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WIENER FUNCTIONALS OF SECOND ORDER AND THEIR LÉVY MEASURES

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Abstract The distributions of Wiener functionals of second order are infinitely divisible. An explicit expression of the associated Lévy measures in terms of the eigenvalues of the corresponding Hilbert–Schmidt operators on the Cameron–Martin subspace is presented. In some special cases, a formula for the densities of the distributions is given. As an application of the explicit expression, an exponential decay property of the characteristic functions of the Wiener functionals is discussed. In three typical examples, complete descriptions are given.

Keywords Wiener functional of second order, Lévy measure, Mellin transform, exponential decay

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Introduction

Let W be a classical Wiener space, and μ be the Wiener measure on it. A Wiener functional F of second order is a measurable functional $F : W \to \mathbb{R}$ with $\nabla^3 F = 0$, ∇ being the Malliavin gradient. It is represented as a sum of Wiener chaos of order at most two. Widely known Wiener functionals of second order are the square of the L^2-norm on an interval of the Wiener process, Lévy's stochastic area, and the sample variance of the Wiener process. The study of Wiener functionals of second order has a history longer than a half century, and many contributions have been made. Among them, pioneering works were made by Cameron-Martin and Lévy [2, 3, 12] for the square of the L^2-norm on an interval of the Wiener process and Lévy's stochastic area. The sample variance plays an important role in the Malliavin calculus (cf. [8]), and it was studied in detail. For example, see [5, 7].

There are a lot of reasons why one studies such Wiener functionals. One is that they are the easiest functionals next to linear ones. This may sound rather nonsensical, but a wide gap between Wiener functionals of first and second orders can be found in recent works by Lyons (for example, see [13]). Recalling roles played by quadratic Lagrange functions in the theories of Feynman path integrals and of semi-classical analysis for Schrödinger operators, one must encounter another reason for studying Wiener functionals of second order. A Wiener functional of second order is one of key ingredients in the asymptotic theories, the Laplace method, the stationary phase method et al, on infinite dimensional spaces.

As was employed by Cameron-Martin and Lévy, a fundamental strategy to investigate Wiener functionals of second order is computing their Laplace transforms or characteristic functions, and then their Lévy measures. In this paper, we give explicit expressions of Lévy measures of Wiener functionals of second order in terms of the eigenvalues and eigenfunctions of the corresponding Hilbert-Schmidt operators. See Theorem 2. Moreover, we extend the result to the case where μ is replaced by a conditional probability (Theorem 4). These explicit representations are essentially based on the splitting property of the Wiener measure μ, in other words, a decomposition of the Brownian motion via the eigenfunctions of the Hilbert-Schmidt operator. Wiener used a decomposition of this kind, the Fourier series expansion, to construct a Brownian motion, and a generalization we use is due to Itô-Nisio [9].

With the help of the explicit expression, we compute the Mellin transform of the Lévy measures. See Proposition 5. Recently Biane, Pitman and Yor ([1, 15]) showed that certain probability distributions corresponding to Wiener functionals of second order are closely related to special functions like Riemann's ζ-function. The general expression given in Proposition 5 will indicate that the relations studied by them are very natural ones. As another application, we shall investigate the order of decay of the characteristic function as the parameter of Fourier transform tends to infinity. If the Lévy-Khintchine representation admits a Gaussian term, then the decay is very fast, but if there is no Gaussian term, then the decay is determined by the behavior of the Lévy measure at the origin. For details, see Theorem 7. A characteristic function of a quadratic Wiener functional is a key object to investigate the principle of stationary phase on the Wiener space, and its exponential decay is indispensable to achieve such a principle on infinite dimensional spaces. The exponential decay also implies that the distribution of the Wiener functional of second order has a density function of Gevrey class with respect to the Lebesgue measure, which relates to the property called transversal analyticity by Malliavin [14]. Another criteria for the distribution to possess a smooth density function will also be given, and a method to compute it by using the residue theorem is shown (Theorem 11).

In Section 3, all our general results are tested for three concrete Wiener functionals of second order mentioned above. Comparisons with known results will be also discussed there.

1 Lévy measures of Wiener functionals of second order

1.1 General Scheme

Throughout this subsection, (W, H, μ) stands for an abstract Wiener space. For the definition, see [11]. The inner product and the norm of H are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|_H$, respectively. Given a symmetric Hilbert-Schmidt operator $A : H \to H$ and $\ell \in H$, decomposing A as $A = \sum_{n=1}^{\infty} a_n h_n \otimes h_n$ with $a_n \in \mathbb{R}$ and an orthonormal basis $\{h_n\}_{n=1}^{\infty}$ of H such that $\|A\|_2^2 = \sum_{n=1}^{\infty} a_n^2 < \infty$, we define

$$Q_A = \sum_{n=1}^{\infty} a_n \left\{\langle\cdot, h_n\rangle^2 - 1\right\}, \tag{1}$$

$$f_{A,\ell}(x) = \begin{cases} \dfrac{1}{2}\displaystyle\sum_{n;a_n>0}\left\{\frac{1}{x} + \frac{\langle\ell, h_n\rangle^2}{a_n^3}\right\}\exp[-x/a_n], & x>0,\\ 0, & x=0,\\ \dfrac{1}{2}\displaystyle\sum_{n;a_n<0}\left\{\frac{1}{-x} + \frac{\langle\ell, h_n\rangle^2}{-a_n^3}\right\}\exp[-x/a_n], & x<0, \end{cases} \tag{2}$$

where $\langle\cdot, h_n\rangle$ stands for the Itô integral of h_n, and if there exists no a_n with required property, the summation is defined to be equal to zero. It is possible to define $f_{A,\ell}$ without using the eigenfunction decomposition of A. Indeed, if $B : H \to H$ is a symmetric non-negative definite Hilbert-Schmidt operator, then, for any $N \in \mathbb{N}$, a bounded linear operator $(B+\varepsilon I)^{-N}\exp\left[-\{B+\varepsilon I\}^{-1}\right]$ on H converges strongly to a linear operator $T_B^{(N)}$ of trace class. Decomposing $A : H \to H$ as $A = A_1 - A_2$ with symmetric non-negative definite Hilbert-Schmidt operators A_1 and A_2 on H such that $A_1A_2 = A_2A_1 = 0$, we obtain

$$f_{A,\ell}(x) = \begin{cases} \dfrac{1}{x}\mathrm{Tr}\, T^{(0)}_{(1/x)A_1} + \dfrac{1}{x^3}\langle\ell, T^{(3)}_{(1/x)A_1}\ell\rangle, & x>0,\\ 0, & x=0,\\ \dfrac{1}{|x|}\mathrm{Tr}\, T^{(0)}_{(1/|x|)A_2} + \dfrac{1}{|x|^3}\langle\ell, T^{(3)}_{(1/|x|)A_2}\ell\rangle, & x<0. \end{cases}$$

Lemma 1. *It holds that*

$$0 \le f_{A,\ell}(x) \le \frac{\|A\|_\infty^{k-2}}{2|x|^{k+1}}\left\{k!\|A\|_2^2 + (k+1)!\|\ell\|_H^2\right\} \tag{3}$$

for any $x \neq 0, k \ge 2$, *and*

$$\int_{-1}^{1} |x|^2 f_{A,\ell}(x)dx \le \frac{1}{2}\|A\|_2^2 + \|\ell\|_H^2, \tag{4}$$

where $\|A\|_\infty = \sup\{|a_n| : n = 1, 2, \dots\}$.

Proof. Since $\exp[-|x|/|a_n|] \le k!(|a_n|/|x|)^k$, (3) follows. (4) is an easy application of the monotone convergence theorem and an estimation

$$\int_0^1 |x|^m \exp[-|x|/|a_n|]dx = |a_n|^{m+1} \int_0^{1/|a_n|} y^m \exp[-y]dy \le m!|a_n|^{m+1},$$

for every $m \in \mathbb{N}$. $\square$

Theorem 2. *Let $A : H \to H$ be a symmetric Hilbert-Schmidt operator, $\ell \in H$, and $\gamma \in \mathbb{R}$. Then, for any $\lambda \in \mathbb{R}$, it holds that*

$$\int_W \exp\left[i\lambda\left(\frac{1}{2}Q_A + \langle\cdot,\ell\rangle + \gamma\right)\right] d\mu$$
$$= \exp\left[-\frac{\lambda^2\|\ell_A\|_H^2}{2} + i\lambda\gamma + \int_{\mathbb{R}}\left(e^{i\lambda x} - 1 - i\lambda x\right) f_{A,\ell}(x)dx\right], \quad (5)$$

where $i = \sqrt{-1}$ and

$$\ell_A = \sum_{n;a_n=0} \langle h_n, \ell\rangle h_n. \quad (6)$$

Remark 3. (i) The integrability of $(e^{i\lambda x} - 1 - i\lambda x)f_{A,\ell}(x)$ in (5) is guaranteed by Lemma 1.
(ii) The theorem asserts that the distribution of $\frac{1}{2}Q_A + \langle\cdot,\ell\rangle + \gamma$ is infinitely divisible and the corresponding Lévy measure is $f_{A,\ell}(x)dx$. Moreover, the distribution of $\frac{1}{2}Q_A + \gamma$ is selfdecomposable. See [16, §8 and §15].
(iii) Let $\mathfrak{C}_n(W)$ be the space of Wiener chaos of order n. A Wiener functional F of second order is a Wiener chaos of order at most two, i.e., $F \in \mathfrak{C}_2(W) \oplus \mathfrak{C}_1(W) \oplus \mathfrak{C}_0(W)$, and is of the form that $F = (1/2)Q_A + \langle\cdot,\ell\rangle + \gamma$ for some symmetric Hilbert-Schmidt operator A, $\ell \in H$, and $\gamma \in \mathbb{R}$, and $Q_A \in \mathfrak{C}_2(W)$, $\langle\cdot,\ell\rangle \in \mathfrak{C}_1(W)$, and $\gamma \in \mathfrak{C}_0(W)$. Moreover, A, ℓ, and γ are determined so that $A = \nabla^2 F$, $\ell = \int_W \nabla F\, d\mu$, and $\gamma = \int_W F\, d\mu$, where ∇ stands for the Malliavin derivative.
(iv) It should be noted that $\mathfrak{C}_2(W) \oplus \mathfrak{C}_1(W) \oplus \mathfrak{C}_0(W)$ is invariant under shifts in the direction of H. Namely, let $F = \frac{1}{2}Q_A + \langle\cdot,\ell\rangle + \gamma$ and $h \in H$. Then

$$F(\cdot + h) = F + \langle\cdot, Ah\rangle + \frac{1}{2}\langle h, Ah\rangle + \langle h,\ell\rangle \in \mathfrak{C}_2(W) \oplus \mathfrak{C}_1(W) \oplus \mathfrak{C}_0(W).$$

In particular, the theorem is applicable to a quadratic form of the form $\frac{1}{2}Q_A(\cdot - h)$, which is one of main ingredients in the study of the principle of stationary phase on W. See [17]

Proof of Theorem 2. The proof is divided into three steps according to the signs of a_n's, the eigenvalues of A.

1st step: the case where $a_n > 0$ for all $n \in \mathbb{N}$.
Let $\lambda > 0$. Note that

$$\frac{1}{2}Q_A + \langle\cdot,\ell\rangle = \sum_{n=1}^{\infty}\left[\frac{a_n}{2}\left\{\langle\cdot,h_n\rangle^2 - 1\right\} - \langle\ell,h_n\rangle\langle\cdot,h_n\rangle\right].$$

Since $\{\langle\cdot,h_n\rangle : n\in\mathbb{N}\}$ is a family of independent Gaussian random variables of mean 0 and variance 1, we obtain the following well known identity:

$$\int_W \exp\left[-\lambda\left(\frac{1}{2}Q_A+\langle\cdot,\ell\rangle+\gamma\right)\right]d\mu$$
$$=\exp[-\lambda\gamma]\prod_{n=1}^{\infty}\left((1+\lambda a_n)^{-1/2}e^{\lambda a_n/2}\exp\left[\frac{\lambda^2}{2}\frac{\langle\ell,h_n\rangle^2}{1+\lambda a_n}\right]\right). \quad (7)$$

Applying the identities

$$\log(1+(\lambda/a))=\int_0^\infty(1-e^{-\lambda x})\frac{e^{-ax}}{x}dx,\quad \int_0^\infty e^{-x/a}dx=a,$$
$$\int_0^\infty\left(e^{-\lambda x}-1+\lambda x\right)e^{-x/a}dx=\frac{a^3\lambda^2}{1+a\lambda},\quad a,\lambda>0,$$

we obtain

$$\log\left(\prod_{n=1}^{\infty}\left\{(1+a_n)^{-1/2}e^{a_n/2}\exp\left[\frac{\lambda^2}{2}\frac{\langle\ell,h_n\rangle^2}{1+a_n}\right]\right\}\right)$$
$$=-\frac{1}{2}\sum_{n=1}^{\infty}\{\log(1+\lambda a_n)-\lambda a_n\}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{\lambda^2\langle\ell,h_n\rangle^2}{1+\lambda a_n}$$
$$=\int_0^\infty(e^{-\lambda x}-1+\lambda x)f_{A,\ell}(x)dx.$$

Plugging this into (7), we obtain

$$\int_W \exp\left[-\lambda\left(\frac{1}{2}Q_A+\langle\cdot,\ell\rangle+\gamma\right)\right]d\mu$$
$$=\exp\left[-\lambda\gamma+\int_0^\infty\left(e^{-\lambda x}-1+\lambda x\right)f_{A,\ell}(x)dx\right]. \quad (8)$$

Note that

$$\left|\frac{d}{d\zeta}\left(e^{\zeta x}-1-\zeta x\right)\right|\le\frac{|\zeta|^2x^3}{2}+2|\zeta|x^2\quad\text{and}\quad\left|e^{\zeta x}-1-\zeta x\right|\le\frac{3|\zeta|^2x^2}{2}$$

for $\zeta\in\mathbb{C}$ with $\mathrm{Re}\zeta<0$ and $x\ge 0$. Hence, continuing (8) holomorphically to $\Omega=\{\zeta\in\mathbb{C}:\mathrm{Re}\zeta<0\}$, and then letting $\mathrm{Re}\zeta\to 0$ in Ω, we we arrive at (5), because $f_{A,\ell}(x)=0$ for $x\le 0$.

