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ALMOST ALL WORDS ARE SEEN IN CRITICAL SITE PERCOLATION ON THE TRIANGULAR LATTICE

H. Kesten

Harry Kesten, Department of Mathematics, White Hall, Cornell University, Ithaca NY 14853, USA, kesten@math.cornell.edu

V. Sidoravicius

Vladas Sidoravicius, IMPA, Estr. Dona Castorina 110, Rio de Janeiro, Brazil, vladas@impa.br

Y. Zhang

Yu Zhang, Department of Mathematics, University of Colorado, Colorado Springs CO 80933, USA, yzhang@vision.uccs.edu

Abstract We consider critical site percolation on the triangular lattice, that is, we choose $X(v) = 0$ or 1 with probability 1/2 each, independently for all vertices v of the triangular lattice. We say that a word $(\xi_1, \xi_2, \dots) \in \{0,1\}^{\mathbb{N}}$ is seen in the percolation configuration if there exists a selfavoiding path $(v_1, v_2, \dots)$ on the triangular lattice with $X(v_i) = \xi_i, i \geq 1$. We prove that with probability 1 'almost all' words, as well as all periodic words, except the two words $(1,1,1,\dots)$ and $(0,0,0,\dots)$, are seen. 'Almost all' words here means almost all with respect to the measure μ_β under which the ξ_i are i.i.d. with $\mu_\beta\{\xi_i = 0\} = 1 - \mu_\beta\{\xi_i = 1\} = \beta$ (for an arbitrary $0 < \beta < 1$).

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1. Introduction.

We take up a problem of Benjamini and Kesten (1995) for site percolation on a graph $\mathcal{G}$ with vertex set $\mathcal{V}$. It is assumed throughout that all vertices are independently occupied or vacant, each with probability 1/2. The corresponding measure on occupancy configurations is denoted by P; E denotes expectation with respect to P. Set $X(v) = 1(0)$ if v is occupied (vacant, respectively). We say that a word $\xi = (\xi_1, \xi_2, \dots)$, with $\xi_i \in \{0, 1\}$, is *seen along a path* $\pi = (v_0, v_1, \dots)$ on $\mathcal{G}$ if π is selfavoiding and $X(v_i) = \xi_i, i \geq 1$ (note that the state of v_0 does not figure in this definition). Benjamini and Kesten (1995) studied which words ξ are seen along some path on $\mathcal{G}$. The classical percolation question is whether the word $(1, 1, \dots)$ with all $\xi_i = 1$ is seen on $\mathcal{G}$. The question whether so-called AB-percolation occurs (see Mai and Halley (1980) and Wierman and Appel (1987)) becomes in the present terminology the question whether the word $(1, 0, 1, 0, \dots)$ is seen. Let

$$\rho(\xi) = P\{\xi \text{ is seen along some path}\}, \tag{1.1}$$

and let

$$\Xi = \{0, 1\}^{\mathbb{N}}$$

be the space of all words. On this space we put the measure

$$\mu = \prod_{i=1}^{\infty} \mu_i, \tag{1.2}$$

where for each $i, \mu_i(\{0\}) = 1 - \mu(\{1\}) = \beta$ for some $0 < \beta < 1$. Benjamini and Kesten (1995) only considered the case $\beta = 1/2$, but W. Thurston asked us what the influence of β is. It turns out that everything in this paper works for $0 < \beta < 1$. On 'nice' graphs, $\rho(\xi)$ can only take the values 0 or 1. Moreover, $\xi \mapsto \rho(\xi)$ is a tail function, that is, it depends only on $\{\xi_i\}_{i \geq n}$ for any n. It therefore follows from Kolmogorov's zero-one law that

$$\mu\{\xi : \rho(\xi) = 1\} = 0 \text{ or } 1. \tag{1.3}$$

In particular this is the case when $\mathcal{G}$ is the triangular lattice $\mathcal{T}$ in the plane (see Proposition 3 in Benjamini and Kesten (1995)). If $\mu\{\xi : \rho(\xi) = 1\} = 1$, we say that 'the random word percolates'. Benjamini and Kesten (1995) raised the question whether for $\mathcal{G} = \mathcal{T}$ this is the case or not. This special case is of interest because site percolation on $\mathcal{T}$ with each vertex occupied with probability 1/2 is critical (see Kesten (1982), Grimmett (1989)), and it is known (see Harris (1960), Fisher (1961), Kesten (1982)) that one does not see the word $(1, 1, \dots)$ (i.e., $\rho(1, 1, \dots) = 0$ or ordinary percolation does not occur), but one does see the word $(1, 0, 1, 0, \dots)$ (i.e., $\rho(1, 0, 1, 0, \dots) = 1$ or AB-percolation does occur)(see Wierman and Appel (1987)). So now one wants to know whether almost all $[\mu]$

words are seen or almost no words are seen; these are the only possibilities by (1.3). Here we prove that almost all words are seen on $\mathcal{T}$ and also generalize considerably the result of Wierman and Appel (1987) that AB-percolation occurs (albeit only when the occupation probability of each vertex equals 1/2, while AB-percolation is known to occur for the occupation probability in a small interval around 1/2). To formulate our result we define the *lengths of the runs*, $r_m = r_m(\xi)$, of a word ξ by the requirement that $\xi_i = 1$ for $\sum_1^{2k} r_m < i \leq \sum_1^{2k+1} r_m$ and $\xi_i = 0$ for $\sum_1^{2k+1} r_m < i \leq \sum_1^{2k+2} r_m, k \geq 0$, and $r_1 \geq 0, r_m > 0$ for $m > 1$. We shall prove the following result.

Theorem. *On $\mathcal{T}$, $\rho(\xi) = 1$ for almost all $[\mu]$ words ξ. Also $\rho(\xi) = 1$ for all words ξ with $r_m(\xi)$ bounded in m.*

It becomes an interesting question to find more explicitly for which words ξ, $\rho(\xi) = 1$ and for which words $\rho(\xi) = 0$ on $\mathcal{T}$. By the theorem $\rho(\xi) = 1$ for all words whose letters ξ_i are eventually periodic with the exception of words which have $\xi_i = 1$ or $\xi_i = 0$ from some point on. The latter words are excluded by boundedness of the r_m, and in fact, as we already stated, $\rho((1, 1, \dots)) = \rho((0, 0, \dots)) = 0$. There certainly are uncountably many ξ's for which $\rho(\xi) = 0$. Indeed, the almost sure existence of infinitely many occupied and vacant circuits surrounding $\mathbf{0}$ on $\mathcal{T}$ quickly shows that $\rho(\xi) = 0$ if ξ has long runs of ones and zeroes. That is, if $r_m(\xi)$ grows fast enough with m, then $\rho(\xi) = 0$.

The idea of the proof of the Theorem is rather combinatorial and strongly uses special properties of the triangular lattice. We do not know whether one sees almost all words in bond percolation on $\mathbb{Z}^2$ at the critical point (or more accurately in the equivalent site percolation on the covering graph of $\mathbb{Z}^2$). We almost explicitly find occupancy configurations inside of which we can see many words. The key idea is that of a *double path*. This is a pair of selfavoiding paths (π', π'') such that $\pi'(\pi'')$ is occupied (vacant) and such that their initial points u' and u'' are neighbors, as well as their final points v' and v''. The 'region between the paths' should also have a certain minimality property (see (5.4)). One shows fairly simply, by means of Proposition 2.2 of Kesten (1982), that for any word $\xi = (\xi_1, \xi_2, \dots)$ there is a path $(w_0, w_1, \dots, w_\nu)$ in the region between π' and π'' along which one sees an initial piece of ξ. This path starts at $w_0 = u'$ and ends at w_ν which is either v' or v''. Thus one sees at least a piece of ξ of length $\min(\|v' - u'\|, \|v'' - u'\|)$. The same is true when u' is replaced by u''. Unfortunately, with probability 1 there do not exists infinite double paths in critical percolation on $\mathcal{T}$; in fact, with probability 1 there are not even infinite occupied or vacant paths. There exist infinitely many occupied and vacant circuits surrounding the origin and these form obstructions to double paths. In order to see a word ξ along a path which crosses an occupied circuit at a vertex v, the word must have a 1 at the position corresponding to the vertex v. On the other hand, Proposition 2.2 of Kesten (1982) easily shows that there exist at least some double paths between successive occupied or vacant circuits $\mathcal{C}'$ and $\mathcal{C}''$ (see Section 5 after Corollary 1). Practically all the work is to show that there exist many suitable double paths between $\mathcal{C}'$ and $\mathcal{C}''$ when these circuits are 'far apart'. These double paths need to have special configurations near their endpoint. These special configurations will allow one to connect in many ways a double path between $\mathcal{C}'$ and $\mathcal{C}''$ and another double path between $\mathcal{C}''$ and the next circuit $\mathcal{C}'''$ in such a way that one sees a random word ξ with high probability along one of the many pairs of connected double paths. There is also considerable technical work needed to show that in general successive occupied or vacant circuits are far apart (see Lemmas 5-7).

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2. Notation and some preliminaries about occupied and vacant circuits. We write $\mathcal{T}$ for the triangular lattice. For the present purposes we shall assume that $\mathcal{T}$ is imbedded in the plane such that its vertex set is $\mathbb{Z}^2$, and its edges are all edges between pairs (i_1, j_1) and (i_2, j_2) with either

$$|i_1 - i_2| + |j_1 - j_2| = 1$$

or

$$i_2 = i_1 + 1, j_2 = j_1 - 1.$$

(These are the usual edges on $\mathbb{Z}^2$ plus the NW diagonals; see Figure 1).

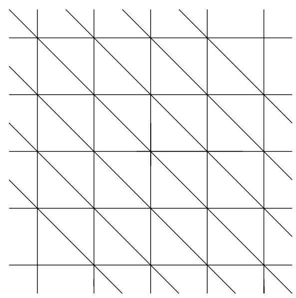

FIGURE 1. A piece of the triangular lattice $\mathcal{T}$.

Most terminology about graphs and paths on graphs follows Kesten (1982) and for seeing words or percolation of words, the terminology is taken from Benjamini and Kesten (1995). We point out that *in this paper a path will always mean a selfavoiding path* even if this is not explicitly indicated. $\|x\|$ will denote the Euclidean norm of x.

Let $S(m)$ be the square

$$S(m) = [-m, m]^2, \tag{2.1}$$

and let A_m be the annulus

$$A_m = S(2^m) \setminus S(2^{m-1}). \tag{2.2}$$

For a subset G of $\mathbb{R}^2$ we denote its topological boundary by ΔG. We shall say that a circuit J on $\mathcal{T}$, or a Jordan curve J in $\mathbb{R}^2$ *surrounds* G, if $J \cap G = \emptyset$ and if there is no continuous path in $\mathbb{R}^2 \setminus J$ from G to ∞ (in other words, if J separates G from ∞).

For any circuit C we denote by $\overset{\circ}{C}$ and C^{ext} the region in the interior and exterior of $\mathbb{R}^2 \setminus C$, respectively. $\overline{C}$ will stand for $C \cup \overset{\circ}{C}$.

We also need a partial order of circuits. If C', C'' are circuits on $\mathcal{T}$ we say that C'' *is larger than* C' (or C' *is smaller than* C'') if $\mathring{C}' \subset \mathring{C}''$, or equivalently, if $C' \subset \overline{C}''$. In accordance with this ordering, we say that C' is the *largest, or outermost occupied circuit in* C'' if C' is occupied, lies inside $\mathring{C}''$ (in particular, C' and C'' should be disjoint), and there is no other occupied circuit $C''' \subset \mathring{C}''$ such that C''' is larger than, but not equal to C'. If there is any occupied circuit in $\mathring{C}''$, then there is a largest such circuit, by the argument of Lemma 1 in Kesten (1980). Similarly if 'occupied' is replaced by 'vacant'.

Remark 1. We shall repeatedly need the following fact. Let $C' \subset \mathring{C}''$ for two given circuits C', C'' such that C' is the largest occupied circuit in C''. Then C' must be occupied, and every vertex $v \in C'$ must have a neighbor $w \in \overline{C}'' \setminus \overline{C}'$ which belongs to C'' or is connected by a vacant path to a neighbor of C''. (It is not important for this what the state of the vertices on C'' is.) Even though the statement is similar to Proposition 2.2 in Kesten (1982), it does not seem to follow from this Proposition. Instead we shall use Corollary 2.2 of Kesten (1982). Assume that C' is occupied, but there is no occupied circuit in $\mathring{C}''$ which is larger than but not equal to C'. Change all vertices in $\mathring{C}'$ (if any) to vacant. This can be done because it influences neither our hypothesis nor conclusion. We also change all vertices in $C'' \cup (C'')^{\text{ext}}$ to vacant. Then there will be no occupied circuit in all of $\mathcal{T}$ larger than but not equal to C'. Now let $v \in C'$. If we also change v to vacant, then there does not exist an occupied circuit in $\mathcal{T}$ which surrounds v and therefore, by Corollary 2.2 of Kesten (1982), the vacant cluster of v is unbounded (in our changed configuration). But this means that in the changed configuration there is a vacant selfavoiding path on $\mathcal{T}$ from v to ∞. Such a path has to stay in $(C')^{\text{ext}}$ except for its initial vertex v, because it cannot exit from C' at any point but v (recall that $C' \setminus \{v\}$ is occupied). This is equivalent to the existence of a neighbor w of v which belongs to C'' or has a vacant connection to a neighbor of C'' in the original configuration, as claimed.

A similar argument applies to show that if C' is an occupied circuit in $\mathring{C}''$ then there is no vacant circuit in $\mathring{C}'' \setminus \mathring{C}'$ if and only if there is an occupied path from some vertex $w \in C'$ to a vertex u adjacent to C''. For the 'only if' direction in this situation, make all vertices in $\mathring{C}' \cup C'' \cup (C'')^{\text{ext}}$ occupied. Then, if there is no vacant circuit in $\mathring{C}'' \setminus \mathring{C}'$, the occupied cluster of C' has to be infinite.

We next define $\mathcal{C}_1^{(N)}, \mathcal{C}_2^{(N)}, \dots$ to be the successive disjoint occupied or vacant circuits in $S(2^N)$, working from the outside in. More precisely, $\mathcal{C}_1^{(N)}$ is the outermost occupied or vacant circuit, as the case may be, in $S(2^N)$. $\mathcal{C}_{k+1}^{(N)}$ is the outermost occupied or vacant circuit inside $\mathcal{C}_k^{(N)}$ and disjoint from $\mathcal{C}_k^{(N)}$. We continue as long as there are such circuits surrounding the origin, $\mathbf{0}$. It is more elegant to let $N \to \infty$, to obtain a system of circuits which does not depend on N, and we shall do this now. Note that in contrast to the $\mathcal{C}_i^{(N)}$, this new system of circuits will be numbered from the inside out, that is we will have $\mathcal{C}_i \subset \overset{\circ}{\mathcal{C}}_{i+1}$. We use the notation $\kappa(C) = 0$ and $\kappa(C) = 1$ for the events $\{C \text{ is vacant}\}$ and $\{C \text{ is occupied}\}$, respectively.

Lemma 1. *With probability 1 there exists a sequence of random disjoint circuits $\mathcal{C}_k$, $k \geq 1$, surrounding $\mathbf{0}$ and so that each of these circuits is occupied or vacant,*

$$\mathcal{C}_k \subset \overset{\circ}{\mathcal{C}}_{k+1}, \tag{2.3}$$

and

$$\mathcal{C}_k \textit{ is the largest circuit in } \overset{\circ}{\mathcal{C}}_{k+1} \textit{ which is entirely occupied or vacant.} \tag{2.4}$$

Moreover, for a fixed circuit C surrounding $\mathbf{0}$ and $\varepsilon \in \{0, 1\}$, the event

$$\bigcup_k \{\mathcal{C}_k = C, \kappa(C) = \varepsilon\} = \{C = \textit{some } \mathcal{C}_k, \kappa(C) = \varepsilon\} \tag{2.5}$$

depends only on the vertices in $C^{\text{ext}} \cup C$.

Proof. For some fixed M, let $\mathcal{C}_1^{(M)}$ be the outermost occupied or vacant circuit in $S(2^M)$. For the sake of argument, let $\mathcal{C}_1^{(M)}$ be occupied. Next let $\mathcal{D}^{(M)}$ be the largest vacant circuit in $\overset{\circ}{\mathcal{C}}{}_1^{(M)}$. We claim that then for all $N \geq M$ the outermost vacant circuit in $S(2^M)$ from the sequence $\mathcal{C}_k^{(N)}, k \geq 1$, is $\mathcal{D}^{(M)}$; in particular, $\mathcal{D}^{(M)}$ will be one of the $\mathcal{C}_k^{(N)}$. Indeed, this outermost vacant circuit in $S(2^M)$ among the $\mathcal{C}_k^{(N)}$ cannot intersect the occupied $\mathcal{C}_1^{(M)}$ and hence must lie entirely inside or outside it. It cannot lie outside $\mathcal{C}_1^{(M)}$, because there is neither an occupied nor a vacant circuit outside $\mathcal{C}_1^{(M)}$ in $S(2^M)$. Thus the outermost vacant circuit in $S(2^M)$ from the sequence $\mathcal{C}_k^{(N)}$ must lie inside $\mathcal{C}_1^{(M)}$ and equal $\mathcal{D}^{(M)}$. This proves our claim.

