euclidean norm or its induced operator norm unless otherwise stated. However, due to the equivalence of all norms none of the results in the paper depends on the choice of the norm. For more information on $\mathbb{R}^d$-valued Lévy bases see [43] and [45].

Following [43], it can be shown that a Lévy basis has a Lévy-Itô decomposition.

Theorem 2.1. *Let* Λ *be a homogeneous and factorisable* $\mathbb{R}^d$*-valued Lévy basis on* $S \times \mathbb{R}$ *with generating quadruple* $(\gamma, \Sigma, \nu, \pi)$. *Then there exists a modification* $\tilde{\Lambda}$ *of* Λ *which is also a Lévy basis with generating quadruple* $(\gamma, \Sigma, \nu, \pi)$ *such that there exists an* $\mathbb{R}^d$*-valued Lévy basis* $\tilde{\Lambda}^G$ *on* $S \times \mathbb{R}^d$ *with generating quadruple* $(0, \Sigma, 0, \pi)$ *and an independent Poisson random measure* μ *on* $(\mathbb{R}^d \times S \times \mathbb{R}, \mathcal{B}(\mathbb{R}^d \times S \times \mathbb{R}))$ *with intensity measure* $\nu \times \pi \times \lambda$ *which satisfy*

$$\tilde{\Lambda}(B) = \gamma(\pi \times \lambda)(B) + \tilde{\Lambda}^G(B) + \int_{\|x\| \leq 1} \int_B x \left(\mu(dx, dA, ds) - ds\pi(dA)\,\nu(dx)\right)$$

$$+ \int_{\|x\|>1} \int_B x \, \mu(dx, dA, ds) \tag{2.2}$$

for all $B \in \mathcal{B}_b(S \times \mathbb{R})$ and all $\omega \in \Omega$.
Provided $\int_{\|x\|\leq 1} \|x\| \nu(dx) < \infty$, it holds that

$$\tilde{\Lambda}(B) = \gamma_0 (\pi \times \lambda)(B) + \tilde{\Lambda}^G(B) + \int_{\mathbb{R}^d} \int_B x \, \mu(dx, dA, ds)$$

for all $B \in \mathcal{B}_b(S \times \mathbb{R})$ with

$$\gamma_0 := \gamma - \int_{\|x\|\leq 1} x \, \nu(dx). \tag{2.3}$$

Furthermore, the integral with respect to μ exists as a Lebesgue integral for all $\omega \in \Omega$.

Here an $\mathbb{R}^d$-valued Lévy basis $\tilde{\Lambda}$ on $S \times \mathbb{R}$ is called a modification of a Lévy basis Λ if $\tilde{\Lambda}(B) = \Lambda(B)$ a.s. for all $B \in \mathcal{B}_b(S \times \mathbb{R})$. We refer the reader to [34, Section 2.1] for further details on the integration with respect to Poisson random measures.

We also recall the following multivariate extension of [45, Theorem 2.7]. We denote by A' the transpose of a matrix A in what follows.

Theorem 2.2. *Let Λ be an $\mathbb{R}^d$-valued Lévy basis with generating quadruple $(\gamma, \Sigma, \nu, \pi)$, $f : S \times \mathbb{R} \to M_{n \times d}(\mathbb{R})$ be a $\mathcal{B}(S \times \mathbb{R})$-measurable function. Then f is Λ-integrable as a limit in probability in the sense of Rajput and Rosiński [45], if and only if*

$$\int_S \int_{\mathbb{R}} \left\| f(A,s)\gamma + \int_{\mathbb{R}^d} f(A,s)x\Big(\mathbb{1}_{[0,1]}(\|f(A,s)x\|) - \mathbb{1}_{[0,1]}(\|x\|)\Big) \, \nu(dx) \right\| ds \, \pi(dA) < \infty, \tag{2.4}$$

$$\int_S \int_{\mathbb{R}} \|f(A,s)\Sigma f(A,s)'\| \, ds \, \pi(dA) < \infty, \tag{2.5}$$

and

$$\int_S \int_{\mathbb{R}} \int_{\mathbb{R}^d} \left(1 \wedge \|f(A,s)x\|^2\right) \nu(dx) \, ds \, \pi(dA) < \infty. \tag{2.6}$$

If f is Λ-integrable, the distribution of $\int_S \int_{\mathbb{R}} f(A,s) \, \Lambda(dA, ds)$ is infinitely divisible with characteristic triplet $(\gamma_{int}, \Sigma_{int}, \nu_{int})$ given by

$$\gamma_{int} = \int_S \int_{\mathbb{R}} \left(f(A,s)\gamma + \int_{\mathbb{R}^d} f(A,s)x\Big(\mathbb{1}_{[0,1]}(\|f(A,s)x\|) - \mathbb{1}_{[0,1]}(\|x\|)\Big) \, \nu(dx)\right) ds \, \pi(dA) \tag{2.7}$$

$$\Sigma_{int} = \int_S \int_{\mathbb{R}} f(A,s)\Sigma f(A,s)' \, ds \, \pi(dA) \tag{2.8}$$

and

$$\nu_{int}(B) = \int_S \int_{\mathbb{R}} \int_{\mathbb{R}^d} \mathbb{1}_B(f(A,s)x) \, \nu(dx) \, ds \, \pi(dA) \tag{2.9}$$

for all Borel sets $B \subseteq \mathbb{R}^n \setminus \{0\}$.

Implicitly, we assume that Σ_{int} or ν_{int} are different from zero throughout the paper to rule out the deterministic case.

When the underlying Lévy process has finite variation we can do ω-wise Lebesgue integration; that is, the integral can be obtained as a Lebesgue integral for each $\omega \in \Omega$.

Corollary 2.1. *Let Λ be an $\mathbb{R}^d$-valued Lévy basis with characteristic quadruple $(\gamma, 0, \nu, \pi)$ satisfying $\int_{\|x\| \le 1} \|x\| \nu(dx) < \infty$, and define γ_0 as in (2.3), that is $\Phi(u) = i\langle u, \gamma_0 \rangle + \int (e^{i\langle u,x \rangle} - 1)\nu(dx)$. Furthermore, let $f : S \times \mathbb{R} \to M_{n \times d}$ be a $\mathcal{B}(S \times \mathbb{R})$-measurable function satisfying*

$$\int_S \int_{\mathbb{R}} \|f(A, s)\gamma_0\| ds\, \pi(dA) < \infty, \tag{2.10}$$

and

$$\int_S \int_{\mathbb{R}} \int_{\mathbb{R}^d} \left(1 \wedge \|f(A, s)x\| \right) \nu(dx)\, ds\, \pi(dA) < \infty. \tag{2.11}$$

Then,

$$\int_S \int_{\mathbb{R}} f(A, s)\Lambda(dA, ds) = \int_S \int_{\mathbb{R}} f(A, s)\, \gamma_0\, ds\pi(dA)$$

$$+ \int_{\mathbb{R}^d} \int_S \int_{\mathbb{R}} f(A, s)x\, \mu(dx, dA, ds), \tag{2.12}$$

and the right hand side is a Lebesgue integral for every $\omega \in \Omega$. Moreover, the distribution $\int_S \int_{\mathbb{R}} f(A, s)\, \Lambda(dA, ds)$ is infinitely divisible with characteristic function

$$\mathbb{E}\Big(\exp\Big(i\langle u, \int_S \int_{\mathbb{R}} f(A, s)\, \Lambda(dA, ds) \rangle \Big) \Big) = e^{i\langle u, \gamma_{int,0} \rangle + \int_{\mathbb{R}^n} (e^{i\langle u,x \rangle} - 1)\nu_{int}(dx)} \quad u \in \mathbb{R}^n,$$

where

$$\gamma_{int} = \int_S \int_{\mathbb{R}} f(A, s)\gamma_0\, ds\, \pi(dA),$$

$$\nu_{int}(B) = \int_S \int_{\mathbb{R}} \int_{\mathbb{R}^d} \mathbb{1}_B(f(A, s)x)\, \nu(dx)\, ds\, \pi(dA) \ \ \forall B \in \mathbb{R}^n \setminus \{0\}.$$

The above corollary follows immediately from the Lévy-Itô decomposition (2.2) and the usual integration theory with respect to a Poisson random measure. We notice that the result (2.12) is an immediate consequence of working with an underlying Lévy process of finite variation, as no compensation for the small jumps is needed if $\int_{\|x\| \le 1} \|x\| \nu(dx) < \infty$.

We can now introduce an MMA process driven by a Lévy basis.

Definition 2.2. *Let Λ be an $\mathbb{R}^d$-valued Lévy basis on $S \times \mathbb{R}$ and let $f : S \times \mathbb{R} \to M_{n \times d}(\mathbb{R})$ be a $\mathcal{B}(S \times \mathbb{R})$-measurable function satisfying assumptions (2.4), (2.5) and (2.6). Then, the process*

$$X_t : = \int_S \int_{\mathbb{R}} f(A, t - s)\, \Lambda(dA, ds) \tag{2.13}$$

is well defined for each $t \in \mathbb{R}$, infinitely divisible and strictly stationary. It is called a n-dimensional mixed moving average process and f its kernel function.

We conclude the section giving sufficient conditions ensuring the finiteness of moments of an MMA process.

Proposition 2.1. *Let X be an n-dimensional MMA process driven by a Lévy basis Λ satisfying the conditions of Theorem 2.2.*

(i) *If $\int_{\|x\|>1} \|x\|^r \nu(dx) < \infty$ and $f \in L^r(S \times \mathbb{R}, \pi \otimes \lambda)$ for $r \in [2, \infty)$, then $\mathbb{E}[\|X_t\|^r] < \infty$.*

(ii) *If $\int_{\|x\|>1} \|x\|^r \nu(dx) < \infty$ and $f \in L^r(S \times \mathbb{R}, \pi \otimes \lambda) \cap L^2(S \times \mathbb{R}, \pi \otimes \lambda)$ for $r \in (0, 2)$, then $\mathbb{E}[\|X_t\|^r] < \infty$.*

Proof. Following [47, Corollary 25.8], we have to show that $\int_{\|x\|>1} \|x\|^r \nu_{int}(dx) < \infty$. Since

$$\int_{\|x\|>1} \|x\|^r \nu_{int}(dx)$$

$$= \int_S \int_{\mathbb{R}} \int_{\mathbb{R}^d} \|f(A,s)x\|^r 1_{(1,\infty)}(\|f(A,s)x\|)\nu(dx)ds\pi(dA)$$

$$\leq \int_S \int_{\mathbb{R}} \int_{\mathbb{R}^d} \|f(A,s)\|^r \|x\|^r 1_{(1,\infty)}(\|x\|)\nu(dx)ds\pi(dA)$$

$$+ \int_S \int_{\mathbb{R}} \int_{\mathbb{R}^d} \|f(A,s)\|^{r\vee 2} \|x\|^{r\vee 2} 1_{(0,1)}(\|x\|)1_{(1,\infty)}(\|f(A,s)x\|)\nu(dx)ds\pi(dA),$$

we can conclude that (i) and (ii) follow, given that ν is a Lévy measure. $\qquad \square$

If the underlying Lévy process L is of finite variation, an analogous proof gives the following results.

Corollary 2.2. *Let X be an n-dimensional MMA process driven by a Lévy basis Λ satisfying the conditions of Corollary 2.1.*

(i) *If $\int_{\|x\|>1} \|x\|^r \nu(dx) < \infty$ and $f \in L^r(S \times \mathbb{R}, \pi \otimes \lambda)$ for $r \in [1, \infty)$, then $\mathbb{E}[\|X_t\|^r] < \infty$.*

(ii) *If $\int_{\|x\|>1} \|x\|^r \nu(dx) < \infty$ and $f \in L^r(S \times \mathbb{R}, \pi \otimes \lambda) \cap L^1(S \times \mathbb{R}, \pi \otimes \lambda)$ for $r \in (0, 1)$, then $\mathbb{E}[\|X_t\|^r] < \infty$.*

3. Weak dependence properties of a mixed moving average process

Let $(\mathcal{A}_t)_{t\in\mathbb{R}}$ be the filtration generated by Λ defined as the σ-algebras $\mathcal{A}_t$ generated by the set of random variables $\{\Lambda(B) : B \in \mathcal{B}(S \times (-\infty, t])\}$ for $t \in \mathbb{R}$. If an MMA process is adapted to $(\mathcal{A}_t)_{t\in\mathbb{R}}$, we call it causal. Otherwise it is referred to as being non-causal.

In the following we will refer by $\mathbb{N}$ to the set of the non negative integers, by $\mathbb{N}^*$ to the set of the positive integers, by $\mathbb{R}^-$ to the set of negative real numbers and by $\mathbb{R}^+$ to the set of the non negative real numbers.

3.1. Non-causal case

Let

$$\mathcal{F} = \bigcup_{u \in \mathbb{N}^*} \mathcal{F}_u$$

where $\mathcal{F}_u$ is the class of bounded functions from $(\mathbb{R}^n)^u$ to $\mathbb{R}$ Lipschitz with respect to a distance δ_1 on $(\mathbb{R}^n)^u$ defined by

$$\delta_1(x^*, y^*) = \sum_{i=1}^{u} \delta(x_i, y_i), \tag{3.1}$$

where $x^* = (x_1, \ldots, x_u)$ and $y^* = (y_1, \ldots, y_u)$ and $x_i, y_i \in \mathbb{R}^n$ for all $i = 1, \ldots, u$. We consider $\mathbb{R}^n$ equipped with the Euclidean norm and then $\delta(x_i, y_i) = \|x_i - y_i\|$.

Definition 3.1. *A process $X = (X_t)_{t \in \mathbb{R}}$ with values in $\mathbb{R}^n$ is called an η-weakly dependent process if there exists a sequence $(\eta(r))_{r \in \mathbb{R}^+}$ converging to 0, satisfying*

$$|Cov(F(X_{i_1}, \ldots, X_{i_u}), G(X_{j_1}, \ldots, X_{j_v}))| \leq c \left(u Lip(F) \|G\|_\infty + v Lip(G) \|F\|_\infty \right) \eta(r) \tag{3.2}$$

for all

$$\begin{cases} (u, v) \in \mathbb{N}^* \times \mathbb{N}^*; \\ r \in \mathbb{R}^+; \\ (i_1, \ldots, i_u) \in \mathbb{R}^u \ and \ (j_1, \ldots, j_v) \in \mathbb{R}^v, \\ with \ i_1 \leq \ldots \leq i_u \leq i_u + r \leq j_1 \leq \ldots \leq j_v; \\ functions \ F \colon (\mathbb{R}^n)^u \to \mathbb{R} \ and \ G \colon (\mathbb{R}^n)^v \to \mathbb{R} \ belonging \ to \ \mathcal{F} \ and \ satisfying \\ \|G\|_\infty \leq 1, \ \|F\|_\infty \leq 1 \ and \ Lip(F) + Lip(G) < \infty, \\ where \ Lip(F) = \sup_{x \neq y} \frac{|F(x) - F(y)|}{\|x_1 - y_1\| + \|x_2 - y_2\| + \ldots + \|x_n - y_n\|}, \end{cases}$$

and where c is a constant independent of r. We call $(\eta(r))_{r \in \mathbb{R}^+}$ the sequence of the η-coefficients.

The above definition makes the asymptotic independence between *past* and *future* explicit, this means that the *past* is progressively forgotten. In terms of the initial process X, *past* and *future* are elementary events respectively defined through finite-dimensional marginals as $A_u = (X_{i_1}, \ldots, X_{i_u})$ and $B_v = (X_{j_1}, \ldots, X_{j_v})$ for $i_1 \leq \ldots \leq i_u \leq i_u + r \leq j_1 \leq \ldots \leq j_v$ and $r \geq 0$.

The weak dependence property, as stated in Definition 3.1, depends upon the class of functions $\mathcal{H} = \{f \in \mathcal{F} : \|f\|_\infty \leq 1\}$. However, it can also be defined in $\mathcal{F}$ as discussed in [23]. Note, a similar definition can be given for the strong mixing property introduced by Rosenblatt [46]. Let $\sigma(A_u)$ and $\sigma(B_v)$ be the σ-algebras generated by the finite-dimensional marginals A_u and B_v and $\mathcal{F}^* = \bigcup_{u \in \mathbb{N}} \mathcal{F}_u^*$ where $\mathcal{F}_u^*$ is the class of bounded functions from $(\mathbb{R}^n)^u$ to $\mathbb{R}$. We define

$$\alpha(\sigma(A_u), \sigma(B_v)) = \sup_{f, g \in \mathcal{H}^*} |Cov(f(A_u), g(B_v))|, \tag{3.3}$$

where $\mathcal{H}^* = \{f \in \mathcal{F}^* : \|f\|_\infty \leq 1\}$, and then α-strong mixing coefficient is

$$\alpha(r) = \sup_{u,v} \max_{\substack{i_1 \leq \ldots \leq i_u \\ j_i \leq \ldots \leq j_v \\ r = j_1 - i_u}} \alpha(\sigma(A_u), \sigma(B_v)).$$

We notice that definition (3.2) holds for a set of functions in $\mathcal{H}$ whereas (3.3) holds for functions belonging to $\mathcal{H}^*$. This means that if a process X is strongly mixing then it is also weakly dependent but the reverse implication does not necessarily hold. The only known case of the equivalence of the two definitions can be found in [22, Proposition 1] where it is shown that an η-weakly dependent integer valued process satisfies the strong mixing condition.

We now show that a non-causal MMA process is an η-weakly dependent process.

Proposition 3.1. *Let Λ be an $\mathbb{R}^d$-valued Lévy basis with characteristic quadruple $(\gamma, \Sigma, \nu, \pi)$ such that $\mathbb{E}[L_1] = 0$ and $\int_{\|x\|>1} \|x\|^2 \nu(dx) < \infty$, $f : S \times \mathbb{R} \to M_{n \times d}(\mathbb{R})$ a $\mathcal{B}(S \times \mathbb{R})$-measurable function and $f \in L^2(S \times \mathbb{R}, \pi \otimes \lambda)$. Then, the resulting MMA process X is an η-weakly dependent process with coefficients*

$$\eta_X(r) = \Big(\int_S \int_{\mathbb{R}} tr(f(A, -s)\Sigma_L f(A, -s)') \mathbb{1}_{(-\infty, -r)}(2s)\, ds\, \pi(dA) \Big)^{\frac{1}{2}} \qquad (3.4)$$

$$+ \Big(\int_S \int_{\mathbb{R}} tr(f(A, -s)\Sigma_L f(A, -s)') \mathbb{1}_{(r, +\infty)}(2s)\, ds\, \pi(dA) \Big)^{\frac{1}{2}},$$

for all $r \geq 0$, where $\mathbb{E}[L_1 L_1'] = \Sigma_L = \Sigma + \int_{\mathbb{R}^d} xx' \nu(dx)$.

Proof. First, we define $\forall t \in \mathbb{R}$ and $m \geq 0$ the truncated sequence

$$X_t^{(m)} = \int_S \int_{\mathbb{R}} f(A, t-s) \mathbb{1}_{[-m,m]}(t-s)\, \Lambda(dA, ds) = \int_S \int_{t-m}^{t+m} f(A, t-s)\, \Lambda(dA, ds).$$

$$(3.5)$$

Since the kernel function f is square integrable, we have that properties (2.5) and (2.6) hold and so f is Λ-integrable (Theorem 2.2) and X is well defined. Moreover, Proposition 2.1 holds, $\mathbb{E}[X_t^2] < \infty$ for all $t \in \mathbb{R}$ and we can determine an upper bound of the expectation

$$\mathbb{E}\|X_t - X_t^{(m)}\| = \mathbb{E}\Big\| \int_S \int_{-\infty}^{t-m} f(A, t-s)\, \Lambda(dA, ds) + \int_S \int_{t+m}^{+\infty} f(A, t-s)\Lambda(dA, ds) \Big\|$$

$$\leq \Big(\mathbb{E}\Big\| \int_S \int_{-\infty}^{t-m} f(A, t-s)\, \Lambda(dA, ds) \Big\|^2 \Big)^{\frac{1}{2}} + \Big(\mathbb{E}\Big\| \int_S \int_{t+m}^{\infty} f(A, t-s)\, \Lambda(dA, ds) \Big\|^2 \Big)^{\frac{1}{2}}.$$

Due to the stationarity of X the above estimation is independent of t and equal to

$$\Big(\int_S \int_{-\infty}^{-m} tr(f(A, -s)\Sigma_L f(A, -s)')\, ds\pi(dA) \Big)^{\frac{1}{2}}$$

$$+ \Big(\int_S \int_m^{\infty} tr(f(A, -s)\Sigma_L f(A, -s)')\, ds\pi(dA) \Big)^{\frac{1}{2}}$$

$$(3.6)$$

Let F and G belong to the class of bounded functions $\mathcal{H}$, $(u,v) \in \mathbb{N}^* \times \mathbb{N}^*$, $r \in \mathbb{R}^+$, $(i_1, \ldots, i_u) \in \mathbb{R}^u$ and $(j_1, \ldots, j_v) \in \mathbb{R}^v$ with $i_1 \leq \ldots \leq i_u \leq i_u + r \leq j_1 \leq \ldots \leq j_v$, $X_i^* = (X_{i_1}, \ldots, X_{i_u})$, $X_j^* = (X_{j_1}, \ldots, X_{j_v})$, and $X_i^{*(m)} = (X_{i_1}^{(m)}, \ldots, X_{i_u}^{(m)})$, $X_j^{*(m)} = (X_{j_1}^{(m)}, \ldots, X_{j_v}^{(m)})$ where for all $m \geq 0$

$$X_{i_u}^{(m)} = \int_S \int_{i_u-m}^{i_u+m} f(A, i_u - s) \, \Lambda(dA, ds)$$

and $\hspace{11cm}$ (3.7)

$$X_{j_1}^{(m)} = \int_S \int_{j_1-m}^{j_1+m} f(A, j_1 - s) \, \Lambda(dA, ds).$$

Then, if $j_1 - m - i_u - m \geq 0$, that can also be expressed as $j_1 - i_u \geq 2m$, $I_u = S \times [i_u - m, i_u + m]$ and $J_1 = S \times [j_1 - m, j_1 + m]$ are disjoint sets or they have intersection $S \times \{j_1 - m\}$ when $j_1 - i_u = 2m$. Noting that $\pi \times \lambda(S \times \{j_1 - m\}) = 0$ by the definition of a Lévy basis, the two sequences $(X_i^{(m)})_{i_1, \ldots, i_u}$ and $(X_j^{(m)})_{j_1, \ldots, j_v}$ are independent and so are $F(X_i^{*(m)})$ and $G(X_j^{*(m)})$. Therefore,

$$|Cov(F(X_i^*), G(X_j^*))|$$

$$\leq |Cov(F(X_i^*) - F(X_i^{*(m)}), G(X_j^*))| + |Cov(F(X_i^{*(m)}), G(X_j^*) - G(X_j^{*(m)}))|$$

$$\leq 2(\mathbb{E}|F(X_i^*) - F(X_i^{*(m)})| + \mathbb{E}|G(X_j^*) - G(X_j^{*(m)})|)$$

the last relation comes from $\|F\|_\infty, \|G\|_\infty \leq 1$

$$\leq 2\Big(Lip(F) \sum_{l=1}^u \mathbb{E}\|X_{i_l} - X_{i_l}^{(m)}\| + Lip(G) \sum_{k=1}^v \mathbb{E}\|X_{j_k} - X_{j_k}^{(m)}\|\Big)$$

using the result (3.6) for $m = \frac{r}{2}$

$$\leq 2(uLip(F) + vLip(G))$$

$$\times \Big\{\Big(\int_S \int_{\mathbb{R}} tr(f(A, -s)\Sigma_L f(A, -s)') \mathbb{1}_{(-\infty, -r)}(2s) \, ds\pi(dA)\Big)^{\frac{1}{2}}$$

$$+ \Big(\int_S \int_{\mathbb{R}} tr(f(A, -s)\Sigma_L f(A, -s)') \mathbb{1}_{(r, +\infty)}(2s) \, ds\pi(dA)\Big)^{\frac{1}{2}}\Big\}$$

$$= 2(uLip(F) + vLip(G))\eta_X(r),$$

which converges to zero as r goes to infinity by applying the dominated convergence theorem. $\hspace{1cm}$ □

The following Corollary establishes the η weak dependence of an MMA when its underlying Lévy process has finite mean possibly different from zero.