2nd step: the case where $a_n<0$ for all $n\in\mathbb{N}$.
Since $-Q_A=Q_{-A}$, applying the result in the first step to $-A$, we obtain

$$\int_W \exp\left[i\lambda\left(\frac{1}{2}Q_A+\langle\cdot,\ell\rangle+\gamma\right)\right]d\mu$$
$$=\int_X \exp\left[i(-\lambda)\left(\frac{1}{2}Q_{(-A)}+\langle\cdot,-\ell\rangle+(-\gamma)\right)\right]d\mu$$
$$=\exp[i\lambda\gamma]\exp\left[\int_{\mathbb{R}}\left(e^{-i\lambda x}-1+i\lambda x\right)f_{-A,-\ell}(x)dx\right].$$

Since $f_{-A,-\ell}(-x) = f_{A,\ell}(x)$, this yields (5).

3rd step: the general case.
Set

$$A_+ = \sum_{n; a_n > 0} a_n h_n \otimes h_n, \quad A_- = \sum_{n; a_n < 0} a_n h_n \otimes h_n,$$
$$\ell_+ = \sum_{n; a_n > 0} \langle \ell, h_n \rangle h_n, \quad \ell_- = \sum_{n; a_n < 0} \langle \ell, h_n \rangle h_n.$$

Then

$$\frac{1}{2} Q_A + \langle \cdot, \ell \rangle = \langle \cdot, \ell_A \rangle + \frac{1}{2} Q_{A_+} + \langle \cdot, \ell_+ \rangle + \frac{1}{2} Q_{A_-} + \langle \cdot, \ell_- \rangle.$$

Moreover, the random variables $\langle \cdot, \ell_A \rangle$, $\frac{1}{2} Q_{A_+} + \langle \cdot, \ell_+ \rangle$, and $\frac{1}{2} Q_{A_-} + \langle \cdot, \ell_- \rangle$ are independent under μ, and

$$f_{A,\ell} = f_{A_+,\ell_+} + f_{A_-,\ell_-}.$$

From the observations made in the 1st and 2nd steps, we obtain

$$\begin{aligned}
&\int_W \exp\left[i\lambda \left(\frac{1}{2} Q_A + \langle \cdot, \ell \rangle + \gamma \right) \right] d\mu \\
&= \int_W \exp\left[i\lambda \langle \cdot, \ell_A \rangle \right] d\mu \times \int_W \exp\left[i\lambda \left(\frac{1}{2} Q_{A_+} + \langle \cdot, \ell_+ \rangle + \gamma \right) \right] d\mu \\
&\qquad\qquad \times \int_W \exp\left[i\lambda \left(\frac{1}{2} Q_{A_-} + \langle \cdot, \ell_- \rangle \right) \right] d\mu \\
&= \exp\left[-\frac{\lambda^2 \|\ell_A\|_H^2}{2} \right] \exp\left[i\lambda\gamma + \int_{\mathbb{R}} \left(e^{i\lambda x} - 1 - i\lambda x \right) f_{A_+,\ell_+}(x) dx \right] \\
&\qquad\qquad \times \exp\left[\int_{\mathbb{R}} \left(e^{i\lambda x} - 1 - i\lambda x \right) f_{A_-,\ell_-}(x) dx \right],
\end{aligned}$$

from which (5) follows. □

1.2 Conditional expectation

Let $\eta = \{\eta_1, \ldots, \eta_m\} \subset W^*$, W^* being the dual space of W, be an orthonormal system in H ; $\langle \eta_i, \eta_j \rangle = \delta_{ij}$. Setting

$$W_0^{(\eta)} = \{ w \in W; \eta(w) = 0 \}, \quad H_0^{(\eta)} = H \cap W_0^{(\eta)},$$

where $\eta(w) = (\eta_1(w), \ldots, \eta_m(w)) \in \mathbb{R}^m$, we define a projection $P^{(\eta)} : W \to W_0^{(\eta)}$ by $P^{(\eta)} w = w - \sum_{n=1}^m \eta_n(w) \eta_n$, and denote by $\mu_0^{(\eta)}$ the induced measure of μ on $W_0^{(\eta)}$ via $P^{(\eta)}$. Then the triplet $(W_0^{(\eta)}, H_0^{(\eta)}, \mu_0^{(\eta)})$ is an abstract Wiener space.

For a symmetric Hilbert-Schmidt operator $A : H \to H$, we define a symmetric Hilbert-Schmidt operator $A^{(\eta)}$ on $H_0^{(\eta)}$ by $A^{(\eta)} = P^{(\eta)} A$. We denote by $\mathbb{E}_\mu[F | \eta(w) = y]$ the conditional expectation of a Wiener functional $F : W \to \mathbb{R}$ given $\eta(w) = y$. For $y = (y^1, \ldots, y^m) \in \mathbb{R}^m$, put $y \cdot \eta = \sum_{n=1}^m y^n \eta_n$.

Theorem 4. *Let $A : H \to H$ be a symmetric Hilbert-Schmidt operator, $\ell \in H$, and $\gamma \in \mathbb{R}$. Then, for every $\lambda \in \mathbb{R}$, it holds that*

$$\begin{aligned}
&\mathbb{E}_\mu\left[\exp\left[i\lambda\left(\frac{1}{2}Q_A + \langle\cdot,\ell\rangle + \gamma\right)\right]\middle|\eta(w) = y\right] \\
&= \exp\Bigg[-\frac{\lambda^2\|\{P^{(\eta)}(A(y\cdot\eta)+\ell)\}_{A^{(\eta)}}\|_H^2}{2} \\
&\qquad + i\lambda\Big\{\frac{1}{2}\Big(\langle A(y\cdot\eta),(y\cdot\eta)\rangle - \sum_{n=1}^m\langle A\eta_n,\eta_n\rangle\Big) + \langle(y\cdot\eta),\ell\rangle + \gamma\Big\} \\
&\qquad + \int_{\mathbb{R}}\left(e^{i\lambda x} - 1 - i\lambda x\right) f_{A^{(\eta)},P^{(\eta)}(A(y\cdot\eta)+\ell)}(x)dx\Bigg],
\end{aligned} \tag{9}$$

where $\{P^{(\eta)}(A(y\cdot\eta)+\ell)\}_{A^{(\eta)}}$ and $f_{A^{(\eta)},P^{(\eta)}(A(y\cdot\eta)+\ell)}(x)$ are defined by (2) and (6), computed on the space $(W_0^{(\eta)}, H_0^{(\eta)}, \mu_0^{(\eta)})$.

Proof. According to the decomposition of $w \in W$ so that $w = w_0 + y\cdot\eta$ with $w_0 = P^{(\eta)}w$ and $y = \eta(w)$, the Wiener measure μ is represented as

$$\mu(dw) = \mu_0^{(\eta)}(dw_0) \otimes \frac{1}{\sqrt{2\pi}^m}e^{-|y|^2/2}dy.$$

Hence we have

$$\mathbb{E}_\mu[F|\eta(w) = y] = \int_{W_0^{(\eta)}} F(w_0 + y\cdot\eta)\mu_0^{(\eta)}(dw_0). \tag{10}$$

Using a finite dimensional approximation argument, we can easily show that

$$\begin{aligned}
\frac{1}{2}Q_A(w) + \langle w,\ell\rangle =& \frac{1}{2}Q_{A^{(\eta)}}(w_0) + \frac{1}{2}\Big\{\langle A(y\cdot\eta),(y\cdot\eta)\rangle - \sum_{n=1}^m\langle A\eta_n,\eta_n\rangle\Big\} \\
&+ \langle w_0, P^{(\eta)}(A(y\cdot\eta)+\ell)\rangle + \langle(y\cdot\eta),\ell\rangle.
\end{aligned} \tag{11}$$

In conjunction with (10), applying Theorem 2 on $(W_0^{(\eta)}, H_0^{(\eta)}, \mu_0^{(\eta)})$, we obtain (9). ☐

1.3 Mellin transform

Proposition 5. *The Mellin transform of $f_{A,\ell}$ defined by (2) is given by*

$$\int_{\mathbb{R}}|x|^s f_{A,\ell}(x)dx = \frac{\Gamma(s)}{2}\sum_{n=1}^\infty|a_n|^s + \frac{\Gamma(s+1)}{2}\sum_{n:a_n\neq 0}|a_n|^{s-2}\langle\ell,h_n\rangle^2 \tag{12}$$

for any $s \geq 2$.

Proof. Rewrite

$$f_{A,\ell}(x) = \begin{cases} \dfrac{1}{2}\displaystyle\sum_{n;a_n>0}\left\{\frac{1}{|x|} + \frac{\langle\ell,h_n\rangle^2}{|a_n|^3}\right\}\exp[-|x|/|a_n|], & x > 0, \\ 0, & x = 0, \\ \dfrac{1}{2}\displaystyle\sum_{n;a_n<0}\left\{\frac{1}{|x|} + \frac{\langle\ell,h_n\rangle^2}{|a_n|^3}\right\}\exp[-|x|/|a_n|], & x < 0, \end{cases}$$

Then, by a change of variables $x \to -x$ on $(-\infty, 0)$, we obtain

$$\begin{aligned}\int_{\mathbb{R}} |x|^s f_{A,\ell}(x)dx &= \frac{1}{2}\sum_{n;a_n<0}\int_{-\infty}^{0} |x|^s \left\{\frac{1}{|x|} + \frac{\langle \ell, h_n\rangle^2}{|a_n|^3}\right\} e^{-|x|/|a_n|}dx \\ &\quad + \frac{1}{2}\sum_{n;a_n>0}\int_{0}^{\infty} |x|^s \left\{\frac{1}{|x|} + \frac{\langle \ell, h_n\rangle^2}{|a_n|^3}\right\} e^{-|x|/|a_n|}dx \\ &= \frac{1}{2}\sum_{n:a_n\neq 0}\int_{0}^{\infty} |x|^s \left\{\frac{1}{|x|} + \frac{\langle \ell, h_n\rangle^2}{|a_n|^3}\right\} e^{-|x|/|a_n|}dx \\ &= \frac{\Gamma(s)}{2}\sum_{n=1}^{\infty}|a_n|^s + \frac{\Gamma(s+1)}{2}\sum_{n:a_n\neq 0}|a_n|^{s-2}\langle \ell, h_n\rangle^2,\end{aligned}$$

which completes the proof. □

Remark 6. A sufficient condition for the Mellin transform of $f_{A,\ell}$ to have a meromorphic extension to $\mathbb{C}$ is given by Jorgenson and Lang ([10]).

2 Exponential decay and density functions

In this section, as an application of Theorems 2 and 4, we first study how fast a stochastic oscillatory integral decays when its phase function is a Wiener chaos of order at most two. Moreover we also show how to compute the density function of its distribution.