Now suppose $\mathcal{D}^{(M)} = \mathcal{C}_{k(N)}^{(N)}$ and let the last circuit which exists in the sequence $\mathcal{C}^{(N)}$ be $\mathcal{C}_{\ell(N)}^{(N)}$. Then we can find the successive circuits $\mathcal{C}_i^{(N)}, k(N) \leq i \leq \ell(N)$, as the sequence of successive occupied or vacant circuits in $\mathcal{D}^{(M)}$. Thus the circuits in the sequence $\mathcal{C}_k^{(N)}$ inside $\mathcal{D}^{(M)}$ are the same for all $N \geq M$. Since, with probability 1, $\mathcal{D}^{(M)}$ lies outside any fixed square for all large M, we can take for the circuits $\mathcal{C}_k$ inside a fixed square S simply the circuits from the sequence $\mathcal{C}_k^{(M)}$ for any M for which $\mathcal{D}^{(M)}$ lies outside S. If we index these circuits so that $\mathcal{C}_k \subset \overset{\circ}{\mathcal{C}}_{k+1}$, then (2.3) automatically holds and (2.4) follows because $\mathcal{C}_k$ and $\mathcal{C}_{k+1}$ are successive circuits in the sequence $\mathcal{C}_k^{(M)}$ for large M. Finally, the event in (2.5) equals

$$\lim_{M\to\infty} \bigcup_k \{\mathcal{C}_k^{(M)} = C, \kappa(C) = \varepsilon\}, \tag{2.6}$$

and for fixed M the event here depends only on the vertices in $C^{\text{ext}} \cup C$. Indeed, the event $\{\mathcal{C}_k^{(M)} = C, \kappa(C) = \varepsilon\}$ is itself a disjoint union of

$$\{\mathcal{C}_k^{(M)} = C, \mathcal{C}_i^{(M)} = C_i, \kappa(C) = \varepsilon, \kappa(C_i) = \varepsilon_i\} \tag{2.7}$$

over $C_i, \varepsilon_i, 1 \le i \le k-1$, with $C_1 \subset S(2^M), C \subset \overset{\circ}{C}_{k-1}, C_{i+1} \subset \overset{\circ}{C}_i, 1 \le i < k$. Moreover, one can show by induction on k that the event in (2.7) depends only on vertices in $C^{\text{ext}} \cup C$. E.g., the event $\{\mathcal{C}_1^{(M)} = C_1, \kappa(C_1) = 1\}$ occurs if and only if C_1 is occupied and there does not exist an occupied circuit C' in $S(2^M)$ which is larger than, but not equal to C_1. Thus the last statement of the lemma also holds. ■

From now on $\{\mathcal{C}_k\}$ will be the 'growing' sequence of circuits constructed in Lemma 1 and no further reference to the $\mathcal{C}_k^{(N)}$ will be made. In the next lemma we formulate some more distributional properties for the vertices between the successive circuits $\mathcal{C}_k$.

Lemma 2. *The events $\{\kappa(\mathcal{C}_k) = \varepsilon_k\}, k \ge 1$, are independent with*

$$P\{\kappa(\mathcal{C}_k) = \varepsilon_k\} = \frac{1}{2} \tag{2.8}$$

for all choices of $\varepsilon_k \in \{0, 1\}$.

Moreover, if $\{C_i : 0 \le i \le n\}$ are fixed circuits surrounding the origin such that $C_i \subset \overset{\circ}{C}_{i+1}$, then conditionally on the event

$$\{\mathcal{C}_i = C_i, \kappa(C_i) = \varepsilon_i, 1 \le i \le n\}, \tag{2.9}$$

the collections of vertices

$$\{v : v \in C_{i-1}^{\text{ext}} \cap \overset{\circ}{C}_i\}, 1 \le i \le n \ (\textit{with } C_0^{\text{ext}} = \mathcal{T}),$$

are independent, and the distribution of the occupancy of $\{v : v \in C_{i-1}^{\text{ext}} \cap \overset{\circ}{C}_i\}$, given (2.9), is simply the conditional distribution of $\{v : v \in C_{i-1}^{\text{ext}} \cap \overset{\circ}{C}_i\}$ given that there is no occupied or vacant circuit in $\mathcal{C}_{i-1}^{\text{ext}} \cap \overset{\circ}{\mathcal{C}}_i$.

Proof. We only prove the independence of the events $\{\kappa(\mathcal{C}_k) = \varepsilon_k\}$. We do this by interchanging occupied and vacant vertices in a certain region. Condition on

$\{\mathcal{C}_{k+i} = C_{k+i}, \kappa(\mathcal{C}_{k+i}) = \varepsilon_{k+i}, 0 \le i \le r\}$ for any $r < \infty$. This event depends only on the vertices in $C_k^{\text{ext}} \cup C_k$. Thus conditionally on this event, the vertices in $\overset{\circ}{C}_k$ are still independently occupied or vacant with probability $1/2$ each. In particular, there is still symmetry between 'occupied' and 'vacant' in $\overset{\circ}{C}_k$. Consequently, the events $\{\mathcal{C}_{k-1}$ is occupied$\}$ and $\{\mathcal{C}_{k-1}$ is vacant$\}$ still have the same conditional probability, which must then be equal to $1/2$. This proves the stated independence and (2.8).

The rest of the lemma follows from Remark 1. We leave the details to the reader.

■

3. Spacings between circuits. This section discusses how close together the circuits $\mathcal{C}_k$ are. Throughout, c_i will denote a strictly positive finite constant. We begin with a few very simple lemmas of this kind.

Lemma 3. *For $0 \le \ell < k$, let $\sigma(\ell, k)$ be the maximal number of disjoint occupied circuits surrounding $\mathbf{0}$ in $S(2^k) \setminus S(2^\ell)$. Then there are constants $0 < c_i < \infty$ such that for $\ell < k$,*

$$P\{\sigma(\ell, k) \le c_1(k - \ell)\} \le 2e^{-c_2(k-\ell)} \tag{3.1}$$

and

$$P\{\sigma(\ell, k) \ge x(k - \ell)\} \le c_3 e^{-c_4 x}, \quad x \ge 0. \tag{3.2}$$

Proof. By the Russo-Seymour-Welsh lemma, the probability that there exists an occupied circuit in A_m is bounded below by a strictly positve constant, c_5 say. Since $S(2^k) \setminus S(2^\ell)$ is the disjoint union of the annuli A_i, $\ell < i \le k$, it follows that $\sigma(\ell, k)$ is stochastically larger than a binomial variable with $(k - \ell)$ trials and success probability c_5. Thus (3.1) with $c_1 = (1/2)c_5$ follows from standard exponential bounds for the binomial distribution.

On the other hand, the argument in Chayes, Chayes and Durrett (1986) shows that $E\sigma(\ell, k) \le c_6(k - \ell)$. Thus by Markov's inequality

$$P\{\sigma(\ell, k) \ge 2c_6(k - \ell)\} \le \frac{1}{2}.$$

By the BK inequality (cf. Grimmett (1989), Sect. 2.3) it then follows that

$$P\{\sigma(\ell, k) \ge 2mc_6(k - \ell)\} \le \left(\frac{1}{2}\right)^m, \quad m \ge 0.$$

The inequality (3.2) easily follows from this. ■

Define the *diameter* of a set of vertices $\mathcal{D}$ by

$$\text{diam } (\mathcal{D}) = \sup\{\|x - y\| : x, y \in \mathcal{D}\}.$$

We shall use without proof that

$$\text{diam}\,(\mathcal{C}_k) \le \text{diam}\,(\mathcal{C}_{k+1});$$

this follows from $\mathcal{C}_k \subset \overset{\circ}{\mathcal{C}}_{k+1}$.

Lemma 4. *There exist constants c_i such that with probability 1*

$$c_7 \le \liminf_{k\to\infty} \frac{\log(\text{diam}\,(\mathcal{C}_k))}{k} \le \limsup_{k\to\infty} \frac{\log(\text{diam}\,(\mathcal{C}_k))}{k} \le c_8. \tag{3.3}$$

Also with probability 1 there exists a $k_0 = k_0(\omega) < \infty$ such that for all $k \ge k_0$ the following properties hold:

$$\min_{x\in\mathcal{C}_k} \|x\| \ge \frac{c_9}{k^{c_{10}}} \max_{x\in\mathcal{C}_k} \|x\|, \tag{3.4}$$

and even

$$\min_{x\in\mathcal{C}_{k-1}} \|x\| \ge \frac{c_9}{k^{c_{10}}} \max_{x\in\mathcal{C}_k} \|x\|. \tag{3.5}$$

Proof. For the upper bound in (3.3), note that if there exists a vacant circuit in A_m and an occupied circuit in A_{m+1}, both surrounding $\mathbf{0}$, then at least one of the vacant circuits among the $\mathcal{C}_k$ has to lie in $A_m \cup A_{m+1}$, by the argument used in the beginning of Lemma 1. Since

$$\begin{aligned} &P\{\text{there exists a vacant circuit in } A_m \text{ and an occupied circuit in } A_{m+1} \\ &\qquad\qquad\qquad \text{both surrounding } \mathbf{0}\} \\ &\ge c_5^2, \end{aligned}$$

it follows from the independence of the configurations in different A_m and the strong law of large numbers that with probability 1 there are, for $\varepsilon > 0$ and n large, at least $(c_5^2 - \varepsilon)n/2$ vacant circuits in $S(2^n)$ among the $\mathcal{C}_k$. This implies the last inequality in (3.3).

For the lower bound in (3.3) let us count the maximal number of disjoint occupied circuits in $S(2^k)$. Let $1 \le m_1 < m_2 < \dots$ be the successive values of m for which A_m contains a vacant circuit surrounding $\mathbf{0}$. Let $\mathcal{D}_i$ denote the smallest vacant circuit surrounding $\mathbf{0}$ in A_{m_i}. No occupied circuit can cross a $\mathcal{D}_i$. Thus all occupied circuits which intersect $\cup_{m<m_i} A_m$ must be contained in $S(2^{m_i})$. In particular, if $\nu = \nu(k)$ is the unique index for which $m_\nu \le k < m_{\nu+1}$, then

$$\begin{aligned} &\text{maximal number of disjoint occupied circuits surrounding } \mathbf{0} \text{ in } S(2^k) \\ &\le \sum_{i=0}^{\nu} \widetilde{\sigma}(m_i, m_{i+1}), \end{aligned} \tag{3.6}$$

where

$$\widetilde{\sigma}(m_i, m_{i+1}) = \text{maximal number of disjoint occupied circuits surrounding } \mathbf{0} \\ \text{inside } \mathcal{D}_{i+1} \text{ and outside } \mathcal{D}_i$$

($\sigma(m_0, m_1) =$ maximal number of disjoint occupied circuits inside $\mathcal{D}_1$.) Obviously, $\nu(k) \le k$ so that the right hand side of (3.6) is at most

$$\sum_{i=0}^{k} \widetilde{\sigma}(m_i, m_{i+1}).$$

Moreover, conditionally on $m_i = n, \mathcal{D}_i = D$ and all vertices inside D, we have essentially by the same argument as for (3.2) (see also Kesten and Zhang (1997), Lemma 3), that

$$\begin{aligned}
&P\{\widetilde{\sigma}(m_i, m_{i+1}) \ge x | m_i = n, \mathcal{D}_i = D, \text{ all vertices inside } D\} \\
&\le P\{m_{i+1} - m_i > p | m_i = n, \mathcal{D}_i = D, \text{ all vertices inside } D\} \\
&\qquad + P\{\widetilde{\sigma}(n-1, n+p) \ge x | m_i = n, \mathcal{D}_i = D, \text{ all vertices inside } D\} \\
&\le \prod_{j=1}^{p} P\{\text{there is no vacant circuit in } A_{n+j}\} + P\{\sigma(n-1, n+p) \ge x\} \\
&\le (1 - c_5)^p + c_3 e^{-c_4 x/(p+1)} \text{ (by (3.2))} \\
&\le c_{11} e^{-c_{12}\sqrt{x}} \text{ (by taking } p = \lfloor \sqrt{x} \rfloor).
\end{aligned}$$

Thus, $\widetilde{\sigma}(m_0, m_1), \widetilde{\sigma}(m_1, m_2), \dots$ are stochastically smaller than a sequence of i.i.d. random variables $\widetilde{\sigma}_1, \widetilde{\sigma}_2, \dots$ with

$$P\{\widetilde{\sigma}_i \ge x\} \le c_{11} e^{-c_{12}\sqrt{x}}.$$

By applying the strong law of large numbers to the $\widetilde{\sigma}_i$ we see that with probability 1

$$\limsup_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k} \widetilde{\sigma}(m_i, m_{i+1}) \le \limsup_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k} \widetilde{\sigma}_i < c_{13} < \infty.$$

Consequently, (3.6) yields that

$$\frac{1}{k}\{\text{maximal number of disjoint occupied circuits surrounding } \mathbf{0} \text{ in } S(2^k)\} \le c_{13}$$

eventually. Adding the same estimate for the vacant circuits we finally see that for large k there are no more than $2c_{13}k$ of the $\{\mathcal{C}_i\}$ in $S(2^k)$. Thus diam $(\mathcal{C}_{\lceil 2c_{13}k \rceil + 1}) \ge 2^k$, which implies the first inequality in (3.3).

Finally we turn to (3.4) and (3.5). Assume that $\mathcal{C}_k$ is an occupied circuit which is not contained in $S(2^j)$. For a suitable choice of the constant c_{14} there exists with probability 1, for all large j, a vacant circuit $\mathcal{C}$ surrounding $\mathbf{0}$ in some A_ℓ with $j - c_{14}\log j < \ell \le j$. Then, by assumption, $\mathcal{C}_k$ contains a point outside $\mathcal{C}$. Since the occupied circuit $\mathcal{C}_k$ cannot cross the vacant $\mathcal{C}$ it must lie entirely outside $\mathcal{C}$ and therefore outside $S(2^{j-c_{14}\log j})$. (3.4) is an immediate consequence of this, (3.3) and the fact that $\mathcal{C}_k$ is not contained in $S(2^j)$ for any j with $2^j < \frac{1}{2}\max_{x\in\mathcal{C}_k}\|x\|$.

Essentially the same argument proves (3.5), because if $\mathcal{C}_k$ and $\mathcal{C}$ are as in the preceding paragraph, then $\mathcal{C}_{k-1}$ cannot contain any points of $\overset{\circ}{\mathcal{C}}$, because $\mathcal{C}$ is already a circuit which surrounds $\mathbf{0}$ and lies inside $\mathcal{C}_k$ and is disjoint from it. ■

Lemma 5. *There exist constants c_{15}, c_{16} such that for any vertex v and $n \ge 1$*

$$\begin{aligned}
&\frac{c_{15}}{n^2}\\
&\le P\{\textit{there exists 5 disjoint paths from some neighbors of } v\\
&\qquad \textit{to } \Delta(v+S(n)),\ \textit{one of which is vacant}\\
&\qquad \textit{and the other four of which are occupied}\}\\
&\le \frac{c_{16}}{n^2}.
\end{aligned}\tag{3.7}$$

More generally, for $1 \le k \le n$

$$\begin{aligned}
&P\{\textit{there exists 5 disjoint paths from } \Delta(v+S(k)) \textit{ to } \Delta(v+S(n)),\\
&\qquad \textit{one of which is vacant and the other four of which are occupied}\}\\
&\le c_{16}\frac{k^2}{n^2}.
\end{aligned}\tag{3.8}$$

Finally,

$$\begin{aligned}
&P\{\textit{there exist at least 6 disjoint paths from } \Delta(v+S(k)) \textit{ to } \Delta(v+S(n)),\\
&\qquad \textit{at least one of which is vacant and at least four of which are occupied}\}\\
&\le c_{16}\left(\frac{k}{n}\right)^{2+c_2/\log 2}.
\end{aligned}\tag{3.9}$$

Proof. We begin with the upper bound in (3.7). Consider a vertex $w \in \overset{\circ}{S}(n)$ and assume that the following event occurs:

$$\begin{aligned}
F(w,n) :=\{&\text{there exist 5 disjoint paths from neighbors of } w,\\
&\text{which consist of two occupied paths with endpoints}\\
&\text{on the left edge of } S(n),\ \text{another occupied path}\\
&\text{with endpoint on the right edge of } S(n) \text{ and two vacant}\\
&\text{paths with endpoints on the upper and lower edge of } S(n),\\
&\text{respectively}\}.
\end{aligned}\tag{3.10}$$

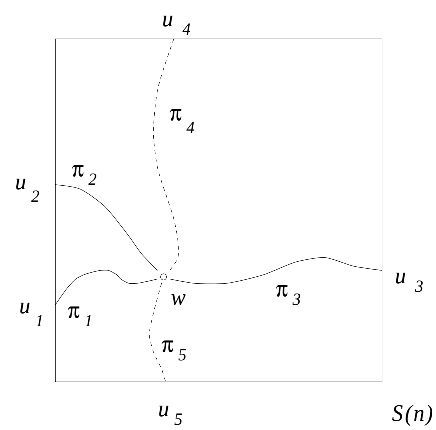

FIGURE 2. Illustration of $F(w, n)$ and the five paths π_i. The solidly drawn paths are occupied and the dashed paths are vacant.