Corollary 3.1. *Let Λ be an $\mathbb{R}^d$-valued Lévy basis with characteristic quadruple $(\gamma, \Sigma, \nu, \pi)$ and $\int_{\|x\|>1} \|x\|^2 \nu(dx) < \infty$, $f : S \times \mathbb{R} \to M_{n \times d}(\mathbb{R})$ a $\mathcal{B}(S \times \mathbb{R})$-measurable function satisfying assumption (2.4) and $f \in L^2(S \times \mathbb{R}, \pi \otimes \lambda)$. Then, the resulting MMA process X is an η-weakly dependent process with coefficients*

$$
\begin{aligned}
\eta_X(r) = \Big(&\int_S \int_{\mathbb{R}} tr(f(A, -s)\Sigma_L f(A, -s)') \mathbb{1}_{(-\infty, -r)}(2s)\, ds\, \pi(dA) \\
&+ \Big\| \int_S \int_{\mathbb{R}} f(A, -s)\mu\, \mathbb{1}_{(-\infty, -r)}(2s)\, ds\, \pi(dA) \Big\|^2 \Big)^{\frac{1}{2}} \\
&+ \Big(\int_S \int_{\mathbb{R}} tr(f(A, -s)\Sigma_L f(A, -s)') \mathbb{1}_{(r, +\infty)}(2s)\, ds\, \pi(dA) \\
&+ \Big\| \int_S \int_{\mathbb{R}} f(A, -s)\mu\, \mathbb{1}_{(r, +\infty)}(2s)\, ds\, \pi(dA) \Big\|^2 \Big)^{\frac{1}{2}}
\end{aligned}
\tag{3.8}
$$

for all $r \geq 0$, where $E[L_1] = \mu = \gamma + \int_{\|x\|>1} x\, \nu(dx)$ and $\mathbb{E}[L_1 L_1'] = \Sigma_L = \Sigma + \int_{\mathbb{R}^d} xx'\, \nu(dx)$.

Proof. Let $X_t^{(m)}$ defined as in Proposition 3.1 for all $t \in \mathbb{R}$ and $m \geq 0$. Then, (3.6) becomes

$$
\begin{aligned}
\Big(&\int_S \int_{\mathbb{R}} tr(f(A, -s)\Sigma_L f(A, -s)') \mathbb{1}_{(-\infty, -m)}(s)\, ds\, \pi(dA) \\
&+ \Big\| \int_S \int_{\mathbb{R}} f(A, -s)\mu \mathbb{1}_{(-\infty, -m)}(s)\, ds\, \pi(dA) \Big\|^2 \Big)^{\frac{1}{2}} \\
&+ \Big(\int_S \int_{\mathbb{R}} tr(f(A, -s)\Sigma_L f(A, -s)') \mathbb{1}_{(m, +\infty)}(s)\, ds\, \pi(dA) \\
&+ \Big\| \int_S \int_{\mathbb{R}} f(A, -s)\mu \mathbb{1}_{(m, +\infty)}(s)\, ds\, \pi(dA) \Big\|^2 \Big)^{\frac{1}{2}}
\end{aligned}
$$

Proceeding as in the proof of Proposition 3.1, the η-coefficients (3.8) are obtained. $\quad\square$

When the underlying Lévy process is of finite variation we can lighten the moment assumptions on the MMA process. The below result applies to all finite variation MMA process with finite mean.

Corollary 3.2. *Let Λ be an $\mathbb{R}^d$-valued Lévy basis with characteristic quadruple $(\gamma, 0, \nu, \pi)$ such that $\int_{\mathbb{R}^d} \|x\| \nu(dx) < \infty$, $f : S \times \mathbb{R} \to M_{n \times d}(\mathbb{R})$ a $\mathcal{B}(S \times \mathbb{R})$-measurable function and $f \in L^1(S \times \mathbb{R}, \pi \otimes \lambda)$ and define γ_0 as in (2.3). Then, the resulting MMA process X is an η-weakly dependent process with coefficients*

$$
\eta_X(r) = \int_S \int_{\mathbb{R}} \int_{\mathbb{R}^d} \|f(A, -s)x\|\, \mathbb{1}_{(-\infty, -r) \cup (r, +\infty)}(2s)\, \nu(dx)\, ds\, \pi(dA)
\tag{3.9}
$$

$$
+ \int_S \int_{\mathbb{R}} \|f(A, -s)\gamma_0\|\, \mathbb{1}_{(-\infty, -r) \cup (r, +\infty)}(2s)\, ds\, \pi(dA),
$$

for all $r \geq 0$.

Proof. As the kernel function f is in L^1, the properties (2.10) and (2.11) are satisfied. Thus, f is Λ-integrable and X well defined. Moreover, because of Corollary 2.2, $\mathbb{E}[X_t] < \infty$. Using the notation of Proposition 3.1, for all $t \in \mathbb{R}$ and $m \geq 0$

$$\mathbb{E}\|X_t - X_t^{(m)}\| \leq \int_S \int_{\mathbb{R}} \int_{\mathbb{R}^d} \|f(A,-s)x\| \mathbb{1}_{(-\infty,-m) \cup (m,+\infty)}(s) \, \nu(dx) \, ds \, \pi(dA),$$

$$+ \int_S \int_{\mathbb{R}} \|f(A,-s)\gamma_0\| \mathbb{1}_{(-\infty,-m) \cup (m,+\infty)}(s) \, ds \, \pi(dA),$$

where $X_t^{(m)}$ is the truncated sequence (3.5). Thus, for $m = \frac{r}{2}$ and F, G, X_i^* and X_j^*

$$|Cov(F(X_i^*), G(X_j^*))| \leq 2(uLip(F) + vLip(G))$$

$$\times \Big\{ \int_S \int_{\mathbb{R}} \int_{\mathbb{R}^d} \|f(A,-s)x\| \mathbb{1}_{(-\infty,-r) \cup (r,+\infty)}(2s) \, \nu(dx) \, ds \, \pi(dA)$$

$$+ \int_S \int_{\mathbb{R}} \|f(A,-s)\gamma_0\| \mathbb{1}_{(-\infty,-r) \cup (r,+\infty)}(2s) \, ds \, \pi(dA) \Big\}$$

$$= 2(uLip(F) + vLip(G))\eta_X(r).$$

Finally, we conclude by applying the dominated convergence theorem. $\qquad\square$

The η coefficients have some hereditary properties. For example, let $h : \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz function, then if the sequence $(X_t)_{t \in \mathbb{R}}$ is η-weakly dependent, the same is true for the sequence $(h(X_t))_{t \in \mathbb{R}}$. The latter can be readily checked directly based on Definition 3.1. Hereditary properties for functions that are not Lipschitz on the whole space $\mathbb{R}^n$ can be found in [2] Lemma 6, for stationary processes. Below follows a multivariate extension of this Lemma for $h : \mathbb{R}^n \to \mathbb{R}^m$.

Proposition 3.2. *Let $(X_t)_{t \in \mathbb{R}}$ be an $\mathbb{R}^n$-valued stationary process and assume there exists some constant $C > 0$ such that $\mathbb{E}[\|X_0\|^p]^{\frac{1}{p}} \leq C$, with $p > 1$, $h : \mathbb{R}^n \to \mathbb{R}^m$ be a function such that $h(0) = 0$, $h(x) = (h_1(x), \ldots, h_m(x))$ and*

$$\|h(x) - h(y)\| \leq c \|x - y\|(1 + \|x\|^{a-1} + \|y\|^{a-1}), \qquad (3.10)$$

for $x, y \in \mathbb{R}^n$, $c > 0$ and $1 \leq a < p$. Define $(Y_t)_{t \in \mathbb{R}}$ by $Y_t = h(X_t)$. If $(X_t)_{t \in \mathbb{R}}$ is an η-weakly dependent process, then $(Y_t)_{t \in \mathbb{R}}$ is an η-weakly dependent process such that

$$\forall r \geq 0, \ \eta_Y(r) = \mathcal{C} \, \eta_X(r)^{\frac{p-a}{p-1}},$$

with the constant $\mathcal{C}$ independent of r.

Proof. For $(u,v) \in \mathbb{N}^* \times \mathbb{N}^*$, $(i_1, \ldots, i_u) \in \mathbb{R}^u$ and $(j_1, \ldots, j_v) \in \mathbb{R}^v$ with $i_1 \leq \ldots \leq i_u \leq i_u + r \leq j_1 \leq \ldots \leq j_v$, let us call

$$Y_i^* = (h(X_{i_1}), \ldots, h(X_{i_u})), \ Y_j^* = (h(X_{j_1}), \ldots, h(X_{j_v})).$$

Let $F : \mathbb{R}^{mu} \to \mathbb{R}$, $G : \mathbb{R}^{mv} \to \mathbb{R}$ bounded, such that $\|F\|_\infty, \|G\|_\infty \leq 1$, and Lipschitz functions with respect to the distance (3.1), then

$$F(Y_i^*) = F(h(X_{i_1}), \ldots, h(X_{i_u})), \ G(Y_j^*) = F(h(X_{j_1}), \ldots, h(X_{j_v})),$$

and

$$F^{(M)}(Y_i^*) = F(h(X_{i_1}^{(M)}), \ldots, h(X_{i_u}^{(M)})), \ G^{(M)}(Y_j^*) = F(h(X_{j_1}^{(M)}), \ldots, h(X_{j_v}^{(M)})),$$

where $X_i^{(M)} = X_i \mathbb{1}_{\|X_i\| \leq M}$ and w.l.o.g $M > 1$. According to Definition 3.1, we start by analyzing

$$
\begin{aligned}
|Cov(F(Y_i^*), G(Y_j^*))| &\leq |Cov(F(Y_i^*) - F^{(M)}(Y_i^*), G(Y_j^*))| \\
&+ |Cov(F^{(M)}(Y_i^*), G(Y_j^*) - G^{(M)}(Y_j^*))| + |Cov(F^{(M)}(Y_i^*), G^{(M)}(Y_j^*))|.
\end{aligned}
\tag{3.11}
$$

We have that

$$|Cov(F(Y_i^*) - F^{(M)}(Y_i^*), G(Y_j^*))| \leq 2\|G\|_\infty \mathbb{E}|F(Y_i^*) - F^{(M)}(Y_i^*)| \tag{3.12}$$

$$\leq 2 Lip(F) \sum_{l=1}^{u} \mathbb{E}\|h(X_{i_l}) - h(X_{i_l}^{(M)})\|.$$

By assumption, for each $l = 1, \ldots, u$

$$\mathbb{E}(\|h(X_{i_l}) - h(X_{i_l}^{(M)})\|) \leq c \, \mathbb{E}(\|X_{i_l} - X_{i_l}^{(M)}\|(1 + \|X_{i_l}\|^{a-1} + \|X_{i_l}^{(M)}\|^{a-1})) \tag{3.13}$$

$$\leq c \, \mathbb{E}(\|X_{i_l}\| \mathbb{1}_{\|X_{i_l}\|>M}) + c \, \mathbb{E}(\|X_{i_l}\|^{a-1}\|X_{i_l}\| \mathbb{1}_{\|X_{i_l}\|>M})$$

$$+ c \, \mathbb{E}(\|X_{i_l}\| \mathbb{1}_{\|X_{i_l}\|>M}) M^{a-1}$$

$$\leq c \, \mathbb{E}\Big(\|X_{i_l}\| \frac{\|X_{i_l}\|^{p-1}}{M^{p-1}} \mathbb{1}_{\|X_{i_l}\|>M}\Big) + c \, \mathbb{E}\Big(\|X_{i_l}\|^{a-1} \frac{\|X_{i_l}\|^{p-a}}{M^{p-a}} \|X_{i_l}\| \mathbb{1}_{\|X_{i_l}\|>M}\Big)$$

$$+ c \, \mathbb{E}\Big(\|X_{i_l}\| \frac{\|X_{i_l}\|^{p-1}}{M^{p-1}} \mathbb{1}_{\|X_{i_l}\|>M}\Big) M^{a-1}$$

$$\leq c \, \mathbb{E}(\|X_{i_l}\|^p \mathbb{1}_{\|X_{i_l}\|>M}) M^{1-p} + 2c \, \mathbb{E}(\|X_{i_l}\|^p \mathbb{1}_{\|X_{i_l}\|>M}) M^{a-p}$$

$$\leq 3c \, \mathbb{E}(\|X_{i_l}\|^p) M^{a-p}.$$

Therefore (3.12) is less than or equal to $6c \, u \, Lip(F) C^p M^{a-p}$. An analogous bound holds for $|Cov(F^{(M)}(Y_i^*), G(Y_j^*) - G^{(M)}(Y_j^*))|$. Moreover, $F^{(M)}$ is a Lipschitz function on the set $\mathcal{A} = \{x = (x_1, \ldots, x_u) \in \mathbb{R}^{nu} : \|x_i\| \leq M \text{ for } i = 1, \ldots, u\}$. Let $Z^{(M)}, W^{(M)} \in \mathcal{A}$, then

$$|F(h(Z_1^{(M)}), \ldots, h(Z_u^{(M)})) - F(h(W_1^{(M)}), \ldots, h(W_u^{(M)}))|$$

$$\leq Lip(F) \sum_{l=1}^{u} \|h(Z_l^{(M)}) - h(W_l^{(M)})\|$$

$$\leq c\,Lip(F)\sum_{l=1}^{u}\|Z_l^{(M)}-W_l^{(M)}\|(1+\|Z_l^{(M)}\|^{a-1}\|W_l^{(M)}\|^{a-1})$$

$$\leq 3c\,Lip(F)M^{a-1}\sum_{l=1}^{u}\|Z_l^{(M)}-W_l^{(M)}\|.$$

The same argument holds also for the function $G^{(M)}$.

$X_t^{(M)}$ is a process with values in $\mathcal{A}$ and η-weakly dependent with the same coefficients as X_t, then

$$|Cov(F^{(M)}(Y_i^*),G^{(M)}(Y_j^*))| =$$

$$\leq 3c\,(uLip(F)+vLip(G))M^{a-1}\eta_X(r).$$

To conclude, (3.11) is less than or equal to

$$6c\,(uLip(F)+vLip(G))\Big(\frac{M^{a-1}}{2}\eta_X(r)+C^pM^{a-p}\Big).$$

By choosing $M=\eta_X(r)^{\frac{1}{1-p}}$ and calling $\mathcal{C}=6c(C^p+\frac{1}{2})$, we obtain that

$$\eta_Y(r)=\mathcal{C}\,\eta_X(r)^{\frac{p-a}{p-1}}. \qquad\qquad \square$$

For a polynomial function $h(x)$ we have

Corollary 3.3. *Let $(X_t)_{t\in\mathbb{R}}$ be an $\mathbb{R}^n$-valued stationary process and assume there exists some constant $C>0$ such that $\mathbb{E}[\|X_0\|^p]^{\frac{1}{p}}\leq C$, with $p>1$, $h\colon\mathbb{R}^n\to\mathbb{R}^m$ be a function such that $h(0)=0$ and $h(x)=(h_1(x),\ldots,h_m(x))$ with $h_s(\cdot)$ for $s=1,\ldots,m$ being a polynomial with degree at most a for $1\leq a<p$. Define $(Y_t)_{t\in\mathbb{R}}$ by $Y_t=h(X_t)$ for $t\in\mathbb{R}$ an $\mathbb{R}^m$-valued process. If $(X_t)_{t\in\mathbb{R}}$ is an η-weakly dependent process, then $(Y_t)_{t\in\mathbb{R}}$ is an η-weakly dependent process such that*

$$\forall\,r\geq 0,\ \ \eta_Y(r)=\mathcal{C}\,\eta_X(r)^{\frac{p-a}{p-1}},$$

with the constant $\mathcal{C}$ independent of r.

Proof. The function h satisfies the assumption (3.10) for each polynomial degree a less than p. Proposition 3.2 can then be applied. $\square$

3.2. Causal case

Definition 3.2. *A process $X=(X_t)_{t\in\mathbb{R}}$ with values in $\mathbb{R}^n$ is called a θ-weakly dependent process if there exists a sequence $(\theta(r))_{r\in\mathbb{R}^+}$ converging to 0, satisfying*

$$|Cov(F(X_{i_1},\ldots,X_{i_u}),G(X_{j_1},\ldots,X_{j_v}))| \leq c\,(vLip(G)\|F\|_\infty)\theta(r) \qquad (3.14)$$

for all

$$
\begin{cases}
(u, v) \in \mathbb{N}^* \times \mathbb{N}^*, \\
r \in \mathbb{R}^+; \\
(i_1, \ldots, i_u) \in \mathbb{R}^u \text{ and } (j_1, \ldots, j_v) \in \mathbb{R}^v, \\
\text{with } i_1 \leq \ldots \leq i_u \leq i_u + r \leq j_1 \leq \ldots \leq j_v; \\
\text{functions } F \colon (\mathbb{R}^n)^u \to \mathbb{R} \text{ and } G \colon (\mathbb{R}^n)^v \to \mathbb{R} \text{ respectively belonging to } \mathcal{H}^* \text{ and } \mathcal{H}, \\
\text{where } Lip(G) = \sup_{x \neq y} \frac{|G(x) - G(y)|}{\|x_1 - y_1\| + \|x_2 - y_2\| + \ldots + \|x_n - y_n\|},
\end{cases}
$$

and where c is a constant independent of r. We call $(\theta(r))_{r \in \mathbb{R}^+}$ the sequence of the θ-coefficients.

The θ-weak dependence condition is stronger than the one for η-weak dependence. Hence, moment conditions and decay demands on the rate of the θ-coefficients for central limit theorems are typically weaker than in the case of η-weak dependence, see [18]. It should be also noticed that $\eta(r) \leq \theta(r)$ and that in the case of integer valued processes, [22], the θ-weak dependence implies the strong mixing condition.

A causal MMA process is defined as follows

Definition 3.3. *Let Λ be an $\mathbb{R}^d$-valued Lévy basis on $S \times \mathbb{R}^+$ and let $f :$ $S \times \mathbb{R}^+ \to M_{n \times d}(\mathbb{R})$ be a $\mathcal{B}(S \times \mathbb{R}^+)$-measurable function satisfying assumptions (2.4), (2.5) and (2.6). Then, the process*

$$
X_t \colon = \int_S \int_{-\infty}^t f(A, t - s) \, \Lambda(dA, ds) \tag{3.15}
$$

is well defined for each $t \in \mathbb{R}$, infinitely divisible and strictly stationary. It is called a causal n-dimensional mixed moving average process and f its kernel function.

Proposition 3.3. *Let Λ be an $\mathbb{R}^d$-valued Lévy basis with characteristic quadruple $(\gamma, \Sigma, \nu, \pi)$ such that $\mathbb{E}[L_1] = 0$ and $\int_{\|x\| > 1} \|x\|^2 \nu(dx) < \infty$, $f : S \times \mathbb{R}^+ \to M_{n \times d}(\mathbb{R})$ a $\mathcal{B}(S \times \mathbb{R}^+)$-measurable function and $f \in L^2(S \times \mathbb{R}^+, \pi \otimes \lambda)$. Then, the resulting causal MMA process X is a θ-weakly dependent process with coefficients*

$$
\theta_X(r) = \Big(\int_S \int_{-\infty}^{-r} tr(f(A, -s) \Sigma_L f(A, -s)') \, ds \, \pi(dA) \Big)^{\frac{1}{2}} \tag{3.16}
$$

for all $r \geq 0$, where $\mathbb{E}[L_1 L_1'] = \Sigma_L = \Sigma + \int_{\mathbb{R}^d} xx' \nu(dx)$.