2.1 Exponential decay

Theorem 7. *Let (W, H, μ) be an abstract Wiener space, $A : H \to H$ be a symmetric Hilbert-Schmidt operator, $\ell \in H$, $y \in \mathbb{R}^m$, and $\eta = \{\eta_1, \dots, \eta_m\} \subset W^*$ be an orthonormal system in H. Suppose that $\ell_A = 0$ when μ is considered, and that $\{P^{(\eta)}(A(y\cdot\eta) + \ell)\}_{A^{(\eta)}} = 0$ when $\mu(\cdot|\eta(w) = y)$ is dealt with. Define $f_{A,\ell}$ by (2), and $f_{A^{(\eta)},P^{(\eta)}(A(y\cdot\eta)+\ell)}$ as described in Theorem 4.*
(i) *For any $\lambda \in \mathbb{R}$, it holds that*

$$\left|\int_W \exp\left[i\lambda\left(\frac{1}{2}Q_A + \langle\cdot,\ell\rangle + \gamma\right)\right] d\mu\right| \geq \exp\left[-\frac{\lambda^2}{2}\int_{\mathbb{R}} x^2 f_{A,\ell}(x)dx\right] \tag{13}$$

$$\begin{aligned}&\left|\mathbb{E}_\mu\left[\exp\left[i\lambda\left(\frac{1}{2}Q_A + \langle\cdot,\ell\rangle + \gamma\right)\right]\middle|\eta(w) = y\right]\right| \\ &\qquad\geq \exp\left[-\frac{\lambda^2}{2}\int_{\mathbb{R}} x^2 f_{A^{(\eta)},P^{(\eta)}(A(y\cdot\eta)+\ell)}(x)dx\right]\end{aligned} \tag{14}$$

(ii) *Let*

$$\begin{aligned}a_- &:= \sup\left\{a > 0 : \limsup_{\lambda\to\infty}\lambda^{-a}\int_{(-\infty,0)}(\cos(\lambda x) - 1)f_{A,\ell}(x)dx < 0,\right\},\\ a_+ &:= \sup\left\{a > 0 : \limsup_{\lambda\to\infty}\lambda^{-a}\int_{(0,\infty)}(\cos(\lambda x) - 1)f_{A,\ell}(x)dx < 0\right\},\end{aligned}$$

where $a_- = 0$, $a_+ = 0$ if $\{\cdots\} = \emptyset$. If $\max\{a_-, a_+\} > 0$, then, for every $a < \max\{a_-, a_+\}$, there exist $C_a > 0$ and $\lambda_a > 0$ such that

$$\left| \int_W \exp\left[i\lambda \left(\frac{1}{2} Q_A + \langle \cdot, \ell \rangle + \gamma \right) \right] d\mu \right| \le \exp[-C_a \lambda^a], \tag{15}$$

for every $\lambda \ge \lambda_a$, $\gamma \in \mathbb{R}$. Moreover, if both supremums a_+, a_- are attained as maximums, then the above assertion holds with $a = \max\{a_-, a_+\}$.
(iii) *Put*

$$b_- := \sup \left\{ b > 0 : \limsup_{\lambda\to\infty} \lambda^{-b} \int_{(-\infty,0)} (\cos(\lambda x) - 1) f_{A^{(\eta)}, P^{(\eta)}(A(y\cdot\eta)+\ell)}(x) dx < 0 \right\},$$
$$b_+ := \sup \left\{ b > 0 : \limsup_{\lambda\to\infty} \lambda^{-b} \int_{(0,\infty)} (\cos(\lambda x) - 1) f_{A^{(\eta)}, P^{(\eta)}(A(y\cdot\eta)+\ell)}(x) dx < 0 \right\},$$

where $b_- = 0$, $b_+ = 0$ if $\{\cdots\} = \emptyset$. If $\max\{b_-, b_+\} > 0$, then, for every $b < \max\{b_-, b_+\}$, there exist $C_b > 0$ and $\lambda_b > 0$ such that

$$\left| \mathbb{E}_\mu \left[\exp\left[i\lambda \left(\frac{1}{2} Q_A + \langle \cdot, \ell \rangle + \gamma \right) \right] \middle| \eta(w) = y \right] \right| \le \exp[-C_b \lambda^b], \tag{16}$$

for any $\lambda \ge \lambda_b$, $\gamma \in \mathbb{R}$. Moreover, if both supremums b_+, b_- are attained as maximums, then the above assertion holds with $b = \max\{b_-, b_+\}$.

Remark 8. (i) If $\ell_A \neq 0$, then we have an exponent $-\lambda^2 \|\ell_A\|_H^2/2$, which gives a much faster decay than the one discussed in Theorem 7. Similarly, if $\{P^{(\eta)}(A(y\cdot\eta)+\ell)\}_{A^{(\eta)}} \neq 0$, then we obtain a much faster decay than the one discussed in the theorem.
(ii) The integrability of $x^2 f_{A,\ell}(x)$ and $x^2 f_{A^{(\eta)}, P^{(\eta)}(A(y\cdot\eta)+\ell)}(x)$ is due to Lemma 1.
(iii) The lower estimates in (13) and (14) are sharp as we shall see in Lemma 10. For example, if we consider $A = \sum_{n=1}^\infty n^{-p} h_n \otimes h_n$ for some $p > 1/2$ and an orthonormal basis $\{h_n\}$ of H, then there exist $C > 0$ and $\lambda_0 > 0$ such that

$$\left| \int_W \exp\left[i\lambda \left(\frac{1}{2} Q_A + \langle \cdot, \ell \rangle + \gamma \right) \right] d\mu \right| \le \exp[-C\lambda^{1/p}] \quad \text{for any } \lambda > \lambda_0.$$

For details, see Lemma 10 and its proof.
(iv) If $a_n > 0$ for some n, then $\lim_{\lambda\to\infty} \int_0^\infty (\cos(\lambda x) - 1) f_{A,\ell}(x) dx = -\infty$. In fact, it holds that $f_{A,\ell} \ge f_{A,0}$ and

$$\int_0^\infty (1 - \cos(\lambda x)) f_{A,0}(x) dx = \int_0^\infty \frac{1 - \cos y}{y} \sum_{n; a_n > 0} \exp[-y/(\lambda a_n)] dy.$$

Then, applying the monotone convergence theorem, we obtain the desired divergence. Similarly, if $a_n < 0$ for some n, then $\lim_{\lambda\to\infty} \int_{-\infty}^0 (\cos(\lambda x) - 1) f_{A,\ell}(x) dx = -\infty$. Thus the assumption made on $a_\pm$ is that only on the order of divergence.

If $\#\{n; a_n \neq 0\} < \infty$, then $a_+ = a_- = 0$. Indeed, in this case, there are $C, C' > 0$ such that $f_{A,\ell}(x) \le C\{(1/|x|) + 1\} \exp[-C'|x|]$ for every $x \in \mathbb{R} \setminus \{0\}$. For each $\delta > 0$, this implies the existence of $C_\delta > 0$ such that $f_{A,\ell}(x) \le C_\delta |x|^{-1-\delta}$ for any $x \in \mathbb{R} \setminus \{0\}$. Hence, for every $\lambda > 0$,

$$0 \le \max\left\{ \int_{-\infty}^{0} (1-\cos(\lambda x)) f_{A,\ell}(x)dx, \int_0^\infty (1-\cos(\lambda x)) f_{A,\ell}(x)dx \right\} \le C_\delta \lambda^\delta \int_0^\infty \frac{1-\cos y}{y^{1+\delta}} dy,$$

from which it follows that $a_\pm = 0$.

Proof of Theorem 7. Assume that $\ell_A = 0$ or $\{P^{(\eta)}(A(y\cdot\eta)+\ell)\}_{A^{(\eta)}} = 0$ accordingly as μ or $\mu(\cdot|\eta(w) = y)$ is considered. By virtue of Theorems 2 and 4, we have

$$\left| \int_W \exp\left[i\lambda \left(\frac{1}{2} Q_A + \langle \cdot, \ell \rangle + \gamma \right) \right] d\mu \right| = \exp\left[\int_{\mathbb{R}} (\cos(\lambda x) - 1) f_{A,\ell}(x) dx \right]$$

$$\left| \mathbb{E}_\mu \left[\exp\left[i\lambda \left(\frac{1}{2} Q_A + \langle \cdot, \ell \rangle + \gamma \right) \right] \middle| \eta(w) = y \right] \right| = \exp\left[\int_{\mathbb{R}} (\cos(\lambda x) - 1) f_{A^{(\eta)}, P^{(\eta)}(A(y\cdot\eta)+\ell)}(x) dx \right]$$

Since $|\cos x - 1| \le x^2/2$ for any $x \in \mathbb{R}$, the assertion (i) follows immediately.
Let $f = f_{A,\ell}$ or $= f_{A^{(\eta)}, P^{(\eta)}(A(y\cdot\eta)+\ell)}$. Since $(\cos(\lambda x) - 1) f(x) \le 0$, we have

$$\int_{\mathbb{R}} (\cos(\lambda x) - 1) f(x) dx \le \min\left\{ \int_{(-\infty,0)} (\cos(\lambda x) - 1) f(x) dx, \int_{(0,\infty)} (\cos(\lambda x) - 1) f(x) dx \right\}.$$

Thus the estimations in (ii) and (iii) also follow. ☐

A function $\varphi \in C^\infty(\mathbb{R})$ is said to belong a Gevrey class of order $a > 1$ ($\varphi \in G^a(\mathbb{R})$ in notation) if, for any compact subset $K \subset \mathbb{R}$, there exists a constant $C_K > 0$ such that

$$\left| \frac{d^n \varphi}{dx^n}(x) \right| \le C_K \left(C_K (n+1)^a \right)^n \quad \text{for every } x \in K,\ n \in \mathbb{N}.$$

A finite Radon measure u on $\mathbb{R}$ admits a density function of class $G^a(\mathbb{R})$ if there is a $C > 0$ such that

$$|\hat{u}(\xi)| \le C \left(\frac{C(n+1)^a}{|\xi|} \right)^n \quad \text{for any } \xi \in \mathbb{R} \setminus \{0\},\ n \in \mathbb{N},$$

where $\hat{u}$ is the Fourier transformation of u (cf. [6, Prop.8.4.2]). Since $e^{-x} \le \alpha^\alpha x^{-\alpha}$ for $\alpha, x > 0$, a sufficient condition for this to hold is that there exist $C_1, C_2 > 0$ such that

$$|\hat{u}(\xi)| \le C_1 \exp[-C_2 |\xi|^{1/a}] \quad \text{for any } \xi \in \mathbb{R}.$$

We obtain the following from Theorem 7.

Corollary 9. *Let $a_\pm, b_\pm$ be as in Theorem 7.*
(i) *If $\ell_A = 0$ and $\max\{a_-, a_+\} > 0$, then the distribution on $\mathbb{R}$ of $\frac{1}{2}Q_A + \langle \cdot, \ell \rangle + \gamma$ under μ admits a density function, which is in $G^{1/a}(\mathbb{R})$ for any $a < \max\{a_-, a_+\}$, with respect to the Lebesgue measure.*
(ii) *If $\{P^{(\eta)}(A(y \cdot \eta) + \ell)\}_{A^{(\eta)}} = 0$ and $\max\{b_-, b_+\} > 0$, then the distribution on $\mathbb{R}$ of $\frac{1}{2}Q_A + \langle \cdot, \ell \rangle + \gamma$ under the conditional probability $\mu(\cdot|\eta(w) = y)$ admits a density function, which is in $G^{1/b}(\mathbb{R})$ for any $b < \max\{b_-, b_+\}$, with respect to the Lebesgue measure.*

We give a sufficient condition for $a_\pm$ to be positive. It follows from (3) that $f_{A,\ell}(x)$ diverges at the order of at most $|x|^{-3}$ as $|x| \to 0$. If we assume a uniform order of divergence, then $\max\{a_-, a_+\} > 0$.

Lemma 10. (i) *If there exist $\delta, C > 0$ and $\varepsilon < 2$ such that*

$$f_{A,\ell}(x) \geq C x^{\varepsilon - 3} \quad \text{for any } x \in (0, \delta),$$

then $a_+ \geq 2 - \varepsilon$. If there exist $\delta, C > 0$ and $\varepsilon < 2$ such that

$$f_{A,\ell}(x) \geq C|x|^{\varepsilon - 3} \text{for any } x \in (-\delta, 0)$$

then $a_- \geq 2 - \varepsilon$.
(ii) *Let $\{a_n\}_{n=1}^\infty$ be eigenvalues of A. Suppose that there exist a subsequence $\{a_{n_k}\}$, $C > 0$, and $p > 1/2$ such that $a_{n_k} \geq C/k^p$ for any $k \in \mathbb{N}$. Then, for any $\delta > 0$,*

$$f_{A,\ell}(x) \geq \left\{ \frac{C^{1/p}}{p} \int_{\delta/C}^\infty z^{(1/p)-1} e^{-z} dz \right\} x^{-1-(1/p)}$$

holds for any $x \in (0, \delta)$. In particular, $a_+ \geq 1/p$.
If there exist a subsequence $\{a_{n_k}\}$, $C > 0$, and $p > 1/2$ such that $a_{n_k} \leq -C/k^p$ for any $k \in \mathbb{N}$. Then, for any $\delta > 0$,

$$f_{A,\ell}(x) \geq \left\{ \frac{C^{1/p}}{p} \int_{\delta/C}^\infty z^{(1/p)-1} e^{-z} dz \right\} x^{-1-(1/p)}$$

holds for any $x \in (-\delta, 0)$. In particular, $a_- \geq 1/p$.

Proof. (i) It follows from (3) that

$$\int_\delta^\infty (\cos(\lambda x) - 1) f_{A,\ell}(x) dx \leq 0.$$

Due to the first assumption, we have

$$\begin{aligned}\int_0^\delta (\cos(\lambda x) - 1) f_{A,\ell}(x) dx &\leq C \int_0^\delta (\cos(\lambda x) - 1) x^{\varepsilon - 3} dx \\ &= C\lambda^{2-\varepsilon} \int_0^{\lambda\delta} (\cos x - 1) x^{\varepsilon - 3} dx.\end{aligned}$$

Hence

$$\limsup_{\lambda\to\infty}\lambda^{-(2-\varepsilon)}\int_0^\infty(\cos(\lambda x)-1)f_{A,\ell}(x)dx\le C\int_0^\infty(\cos x-1)x^{3-\varepsilon}dx<0.$$

Thus the first half has been verified. The latter half can be seen in exactly the same way.