Let π_1, π_2 (π_3) be the three occupied paths with their endpoint on the left edge (respectively, right edge) of $S(n)$, and let π_4, π_5 be the vacant paths with endpoints on the upper and lower edge, respectively (see Figure 2). Let u_i be the endpoint of π_i and index the paths π_1, π_2 such that u_2 lies above u_1 on the left edge. Then the existence of $\pi_1, \pi_3, \pi_4, \pi_5$ shows that w is pivotal for the existence of an occupied left-right crossing of $S(n)$. That is, if w is occupied (vacant) then such a crossing exists (respectively, does not exist). Let us now take w occupied; this can be done because the state of w does not influence the occurrence of $F(w, n)$. Then π_1, w and π_3 together form a left-right crossing, and in fact the existence of the vacant π_4, π_5 shows that any such crossing has to pass through w. Let $\gamma = (v_1, v_2, \dots, v_\lambda)$ be the lowest occupied left-right crossing, with the vertices numbered such that $v_1 (v_\lambda)$ lies in the left (respectively, right) edge of $S(n)$ and such that $v_i \in \overset{\circ}{S}(n)$ if $1 < i < \lambda$. In particular, w has to be one of the v_i. The existence of π_2, π_4 says that w is the first of the v_i which has a vacant connection to the top edge. This uniquely locates w in $S(n)$. In other words, $F(w, n)$ can occur for at most one $w \in S(n)$ and consequently

$$\sum_{w \in S(n/2)} P\{w \text{ is occupied and } F(w, n)\} = \frac{1}{2} \sum_{w \in S(n/2)} P\{F(w, n)\} \leq 1. \tag{3.11}$$

One next has to prove that for any fixed $0 < \delta < 1$, $P\{F(w, n)\}$ is of the same order for all $w \in S(n(1-\delta))$. More precisely, there exists a constant $0 < c_{17} < \infty$ such that

$$\frac{1}{c_{17}} \leq \frac{P\{F(w', n)\}}{P\{F(w'', n)\}} \leq c_{17} \text{ for all } w', w'' \in S(n(1-\delta)). \tag{3.12}$$

(3.12) with $\delta = 1/2$ and (3.11) combined will show that for any $w' \in S(n/2)$,

$$P\{F(w',n)\} \le c_{17} \frac{1}{\text{number of vertices in } S(n/2)} \sum_{w \in S(n/2)} P\{F(w,n)\} \le \frac{c_{18}}{n^2}. \tag{3.13}$$

The proof of (3.12) is quite long and involved and we shall not give it here. It is very similar to the proof of Lemma 4 in Kesten (1987), which deals with the probability of four disjoint paths to $\Delta S(n)$, two occupied ones and two vacant ones, with the occupied paths separating the vacant ones. An extra complication arises in the case of five paths, considered here, because the occupied and vacant paths cannot separate each other in this case. One needs to modify the definition of an (η, k) fence in Kesten (1987) to deal with this. In the appendix we shall give a brief indication how to do this.

Next consider the event

$$\begin{aligned} G(w,n) :=&\{\text{there exist 5 disjoint paths from neighbors of } w, \\ &\text{which consist of two occupied paths with endpoints} \\ &\text{on the left edge of } S(n), \text{ two more occupied paths} \\ &\text{with endpoints on the top and right edge of } S(n), \\ &\text{respectively, and one vacant path with endpoint} \\ &\text{on the bottom edge of } S(n)\}. \end{aligned} \tag{3.14}$$

$G(w,n)$ is quite similar to $F(w,n)$; only the path to the top edge is now assumed occupied instead of vacant. Denote the path to the top edge still by π_4, though, and leave the meaning of the π_i with $i \neq 4$ unchanged. We shall show that in fact $G(w,n)$ and $F(w,n)$ have the same probability, by changing the path to the top edge from occupied to vacant, as we now explain. Assume that $G(w,n)$ occurs. Again the state of w does not influence the occurrence of $F(w,n)$ or $G(w,n)$, so we take it as occupied. Then w again lies on the lowest occupied left-right crossing of $S(n)$. Again denote this lowest crossing by $\gamma = (v_1, \dots, v_\lambda)$ with $v_1(v_\lambda)$ on the left (right) edge of $S(n)$. γ is a crosscut of $\overset{\circ}{S}(n)$ and $\overset{\circ}{S}(n) \setminus \gamma$ consists of two components. (A crosscut of an open connected set D is a simple continuous curve C, such that C minus its endpoints lies in D and the endpoints of C lie on the boundary of D; see Newman (1951), Section V.11.) We denote the component of $\overset{\circ}{S}(n) \setminus \gamma$ which lies 'above' γ (that is the one which has the upper edge of $S(n)$ as part of its boundary) by $U = U(\gamma)$ and the one which has the lower edge in its boundary by $L = L(\gamma)$. Similarly, w lies on the occupied left-right crossing π which consists of (the reverse of) $\pi_1, \{w\}$ and π_3. The corresponding components of $\overset{\circ}{S}(n) \setminus \pi$ are denoted by $U(\pi)$ and $L(\pi)$. Because γ is the *lowest* left-right crossing, π cannot contain any points of $L(\gamma)$, i.e., $\pi \subset \overline{U}(\gamma) := U(\gamma) \cup \gamma$. Equivalently, $U(\pi) \subset U(\gamma)$. Then π_2, which is disjoint from π and has its endpoint on the left edge above that of π_1, must lie in $U(\pi) \subset U(\gamma)$. In particular the path consisting of the edge from w to the initial point of π_2, followed by π_2 itself is an occupied crosscut of $U(\gamma)$ with one endpoint at w. For any such crosscut δ let $V = V(\delta, \gamma)$ be the component of $U(\gamma) \setminus \delta$ which has the left endpoint of γ, v_1, in its boundary, and let $W(\delta, \gamma)$ be the other component of $U(\gamma) \setminus \delta$, with v_λ in its boundary. By Proposition 2.3 of Kesten (1982) there then exists an occupied crosscut of $U(\gamma), \overline{\delta}$ say, with one endpoint at w, so that $V(\overline{\delta}, \gamma)$ is minimal among all such crosscuts.

Finally, define the map Φ on the occupancy configurations in $S(n)$ which have a lowest occupied crossing γ through w and a further crosscut $\overline{\delta}$ as above. For a given configuration $\omega, \Phi(\omega)$ is the configuration obtained from ω by changing all occupied vertices in $W(\overline{\delta}, \gamma)$ to vacant and vice versa. Since the lowest crossing γ and after that the minimal crosscut $\overline{\delta}$ are uniquely determined, the map Φ is well defined. Since it leaves the state of the vertices on γ and on $\overline{\delta}$ unchanged, one easily sees that the lowest occupied crossing in $\Phi(\omega)$ is still γ, and also the occupied crosscut minimizing $V(\delta, \gamma)$ for $\Phi(\omega)$ is still $\overline{\delta}$. Thus one can recognize in the configuration $\Phi(\omega)$ the states of which vertices have been flipped by Φ, so that Φ is one-to-one. Moreover, the event $G(w, n)$ is taken onto the event $F(w, n)$ by Φ, because the occupied path π_4 in $G(w, n)$ is turned into a vacant one by applying Φ. Indeed, $\pi_4 \setminus \{u_4\}$ must lie in $W(\overline{\delta}, \gamma)$, because its endpoint u_4 lies on the upper edge of $S(n)$ which belongs to $\Delta W(\overline{\delta}, \gamma)$ and not to $\Delta V(\overline{\delta}, \gamma)$ whenever the endpoint of $\overline{\delta}$ lies on the left edge of $S(n)$. Since the probability of the configuration $\Phi(\omega)$ is the same as the probability of ω, it follows that

$$P\{G(w, n)\} = P\{F(w, n)\} \leq \frac{c_{18}}{n^2}, \quad w \in S(n/2) \text{ (by (3.13))}. \tag{3.15}$$

The last inequality seems almost what we want. What must still be shown is that the probability in (3.7) (which is independent of v) is at most a constant (independent of n) times $P\{G(\mathbf{0}, n)\}$. The difference between $G(\mathbf{0}, n)$ and the event in (3.7) for $v = \mathbf{0}$ lies in the restrictions on the endpoints of the paths π_i imposed in $G(\mathbf{0}, n)$. For $G(\mathbf{0}, n)$ to occur these endpoints have to lie on certain edges of $S(n)$, while there is no such restriction in (3.7). The proof that removing these restrictions on the endpoints only increases the probability by a bounded factor is basically the same as the proof of (3.12) and is again based on Kesten (1987). We skip this proof, but make some remarks about this in the Appendix. This proves the upper bound in (3.7).

For the lower bound in (3.7) we note that by (3.15) and (3.12)

$$P\{G(\mathbf{0}, n)\} = P\{F(\mathbf{0}, n)\} \geq \frac{1}{c_{17} n^2} \sum_{w \in S(n(1-\delta))} P\{F(w, n)\}. \tag{3.16}$$

Furthermore, the interpretation of $F(w,n)$ as the event that the first v_i on γ with a vacant connection to the top edge of $S(n)$ is w, shows that the sum in the right hand side here equals

$$P\{\text{first vertex } v_i \in \gamma \text{ with a vacant connection to the top edge of } S(n) \text{ lies in } S(n(1-\delta))\}. \tag{3.17}$$

Thus the lower bound in (3.7) for $v = \mathbf{0}$, and hence for any v, will follow once we prove that for some small fixed $0 < \delta < 1$, the probability (3.17) is at least $c_{19} > 0$. This follows by combining fairly standard arguments, but since it is not entirely trivial we give most of the proof anyway. First we want to show that for some $\delta > 0$, the probability of the event

$$\begin{aligned}&\{\text{the lowest occupied left-right crossing } \gamma \text{ of } S(n) \text{ exists,}\\ &\text{lies in } [-n,n]\times[-(1-5\delta)n,(1-5\delta)n],\\ &\text{and does neither return to the vertical line } \{x=-(1-5\delta)n\}\\ &\text{after it first reaches the vertical line } \{x=(1-5\delta)n\},\\ &\text{nor returns to the vertical line } \{x=(1-3\delta)n\} \text{ after it}\\ &\text{first reaches the vertical line } \{x=(1-\delta)n\}\\ &(\text{starting from the left edge of } S(n))\}\end{aligned} \tag{3.18}$$

is bounded away from 0 (as $n \to \infty$). To see this we take $\delta < 1/8$ without loss of generality, and assume that the following two paths exist (see Figure 3):
(i) a vacant left-right crossing π_6' in $[-n,n]\times[-n(1-5\delta),-n(1-6\delta))$;
(ii) a vacant connection π_7' from π_6' to the bottom edge of $S(n)$.
One easily sees that in this case, the lowest occupied left-right crossing of $S(n)$ (if it exists) must lie above π_6' and hence in $[-n,n]\times[-(1-5\delta)n,n]$. The probability that such π_6', π_7' exist is at least c_{20} (by the Russo-Seymour-Welsh lemma and the Harris-FKG inequality). If such π_6', π_7' exist, then we take π_6 to be the lowest vacant left-right crossing of $[-n,n]\times[-n(1-5\delta),-n(1-6\delta))$. The existence of π_7' then guarantees that also π_6 has a vacant connection to the bottom edge of $S(n)$. Thus we have a probability of at least $c_{21} = c_{21}(\delta) > 0$ that the lowest vacant left-right crossing of $[-n,n]\times[-n(1-5\delta),-n(1-6\delta))$ exists and has a vacant connection to the bottom edge of $S(n)$. We shall now condition on this last event and even on the specific value of π_6. Then we still know nothing about the vertices 'above π_6' and they are independently occupied or vacant with probability 1/2. Here and in the rest of this lemma, 'above π_6' means in the component of $\overset{\circ}{S}(n)\setminus\pi_6$ which has the upper edge of $S(n)$ in its boundary. Define

$$a(\pi_6) = P\{\text{there exists an occupied left-right crossing of } [-n,n]\times[-(1-5\delta)n,(1-5\delta)n] \text{ above } \pi_6 | \pi_6\}.$$

Then it is easy to see that there is some constant $c_{22} > 0$ (independent of π_6 and of $0 < \delta < 1/8$) such that

$$c_{22} \le P\{\text{there exists an occupied left-right crossing of } [-n, n] \times [-(1-6\delta)n, (1-6\delta)n]\} \le a(\pi_6) \le 1 - c_{22}. \tag{3.19}$$

We also need the following probabilities:

$$b(\pi_6, \delta) := P\{\text{there exists an occupied left-right crossing of } [-n, (1-\delta)n] \times [-(1-5\delta)n, (1-5\delta)n] \text{ above } \pi_6 | \pi_6\}.$$

and

$$c(\pi_6, \delta) := P\{\text{there exists an occupied left-right crossing of } [-(1-5\delta)n, (1-5\delta)n] \times [-(1-5\delta)n, (1-5\delta)n] \text{ or of } [(1-3\delta)n, (1-\delta)n] \times [-(1-5\delta)n, (1-5\delta)n] \text{ above } \pi_6 | \pi_6\}.$$

We claim that uniformly in π_6,

$$\lim_{\delta \downarrow 0} \left| \frac{b(\pi_6, \delta)}{a(\pi_6)} - 1 \right| = 0, \tag{3.20}$$

and

$$c(\pi_6, \delta) \le 1 - c_{23}. \tag{3.21}$$

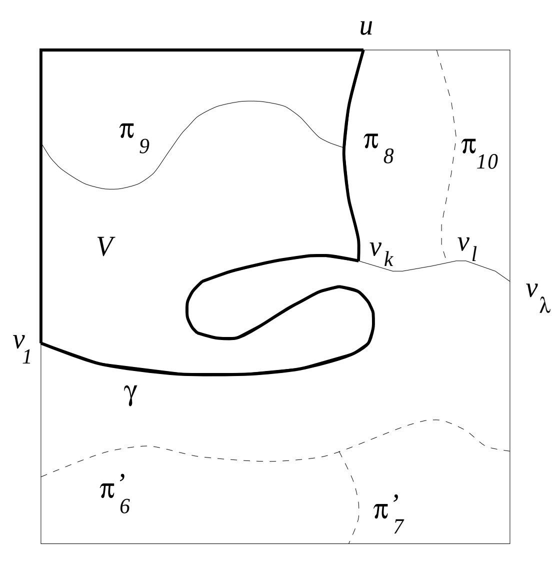

FIGURE 3. Illustration of $\pi_6 - \pi_{10}$ and γ. Again solidly drawn (dashed) paths are occupied (vacant). The boundary of V is boldly drawn.

Before proving (3.20) and (3.21) we point out that they imply that the probability of the event in (3.18) is bounded away from 0. Indeed, given π_6 and that π_6 has a vacant connection to the bottom edge of $S(n)$, the conditional probability that there exists a lowest occupied crossing γ of $S(n)$ in $[-n,n]\times[-(1-5\delta)n,(1-5\delta)n]$ is $a(\pi_6)$. If γ returns to the vertical line $\{x=-(1-5\delta)n\}$ after first hitting the line $\{x=(1-5\delta)n\}$ (starting from the left edge of $S(n)$), then γ must contain an occupied left-right crossing γ_1 of $[-n,(1-5\delta)n]\times[-(1-5\delta)n,(1-5\delta)n]$ above π_6, and in addition two occupied left-right crossings of $[-(1-5\delta)n,(1-5\delta)n]\times[-(1-5\delta)n,(1-5\delta)n]$ which are disjoint from γ_1. Indeed, the piece of γ from the left edge of $S(n)$ to the first visit to the vertical line $\{x=(1-5\delta)n\}$ contains such a crossing γ_1, then the return from $\{x=(1-5\delta)n\}$ to $\{x=-(1-5\delta)n\}$ contains another occupied crossing of $[-(1-5\delta)n,(1-5\delta)n]\times[-(1-5\delta)n,(1-5\delta)n]$, and finally there is a third crossing between the return to $\{x=-(1-5\delta)\}n$ and the final point on the right edge of $S(n)$. Similarly, if γ returns to the line $\{x=(1-3\delta)n\}$ after reaching the line $\{x=(1-\delta)n\}$, then there are two occupied left-right crossings of $[(1-3\delta)n,(1-\delta)n]\times[(1-5\delta)n,(1-5\delta)n]$, disjoint from γ_1. Hence, by the BK inequality, the conditional probability that γ exists but returns to $\{x=-(1-5\delta)n\}$ after first visiting $\{x=(1-5\delta)n\}$ or that γ returns to $\{x=(1-3\delta)n\}$ after first visiting $\{x=(1-\delta)n\}$ is for small δ at most

$$\begin{aligned}&b(\pi_6,\delta)c^2(\pi_6,\delta)\le(1-c_{23})^2b(\pi_6,\delta)\ \text{(by (3.21))}\\&\le(1-c_{23})a(\pi_6)\ \text{(by (3.20) for } \delta \text{ small enough).}\end{aligned}$$

Finally the conditional probability of the event (3.18), given π_6, is at least

$$a(\pi_6)-b(\pi_6,\delta)c^2(\pi_6,\delta)\ge c_{23}a(\pi_6)\ge c_{23}c_{22}.$$

Since π_6 exists with probability at least c_{21} we find that for small δ

$$P\{(3.18)\text{ occurs}\}\ge c_{21}c_{22}c_{23}.$$

We still have to prove (3.20) and (3.21). For (3.20) we note first that by definition, $b(\pi_6,\delta)\ge a(\pi_6)$. On the other hand, it is also clear that

$$\begin{aligned}b(\pi_6,\delta)\le P\{&\text{there exists an occupied left-right crossing of}\\&[-n,(1-\delta)n]\times[-n,n]\}.\end{aligned}\tag{3.22}$$

To see that the last probability on the right here is not much larger than $P\{$there exists an occupied left-right crossing of $S(n)\}$ when δ is small, we can use the argument in the beginning of Lemma 2 of Kesten (1987). If there is a crossing as in the right hand side of (3.22), then the lowest such crossing can with high probability be extended on the right to give a left-right crossing of $S(n)$. This is ilustrated in Figure 7 of Kesten (1987) and we leave the details to the reader. Then we have for any fixed $\eta > 0$ and $0 < \delta < \delta_0$ for some $\delta_0 = \delta_0(\eta) > 0$, that

$$\begin{aligned} b(\pi_6, \delta) &\leq (1+\eta)P\{\text{there exists an occupied left-right crossing of } S(n)\} \\ &\leq (1+\eta)^2 P\{\text{there exists an occupied left-right crossing of} \\ &\qquad\qquad [-n,n] \times [-(1-6\delta)n, (1-6\delta)n]\} \\ &\leq (1+\eta)^2 a(\pi_6); \end{aligned} \tag{3.23}$$

the second inequality here holds by a similar argument as for the first inequality, this time applied to the existence of vacant top-bottom crossings. Thus (3.20) holds.