Proof. First, we define $\forall t \in \mathbb{R}$ and $m \geq 0$ the truncated sequence

$$
X_t^{(m)} = \int_S \int_{-\infty}^t f(A, t-s) \mathbb{1}_{[0,m]}(t-s) \, \Lambda(dA, ds) = \int_S \int_{t-m}^t f(A, t-s) \, \Lambda(dA, ds). \tag{3.17}
$$

Since the kernel function f is square integrable, we have that properties (2.5) and (2.6) hold and then f is Λ-integrable (Theorem 2.2) and X is well defined.

Thus, because of Proposition 2.1, $\mathbb{E}[X_t^2] < \infty$ for all $t \in \mathbb{R}$ and we can determine an upper bound of the expectation

$$\mathbb{E}\|X_t - X_t^{(m)}\| = \mathbb{E}\Big\|\int_S \int_{-\infty}^{t-m} f(A, t-s)\,\Lambda(dA, ds)\Big\|$$

$$\leq \Big(\mathbb{E}\Big\|\int_S \int_{-\infty}^{t-m} f(A, t-s)\,\Lambda(dA, ds)\Big\|^2\Big)^{\frac{1}{2}}.$$

Due to the stationarity of X the above estimation is independent of t and equal to

$$\Big(\int_S \int_{-\infty}^{-m} tr(f(A, -s)\Sigma_L f(A, -s)')\,ds\pi(dA)\Big)^{\frac{1}{2}} \tag{3.18}$$

Let F and G belong respectively to the class of bounded functions $\mathcal{H}^*$ and $\mathcal{H}$, $(u, v) \in \mathbb{N}^* \times \mathbb{N}^*$, $r \in \mathbb{R}^+$, $(i_1, \ldots, i_u) \in \mathbb{R}^u$ and $(j_1, \ldots, j_v) \in \mathbb{R}^v$ with $i_1 \leq \ldots \leq i_u \leq i_u + r \leq j_1 \leq \ldots \leq j_v$, $X_i^* = (X_{i_1}, \ldots, X_{i_u})$ and $X_j^{*(m)} = (X_{j_1}^{(m)}, \ldots, X_{j_v}^{(m)})$ where for all $m \geq 0$

$$X_{i_u} = \int_S \int_{-\infty}^{i_u} f(A, i_u - s)\,\Lambda(dA, ds)$$

and $\tag{3.19}$

$$X_{j_1}^{(m)} = \int_S \int_{j_1-m}^{j_1} f(A, j_1 - s)\,\Lambda(dA, ds).$$

Then, if $j_1 - m - i_u \geq 0$, which can also be expressed as $j_1 - i_u \geq m$, $I_u = S \times (-\infty, i_u]$ and $J_1 = S \times [j_1 - m, j_1]$ are disjoint sets or they have intersection $S \times \{j_1 - m\}$ when $j_1 - m = i_u$. Noting that $\pi \times \lambda(S \times \{j_1 - m\}) = 0$, by the definition of a Lévy basis, the two sequences $(X_i)_{i_1, \ldots, i_u}$ and $(X_j^{(m)})_{j_1, \ldots, j_v}$ are independent and so are $F(X_i^*)$ and $G(X_j^{*(m)})$. Therefore, let $m = r$

$$|Cov(F(X_i^*), G(X_j^*))| \leq |Cov(F(X_i^*), G(X_j^*) - G(X_j^{*(m)}))|$$

$$+|Cov(F(X_i^*), G(X_j^{*(m)}))| \leq 2\mathbb{E}|G(X_j^*) - G(X_j^{*(m)})|$$

the last relation comes from $\|F\|_\infty \leq 1$

$$\leq 2Lip(G)\sum_{k=1}^{v}\mathbb{E}\|X_{j_k} - X_{j_k}^{(m)}\|$$

using the result (3.18)

$$\leq 2vLip(G)\Big(\int_S \int_{-\infty}^{-r} tr(f(A, -s)\Sigma_L f(A, -s)')\,ds\pi(dA)\Big)^{\frac{1}{2}}$$

$$= 2vLip(G)\,\theta_X(r),$$

which converges to zero as r goes to infinity by applying the dominated convergence theorem. $\qquad\square$

Also in the case of θ-weak dependence, the θ-coefficients change when the underlying Lévy process has mean different from zero.

Corollary 3.4. *Let Λ be an $\mathbb{R}^d$-valued Lévy basis with characteristic quadruple $(\gamma, \Sigma, \nu, \pi)$ such that $\int_{\|x\|>1} \|x\|^2 \nu(dx) < \infty$, $f : S \times \mathbb{R}^+ \to M_{n \times d}(\mathbb{R})$ a $\mathcal{B}(S \times \mathbb{R}^+)$-measurable function satisfying assumption (2.4) and $f \in L^2(S \times \mathbb{R}^+, \pi \otimes \lambda)$. Then, the resulting causal MMA process X is a θ-weakly dependent process with coefficients*

$$\theta_X(r) = \left(\int_S \int_{-\infty}^{-r} tr(f(A, -s) \Sigma_L f(A, -s)') \, ds \, \pi(dA) \right.$$
$$\left. + \| \int_S \int_{-\infty}^{-r} f(A, -s) \mu \, ds \, \pi(dA) \|^2 \right)^{\frac{1}{2}} \tag{3.20}$$

for all $r \geq 0$, where $\mathbb{E}[L_1] = \mu = \gamma + \int_{\|x\|>1} x \, \nu(dx)$ and $\mathbb{E}[L_1 L_1'] = \Sigma_L = \Sigma + \int_{\mathbb{R}^d} xx' \, \nu(dx)$.

We conclude the study of the θ-weak dependence properties of an MMA process with the computation of the θ-coefficients for an underlying Lévy process of finite variation.

Corollary 3.5. *Let Λ be an $\mathbb{R}^d$-valued Lévy basis with characteristic quadruple $(\gamma, 0, \nu, \pi)$ such that $\int_{\mathbb{R}^d} \|x\| \nu(dx) < \infty$, $f : S \times \mathbb{R}^+ \to M_{n \times d}(\mathbb{R})$ a $\mathcal{B}(S \times \mathbb{R}^+)$-measurable function and $L^1(S \times \mathbb{R}^+, \pi \otimes \lambda)$ and define γ_0 as in (2.3). Then, the resulting causal MMA process X is a θ-weakly dependent process with coefficients*

$$\theta_X(r) = \int_S \int_{-\infty}^{-r} \int_{\mathbb{R}^d} \|f(A, -s)x\| \, \nu(dx) \, ds \, \pi(dA) \tag{3.21}$$

$$+ \int_S \int_{-\infty}^{-r} \|f(A, -s)\gamma_0\| \, ds \, \pi(dA),$$

for all $r \geq 0$.

Proof. The kernel function f is in L^1 then the properties (2.10) and (2.11) are satisfied and then f is Λ-integrable and X well defined and with finite mean by Corollary 2.2. Using the notation in Proposition 3.3, for all $t \in \mathbb{R}$ and $m \geq 0$

$$\mathbb{E}\|X_t - X_t^{(m)}\| \leq \int_S \int_{-\infty}^{-m} \int_{\mathbb{R}^d} \|f(A, -s)x\| \, \nu(dx) \, ds \, \pi(dA)$$
$$+ \int_S \int_{-\infty}^{-m} \|f(A, -s)\gamma_0\| \, ds \, \pi(dA), \tag{3.22}$$

where $X_t^{(m)}$ is the truncated sequence (3.17). Thus, for $m = r$ and F, G, X_i^* and X_j^*

$$|Cov(F(X_i^*), G(X_j^*))|$$
$$\leq 2vLip(G)\Big(\int_S \int_{-\infty}^{-r} \int_{\mathbb{R}^d} \|f(A, -s)x\| \, \nu(dx) \, ds \, \pi(dA)$$

$$+ \int_S \int_{-\infty}^{-r} \|f(A, -s)\gamma_0\| \, ds \, \pi(dA) \Big)$$

$$= 2v Lip(G) \, \theta_X(r).$$

Finally, we conclude by applying the dominated convergence theorem. □

Remark 3.1. *The η-coefficients of a causal MMA process can be chosen to be equal to the θ-coefficients for each $r \geq 0$. This can be easily seen by noticing that the truncated sequences (3.7) in Proposition 3.1 are equal to the truncated sequences (3.19) in Proposition 3.3. This leads to select the parameter $m = r$ in both proofs. Moreover, (3.6) is equal to (3.18) and then $\eta_X(r) = \theta_X(r)$. The same observations hold when comparing the results in Corollary 3.1 or Corollary 3.2 with Corollary 3.4 or Corollary 3.5.*

An example of a causal MMA process is the supOU process studied in [3] and [10]. Let us analyze its weak dependence properties in the univariate case. We consider the kernel function $f(A, s) = e^{As}1_{[0,\infty)}(s)$, $A \in \mathbb{R}^-$, $s \in \mathbb{R}$ and Λ a 1-dimensional Lévy basis on $\mathbb{R}^- \times \mathbb{R}$ with generating quadruple $(\gamma, \Sigma, \nu, \pi)$ such that

$$\int_{|x|>1} \log(|x|) \, \nu(dx) < \infty, \text{ and } \int_{\mathbb{R}^-} -\frac{1}{A}\pi(dA) < \infty, \tag{3.23}$$

then the process

$$X_t = \int_{\mathbb{R}^-} \int_{-\infty}^{t} e^{A(t-s)} \, \Lambda(dA, ds) \tag{3.24}$$

is well defined for each $t \in \mathbb{R}$ and strictly stationary. For the supOU process, A represents a random mean reversion parameter.

If $\mathbb{E}[L_1] = 0$ and $\int_{|x|>1} |x|^2\nu(dx) < \infty$, (3.24) is θ-weakly dependent with coefficients

$$\theta_X(r) = \Big(\int_{\mathbb{R}^-} \int_{-\infty}^{-r} e^{-2As}\sigma^2 \, ds \, \pi(dA) \Big)^{\frac{1}{2}} = \Big[-\sigma^2 \int_{\mathbb{R}^-} \frac{e^{2Ar}}{2A} \, \pi(dA) \Big]^{\frac{1}{2}} \tag{3.25}$$

$$= Cov(X_0, X_{2r})^{\frac{1}{2}},$$

by using [10, Theorem 3.11] and where $\sigma^2 = \Sigma + \int_{\mathbb{R}} x^2\nu(dx)$.

If $\mathbb{E}[L_1] = \mu$ and $\int_{|x|>1} |x|^2\nu(dx) < \infty$, the supOU process is θ-weakly dependent with coefficients

$$\theta_X(r) = \Big(Cov(X_0, X_{2r}) + \frac{4\mu^2}{\sigma^4}Cov(X_0, X_r)^2 \Big)^{\frac{1}{2}}. \tag{3.26}$$

If $\int_{\mathbb{R}} |x|\nu(dx) < \infty$, $\gamma_0 = \gamma - \int_{|x|\leq 1} x \, \nu(dx) > 0$ and $\nu(\mathbb{R}^-) = 0$, i.e. the underlying Lévy process is a subordinator, then (3.24) admits θ-coefficients

$$\theta_X(r) = -\mu \int_{\mathbb{R}^-} \frac{e^{Ar}}{A} \, \pi(dA), \tag{3.27}$$

and when in addition $\int_{|x|>1} |x|^2 \nu(dx) < \infty$

$$\theta_X(r) = \frac{2\mu}{\sigma^2} Cov(X_0, X_r). \tag{3.28}$$

Note that in the finite superposition case strong mixing of the supOU process has been shown in [36, 37] based on Masuda's result [40]. As this crucially hinges on an embedding into a finite dimensional Markov process this does not readily extend to the general case.

Remark 3.2. *The necessary and sufficient condition* $\int_{\mathbb{R}^-} -\frac{1}{A} \pi(dA)$ *for the supOU process to exist is satisfied by many continuous and discrete distributions* π*, see [48, Section 2.4] for more details. For example, a probability measure* π *being absolutely continuous with density* $\pi' = (-x)^\alpha l(x)$ *and regularly varying at zero from the left with* $\alpha > 0$ *(see [13]), i.e. l is slowly varying at zero, satisfies the above condition. If moreover,* $l(x)$ *is continuous in* $(-\infty, 0)$ *and* $\lim_{x \to 0^-} l(x) > 0$ *exists, it holds that*

$$Cov(X_0, X_r) \sim \frac{C}{r^\alpha}, \text{ with a constant } C > 0 \text{ and } r \in \mathbb{R}^+$$

where for $\alpha \in (0, 1)$ *the supOU process exhibits long memory and for* $\alpha > 1$ *short memory, see [28, Definition 3.1.2]. Concrete examples where the covariances are calculated explicitely, in this set-up, can be found in [7].*

Remark 3.3. *A natural question is whether one can improve the weak dependence coefficients that we obtain.*

[24, Lemma 4.1] shows that for stationary processes with finite m-moments $(m > 2 + \delta$*, for* $\delta > 0)$ *being* λ*-weakly dependent (cf. [24, Definition 2.1]) and thus* η*-weakly dependent*

$$|Cov(X_0, X_r)| \le 9 \, \mathbb{E}[\|X_0\|^m]^{\frac{1}{m-1}} \, \lambda(r)^{\frac{m-2}{m-1}}.$$

The above arguments can be easily adapted to the causal case and θ*-weak dependence where we likewise get*

$$|Cov(X_0, X_r)| \le 9 \, \mathbb{E}[\|X_0\|^m]^{\frac{1}{m-1}} \, \theta(r)^{\frac{m-2}{m-1}}.$$

If the stationary process has finite moments of any order we thus obtain the inequalities

$$|Cov(X_0, X_r)| \le 9 \, \eta(r) \text{ and } |Cov(X_0, X_r)| \le 9 \, \theta(r). \tag{3.29}$$

Equation (3.28) *shows that our weak dependence coefficients are sharp for a supOU process having as underlying Lévy process a subordinator with finite second moment. Note that "sharp" here means that the right and left hand side of the inequalities* (3.29) *only differ by a constant, as for the weak dependence coefficients one usually - like in the upcoming CLTs - only cares about their summability/integrability in r. The inequalities* (3.29) *compared to* (3.25) *and*

([3.26](#)) *show that we might obtain smaller weak dependence coefficients for the supOU process having an underlying Lévy process of infinite variation. In fact following Remark [3.2](#), if $Cov(X_0, X_r) \sim r^{-\alpha}$ for $\alpha > 0$, then the left hand side in ([3.29](#)) decays like $r^{-\alpha}$ whereas the right hand side decays like $r^{-\alpha/2}$.*

Inspecting the proofs of Corollaries [3.2](#) and [3.5](#) and Propositions [3.1](#) and [3.3](#), where the η and θ-coefficients are determined, the crucial issue is that we use the equality ([2.12](#)) to compute a bound of the term $\mathbb{E}\|X_t - X_t^{(m)}\|$ in the finite variation case, whereas we bound $\mathbb{E}\|X_t - X_t^{(m)}\|$ by means of a second moment in the infinite variation one. We do the latter because to the best of our knowledge there are no sharper bounds known for the first absolute moment of an infinitely divisible distribution that are suitably expressible in terms of the characteristic triplet in this set-up.

To conclude, we state the hereditary property of a θ-weakly dependent process.

Proposition 3.4. *Let $(X_t)_{t \in \mathbb{R}}$ be an $\mathbb{R}^n$-valued stationary process and assume there exists some constant $C > 0$ such that $\mathbb{E}[\|X_0\|^p]^{\frac{1}{p}} \leq C$, with $p > 1$, $h \colon \mathbb{R}^n \to \mathbb{R}^m$ be a function such that $h(0) = 0$, $h(x) = (h_1(x), \ldots, h_m(x))$ and*

$$\|h(x) - h(y)\| \leq c \, \|x - y\|(1 + \|x\|^{a-1} + \|y\|^{a-1}),$$

for $x, y \in \mathbb{R}^n$, $c > 0$ and $1 \leq a < p$. Define $(Y_t)_{t \in \mathbb{R}}$ by $Y_t = h(X_t)$. If $(X_t)_{t \in \mathbb{R}}$ is a θ-weakly dependent process, then $(Y_t)_{t \in \mathbb{R}}$ is a θ-weakly dependent process such that

$$\forall \, r \geq 0, \;\; \theta_Y(r) = \mathcal{C} \, \theta_X(r)^{\frac{p-a}{p-1}},$$

with the constant $\mathcal{C}$ independent of r.

Proof. Analogous to Proposition [3.2](#). $\qquad \square$

4. Sample moments of an MMA process

We consider a sample of N observations of a univariate MMA process $\{X_\Delta, \ldots, X_{N\Delta}\}$, where Δ is a positive integer.

$$X_{i\Delta} \colon = \int_S \int_{\mathbb{R}} f(A, i\Delta - s)\Lambda(dA, ds), \;\; \text{for } i \in \mathbb{Z}. \tag{4.1}$$

If the underlying Lévy process L has finite first moment, we define $\tilde{X}_{i\Delta} = X_{i\Delta} - \mathbb{E}[X_0]$.

The sample mean of the process X is defined as

$$\frac{1}{N} \sum_{i=1}^{N} X_{i\Delta} \tag{4.2}$$

and its sample autocovariance function at lag $k \in \mathbb{N}$.

$$\frac{1}{N} \sum_{j=1}^{N} \tilde{X}_{j\Delta}\tilde{X}_{(j+k)\Delta}. \tag{4.3}$$

W.l.o.g, we consider below $\mathbb{E}[X_0] = 0$ and $\Delta = 1$ in order to lighten the notations and, when the asymptotic properties of the sample auto-covariance functions are investigated, we focus on the features of the processes

$$Y_{j,k} = X_j X_{j+k} - D(k) \text{ for all } k \in \mathbb{N}, \tag{4.4}$$

where we denote by $D(k)$ the covariances at lag k defined, when $\mathbb{E}[X_0] = 0$, by

$$D(k) = Cov(X_0, X_k) = \mathbb{E}[X_0 X_k] = \int_S \int_{\mathbb{R}} f(A, -s)\Sigma_L f(A, k-s)' \, ds \, \pi(dA), \tag{4.5}$$

for $k \in \mathbb{Z}$, where $\mathbb{E}[L_1 L_1'] = \Sigma_L = \Sigma + \int_{\mathbb{R}^d} xx' \nu(dx)$.

We start by analyzing the asymptotic properties of the sample mean (4.2) for a non-causal and a causal MMA process.

Theorem 4.1. *Let Λ be an $\mathbb{R}^d$-valued Lévy basis with characteristic quadruple $(\gamma, \Sigma, \nu, \pi)$ such that $\mathbb{E}[L_1] = 0$ and $\int_{\|x\|>1} \|x\|^{2+\delta}\nu(dx) < \infty$, for some $\delta > 0$, $f : S \times \mathbb{R} \to M_{1 \times d}(\mathbb{R})$ a $\mathcal{B}(S \times \mathbb{R})$-measurable function and $f \in L^{2+\delta}(S \times \mathbb{R}, \pi \otimes \lambda) \cap L^2(S \times \mathbb{R}, \pi \otimes \lambda)$. If $(X_i)_{i \in \mathbb{Z}}$ as defined in (4.1) is an η-weakly dependent process with coefficients $\eta_X(r) = O(r^{-\beta})$ and $\beta > 4 + \frac{2}{\delta}$, then*

$$\sigma_\eta^2 = \sum_{k \in \mathbb{Z}} Cov(X_0, X_k) \tag{4.6}$$

is finite, non-negative and as $N \to \infty$

$$\frac{1}{\sqrt{N}} \sum_{i=1}^{N} X_i \xrightarrow{d} \mathcal{N}(0, \sigma_\eta^2). \tag{4.7}$$

Proof. We have that $\mathbb{E}[X_0^{2+\delta}] < \infty$ for $\delta > 0$ because of Proposition 2.1. Moreover, the η-weakly dependent process X satisfies the sufficient conditions of [24, Theorem 2.2]. The absolute summability of the series (4.6) follows and so the asymptotic normality (4.7). $\quad\square$

In the case of a causal MMA process, the required decay rate of the θ coefficients is lower than in the η-weak dependence case.

Theorem 4.2. *Let Λ be an $\mathbb{R}^d$-valued Lévy basis with characteristic quadruple $(\gamma, \Sigma, \nu, \pi)$ such that $\mathbb{E}[L_1] = 0$ and $\int_{\|x\|>1} \|x\|^{2+\delta}\nu(dx) < \infty$, for some $\delta > 0$, $f : S \times \mathbb{R}^+ \to M_{1 \times d}(\mathbb{R})$ a $\mathcal{B}(S \times \mathbb{R}^+)$-measurable function and $f \in L^{2+\delta}(S \times \mathbb{R}^+, \pi \otimes \lambda) \cap L^2(S \times \mathbb{R}^+, \pi \otimes \lambda)$. If $(X_i)_{i \in \mathbb{Z}}$ as defined in (4.1) is a θ-weakly dependent process with coefficients $\theta_X(r) = O(r^{-\alpha})$ and $\alpha > 1 + \frac{1}{\delta}$, then*

$$\sigma_\theta^2 = \sum_{k \in \mathbb{Z}} Cov(X_0, X_k) \tag{4.8}$$

is finite, non-negative and as $N \to \infty$

$$\frac{1}{\sqrt{N}} \sum_{i=1}^{N} X_i \xrightarrow{d} \mathcal{N}(0, \sigma_\theta^2). \tag{4.9}$$

Proof. The MMA process has finite $2 + \delta$-moment for $\delta > 0$ (Proposition 2.1) and is ergodic, as shown in [27]. By Lemma 2 in [18], the condition $D(2, \theta/2, X_0)$ holds. Then, by Corollary 1 in [18] and the ergodicity of the process X, (4.9) follows by applying [20, Theorem 1]. $\qquad\square$

Remark 4.1. *In the special case of the supOU process, being representable as a finite sum of independent Ornstein-Uhlenbeck processes with gamma or inverse gaussian marginals, a comparable result can be found in [37, Theorem 2].*

Remark 4.2. *Theorem 4.1 and 4.2 as well as all the upcoming central limit theorems can be also formulated as functional central limit theorems, following [24] and [18] respectively. However, we state the theorems just for the sample moments we are interested in (and which we are using in Section 6) to lighten the notations.*

For example, denote for $t \in [0, 1]$ and $n \geq 1$

$$S_n(t) = X_1 + \cdots + X_{[nt]},$$

in the case of a non-causal MMA process, and

$$S_n^*(t) = X_1 + \cdots + X_{[nt]} + (nt - [nt])X_{[nt]+1}$$

for a causal MMA process. Then, under the assumptions of Theorem 4.1 or 4.2, $n^{-\frac{1}{2}}S_n(t)$ converges in distribution in the Skorohod space $D[0,1]$ to $\sigma_\eta W$ and $n^{-\frac{1}{2}}S_n^*(t)$ converges in distribution in the space $C[0,1]$ to $\sigma_\theta W$, respectively. Here, W denotes a standard Brownian motion.