(ii) Suppose the first assumption. Then it holds that

$$f_{A,\ell}(x)\ge\sum_{k=1}^\infty\frac{e^{-xk^p/C}}{x}\ge\int_1^\infty\frac{e^{-xy^p/C}}{x}dy=x^{-1-(1/p)}\frac{C^{1/p}}{p}\int_{x/C}^\infty z^{(1/p)-1}e^{-z}dz$$

for any $x>0$. This yields the first estimation. Since $-1-(1/p)=\{2-(1/p)\}-3$, by the assertion (i), we have that $a_+\ge 1/p$. Thus the first half has been verified.

The latter half can be seen similarly. □

2.2 Density functions

Corollary 9 gives a sufficient condition for the distribution of $\frac{1}{2}Q_A+\langle\cdot,\ell\rangle+\gamma$ under μ or $\mu(\cdot|\eta(w)=y)$ to have a smooth density function with respect to the Lebesgue measure. We now show another condition for the distribution to possess a smooth density function, and also a method to compute it.

Theorem 11. *Let (W,H,μ) be an abstract Wiener space, $A:H\to H$ be a symmetric Hilbert-Schmidt operator, and decompose as $A=\sum_{n=1}^\infty a_nh_n\otimes h_n$ with an orthonormal basis $\{h_n\}_{n=1}^\infty$ of H.*

(i) *Suppose that $\#\{n:a_n\ne0\}=\infty$. Then there exists a $p_A\in C^\infty(\mathbb{R})$ such that $\mu\big(Q_A/2\in dx\big)=p_A(x)dx$.*

(ii) *Suppose that $a_{2n-1}=a_{2n}$ for any $n\in\mathbb{N}$ and $\#\{n:a_n\ne0\}=\infty$. Fix $x\in\mathbb{R}$, and assume that there exists a family of simple C^1 curves $\Gamma_n=\{\gamma_n(t):t\in[\alpha_n,\beta_n]\}$ in $\mathbb{C}$ such that* (a1) *$\gamma_n(\alpha_n)\in(-\infty,0)$, $\gamma_n(\beta_n)\in(0,\infty)$,* (a2) *$\inf\{|\gamma_n(t)|:t\in[\alpha_n,\beta_n]\}\to\infty$ as $n\to\infty$, and* (a3) *$\int_{\Gamma_n}\{e^{-i\zeta x}/\det_2(I-i\zeta\widehat{A})\}d\zeta\to0$ as $n\to\infty$, where $\widehat{A}=\sum_{n=1}^\infty a_{2n}h_{2n}\otimes h_{2n}$.*

(ii-1) *If $\operatorname{Im}\gamma_n(t)>0$ and $-i/a_m\notin\Gamma_n$ for any $n\in\mathbb{N}$, $t\in(\alpha_n,\beta_n)$, and $m\in\mathbb{N}$ with $a_m<0$, then*

$$p_A(x)=i\sum_{n:a_n<0}\operatorname{Res}\left(\frac{e^{-i\zeta x}}{\det_2(I-i\zeta\widehat{A})};-\frac{i}{a_n}\right),\tag{17}$$

where $\operatorname{Res}(f(\zeta);z)$ denotes the residue of f at z.

(ii-2) *If $\operatorname{Im}\gamma_n(t)<0$ and $-i/a_m\notin\Gamma_n$ for any $n\in\mathbb{N}$, $t\in(\alpha_n,\beta_n)$, and $m\in\mathbb{N}$ with $a_m>0$, then*

$$p_A(x)=-i\sum_{n:a_n>0}\operatorname{Res}\left(\frac{e^{-i\zeta x}}{\det_2(I-i\zeta\widehat{A})};-\frac{i}{a_n}\right).\tag{18}$$

(iii) *Suppose $\sum_{n=1}^\infty|a_n|<\infty$, and set $q_A=Q_A+\sum_{n=1}^\infty a_n$. Then all assertions in* (i) *and* (ii) *hold, replacing Q_A and $\det_2(I-i\zeta\widehat{A})$ by q_A and $\det(I-i\zeta\widehat{A})$, respectively.*

Remark 12. (i) The mapping $\mathbb{C} \ni \zeta \mapsto \det_2(I - i\zeta \widehat{A})$ is holomorphic ([4]).
(ii) Let $\eta = \{\eta_1, \dots, \eta_m\}$ be an orthonormal system of H. By (11), it holds

$$\mu\Big(\frac{1}{2}Q_A \in dx \Big| \eta(w) = 0\Big) = \mu_0^{(\eta)}\Big(\frac{1}{2}\Big\{Q_{A^{(\eta)}} - \sum_{n=1}^{m}\langle A\eta_i, \eta_i\rangle\Big\} \in dx\Big),$$
$$\mu\Big(\frac{1}{2}q_A \in dx \Big| \eta(w) = 0\Big) = \mu_0^{(\eta)}\Big(\frac{1}{2}q_{A^{(\eta)}} \in dx\Big).$$

Thus, we can compute the density functions of the distributions of $Q_A/2$ and $q_A/2$ under $\mu(\cdot|\eta(w) = 0)$, by applying Theorem 11 to $Q_{A^{(\eta)}}$ and $q_{A^{(\eta)}}$ on $W^{(\eta)}$.
(iii) The method to compute the density with the help of the residue theorem has been already applied by Cameron-Martin ([2]) more than a half century ago to the square of the L^2-norm on an interval of the one-dimensional Wiener process.

Proof. Since $\|\nabla(Q_A/2)\|_H^2 = \sum_{n=1}^{\infty} a_n^2 \langle \cdot, h_n\rangle^2$, $\|\nabla(Q_A/2)\|_H^{-1} \in \bigcap_{p>0} L^p(\mu)$ if $\#\{n : a_n \neq 0\} = \infty$ (cf.[18]). Thus the assertion (i) follows as an fundamental application of the Malliavin calculus.

By (7) and the assumption that $a_{2n-1} = a_{2n}$, we have

$$\int_{\mathbb{R}} e^{i\lambda x} p_A(x) dx = \frac{1}{\det_2(I - i\lambda \widehat{A})}, \tag{19}$$

and hence

$$p_A(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{-i\lambda x}}{\det_2(I - i\lambda\widehat{A})} d\lambda.$$

Then the assertions in (ii) are immediate consequences of the residue theorem.

To see the last assertion, it suffices to mention that (19) implies that, if we denote by $\tilde{p}_A$ the density function of $q_A/2$, then

$$\int_{\mathbb{R}} e^{i\lambda x} \tilde{p}_A(x) dx = \frac{1}{\det(I - i\lambda\widehat{A})}.$$

□

3 Typical quadratic Wiener functionals

In this section, we investigate how our results work for typical quadratic Wiener functionals. Some of the computations below have been carried out in Ikeda-Manabe [7], but we give all the results for convenience of the reader. Moreover, when we do not consider the first order terms of the Wiener chaos, that is, when $\ell = 0$ in (5), explicit expressions of the Fourier or Laplace transforms of the distributions are well known for the examples considered in the following and, from them, we can obtain the same results after some elementary calculations.

3.1 The square of the L^2-norm on an interval

Let $T > 0$ and consider the classical one-dimensional Wiener space (W_T^1, H_T^1, μ_T^1) over $[0, T]$; W_T^1 is the space of continuous functions $w : [0, T] \to \mathbb{R}$ with $w(0) = 0$, H_T^1 consists of $h \in W_T^1$ which is absolutely continuous and has a square integrable derivative dh/dt, and μ_T^1 is the Wiener measure. The inner product in H_T^1 is given by

$$\langle h, k\rangle = \int_0^T \frac{dh}{dt}(t)\frac{dk}{dt}(t)dt, \quad h, k \in H_T^1.$$

In this subsection we consider

$$\mathfrak{h}_T(w) = \int_0^T w(t)^2 dt, \quad w \in W_T^1.$$

3.1.1

We first compute the Lévy measure of $\frac{1}{2}\mathfrak{h}_T + \langle\cdot, \ell\rangle + \gamma$ under μ_T^1 by applying Theorem 2, where $\ell \in H_T^1$ and $\gamma \in \mathbb{R}$.

Define a symmetric Hilbert-Schmidt operator $A : H_T^1 \to H_T^1$ by

$$\frac{d(Ah)}{dt}(t) = \int_t^T h(s)ds, \quad h \in H_T^1, t \in [0, T].$$

Note that $w(t)^2$ is in $\mathfrak{C}_2(W_T^1) \oplus \mathfrak{C}_0(W_T^1)$, and so is $\mathfrak{h}_T$. It is easily seen that $\nabla^2\mathfrak{h}_T = 2A$ and that $\int_{W_T^1} \mathfrak{h}_T d\mu_T^1 = T^2/2$. Then, by virtue of Remark 3 (iii), we observe that

$$\mathfrak{h}_T = Q_A + \frac{T^2}{2}. \tag{20}$$

It is easy to see that

$$A = \sum_{n=0}^{\infty} \left(\frac{T}{\left(n + \frac{1}{2}\right)\pi}\right)^2 h_n^A \otimes h_n^A, \tag{21}$$

where

$$h_n^A(t) = \frac{\sqrt{2T}}{\left(n + \frac{1}{2}\right)\pi} \sin\left(\frac{\left(n + \frac{1}{2}\right)\pi t}{T}\right).$$

In particular, we have

$$\ell_A = 0 \quad \text{and} \quad f_{A,\ell}(x) = 0, \quad x \le 0.$$

Set

$$\ell_{(n)} = \langle \ell, h_n^A\rangle = \sqrt{\frac{2}{T}} \int_0^T \frac{d\ell}{dt}(t) \cos\left(\frac{\left(n + \frac{1}{2}\right)\pi t}{T}\right) dt,$$

$$g_H(x; T, \ell) = \frac{\pi^6}{2^7 T^6} \sum_{n=0}^{\infty} (2n + 1)^6 |\ell_{(n)}|^2 \exp\left[-\frac{(2n + 1)^2\pi^2 x}{4T^2}\right]. \tag{22}$$

Since Jacobi's theta function $\Theta(u) = \sum_{n\in\mathbb{Z}} \exp[-n^2 u]$ enjoys the relation

$$\Theta(u) - \Theta(4u) = 2\sum_{n=0}^{\infty} e^{-(2n+1)^2 u},$$

it is straightforward to see that

$$f_{A,\ell}(x) = \frac{1}{4x}\left\{\Theta\left(\frac{\pi^2 x}{4T^2}\right) - \Theta\left(\frac{\pi^2 x}{T^2}\right)\right\} + g_H(x;T,\ell), \quad x > 0.$$

By virtue of this and (20), applying Theorem 2, we arrive at:

Proposition 13. *It holds that*

$$\int_{W_T^1} \exp\left[i\lambda\left(\frac{1}{2}\mathfrak{h}_T + \langle\cdot,\ell\rangle + \gamma\right)\right] d\mu_T^1$$
$$= \exp\left[i\lambda\left(\gamma + \frac{T^2}{4}\right) + \int_0^\infty \left(e^{i\lambda x} - 1 - i\lambda x\right)\left(\frac{1}{4x}\left\{\Theta\left(\frac{\pi^2 x}{4T^2}\right) - \Theta\left(\frac{\pi^2 x}{T^2}\right)\right\} + g_H(x;T,\ell)\right) dx\right],$$

where g_H is defined by (22).