The inequality (3.21) is completely standard. It follows from the fact that for fixed $0 < \delta_1 < \delta_2$,

$$\begin{aligned} P\{&\text{there exists a vacant top-bottom crossing of } [\delta_1 n, \delta_2 n] \times [-n,n]\} \\ &\geq c_{24} = c_{24}(\delta_1, \delta_2) > 0 \end{aligned}$$

(by the Russo-Seymour-Welsh lemma; compare Kesten (1982), Theorem 6.1). Again we leave the details to the reader.

The inequalities (3.20) and (3.21) finally complete the proof of that (3.18) has a probability bounded away from 0. We still have to deduce (3.17) from this lower bound for the probability of (3.18). Fortunately this is fairly easy. Assume that the event (3.18) occurs . We now condition on the lowest occupied left-right crossing γ, say we condition on $\gamma = (v_1, v_2, \dots, v_\lambda)$. Again, under this condition the vertices in $S(n)$ above γ are still independently occupied or vacant with probability 1/2. Therefore (even conditionally on γ) there is a probability at least $c_{25} > 0$ that the following paths occur (see Figure 3):
(iii) an occupied connection π_8 from a vertex $v_k \in \gamma$ to a vertex u in the top edge of $S(n)$ in the vertical strip $[(1-5\delta)n, (1-4\delta)n) \times [-n,n]$;
(iv) an occupied connection π_9 from π_8 to the left edge of $S(n)$; π_9 lies above γ and is disjoint from γ;
(v) a vacant connection π_{10} from a neighbor of a vertex $v_\ell \in \gamma$ to the top edge of $S(n)$ in the vertical strip $[(1-4\delta)n, (1-3\delta)n]$.
Now $S(n) \setminus \gamma$ consists of the two components with the bottom edge and the top edge of $S(n)$ in their boundary, respectively. Let $U = U(\gamma)$ be the component with the top edge in its boundary. π_8 is a crosscut of U. We consider the component V of $U \setminus \pi_8$ whose boundary consists of the following pieces: the piece of γ from v_1 on the left edge of $S(n)$ to v_k, π_8, the piece of the top edge of $S(n)$ from u to the upper left corner of $S(n)$, and finally the piece of the left edge of $S(n)$ from the top left corner back to v_1. Now there cannot be a vacant connection from a vertex adjacent to γ and located in V to the top edge of $S(n)$, because

any such connection would have to cross $\pi_8 \cup \pi_9$ which is impossible. Therefore, any vacant connection from a vertex adjacent to γ to the top edge of $S(n)$ has to start at a vertex adjacent to one of $\{v_k, v_{k+1}, \dots, v_\lambda\}$. However, π_8 lies in the vertical strip $[(1-5\delta)n, (1-4\delta)n) \times [-n, n]$, so that also v_k lies in this strip and hence comes at or after γ's first intersection with the vertical line $\{x = (1-5\delta)n\}$. But then, on the event (3.18), $v_k, v_{k+1}, \dots, v_\lambda$ lie to the right of the vertical line $\{x = -(1-5\delta)n\}$. Thus the first vertex on γ with a vacant connection to the top edge (if any) lies in $[-(1-5\delta)n, n] \times [-(1-5\delta), (1-5\delta)n]$ (for the vertical restriction we used the vertical restriction of γ on (3.18)). The path π_{10} guarantees that there is at least one vacant connection from γ to the top edge. The restrictions on the location of π_{10} imply that $v_\ell \in [(1-4\delta)n, (1-3\delta)n]$. Unfortunately v_ℓ is not necessarily the first point on γ with a vacant connection to the top edge of $S(n)$. If this first vertex on γ is v_f, then we know from the preceding argument that $k \le f \le \ell$ and that $v_f \in [-(1-5\delta)n, n] \times [-(1-5\delta)n, (1-5\delta)n]$, but a priori, v_f could still lie arbitrarily close to the right edge of $S(n)$. However, if $v_f \in [(1-\delta)n, n] \times [-(1-5\delta)n, (1-5\delta)n]$, then γ first reaches the line $\{x = (1-\delta)n\}$ (at or before v_f) and then returns to the line $\{x = (1-3\delta)n\}$ (because v_ℓ lies on or to the left of this line). This cannot occur on (3.18). Therefore

$$
\begin{aligned}
&P\{\gamma \text{ and } v_f \text{ exist and } v_f \in [-(1-\delta)n, (1-\delta)n] \times [-(1-5\delta)n, (1-5\delta)n]\} \\
&\qquad \ge P\{(3.18) \text{ occurs}\} c_{25}.
\end{aligned}
$$

This finally proves that (3.17) is bounded below by some c_{19} as desired.

Now that (3.7) has been established, (3.8) can be deduced by the method of Lemma 6 in Kesten (1987). This method allows one to show that the probability appearing in (3.7) equals at least $c_{26} P\{G(v, k/2)\} \times$ (the probability in (3.8)). Thus if we combine the upper bound of (3.7) with the lower bound

$$
P\{G(v, k/2)\} \ge \frac{c_{27}}{k^2}
$$

(use (3.16) and the fact that (3.17) is bounded below), we obtain (3.8).

Finally, (3.9) follows immediately from (3.8) and Reimer's inequality (Reimer (1996)), or even by Theorem 4.2 of van den Berg and Fiebig (1987). Indeed, Reimer's inequality shows that the left hand side of (3.9) is at most equal to the left hand side of (3.8) times

$$
P\{\text{there exists an occupied or vacant path from } \Delta(v + S(k)) \text{ to } \Delta(v + S(n))\}. \tag{3.24}
$$

But if $k \le 2^p < 2k, n/2 < 2^q \le n$, then

$$
\begin{aligned}
&P\{\text{there exists a vacant path from } \Delta(v + S(k)) \text{ to } \Delta(v + S(n))\} \\
&\le P\{\text{there is no occupied circuit in } A_m \text{ for any } A_m \subset S(n) \setminus S(k)\} \\
&\le P\{\sigma(p, q) = 0\} \le 2e^{-c_2(q-p)} \text{ (see Lemma 3) } \le 2\left(\frac{4k}{n}\right)^{c_2/\log 2}.
\end{aligned}
$$

Adding to this the same estimate for occupied paths we obtain that (3.24) is bounded by $4(4k/n)^{c_2/\log 2}$, so that (3.9) indeed follows from (3.8). ■

Lemma 6. *For $0 < c < 1$ and $j \geq 1$ define*

$$T(r,s) = T(r,s;c,j) = [r2^{c(j+1)}, (r+3)2^{c(j+1)}] \times [s2^{c(j+1)}, (s+3)2^{c(j+1)}] \quad (3.25)$$

and

$N(j,c) =$ *number of squares $T(r,s)$ which intersect $S(2^{j+1})$ and which, for some k, intersect two successive circuits $\mathcal{C}_k$ and $\mathcal{C}_{k+1}$ with diam $(\mathcal{C}_k) \geq 2^j$.*

Then, for fixed $c \in (0,1)$, with probability 1,

$$N(j,c) \leq j^2 \text{ for all large } j. \quad (3.26)$$

Furthermore, if

$N^{(3)}(j,c) =$ *number of squares $T(r,s)$ which intersect $S(2^{j+1})$ and which, for some k, intersect three successive circuits $\mathcal{C}_k - \mathcal{C}_{k+2}$ with diam $(\mathcal{C}_k) \geq 2^{j-(\log j)^2}$,*

then, with probability 1,

$$N^{(3)}(j,c) = 0 \text{ for all large } j. \quad (3.27)$$

Finally, with probability 1,

$$(\text{number of vertices of } \mathcal{C}_k \text{ which are adjacent to } \mathcal{C}_{k+1}) \leq k^2 \text{ eventually.} \quad (3.28)$$

Proof. A trivial extension of the argument used for (2.8) shows that,

$$\begin{aligned} &P\{\text{for some } k,\ T(r,s) \text{ intersects } \mathcal{C}_k \text{ and } \mathcal{C}_{k+1} \text{ with diam } (\mathcal{C}_k) \geq 2^j\} \\ &= 8P\{\text{for some } k,\ T(r,s) \text{ intersects } \mathcal{C}_k \text{ and } \mathcal{C}_{k+1} \text{ and both these circuits} \\ &\text{are occupied and } \mathcal{C}_{k+2} \text{ is vacant and diam } (\mathcal{C}_k) \geq 2^j\}. \end{aligned} \quad (3.29)$$

Now assume that $T(r,s)$ intersects $S(2^{j+1})$. If $T(r,s)$ intersects the two occupied circuits $\mathcal{C}_k, \mathcal{C}_{k+1}$, and $\mathcal{C}_{k+2}$ is vacant, and diam $(\mathcal{C}_k) \geq 2^j$, then there are 5 disjoint paths starting on or adjacent to $\Delta T(r,s)$ and reaching points at distance $2^{j-1} - 6 \cdot 2^{c(j+1)} \geq 2^{j-2}$ from $T(r,s)$. Four of these are occupied and consist of disjoint arcs of $\mathcal{C}_k$ and $\mathcal{C}_{k+1}$. In addition,by definition, $\mathcal{C}_{k+1}$ is the largest circuit in $\overset{\circ}{\mathcal{C}}_{k+2}$. By Remark 1 and the fact that $\mathcal{C}_{k+1}$ is assumed occupied and $\mathcal{C}_{k+2}$ is assumed vacant, there is for any vertex $w \in T(r,s) \cap \mathcal{C}_{k+1}$ a vacant connection from a neighbor u of w to $\mathcal{C}_{k+2}$. This vacant connection together with $\mathcal{C}_{k+2}$ contains a vacant path of diameter $\geq \frac{1}{2}$ diam $(\mathcal{C}_{k+2}) \geq 2^{j-1}$. This is the fifth path starting from a vertex on or adjacent to $\Delta T(r,s)$; it is disjoint from the other four because it is vacant.

It follows from the preceding argument and (3.8) that for fixed r, s and large j, the probability of the event in the left hand side of (3.29) is at most $c_{28} 2^{-2(1-c)(j+1)}$. The possible number of choices for r, s is at most $10 \cdot 2^{2(1-c)(j+1)}$, because we only consider $T(r,s)$ which intersect $S(2^{j+1})$. Therefore the expected number of pairs r, s for which the event in the left hand side of (3.29) occurs is at most c_{29}. By Markov's inequality the probability that there are more than j^2 such pairs r, s is then at most c_{29}/j^2. The Borel-Cantelli lemma now proves (3.26).

The proof of (3.27) is similar. If $\mathcal{C}_k, \mathcal{C}_{k+1}$ and $\mathcal{C}_{k+2}$ are occupied and intersect $T(r,s)$ and $\mathcal{C}_{k+3}$ is vacant, then there are seven disjoint paths starting on or adjacent to $\Delta T(r,s)$ and reaching points at distance 2^{j-2} from $\Delta T(r,s)$. Now use (3.9) instead of (3.8).

We shall not prove (3.28). Its proof is essentially the same as that of (3.26), but simpler. One now uses (3.7) directly, instead of (3.8). ■

The next lemma is a simple deterministic result. For two sets of vertices π', π'' we define

$$d(\pi', \pi'') = \inf\{\|x-y\| : x \in \pi', y \in \pi''\}.$$

Lemma 7. *Let $\mathcal{C}_{k+1}$ and $\mathcal{C}_{k+2}$ be two successive circuits. Assume*

$$2^j \leq \text{ diam } (\mathcal{C}_{k+1}) < 2^{j+1}.$$

Let $V_1^{(k+1)}, V_2^{(k+1)}, \ldots, V_M^{(k+1)}$ be M arbitrary subsets of $\mathcal{C}_{k+1}$ for which

$$d(V_p^{(k+1)}, V_q^{(k+1)}) := \min\{\|x-y\| : x \in V_p^{(k+1)}, y \in V_q^{(k+1)}\} \geq 8 \cdot 2^{cj}, p \neq q. \quad (3.30)$$

If $N(j,c) \leq j^2$, then at least $M - j^2$ of the sets $V_m^{(k+1)}$ satisfy

$$d(V_m^{(k+1)}, \mathcal{C}_{k+2}) > 2^{cj}. \quad (3.31)$$

In particular, if $M > j^2$, and $N(j,c) \leq j^2$, then there is at least one set $V_m^{(k+1)}$ which satisfies (3.31)

Proof. Assume that for some m_0 the set $V_{m_0}^{(k+1)}$ contains a vertex $x \in \mathcal{C}_{k+1}$ for which there exist a $y \in \mathcal{C}_{k+2}$ with $\|x-y\| \leq 2^{cj}$. Since diam $(\mathcal{C}_{k+1}) < 2^{j+1}$, we have $\mathcal{C}_{k+1} \subset S(2^{j+1})$. Thus x belongs to some square $[(r+1)2^{cj}, (r+2)2^{cj}] \times [(s+1)2^{cj}, (s+2)2^{cj}]$ and this square must intersect $S(2^{j+1})$. It then follows from $\|x-y\| \leq 2^{cj}$ that $y \in T(r,s;c,j))$ (see (3.25) for $T(r,s)$). Thus for each $V_{m_0}^{(k+1)}$

which contains an x as just mentioned, there is a $T(r,s)$ which intersects $S(2^{j+1})$ as well as $\mathcal{C}_{k+1}, \mathcal{C}_{k+2}$, and which contains a point of $V_{m_0}^{(k+1)}$. Different sets $V_m^{(k+1)}$ lead to different $T(r,s)$, because two different $V_m^{(k+1)}$ cannot intersect the same $T(r,s)$ by (3.30).

By definition, $N(j,c) \le j^2$ means that there are at most j^2 squares $T(r,s)$ of the form (3.25) which intersect $S(2^{j+1}), \mathcal{C}_{k+1}$ and $\mathcal{C}_{k+2}$. By the preceding paragraph this implies that there are at most $N(j,c) \le j^2$ sets $V_m^{(k+1)}$ with $d(V_m^{(k+1)}, \mathcal{C}_{k+2}) \le 2^{cj}$. The remaining $M - N \ge M - j^2$ sets $V_m^{(k+1)}$ satisfy $d(V_m^{(k+1)}, \mathcal{C}_{k+2}) > 2^{cj}$. ■

4. Almost all circuits are 'good'.

Let C be a circuit surrounding $\mathbf{0}$. For certain constants $0 < c_{30}, c_{31}, c_{33} < 1$, $c_{32} > 0$, still to be determined, we consider vacant connected (on $\mathcal{T}$) sets $\mathcal{D}$ with some or all of the following properties:

$$\mathcal{D} \subset \overset{\circ}{C} \text{ and } \mathcal{D} \text{ contains exactly one vertex adjacent to } C; \tag{4.1}$$

$$\operatorname{diam}(\mathcal{D}) \ge (\operatorname{diam}(C))^{c_{30}}; \tag{4.2}$$

the vacant cluster of $\mathcal{D}$ in $\overset{\circ}{C}$ contains at least $[\text{diam }(C)]^{c_{31}}$
selfavoiding paths θ_m which are adjacent to C, have
length $(\theta_m) \ge c_{32} \log\log(\operatorname{diam}(C))$ and satisfy
$d(\theta_p, \theta_q) \ge [\text{diam }(C)]^{c_{33}c_{30}}$ for $p \ne q$;
moreover, there exists a vertex $z \in \mathcal{D}$ and for each of the θ_m
a vacant path from z to θ_m and such that only its
endpoint on θ_m is adjacent to C. (4.3)

We call an occupied circuit C *good* (or more explicitly $(c_{30} - c_{33})$-good) if it has the following property:

Any vacant connected (on $\mathcal{T}$) set $\mathcal{D}$ with the
properties (4.1) and (4.2) also satisfies (4.3). (4.4)

We note that 'θ adjacent to C' means that for each vertex u of θ, there exists a vertex v in C which is adjacent to u. This is *not a symmetric relation*; θ adjacent to C does not imply that C is adjacent to θ.

A *good vacant* circuit C is defined by interchanging 'occupied' and 'vacant' in this definition.

Here is our principal estimate.

Proposition 1. *For any $0 < c_{33} < 1$, and $0 < c_{30} < 1$, but c_{30} sufficiently close to 1, we have*

$$P\{\textit{there exists an occupied circuit } \mathcal{C}_k \textit{ with } 2^j \le \textit{diam } (\mathcal{C}_k) < 2^{j+1} \\ \textit{and which is not good}\} \\ \le c_{34} \exp(-c_{35} j). \tag{4.5}$$

Here c_{34}, c_{35} depend on $c_{30} - c_{33}$ only.

Proof. We shall break down the proof into five steps. c_i will denote a strictly positive finite constant throughout this proof. All these constants with $i \ge 37$ will be independent of c_{30}, so that c_{30} still can be adjusted at the end of the proof. The proof works for any choice of $c_{32} > 0$. Throughout this proof we tacitly assume that j is large; our estimates may fail for a finite number of j's.