Remark 4.3. *In the finite variation case, (4.7) or (4.9), respectively, hold under $\int_{\|x\|>1} \|x\|^{2+\delta}\nu(dx) < \infty$, for some $\delta > 0$, and $f \in L^{2+\delta}(S \times \mathbb{R}, \pi \otimes \lambda) \cap L^1(S \times \mathbb{R}, \pi \otimes \lambda)$ or $f \in L^{2+\delta}(S \times \mathbb{R}^+, \pi \otimes \lambda) \cap L^1(S \times \mathbb{R}^+, \pi \otimes \lambda)$, respectively.*

Remark 4.4. *Assuming that f is not equal to zero π-almost everywhere and that $f \geq 0$ or $f \leq 0$, the asymptotic variance in Theorems 4.1 and 4.2 is not degenerate. This is the case for example when working with the supOU process (3.24). Moreover, it is worthy to observe that the assumptions in Theorem 4.2 clearly indicate that we obtain asymptotic normality of the sample mean for a causal MMA process just in the short memory case.*

To find an asymptotic distribution for the sample autocovariance functions (4.3), we first show that $(Y_{j,k})_{j\in\mathbb{Z}}$ are η-weakly or θ-weakly dependent processes. In addition to the hereditary properties in Proposition 3.2 and 3.4, we need to establish when the weak dependence properties of a univariate MMA process are inherited by the process $Z_t = (X_t, X_{t+1}, \ldots, X_{t+k})$ for all $k \in \mathbb{N}$.

Proposition 4.1. *Let Λ be an $\mathbb{R}^d$-valued Lévy basis with generating quadruple $(\gamma, \Sigma, \nu, \pi)$ and $f : S \times \mathbb{R} \to M_{1\times d}(\mathbb{R})$ be a $\mathcal{B}(S \times \mathbb{R})$-measurable function satisfying the assumptions of Theorem 2.2. If for all $t \in \mathbb{R}$, X is a non-causal or causal MMA as defined in (2.13) or (3.15) respectively, then*

$$Z_t = \int_S \int_{\mathbb{R}} g(A, t - s)\,\Lambda(dA, ds),$$

where $g(A,s) = \begin{pmatrix} f(A,s) \\ f(A,s-1) \\ \dots \\ f(A,s-k) \end{pmatrix}$ *is a $\mathcal{B}(S \times \mathbb{R})$-measurable function with values*

in $M_{k+1 \times d}(\mathbb{R})$ and $k \in \mathbb{N}$, is an MMA process. Moreover, if X satisfies the assumptions of Proposition 3.1 (Corollary 3.1) or Proposition 3.3 (Corollary 3.4) then Z is η or $\theta-$weakly dependent respectively with coefficients

$$\eta_Z(r) = \mathcal{D}\,\eta_X(r - 2k)\,\text{for}\,r \geq 2k \;\; or \;\; \theta_Z(r) = \mathcal{D}\,\theta_X(r - k)\,\text{for}\,r \geq k, \quad (4.10)$$

where $\mathcal{D} = (k+1)^{\frac{1}{2}}$. In the case when the assumptions of Corollaries 3.2 or 3.5 hold, the process Z is respectively η or θ-weakly dependent with coefficients

$$\eta_Z(r) = \mathcal{D}\,\eta_X(r - 2k)\,\text{for}\,r \geq 2k \;\; or \;\; \theta_Z(r) = \mathcal{D}\,\theta_X(r - k)\,\text{for}\,r \geq k, \quad (4.11)$$

and $\mathcal{D} = k + 1$.

Proof. For $k = 1$, the first step of the proof consists of checking that g is a Λ-integrable function as prescribed by Theorem 2.2. W.l.o.g, we consider in our calculations the norm

$$\|(x,y)\| = \|x\| + \|y\|$$

for $x, y \in M_{1 \times d}(\mathbb{R})$. We have that

$$\int_S \int_{\mathbb{R}} \left\| g(A,s)\gamma + \int_{\mathbb{R}^d} g(A,s)x \Big(\mathbb{1}_{[0,1]}(\|g(A,s)x\|) - \mathbb{1}_{[0,1]}(\|x\|)\Big)\,\nu(dx) \right\| ds\,\pi(dA)$$
$$(4.12)$$
$$= \int_S \int_{\mathbb{R}} \left\| \begin{pmatrix} f(A,s) \\ f(A,s-1) \end{pmatrix} \gamma \right.$$

$$+ \int_{\mathbb{R}^d} \begin{pmatrix} f(A,s) \\ f(A,s-1) \end{pmatrix} x \Big(\mathbb{1}_{[0,1]}\Big(\left\| \begin{pmatrix} f(A,s) \\ f(A,s-1) \end{pmatrix} x \right\|\Big) - \mathbb{1}_{[0,1]}(\|x\|)\Big)\nu(dx) \Big\| \,ds\,\pi(dA)$$

$$= \int_S \int_{\mathbb{R}} \left\| f(A,s)\gamma + \int_{\mathbb{R}^d} f(A,s)x \Big(\mathbb{1}_{[0,1]}(\|g(A,s)x\|) - \mathbb{1}_{[0,1]}(\|x\|)\Big)\,\nu(dx) \right\| ds\,\pi(dA)$$

$$+ \int_S \int_{\mathbb{R}} \left\| f(A,s-1)\gamma + \int_{\mathbb{R}^d} f(A,s-1)x \Big(\mathbb{1}_{[0,1]}(\|g(A,s)x\|) - \mathbb{1}_{[0,1]}(\|x\|)\Big)\,\nu(dx) \right\| ds\,\pi(dA).$$

Noting that
$$\mathbb{1}_{[0,1]}(\|g(A,s)x\|) \leq \mathbb{1}_{[0,1]}(\|f(A,s)x\|)$$

and

$$\mathbb{1}_{[0,1]}(\|g(A,s)x\|) \leq \mathbb{1}_{[0,1]}(\|f(A,s-1)x\|),$$

it then holds that (4.12) is finite.

Let us pass to the second condition

$$\int_S \int_{\mathbb{R}} \|g(A,s)\Sigma g(A,s)'\|\,ds\,\pi(dA)$$

$$\leq \int_S \int_{\mathbb{R}} \Big(\|f(A,s)\Sigma f(A,s)'\| + \|f(A,s-1)\Sigma f(A,s-1)'\| \Big) \, ds \, \pi(dA).$$

Therefore, f being a kernel of an MMA process, the above integral is finite. Finally, we have

$$\int_S \int_{\mathbb{R}} \int_{\mathbb{R}^d} \Big(1 \wedge \|g(A,s)x\|^2 \Big) \, \nu(dx) \, ds \, \pi(dA)$$

$$\leq 2 \int_S \int_{\mathbb{R}} \int_{\mathbb{R}^d} \Big(1 \wedge \|f(A,s)x\|^2 \Big) \, \nu(dx) \, ds \, \pi(dA)$$

$$+ 2 \int_S \int_{\mathbb{R}} \int_{\mathbb{R}^d} \Big(1 \wedge \|f(A,s-1)x\|^2 \Big) \, \nu(dx) \, ds \, \pi(dA) < \infty.$$

Thus the kernel function g is a Λ-integrable function. By induction the statement can be shown for each $k \in \mathbb{N}$. Because all the assumptions of Theorem 2.2 hold, we have that Z is an MMA process.

Depending now on the properties of the underlying Lévy process, we can distinguish three different scenarios for the η and θ-weak dependence. When X satisfies the assumptions of Proposition 3.1,

$$\eta_Z(r) = \Big(\int_S \int_{\mathbb{R}} tr(g(A,-s)\Sigma_L g(A,-s)') \, \mathbb{1}_{(-\infty,-r)}(2s) \, ds\pi(dA) \Big)^{\frac{1}{2}}$$

$$+ \Big(\int_S \int_{\mathbb{R}} tr(g(A,-s)\Sigma_L g(A,-s)') \, \mathbb{1}_{(r,+\infty)}(2s) \, ds\pi(dA) \Big)^{\frac{1}{2}}$$

$$\leq \Big(\int_S \int_{\mathbb{R}} tr(f(A,-s)\Sigma_L f(A,-s)') \, \mathbb{1}_{(-\infty,-r)}(2s) \, ds\pi(dA) + \ldots$$

$$+ \int_S \int_{\mathbb{R}} tr(f(A,k-s)\Sigma_L f(A,k-s)') \, \mathbb{1}_{(-\infty,-r)}(2s) \, ds\pi(dA) \Big)^{\frac{1}{2}}$$

$$+ \Big(\int_S \int_{\mathbb{R}} tr(f(A,-s)\Sigma_L f(A,-s)') \, \mathbb{1}_{(r,+\infty)}(2s) \, ds\pi(dA) + \ldots$$

$$+ \int_S \int_{\mathbb{R}} tr(f(A,k-s)\Sigma_L f(A,k-s)') \, \mathbb{1}_{(r,+\infty)}(2s) \, ds\pi(dA) \Big)^{\frac{1}{2}}$$

$$\leq (k+1)^{\frac{1}{2}} \Big(\int_S \int_{\mathbb{R}} tr(f(A,-s)\Sigma_L f(A,-s)') \, \mathbb{1}_{(-\infty,-r+2k)}(2s) \, ds\pi(dA) \Big)^{\frac{1}{2}}$$

$$+ \Big(\int_S \int_{\mathbb{R}} tr(f(A,-s)\Sigma_L f(A,-s)') \, \mathbb{1}_{(r-2k,+\infty)}(2s) \, ds\pi(dA) \Big)^{\frac{1}{2}}$$

$$\leq (k+1)^{\frac{1}{2}} \eta_X(r-2k),$$

for each $r > 2k$. Thus, Z is a $k+1$-dimensional MMA process with η coefficients

$$\eta_Z(r) = (k+1)^{\frac{1}{2}} \eta_X(r-2k).$$

If X satisfies the assumptions of Proposition 3.3, it can be shown similarly that Z is θ-weakly dependent with coefficients

$$\theta_Z(r) = (k+1)^{\frac{1}{2}}\theta_X(r-k).$$

Similar calculations follow in the finite variation case leading to the statements (4.11).   □

Proposition 4.2. *Let Λ be an $\mathbb{R}^d$-valued Lévy basis with characteristic quadruple $(\gamma, \Sigma, \nu, \pi)$ and $\int_{\|x\|>1} \|x\|^{2+\delta}\nu(dx) < \infty$, for some $\delta > 0$, $f : S \times \mathbb{R} \to M_{1\times d}(\mathbb{R})$ a $\mathcal{B}(S \times \mathbb{R})$-measurable function and $f \in L^{2+\delta}(S \times \mathbb{R}, \pi \otimes \lambda) \cap L^2(S \times \mathbb{R}, \pi \otimes \lambda)$. If $(X_i)_{i\in\mathbb{Z}}$ as defined in (4.1) is η or θ-weakly dependent respectively with coefficients η_X or θ_X, then for all $k \geq 0$ the processes $(Y_{j,k})_{j\in\mathbb{Z}}$ are respectively η or θ-weakly dependent with coefficients*

$$\eta_Y(r) = \mathcal{C}(\sqrt{2}\eta_X(r-2k))^{\frac{\delta}{1+\delta}} \ \text{ or } \ \theta_Y(r) = \mathcal{C}(\sqrt{2}\theta_X(r-k))^{\frac{\delta}{1+\delta}}.$$

If L is a process of finite variation and $f \in L^{2+\delta}(S \times \mathbb{R}, \pi \otimes \lambda) \cap L^1(S \times \mathbb{R}, \pi \otimes \lambda)$, then

$$\eta_Y(r) = \mathcal{C}(2\eta_X(r-2k))^{\frac{\delta}{1+\delta}} \ \text{ or } \ \theta_Y(r) = \mathcal{C}(2\theta_X(r-k))^{\frac{\delta}{1+\delta}}.$$

The constant $\mathcal{C}$, appearing in the η and θ-coefficients, is independent of r.

Proof. Let us consider a 2-dimensional process $Z = (X_j, X_{j+k})_{j\in\mathbb{Z}}$ with $k \in \mathbb{N}$. The η or θ coefficients of the process Z are

$$\eta_Z(r) = \sqrt{2}\eta_X(r-2k) \ \text{ or } \ \theta_Z(r) = \sqrt{2}\theta_X(r-k)$$

by Proposition 4.1. The $2+\delta$ moment, for $\delta > 0$, of the MMA process exists because of Proposition 2.1. Let us now consider $h \colon \mathbb{R}^2 \to R$ as $h(x_1, x_2) = x_1 x_2$. The function h satisfies assumption (3.10), for $p = 2+\delta$, $c = 1$ and $a = 2$. Then, Proposition 3.2 or 3.4 applies and $h(Z) = X_j X_{j+k}$, as well as $Y_{j,k}$, has either coefficients

$$\eta_Y(r) = \mathcal{C}(\sqrt{2}\eta_X(r-2k))^{\frac{\delta}{1+\delta}} \ \text{ or } \ \theta_Y(r) = \mathcal{C}(\sqrt{2}\theta_X(r-k))^{\frac{\delta}{1+\delta}}.$$

The finite variation case follows easily by applying Proposition 4.1 and using the coefficients (4.11).   □

We can now give a distributional limit theorem for the processes $Y_{j,k}$, namely determining the asymptotic distribution of

$$\frac{1}{N}\sum_{j=1}^{N} Y_{j,k} \ \text{ for all } k \in \mathbb{N}.$$

Corollary 4.1. *Let Λ be an $\mathbb{R}^d$-valued Lévy basis with characteristic quadruple $(\gamma, \Sigma, \nu, \pi)$ such that $\mathbb{E}[L_1] = 0$, $\int_{\|x\|>1} \|x\|^{4+\delta}\nu(dx) < \infty$, for some $\delta > 0$, $f : S \times \mathbb{R} \to M_{1\times d}(\mathbb{R})$ a $\mathcal{B}(S \times \mathbb{R})$-measurable function and $f \in L^{4+\delta}(S \times \mathbb{R}, \pi \otimes \lambda) \cap L^2(S \times \mathbb{R}, \pi \otimes \lambda)$. Let $(Y_{j,k})_{j\in\mathbb{Z}}$ be defined as in (4.4) for $k \in \mathbb{N}$. If $(X_i)_{i\in\mathbb{Z}}$*

as defined in ([4.1](#)) is η-weakly dependent with coefficients $\eta_X(r) = O(r^{-\beta})$ such that $\beta > (4 + \frac{2}{\delta})(\frac{3+\delta}{2+\delta})$ or it is θ-weakly dependent with coefficients $\theta_X(r) = O(r^{-\alpha})$ such that $\alpha > (1 + \frac{1}{\delta})(\frac{3+\delta}{2+\delta})$, then

$$\gamma_k^2 = \sum_{l\in\mathbb{Z}} Cov(Y_{0,k}, Y_{l,k}) = \sum_{l\in\mathbb{Z}} Cov(X_0 X_k, X_l X_{l+k})$$

is finite, non-negative and as $N \to \infty$

$$\frac{1}{\sqrt{N}} \sum_{j=1}^{N} Y_{j,k} \xrightarrow{d} \mathcal{N}(0, \gamma_k^2) \tag{4.13}$$

Proof. Since the results of Proposition [2.1](#) apply, by using [24, Theorem 2.2] in the case of an η-weakly dependent process or [20, Theorem 1] when the process X is θ-weakly dependent, the distributional limit ([4.13](#)) holds. $\qquad\square$

Remark 4.5. *The asymptotic variance γ_k^2 can be expressed in terms of the fourth order cumulant of a zero mean MMA process and its covariances as follows. Let us consider an $\mathbb{R}^4$-valued MMA process $X = (X_i, X_j, X_k, X_l)$ with $(i, j, k, l) \in \mathbb{R}^4$ and kernel function $g(A, s) = [f(A, s - i), f(A, s - j), f(A, s - k), f(A, s - l)]'$ with values in $M_{4\times d}(\mathbb{R})$. The Lévy basis Λ, underlying the definition of X, satisfies the assumptions of Corollary [4.1](#). Thus, X is infinitely divisible with characteristic triplet $(\gamma_{int}, \Sigma_{int}, \nu_{int})$ as given in Theorem [2.2](#) and characteristic exponent*

$$log(\mathbb{E}[e^{i\langle u, X\rangle}]) = i\langle \gamma_{int}, u\rangle - \frac{1}{2}\langle u, \Sigma_{int} u\rangle + \int_{\mathbb{R}^d} e^{i\langle u, x\rangle} - 1 - i\langle u, x\rangle \mathbb{1}_{[0,1]}(\|x\|) \ \nu_{int}(dx).$$

We denote by $\kappa(i, j, k, l)$ the fourth order cumulant of X. By [28, Proposition 4.2.2.]

$$\kappa(i, j, k, l) = \mathbb{E}[X_i X_j X_k X_l] - \mathbb{E}[X_i X_j]\mathbb{E}[X_k X_l] - \mathbb{E}[X_i X_k]\mathbb{E}[X_j X_l]$$
$$- \mathbb{E}[X_i X_l]\mathbb{E}[X_j X_k].$$

On the other hand, cf. [28, Definition 4.2.1],

$$\kappa(i, j, k, l) = \frac{\partial^4}{\partial u_1 \partial u_2 \partial u_3 \partial u_4} log(\mathbb{E}[e^{i\langle u, X\rangle}])\Big|_{u_1 = u_2 = u_3 = u_4 = 0}$$

$$= \int_S \int_{\mathbb{R}} \int_{\mathbb{R}^d} f(A, s-i) \, x \, x' \, f(A, s-j)' \, f(A, s-k) \, x \, x' \, f(A, s-l)' \, \nu(dx) \, ds \, \pi(dA). \tag{4.14}$$

Then, $\forall (l, k) \in \mathbb{R}^2$

$$Cov(Y_{0,k}, Y_{l,k}) = \kappa(0, k, l, l+k) + D(l)^2 + D(k+l)D(k-l).$$

where $D(l)$ is defined in ([4.5](#)).

Analogously, the formula to compute the third order cumulant $\kappa(i,j,k)$ can be derived. In fact,

$$\kappa(i,j,k) = \int_S \int_{\mathbb{R}} \int_{\mathbb{R}^d} f(A, s-i)\, x\, x'\, f(A, s-j)'\, f(A, s-k)\, x\, \nu(dx)\, ds\, \pi(dA)$$

$$(4.15)$$

and for a zero mean MMA process holds that $\mathbb{E}[X_i X_j X_k] = \kappa(i,j,k)$. This computation is useful in Section 6.

Corollary 4.2. *Let Λ be an $\mathbb{R}^d$-valued Lévy basis with characteristic quadruple $(\gamma, \Sigma, \nu, \pi)$ such that $\mathbb{E}[L_1] = 0$, $\int_{\|x\|>1} \|x\|^{4+\delta}\nu(dx) < \infty$, for some $\delta > 0$, $f : S \times \mathbb{R} \to M_{1\times d}(\mathbb{R})$ a $\mathcal{B}(S \times \mathbb{R})$-measurable function and $f \in L^{4+\delta}(S \times \mathbb{R}, \pi \otimes \lambda) \cap L^2(S \times \mathbb{R}, \pi \otimes \lambda)$. Let $\mathcal{Z}_j = (Y_{j,0}, \ldots, Y_{j,k})$ for all $j \in \mathbb{Z}$. If $(X_i)_{i\in\mathbb{Z}}$ is η-weakly dependent with coefficients $\eta_X(r) = O(r^{-\beta})$ such that $\beta > (4 + \frac{2}{\delta})(\frac{3+\delta}{2+\delta})$ or it is θ-weakly dependent with coefficients $\theta_X(r) = O(r^{-\alpha})$ such that $\alpha > (1+\frac{1}{\delta})(\frac{3+\delta}{2+\delta})$, then respectively for each $p, q \in \{0, \ldots, k\}$ with $k \in \mathbb{N}$,*

$$\sum_{l\in\mathbb{Z}} Cov(X_0 X_p, X_l X_{l+q}) = \sum_{l\in\mathbb{Z}} k(0, p, l, l+q) + D(l)D(l+q-p) + D(q+l)D(l-p),$$

where $k(l, i, j, k)$ is defined in (4.14) and $D(k)$ in (4.5) for each $l, i, j, k \in \mathbb{Z}$, is finite and as $N \to \infty$

$$\frac{1}{\sqrt{N}} \sum_{j=1}^{N} \mathcal{Z}_j \xrightarrow{d} \mathcal{N}_{k+1}(0, \Xi)$$

where Ξ is equal to

$$\begin{bmatrix} \sum_{l\in\mathbb{Z}} Cov(X_0^2, X_l^2) & \sum_{l\in\mathbb{Z}} Cov(X_0^2, X_l X_{l+1}) & \ldots & \sum_{l\in\mathbb{Z}} Cov(X_0^2, X_l X_{l+k}) \\ \ldots & \sum_{l\in\mathbb{Z}} Cov(X_0 X_1, X_l X_{l+1}) & \ldots & \sum_{l\in\mathbb{Z}} Cov(X_0 X_1, X_l X_{l+k}) \\ \ldots & \ldots & \ldots & \sum_{l\in\mathbb{Z}} Cov(X_0 X_k, X_l X_{l+k}) \end{bmatrix}$$

and positive semidefinite.