3.1.2

We next compute the Lévy measure of $\frac{1}{2}\mathfrak{h}_T + \langle\cdot,\ell\rangle + \gamma$ under the conditional probability $\mu_T^1(\cdot|w(T) = y)$ given $w(T) = y$, where $\ell \in H_T^1$, $\gamma \in \mathbb{R}$, and $y \in \mathbb{R}$.
Set $\eta_1(w) = w(T)/\sqrt{T}$, $w \in W_T^1$, and $\eta = \{\eta_1\}$. Note that

$$\langle A(y\cdot\eta),(y\cdot\eta)\rangle - \langle A\eta_1,\eta_1\rangle = (y^2-1)\langle A\eta_1,\eta_1\rangle = \frac{(y^2-1)T^2}{3}, \quad \langle (y\cdot\eta),\ell\rangle = \frac{y\ell(T)}{\sqrt{T}}.$$

By a straightforward computation, we obtain

$$\frac{d(A^{(\eta)}h)}{dt}(t) = \int_t^T h(s)ds - \frac{1}{T}\int_0^T\left(\int_s^T h(u)du\right)ds, \quad h \in (H_T^1)_0^{(\eta)},$$

and hence

$$A^{(\eta)} = \sum_{n=1}^{\infty}\Big(\frac{T}{n\pi}\Big)^2 k_n^A \otimes k_n^A, \quad \text{where} \quad k_n^A(t) = \frac{\sqrt{2T}}{n\pi}\sin\Big(\frac{n\pi t}{T}\Big). \tag{23}$$

In particular,

$$\{P^{(\eta)}(A(y\cdot\eta)+\ell)\}_{A^{(\eta)}} = 0 \quad \text{and} \quad f_{A^{(\eta)},P^{(\eta)}(A(y\cdot\eta)+\ell)}(x) = 0, \quad x \le 0.$$

Note that

$$\begin{aligned}\langle P^{(\eta)}(A(y\cdot\eta)+\ell),k_n^A\rangle &= y\langle A\eta,k_n^A\rangle + \langle\ell,k_n^A\rangle\\ &= (-1)^{n+1}\sqrt{2}y\Big(\frac{T}{n\pi}\Big)^2 + \tilde{\ell}_{(n)},\end{aligned} \tag{24}$$

where

$$\tilde{\ell}_{(n)} = \langle \ell, k_n^A \rangle = \sqrt{\frac{2}{T}} \int_0^T \frac{d\ell}{dt}(t) \cos\Big(\frac{n\pi t}{T}\Big) dt.$$

Hence, for $x > 0$,

$$\begin{aligned}
&f_{A^{(\eta)},P^{(\eta)}(A(y\cdot\eta)+\ell)}(x) \\
&\quad = \frac{1}{2}\sum_{n=1}^{\infty}\Big\{\frac{1}{x} + 2y^2\Big(\frac{n\pi}{T}\Big)^2 + (-1)^{n+1}2^{3/2}y\Big(\frac{n\pi}{T}\Big)^4\tilde{\ell}_{(n)} \\
&\qquad\qquad + \Big(\frac{n\pi}{T}\Big)^6\tilde{\ell}^2_{(n)}\Big\}\exp\Big[-\frac{n^2\pi^2}{T^2}x\Big] \\
&\quad = \frac{1}{4x}\Big\{\Theta\Big(\frac{\pi^2 x}{T^2}\Big) - 1\Big\} - \frac{\pi^2 y^2}{2T^2}\Theta'\Big(\frac{\pi^2 x}{T^2}\Big) + \tilde{g}_H(x;T,\ell,y),
\end{aligned} \tag{25}$$

where

$$\begin{aligned}
&\tilde{g}_H(x;T,\ell,y) \\
&\quad = \frac{1}{2}\sum_{n=1}^{\infty}\Big\{(-1)^{n+1}2^{3/2}y\Big(\frac{n\pi}{T}\Big)^4\tilde{\ell}_{(n)} + \Big(\frac{n\pi}{T}\Big)^6\tilde{\ell}^2_{(n)}\Big\}\exp\Big[-\frac{n^2\pi^2 x}{T^2}\Big].
\end{aligned} \tag{26}$$

Due to Theorem 4, we obtain

$$\begin{aligned}
&\mathbb{E}_{\mu_T^1}\left[\exp\left[i\lambda\left(\frac{1}{2}Q_A + \langle\cdot,\ell\rangle + \gamma\right)\right]\Big|\eta(w) = y\right] \\
&= \exp\Big[i\lambda\Big(\frac{(y^2-1)T^2}{6} + \frac{y\ell(T)}{\sqrt{T}} + \gamma\Big) + \int_0^\infty \Big(e^{i\lambda x} - 1 - i\lambda x\Big)\Big[\frac{1}{4x}\Big\{\Theta\Big(\frac{\pi^2 x}{T^2}\Big) - 1\Big\} \\
&\qquad - \frac{\pi^2 y^2}{2T^2}\Theta'\Big(\frac{\pi^2 x}{T^2}\Big) + \tilde{g}_H(x;T,\ell,y)\Big]dx\Big].
\end{aligned}$$

Since $\eta(w) = w(T)/\sqrt{T}$, combined with (20), we conclude from this:

Proposition 14. *It holds that*

$$\begin{aligned}
&\mathbb{E}_{\mu_T^1}\left[\exp\left[i\lambda\left(\frac{1}{2}\mathfrak{h}_T + \langle\cdot,\ell\rangle + \gamma\right)\right]\Big|w(T) = y\right] \\
&\qquad = \exp\Big[i\lambda\Big\{\frac{Ty^2}{6} + \frac{T^2}{12} + \frac{y\ell(T)}{T} + \gamma\Big\} + \int_0^\infty \Big(e^{i\lambda x} - 1 - i\lambda x\Big) f_H(x;T,\ell,y)dx\Big],
\end{aligned}$$

where

$$f_H(x;T,\ell,y) = \frac{1}{4x}\Big(\Theta\Big(\frac{\pi^2 x}{T^2}\Big) - 1\Big) - \frac{\pi^2 y^2}{2T^3}\Theta'\Big(\frac{\pi^2 x}{T^2}\Big) + \tilde{g}_H(x;T,\ell,y/\sqrt{T}),$$

and $\tilde{g}_H$ *is given by* (26).

3.1.3

We finally study the exponential decay of the characteristic function of $\frac{1}{2}\mathfrak{h}_T + \langle\cdot,\ell\rangle + \gamma$ under μ_T^1 and $\mu_T^1(\cdot|w(T) = y)$.

As was seen in §3.1.1, the Hilbert-Schmidt operator A associated with $\mathfrak{h}_T$ has eigenvalues $\{T^2/[(n+\frac{1}{2})\pi]^2; n\in\mathbb{N}\cup\{0\}\}$, each of them being of multiplicity one. By Theorem 7 and Lemma 10, there exist $C_1>0$ and $\lambda_1>0$ such that

$$\left|\int_{W_T^1}\exp\left[i\lambda\left(\frac{1}{2}\mathfrak{h}_T+\langle\cdot,\ell\rangle+\gamma\right)\right]\,d\mu\right|\le\exp[-C_1\lambda^{1/2}] \tag{27}$$

for any $\lambda\ge\lambda_1,\ \gamma\in\mathbb{R}$.

Let $\eta_1(t)=t/\sqrt{T}$ and $\eta=\{\eta_1\}$. As was shown in §3.1.2, the Hilbert-Schmidt operator $A^{(\eta)}$ possesses eigenvalues $\{(T/(n\pi))^2; n\in\mathbb{N}\}$, each of them being of multiplicity 1. Since $\eta(w)=w(T)/\sqrt{T}$, by Theorem 7 and Lemma 10, there exist $C_2>0$ and $\lambda_2>0$ such that

$$\left|\mathbb{E}_{\mu_T^1}\left[\exp\left[i\lambda\left(\frac{1}{2}\mathfrak{h}_T+\langle\cdot,\ell\rangle+\gamma\right)\right]\Big|w(T)=y\right]\right|\le\exp[-C_2\lambda^{1/2}] \tag{28}$$

for any $\lambda\ge\lambda_2,\ \gamma\in\mathbb{R}$.

When $\ell=0$ and $\gamma=0$, it is well known ([3, 12] and [8, pp.470–473]) that

$$\int_{W_T^1}\exp\left[-\frac{\lambda}{2}\mathfrak{h}_T\right]d\mu_T^1=\frac{1}{(\cosh(\sqrt{\lambda}T))^{1/2}}$$

and

$$\begin{aligned}&\mathbb{E}_{\mu_T^1}\left[\exp\left[-\frac{\lambda}{2}\mathfrak{h}_T\right]\Big|w(T)=y\right]\\&=\left(\frac{\sqrt{\lambda}T}{\sinh(\sqrt{\lambda}T)}\right)^{1/2}\exp\left[\left(1-\sqrt{\lambda}T\coth(\sqrt{\lambda}T)\right)\frac{y^2}{2T}\right]\end{aligned} \tag{29}$$

hold for $\lambda>0$. Thus, continuing holomorphically, we see that our estimations (27) and (28) coincide with the order obtained from these precise expressions.

Starting from these well-known expressions, and recalling the elementary formulae

$$\begin{aligned}\cosh(x)&=\prod_{k=0}^{\infty}\left(1+\frac{4x^2}{(2k+1)^2\pi^2}\right),\quad \sinh(x)=x\prod_{k=1}^{\infty}\left(1+\frac{x^2}{k^2\pi^2}\right),\\ \coth(\pi x)&=\frac{1}{\pi x}+\frac{2x}{\pi}\sum_{k=1}^{\infty}\frac{1}{x^2+k^2},\end{aligned}$$

we can also show explicit expressions for the Lévy measures $\nu_T(dx)$ and $\nu_{T,y}(dx)$ of the distribution of $\mathfrak{h}_T/2$ under μ_T^1 and the conditional probability measure $\mu_T^1(\,\cdot\,|w(T)=y)$ as described in Propositions 13 and 14 with $\ell=0$.

Moreover, by using the Riemann zeta function $\zeta(s)=\sum_{n=1}^{\infty}n^{-s}$, we can give explicit forms of the Mellin transforms of ν_T and $\nu_{T,y}$. Namely, noting that

$$\nu_T(dx)=f_{A,0}(x)dx,\quad \nu_{T,y}(dx)=f_{A^{(\eta)},P^{(\eta)}(A((y/\sqrt{T})\cdot\eta))}(x)dx,$$

and then plugging (21), (23), and (24) into (12), we obtain:

Proposition 15. *The Mellin transform of $\nu_T(dx)$ and $\nu_{T,y}(dx)$ are given by*

$$\int_0^\infty x^s \nu_T(dx) = \left(\frac{4T^2}{\pi^2}\right)^s \frac{2^{2s}-1}{2^{2s+1}} \Gamma(s)\zeta(2s)$$

and

$$\int_0^\infty x^s \nu_{T,y}(dx) = \frac{1}{2}\left(\frac{T^2}{\pi^2}\right)^s \Gamma(s)\zeta(2s) + \frac{y^2}{T}\left(\frac{T^2}{\pi^2}\right)^s \Gamma(s+1)\zeta(2s), \quad s \geq 2,$$

respectively, where Γ is the usual gamma function.

Recently, Biane-Pitman-Yor [1] and Pitman-Yor [15] have discussed the related topics and shown similar formulae.

3.2 Lévy's stochastic area

Let $T > 0$ and consider the classical two-dimensional Wiener space (W_T^2, H_T^2, μ_T^2) over $[0, T]$; W_T^2 is the space of continuous functions $w : [0, T] \to \mathbb{R}^2$ with $w(0) = 0$, H_T^2 consists of $h \in W_T^2$ which is absolutely continuous and has a square integrable derivative dh/dt, and μ_T^2 is the Wiener measure. The inner product in H_T^2 is given by

$$\langle h, k \rangle = \int_0^T \left\langle \frac{dh}{dt}(t), \frac{dk}{dt}(t) \right\rangle_{\mathbb{R}^2} dt, \quad h, k \in H_T^2.$$

Define Lévy's stochastic area by

$$\mathfrak{s}_T(w) = \frac{1}{2}\int_0^T \langle Jw(t), dw(t) \rangle_{\mathbb{R}^2}^2 = \frac{1}{2}\int_0^T \left\{ w^1(t)dw^2(t) - w^2(t)dw^1(t) \right\},$$

where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $dw(t)$ denotes the Itô integral.

3.2.1

We first compute the Lévy measure of $\mathfrak{s}_T + \langle \cdot, \ell \rangle + \gamma$ under μ_T^2 by applying Theorem 2, where $\ell \in H_T^2$ and $\gamma \in \mathbb{R}$.
Define a symmetric Hilbert-Schmidt operator $B : H_T^2 \to H_T^2$ by

$$\frac{d(Bh)}{dt}(t) = J\left(h(t) - \frac{1}{2}h(T)\right), \quad h \in H_T^2, t \in [0, T].$$

Since $w^i(s)\{w^j(t) - w^j(s)\}$ is in $\mathfrak{C}_2(W_T^2)$ for $(i, j) \in \{(1, 2), (2, 1)\}$ and $s < t$, so is $\mathfrak{s}_T$. It is easily seen that $\nabla^2 s_T = B$, and hence, due to Remark 3 (iii), we have

$$\mathfrak{s}_T = \frac{1}{2}Q_B. \tag{30}$$

By a direct computation, we see that

$$B = \sum_{n\in\mathbb{Z}} \frac{T}{(2n+1)\pi} \left\{ h_n^B \otimes h_n^B + \tilde{h}_n^B \otimes \tilde{h}_n^B \right\}, \tag{31}$$

where

$$h_n^B(t) = \frac{\sqrt{T}}{(2n+1)\pi} \begin{pmatrix} \cos((2n+1)\pi t/T) - 1, \\ \sin((2n+1)\pi t/T) \end{pmatrix}, \quad \tilde{h}_n^B(t) = J h_n^B(t).$$