In the first step we show how to replace the random $\mathcal{C}_k$ by a fixed circuit C from a specified class. The second step is largely topological and serves to find a certain circuit $J \subset \overset{\circ}{C}$ adjacent to C. Step (iii) in conjunction with Step (ii) bounds the number of circuits J which have to be considered. Steps (iv) and (v) together estimate the probability that (4.3) fails for $\mathcal{D}$ equal to some vacant path with one endpoint on a specific circuit J as constructed in Step (ii).

Step (i) Any circuit C which surrounds $\mathbf{0}$ and has $2^j \le$ diam $(C) < 2^{j+1}$ must be contained in $S(2^{j+1})$, but must also contain points outside $S(2^{j-2})$. Now let $\mathcal{C}_\tau$ be the last circuit in our sequence $\mathcal{C}_1, \mathcal{C}_2, \dots,$ which is contained in $S(2^{j+1})$. Then the probability in (4.5) is (for any choice of $u \ge 0$) at most

$$\sum_{0 \le p < u} P\{p < \tau, \, \mathcal{C}_{\tau-p} \text{ is occupied, has diameter } \ge 2^j, \text{ but is not good}\} \\ + P\{u < \tau, \, \mathcal{C}_{\tau-u} \text{ is not contained in } S(2^{j-2})\}. \tag{4.6}$$

The second term in (4.6) can be estimated by the argument of Lemma 3. If $\mathcal{C}_{\tau-u}$ is not contained in $S(2^{j-2})$, then at least one of the following three events must occur (for any choice of $1 \le q \le j-2$):

$$\text{there exists no vacant circuit in } S(2^{j-2}) \setminus S(2^{j-2-q}),$$
$$\text{there exists no occupied circuit in } S(2^{j-2}) \setminus S(2^{j-2-q}),$$

or

$$\mathcal{C}_{\tau-u} \text{ lies outside } S(2^{j-2-q}), \text{ so that } \mathcal{C}_\tau, \mathcal{C}_{\tau-1}, \dots, \mathcal{C}_{\tau-u} \subset S(2^{j+1}) \setminus S(2^{j-2-q}). \tag{4.7}$$

If (4.7) holds, then $\sigma(j-2-q, j+1) \geq (u+1)/2$ or there are at least $(u+1)/2$ disjoint vacant circuits in $S(2^{j+1}) \setminus S(2^{j-2-q})$. Therefore, with c_5 as in the proof of Lemma 3,

$$\begin{aligned} P\{\mathcal{C}_{\tau-u} \not\subset S(2^{j-2})\} &\leq 2(1-c_5)^q + 2P\{\sigma(j-2-q, j+1) \geq (u+1)/2\} \\ &\leq 2(1-c_5)^q + 2c_3 e^{-c_4(u+1)/(2q+6)} \text{ (by (3.2))}. \end{aligned}$$

By choosing $q = j-2$ and $u = j^2$ we find that the second term in (4.6) is at most

$$(2c_3+1)e^{-c_{36}j}. \tag{4.8}$$

Each term in the sum in (4.6) will be decomposed into two pieces. We fix

$$\ell = \lfloor c_{30}j - 2\log j/\log 2 \rfloor - 4, \tag{4.9}$$

and divide $S(2^{j+1})$ into the at most

$$2^{2(j+2-\ell)} \leq c_{37} j^4 2^{2(1-c_{30})j} \tag{4.10}$$

squares

$$S(r,s) := [r2^\ell, (r+1)2^\ell] \times [s2^\ell, (s+1)2^\ell], \quad -2^{j+1-\ell} \leq r, s < 2^{j+1-\ell}. \tag{4.11}$$

For each of these squares we take

$$\widetilde{S}(r,s) = [(r-1)2^\ell, (r+2)2^\ell] \times [(s-1)2^\ell, (s+2)2^\ell]; \tag{4.12}$$

its topological boundary is denoted by $\Delta\widetilde{S}(r,s)$. Note that by our imbedding of $\mathcal{T}$, this boundary is a circuit on the triangular lattice. Now if $\mathcal{C}_{\tau-r} = C$ for some fixed circuit C surrounding the origin and of diameter $\geq 2^j$, and if C intersects a given $S(r,s)$, then C must also contain points outside $\widetilde{S}(r,s)$ (because its diameter exceeds that of $\widetilde{S}(r,s)$). Thus in this case, C must contain a crossing of the annulus $\widetilde{S}(r,s) \setminus S(r,s)$. Here, and in the sequel we define for $m' < m$, a *crossing of the annulus* $S(m) \setminus S(m')$ to be a path in this annulus from its outer boundary $\Delta S(m)$ to a point adjacent to its inner boundary $\Delta S(m')$.

We shall first split off the probability of the event that there are 'too many' such crossings. Specifically, we shall prove that for some c_i

$$\begin{aligned} &P\{\text{there exist more than } c_{38} j^4 2^{2(1-c_{30})j} \text{ disjoint occupied crossings} \\ &\qquad \text{of some } \widetilde{S}(r,s) \setminus S(r,s),\ -2^{j+1-\ell} \leq r, s < 2^{j+1-\ell}\} \\ &\leq c_{39} \exp[-c_{40} j^4 2^{2(1-c_{30})j}]. \end{aligned} \tag{4.13}$$

To see (4.13), note that again by the Russo-Seymour-Welsh lemma, for any fixed r, s,

$$P\{\text{there exists an occupied crossing of } \widetilde{S}(r,s) \setminus S(r,s)\} \le (1 - c_{41}).$$

By the BK-inequality we therefore also have, still for fixed r, s,

$$P\{\text{there exist } q \text{ disjoint occupied crossings of } \widetilde{S}(r,s) \setminus S(r,s)\} \le (1 - c_{41})^q,\ q \ge 0. \tag{4.14}$$

Thus the number of crossings of each of our annuli is bounded by a geometric variable. Moreover, the number of crossings of two annuli $\widetilde{S}(r', s') \setminus S(r', s')$ and $\widetilde{S}(r'', s'') \setminus S(r'', s'')$ are independent as soon as $|r'' - r'| \vee |s'' - s'| \ge 3$. The estimate (4.13) now follows by standard methods for exponential bounds. Combining (4.6) with the estimates (4.8) and (4.13) we find that the probability in (4.5) is at most

$$\begin{aligned}
&(2c_3 + 1)e^{-c_{36}j} + j^2 c_{39} \exp[-c_{40} j^4 2^{2(1-c_{30})j}] \\
&\quad + \sum_{0 \le p < j^2} \sum_C P\{p < \tau,\ \mathcal{C}_{\tau-p} \text{ is occupied and equals } C \text{ and } \mathcal{C}_{\tau-p} \text{ is not good}\} \\
&= (2c_3 + 1)e^{-c_{36}j} + j^2 c_{39} \exp[-c_{40} j^4 2^{2(1-c_{30})j}] \\
&\quad + \sum_{0 \le p < j^2} \sum_C P\{C \text{ is not good} | p < \tau,\ \mathcal{C}_{\tau-p} \text{ is occupied and equals } C\} \\
&\qquad\qquad \times P\{p < \tau, \mathcal{C}_{\tau-p} \text{ is occupied and equals } C\},
\end{aligned} \tag{4.15}$$

where the sum over C runs over the circuits contained in $S(2^{j+1})$, with diameter $\ge 2^j$ and with at most $c_{38} j^4 2^{2(1-c_{30})j}$ disjoint crossings of all $\widetilde{S}(r,s) \setminus S(r,s)$ together.

Step (ii) In this step we make a number of topological preparations. Basically we need a circuit J on $\mathcal{T}$ which is 'just inside' C and we need some relation between crossings of squares by C and J. The statements here are fairly intuitive but the arguments are finicky; one has to make sure that they don't rely on pictures.

We need some definitions. These follow Kesten (1982). If W is a set of vertices of $\mathcal{T}$, then its *boundary* is

$$\partial W := \{v : v \notin W \text{ but } v \text{ is adjacent to } W\};$$

the *exterior boundary* is denoted by ∂_{ext} and defined as

$$\begin{aligned}
\partial_{\text{ext}} W = &\{v \in \partial W : \exists \text{ a path } \pi \text{ on } \mathcal{T} \text{ from } v \text{ to } \infty \\
&\quad \text{such that the only point of } \pi \text{ in } W \cup \partial W \text{ is } v\}.
\end{aligned}$$

The following separation property is basic for us:

$$\begin{aligned}&\text{If } W \text{ is a nonempty, finite connected set of vertices of } \mathcal{T},\\ &\text{then there exists a circuit } J \text{ on } \mathcal{T} \text{ which passes}\\ &\text{exactly through all vertices of } \partial_{\text{ext}} W \text{ and surrounds } W.\end{aligned} \tag{4.16}$$

Basically this is what the proof of Proposition 2.1 in the Appendix of Kesten (1982) shows. There are two differences. Firstly, Kesten (1982) works on a graph $\mathcal{M}_{\text{pl}}$ and secondly, Kesten (1982) only proves that $J \subset \partial_{\text{ext}} W$, but not the converse. However, the only property of $\mathcal{M}_{\text{pl}}$ which is used in Kesten (1982) is that it is planar and has triangular faces. Thus the proof there applies to $\mathcal{T}$ without changes and we only need to show that

$$\partial_{\text{ext}} W \subset J. \tag{4.17}$$

Assume that this fails and that there exists a vertex $u \in \partial_{\text{ext}} W \setminus J$. Let u be adjacent to $w \in W$ and let π be a path on $\mathcal{T}$ from u to ∞ such that $\pi \cap (W \cup \partial W) = u$. Let $\widetilde{\pi}$ be the path consisting of the edge from w to u followed by π. Then $\widetilde{\pi}$ is a path from W to ∞ which does not intersect J (because $u \notin J$ and $J \subset \partial_{\text{ext}} W \subset \partial W$ and hence $(\pi \setminus \{u\}) \cap J = \emptyset$). But then J does not surround W, contrary to what we know about J already. Hence there is no $u \in \partial_{\text{ext}} W \setminus J$ and (4.17) holds. This also proves (4.16).

It is standard in topological considerations of this type to interchange exterior and interior by letting a finite point play the role of ∞. We shall do this here for the situation where $W = C$, a fixed circuit on $\mathcal{T}$. We let a point $z \in \overset{\circ}{C}$ which is not adjacent to C play the role of ∞. We define

$$\begin{aligned}\partial^z_{\text{int}} C =&\{v \in \partial C : \exists \text{ a path } \pi \text{ on } \mathcal{T} \text{ from } v \text{ to } z\\ &\qquad \text{such that the only point of } \pi \text{ in } C \cup \partial C \text{ is } v\}.\end{aligned} \tag{4.18}$$

It is easy to see that if $z \in \overset{\circ}{C}$, then $\partial^z_{\text{int}} C \subset \overset{\circ}{C}$. The proof in Kesten (1982) of (4.16) above now yields

$$\begin{aligned}&\text{there exists a circuit } J^z \text{ on } \mathcal{T} \text{ such that } J^z \subset \overset{\circ}{C} \cap \partial C\\ &\text{and such that the vertices on } J^z \text{ are exactly the vertices of } \partial^z_{\text{int}} C;\\ &\text{moreover any continuous path on } \mathbb{R}^2 \text{ from } C \text{ to } z \text{ must intersect } J^z.\end{aligned} \tag{4.19}$$

We now want to show that if C has at most $c_{38} j^4 2^{2(1-c_{30})j}$ crossings of all $\widetilde{S}(r,s) \setminus S(r,s)$ together, then we only need to consider (in a sense to be made precise) $c_{38}j^4 2^{2(1-c_{30})j}$ crossings by the circuit J^z of all $\widetilde{S}(r,s) \setminus S(r,s)$. For the time being fix (r,s) and abbreviate $S(r,s)$ and $\widetilde{S}(r,s)$ to S and $\widetilde{S}$, respectively. We assume that C has the properties listed at the end of Step (i) and that

$$z \in \overset{\circ}{C} \setminus \widetilde{S}, \text{ and } z \text{ is not adjacent to } C. \tag{4.20}$$

For the time being we further suppress the superscript z in the notation and write $\partial_{\text{int}}C$ for $\partial^z_{\text{int}}C$ and J for J^z.

Let w be a vertex of C which lies in S or is adjacent to S. Since diam $(C) > 2\cdot$diam $(\widetilde{S})$, C cannot be contained in $\widetilde{S}$. We can then move counterclockwise and clockwise from w along C until we reach $\Delta\widetilde{S}$ for the first time in u' and u'' say. Write $E = E(w)$ for the piece of C from u' to u'' through w. Then the first piece of E from u' to the first vertex of E adjacent to ΔS is a crossing of $\widetilde{S} \setminus S$. Conversely, for any piece of C which forms a crossing from a point u' on $\Delta\widetilde{S}$ to a point u''' adjacent to ΔS, there is only one piece E of C, as above, which contains the crossing from u' to u''', to wit $E(u''')$ (this is found by simply continuing from u''' along C until we reach $\Delta\widetilde{S}$ again at some point u''). Thus the number of different pieces $E(w)$ which can be found is at most equal to the number of crossings by C of $\widetilde{S} \setminus S$ (even though there may be many more points w of C in or adjacent to S).

We want to establish a similar statement for J. We will be considering vertices v of J in S. Let $v \in J$. We follow J counterclockwise and clockwise from v until we first reach vertices in the exterior of $\widetilde{S}$, say the vertices x' and x'', respectively. Such points must exist, because we cannot have $J \subset \widetilde{S}$ if J separates z from C – and hence from ∞ – and $z \in \overset{\circ}{C} \setminus \widetilde{S}$ (see (4.20)). Denote the piece of J from x' to x'' through v by $F = F(v)$. Note that v is adjacent to some vertex w of C, because $J \subset \partial C$. $F(v)$ is an obvious analogue of $E(w)$. We are going to show that

$$\begin{aligned}&\text{the number of different } F(v) \text{ with } v \in J \cap S \text{ which}\\ &\text{can arise is at most equal to the number of different } E(w).\end{aligned} \tag{4.21}$$

It suffices for this to show that if v' and v'' are two distinct vertices of J in S which are adjacent to w' and w'' and if $E(w') = E(w'')$, then also $F(v') = F(v'')$. To prove this, fix $v', v'' \in J \cap S$ and let $w', w'' \in C$ be adjacent to v' and v'', respectively and assume $E(w') = E(w'')$. Then, by our construction of the E's, there is a piece of C from w' to w'' which is entirely contained in the interior of $\widetilde{S}$. Denote this piece of C by B. Further let π' be a path on $\mathcal{T}$ from $v' \in J \subset \partial_{\text{int}}C$ to z such that the only point of π' in $C \cup \partial C$ is v'. Write $\widetilde{\pi}'$ for the path consisting of the edge $\{w', v'\}$ followed by π'. Choose π'' and $\widetilde{\pi}''$ in a similar way for v'' (see Figure 4). Let y be the first point of π'' which is also on π'. We can then replace the part of π'' from y to z by the piece of π' from y to z, so that we may assume without loss of generality that π' and π'' coincide from y to z (of course $y = z$ is possible). Finally, let $J_B \subset \partial_{\text{ext}} B$ be the circuit on $\mathcal{T}$ which surrounds B as in (4.16) with W replaced by B. Now $v' \in \partial_{\text{int}} C$ also belongs to $\partial_{\text{ext}} B$ because v' is adjacent to B and $B \subset C$ (hence $B \cup \partial B \subset C \cup \partial C$); moreover, a path on $\mathcal{T}$ from v' to z can be extended to a path on $\mathcal{T}$ to ∞ without hitting $B \cup \partial B$, because $B \subset$ interior of $\widetilde{S}$, hence $B \cup \partial B \subset \widetilde{S}$, and $z \notin \widetilde{S}$. Thus $v' \in J_B$ and the same holds for v''. Hence there are two arcs of J_B from v' to v''. We next consider the path σ on $\mathcal{T}$ from w' to w'' which consists of the piece of $\widetilde{\pi}'$ from w' to y followed by the reverse of the part of $\widetilde{\pi}''$ from y to w''. It is immediate that this path intersects C only at its endpoints. This path has no double points, unless $w' = w''$, in which case w' is the unique double point on this path. Moreover, since $z \in \overset{\circ}{C}$, also $y \in \overset{\circ}{C}$ and $\sigma \setminus \{w', w''\} \subset \overset{\circ}{C}$. Thus σ is a crosscut of $\overset{\circ}{C}$ and divides it into two components (see Figure 5). Let G be the component of $\overset{\circ}{C} \setminus \sigma$ which has boundary consisting of σ and B. The other component has boundary consisting of σ and $C \setminus B$ (see Newman (1951), Theorem V.11.8). The part $C \setminus B$ of this boundary lies in the exterior of $\Delta G =$ (σ concatenated with B), since we can connect a point of $C \setminus B$ to ∞ without hitting $C \cup \overset{\circ}{C} \supset G \cup \Delta G$.

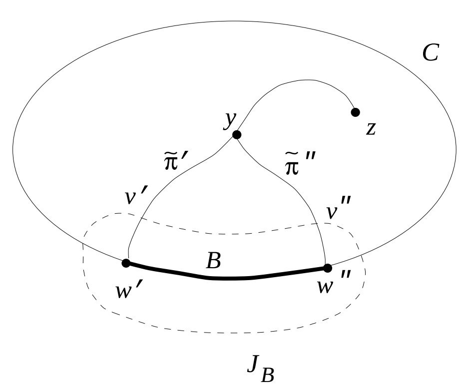

FIGURE 4. The circuits C (solidly drawn) and J_B (dashed). Also shown are the arc B of C (boldly drawn) and the paths $\widetilde{\pi}', \widetilde{\pi}''$.