Proof. Let us consider the vector Z as defined in Proposition 4.1. Given the assumptions of the Corollary, Z is η-weakly dependent with coefficients $\eta_Z(r) = \mathcal{D}\eta_X(r-2k)$ or θ-weakly dependent with coefficients $\theta_Z(r) = \mathcal{D}\theta_X(r-k)$ because of Proposition 4.1 and given that the results of Proposition 2.1 apply. We apply now the function $f : \mathbb{R}^{k+1} \to \mathbb{R}^{k+1}$ to the vector Z such that

$$f(Z_j) = \begin{pmatrix} X_j^2 \\ \vdots \\ X_j X_{j+k} \end{pmatrix} = \mathcal{Z}_j + \begin{pmatrix} D(0) \\ \vdots \\ D(k) \end{pmatrix}.$$

The assumptions of Proposition 3.2 hold with $p = 4 + \delta$, $c = 1$, $a = 2$, then $f(Z_t)$ is η or θ-weakly dependent with coefficients $\mathcal{C}(\mathcal{D}\eta_X(r-2k))^{\frac{2+\delta}{3+\delta}}$ or $\mathcal{C}(\mathcal{D}\theta_X(r-k))^{\frac{2+\delta}{3+\delta}}$. The process $\mathcal{Z}$ is then a process with the same weak dependence coefficients as $f(Z_t)$. For all $a \in \mathbb{R}^{k+1}$, $a'\mathcal{Z}$ is an η or a θ-weakly dependent

process, because a linear function is Lipschitz, having the same coefficients as the process $\mathcal{Z}$. By [24, Theorem 2.2] or [20, Theorem 1], then

$$\frac{1}{\sqrt{N}} \sum_{j=1}^{N} a' \mathcal{Z}_j \xrightarrow{d} \mathcal{N}(0, a'\Sigma a)$$

as $N \to \infty$. Applying the Cramer-Wold device, the asymptotic normality of the vector $\mathcal{Z}$ is shown. □

Remark 4.6. *In this paper we consider the classical case of equidistant observations. In many applications different sampling schemes are also highly relevant and for some special cases results have been obtained. For example, [14] considers the asymptotics of the autocovariance function for Lévy-driven moving average processes sampled at an independent renewal sequence and [26] consider the asymptotics of the pathwise Fourier transform/periodogram for Lévy-driven CARMA processes sampled at deterministic irregular grids. Considering independent renewal sampled Lévy-driven MMA processes is beyond the scope of the present paper and the content of future research just starting in [15] where the preservation of strong mixing and weak dependence properties is discussed in general.*

5. Sample moments of an MMA SV model

Let us consider a Lévy basis with characteristic quadruple $(\gamma, \sigma^2, \nu, \pi)$ and values in $\mathbb{R}$ and the respective univariate casual MMA process X with kernel function $f : (S \times \mathbb{R}^+) \to \mathbb{R}$. Its dependence structure is given by

$$Cov(X_0, X_t) = \int_S \int_{-\infty}^{0} f(A, -s)\sigma^2 f(A, t-s) \, ds \, d\pi \ \text{ with } t \in \mathbb{R}$$

and controlled by the probability measure π. If we choose a causal MMA process as the model for the volatility of a logarithmic asset price, its dependence structure can be modeled in a versatile way by choosing the distribution π. Then, the typical decay of the autocovariances of the squared returns, see [17], can be more easily reproduced.

Let the logarithmic asset price $(J_t)_{t \in \mathbb{R}^+}$ be

$$J_t = \int_0^t \sqrt{X_s} dW_s, \ \ J_0 = 0, \tag{5.1}$$

where $(W_t)_{t \in \mathbb{R}^+}$ is a standard Brownian motion and $(X_t)_{t \in \mathbb{R}^+}$ is an adapted, stationary and square-integrable causal MMA process with values in $\mathbb{R}^+$ being independent of W. We call (5.1) an MMA SV model. In the literature, stochastic volatility models where X is given by a sum of independent non-Gaussian OU type process are given in [6, 9] which have been later extended to supOU processes in [10, 11]. The latter is an example of an MMA SV model whose

dependence structure is going to be analysed in this section. Other financial market models using supOU processes as building blocks and allowing for short and long range dependence can be found in [33, 36, 37].

We show the θ-weak dependence of the return process, over equidistant time intervals $[(t-1)\Delta, t\Delta]$

$$Y_t = J_{t\Delta} - J_{(t-1)\Delta} = \int_{(t-1)\Delta}^{t\Delta} \sqrt{X_s} dW_s, \qquad (5.2)$$

where for convenience of notation we consider $t \in \mathbb{R}$, and the asymptotic normality of its related sample moments. To this aim, the moments of the return process by using the Itô isometry, as in [44], turn out to be determined as a function of the moments of the integrated process

$$V_t = \int_{(t-1)\Delta}^{t\Delta} X_s \, ds, \qquad (5.3)$$

for $t \in \mathbb{R}$ and Δ a positive constant. Note that, $(V_t)_{t\in\mathbb{R}}$ corresponds to the integrated volatility process computed over a time interval $[(t-1)\Delta, t\Delta]$. It is immediate from the definitions (5.2) and (5.3) that the strict stationarity of the process $(X_s)_{s\in\mathbb{R}}$ and its square-integrability imply the stationarity and the square-integrability of the processes $(Y_t)_{t\in\mathbb{R}}$, $(Y_t^2)_{t\in\mathbb{R}}$ and $(V_t)_{t\in\mathbb{R}}$. Note that, under the square integrability assumption, the moments of the return process can be determined up to the 4th order.

In general, we consider all processes adapted with respect to the filtration $(\mathcal{A}_t)_{t\in\mathbb{R}}$ generated by the set of random variables $\{\Lambda(B) : B \in \mathcal{B}(S \times (-\infty, t])\}$ and the increments of the Brownian motion $\{(W_u - W_s)_{s\leq u\leq t}\}$ for all $t \in \mathbb{R}$.

In order to ensure that the stochastic integrals involving an MMA process as integrand are well defined we assume throughout that

$$(H): \begin{cases} \text{The Lévy basis } \Lambda \text{ has generating quadruple } (\gamma, 0, \nu, \pi) \text{ such that} \\ \int_{\mathbb{R}} |x|\nu(dx) < \infty, \ \ \gamma - \int_{|x|\leq 1} x\nu(dx) \geq 0 \text{ and } \nu(\mathbb{R}^-) = 0; \\ \text{the kernel function } f \text{ is } \mathcal{B}(S \times \mathbb{R}^+)\text{-measurable, non-negative} \\ \text{and satisfies the assumptions of Corollary 2.1;} \\ X_s = \int_{-\infty}^{s} f(A, s-u) \, \Lambda(dA, du) \text{ is adapted and càdlàg.} \end{cases}$$

Sufficient conditions for an MMA process to have càdlàg sample paths can be found in [42, Theorem 3.1] and the references therein.

We now show the weak dependence properties of the return process.

Proposition 5.1. *Let W be a standard Brownian motion independent of the Lévy basis Λ and assume Assumptions (H) are satisfied. Then, the return process defined in (5.2) is θ-weakly dependent with coefficients*

$$\theta_Y(r) = \sqrt{\Delta\, \theta_X((r-1)\Delta)}, \qquad (5.4)$$

where $(\theta_X(r))_{r\in\mathbb{R}^+}$ are the coefficients (3.21), for all $r \geq 1$.

Proof. The assumptions (H) imply that the resulting MMA process is non-negative and the process Y_t is well defined and square-integrable by Corollary 2.2. Let $Y_t^{(m)}$ be defined for $m \geq 0$

$$Y_t^{(m)} = \int_{(t-1)\Delta}^{t\Delta} \sqrt{X_s^{(m)}} \, dW_s$$

where $X_s^{(m)}$ is defined in (3.17). Then,

$$\mathbb{E}[|Y_t - Y_t^{(m)}|] = \mathbb{E}\Big[\Big|\int_{(t-1)\Delta}^{t\Delta} \sqrt{X_s} - \sqrt{X_s^{(m)}} \, dW_s\Big|\Big] \tag{5.5}$$

$$\leq \mathbb{E}\Big[\int_{(t-1)\Delta}^{t\Delta} (\sqrt{X_s} - \sqrt{X_s^{(m)}})^2 \, ds\Big]^{\frac{1}{2}}$$

by using the inequality $\sqrt{a+b} - \sqrt{a} \leq \sqrt{b}$ for all $a, b \in \mathbb{R}^+$

$$\leq \mathbb{E}\Big[\int_{(t-1)\Delta}^{t\Delta} |X_s - X_s^{(m)}| \, ds\Big]^{\frac{1}{2}}$$

$$\leq \mathbb{E}\Big[\int_{(t-1)\Delta}^{t\Delta} \Big|\int_S \int_{-\infty}^{s-m} f(A, s-u) \, \Lambda(dA, du)\Big|\Big]^{\frac{1}{2}} \leq \sqrt{\Delta \theta_X(m)}.$$

The last inequality follows by (3.22).

Let F and G belong respectively to the class of bounded functions $\mathcal{H}^*$ and $\mathcal{H}$, $(u, v) \in \mathbb{N}^* \times \mathbb{N}^*$, $r \in \mathbb{R}^+$, $(i_1, \ldots, i_u) \in \mathbb{R}^u$ and $(j_1, \ldots, j_v) \in \mathbb{R}^v$ with $i_1 \leq \ldots \leq i_u \leq i_u + r \leq j_1 \leq \ldots \leq j_v$, $Y_i^* = (Y_{i_1}, \ldots, Y_{i_u})$ and $Y_j^{*(m)} = (Y_{j_1}^{(m)}, \ldots, Y_{j_v}^{(m)})$ where for all $m \geq 0$

$$Y_{i_u} = \int_{(i_u-1)\Delta}^{i_u\Delta} \sqrt{X_s} \, dW_s \text{ and } Y_{j_1}^{(m)} = \int_{(j_1-1)\Delta}^{j_1\Delta} \sqrt{X_s^{(m)}} \, dW_s.$$

Then, if $(j_1 - 1)\Delta - m - i_u\Delta \geq 0$ that can also be expresses as $j_1 - i_u \geq \frac{m}{\Delta} + 1$, $I_u = S \times (-\infty, i_u\Delta]$ and $J_1 = S \times [(j_1 - 1)\Delta - m, j_1\Delta]$ are disjoint sets or have as intersection a set of measure zero when $i_u\Delta = (j_1 - 1)\Delta - m$, i.e. $\pi \times \lambda(S \times \{i_u\Delta\}) = 0$. Then, by the definition of a Lévy basis, the integrands of Y_{i_u} and $Y_{j_1}^{(m)}$ are independent and so are Y_{i_u} and $Y_{j_1}^{(m)}$ because of the independence of W and the Lévy basis. Then, the sequences $(Y_i)_{i \leq i_u}$ and $(Y_j^{(m)})_{j \geq j_1}$ are independent and so are $F(Y_i^*)$ and $G(Y_j^{*(m)})$. Therefore, let $m = (r-1)\Delta$

$$|Cov(F(Y_i^*), G(Y_j^*))|$$

$$\leq |Cov(F(Y_i^*), G(Y_j^*) - G(Y_j^{*(m)}))| + |Cov(F(Y_i^*), G(Y_j^{*(m)}))|$$

$$\leq 2\,\mathbb{E}|G(Y_j^*) - G(Y_j^{*(m)})|$$

the last relation comes from $\|F\|_\infty \le 1$

$$\le 2\,Lip(G) \sum_{k=1}^{v} \mathbb{E}|Y_{j_k} - Y_{j_k}^{(m)}|$$

using (5.5)

$$= 2vLip(G)\sqrt{\Delta\,\theta_X((r-1)\Delta)},$$

which converges to zero as r goes to infinity by applying the dominated convergence theorem. $\qquad\Box$

Let us consider a supOU process X defined as in (3.24) such that the underlying Lévy process L is a subordinator. It can be shown that the process is adapted and càdlàg under the assumptions (ii) and (iii) of [10, Theorem 3.12]. Then, Assumptions (H) are satisfied and we can define a supOU SV model and the resulting return process

$$Y_t = \int_{(t-1)\Delta}^{t\Delta} \sqrt{\int_{\mathbb{R}^-} \int_{-\infty}^{s} e^{A(s-u)}\,\Lambda(dA, du)} dW_s. \tag{5.6}$$

By applying Proposition 5.1, Y is θ-weakly dependent with coefficients

$$\theta_Y(r) = \sqrt{-\Delta\mu \int_{\mathbb{R}^-} \frac{e^{A\Delta(r-1)}}{A}\,\pi(dA)}, \tag{5.7}$$

where μ is the mean of the underlying Lévy process.

We consider a sample of N observations of Y and we define the following sample moments for the return process.

The sample mean

$$\frac{1}{N}\sum_{i=1}^{N} Y_{1+i}, \tag{5.8}$$

the sample autocovariance function for $k \in \mathbb{N}$

$$\frac{1}{N}\sum_{j=1}^{N} Y_{1+j}Y_{1+j+k}, \tag{5.9}$$

and the fourth order (non-centered) sample moments for $k \in \mathbb{N}$

$$\frac{1}{N}\sum_{j=1}^{N} Y_{1+j}^2 Y_{1+j+k}^2. \tag{5.10}$$

When the asymptotic properties of the sample autocovariance functions are investigated, we focus on the processes

$$W_{j,k} = Y_{1+j}Y_{1+j+k} - T(k), \tag{5.11}$$

where we denote by $T(k)$ the covariances of order k defined by

$$T(k) = \mathbb{E}[Y_1 Y_{1+k}],$$

whereas in the case of the fourth order sample moments on

$$\tilde{W}_{j,k} = Y_{1+j}^2 Y_{1+j+k}^2 - D^*(k) - \mathbb{E}[V_1]^2, \tag{5.12}$$

where we denote by $D^*(k)$ the covariances of order k defined by

$$D^*(k) = Cov(V_1, V_{1+k})$$

where V is the integrated process defined in (5.3).

Analogous to Proposition 4.1, we show that the θ-weak dependence of the return process is inherited by the process $Z_t = (Y_t, Y_{t+1}, \ldots, Y_{t+k})$ for all $k \in \mathbb{N}$.

Lemma 5.1. *Let Λ be a Lévy basis and X an MMA process satisfying Assumptions (H) and W be a standard Brownian motion independent of Λ. We consider the process*

$$Z_t = \begin{pmatrix} Y_t \\ \vdots \\ Y_{t+k} \end{pmatrix} = \begin{pmatrix} \int_{(t-1)\Delta}^{(t)\Delta} \sqrt{X_s}\, dW_s \\ \vdots \\ \int_{(t+k-1)\Delta}^{(t+k)\Delta} \sqrt{X_s}\, dW_s \end{pmatrix} = \int_{(t-1)\Delta}^{(t+k)\Delta} G_s\, dW_s$$

where Y_t is the return process defined in (5.2) and G_s is an $\mathbb{R}^{k+1} \times \mathbb{R}$ valued process defined as $G_s = \begin{pmatrix} \sqrt{X_s}\mathbb{1}_{((t-1)\Delta, t\Delta]}(s) \\ \sqrt{X_s}\mathbb{1}_{(t\Delta, (t+1)\Delta]}(s) \\ \vdots \\ \sqrt{X_s}\mathbb{1}_{((t+k-1)\Delta, (t+k)\Delta]}(s) \end{pmatrix}$. Then, Z is θ-weakly dependent with coefficients

$$\theta_Z(r) = \mathcal{D}^* \sqrt{\Delta \theta_X((r-k-1)\Delta)}$$

for $r \geq k+1$ being $\mathcal{D}^* = (k+1)$ and θ_X given in (3.21).

Proof. For $m \geq 0$, let $Z_t^{(m)}$ be

$$Z_t^{(m)} = \int_{(t-1)\Delta}^{(t+k)\Delta} \sqrt{G_s^{(m)}}\, dW_s$$

where $G_s^{(m)} = \begin{pmatrix} \sqrt{X_s^{(m)}}\mathbb{1}_{((t-1)\Delta, t\Delta]}(s) \\ \sqrt{X_s^{(m)}}\mathbb{1}_{(t\Delta, (t+1)\Delta]}(s) \\ \vdots \\ \sqrt{X_s^{(m)}}\mathbb{1}_{((t+k-1)\Delta, (t+k)\Delta]}(s) \end{pmatrix}$. Then,

$$\mathbb{E}[\|Z_t - Z_t^{(m)}\|] = \mathbb{E}\Big[\Big\| \int_{(t-1)\Delta}^{(t+k)\Delta} G_s - G_s^{(m)} \, dW_s \Big\|\Big] \tag{5.13}$$

$$\leq \mathbb{E}\Big[\int_{(t-1)\Delta}^{(t+k)\Delta} tr((G_s - G_s^{(m)})(G_s - G_s^{(m)})') \, ds \Big]^{\frac{1}{2}}$$

by means of the triangular inequality

$$\leq \mathbb{E}\Big[\int_{(t-1)\Delta}^{t\Delta} (\sqrt{X_s} - \sqrt{X_s^{(m)}})^2 \, ds \Big]^{\frac{1}{2}} + \ldots + \mathbb{E}\Big[\int_{(t+k-1)\Delta}^{(t+k)\Delta} (\sqrt{X_s} - \sqrt{X_s^{(m)}})^2 \, ds \Big]^{\frac{1}{2}}$$

$$\leq (k+1)\sqrt{\Delta\theta_X(m)}.$$

Proceeding as in Proposition 4.1 and in Proposition 5.1 the claim follows. $\square$

It can also be shown that the process Z_t is mixing, and thus ergodic, proceeding as in the proof of [27, Theorem 4.2].

The following asymptotic result holds for (5.8).

Theorem 5.1. *We assume that Assumptions (H) hold and that $\int_{|x|>1} |x|^{1+\delta}\nu(dx) < \infty$, for some $\delta > 0$, and f belongs to $L^{1+\delta}(S \times \mathbb{R}^+, \pi \otimes \lambda) \cap L^1(S \times \mathbb{R}^+, \pi \otimes \lambda)$. If $(Y_i)_{i\in\mathbb{R}}$ as defined in (5.2) is a θ-weakly dependent process such that the volatility process X admits coefficients $\theta_X(r) = O(r^{-\alpha})$ with $\alpha > 2\left(1 + \frac{1}{\delta}\right)$, then $\sigma_Y^2 = Var(Y_1)$ is non-negative and as $N \to \infty$*

$$\frac{1}{\sqrt{N}} \sum_{i=1}^{N} Y_{1+i} \xrightarrow{d} \mathcal{N}(0, \sigma_Y^2). \tag{5.14}$$

Proof. Corollary 2.2 applies and the return process is ergodic because of its mixing properties shown in [27, Theorem 4.2]. [20, Theorem 1] can be applied, analogously as in Theorem 4.2, assuring the result (5.14) where the asymptotic variance of the sample mean is given by the absolute summable series $\sum_{l\in Z} Cov(Y_1, Y_{1+l}) = Var(Y_1)$. $\square$

Applying Proposition 5.1 and Proposition 3.4, the following can be easily shown.

Proposition 5.2. *We assume that Assumptions (H) hold and that $\int_{\|x\|>1} \|x\|^{2+\delta}\nu(dx) < \infty$, for some $\delta > 0$, and f belongs to $L^{2+\delta}(S\times\mathbb{R}^+, \pi\otimes\lambda)\cap L^1(S \times \mathbb{R}^+, \pi \otimes \lambda)$. If $(Y_i)_{i\in\mathbb{Z}}$ as defined in (5.2) is θ-weakly dependent process with coefficients θ_Y as in (5.4), then, for all $k > 0$ $\mathcal{Z} = (W_{j,0}, W_{j,1}, \ldots, W_{j,k})_{j\in\mathbb{Z}}$ is θ-weakly dependent with coefficients*

$$\theta_{\mathcal{Z}}(r) = \mathcal{C}(\mathcal{D}^*\sqrt{\Delta\theta_X((r-k-1)\Delta)})^{\frac{\delta}{1+\delta}},$$

Moreover, if we assume that $\int_{\|x\|>1} \|x\|^{4+\delta}\nu(dx) < \infty$, for some $\delta > 0$, and f belongs to $L^{4+\delta}(S \times \mathbb{R}, \pi \otimes \lambda) \cap L^1(S \times \mathbb{R}, \pi \otimes \lambda)$, then the process $\tilde{\mathcal{Z}} = (\tilde{W}_{j,0}, \tilde{W}_{j,1}, \ldots, \tilde{W}_{j,k})_{j\in\mathbb{Z}}$ is θ-weakly dependent with coefficients

$$\theta_{\tilde{\mathcal{Z}}}(r) = \mathcal{C}(\mathcal{D}^* \sqrt{\Delta \theta_X((r-k-1)\Delta)})^{\frac{\delta}{3+\delta}}.$$

The constant $\mathcal{C}$ is independent of r and $\mathcal{D}^* = k+1$ in the above formulas.

Corollary 5.1. *We assume that Assumptions (H) hold and that $\int_{\|x\|>1} \|x\|^{2+\delta} \nu(dx) < \infty$, for some $\delta > 0$, and f belongs to $L^{2+\delta}(S \times \mathbb{R}^+, \pi \otimes \lambda) \cap L^1(S \times \mathbb{R}^+, \pi \otimes \lambda)$. If $(\mathcal{Z}_i)_{i \in \mathbb{Z}}$ as defined in Proposition 5.2 is θ-weakly dependent such that the volatility process X admits coefficients $\theta_X(r) = O(r^{-\alpha})$ with $\alpha > (1 + \frac{1}{\delta})(\frac{2+2\delta}{\delta})$, then for each $p, q \in \{0, \dots, k\}$ with $k \in \mathbb{N}$,*

$$\sum_{l \in \mathbb{N}} Cov(W_{0,p}, W_{l,q})$$

are finite and as $N \to \infty$

$$\frac{1}{\sqrt{N}} \sum_{j=1}^N \mathcal{Z}_j \xrightarrow{d} \mathcal{N}_{k+1}(0, \Psi)$$

where Ψ is equal to

$$\begin{bmatrix} \sum_{l \in \mathbb{Z}} Cov(Y_1^2, Y_{l+1}^2) & \sum_{l \in \mathbb{Z}} Cov(Y_1^2, Y_{l+1}Y_{l+2}) & \cdots & \sum_{l \in \mathbb{Z}} Cov(Y_1^2, Y_{l+1}Y_{l+k+1}) \\ \cdots & \sum_{l \in \mathbb{Z}} Cov(Y_1 Y_2, Y_{l+1}Y_{l+2}) & \cdots & \sum_{l \in \mathbb{Z}} Cov(Y_1 Y_2, Y_{l+1}Y_{l+k+1}) \\ \cdots & \cdots & \cdots & \sum_{l \in \mathbb{Z}} Cov(Y_1 Y_{k+1}, Y_{l+1}Y_{l+k+1}) \end{bmatrix}.$$

and positive semidefinite.