In particular

$$\ell_B = 0.$$

Set

$$\begin{aligned} \ell^1_{(n)} &= \langle \ell, h_n^B \rangle = \frac{1}{\sqrt{T}} \int_0^T \left\langle \frac{d\ell}{dt}(t), \begin{pmatrix} -\sin((2n+1)\pi t/T) \\ \cos((2n+1)\pi t/T) \end{pmatrix} \right\rangle_{\mathbb{R}^2} dt, \\ \ell^2_{(n)} &= \langle \ell, \tilde{h}_n^B \rangle = -\frac{1}{\sqrt{T}} \int_0^T \left\langle \frac{d\ell}{dt}(t), \begin{pmatrix} \cos((2n+1)\pi t/T) \\ \sin((2n+1)\pi t/T) \end{pmatrix} \right\rangle_{\mathbb{R}^2} dt, \\ \ell_{(n)} &= \begin{pmatrix} \ell^1_{(n)} \\ \ell^2_{(n)} \end{pmatrix}, \end{aligned}$$

and

$$g_L(x;T,\ell) = \begin{cases} \dfrac{\pi^3}{2T^3} \displaystyle\sum_{n=0}^{\infty} (2n+1)^3 |\ell_{(n)}|^2 \exp\left[-(2n+1)\pi x/T\right], & x > 0 \\ 0, & x = 0, \\ \dfrac{\pi^3}{2T^3} \displaystyle\sum_{n=1}^{\infty} (2n-1)^3 |\ell_{(-n)}|^2 \exp\left[-(2n-1)\pi x/T\right], & x < 0. \end{cases} \tag{32}$$

Then it is easily seen that

$$f_{B,\ell}(x) = \frac{1}{2x\sinh(\pi x/T)} + g_L(x;T,\ell), \quad x \in \mathbb{R}.$$

By virtue of this and (30), applying Theorem 2, we arrive at;

Proposition 16. *It holds that*

$$\begin{aligned} &\int_{W_T^2} \exp\left[i\lambda\left(\mathfrak{s}_T + \langle \cdot, \ell \rangle + \gamma\right)\right] d\mu_T^2 \\ &\qquad = \exp\left[i\lambda\gamma + \int_{\mathbb{R}} \left(e^{i\lambda x} - 1 - i\lambda x\right) \left(\frac{1}{2x\sinh(\pi x/T)} + g_L(x;T,\ell) \right) dx \right], \end{aligned} \tag{33}$$

where g_L is defined by (32)

3.2.2

We next compute the Lévy measure of $\mathfrak{s}_T + \langle \cdot, \ell \rangle + \gamma$ under the conditional probability $\mu_T^2(\cdot | w(T) = y)$ given $W(T) = y$, where $\ell \in H_T^2$, $\gamma \in \mathbb{R}$, and $y \in \mathbb{R}^2$.
Let $\eta = \{\eta_1, \eta_2\} \subset W^*$, where $\eta_i(w) = w^i(T)/\sqrt{T}$. Since $y \cdot \eta = ty/\sqrt{T}$ and $\langle Jz, z \rangle_{\mathbb{R}^2} = 0$ for any $z \in \mathbb{R}^2$,

$$\langle B(y \cdot \eta), (y \cdot \eta) \rangle - \sum_{n=1}^{2} \langle B\eta_n, \eta_n \rangle = 0, \quad \langle (y \cdot \eta), \ell \rangle = \frac{\langle y, \ell(T) \rangle_{\mathbb{R}^2}}{\sqrt{T}}.$$

For any $h, g \in (H_T^2)_0^{(\eta)}$, the identity $\langle B^{(\eta)} h, g \rangle = \langle Bh, g \rangle$ holds, and hence

$$\frac{d(B^{(\eta)} h)}{dt}(t) = J(h(t) - \bar{h}), \quad \text{where } \bar{h} = \frac{1}{T} \int_0^T h(t) dt.$$

Then it is straightforward to see that

$$B^{(\eta)} = \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{T}{2n\pi} \left(k_n^B \otimes k_n^B + \tilde{k}_n^B \otimes \tilde{k}_n^B \right), \tag{34}$$

where

$$k_n^B(t) = \frac{\sqrt{T}}{2n\pi} \begin{pmatrix} \cos(2n\pi t/T) - 1 \\ \sin(2n\pi t/T) \end{pmatrix}, \quad \tilde{k}_n^B(t) = J k_n^B(t).$$

Hence

$$\{P^{(\eta)}(B(y \cdot \eta) + \ell)\}_{B^{(\eta)}} = 0,$$

and it holds that

$$\langle P^{(\eta)}(B(y \cdot \eta) + \ell), k_n^B \rangle = \langle (y \cdot \eta), B k_n^B \rangle + \tilde{\ell}_{(n)}^1 = -\frac{Ty^2}{2n\pi} + \tilde{\ell}_{(n)}^1,$$
$$\langle P^{(\eta)}(B(y \cdot \eta) + \ell), \tilde{k}_n^B \rangle = \langle (y \cdot \eta), B \tilde{k}_n^B \rangle + \tilde{\ell}_{(n)}^2 = \frac{Ty^1}{2n\pi} + \tilde{\ell}_{(n)}^2,$$

where

$$\tilde{\ell}_{(n)}^1 = \langle \ell, k_n^B \rangle = \int_0^T \left\langle \frac{d\ell}{dt}(t), \frac{1}{\sqrt{T}} \begin{pmatrix} -\sin(2n\pi t/T) \\ \cos(2n\pi t/T) \end{pmatrix} \right\rangle_{\mathbb{R}^2} dt,$$
$$\tilde{\ell}_{(n)}^2 = \langle \ell, \tilde{k}_n^B \rangle = \int_0^T \left\langle \frac{d\ell}{dt}(t), \frac{1}{\sqrt{T}} \begin{pmatrix} -\cos(2n\pi t/T) \\ -\sin(2n\pi t/T) \end{pmatrix} \right\rangle_{\mathbb{R}^2} dt.$$

Setting $\tilde{\ell}_{(n)} = \begin{pmatrix} \tilde{\ell}_{(n)}^1 \\ \tilde{\ell}_{(n)}^2 \end{pmatrix}$, we obtain

$$\begin{aligned} &\langle P^{(\eta)}(B(y \cdot \eta) + \ell), k_n^B \rangle^2 + \langle P^{(\eta)}(B(y \cdot \eta) + \ell), \tilde{k}_n^B \rangle^2 \\ &\qquad = \left(\frac{T}{2n\pi} \right)^2 |y|^2 + \frac{T}{n\pi} \langle \tilde{\ell}_{(n)}, Jy \rangle_{\mathbb{R}^2} + |\tilde{\ell}_{(n)}|^2. \end{aligned} \tag{35}$$

Thus, if we put

$$
\tilde{g}_L(x;T,\ell,y)
= \begin{cases}
\frac{1}{2}\sum_{n=1}^{\infty}\Big\{2\Big(\frac{2n\pi}{T}\Big)^2\langle\tilde{\ell}_{(n)},Jy\rangle_{\mathbb{R}^2}+\Big(\frac{2n\pi}{T}\Big)^3|\tilde{\ell}_{(n)}|^2\Big\}\exp\Big[-\frac{2n\pi x}{T}\Big], & x>0,\\
0, & x=0,\\
\frac{1}{2}\sum_{n=1}^{\infty}\Big\{-2\Big(\frac{2n\pi}{T}\Big)^2\langle\tilde{\ell}_{(-n)},Jy\rangle_{\mathbb{R}^2}+\Big(\frac{2n\pi}{T}\Big)^3|\tilde{\ell}_{(-n)}|^2\Big\}\exp\Big[\frac{2n\pi x}{T}\Big], & x<0,
\end{cases} \tag{36}
$$

then, for $x>0$, it holds that

$$
\begin{aligned}
&f_{B^{(\eta)},P^{(\eta)}(B(y\cdot\eta)+\ell)}(x)\\
&=\frac{1}{2}\sum_{n=1}^{\infty}\frac{2}{x}\exp[-2n\pi x/T]+\frac{1}{2}\sum_{n=1}^{\infty}\frac{\{T/(2n\pi)\}^2|y|^2}{\{T/(2n\pi)\}^3}\exp[-2n\pi x/T]+\tilde{g}_L(x;T,\ell,y)\\
&=\frac{1}{x}\frac{\exp[-2\pi x/T]}{1-\exp[-2\pi x/T]}+\frac{\pi|y|^2}{T}\sum_{n=1}^{\infty}n\exp[-2n\pi x/T]+\tilde{g}_L(x;T,\ell,y)\\
&=\frac{1}{x}\frac{1}{\exp[2\pi x/T]-1}+\frac{\pi|y|^2}{4T}\frac{1}{\sinh^2(\pi x/T)}+\tilde{g}_L(x;T,\ell,y),
\end{aligned}
$$

where we have used the identity $\sum_{n=1}^{\infty}ne^{-nx}=1/\{4\sinh^2(x/2)\}$. Similarly, for $x<0$, we have

$$
f_{B^{(\eta)},P^{(\eta)}(B(y\cdot\eta)+\ell)}(x)=\frac{1}{|x|}\frac{1}{\exp[2\pi|x|/T]-1}+\frac{\pi|y|^2}{4T}\frac{1}{\sinh^2(\pi x/T)}+\tilde{g}_L(x;T,\ell,y).
$$

Applying Theorem 4, we get to

$$
\begin{aligned}
&\mathbb{E}_{\mu_T^2}\left[\exp\left[i\lambda\left(\frac{1}{2}Q_B+\langle\cdot,\ell\rangle+\gamma\right)\right]\middle|\eta(w)=y\right]\\
&\quad=\exp\Big[i\lambda\Big(\frac{\langle y,\ell(T)\rangle_{\mathbb{R}^2}}{\sqrt{T}}+\gamma\Big)+\int_{\mathbb{R}}\Big(e^{i\lambda x}-1-i\lambda x\Big)\Big\{\frac{1}{|x|}\frac{1}{\exp[2\pi|x|/T]-1}\\
&\qquad\qquad+\frac{\pi|y|^2}{4T}\frac{1}{\sinh^2(\pi x/T)}+\tilde{g}_L(x;T,\ell,y)\Big\}dx\Big].
\end{aligned}
$$

Since $\eta(w)=w(T)/\sqrt{T}$, combined with (30), this yields:

Proposition 17. *It holds that*

$$
\begin{aligned}
&\mathbb{E}_{\mu_T^2}\left[\exp\left[i\lambda\left(\mathfrak{s}_T+\langle\cdot,\ell\rangle+\gamma\right)\right]\middle|w(T)=y\right]\\
&\quad=\exp\Big[i\lambda\Big\{\frac{\langle y,\ell(T)\rangle_{\mathbb{R}^2}}{T}+\gamma\Big\}+\int_{\mathbb{R}}\Big(e^{i\lambda x}-1-i\lambda x\Big)f_L(x;T,\ell,y)dx\Big],
\end{aligned}
$$

where

$$
f_L(x;T,\ell,y)=\frac{1}{|x|\{\exp[2\pi|x|/T]-1\}}+\frac{\pi|y|^2}{4T^2}\frac{1}{\sinh^2(\pi x/T)}+\tilde{g}_L(x;T,\ell,y/\sqrt{T}),
$$

and $\tilde{g}_L$ *is given by* (36)

3.2.3

We finally consider the exponential decay of the characteristic function of $\mathfrak{s}_T + \langle \cdot, \ell \rangle + \gamma$ under μ_T^2 and $\mu_T^2(\cdot | w(T) = y)$.

As was seen in §3.2.1, the corresponding Hilbert-Schmidt operator B possesses eigenvalues $\{T/[(2n+1)\pi]; n \in \mathbb{Z}\}$ and each of them is of multiplicity 2. By Theorem 7 and Lemma 10, there exist $C_1 > 0$ and $\lambda_1 > 0$ such that

$$\left| \int_{W_T^2} \exp\left[i\lambda \left(\mathfrak{s}_T + \langle \cdot, \ell \rangle + \gamma \right) \right] d\mu \right| \leq \exp[-C_1 \lambda] \tag{37}$$

for every $\lambda \geq \lambda_1,\ \gamma \in \mathbb{R}$.

Let $\eta_1(t) = \begin{pmatrix} t/\sqrt{T} \\ 0 \end{pmatrix}$, $\eta_2(t) = \begin{pmatrix} 0 \\ t/\sqrt{T} \end{pmatrix}$, and $\eta = \{\eta_1, \eta_2\}$. As was shown in §3.2.2, the Hilbert-Schmidt operator $B^{(\eta)}$ has eigenvalues $\{(T/(2n\pi)); n \in \mathbb{Z} \setminus \{0\}\}$, each of them being of multiplicity 2. Since $\eta(w) = w(T)/\sqrt{T}$, by Theorem 7 and Lemma 10, there exist $C_2 > 0$ and $\lambda_2 > 0$ such that

$$\left| \mathbb{E}_{\mu_T^2} \left[\exp\left[i\lambda \left(\mathfrak{s}_T + \langle \cdot, \ell \rangle + \gamma \right) \right] \middle| w(T) = y \right] \right| \leq \exp[-C_2 \lambda] \tag{38}$$

for every $\lambda \geq \lambda_2, \gamma \in \mathbb{R}$.

As in the previous subsection, when $\ell = 0$ and $\gamma = 0$, it is well known ([12] and [8, pp.470–473]) that

$$\int_{W_T^2} \exp[i\lambda \mathfrak{s}_T] d\mu_T^2 = \frac{1}{\cosh(\lambda T/2)},$$

and

$$\mathbb{E}_{\mu_T^2} \left[\exp[i\lambda \mathfrak{s}_T] \middle| w(T) = y \right] = \frac{\lambda T/2}{\sinh(\lambda T/2)} \exp\left[\left(1 - \frac{\lambda T}{2} \coth\left(\frac{\lambda T}{2} \right) \right) \frac{|y|^2}{2T} \right].$$

Thus our estimations (37) and (38) coincide with the order obtained from these precise expressions.