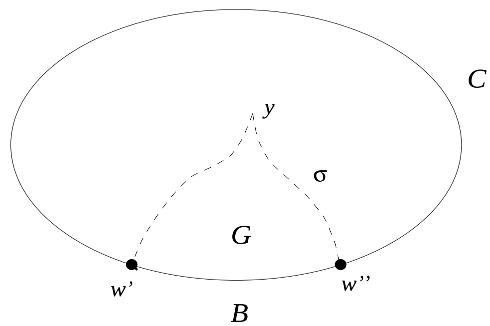

FIGURE 5. σ (the dashed path) is a crosscut of $\overset{\circ}{C}$. C is the solidly drawn circuit.

We further claim that the interior of one of the arcs of J_B between v' and v'' is contained in G. Indeed, the only points of σ in ∂C are the two points v' and v''. Thus also $\sigma \cap J_B = \{v', v''\}$ (recall $J_B \subset \partial B \subset C \cup \partial C$). Moreover, B is disjoint from J_B. Thus $J_B \cap (\sigma \cup B) = \{v', v''\}$. It follows that each of the two arcs of J_B from v' to v'' (minus their endpoints) lie in the interior or exterior of the Jordan curve made up from B and σ. If both these arcs would lie in the exterior, then J_B could not separate B from ∞, because there would be a path φ from any point of B to y inside G (except for the endpoints of φ). φ would be disjoint from J_B. Moreover, φ can be continued by the common piece of π' and π'' from y to z; this is still disjoint from J_B, because it is disjoint from $(C \cup \partial C) \supset \partial_{\text{ext}} B$. Therefore, J_B would not separate B from z. But $B \subset$ interior of $\widetilde{S}$ and hence

$$J_B \subset \partial B \subset \widetilde{S} \tag{4.22}$$

and $z \notin \widetilde{S}$ (see (4.20)). Therefore z also lies in the exterior of J_B. But then J_B would also not separate B from ∞, contrary to the choice of J_B. This proves that at least one of the arcs of J_B from v' to v'' lies in G. We shall denote such an arc by τ. (In fact, τ is unique, because the other arc of J_B from v' to v'' has to lie outside G, but we shall not need this.)

We next show that τ is actually also one of the arcs of J from v' to v''. To begin with let u be a vertex on $\tau \setminus \{v', v''\}$ and let ψ be a path on $\mathcal{T}$ from u to z which intersects $B \cup \partial B$ only in u; such a path exists, because $u \in J_B = \partial_{\text{ext}} B$. We have to consider two cases. First consider the case when $z \in G$. In this case we may assume that

$$\psi \subset G \cup \pi' \cup \pi''. \tag{4.23}$$

Indeed, ψ starts at $u \in G$. If it hits ∂G this must be in a point of $\pi' \cup \pi'' \setminus \{v', v''\}$, since ψ was chosen disjoint from B, and because it is a path on $\mathcal{T}$, it does not intersect the edges $\{w', v'\}$ and $\{w'', v''\}$ (whose endpoints lie in $B \cup \partial B$ and not on ψ). But then we can replace the piece of ψ between its first intersection with

$\pi' \cup \pi''$ and z by a piece of $\pi' \cup \pi''$, to get (4.23). Note that the change in ψ will preserve the property

$$\psi \cap (B \cup \partial B) = \{u\}, \tag{4.24}$$

because $(\pi' \cup \pi'') \cap (B \cup \partial B) \subset (\pi' \cup \pi'') \cap (C \cup \partial C) = \{v', v''\}$, by choice of π', π''. The second case is when $z \notin G$. Then ψ begins in G, but ends at $z \notin G$ and must therefore hit the (topological) boundary of G somewhere. As in the first case this intersection point must lie on $\pi' \cup \pi'' \setminus \{v', v''\}$. We may replace the piece of ψ after the intersection with $\pi' \cup \pi''$ by a piece of π' or of π''. Again this will preserve property (4.24). Thus, (4.23) holds in both cases. We now show that this implies

$$\psi \cap (C \cup \partial C) = \{u\}. \tag{4.25}$$

Indeed, we proved before that $C \setminus B$ is in the exterior of $\sigma \cup B$, so that no vertex on $C \setminus B$ can be equal to or adjacent to a vertex in $G =$ the interior of $\sigma \cup B$. In particular, (4.23) then tells us that ψ contains no vertices on or adjacent to $C \setminus B$. Together with (4.24) this gives (4.25).

The result (4.25) together with the fact that $u \in \tau \subset J_B$ is adjacent to $B \subset C$ implies that

$$u \in \partial_{\text{int}} C. \tag{4.26}$$

Thus all vertices on τ are vertices of J. This almost shows that τ is an arc of J. What we still have to rule out is that vertices occur in different order on τ than on J. To show that this is not the case we prove that if $J = (x_0, x_1, \dots, x_{\nu-1}, x_\nu = x_0)$, then the only neighbors of x_i on J are x_{i-1} and x_{i+1} (with $x_{-1} = x_{\nu-1}, x_{\nu+1} = x_1$). If this were not the case, there would exist an edge e between some x_i and x_j for which there exists vertices y' and y'' on each of the two arcs of J between x_i and x_j. e would be a crosscut of $\overset{\circ}{J}$ or of J^{ext}. Both of these possibilities lead to a contradiction. Assume first that e is a crosscut of $\overset{\circ}{J}$. Then $\overset{\circ}{J} \setminus e$ consists of two components, one of which contains z (because of the fact that J separates z from C and $J \subset \overset{\circ}{C}$ implies $z \in \overset{\circ}{J}$; moreover, e contains only the vertices x_i and x_j, so that $z \notin e$). Let z lie in the component bounded by e and the arc of J from x_i to x_j through y'. Then any path on $\mathcal{T}$ from y'' to z would have to intersect e or $J \subset \partial C$ in a point other than y''. A path on $\mathcal{T}$ can intersect e only in x_i or x_j. Thus there would not exist a path on $\mathcal{T}$ from y'' to z which only intersects $C \cup \partial C$ in y''. This contradicts the fact that $y'' \in \partial^z_{\text{int}} C$. If, on the other hand, e is a crosscut of J^{ext}, then it divides J^{ext} into two components, one of which contains all of C. Let this again be the component bounded by e and the arc of J from x_i to x_j through y'. Then any path on $\mathcal{T}$ from y'' to C must intersect e or J in a point different from y'', contrary to the fact that y'' is adjacent to C. Thus no neighbors x_i, x_j as above can exist and the only neighbors of x_i are x_{i-1}, x_{i+1}, as claimed. But then, if τ runs successively through the vertices

$$x_{i_0} = v', x_{i_1}, \dots, x_{i_\rho} = v''$$

of J, we must have $i_{\ell+1} = i_\ell + 1 \pmod{\nu}$ or $i_{\ell+1} = i_\ell - 1 \pmod{\nu}$ for $0 \le \ell \le \rho - 1$, so that τ is indeed an arc of J.

This finally proves (4.21). Indeed, since $\tau \subset J_B$ is adjacent to B, and $B \subset$ interior of $\widetilde{S}$, we have $\tau \subset \widetilde{S}$. Thus, there is an arc of J from v' to v'' which stays in $\widetilde{S}$. Hence, when forming $F(v')$ we must pass through v'' so that $F(v') = F(v'')$.

Step (iii) The result (4.21) has been proven for a single square $\widetilde{S}(r,s)$ and a single $z \notin \widetilde{S}(r,s)$, or rather a single J^z, only. We already know that there are at most $c_{37} j^4 2^{2(1-c_{30})j}$ different pairs (r,s) (see (4.10)). In this step we show how to control the number of different circuits J^z which can arise.

If $\mathcal{C}_{\tau-p} = C$ and $\mathcal{C}_{\tau-p}$ is occupied but not good, then there exists a vacant set $\mathcal{D}$ which satisfies (4.1) and (4.2), but not (4.3). It is easily seen that there then exists a selfavoiding vacant path, $\zeta = (v_1, \dots, v_q) \subset \mathcal{D}$, such that

$$v_1 \text{ is the unique vertex of } \zeta \text{ adjacent to } C \text{ and } \zeta \subset \overset{\circ}{C}; \tag{4.27}$$

$$\operatorname{diam}(\zeta) \ge \frac{1}{2}(\operatorname{diam}(C))^{c_{30}} \tag{4.28}$$

and such that (4.3) with $\mathcal{D}$ replaced by ζ fails. The vertex v_1 must lie in some $S(r,s)$. Since $\operatorname{diam}(\zeta) > \operatorname{diam}(\widetilde{S}(r,s))$, ζ must contain points outside $\widetilde{S}(r,s)$. We shall replace ζ by its initial piece from v_1 to the first exit from $\widetilde{S}$. Then (4.27) holds as well as

$$v_1 \in S(r,s), v_q \notin \widetilde{S}(r,s), \text{ but } v_i \in \widetilde{S}(r,s) \text{ for } 1 \le i \le q-1. \tag{4.29}$$

We further introduce the square

$$\widehat{S} = \widehat{S}(r,s) = [(r-1)2^\ell - 1, (r+2)2^\ell + 1] \times [(s-1)2^\ell - 1, (s+2)2^\ell + 1]$$

which just surrounds $\widetilde{S}$. Its topological boundary is denoted by $\Delta\widehat{S} = \Delta\widehat{S}(r,s)$. Since v_q is adjacent to $v_{q-1} \in \widetilde{S}(r,s) \subset$ interior of $\widehat{S}(r,s)$, we must have

$$v_q \in \Delta\widehat{S}. \tag{4.30}$$

Again we fix (r,s) so that this holds, but suppress (r,s) in the notation in this step.

We claim that (4.27) implies that

$$v_1 \in J^{v_q}. \tag{4.31}$$

To see this, note that v_1 is adjacent to some vertex $w \in C$ and (the reverse of) the path ζ followed by the edge $\{v_1, w\}$ is a path from v_q to C, which starts at

$v_q \in \overset{\circ}{C}$. It therefore must intersect $\partial_{\text{int}}^{v_q} C = J^{v_q}$ (see (4.19)). But the only possible intersection point is v_1, because that is the only point on $\zeta \cap \partial C$. Hence (4.31) holds. Thus if we want to bound the number of possible choices for J^z, then we only need to consider J^z with $z \in \partial \widehat{S}$ for some $\widehat{S}(r,s)$ and such that J^z contains a point of $S(r,s)$.

Now consider two vertices z' and z'' on $\partial \widehat{S}$ with corresponding paths ζ', ζ'' to vertices $v_1' \in J^{z'} \cap S, v_1'' \in J^{z''} \cap S$, respectively. We may assume that these paths satisfy (4.27) and (4.29)-(4.31) with the prime or double prime added at the appropriate places. In particular,

$$\zeta' \setminus \{v_1'\} \text{ and } \zeta'' \setminus \{v_1''\} \text{ are disjoint from } C \cup \partial C. \tag{4.32}$$

If there exists a path on $\mathcal{T}$ from ζ' to ζ'' which is disjoint from $C \cup \partial C$, then the definition (4.18) shows that $\partial_{\text{int}}^{z'} C = \partial_{\text{int}}^{z''} C$. In particular, this is the case if $C \cup \partial C$ does not intersect a piece of ∂U which connects ζ' to ζ'', where

$$U = U(r,s) := [r2^\ell - 2^{\ell-1}, (r+1)2^\ell + 2^{\ell-1}] \times [s2^\ell - 2^{\ell-1}, (s+1)2^\ell + 2^{\ell-1}].$$

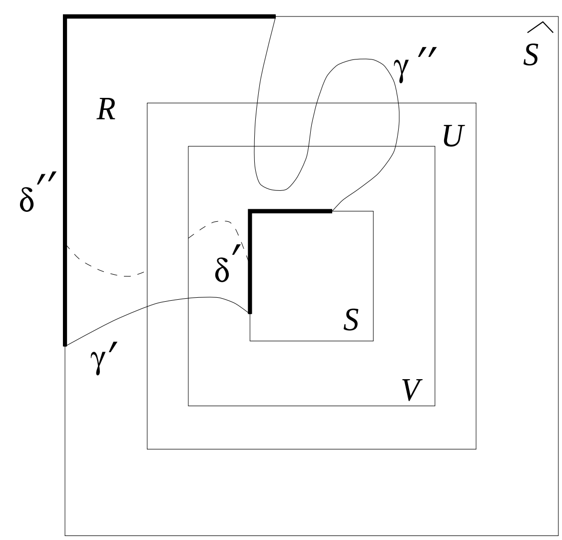

FIGURE 6. The successive squares from the inside out are $S, V, U, \widehat{S}$. δ'' is the boldly drawn arc on $\Delta \widehat{S}$ and δ' is just outside S. Also shown are γ' and γ''. R is the upper left hand corner of $\widehat{S} \setminus S$ between γ' and γ''. C must contain a crossing like one of the dashed ones to separate γ' from γ''.

Note that ΔU is the boundary of a square intermediate between S and $\widetilde{S}$, and ζ' and ζ'' both intersect ΔU. In fact there exists a piece, γ' say of ζ' of the form

$(v'_{t'}, \ldots, v'_{q'})$ which connects a vertex in ΔS to $\Delta \widehat{S}$ and which lies in $\widehat{S} \setminus S$. We will have

$$v'_{t'} \in \Delta[r2^\ell - 1, (r+1)2^\ell + 1] \times [s2^\ell - 1, (s+1)2^\ell + 1]. \tag{4.33}$$

Also ζ'' contains a similar piece γ''. If γ' and γ'' intersect, then ζ' and ζ'' are clearly connected outside $C \cup \partial C$ (see (4.32) and note that $v'_1 \neq v'_{t'}$ because $v'_1 \in S$ and $v'_{t'}$ lies outside S by (4.33); similarly $v''_1 \neq v''_{t''}$). If γ' and γ'' are disjoint, then γ', γ'' partition the interior of

$$\widehat{S} \setminus [r2^\ell - 1, (r+1)2^\ell + 1] \times [s2^\ell - 1, (s+1)2^\ell + 1]$$

into two components (see Newman (1951), exercise V.11.3). Let R be one of these components. Its boundary consists of γ', γ'' and an arc δ' of $\Delta[r2^\ell - 1, (r+1)2^\ell + 1] \times [s2^\ell - 1, (s+1)2^\ell + 1]$ between $v'_{t'}$ and $v''_{t''}$ and an arc δ'' of $\Delta\widehat{S}$ between $v'_{q'}$ and $v''_{q''}$ (see Figure 6). As observed above, if $C \cup \partial C$ does not intersect $(R \cup \Delta R) \cap \Delta U$, then $J^{z'} = J^{z''}$. However, the only way $C \cup \partial C$ can intersect $(R \cup \Delta R) \cap \Delta U$ is when C contains a piece in R which connects $\delta' \subset \Delta[r2^\ell - 1, (r+1)2^\ell + 1] \times [s2^\ell - 1, (s+1)2^\ell + 1]$ with a point adjacent to ΔU, or C contains a piece in R which connects $\delta'' \subset \Delta\widehat{S}$ with a point adjacent to ΔU (recall that C contains points outside $\widehat{S}$, because $\text{diam}(C) > 2{\cdot}\text{diam}(\widehat{S})$ and $C \cup \partial C$ cannot intersect the pieces γ', γ'' of ΔR by (4.27)). Thus for any $J^{z'}, J^{z''}$ which are different, C has to have a crossing of $V \setminus S$ or a crossing of $\widehat{S} \setminus U$ in the corresponding region R, where

$$V = V(r,s) := [r2^\ell - 2^{\ell-1} + 1, (r+1)2^\ell + 2^{\ell-1} - 1] \times [s2^\ell - 2^{\ell-1} + 1, (s+1)2^\ell + 2^{\ell-1} - 1].$$

Now choose a maximal collection $z_1, \ldots, z_n \subset \partial\widehat{S}$ for which the J^{z_i} are distinct. Index the z_i so that we meet $z_1, z_2, \ldots, z_n$ in this order as we traverse $\partial\widehat{S}$ clockwise, starting from z_1. Let ζ_i be a path from z_i to some vertex in $J^{z_i} \cap S$, satisfying (4.27) and (4.29)-(4.31). Then C must have a crossing of the above form between any two successive ζ_i. Therefore,

$$\begin{aligned} &\text{the maximal number of distinct } J^z \\ &\le \text{ the maximal number of disjoint crossings by } C \text{ of } V \setminus S \text{ or of } \widehat{S} \setminus U. \end{aligned} \tag{4.34}$$

We just proved that (for fixed (r,s)) the total number of circuits J^z and arcs $F(v)$ of J^z which can arise as we vary the endpoint v of a path ζ which satisfies (4.27) and (4.29)-(4.31), is at most

$$\begin{aligned} &[\text{maximal number of disjoint occupied crossings of } V(r,s) \setminus S(r,s) \\ &\qquad\qquad \text{or of } \widehat{S}(r,s) \setminus U(r,s)] \\ &\times [\text{maximal number of disjoint occupied crossings of } \widetilde{S}(r,s) \setminus S(r,s)]. \end{aligned} \tag{4.35}$$

Finally we take the union over $-2^{j+1-\ell} \le r, s < 2^{j+1-\ell}$. We find that the total number of choices for $J^z, F(v)$ is the sum of (4.35) over (r, s).