Proof. Since Corollary 2.2 holds, Z is a θ-weakly dependent process with coefficients given in Proposition 5.2, $\theta_Z(r) = \mathcal{C}(\mathcal{D}^* \sqrt{\Delta \theta_X((r-k-1)\Delta)})^{\frac{\delta}{1+\delta}}$. For all $a \in \mathbb{R}^{k+1}$, $a'Z$ is a θ-weakly dependent process, because a linear function is Lipschitz, and ergodic having the same θ-coefficients as the process $\mathcal{Z}$. Under the assumptions of the Corollary, [20, Theorem 1] is then applied and

$$\frac{1}{\sqrt{N}} \sum_{j=1}^N a' \mathcal{Z}_j \xrightarrow{d} \mathcal{N}(0, a'\Psi a)$$

as $N \to \infty$. Applying the Cramer-Wold device, the asymptotic normality of the vector $\mathcal{Z}$ is shown. □

Corollary 5.2. *We assume that Assumptions (H) hold and that $\int_{\|x\|>1} \|x\|^{4+\delta} \nu(dx) < \infty$, for some $\delta > 0$, and f belongs to $L^{4+\delta}(S \times \mathbb{R}^+, \pi \otimes \lambda) \cap L^1(S \times \mathbb{R}^+, \pi \otimes \lambda)$. If $(\tilde{\mathcal{Z}}_i)_{i \in \mathbb{Z}}$ as defined in Proposition 5.2 is θ-weakly dependent such that the volatility process X admits coefficients $\theta_X(r) = O(r^{-\alpha})$ with $\alpha > (1 + \frac{1}{\delta})(\frac{6+2\delta}{\delta})$, then for each $p, q \in \{0, \dots, k\}$ with $k \in \mathbb{N}$,*

$$\sum_{l \in \mathbb{N}} Cov(\tilde{W}_{0,p}, \tilde{W}_{l,q})$$

are finite and as $N \to \infty$

$$\frac{1}{\sqrt{N}} \sum_{j=1}^N \tilde{\mathcal{Z}}_j \xrightarrow{d} \mathcal{N}_{k+1}(0, \mathcal{M})$$

where $\mathcal{M}$ is equal to

$$\begin{bmatrix} \sum_{l\in\mathbb{Z}} Cov(Y_1^4, Y_{l+1}^4) & \sum_{l\in\mathbb{Z}} Cov(Y_1^4, Y_{l+1}^2 Y_{l+2}^2) & \cdots & \sum_{l\in\mathbb{Z}} Cov(Y_1^4, Y_{l+1}^2 Y_{l+k+1}^2) \\ \cdots & \sum_{l\in\mathbb{Z}} Cov(Y_1^2 Y_2^2, Y_{l+1}^2 Y_{l+2}^2) & \cdots & \sum_{l\in\mathbb{Z}} Cov(Y_1^2 Y_2^2, Y_{l+1}^2 Y_{l+k+1}^2) \\ \cdots & \cdots & \cdots & \sum_{l\in\mathbb{Z}} Cov(Y_1^2 Y_{k+1}^2, Y_{l+1}^2 Y_{l+k+1}^2) \end{bmatrix}.$$

is positive semidefinite.

Proof. The proof follows as in Corollary 5.1, using the θ coefficients of the process $\tilde{\mathcal{Z}}$ as determined in Proposition 5.2. $\qquad\square$

Remark 5.1. *In view of Section 6, let us give explicit formulas of the third and fourth order cumulant of an integrated process V and of the covariances $Cov(\tilde{W}_{0,p}, \tilde{W}_{l,q})$ for $p, q \in \{0, \ldots, k\}$ and $k \in \mathbb{N}$ under the assumptions of Corollary 5.2. Let us consider an integrated process as defined in (5.3) with mean $\mathbb{E}[V_1] = C^*$. For all $(i, j, k, l) \in \mathbb{R}^4$, we call $K(i, j, k)$ and $K(i, j, k, l)$ the centered cumulant or order three and four which are respectively equal to*

$$K(i, j, k) = \int_{i\Delta}^{(i+1)\Delta} \int_{j\Delta}^{(j+1)\Delta} \int_{k\Delta}^{(k+1)\Delta} \kappa(s, t, u)\, ds\, dt\, du, \qquad (5.15)$$

with $\kappa(s, t, u)$ given in (4.15), and

$$K(i, j, k, l) = \int_{i\Delta}^{(i+1)\Delta} \int_{j\Delta}^{(j+1)\Delta} \int_{k\Delta}^{(k+1)\Delta} \int_{l\Delta}^{(l+1)\Delta} \kappa(s, t, u, z)\, ds\, dt\, du\, dz, \quad (5.16)$$

where $\kappa(s, t, u, z)$ is defined in (4.14). Moreover,

$$D^*(l) = Cov(V_1, V_{1+l}) = \int_0^\Delta \int_{l\Delta}^{(l+1)\Delta} D(u - s)\, ds\, du, \qquad (5.17)$$

where $D(\cdot)$ is the covariance function, as in (4.5), of the centered MMA process underlying the definition of the integrated process. Hence, by means of the Itô formula, the independence of the process $(X_t)_{t\in\mathbb{R}}$ with $(W_t)_{t\in\mathbb{R}}$ and using arguments similar to the formula (40) in [44]

$$Cov(\tilde{W}_{0,p}, \tilde{W}_{l,q})$$

$$= K(0, p, l, l+q) + C^*(K(0, p, l) + K(0, p, l+q) + K(p, l, l+q) + K(0, l, l+q))$$

$$+ C^{*2}(D^*(l - p) + D^*(l + q - p) + D^*(l) + D^*(l + q)) + D^*(l)D^*(l + q - p)$$

$$+ D^*(l + q)D^*(l - p) + A(0, p, l, l+q),$$

where $A(i, j, k, l)$ is defined in Table 1 for $(i, j, k, l) \in \mathbb{R}^4$.

TABLE 1
Explicit closed formula for the summand $A(i,j,k,l)$ for $(i,j,k,l) \in \mathbb{Z}^4$.

(i,j,k,l)	$A(i,j,k,l)$
$\{l \neq i \neq j \neq k\}$	0
$\{i \neq j\} \wedge \{j = k = l\}$	$12 \int_{i\Delta}^{(i+1)\Delta} \int_{j\Delta}^{(j+1)\Delta} \int_{j\Delta}^{(j+1)\Delta} \int_{j\Delta}^{s} \mathbb{E}[X_t X_z X_s X_u] du ds dz dt$
$\{i \neq j \neq k\} \wedge \{k = l\}$	$4 \int_{i\Delta}^{(i+1)\Delta} \int_{j\Delta}^{(j+1)\Delta} \int_{k\Delta}^{(k+1)\Delta} \int_{k\Delta}^{s} \mathbb{E}[X_t X_z X_s X_u] du ds dz dt$
$\{i = j\} \wedge \{k = l\} \wedge \{i \neq k\}$	$4 \int_{i\Delta}^{(i+1)\Delta} \int_{i\Delta}^{(i+1)\Delta} \int_{k\Delta}^{(k+1)\Delta} \int_{k\Delta}^{s} \mathbb{E}[X_t X_z X_s X_u] du ds dz dt$ $+4 \int_{i\Delta}^{(i+1)\Delta} \int_{i\Delta}^{t} \int_{k\Delta}^{(k+1)\Delta} \int_{k\Delta}^{(k+1)\Delta} \mathbb{E}[X_t X_z X_s X_u] du ds dz dt$ $+16 \int_{i\Delta}^{(i+1)\Delta} \int_{k\Delta}^{(k+1)\Delta} \int_{i\Delta}^{t} \int_{k\Delta}^{s} \mathbb{E}[X_t X_s X_z X_u] du ds dz dt$
$\{i = j = k = l\}$	$\mathbb{E}\left[24 \left(\int_{i\Delta}^{(i+1)\Delta} X_t \, dt \right)^2 \int_{i\Delta}^{(i+1)\Delta} X_s \int_{i\Delta}^{s} X_u du ds \right]$ $+\mathbb{E}\left[96 \int_{i\Delta}^{(i+1)\Delta} \int_{i\Delta}^{s} X_u \int_{i\Delta}^{u} X_z \, dz du ds \right]$

6. Generalized method of moments for the supOU process and supOU SV model

In this section, we apply the developed asymptotic theory to determine the asymptotic normality of GMM estimators of the supOU process and of the supOU SV model defined in [48].

Let X and Y be a supOU process and a return process as respectively defined in (3.24) and (5.6). We assume,

$$\int_{|x|>1} x^2 \nu(dx) < \infty,$$

then, as shown in [48, Theorem 2.3 and Theorem 2.8], the supOU process X and the return process Y have known moments given by

$$\mathbb{E}[X_0] = - \mu \int_{\mathbb{R}^-} \frac{1}{A} \pi(dA), \ \ \mathrm{Var}[X_0] = -\sigma^2 \int_{\mathbb{R}^-} \frac{1}{2A} \pi(dA),$$

$$\mathrm{Cov}[X_0, X_{k\Delta}] = -\sigma^2 \int_{\mathbb{R}^-} \frac{e^{Ak\Delta}}{2A} \pi(dA),$$

and

$$\mathbb{E}[Y_1] = 0, \ \ Var[Y_1] = -\Delta \mu \int_{\mathbb{R}^-} \frac{1}{A} \pi(dA), \ \ Cov(Y_1, Y_{1+k}) = 0,$$

$$\mathbb{E}[Y_1^2] = -\Delta \mu \int_{\mathbb{R}^-} \frac{1}{A} \pi(dA),$$

$$Var(Y_1^2) = -3\sigma^2 \int_{\mathbb{R}^-} \frac{1}{A^2} \left(\frac{e^{A\Delta}}{A} - \frac{1}{A} - \Delta \right) \pi(dA) + 2 \left(-\Delta \mu \int_{\mathbb{R}^-} \frac{1}{A} \pi(dA) \right)^2,$$

$$Cov(Y_1^2, Y_{1+k}^2) = -\sigma^2 \int_{\mathbb{R}^-} \frac{1}{2A^3} (s_{k+1} - 2s_k + s_{k-1}) \pi(dA),$$

where $\mu = \mathbb{E}[L_1]$ and $\sigma^2 = Var[L_1]$ are the mean and variance of the underlying Lévy process and $s_k := e^{A\Delta k}$.

Assumption 6.1. *Let us assume that the mean reversion parameter A is Gamma distributed. That is, we assume that π is the distribution of $B\xi$ where $B \in \mathbb{R}^-$ and ξ is $\Gamma(\alpha_\pi, 1)$ distributed with $\alpha_\pi > 2$.*

We emphasize that setting the second parameter of the Gamma distribution equal to one does not restrict the model since this is equivalent to varying B. Under Assumption 6.1, we observe the decay of the autocovariances of the supOU process as given in Remark 3.2, notice that in this set-up $\alpha = \alpha_\pi - 1$.

The moments of the supOU process X and of the return process Y, under Assumption 6.1, have been computed in [48, Section 2.2]

$$\mathbb{E}[X_0] = -\frac{\mu}{B(\alpha_\pi - 1)}, \ \ \mathrm{Var}[X_0] = -\frac{\sigma^2}{2B(\alpha_\pi - 1)},$$
$$\mathrm{Cov}[X_0, X_{k\Delta}] = -\frac{\sigma^2(1 - Bk\Delta)^{1-\alpha_\pi}}{2B(\alpha_\pi - 1)}, \tag{6.1}$$

$$\mathbb{E}[Y_1] = 0 \ \ Var(Y_1) = -\frac{\Delta\mu}{B(\alpha_\pi - 1)}, \ \ Cov(Y_1, Y_{1+k}) = 0,$$

$$\mathbb{E}[Y_1^2] = -\frac{\Delta\mu}{B(\alpha_\pi - 1)},$$

$$Var(Y_1^2) = -3\sigma^2\frac{(1 - B\Delta)^{3-\alpha_\pi} - 1 - \Delta B(\alpha_\pi - 3)}{B^3(\alpha_\pi - 1)(\alpha_\pi - 2)(\alpha_\pi - 3)} + 2\Big(-\frac{\Delta\mu}{B(\alpha_\pi - 1)}\Big)^2,$$

$$Cov(Y_1^2, Y_{1+k}^2) = -\frac{\sigma^2(f_{k+1} - 2f_k + f_{k-1})}{2B^3(\alpha_\pi - 1)(\alpha_\pi - 2)(\alpha_\pi - 3)}, \tag{6.2}$$

where $f_k := (1 - B\Delta k)^{3-\alpha_\pi}$. Therefore, the moment structure, up to the 2nd order for the supOU process and up to the 4th order for the return process, depends only on the parameter vector $\theta := (\mu, \sigma^2, \alpha_\pi, B)$.

6.1. SupOU process

Suppose we observe a sample $\{X_t : t = 1\Delta, \ldots, N\Delta\}$ for the supOU process with Δ a positive constant. We construct the following moment functions, as in [48], by using the auto-covariances up to a lag $m \geq 2$ of X.

First, let us define $X_t^{(m)} = (X_{t\Delta}, X_{(t+1)\Delta}, \ldots, X_{(t+m)\Delta})$ for $t = 1, \ldots, N - m$ and the measurable function $h : \mathbb{R}^{m+1} \times \Theta \to \mathbb{R}^{m+2}$ as

$$h(X_t, \theta) = \begin{pmatrix} h_{\mathbb{E}}(X_t^{(m)}, \theta) \\ h_0(X_t^{(m)}, \theta) \\ \vdots \\ h_k(X_t^{(m)}, \theta) \\ \vdots \\ h_m(X_t^{(m)}, \theta) \end{pmatrix}$$

$$
= \begin{pmatrix}
X_{t\Delta} + \frac{\mu}{B(\alpha_\pi - 1)} \\
X_{t\Delta}^2 - \left(\frac{\mu}{B(\alpha_\pi - 1)}\right)^2 + \frac{\sigma^2}{2B(\alpha_\pi - 1)} \\
X_{t\Delta} X_{(t+1)\Delta} - \left(\frac{\mu}{B(\alpha_\pi - 1)}\right)^2 + \frac{\sigma^2 (1 - B\Delta)^{1-\alpha_\pi}}{2B(\alpha_\pi - 1)} \\
\vdots \\
X_{t\Delta} X_{(t+k)\Delta} - \left(\frac{\mu}{B(\alpha_\pi - 1)}\right)^2 + \frac{\sigma^2 (1 - Bk\Delta)^{1-\alpha_\pi}}{2B(\alpha_\pi - 1)} \\
\vdots \\
X_{t\Delta} X_{(t+m)\Delta} - \left(\frac{\mu}{B(\alpha_\pi - 1)}\right)^2 + \frac{\sigma^2 (1 - Bm\Delta)^{1-\alpha_\pi}}{2B(\alpha_\pi - 1)}
\end{pmatrix}. \tag{6.3}
$$

We can now define the sample moment function for the supOU process as

$$
g_{N,m}(X, \theta) = \begin{pmatrix}
\frac{1}{N-m} \sum_{t=1}^{N-m} h_{\mathbb{E}}(X_t^{(m)}, \theta) \\
\frac{1}{N-m} \sum_{t=1}^{N-m} h_0(X_t^{(m)}, \theta) \\
\vdots \\
\frac{1}{N-m} \sum_{t=1}^{N-m} h_k(X_t^{(m)}, \theta) \\
\vdots \\
\frac{1}{N-m} \sum_{t=1}^{N-m} h_m(X_t^{(m)}, \theta)
\end{pmatrix} \tag{6.4}
$$

$$
= \begin{pmatrix}
\frac{1}{N-m} \sum_{t=1}^{N-m} \left(X_{t\Delta} + \frac{\mu}{B(\alpha_\pi - 1)} \right) \\
\frac{1}{N-m} \sum_{t=1}^{N-m} \left(X_{t\Delta}^2 - \left(\frac{\mu}{B(\alpha_\pi - 1)}\right)^2 + \frac{\sigma^2}{2B(\alpha_\pi - 1)} \right) \\
\frac{1}{N-m} \sum_{t=1}^{N-m} \left(X_{t\Delta} X_{(t+1)\Delta} - \left(\frac{\mu}{B(\alpha_\pi - 1)}\right)^2 + \frac{\sigma^2 (1 - B\Delta)^{1-\alpha_\pi}}{2B(\alpha_\pi - 1)} \right) \\
\vdots \\
\frac{1}{N-m} \sum_{t=1}^{N-m} \left(X_{t\Delta} X_{(t+k)\Delta} - \left(\frac{\mu}{B(\alpha_\pi - 1)}\right)^2 + \frac{\sigma^2 (1 - Bk\Delta)^{1-\alpha_\pi}}{2B(\alpha_\pi - 1)} \right) \\
\vdots \\
\frac{1}{N-m} \sum_{t=1}^{N-m} \left(X_{t\Delta} X_{(t+m)\Delta} - \left(\frac{\mu}{B(\alpha_\pi - 1)}\right)^2 + \frac{\sigma^2 (1 - Bm\Delta)^{1-\alpha_\pi}}{2B(\alpha_\pi - 1)} \right)
\end{pmatrix},
$$

and estimate θ_0 by minimizing the objective function

$$
\hat{\theta}_0^{N,m} = \operatorname{argmin} g_{N,m}(X, \theta)' A_{N,m} g_{N,m}(X, \theta) \tag{6.5}
$$

where $A_{N,m}$ is a positive definite matrix to weight the $m+2$ different moments collected in $g_{N,m}(X, \theta)$.

We aim to show the asymptotic normality of the GMM estimator (6.5). Hence, we first need to show that the moment function $h(X_t, \theta_0)$ satisfies a central limit theorem.

Theorem 6.1. *Let Λ be a real valued Lévy basis with generating quadruple $(\gamma, \Sigma, \nu, \pi)$ satisfying assumptions (3.23) such that $\int_{|x|>1} |x|^{4+\delta} \nu(dx) < \infty$ for*

some $\delta > 0$ and let Assumption 6.1 hold with $\alpha_\pi - 1 > (1 + \frac{1}{\delta})(\frac{6+2\delta}{2+\delta})$. Let X be the resulting supOU process, then $h(X_t, \theta_0)$ is a θ-weakly dependent process,

$$H_\Sigma = \sum_{l \in \mathbb{Z}} Cov(h(X_0, \theta_0), h(X_l, \theta_0)) \tag{6.6}$$

is finite, positive semidefinite and as $N \to \infty$

$$\sqrt{N} g_{N,m}(X, \theta_0) \xrightarrow{d} \mathcal{N}(0, H_\Sigma). \tag{6.7}$$

Proof. Proposition 2.1 shows that the $4 + \delta$-th moments of the supOU process exist. We call $C := -\frac{\mu}{B(\alpha_\pi - 1)}$ and, following the notations of Section 4, $D(k) = -\frac{\sigma^2 (1 - B\Delta k)^{1-\alpha_\pi}}{2B(\alpha_\pi - 1)}$ for $k = 0, \ldots, m$. We then consider the vector $Z = (X_{t\Delta}, X_{(t+1)\Delta}, \ldots, X_{(t+m)\Delta})$ and the function $f : \mathbb{R}^{m+1} \to \mathbb{R}^{m+2}$ such that

$$f(Z_t) = f \begin{pmatrix} X_{t\Delta} \\ X_{(t+1)\Delta} \\ \cdots \\ X_{(t+k)\Delta} \\ \cdots \\ X_{(t+m)\Delta} \end{pmatrix} = h(X_t, \theta_0) + \begin{pmatrix} C \\ D(0) + C^2 \\ \cdots \\ D(k) + C^2 \\ \cdots \\ \cdots \\ D(m) + C^2 \end{pmatrix}.$$

Z is a θ-weakly dependent process with coefficients

$$\theta_Z(r) = \mathcal{D}\theta_X(r - m\Delta)$$

where θ_X is given in Formula (3.26). The assumptions of Proposition 3.4 hold with $p = 4 + \delta$, $c = 1$, $a = 2$, thus $f(Z_t)$ is a θ-weakly dependent process with coefficients $\mathcal{C}(\mathcal{D}\theta_X(r - m\Delta))^{\frac{2+\delta}{3+\delta}}$ for $r \in \mathbb{N}$. Hence, $h(X_t, \theta_0)$ is a θ-weak dependent process with the same coefficients and mean zero. We have that $\theta_h(r) = \mathcal{C}\mathcal{D}^{\frac{2+\delta}{3+\delta}} \left(-\frac{\sigma^2 (1 - B(2r - 2m\Delta))^{1-\alpha_\pi}}{2B(\alpha_\pi - 1)} + \left(\frac{-2\mu(1 - B(r - m\Delta))^{1-\alpha_\pi}}{2B(\alpha_\pi - 1)} \right)^2 \right)^{\frac{2+\delta}{6+2\delta}}$, where α_π satisfies the inequality $(\alpha_\pi - 1) > 2(1 + \frac{1}{\delta})(\frac{3+\delta}{2+\delta})$ by assumption. Analogously to the proof of Corollary 4.2, by applying [20, Theorem 1] and the Cramer-Wold device, we obtain the distributional result (6.7). $\qquad\square$

Remark 6.1. *The coefficients of the matrix $Cov(h(X_0, \theta_0), h(X_l, \theta_0))$ for $l \in \mathbb{Z}$ are*

$$Cov(h_\mathbb{E}(X_0^{(m)}, \theta_0), h_\mathbb{E}(X_l^{(m)}, \theta_0) = D(l), \tag{6.8}$$

$$Cov(h_\mathbb{E}(X_0^{(m)}, \theta_0), h_p(X_l^{(m)}, \theta_0)) = \kappa(0, l, l+p) + C(D(l) + D(l+p)), \tag{6.9}$$

$$Cov(h_p(X_0^{(m)}, \theta_0), h_q(X_l^{(m)}, \theta_0)) = \kappa(0, p, l, l+q) \tag{6.10}$$

$$+ C(\kappa(0, p, l) + \kappa(0, p, l+q) + \kappa(p, l, l+q) + \kappa(0, l, l+q))$$

$$+ C^2(D(l-p) + D(l+q-p) + D(l) + D(l+q)) + D(l)D(l+q-p) + D(l+q)D(l-p),$$

for $p, q \in \{0, \ldots, m\}$, where C and $D(l)$ are defined in the proof of Theorem 6.1 and $\kappa(i,j,k)$ and $\kappa(i,j,k,l)$ are respectively the cumulants of the supOU process of order three and four for $i, j, k, l \in \{0, \ldots, m\}$ defined in (4.15) and (4.14).