We can give explicit expressions for the Mellin transforms of the Lévy measures σ_T and $\sigma_{T,y}$ of the distributions of $\mathfrak{s}_T$ under μ_T^2 and the conditional probability measure $\mu_T^2(\ \cdot\ | w(T) = y)$, respectively. Namely, noting that

$$\sigma_T(dx) = f_{B,0}(x)dx, \quad \sigma_{T,y}(dx) = f_{B^{(\eta)}, P^{(\eta)}(B((y/\sqrt{T}) \cdot \eta))}(x)dx,$$

and then plugging (31), (34), and (35) into (12), we obtain:

Proposition 18. *The Mellin transforms of σ_T and $\sigma_{T,y}$ are given by*

$$\int_{\mathbb{R}} |x|^s \sigma_T(dx) = \left(\frac{T}{\pi} \right)^s \frac{2^s - 1}{2^{s-1}} \Gamma(s)\zeta(s)$$

and

$$\int_{\mathbb{R}} |x|^s \sigma_{T,y}(dx) = 2 \left(\frac{T}{2\pi} \right)^s \Gamma(s)\zeta(s) + \frac{|y|^2}{T} \left(\frac{T}{2\pi} \right)^s \Gamma(s+1)\zeta(s), \quad s \geq 2,$$

respectively.

See [1, 15] for the related topics.

3.3 Sample variance

Let $T > 0$ and (W_T^1, H_T^1, μ_T^1) be the two-dimensional classical Wiener space over $[0, T]$. In this subsection, we consider the sample variance

$$\mathfrak{v}_T(w) = \int_0^T (w(t) - \bar{w})^2\, dt, \quad w \in W_T^1, \qquad \text{where} \quad \bar{w} = \frac{1}{T}\int_0^T w(t)dt.$$

3.3.1

We first compute the Lévy measure of $\frac{1}{2}\mathfrak{v}_T + \langle \cdot, \ell\rangle + \gamma$ under μ_T^1 by applying Theorem 2, where $\ell \in H_T^1$ and $\gamma \in \mathbb{R}$.

Define a symmetric Hilbert-Schmidt operator $C : H_T^1 \to H_T^1$ by

$$\frac{d(Ch)}{dt}(t) = \int_t^T (h(s) - \bar{h})ds \quad h \in H_T^1, t \in [0, T].$$

Since $\mathfrak{v}_T(w) = \mathfrak{h}_T(w) - T\bar{w}^2$, due to the observation made at the beginning of §3.1.1, we have that $\mathfrak{v}_T \in \mathfrak{C}_2(W_T^1) \oplus \mathfrak{C}_0(W_T^1)$. It is easily seen that $\nabla^2\mathfrak{v}_T = 2C$ and $\int_{W_T^1} \mathfrak{v}_T d\mu_T^1 = T^2/6$. By Remark 3 (iii), we have

$$\mathfrak{v}_T = Q_C + \frac{T^2}{6}. \tag{39}$$

It is a straightforward computation to see that

$$C = \sum_{n=1}^{\infty} \left(\frac{T}{n\pi}\right)^2 h_n^C \otimes h_n^C, \quad \text{where } h_n^C(t) = \frac{\sqrt{2T}}{n\pi}\left\{\cos\left(\frac{n\pi t}{T}\right) - 1\right\}.$$

Hence $\ell_C = 0$. Define

$$\widetilde{\ell}(t) = \sum_{n=1}^{\infty} \langle \ell, h_n^C\rangle k_n^A(t), \quad t \in [0, T], \tag{40}$$

where $\{k_n^A\}_{n=1}^\infty$ is the orthonormal basis of $(H_T^1)_0^{(\eta)}$ defined in (23). Comparing the above expansion of C with that of $A^{(\eta)}$ in (23), and recalling the definition of $f_{A,\ell}$, we obtain

$$f_{C,\ell} = f_{A^{(\eta)},\widetilde{\ell}} = f_{A^{(\eta)},P^{(\eta)}\widetilde{\ell}}.$$

In conjunction with (25) and (39), Theorem 2 and Proposition 14 lead us to:

Proposition 19. *It holds that*

$$\int_{W_T^1} \exp\left[i\lambda\left(\frac{1}{2}\mathfrak{v}_T + \langle \cdot, \ell\rangle + \gamma\right)\right] d\mu_T^1$$
$$= \exp\left[i\lambda\left(\gamma + \frac{T^2}{12}\right) + \int_0^\infty \left(e^{i\lambda x} - 1 - i\lambda x\right) f_H(x; T, \widetilde{\ell}, 0)dx\right],$$

where f_H is the function defined in Proposition 14, and $\widetilde{\ell}$ is given by (40)*. In particular, the distribution of $\frac{1}{2}\mathfrak{v}_T + \langle \cdot, \ell\rangle + \gamma$ under μ_T^1 coincides with that of $\frac{1}{2}\mathfrak{h}_T + \langle \cdot, \widetilde{\ell}\rangle + \gamma$ under $\mu_T^1(\cdot|w(T) = 0)$.*

3.3.2

We next compute the Lévy measure of $\frac{1}{2}\mathfrak{v}_T + \langle\cdot,\ell\rangle + \gamma$ under the conditional probability $\mu_T^2(\cdot|w(T)=y)$ given $W(T)=y$, where $\ell \in H_T^1$, $\gamma \in \mathbb{R}$, and $y \in \mathbb{R}^1$.
Set $\eta_1(w) = w(T)/\sqrt{T}$, $w \in W_T^1$, and $\eta = \{\eta_1\}$. Observe that

$$\langle C(y\cdot\eta),(y\cdot\eta)\rangle - \langle C\eta_1,\eta_1\rangle = \frac{(y^2-1)T^2}{12}, \quad \langle (y\cdot\eta),\ell\rangle = \frac{y\ell(T)}{\sqrt{T}}.$$

By straightforward computations, we obtain

$$\frac{d(C^{(\eta)}h)}{dt}(t) = \int_t^T (h(s)-\bar{h})ds - \frac{1}{T}\int_0^T \left(\int_s^T (h(u)-\bar{h})du\right)ds, \quad h \in (H_T^1)_0^{(\eta)},$$

and

$$C^{(\eta)} = \sum_{n=1}^{\infty}\Big(\frac{T}{2n\pi}\Big)^2 \left\{k_n^C \otimes k_n^C + \tilde{k}_n^C \otimes \tilde{k}_n^C\right\},$$

where

$$k_n^C(t) = \frac{\sqrt{2T}}{2n\pi}\sin\Big(\frac{2n\pi t}{T}\Big), \quad \tilde{k}_n^C(t) = \frac{\sqrt{2T}}{2n\pi}\left(\cos\Big(\frac{2n\pi t}{T}\Big)-1\right).$$

In particular,

$$\{P^{(\eta)}(C(y\cdot\eta)+\ell)\}_{C^{(\eta)}} = 0 \quad \text{and} \quad f_{C^{(\eta)},P^{(\eta)}(C(y\cdot\eta)+\ell)}(x) = 0, \quad x \le 0.$$

Moreover it holds that

$$\langle P^{(\eta)}(C(y\cdot\eta)+\ell), k_n^C\rangle = -\sqrt{2}y\left(\frac{T}{2n\pi}\right)^2 + \langle \ell, k_n^C\rangle, \;\; \langle P^{(\eta)}(C(y\cdot\eta)+\ell), \tilde{k}_n^C\rangle = \langle \ell, \tilde{k}_n^C\rangle.$$

Hence, for $x>0$,

$$\begin{aligned} f_{C^{(\eta)},P^{(\eta)}(C(y\cdot\eta)+\ell)}(x) &= \frac{1}{2}\sum_{n=1}^{\infty}\Bigg\{\frac{2}{x} + 2y^2\left(\frac{2n\pi}{T}\right)^2 - 2^{3/2}y\left(\frac{2n\pi}{T}\right)^4\langle \ell, k_n^C\rangle \\ &\qquad + \left(\frac{2n\pi}{T}\right)^6\left(\langle \ell, k_n^C\rangle^2 + \langle \ell, \tilde{k}_n^C\rangle^2\right)\Bigg\}e^{-(2n\pi)^2x/T^2} \\ &= \frac{1}{2x}\Big\{\Theta\Big(\frac{4\pi^2x}{T^2}\Big)-1\Big\} - \frac{2\pi^2y^2}{T^2}\Theta'\Big(\frac{4\pi^2x}{T^2}\Big) + g_V(x;T,\ell,y), \end{aligned}$$

where

$$\begin{aligned} g_V(x;T,\ell,y) = \frac{1}{2}\sum_{n=1}^{\infty}\Bigg\{&-2^{3/2}y\left(\frac{2n\pi}{T}\right)^4\langle \ell, k_n^C\rangle \\ &+ \left(\frac{2n\pi}{T}\right)^6\left(\langle \ell, k_n^C\rangle^2 + \langle \ell, \tilde{k}_n^C\rangle^2\right)\Bigg\}e^{-(2n\pi)^2x/T^2}. \end{aligned} \tag{41}$$

Recalling (39), and applying Theorem 4 and Proposition 14, we can conclude:

Proposition 20. *It holds that*

$$\mathbb{E}_{\mu_T^1}\left[\exp\left[i\lambda\left(\frac{1}{2}\mathfrak{v}_T+\langle\cdot,\ell\rangle+\gamma\right)\right]\middle|w(T)=y\right]$$
$$=\exp\left[i\lambda\left(\frac{Ty^2}{24}+\frac{T^2}{24}+\frac{y\ell(T)}{T}+\gamma\right)+\int_0^\infty\left(e^{i\lambda x}-1-i\lambda x\right)f_V(x;T,\ell,y)dx\right],$$

where

$$f_V(x;T,\ell,y)=\frac{1}{2x}\Big\{\Theta\Big(\frac{4\pi^2x}{T^2}\Big)-1\Big\}-\frac{2\pi^2y^2}{T^3}\Theta'\Big(\frac{4\pi^2x}{T^2}\Big)+g_V(x;T,\ell,y/\sqrt{T}),$$

and g_V is given by (41). Moreover, the distribution of $\mathfrak{v}_T/2$ under the conditional probability $\mu_T^1(\cdot|w(T)=y)$ coincides with the one of $\{\mathfrak{h}_{T/2}+\mathfrak{h}'_{T/2}\}/2$ under the product measure $\mu_{T/2}^1(\cdot|w(T/2)=0)\otimes\mu_{T/2}^1(\cdot|w(T/2)=y/\sqrt{2})$, where $\mathfrak{h}'_{T/2}$ denotes an independent copy of $\mathfrak{h}_{T/2}$.

3.3.3

We finally consider the exponential decay of the characteristic function of $\frac{1}{2}\mathfrak{v}_T+\langle\cdot,\ell\rangle+\gamma$ under μ_T^1 and $\mu_T^1(\cdot|w(T)=y)$.

As was seen in §3.3.1, the corresponding Hilbert-Schmidt operator C possesses eigenvalues $\{(T/[n\pi])^2;n\in\mathbb{N}\}$, each of them being of multiplicity 1, and $\ell_C=0$. By Theorem 7, Lemma 10, and (39), there exist $C_1>0$ and $\lambda_1>0$ such that

$$\left|\int_{W_T^1}\exp\left[i\lambda\left(\frac{1}{2}\mathfrak{v}_T+\langle\cdot,\ell\rangle+\gamma\right)\right]d\mu\right|\le\exp[-C_1\lambda^{1/2}] \tag{42}$$

for every $\lambda\ge\lambda_1,\ \gamma\in\mathbb{R}$.

Let $\eta_1(t)=t/\sqrt{T}$ and $\eta=\{\eta_1\}$. As was shown in §3.3.2, the Hilbert-Schmidt operator $C^{(\eta)}$ has eigenvalues $\{(T/(2n\pi))^2;n\in\mathbb{N}\}$, each of them being of multiplicity 2, and $\{P^{(\eta)}(C(y\cdot\eta)+\ell)\}_{C^{(\eta)}}=0$. Since $\eta(w)=w(T)/\sqrt{T}$, by Theorem 7, Lemma 10, and (39), there exist $C_2>0$ and $\lambda_2>0$ such that

$$\left|\mathbb{E}_{\mu_T^1}\left[\exp\left[i\lambda\left(\frac{1}{2}\mathfrak{v}_T+\langle\cdot,\ell\rangle+\gamma\right)\right]\middle|w(T)=y\right]\right|\le\exp[-C_2\lambda^{1/2}] \tag{43}$$

for every $\lambda\ge\lambda_2,\gamma\in\mathbb{R}$.