Step (iv) We now pick up the estimates from Step (i) again. Essentially the same argument as for (4.14) shows that the random variable in (4.35) has all moments. Therefore, by Chebyshev's inequality we have

$$P\{[\text{sum of (4.35) over } -2^{j+1-\ell} \le r, s < 2^{j+1-\ell}] > c_{42} j^4 2^{2(1-c_{30})j}\} \\ \le c_{43} j^{-4} 2^{-2(1-c_{30})j}. \tag{4.36}$$

Thus after an adjustment of c_{36}, c_{38} the estimate (4.15) remains valid even if we restrict C further so that it contains at most $c_{42} j^4 2^{2(1-c_{30})j}$ pairs of crossings, the first one of $V(r,s) \setminus S(r,s)$ or of $\widehat{S}(r,s) \setminus U(r,s)$ for a certain (r,s), and the second one of $\widetilde{S}(r,s) \setminus S(r,s)$ for the same (r,s). (Note that this makes c_{36} dependent on c_{30}.)

The last sum in (4.15) is therefore bounded by

$$j^2 \sup_{p,C} P\{C \text{ is not good} | p < \tau, \mathcal{C}_{\tau-p} \text{ is occupied and equals } C\},$$

where the sup is over $0 \le p < j^2$ and C satisfying the restrictions at the end of Step (i) and the ones just mentioned. For the remainder of this proof C will be a fixed circuit satisfying these conditions. For brevity we define the event

$$E(p, C) = \{p < \tau, \mathcal{C}_{\tau-p} \text{ is occupied and equals } C\}.$$

It is important to realize that for a given circuit C, the event $E(p,C)$ depends only on the vertices in $C^{\text{ext}} \cup C$. (Note that the condition $\{p < \tau, \mathcal{C}_{\tau-p} = C\}$ merely says that C is one of our $\mathcal{C}_k$ and there are exactly p of the circuits $\mathcal{C}_k$ outside C but inside $S(2^{j+1})$; this only involves vertices on or outside C.) Given $E(p,C)$, the further conditions for $\mathcal{C}_{\tau-p}$ to be not good, depend only on the vertices in $\overset{\circ}{C}$. Therefore, for calculating

$$P\{C \text{ is not good} | E(p, C)\}, \tag{4.37}$$

even with the conditioning on $E(p,C)$, we may assume that all vertices in the interior of C are still independently occupied or vacant with probability 1/2. Now we saw that if $E(p,C)$ occurs, but C is not good, then there exists a pair (r,s), one of the possible arcs $F(v)$, and a vacant path $\zeta = (v_1, \dots, v_q)$ which satisfy (4.27) and (4.29)-(4.31), but such that (4.3) with $\mathcal{D}$ replaced by ζ fails. If C is fixed, we shall estimate the probability in (4.37) by first picking a pair (r,s) so that C contains a point in or adjacent to $S(r,s)$, and then one of the possible interior boundaries $J = J^z$ and finally one of the possible arcs $F(v)$ of J^z in $\widehat{S}(r,s)$. Once $(r,s), J$ and $F = F(v)$ have been chosen we estimate

$$\sup P\{\exists \text{ a vacant path } \zeta = (v_1, \dots, v_q) \text{ which satisfies (4.27), (4.29), (4.30)} \\ \text{and } v_q \in \overset{\circ}{J}, v_1 \in F, \text{ but for which (4.3) fails}\}. \tag{4.38}$$

Here the sup is over all choices of C as above, over (r,s), and over the possible choices of J and F (for the chosen C and (r,s)). For the class of circuits C which we allow, the total number of choices for $(r,s), J$, and F is at most $c_{42}j^4 2^{2(1-c_{30})j}$, so that the last sum in (4.15) is bounded by

$$c_{42}j^6 2^{2(1-c_{30})j} \times \text{ the sup in (4.38)}. \tag{4.39}$$

We are going to imitate the proof of Lemma 8.2 in Kesten (1982) to estimate (4.38). For the remainder of this step $C, (r,s)$ and F remain fixed. This of course also tells us which part of $\widehat{S}$ belongs to $\overset{\circ}{C}$. By our construction in Step (i), F is a crosscut of the interior of $\widehat{S}$ which contains some vertex in S. Let $F = (w_1, w_2, \dots, w_\nu)$, with $w_1, w_\nu \in \Delta\widehat{S}$, where the vertices are indexed in such a way that we move along J in the counterclockwise direction as we run successively through $w_1, w_2, \dots, w_\nu$. We number the remaining vertices of J, still as we meet them by continuing in the counterclockwise direction along J, $w_{\nu+1}, w_{\nu+2}, \dots, w_\rho$. w_ρ is adjacent to w_1.

Let I be the component of $\overset{\circ}{J} \cap (\text{interior of } \widehat{S})$ which has F as part of its boundary. There is only one such component. This is relatively easy to see in our situation, because F is a piecewise linear path, built up from edges of $\mathcal{T}$, and F is a crosscut of (interior of $\widehat{S}$). The topological boundary of I consists of F and alternating pieces of $\Delta\widehat{S}$ and J (see Figure 7). We need a more precise statement, though, which describes the relative order of some of the pieces of ΔI. First let us prove a general result. Let J be a Jordan curve and let it be oriented counterclockwise (i. e., with $\overset{\circ}{J}$ 'on the left'). Let $K_1, K_2, \dots, K_N$ be a number of crosscuts of $\overset{\circ}{J}$ with disjoint interiors, i. e., for which $\overset{\circ}{J} \cap K_i \cap K_j = \emptyset$ when $i \neq j$. Let I be one of the components of $\overset{\circ}{J} \setminus \cap_{i=1}^N K_i$. We claim that then ΔI consists of a number of disjoint arcs α_i from J and some of the crosscuts K_i, and we can traverse ΔI in one direction so that we traverse all the α_i in ΔI in the same order and in the same direction as they are traversed when J is traversed in the counterclockwise direction. This claim is easily proven by induction on N. It is trivially true when $N = 0$. Now assume it is already proven when $N = n$ and let K_{n+1} be a crosscut of $\overset{\circ}{J}$ such that $K_{n+1} \cap \overset{\circ}{J}$ is disjoint from $\cup_{i=1}^n K_i$. Finally, let I_n be one of the components of $\overset{\circ}{J} \setminus \cup_{i=1}^n K_i$. If K_{n+1} is disjoint from I_n, then this component and its boundary is unchanged by the addition of K_{n+1}, so our claim is valid for this component. If K_{n+1} intersects I_n, then it is easily seen that K_{n+1} is a crosscut of I_n with its endpoints, a_1 and a_2 say on two of the arcs α_i, α_j ($i = j$ is possible). Then K_{n+1} divides I_n into two new components. Call them I'_n and I''_n. The boundary of each of I'_n, I''_n consists of K_{n+1} plus one of the arcs of ΔI_n from a_1, a_2. Since our claim was known to be true for ΔI_n it is also true for each of the arcs of ΔI_n from a_2 to a_1 followed by K_{n+1}, that is for the boundaries of I'_n and I''_n. This proves our claim for $N = n + 1$.

We now apply the claim of the last statement to I, the component of $\overset{\circ}{J}\cap$ (interior of $\widehat{S}$) defined above. For the K_i we take all segments of $\Delta\widehat{S}$ whose interior lies in $\overset{\circ}{J}$. I is one of the components of $\overset{\circ}{J}$ minus these segments. The claim from the preceding paragraph then tells us that ΔI consists of arcs of J and segments of $\Delta\widehat{S}$, and that we can traverse ΔI in such a way that all the arcs of J in ΔI are traversed in the same direction and order as when J is traversed counterclockwise. In fact this will necessarily be when we traverse ΔI counterclockwise, because when traversing the arc F which belongs to J and to ΔI, any vector pointing into $\overset{\circ}{J}$ also points into I.

Next, we shall introduce a partial order of the paths ζ which we are considering. Call a path $\zeta = (v_1, \dots, v_q)$ *permissible*, if it satisfies (4.27), (4.29), (4.30) and

$$v_q \in \overset{\circ}{J}, v_1 \in F \cap S. \tag{4.40}$$

A permissible path minus its endpoints lies in I; in fact it is a crosscut of I (note that $\zeta \setminus \{v_1\} \subset \overset{\circ}{J}$, by virtue of (4.27) and the fact that $v_q \in \overset{\circ}{J}$; thus, as we move from $v_1 \in F$ along ζ we enter a component of $\overset{\circ}{J}\cap$ (interior of $\widehat{S}$) with F in its boundary, and this must be I). We shall also need to consider a slightly larger class of paths which includes the permissible paths. This is the collection of paths $\gamma = (y_1, \dots, y_q)$ with the following three properties:

$$y_1 \in (J \cap \text{ interior of } \widehat{S}) \text{ and } y_1 \text{ is the only vertex of } \gamma \text{ adjacent to } C, \tag{4.41}$$

$$\gamma \setminus \{y_1, y_q\} \subset \overset{\circ}{I} \subset \overset{\circ}{J}, \tag{4.42}$$

$$y_q \in \Delta\widehat{S} \cap \overset{\circ}{J}, \text{ and } y_1 \text{ not adjacent to } y_q. \tag{4.43}$$

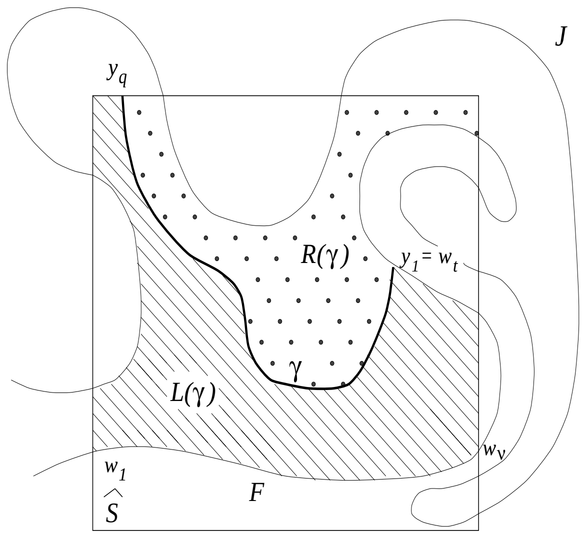

FIGURE 7. Illustration of the regions $I, L(\gamma)$ (hatched) and $R(\gamma)$ (dotted) for a path γ as in (4.41)-(4.43). γ is the boldly drawn curve and I consists of $R(\gamma) \cup L(\gamma) \cup \gamma$ (minus the endpoints of γ).

Basically only the requirement (4.40) for permissible paths has been weakened to $y_1 \in J \cap ($ interior of $\widehat{S})$. We shall write w_t for this initial point y_1. Such a path γ is still a crosscut of I so that $I \setminus \gamma$ consists of two components which we denote as $L(\gamma)$ and $R(\gamma)$. $L(\gamma)$ will be the component of $I \setminus \gamma$ whose boundary contains the endpoint w_1 of F, and $R(\gamma)$ will be the other component. This implies that the whole arc of ΔI in the counterclockwise direction starting from w_1 till it first reaches a point of γ, belongs to $\Delta L(\gamma)$, and except for w_t no point of this arc belongs to $\Delta R(\gamma)$. Call this arc $A(\gamma)$. For a permissible path γ, $A(\gamma)$ is the arc of F from w_1 to $w_t = y_1 \in F$. This means that this arc of F belongs to $\Delta L(\gamma)$ and

$$y_q \notin A(\gamma). \tag{4.44}$$

By the property of ΔI proven above, we have that all vertices w_i of ΔI with $i < t$ must be passed before we get to w_t as we move counterclockwise from w_1. Thus (4.44) implies

$$\begin{aligned}&\text{all } w_i \text{ which belong to } \Delta I \text{ and which have } i < t,\\ &\text{lie in } A(\gamma) \text{ and belong to } \Delta L(\gamma) \setminus \Delta R(\gamma);\\ &\text{in particular } w_i \in \Delta R(\gamma) \text{ implies } i \geq t.\end{aligned} \tag{4.45}$$

(This deduction of (4.45) from (4.44) does not use that γ is permissible, but only the properties (4.41)-(4.43).) We shall soon see that all the paths in which we are interested have the properties (4.44) and (4.45).

It may help the reader to form a mental picture by considering the case when F is a left-right crossing of $\widehat{S}$ with w_1 and w_ν on the left and right edge of $\Delta\widehat{S}$, respectively. Then according to our numbering of the vertices of F, $\overset{\circ}{J}$ contains the points just above F. If ζ has its last point v_q on the top edge of $\Delta\widehat{S}$, then $L(\zeta)$ and $R(\zeta)$ are on the left and right of ζ, repectively. $\zeta^{(1)}$ can be thought of as the 'leftmost' vacant permissible path. Of course, the general picture may be much more complicated.

We now define a partial ordering of permissible paths. (Note that we only order *permissible* paths, not the more general paths γ of the preceding paragraph.) If ζ' and ζ'' are two permissible paths, then we say that

$$\zeta' \text{ precedes } \zeta'' \text{ if } L(\zeta') \subset L(\zeta''). \tag{4.46}$$

It follows from the argument of Lemma 1 in Kesten (1980) or Proposition 2.3 in Kesten (1982) that if ζ', ζ'' are permissible, then there exists a permissible ζ''' which precedes both of them. If ζ', ζ'' are both vacant, then we can take ζ''' also vacant. In what follows we assume that there exists at least one vacant permissible path. From the preceding remarks it then follows that there also exists a first path (in the above order) among all the vacant permissible paths. This path will be denoted by

$$\zeta^{(1)} = (v_1^{(1)}, \dots, v_{q^{(1)}}^{(1)}).$$

It is an important fact that for a fixed permissible path γ (with initial point on F equal to w_t), the event $\{\zeta^{(1)} = \gamma\}$ depends only on the vertices in $L(\gamma) \cup \gamma \cup \{w_1, \dots, w_t\} \cup (\Delta L(\gamma) \cap \overset{\circ}{J})$. Indeed this event occurs if and only if γ is vacant, but any permissible $\gamma' \neq \gamma$ with $L(\gamma') \subset L(\gamma)$ contains some occupied vertex; any such γ' lies in $L(\gamma) \cup \gamma \cup \{w_1, \dots, w_t\} \cup (\Delta L(\gamma) \cap \overset{\circ}{J})$. This is so because γ' minus its endpoints lies in $L(\gamma)$; moreover, the initial point of γ' has to lie in $F \cap \Delta L(\gamma) = \{w_1, \dots, w_t\}$ (see (4.40), (4.44), (4.45)), and the endpoint $v'_{q'}$ of γ' lies in $\overset{\circ}{J}$, hence not on J itself.

We now wish to estimate first

$$P\{\zeta^{(1)} \text{ exists but does not satisfy (4.3)} | E(p, C)\} \tag{4.47}$$

(of course we mean here again that (4.3) with $\zeta^{(1)}$ substituted for $\mathcal{D}$ fails). It is convenient for this to introduce the following definitions. Let $\gamma = (y_1, \dots, y_q)$ be a path which has the properties (4.41)-(4.43). Suppose that $y_1 = w_t \in J\cap$ (interior of $\widehat{S}$) and that G is some subset of $\mathbb{R}^2$. Then we say that a vertex $w_m \in J\cap$ (interior of $\widehat{S}$) *has a vacant connection in the set* G *to* γ if $m = t$ or if there exists a path $\delta = (z_1, \dots, z_u)$ with

$$\delta \subset G, \tag{4.48}$$

$$z_1 = w_m,\ z_u \in \gamma,\ z_u \neq w_t, \tag{4.49}$$

$$\delta \setminus \{z_1, z_u\} \subset R(\gamma), \tag{4.50}$$

and

$$z_1, \dots, z_{u-1} \text{ are vacant} \tag{4.51}$$

(see Figure 8). Note that we do not require z_u to be vacant here (although it will be vacant in our applications); also we always count w_t (the initial point of γ) as having a vacant connection to γ. However, if $w_m \neq w_t$, then (4.51) requires that $w_m = z_1$ be vacant. Note further that $z_1 = w_m \in J\cap$ (interior of $\widehat{S}$) and $\delta \setminus \{z_1, z_u\} \subset I$ by (4.50). Thus each neighborhood of w_m contains points of δ in $R(\gamma) \subset I$. Of course each neighborhood of $w_m \in J$ also contains points of $J^{\text{ext}} \supset I^{\text{ext}}$. Thus $w_m \in \Delta I$. By (4.50) we then have

$$w_m \in \Delta I \cap \Delta R(\gamma).$$

Now assume that γ satisfies also the conditions (4.44) and hence (4.45), in addition to (4.41)-(4.43). Then we must have $t \leq m \leq \rho$. If δ is a vacant connection from w_m to γ and $m > t$, define $\gamma^*(\delta)$ as the path which consists of δ, followed by the piece of γ from z_u (= the endpoint of δ), to y_q (= the endpoint of γ). It follows from (4.49), (4.50), (4.42) and (4.43) that $\gamma^*(\delta)$ is again a crosscut of I and that its initial point $z_1 = w_m$ is the only point of $\gamma^*(\delta)$ which is adjacent to C. (Note that $z_2, \dots, z_{u-1} \in I \subset \overset{\circ}{J}$ by (4.50), $z_u \neq w_t$ by (4.49), so that also $z_u \in \gamma \setminus \{y_1\} \subset \overset{\circ}{J}$.) Therefore, if $z_1 = w_m$ is not adjacent to y_q, then $\gamma^*(\delta)$ again satisfies (4.41)-(4.43) with y_1 replaced by z_1. We want to show that $\gamma^*(\delta)$ also satisfies (4.44) and (4.45). To this end we note that as we move around ΔI counterclockwise from w_1, we first traverse the arc $A(\gamma)$ till we get to w_t. As we continue traversing ΔI we go through an arc of ΔI, A' say, from w_t to y_q. This arc must be the part of ΔI which belongs to $\Delta R(\gamma)$. Finally we will go from y_q back to w_1 along another arc A'' of ΔI which belongs to $\Delta L(\gamma)$ again (because it ends in w_1). Since $w_m \in \Delta I \cap \Delta R(\gamma)$ we must have $w_m \in A'$ and the counterclockwise arc of ΔI from w_t to w_m is part of A' and does not contain y_q. (Note that $w_m \in J, y_q \in \overset{\circ}{J}$, so $w_m \neq y_q$.) Thus $A(\gamma^*(\delta))$, which is the counterclockwise arc of ΔI from w_1 to w_m, consists of $A(\gamma)$ plus the counterclockwise subarc of A' from w_t to w_m, and does not contain y_q. This is precisely the desired analogue of (4.44) for γ^*, namely

$$y_q \notin A(\gamma^*).$$

As we already pointed out, (4.45) for γ^* then follows.