In Table 2, the explicit expressions of the autocovariances and cumulants of the supOU process are reported.

Corollary 6.1. *Let Λ be a real valued Lévy basis with generating quadruple $(\gamma, 0, \nu, \pi)$ such that $\int_{\mathbb{R}} |x| \, \nu(dx) < \infty$ and $\int_{|x|>1} |x|^{4+\delta} \, \nu(dx) < \infty$ for some $\delta > 0$ and let Assumption 6.1 hold with $\alpha_\pi - 1 > (1 + \frac{1}{\delta})(\frac{3+\delta}{2+\delta})$. Let $(X_t)_{t \in \mathbb{R}}$ be the resulting supOU process, then $h(X_t, \theta_0)$ is a θ-weakly dependent process,*

$$H_\Sigma = \sum_{l \in \mathbb{Z}} Cov(h(X_0, \theta_0), h(X_l, \theta_0))$$

is finite, positive semidefinite and as $N \to \infty$

$$\sqrt{N} g_{N,m}(X, \theta_0) \xrightarrow{d} \mathcal{N}(0, H_\Sigma).$$

The proof of the Corollary follows the same steps as Theorem 6.1.

Remark 6.2. *Under the assumptions of Corollary 6.1, a slower decay of the autocovariances of a supOU process is required to obtain asymptotic normality compared to Theorem 6.1. Moreover, if all the moments of the underlying Lévy process exist then the asymptotic result of Corollary 6.1 holds assuming that $\alpha_\pi > 2$. The latter assumption results in the slowest decay of the autocovariances of X that can be reached under short memory, see Remark 3.2 remembering that in the notations of this section $\alpha = \alpha_\pi - 1$, whereas the asymptotic result of Theorem 6.1 holds, under these assumptions, for $\alpha_\pi > 3$.*

Several assumptions have to be made to show the asymptotic normality of the GMM estimator (6.5):

Assumption 6.2. *The parameter space Θ is compact and large enough to include the true parameter vector θ_0.*

Assumption 6.3. *The matrix $A_{N,m}$ converges in probability to a positive definite matrix of constants A.*

Assumption 6.4. *The matrix H_Σ is positive definite.*

In our set-up we always need to choose a parameter space such that $\mu \geq 0$, $\sigma^2 > 0$, $\alpha_\pi > 2$ and $B < 0$. However, Assumption 6.2 remains reasonable, due to the fact that typically an optimization procedure is used to determine $\hat{\theta}^{N,m}$ and then some parameter bounds are always imposed in practice.

Theorem 6.2. *Let Λ be a real valued Lévy basis with generating quadruple $(\gamma, \Sigma, \nu, \pi)$ and X a supOU process satisfying assumptions (3.23) such that $\int_{|x|>1} |x|^{4+\delta} \, \nu(dx) < \infty$ for some $\delta > 0$. Let Assumption 6.1 hold with $\alpha_\pi - 1 > (1+\frac{1}{\delta})(\frac{6+2\delta}{2+\delta})$ or $\alpha_\pi - 1 > (1+\frac{1}{\delta})(\frac{3+\delta}{2+\delta})$ if in addition $\int_{\mathbb{R}} |x| \, \nu(dx) < \infty$. Moreover, if Assumption 6.2, 6.3 and 6.4 hold, then as N goes to infinity*

$$\sqrt{N}(\hat{\theta}_0^{N,m} - \theta_0) \xrightarrow{d} \mathcal{N}(0, M H_\Sigma M')$$

TABLE 2. *Explicit closed formula for the summands appearing in (6.8),(6.9) and (6.10), where $p \leq q$ and $\mu = \mathbb{E}[L_1]$, $\sigma^2 = Var[L_1]$, $s = \sigma^{-3}\mathbb{E}[(L_1 - \mu)^3]$ and $\eta = \sigma^{-4}\mathbb{E}[(L_1 - \mu)^4]$.*

$\{l \geq 0\}$	$\{l < 0\}$
$Cov(X_0, X_{l\Delta}) = \frac{-\sigma^2(1-B\Delta l)^{1-\alpha_\pi}}{2B(\alpha_\pi - 1)}$	$Cov(X_0, X_{l\Delta}) = \frac{-\sigma^2(1+B\Delta l)^{1-\alpha_\pi}}{2B(\alpha_\pi - 1)}$
$\{l \geq 0\}$	$\{l < 0\}$
$\kappa(0, l, l+p) = \frac{-s\sigma^3(1-B\Delta(2l+p))^{1-\alpha_\pi}}{3B(\alpha_\pi - 1)}$	$\kappa(0, p, l) = \frac{-s\sigma^3(1-B\Delta(p-l))^{1-\alpha_\pi}}{3B(\alpha_\pi - 1)}$
$\{l \geq p\}$	$\{0 < l < p\} \vee \{l \leq 0\}$
$\kappa(p, l, l+q) = \frac{-s\sigma^3(1-B\Delta(2l-2p+q))^{1-\alpha_\pi}}{3B(\alpha_\pi - 1)}$	$\kappa(p, l, l+q) = \frac{-s\sigma^3(1-B\Delta(p-l+q))^{1-\alpha_\pi}}{3B(\alpha_\pi - 1)}$
$\{l \geq 0\}$	$\{l < 0\}$
$\kappa(0, p, l, l+q) = -\frac{(\eta-3)\sigma^4}{4B(\alpha_\pi - 1)}(1 - B\Delta(p+2l+q))^{1-\alpha_\pi}$	$\kappa(0, p, l, l+q) = -\frac{(\eta-3)\sigma^4}{4B(\alpha_\pi - 1)}(1 - B\Delta(p+q-2l))^{1-\alpha_\pi}$

where

$$M = (G_0'AG_0)^{-1}G_0'A, \ \ G_0 = \mathbb{E}[\frac{\partial h(X_t, \theta)}{\partial \theta'}]_{\theta=\theta_0},$$
$$and \ H_\Sigma = \sum_{l \in \mathbb{Z}} Cov(h(X_0, \theta_0), h(X_l, \theta_0)).$$

Proof. We follow the steps of the proof of [41, Theorem 1.2]. This involves checking Assumption 1.1-1.9 in [41]. Note that Assumption 1.1-1.3 in [41] are sufficient to give the consistency of the estimator, as obtained in [48, Theorem 3.2]. We show them for completeness. Assumption 1.1(i) is fulfilled by the function $h(X_t, \theta)$ by construction. Assumption 1.1.(ii) is satisfied since the true parameter vector θ_0 is identifiable as shown in [48, Proposition 3.3]. Asssumption 1.2 in [41] requires that $\sup_{\theta \in \Theta} |g_{N,m}^{(i)}(X, \theta) - \mathbb{E}[h^{(i)}(X_t, \theta)]| \xrightarrow{p} 0$ as $N \to \infty$, for all $i = 1, \ldots, m+2$, i.e. each vector component of $g_{N,m}(X, \theta) - \mathbb{E}[h(X_t, \theta)]$ converges uniformly in probability to zero for each $\theta \in \Theta$. Assumption 1.4-1.6 in [41] represent three sufficient conditions that if fulfilled imply as consequence Assumption 1.2 in [41]. We then show that all three of them hold in our set-up. Assumption 1.4 in [41] corresponds to our Assumption 6.2 and Assumption 1.5 in [41] follows from the ergodicity of the supOU process. Showing Assumption 1.6 in [41] means to prove that a stochastic Lipshitz-type assumption holds for each component of the function $h(X_t, \theta)$. Let $\theta_i = (\mu_i, \sigma_i^2, \alpha_\pi^i, B_i)$ be parameter vectors belonging to Θ for $i = 1, 2$. Then, for example for the first component

$$\left| h_\mathbb{E}(X_t^{(m)}, \theta_1) - h_\mathbb{E}(X_t^{(m)}, \theta_2) \right| = \left| \frac{\mu_1}{B_1(\alpha_\pi^1 - 1)} - \frac{\mu_2}{B_2(\alpha_\pi^2 - 1)} \right|.$$

That means, by construction, the terms where X_t appears cancel out and the stochastic Lipschitz-type condition reduces to a Lipschitz continuity condition on the non-random terms in each component of $h(X_t, \theta)$. Since the terms have bounded first partial derivatives, they are Lipshchitz continuous and Assumption 1.6 in [41] holds. Assumption 1.3 in [41] is implied by our Assumption 6.3. Assumption 1.7 in [41] is fulfilled by construction, then $h(X_t, \theta)$ is continuously differentiable w.r.t. θ in Θ. Let $G_{N,m}(X, \theta) = \frac{1}{N-m} \sum_{t=1}^{N-m} \frac{\partial h(X_t, \theta)}{\partial \theta'}$, Assumption 1.8 in [41] requires that a weak law of large numbers applies to $\frac{\partial h(X_t, \theta)}{\partial \theta'}$ in a neighborhood of θ_0. That is, for each sequence θ_N^* such that $\theta_N^* \xrightarrow{p} \theta_0$ then $G_{N,m}(X, \theta_N^*) \to G_0$. We have that the matrix

$$\frac{\partial h(X_t, \theta)}{\partial \theta'} =$$

$$\left[\begin{array}{cccc} \frac{1}{B(\alpha_\pi - 1)} & 0 & -\frac{\mu}{B(\alpha_\pi - 1)^2} & -\frac{\mu}{B^2(\alpha_\pi - 1)} \\ -\frac{2\mu}{B^2(\alpha_\pi - 1)^2} & \frac{1}{2B(\alpha_\pi - 1)} & \frac{2\mu^2}{B^2(\alpha_\pi - 1)^3} - \frac{\sigma^2}{2B(\alpha_\pi - 1)^2} & \frac{2\mu^2}{B^3(\alpha_\pi - 1)^2} - \frac{\sigma^2}{2B^2(\alpha_\pi - 1)} \\ \cdots & \cdots & \cdots & \cdots \\ -\frac{2\mu}{B^2(\alpha_\pi - 1)^2} & \frac{(1 - B\Delta k)^{1-\alpha_\pi}}{2B(\alpha_\pi - 1)} & \tilde{a}(\Delta, \mu, \sigma^2, \alpha_\pi, B, k) & \tilde{b}(\Delta, \mu, \sigma^2, \alpha_\pi, B, k) \\ \cdots & \cdots & \cdots & \cdots \\ -\frac{2\mu}{B^2(\alpha_\pi - 1)^2} & \frac{(1 - B\Delta m)^{1-\alpha_\pi}}{2B(\alpha_\pi - 1)} & \tilde{a}(\Delta, \mu, \sigma^2, \alpha_\pi, B, m) & \tilde{b}(\Delta, \mu, \sigma^2, \alpha_\pi, B, m) \end{array} \right],$$

where

$$\tilde{a}(\Delta, \mu, \sigma^2, \alpha_\pi, B, k) = \frac{2\mu^2}{B^2(\alpha_\pi - 1)^3} - \frac{\sigma^2((\alpha_\pi - 1)ln(1 - B\Delta k) + 1)}{2B(\alpha_\pi - 1)^2(1 - B\Delta k)^{\alpha_\pi - 1}}$$

and

$$\tilde{b}(\Delta, \mu, \sigma^2, \alpha_\pi, B, k) = \frac{2\mu^2}{B^3(\alpha_\pi - 1)^2} + \sigma^2 \frac{B\Delta k(\alpha_\pi - 1) - (1 - B\Delta k)}{2B^2(\alpha_\pi - 1)(1 - B\Delta k)^{\alpha_\pi}}$$

for $k = 1\ldots, m$. Then, $\frac{\partial h(X_t, \theta)}{\partial \theta'}$ does not depend on X_t, $G_{N,m}(X, \theta) = \frac{\partial h(X_t, \theta)}{\partial \theta'}$ and $G_0 = \frac{\partial h(X_t, \theta_0)}{\partial \theta'}$. By the continuous mapping theorem Assumption 1.8 in [41] then follows. Assumption 1.9 in [41] follows by Theorem 6.1 or Corollary 6.1. Then, because of Assumption 6.4, the asymptotic normality of the estimator follows from the same steps as in the proof of [41, Theorem 1.2] when we replace f_T and F_T by $g_{N,m}$ and $G_{N,m}$. $\qquad\square$

6.2. SupOU SV model

We work now with a sample $\{Y_t : t = 1, \ldots, N\}$ of the return process and define $Y_t^{(m)} = (Y_{t+1}, Y_{t+2}, \ldots, Y_{t+m+1})$ for $t = 1, \ldots, N - m$.

The moment function is given by the measurable function $\tilde{h} : \mathbb{R}^{m+1} \times \Theta \to \mathbb{R}^{m+2}$ as

$$\tilde{h}(Y_t, \theta) = \begin{pmatrix} \tilde{h}_{Var}(Y_t^{(m)}, \theta) \\ \tilde{h}_0(Y_t^{(m)}, \theta) \\ \tilde{h}_1(Y_t^{(m)}, \theta) \\ \vdots \\ \tilde{h}_m(Y_t^{(m)}, \theta) \end{pmatrix}$$

$$= \begin{pmatrix} Y_{t+1}^2 + \frac{\mu\Delta}{B(\alpha_\pi - 1)} \\ Y_{t+1}^4 - 3\left(\frac{\Delta\mu}{B(\alpha_\pi - 1)}\right)^2 + 3\sigma^2 \frac{(1 - B\Delta)^{3-\alpha_\pi} - 1 - \Delta B(\alpha_\pi - 3)}{B^3(\alpha_\pi - 1)(\alpha_\pi - 2)(\alpha_\pi - 3)} \\ Y_{t+1}^2 Y_{t+2}^2 - \left(\frac{\Delta\mu}{B(\alpha_\pi - 1)}\right)^2 + \sigma^2 \frac{f_2 - 2f_1 + f_0}{2B^3(\alpha_\pi - 1)(\alpha_\pi - 2)(\alpha_\pi - 3)} \\ \vdots \\ Y_{t+1}^2 Y_{t+1+m}^2 - \left(\frac{\Delta\mu}{B(\alpha_\pi - 1)}\right)^2 + \sigma^2 \frac{f_{m+1} - 2f_m + f_{m-1}}{2B^3(\alpha_\pi - 1)(\alpha_\pi - 2)(\alpha_\pi - 3)} \end{pmatrix}. \qquad (6.11)$$

In this case, the sample moment function of the return process is

$$g_{N,m}(Y, \theta) = \begin{pmatrix} \frac{1}{N-m} \sum_{t=1}^{N-m} \tilde{h}_{Var}(Y_t^{(m)}, \theta) \\ \frac{1}{N-m} \sum_{t=1}^{N-m} \tilde{h}_0(Y_t^{(m)}, \theta) \\ \frac{1}{N-m} \sum_{t=1}^{N-m} \tilde{h}_1(Y_t^{(m)}, \theta) \\ \vdots \\ \frac{1}{N-m} \sum_{t=1}^{N-m} \tilde{h}_m(Y_t^{(m)}, \theta) \end{pmatrix} \qquad (6.12)$$

TABLE 3
Explicit closed formula for the summand $A(i,j,k)$ for $(i,j,k) \in \mathbb{Z}^3$.

(i,j,k)	$A(i,j,k)$
$\{i \neq j \neq k\}$	0
$\{i \neq j\} \wedge \{j = k\}$	$4 \int_{i\Delta}^{(i+1)\Delta} \int_{j\Delta}^{(j+1)\Delta} \int_{j\Delta}^{s} \mathbb{E}[X_t X_s X_u] du ds dt$
$\{i = j = k\}$	$12 \int_{i\Delta}^{(i+1)\Delta} \int_{i\Delta}^{(i+1)\Delta} \int_{i\Delta}^{s} \mathbb{E}[X_t X_s X_u] du ds dt$

$$
= \begin{pmatrix}
\frac{1}{N-m} \sum_{t=1}^{N-m} \left(Y_{t+1}^2 + \frac{\mu\Delta}{B(\alpha_\pi - 1)} \right) \\
\frac{1}{N-m} \sum_{t=1}^{N-m} \left(Y_{t+1}^4 - 3\left(\frac{\Delta\mu}{B(\alpha_\pi - 1)} \right)^2 + 3\sigma^2 \frac{(1-B\Delta)^{3-\alpha_\pi} - 1 - \Delta B(\alpha_\pi - 3)}{B^3(\alpha_\pi - 1)(\alpha_\pi - 2)(\alpha_\pi - 3)} \right) \\
\frac{1}{N-m} \sum_{t=1}^{N-m} \left(Y_{t+1}^2 Y_{t+2}^2 - \left(\frac{\Delta\mu}{B(\alpha_\pi - 1)} \right)^2 + \sigma^2 \frac{f_2 - 2f_1 + f_0}{2B^3(\alpha_\pi - 1)(\alpha_\pi - 2)(\alpha_\pi - 3)} \right) \\
\vdots \\
\frac{1}{N-m} \sum_{t=1}^{N-m} \left(Y_{t+1}^2 Y_{t+1+m}^2 - \left(\frac{\Delta\mu}{B(\alpha_\pi - 1)} \right)^2 + \sigma^2 \frac{f_{m+1} - 2f_m + f_{m-1}}{2B^3(\alpha_\pi - 1)(\alpha_\pi - 2)(\alpha_\pi - 3)} \right)
\end{pmatrix},
$$

and θ_0 can be estimated by minimizing the objective function

$$
\hat{\theta}_0^{*N,m} = \operatorname{argmin} g_{N,m}(Y, \theta)' A_{N,m} g_{N,m}(Y, \theta) \tag{6.13}
$$

where $A_{N,m}$ is a positive definite matrix to weight the $m+2$ different moments collected in $g_{N,m}(Y, \theta)$.

The consistency of the estimator (6.13) is shown in [48, Theorem 3.2], and as before we need to show that the moment function $\tilde{h}(Y, \theta)$ satisfies a central limit theorem.

Theorem 6.3. *Let Λ be a real valued Lévy basis with generating quadruple $(\gamma, 0, \nu, \pi)$, Assumptions (H) be satisfied such that $\int_{|x|>1} |x|^{4+\delta} \nu(dx) < \infty$, for some $\delta > 0$, and let Assumption 6.1 hold with $\alpha_\pi - 1 > (1 + \frac{1}{\delta})(\frac{6+2\delta}{\delta})$. Let $(Y_t)_{t \in R}$ be the resulting return process of a supOU SV model, then*

$$
W_\Sigma = \sum_{l \in \mathbb{Z}} Cov(\tilde{h}(Y_0, \theta_0), \tilde{h}(Y_l, \theta_0))
$$

is finite, positive semidefinite and as $N \to \infty$

$$
\sqrt{N} g_{N,m}(Y, \theta_0) \xrightarrow{d} \mathcal{N}(0, W_\Sigma).
$$

Proof. Proceeding as in Theorem 6.1, it can be shown that $\tilde{h}(Y_t, \theta_0)$ is a θ-weakly dependent process with zero mean, by using Lemma 5.1 and Proposition 3.4. Given the θ-coefficients (3.28), we have $\theta_{\tilde{h}}(r) = \mathcal{C}\mathcal{D}^{*\frac{\delta}{3+\delta}} \times \left(-\Delta \frac{\mu(1-B\Delta(r-m-1))^{1-\alpha_\pi}}{B(\alpha_\pi - 1)} \right)^{\frac{\delta}{6+2\delta}}$, where $\alpha_\pi - 1 > (1 + \frac{1}{\delta})(\frac{6+2\delta}{\delta})$ by assumption. Then applying [20, Theorem 1] and the Cramer-Wold device the result follows. $\square$

TABLE 4. *Explicit closed formula for the summands* $(6.14),(6.15)$ *and* (6.16), *where* $p \geq q$, $f_k = (1 - B\Delta k)^{3-\alpha_\pi}$, $g_k = (1 - B\Delta k)^{4-\alpha_\pi}$, $h_k = (1 - B\Delta k)^{5-\alpha_\pi}$ *and* $\mu = \mathbb{E}[L_1]$, $\sigma^2 = Var[L_1]$, $s = \sigma^{-3}\mathbb{E}[(L_1 - \mu)^3]$ *and* $\eta = \sigma^{-4}\mathbb{E}[(L_1 - \mu)^4]$.