When $\ell=0$ and $\gamma=0$, combining the results in Propositions 19 and 20 with (29), we can show the following explicit expressions of the Laplace transforms for the distributions of $\mathfrak{v}_T$; for $\lambda>0$,

$$\int_{W_T^1}\exp\left[-\frac{1}{2}\lambda\mathfrak{v}_T\right]d\mu_T^1=\left(\frac{\sqrt{\lambda}T}{\sinh(\sqrt{\lambda}T)}\right)^{1/2}$$

and

$$\mathbb{E}_{\mu_T^1}\left[\exp\left[-\frac{1}{2}\lambda\mathfrak{v}_T\right]\middle|w(T)=y\right]$$
$$=\frac{\sqrt{\lambda}T/2}{\sinh(\sqrt{\lambda}T/2)}\exp\left[\left(1-\frac{\sqrt{\lambda}T}{2}\coth\left(\frac{\sqrt{\lambda}T}{2}\right)\right)\frac{y^2}{2T}\right].$$

From these expressions, we see, in the same way as in §3.1.3, that our estimates (42) and (43) coincide with the order obtained from the explicit expressions.

3.4 Density functions

Let $T>0$ and consider the classical two-dimensional Wiener space (W_T^2, H_T^2, μ_T^2) over $[0,T]$. In this subsection, as an application of Theorem 11, we show a way to obtain the explicit expressions of the densities of the distributions of Lévy's stochastic area and the square of the L^2-norm on an interval of the two-dimensional Wiener process. We also compute the Mellin transforms of the distributions. For the related topics, see [1, 15].

3.4.1 Lévy's stochastic area

By (30), (31), and Theorem 11(i), we see that the distribution of $\mathfrak{s}_T$ under μ_T^2 admits a smooth density function p_L with respect to the Lebesgue measure on $\mathbb{R}$, and that the corresponding Hilbert-Schmidt operator B satisfies

$$\det{}_2(I - i\zeta\widehat{B}) = \prod_{n=0}^{\infty}\left\{1+\frac{\zeta^2T^2}{(2n+1)^2\pi^2}\right\} = \cos(i\zeta T/2).$$

Define a simple curve $\Gamma_n = \{\gamma_n(t) : t\in[0,4R_n]\}$ in $\mathbb{C}$ with $R_n = 4n\pi/T$ by

$$\gamma_n(t) = \begin{cases} -R_n + it, & t\in[0,R_n],\\ t-2R_n+iR_n, & t\in[R_n,3R_n],\\ R_n + i\{4R_n - t\}, & t\in[3R_n,4R_n].\end{cases}$$

Let $x<0$. By a straightforward computation, we can show that Γ_n satisfies the conditions in Theorem 11(ii) and conclude that

$$\begin{aligned} p_L(x) &= i\sum_{n=0}^{\infty}\operatorname{Res}\left(\frac{e^{-i\zeta x}}{\cos(i\zeta T/2)}; i\frac{(2n+1)\pi}{T}\right)\\ &= \frac{2}{T}\sum_{n=0}^{\infty}(-1)^n e^{(2n+1)\pi x/T} = \frac{1}{T\cosh(\pi x/T)}.\end{aligned}$$

For $x>0$, the complex conjugate $\overline{\Gamma_n}$ plays the same role as Γ_n, and we obtain

$$p_L(x) = -i\sum_{n=0}^{\infty}\operatorname{Res}\left(\frac{e^{-i\zeta x}}{\cos(i\zeta T/2)}; -i\frac{(2n+1)\pi}{T}\right) = \frac{1}{T\cosh(\pi x/T)}.$$

Let $\eta=\{\eta_1,\eta_2\}$, where $\eta_1(t) = \begin{pmatrix} t/\sqrt{T}\\ 0\end{pmatrix}$ and $\eta_2(t) = \begin{pmatrix} 0\\ t/\sqrt{T}\end{pmatrix}$. Then $\eta(w) = w(T)/\sqrt{T}$. By (34) and Remark 12(ii), the distribution of $\mathfrak{s}_T$ under $\mu_T^2(\cdot|w(T)=0)$ admits a smooth density function $\tilde{p}_L$ with respect to the Lebesgue measure on $\mathbb{R}$, and it holds that

$$\det{}_2(I - i\zeta\widehat{B^{(\eta)}}) = \prod_{n=1}^{\infty}\left\{1+\frac{\zeta^2T^2}{4n^2\pi^2}\right\} = \frac{\sin(i\zeta T/2)}{i\zeta T/2}.$$

Using the same Γ_n's as above, this time with $R_n = (4n+1)\pi/T$, and then applying Theorem 11(ii), we can show that

$$\begin{aligned}\tilde{p}_L(x) &= i\sum_{n=1}^{\infty} \operatorname{Res}\left(\frac{e^{-i\zeta x}(i\zeta T/2)}{\sin(i\zeta T/2)}; i\frac{2n\pi}{T}\right) \\ &= \frac{2\pi}{T}\sum_{n=1}^{\infty}(-1)^{n+1} n e^{-2n\pi|x|/T} = \frac{\pi}{2T\cosh^2(\pi x/T)}, \quad x<0.\end{aligned}$$

For $x>0$, the complex conjugate $\overline{\Gamma_n}$ plays the same role as Γ_n, and we obtain

$$\tilde{p}_L(x) = -i\sum_{n=1}^{\infty} \operatorname{Res}\left(\frac{e^{-i\zeta x}(i\zeta T/2)}{\sin(i\zeta T/2)}; -i\frac{2n\pi}{T}\right) = \frac{\pi}{2T\cosh^2(\pi x/T)}.$$

Thus we have:

Proposition 21. *It holds that*

$$\mu_T^2(\mathfrak{s}_T \in dx) = \frac{1}{T\cosh(\pi x/T)}dx, \quad \mu_T^2(\mathfrak{s}_T \in dx | w(T)=0) = \frac{\pi}{2T\cosh^2(\pi x/T)}dx.$$

Moreover, their Mellin transforms are given by

$$\begin{aligned}\int_{W_T^2} |\mathfrak{s}_T|^s d\mu_T^2 &= \frac{4T^s}{\pi^{s+1}}\Gamma(s+1)L_{\chi_4}(s+1), \quad s > -\frac{1}{2} \\ \mathbb{E}_{\mu_T^2}[|\mathfrak{s}_T|^s | w(T)=0] &= \left(\frac{T}{\pi}\right)^s \frac{2^{s-1}-1}{2^{2(s-1)}}\Gamma(s+1)\zeta(s), \quad s > \frac{1}{2},\end{aligned}$$

where L_{χ_4} is Dirichlet's L-function given by $L_{\chi_4}(s) = \sum_{n=0}^{\infty}(-1)^n(2n+1)^{-s}$.

Proof. We have already seen the first half. The last half is immediate consequence of the representations of p_L and $\tilde{p}_L$ in the form of infinite sums. □

The densities of $\mathfrak{s}_T$ under μ_T^2 and $\mu_T^2(\cdot | w(T)=0)$ are first computed by Lévy ([12]).

3.4.2 L^2-norm on an interval of the two-dimensional Wiener process

Put

$$\mathfrak{h}_T^{(2)}(w) = \int_0^T |w(t)|^2 dt, \quad w \in W_T^2.$$

Since $\mathfrak{h}_T^{(2)}$ is a sum of two independent copies of $\mathfrak{h}_T$ coming from w^1 and w^2, due to the observations made in §3.1.1, we see that the Hilbert-Schmidt operator $D : H_T^2 \to H_T^2$ associated with $\mathfrak{h}_T^{(2)}/2$ has eigenvalues $\{(T/[(n+\frac{1}{2})\pi])^2 : n = 0, 1, \dots\}$, each being of multiplicity 2, and hence

$$\det(I - i\zeta\widehat{D}) = \prod_{n=0}^{\infty}\left\{1 - i\zeta\left(\frac{T}{\left(n+\frac{1}{2}\right)\pi}\right)^2\right\} = \cos(\sqrt{i\zeta}\,T),$$

and that $\mathfrak{h}_T^{(2)}/2 = q_D/2$. By Theorem 11(iii), the distribution of $\mathfrak{h}_T^{(2)}/2$ under μ_T^2 admits a smooth density function p_H with respect to the Lebesgue measure on $\mathbb{R}$. Consider a simple curve $\Gamma_n = \{-i(R_n + it)^2 : t \in [-R_n, R_n]\}$ in $\mathbb{C}$ with $R_n = 2n\pi/T$. Let $x > 0$. By a straightforward computation, we can show that Γ_n satisfies the conditions in Theorem 11(ii) and conclude that

$$\begin{aligned} p_H(x) &= -i \sum_{n=0}^{\infty} \operatorname{Res}\left(\frac{e^{-i\zeta x}}{\cos(\sqrt{i\zeta}\,T)}; -i\left(\frac{\left(n+\frac{1}{2}\right)\pi}{T}\right)^2\right) \\ &= \frac{\pi}{T^2} \sum_{n=0}^{\infty} (-1)^n (2n+1) e^{-(2n+1)^2\pi^2 x/(4T^2)} = \frac{1}{2T^2}\vartheta_1'(0|ix\pi/T^2), \end{aligned}$$

where $\vartheta_1(u|\tau)$ denotes the theta function of the first kind with parameter τ,

$$\vartheta_1(u|\tau) = 2\sum_{n=0}^{\infty} (-1)^n e^{\tau\pi i(n+1/2)^2} \sin[(2n+1)\pi u],$$

and ϑ_1' stands for the first derivative in u. Obviously, $p_H(x) = 0$ if $x < 0$.

Let $\eta = \{\eta_1, \eta_2\}$, where $\eta_1(t) = \begin{pmatrix} t/\sqrt{T} \\ 0 \end{pmatrix}$ and $\eta_2(t) = \begin{pmatrix} 0 \\ t/\sqrt{T} \end{pmatrix}$. Due to the observations made in §3.1.2, $D^{(\eta)} : H_T^2 \to H_T^2$ has eigenvalues $\{(T/[n\pi])^2 : n = 1, 2, \dots\}$, each being of multiplicity 2, and hence

$$\det(I - i\zeta\widehat{D^{(\eta)}}) = \prod_{n=1}^{\infty} \left\{1 - i\zeta\left(\frac{T}{n\pi}\right)^2\right\} = \frac{\sin(\sqrt{i\zeta}\,T)}{\sqrt{i\zeta}\,T}.$$

By Remark 12(iii), the distribution of $\mathfrak{h}_T^{(2)}/2$ under $\mu_T^2(\cdot|w(T) = 0)$ admits a smooth density function $\tilde{p}_H$ with respect to the Lebesgue measure on $\mathbb{R}$. Let $R_n = \left(2n + \frac{1}{2}\right)\pi/T$, and consider the same curves $\{\Gamma_n; n \in \mathbb{N}\}$ as above and $x > 0$. By a straightforward computation, we can show that Γ_n satisfies the conditions in Theorem 11(ii) and conclude that

$$\begin{aligned} \tilde{p}_H(x) &= -i \sum_{n=1}^{\infty} \operatorname{Res}\left(\frac{\sqrt{i\zeta}\,Te^{-i\zeta x}}{\sin(\sqrt{i\zeta}\,T)}; -i\left(\frac{n\pi}{T}\right)^2\right) \\ &= 2\left(\frac{\pi}{T}\right)^2 \sum_{n=1}^{\infty} (-1)^{n+1} n^2 e^{-(n\pi/T)^2 x} = \frac{1}{4T^2}\vartheta_4''(0|ix\pi/T^2), \end{aligned}$$

where $\vartheta_4(u|\tau)$ denotes the theta function of the fourth kind with parameter τ,

$$\vartheta_4(u|\tau) = 1 + 2\sum_{n=1}^{\infty} (-1)^n e^{\tau\pi i n^2} \cos(2n\pi u),$$

and ϑ_4'' stands for the second derivative in u. Obviously, $\tilde{p}_H(x) = 0$ if $x < 0$.

Thus we have:

Proposition 22. *It holds that*

$$\begin{aligned} \mu_T^2\big(\mathfrak{h}_T^{(2)}/2 \in dx\big) &= \frac{1}{2T^2}\vartheta_1'(0|ix\pi/T^2)\mathcal{X}_{(0,\infty)}(x)dx, \\ \mu_T^2\big(\mathfrak{h}_T^{(2)}/2 \in dx \big| w(T) = 0\big) &= \frac{1}{4T^2}\vartheta_4''(0|ix\pi/T^2)\mathcal{X}_{(0,\infty)}(x)dx. \end{aligned}$$

Moreover, their Mellin transforms are given by

$$\int_{W_T^2} \big(\mathfrak{h}_T^{(2)}/2\big)^s d\mu = \frac{2^{2s+2}T^{2s}}{\pi^{2s+1}}\Gamma(s+1)L_{\chi_4}(2s+1), \quad s > -\frac{1}{4},$$

$$\mathbb{E}_{\mu_T^2}\Big[\big(\mathfrak{h}_T^{(2)}/2\big)^s \big| w(T)=0\Big] = 2\Big(\frac{T}{\pi}\Big)^{2s}\Gamma(s+1)(1-2^{1-2s})\zeta(2s), \quad s > \frac{1}{4}.$$

Proof. We have already seen the first half. The last half is immediate consequence of the representations of p_H and $\tilde{p}_H$ in the form of infinite sums. □

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