$\{l \geq 1\}$
$Cov(Y_1^2, Y_{l+1}^2) = \frac{-\sigma^2(f_{l+1} - 2f_l + f_{l-1})}{2B^3(\alpha_\pi - 1)(\alpha_\pi - 2)(\alpha_\pi - 3)}$
$\{l \geq p\} \wedge \{2l - 2p + q \geq 2\}$
$K(p, l, l+q) = \frac{-s\sigma^3(2g_{2l-2p+q-1} - g_{2l-2p+q-2} + g_{2l-2p+q+2} - 2g_{2l-2p+q+1})}{6B^4(\alpha_\pi - 1)(\alpha_\pi - 2)(\alpha_\pi - 3)(\alpha_\pi - 4)}$
$\{l \geq 0\} \wedge \{2l + q \geq 2\}$
$K(0, l, l+q) = \frac{-s\sigma^3(2g_{2l+q-1} - g_{2l+q-2} + g_{2l+q+2} - 2g_{2l+q+1})}{6B^4(\alpha_\pi - 1)(\alpha_\pi - 2)(\alpha_\pi - 3)(\alpha_\pi - 4)}$
$\{l \geq 0\} \wedge \{2l + q + p \geq 3\}$
$K(0, p, l, l+q) = -\frac{(\eta - 3)\sigma^4(-2h_{2l+q+p} + 3h_{2l+q+p-1} - 3h_{2l+q+p-2} + h_{2l+q+p-3} + h_{2l+q+p+3} - 3h_{2l+q+p+2} + 3h_{2l+q+p+1})}{12B^5(\alpha_\pi - 1)(\alpha_\pi - 2)(\alpha_\pi - 3)(\alpha_\pi - 4)(\alpha_\pi - 5)}$

Remark 6.3. *We observe that,*

$$\mathbb{E}[Y_1^2] = \mathbb{E}[V_1] = -\frac{\Delta\mu}{B(\alpha - 1)} := C^*,$$

$$Var[Y_1^2] = 3Var(V_1) + 2\mathbb{E}(V_1)^2 = -3\sigma^2 \frac{(1 - B\Delta)^{3-\alpha_\pi} - 1 - \Delta B(\alpha_\pi - 3)}{B^3(\alpha_\pi - 1)(\alpha_\pi - 2)(\alpha_\pi - 3)}$$

$$+ 2\Big(-\frac{\Delta\mu}{B(\alpha_\pi - 1)}\Big)^2 := D^*(0),$$

and

$$Cov(Y_1^2, Y_{1+k}^2) = Cov(V_1, V_{1+k}) = -\sigma^2 \frac{f_{k+1} - 2f_k + f_{k-1}}{2B^3(\alpha - 1)(\alpha - 2)(\alpha - 3)} := D^*(k),$$

where V is the integrated process as defined in (5.3) and C^* and $D^*(k)$ are defined in Remark 5.1. Thus, the coefficients of the matrix $Cov(\tilde{h}(Y_0, \theta_0), \tilde{h}(Y_l, \theta_0)$ for $l \in \mathbb{Z}$ and $p, q \in \{0, \dots, m\}$ are

$$Cov(\tilde{h}_{Var}(Y_0^{(m)}, \theta_0), \tilde{h}_{Var}(Y_l^{(m)}, \theta_0) = D^*(l), \tag{6.14}$$

$$Cov(\tilde{h}_{Var}(Y_0^{(m)}, \theta_0), \tilde{h}_p(Y_l^{(m)}, \theta_0)) = K(0, l, l + p) + A(0, l, l + p) \\ + C^*(D^*(l) + D^*(l + p)), \tag{6.15}$$

$$\begin{aligned} Cov(\tilde{h}_p(Y_0^{(m)}, \theta_0), \tilde{h}_q(Y_l^{(m)}, \theta_0)) &= K(0, p, l, l + q) + A(0, p, l, l + q) \\ &+ C^*(K(0, p, l) + K(0, p, l + q) + K(p, l, l + q) + K(0, l, l + q)) \\ &+ C^{*2}(D^*(l + q - p) + D^*(l - p) + D^*(l + q) + D^*(l)) \\ &+ D^*(l)D^*(l + q - p) + D^*(l + q)D^*(l - p), \end{aligned} \tag{6.16}$$

where $A(i, j, k)$ and $A(i, j, k, l)$ are defined in Table 3 and Table 1, respectively, and $K(i, j, k)$ and $K(i, j, k, l)$ in (5.15) and (5.16). In Table 4, the explicit expressions of the summands in (6.14), (6.15) and (6.16) can be found for a selection of indices l, p, q. The remaining cases can be easily derived by using the calculations in Table 2.

Remark 6.4. *Note that if all the moments of the underlying Lévy process exist than the asymptotic result of Theorem 6.3 holds assuming that $\alpha_\pi > 3$.*

Additional assumptions have to be made before showing the asymptotic normality of the GMM estimator (6.13):

Assumption 6.5. *The parameter vector θ_0 is identifiable, i.e. $\mathbb{E}[\tilde{h}(Y, \theta)] = 0$ for all Y if and only if $\theta = \theta_0$.*

Assumption 6.6. *The matrix W_Σ is positive definite.*

Regarding Assumption 6.5, it has been shown in [48, Corollary 3.6] that identifiability holds if the number of lags m in the definition of the moment function is infinity, the so called asymptotic identifiability. In practice, we always work with a finite number of lags. Although proving identifiability rigorously seems to be out of reach, the asymptotic identifiability suggests that Assumption 6.5 should be satisfied if m is sufficiently large. It will be interesting to analyze how this affects the precision of our estimator in a simulation study. This is, however, beyond the scope of the present paper.

Finally, we recall that in our set-up $\mu > 0$, $\sigma^2 > 0$, $\alpha_\pi > 2$ and $B < 0$ and the parameter space Θ is large enough to contain the true parameter vector.

Theorem 6.4. *Let Λ be a real valued Lévy basis with generating quadruple $(\gamma, 0, \nu, \pi)$, Assumptions (H) be satisfied such that $\int_{|x|>1} |x|^{4+\delta} \nu(dx) < \infty$, for some $\delta > 0$, and let Assumption 6.1 hold with $\alpha_\pi - 1 > (1 + \frac{1}{\delta})(\frac{6+2\delta}{\delta})$. If, moreover, Assumptions 6.2, 6.3, 6.5 and 6.6 hold, then as N goes to infinity*

$$\sqrt{N}(\hat{\theta}_0^{*N,m} - \theta_0) \xrightarrow{d} \mathcal{N}(0, MW_\Sigma M')$$

where

$$M = \mathbb{E}[G_0^{*\prime} A G_0^*]^{-1} G_0^{*\prime} A, \ G_0^* = \mathbb{E}[\frac{\partial \tilde{h}(Y_t, \theta)}{\partial \theta'}]_{\theta=\theta_0},$$

$$and \ W_\Sigma = \sum_{l \in \mathbb{Z}} Cov(\tilde{h}(Y_0, \theta_0), \tilde{h}(Y_l, \theta_0)).$$

Proof. We check that Assumptions 1.1-1.9 in [41] hold. Assumptions 1.1-1.7 follow by Assumptions 6.2, 6.3 and 6.5 and by construction of the moment function. The line of the proof, in this case, is exactly the same as in Theorem 6.2. The matrix

$$\frac{\partial \tilde{h}(Y_t, \theta)}{\partial \theta'}$$

is equal to

$$\begin{bmatrix} \frac{\Delta}{B(\alpha_\pi-1)} & 0 & -\frac{\Delta\mu}{B(\alpha_\pi-1)^2} & -\frac{\Delta\mu}{B^2(\alpha_\pi-1)} \\ -\frac{6\Delta^2\mu}{B^2(\alpha_\pi-1)^2} & \frac{3f_1-1-\Delta B(\alpha_\pi-3)}{B^3(\alpha_\pi-1)(\alpha_\pi-2)(\alpha_\pi-3)} & a(\Delta,\mu,\sigma^2,\alpha_\pi,B) & b(\Delta,\mu,\sigma^2,\alpha_\pi,B) \\ \cdots & \cdots & \cdots & \cdots \\ -\frac{2\Delta^2\mu}{B^2(\alpha_\pi-1)^2} & \frac{f_{m+1}-2f_m-f_{m-1}}{2B^3(\alpha_\pi-1)(\alpha_\pi-2)(\alpha_\pi-3)} & c(\Delta,\mu,\sigma^2,\alpha_\pi,B,m) & d(\Delta,\mu,\sigma^2,\alpha_\pi,B,m) \end{bmatrix}$$

where

$$a(\Delta,\mu,\sigma^2,\alpha_\pi,B) = \frac{6\Delta^2\mu^2}{B^2(\alpha_\pi-1)^3}$$

$$+ 3\frac{\sigma^2}{B^3}\frac{(\alpha_\pi-2)(\alpha_\pi-3)+(\alpha_\pi-1)(\alpha_\pi-2)+(\alpha_\pi-1)(\alpha_\pi-2)}{(\alpha_\pi-1)^2(\alpha_\pi-2)^2(\alpha_\pi-3)^2}$$

$$- \frac{3\sigma^2}{B^3}f_1 l_1 + \frac{3\sigma^2\Delta((\alpha_\pi-1)+(\alpha_\pi-2))}{B^2(\alpha_\pi-1)^2(\alpha_\pi-2)^2},$$

$$b(\Delta, \mu, \sigma^2, \alpha_\pi, B) = \frac{6\Delta^2 \mu^2}{B^3(\alpha_\pi - 1)^2}$$
$$+ \frac{3\sigma^2}{(\alpha_\pi - 1)(\alpha_\pi - 2)(\alpha_\pi - 3)} \Big(\frac{(\alpha_\pi - 3)(2\Delta + \Delta r_1)}{B^3} - 3\frac{f_1 - 1}{B^4} \Big),$$
$$c(\Delta, \mu, \sigma^2, \alpha_\pi, B, k) = \frac{2\Delta^2 \mu^2}{B^2(\alpha_\pi - 1)^3} - \frac{\sigma^2}{2B^3}(f_{k+1} l_{k+1} - 2f_k l_k + f_{k-1} l_{k-1}) \text{ for } k = 1, \ldots, m,$$

and

$$d(\Delta, \mu, \sigma^2, \alpha_\pi, B, k) = \frac{2\Delta^2 \mu^2}{B^3(\alpha_\pi - 1)^2} + \frac{\sigma^2}{2B^3(\alpha_\pi - 1)(\alpha_\pi - 2)}(r_{k+1}\Delta(k+1) - 2r_k\Delta k$$
$$+ r_{k-1}\Delta(k-1))$$
$$- \frac{3\sigma^2}{2B^4(\alpha_\pi - 1)(\alpha_\pi - 2)(\alpha_\pi - 3)}(f_{k+1} - 2f_k + f_{k-1}),$$

for $k = 1, \ldots, m$, with $r_k := (1 - B\Delta k)^{2-\alpha_\pi}$, $f_k := (1 - B\Delta k)^{3-\alpha_\pi}$ and

$$l_k = \frac{1}{(\alpha_\pi - 1)^2(\alpha_\pi - 2)^2(\alpha_\pi - 3)^2} \big[\ln(1 - B\Delta k)(\alpha_\pi - 1)(\alpha_\pi - 2)(\alpha_\pi - 3)$$
$$+ (\alpha_\pi - 2)(\alpha_\pi - 3) + (\alpha_\pi - 1)(\alpha_\pi - 3) + (\alpha_\pi - 1)(\alpha_\pi - 2) \big],$$

for $k \in \{0, \ldots, m+1\}$. Therefore, $\frac{\partial \tilde{h}(Y_t, \theta)}{\partial \theta'}$ does not depend on Y_t and, as in the proof of Theorem 6.2, the continuous mapping theorem can be applied to show that Assumption 1.8 in [41] holds. Assumption 1.9 follows by the proof of Theorem 6.3. Then, because of Assumption 6.6, the normality of the GMM estimator follows from the same steps as in the proof of [41, Theorem 1.2]. $\square$

In [48, Section 4] a simulation study on the estimators (6.5) and (6.13) looks at their finite sample performances (Theorem 6.2 and 6.4 are applicable to the set-up of the study). The analysis performed shows results in line with asymptotic normality derived theoretically in this paper. To obtain reliable estimation of supOU processes and supOU SV model, a substantial amount of data is needed.

Acknowledgements

References

[1] ANDREWS, D. W. K. (1984). Non-strong mixing autoregressive processes. *J. Appl. Probab.* **21** 930–934. MR0766830

[2] BARDET, J. M., DOUKHAN, P. and LEÓN, J. R. (2008). A uniform central limit theorem for the periodogram and its applications to Whittle parametric estimation for weakly dependent time series. *J. Time Series Anal.* **29** 906–945. MR2450902

[3] BARNDORFF-NIELSEN, O. E. (2001). Superposition of Ornstein-Uhlenbeck type processes. *Theory Probab. Appl.* **45** 175–194. MR1967758

[4] BARNDORFF-NIELSEN, O. E, BENTH, F. and VERAART, E. D. (2015). Cross-commodity modelling by multivariate ambit fields, in *Commodities, Energy and Environmental Finance, edited by R. Aïd, M. Ludkovski and R. Sircar*. Springer, New York. MR3380393

[5] BARNDORFF-NIELSEN, O. E, HEDEVANG, E., SCHMIEGEL, J. and SZOZDA, B. (2016). Some recent developments in ambit stochastics, in *Stochastics of Environmental and Financial Economics, edited by F. Benth and G. Di Nunno*. Springer, Cham. MR3451167

[6] BARNDORFF-NIELSEN, O. E., NICOLATO, E. and SHEPARD, N. (2002). Some recent developments in stochastic volatility modeling. Special issue on volatility modelling. *Quant. Finance* **2** 11–23. MR1919582

[7] BARNDORFF-NIELSEN, O. E. and LEONENKO, N. N. (2005). Spectral properties of superpositions of Ornstein-Uhlenbeck type processes, methodology and computing. *Methodol. Comput. Appl. Probab.* **7** 335–352. MR2210585

[8] BARNDORFF-NIELSEN, O. E. and SCHMIEGEL, J. (2007). Ambit processes: with applications to turbulence and cancer growth, in *Stochastic Analysis and Applications: The Abel Symposium 2005, Abel Symposia, vol. 2, edited by F.E. Benth, G. Di Nunno, T. Lindstrom, B. Øksendal and T. Zhang*. Springer, Berlin. MR2397785

[9] BARNDORFF-NIELSEN, O. E. and SHEPARD, N. (2001). Non-Gaussian Ornstein-Uhlenbeck based models and some of their uses in financial economics. *J. R. Stat. Soc. Ser. B Stat. Methodol.* **63** 167–241. MR1841412

[10] BARNDORFF-NIELSEN, O. E. and STELZER, R. (2011). Multivariate supOU processes. *Ann. Appl. Probab.* **21** 140–182. MR2759198

[11] BARNDORFF-NIELSEN, O. E. and STELZER, R. (2013). The multivariate supOU stochastic volatility model. *Math. Finance* **23** 275–296. MR3034078

[12] BARNDORFF-NIELSEN, O. E. and VERAART, A. E. D. (2013). Stochastic volatility of volatility and variance risk premia. *J. Fin. Econometrics* **11** 1–46.

[13] BINGHAM, N. H., GOLDIE, C. M. and TEUGELS, J. L. (1987). *Regular Variation*. Cambridge University Press, Cambridge. MR1015093

[14] BRANDES, D-P. and CURATO I. V. (2018). On the sample autocovariance of a Lévy driven moving average process when sampled at a renewal sequence. *arxiv:1804.02254v1*.

[15] BRANDES, D-P., CURATO I. V. and STELZER, R. (2019). Preservation of strong mixing and weak dependence under renewal sampling. *Work in progress*.

[16] BROCKWELL, P.J. (2001). Lévy driven CARMA processes. Non-linear non-Gaussian models and related filtering methods (Tokyo, 2000). *Ann. Inst. Statist. Math.* **53** 113–124. MR1820952

[17] CONT, R. and TANKOV, P. (2004). *Financial Modeling With Jump Processes*. Chapman & Hall/CRC, Boca Raton. MR2042661

[18] DEDECKER, J. and DOUKHAN, P. (2003). A new covariance inequality and applications. *Stochastic Process Appl.* **106** 63–80. MR1983043

[19] DEDECKER, J., DOUKHAN, P., LANG, G., LÉON, J. R., LOUHICHI, S.

and PRIEUR, C. (2008). *Weak dependence: with examples and applications.* Springer. MR2338725

[20] DEDECKER, J. and RIO, E. (2000). On the functional central limit theorem for stationary processes. *Ann. Inst. H. Poincaré Probab. Statist.* **36** 1–34. MR1743095

[21] DOUKHAN, P. (1994). *Mixing: properties and examples. Lecture Notes in Statistics, 85.* Springer-Verlag, New York. MR1312160

[22] DOUKHAN, P., FOKIANOS, K. and LI, X. (2012). On weak dependence conditions: the case of discrete valued processes. *Statist. Probab. Lett.* **82** 1941–1948. MR2970296

[23] DOUKHAN, P. and LOUHICHI, S. (1999). A new weak dependence condition and applications to moment inequalities. *Stochastic Process. Appl.* **84** 313–342. MR1719345

[24] DOUKHAN, P. and WINTENBERGER, O. (2007). An invariance principle for weakly dependent stationary general models. *Probab. Math. Statist.* **27** 45–73. MR2353271

[25] FASEN, V. and KLÜPPELBERG, C. (2007). Extremes of supOU processes, in *Stochastic Analysis and Applications: The Abel Symposium 2005, Abel Symposia, vol. 2, edited by F.E. Benth, G. Di Nunno, T. Lindstrom, B. Øksendal and T. Zhang.* Springer, Berlin. MR2397794

[26] FECHNER, Z. and STELZER, R. (2018). Limit behavior of the truncated pathwise Fourier-transformation of Lévy driven CARMA processes for non-equidistant discrete time observations. *Statist. Sinica* **28** 1633–1650. MR3821021

[27] FUCHS, F. and STELZER, R. (2013). Mixing conditions for multivariate infinitely divisible processes with an application to mixed moving averages and the supOU stochastic volatility model. *ESAIM Probab. Stat.* **17** 455–471. MR3070886

[28] GIRAITIS, L., KOUL, H. L. and SURGAILIS, D. (2012). *Large Sample Inference for Long Memory Processes.* Imperial College Press, London. MR2977317

[29] GRAHOVAC, D., LEONENKO, N., SIKORSKII, A. and TAQQU, M. S. (2018). The unusual properties of aggregated superpositions of Ornstein-Uhlenbeck type processes. *Bernoulli, to appear.*

[30] GRAHOVAC, D., LEONENKO, N., SIKORSKII, A. and TEŠNJAK, I. (2016). Intermittency of Superposition of Ornstein-Uhlenbeck type processes. *J. Stat. Phys.* **165** 390–408. MR3554387

[31] GRAHOVAC, D., LEONENKO, N. and TAQQU, M. S. (2017). Limit theorems,scaling of moments and intermittency for integrated finite variance supOU processes. *ArXiv:1711.09623v1.*

[32] GRIFFIN, J. E. and STEEL, M. F. (2010). Bayesian inference with stochastic volatility models using continuous superpositions of non-Gaussian Ornstein-Uhlenbeck processes. *Comput. Statist. Data Anal.* **11** 2594–2608. MR2720463

[33] HEYDE, C. C. and LEONENKO, N. N. (2005). Student processes. *Adv. in Appl. Probab.* **37** 342–365. MR2144557

[34] JACOD, J. and SHIRYAEV, A. N. (2003). *Limit theorems for stochastic processes, 2nd ed.* Springer, Berlin. MR1943877

[35] KELLY, B. C., TREU, T. , MALKAN, M., PANCOAST, A. and WOO, J-H. (2013). Active Galactic Nucleus Black Hole Mass Estimates in the Era of Time Domain Astronomy. *Astrophys. J.* **779** 1–18.

[36] KERSS, A. D. J., LEONENKO, N. N. and SIKORSKII, A. (2014). Risky asset models with tempered stable fractal activity time. *Stoch. Anal. Appl.* **32** 642–663. MR3219698

[37] LEONENKO, N. N, PETHERICK, S. and SIKORSKII, A. (2012). Fractal activity time models for risky asset with dependence and generalized hyperbolic distributions. *Stoch. Anal. Appl.* **30** 476–492. MR2916865

[38] MARQUARDT, T. (2006). Fractional Lévy processes with an application to long memory moving average processes. *Bernoulli* **12** 1099–1126. MR2274856

[39] MARQUARDT, T. and STELZER, R. (2007). Multivariate CARMA process. *Stochastic Processes Appl.* **117** 96–120. MR2287105

[40] MASUDA, H. (2007). Ergodicity and exponential β-mixing bounds for multidimensional diffusions with jumps. Stochastic Process. Appl. **117** 35–56. MR2287102

[41] MÁTYÁS, L. (1999). *Generalized method of moments estimation, Vol. 5.* Cambridge University Press. MR1688695

[42] MOSER, M. and STELZER, R. (2013). Functional regular variation of Lévy-driven multivariate mixed moving average processes. *Extremes* **16** 351–382. MR3101877

[43] PEDERSEN, J. (2003). The Lévy-Itô decomposition of an independently scattered random measure. *MaPhySto Research Paper 2, available at* *http: // www. maphysto. dk* .

[44] PIGORSCH, C. and STELZER, R. (2009). A multivariate Ornstein-Uhlenbeck type stochastic volatility model. *preprint available on* *www. uni-ulm. de/ en/ mawi/ finmath/ people/ stelzer/ publications. html* . MR2555198

[45] RAJPUT, B. S. and ROSIŃSKI, J. (1989). Spectral representations of infinitely divisible processes. *Probab. Theory and Relat. Fields* **82** 451–487. MR1001524

[46] ROSENBLATT, M. (1956). A central limit theorem and a strong mixing condition. *Proc. Nat. Acad. Sci. U.S.A.* **42** 43–47. MR0074711

[47] SATO, K. I. (1999). *Lévy Processes and Infinitely Divisible Distributions, Cambridge Studies in Advanced Mathematics 68.* Cambridge Univ. Press, Cambridge. MR1739520

[48] STELZER, R., TOSSDORF, T. and WITTILINGER, M. (2015). Moment based estimation of supOU processes and a related stochastic volatility model. *Stat. Risk Model.* **32** 1–24. MR3331779

[49] STELZER, R. and ZAVĬSIN, J. (2015). Derivative pricing under the possibility of long memory in the supOU stochastic volatility model, in *Innovations in Quantitative Risk Management edited by K. Glau, M. Scherer and R. Zagst.* Springer.