We now define for $i + \lceil 2c_{32} \log j \rceil \leq \rho$ the event

$$\begin{aligned} Y(w_i, \gamma, h) = &\{w_i \text{ has a vacant connection to } \gamma \text{ in } w_t + S(2^h), \\ &\text{and } w_{i+1}, w_{i+2}, \dots, w_{i+\lceil 2c_{32} \log j \rceil} \text{ are vacant}\}. \end{aligned} \tag{4.52}$$

Note that in (4.52) we look for a connection from w_i to γ in the square $w_t + S(2^h)$, centered at the initial point w_t of γ. Such a connection can exist only if $w_i \in w_t + S(2^h)$. We further define for $D \geq 1$

$$\begin{aligned} Z(\gamma, h, D) = &\text{maximal number of indices } m_i \in [t, \rho - \lceil 2c_{32} \log j \rceil] \\ &\text{for which } \|w_{m_i} - w_{m_k}\| > D \text{ for } i \neq k \text{ and for which} \\ &w_{m_i} \in \text{ (interior of } \widehat{S}) \text{ and } Y(w_{m_i}, \gamma, h) \text{ occurs.} \end{aligned} \tag{4.53}$$

When γ is vacant and

$$Z(\gamma, h, D) \geq \exp[c_{31} j]$$

for

$$D = 2^{c_{33} c_{30} (j+1)}, \tag{4.54}$$

then (4.3) with $\mathcal{D}$ replaced by γ and z by y_q certainly holds. Indeed, for each of the m's for which $Y(w_m, \gamma, h)$ occurs, w_m has a vacant connection, δ say, to γ and is therefore connected to y_q by the vacant path $\gamma^*(\delta)$. The only vertex adjacent to C on this path is w_m. The path $(w_m, w_{m+1}, \dots, w_{m+\lceil 2c_{32} \log j \rceil})$ is therefore one of the possible paths θ required in (4.3). The probability in (4.47) can therefore be bounded by

$$\sum_{\gamma} P\{Z(\gamma, h, D) < \exp[c_{31} j] | E(p, C), \zeta^{(1)} = \gamma\} P\{\zeta^{(1)} = \gamma | E(p, C)\}, \tag{4.55}$$

with D as in (4.54) and the sum over γ running over all permissible paths. We note that for a fixed path γ, $Z(\gamma, h, D)$ depends only on the vertices in $R(\gamma) \cup \{w_m : t < m \leq \rho\} \subset \mathring{J} \cup J \subset \mathring{C}$, because any vacant connection has to lie in $R(\gamma)$ except for its endpoints (by virtue of (4.50)), and no requirements are made on $w_t = y_1$ or z_u in (4.51). Moreover, the initial point w_m of any such vacant connection has to satisfy $t \leq m \leq \rho$, as we saw above. On the other hand, as we observed before, $E(p, C)$ depends only on the vertices in $C^{\text{ext}} \cup C$, while $\{\zeta^{(1)} = \gamma\}$ depends only on the vertices in $L(\gamma) \cup \gamma \cup \{w_1, \dots, w_t\} \cup (\Delta L(\gamma) \cap \mathring{J})$. Since $C^{\text{ext}} \cup C \cup L(\gamma) \cup \gamma \cup \{w_1, \dots, w_t\} \cup (\Delta L(\gamma) \cap \mathring{J})$ and $R(\gamma) \cup \{w_m : t < m \leq \rho\}$ are disjoint, the conditioning in the first factor in (4.55) has no influence. For similar reasons, the conditioning has no influence on the second factor (recall that C is fixed). Therefore (4.55) equals

$$\begin{aligned} &\sum_{\gamma} P\{Z(\gamma, h, D) < \exp[c_{31} j]\} P\{\zeta^{(1)} = \gamma\} \\ &\qquad \leq P\{\zeta^{(1)} \text{ exists}\} \sup_{\gamma} P\{Z(\gamma, h, D) < \exp[c_{31} j]\}, \end{aligned} \tag{4.56}$$

where γ still ranges over the permissible paths. In particular, these have initial vertex $y_1 \in S$, and therefore the distance of y_1 to $\Delta\widehat{S}$ exceeds 2^ℓ. To estimate (4.56) we introduce further

$$\Lambda(\alpha, h, D) = \sup_\gamma E \exp[-\alpha Z(\gamma, h, D)], \tag{4.57}$$

where we now let γ range over all paths satisfying (4.41)-(4.45) and

$$\text{distance of } y_1 \text{ to } \Delta\widehat{S} > 2^h. \tag{4.58}$$

We shall choose $\alpha \geq 0$ and h later, but from now on we restrict h to satisfy

$$4 \leq h \leq \ell, \text{ and } 2^h \geq 8D \vee 32c_{32} \log j. \tag{4.59}$$

We then obviously have for any $\alpha \geq 0$ that (4.56), and hence (4.47) is bounded by

$$P\{\zeta^{(1)} \text{ exists}\} \exp[\alpha \exp\{c_{31} j\}] \Lambda(\alpha, h, D). \tag{4.60}$$

The remainder of this step is devoted to proving that there exists some constant $c_{44} \in (0, 1]$, independent of α, h, D such that under (4.59)

$$\Lambda(\alpha, h, D) \leq (1 - c_{44})\Lambda(\alpha, h-4, D) + c_{44}\Lambda^2(\alpha, h-4, D). \tag{4.61}$$

We shall prove (4.61) by proving that for fixed γ, under (4.58), (4.59), $Z(\gamma, h, D)$ is stochastically larger than

$$Z(\gamma, h-4, D) + \xi \widetilde{Z}(h-4, D), \tag{4.62}$$

where $Z(\gamma, h-4, D), \widetilde{Z}(h-4, D)$ and ξ are independent with

$$P\{\xi = 1\} = 1 - P\{\xi = 0\} \geq c_{44} > 0 \tag{4.63}$$

for a constant $c_{44} \in (0, 1]$ which is independent of γ, h, D, and where $\widetilde{Z}$ has a distribution of the form

$$P\{\widetilde{Z}(h-4, D) \in B\} = \sum_{\overline{\gamma}} q(\overline{\gamma}) P\{Z(\overline{\gamma}, h-4, D) \in B\} \quad (B \text{ a Borel set}). \tag{4.64}$$

Here $q(\overline{\gamma}) \geq 0$, $\sum_{\overline{\gamma}} q(\overline{\gamma}) = 1$, and $\overline{\gamma}$ runs over all paths of with the properties (4.41)-(4.45) and whose initial point $\overline{y}_1$ satisfies

$$\text{distance of } \overline{y}_1 \text{ to } \Delta\widehat{S} > 2^{h-4}. \tag{4.65}$$

In other words, the distribution of $\widetilde{Z}$ is a mixture of distributions of $Z(\overline{\gamma}, h-4, D)$ over $\overline{\gamma}$ which satisfy (4.41)-(4.45) and (4.65). In particular

$$E\exp[-\alpha\widetilde{Z}(h-4,D)] \le \Lambda(\alpha, h-4, D), \quad \alpha \ge 0,$$

and the bound (4.62) implies (under (4.58))

$$\begin{aligned}&E\exp[-\alpha Z(\gamma, h, D)]\\ &\le E\exp[-\alpha Z(\gamma, h-4, D)]\{1 - c_{44} + c_{44}E\exp[-\alpha\widetilde{Z}(h-4,D)]\}\\ &\le (1-c_{44})\Lambda(\alpha, h-4, D) + c_{44}\Lambda^2(\alpha, h-4, D).\end{aligned}$$

Thus it will indeed suffice for (4.61) to show $Z(\gamma, h, D)$ stochastically larger than (4.62). The proof of this stochastic domination is essentially given in Lemma 8.2 of Kesten (1982) so we will skip some details. Let $\gamma = (y_1, \dots, y_q)$ be fixed so that (4.41)-(4.45) hold and assume that (4.58) and (4.59) hold. Let $y_1 = w_t$. We now also need a partial order of the paths $\delta = (z_1, \dots, z_u)$ with $z_1 = w_m \in J \cap$ (interior of $\widehat{S}$) and which satisfy (4.48)-(4.50) with

$$G = w_t + S(5 \cdot 2^{h-3}) \setminus S(4 \cdot 2^{h-3}). \tag{4.66}$$

G is an annulus centered at w_t. Note that $z_1 \in G$ forces $z_1 \in$ (interior of $\widehat{S}$) by (4.58) and (4.59) (recall that $w_t = y_1$). Furthermore, $\|z_1 - y_1\| = \|z_1 - w_t\| > 4 \cdot 2^{h-3} > 2$, so that z_1 is not adjacent to y_1. Also z_1 is not adjacent to y_q because $\|z_1 - y_q\| \ge \|y_1 - y_q\| - \|z_1 - y_1\| > 2^h - 5 \cdot 2^{h-3} \ge 2$ (see (4.58)). Therefore the case that δ consists of a single edge in $\Delta R(\gamma)$ is excluded. These paths δ are therefore crosscuts of $R(\gamma)$ and consequently $R(\gamma) \setminus \delta$ consists of two components which we denote by $U(\delta) = U(\delta, \gamma)$ and $V(\delta) = V(\delta, \gamma)$ (these U, V are unrelated to the U, V of Step (iii) which will not be used anymore). $U(\delta)$ is the component of $R(\gamma) \setminus \delta$ whose boundary consists of δ, the piece of γ from z_u to $y_1 = w_t$ plus that arc of ΔI from w_t to $w_m = z_1$ which is contained in $\Delta R(\gamma)$; see Figure 8.

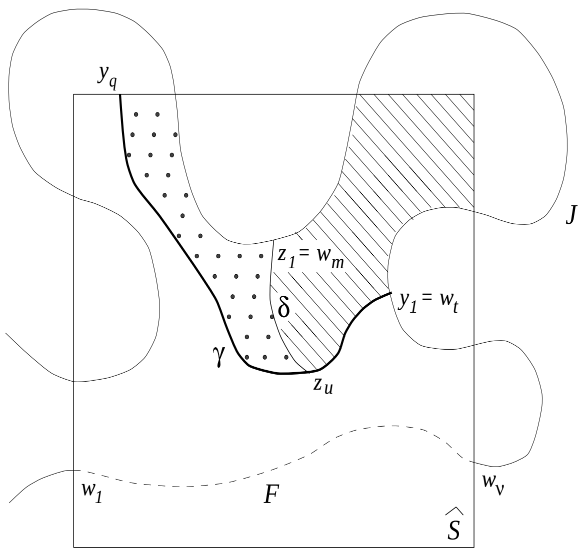

FIGURE 8. Illustration of γ (boldly drawn), δ (solidly drawn) and the components $U(\delta)$ (the hatched region) and $V(\delta)$ (the dotted region). F is the dashed part of J.

The last arc is a subarc of the arc A' introduced in the paragraph following (4.51). As we traverse ΔI, vertices of J are met in the same order as on J, so that this boundary arc of $U(\delta)$ contains only vertices $w_i \in J$ with $t \le i \le m$ (note that A' does not contain w_1, so this subarc of A' cannot be the one which contains w_i with $m \le i \le \rho$ and $1 \le i \le m$). $V(\delta)$ is the other component of $R(\gamma) \setminus \delta$; its boundary consists of δ followed by the part of γ from z_u to y_q, and the part of A' which is not in $\Delta U(\delta)$, i.e., the (reverse of) the subarc from w_m to y_q. We say that δ' *precedes* δ'' (*with respect to* γ) if $U(\delta', \gamma) \subset U(\delta'', \gamma)$. Again, if δ', δ'' lie in a set G, then there exists a path $\delta''' \subset G$ which precedes δ' and δ''. Also if δ' and δ'' are vacant, with the possible exception of their endpoint on γ, then δ''' minus its endpoint on γ can also be taken vacant. Assume now that

$$\text{there exists a path } \delta \text{ satisfying (4.48)-(4.51) with } G \text{ given by (4.66).} \tag{4.67}$$

In this case there is a minimal δ with respect to γ (in the above order) with the properties (4.48)-(4.51). Let this minimal path be $\widetilde{\delta} = (\widetilde{z}_1, \dots, \widetilde{z}_u)$. Let $\widetilde{z}_u = y_v$. Then consider the path $\widetilde{\gamma} := \gamma^*(\widetilde{\delta})$. As we saw before, $\widetilde{\gamma}$ also has the properties (4.41)-(4.45) with the obvious replacements (e.g., y_1 is replaced by the first vertex of $\widetilde{\gamma}$, that is, by $\widetilde{z}_1$).

Now write again w_m for $\widetilde{z}_1$, and consider a vertex w_i for which $Y(w_i, \widetilde{\gamma}, h-4)$ occurs. This means that w_i has a vacant connection to $\widetilde{\gamma}$ in $w_m + S(2^{h-4})$ and that all the vertices $w_i, w_{i+1}, \dots, w_{i+\lceil 2c_{32} \log j \rceil}$ are vacant. Note that this can happen only for

$$w_i \in \big(w_m + S(2^{h-4})\big) \subset [w_t + S(5 \cdot 2^{h-3}) \setminus S(4 \cdot 2^{h-3})] + S(2^{h-4})$$
$$\text{(because } w_m \in G) \subset w_t + S(6 \cdot 2^{h-3}) \subset w_t + S(2^h) \subset \text{ (interior of } \widehat{S}),$$

so that automatically $w_i \in$ (interior of $\widehat{S}$). In addition, it must be the case that $m \le i \le \rho$, as in the lines following (4.51), because $\widetilde{\gamma}$ satisfies (4.44), (4.45). For the sake of argument, let w_i have the vacant connection $\widehat{\delta}$ to $\widetilde{\gamma}$ in $w_m + S(2^{h-4})$. Then one can see that $\widehat{\delta}$, possibly followed by a piece of $\widetilde{\delta}$, forms a vacant connection from w_i to γ (see Kesten (1982), equation (8.73) and its proof). Since this vacant connection is made up from $\widehat{\delta}$ and possibly a piece of $\widetilde{\delta}$, it lies in

$$\big(w_m + S(2^{h-4})\big) \cup G \subset [w_t + S(5 \cdot 2^{h-3}) \setminus S(4 \cdot 2^{h-3})] + S(2^{h-4})$$
$$\text{(because } w_m \in G) \subset w_t + S(2^h).$$

Thus w_i has a vacant connection to γ in $w_t + S(2^h)$. If further w_p is vacant for all $i \le p \le i + \lceil 2c_{32} \log j \rceil$ then $Y(w_i, \gamma, h)$ occurs. In other words, $Y(w_i, \gamma, h)$ occurs if (4.67) holds and $Y(w_i, \widetilde{\gamma}, h-4)$ occurs.

It is further immediate from the definitions that for any w_i, the occurrence of $Y(w_i, \gamma, h-4)$ implies the occurrence of $Y(w_i, \gamma, h)$. This is true whether (4.67) holds or not.

Now by definition of Z, there exist $Z(\gamma, h-4, D)$ indices $m_i \in [t, \rho - \lceil 2c_{32} \log j \rceil]$ with $w_{m_i} \in$ (interior of $\widehat{S}$) and which satisfy

$$\|w_{m_i} - w_{m_k}\| > D \text{ for } i \neq k \tag{4.68}$$

and

$$Y(w_{m_i}, \gamma, h-4) \text{ occurs.} \tag{4.69}$$

Similarly, if (4.67) holds, then there are $Z(\widetilde{\gamma}, h-4, D)$ indices $n_i \ge m$ with $w_{n_i} \in$ (interior of $\widehat{S}$) which satisfy

$$\|w_{n_i} - w_{n_k}\| > D \text{ for } i \neq k \tag{4.70}$$

and

$$Y(w_{n_i}, \widetilde{\gamma}, h-4) \text{ occurs, and hence } Y(w_{n_i}, \gamma, h), \text{ occurs.} \tag{4.71}$$

Also $Y(w_{m_i}, \gamma, h-4)$ can occur only when $w_{m_i} - w_t \in S(2^{h-4})$, while $Y(w_{n_i}, \widetilde{\gamma}, h-4)$ can occur only when $w_{n_i} - w_m \in S(2^{h-4})$. But $w_m = z_1 \in G$, so that

$$w_m - w_t \in S(5 \cdot 2^{h-3}) \setminus S(4 \cdot 2^{h-3}),$$

and in particular $\|w_m - w_t\| > 4 \cdot 2^{h-3}$. Thus for any m_i, n_k which satisfy (4.69) and (4.71), respectively, we must have

$$\begin{aligned}\|w_{m_i} - w_{n_k}\| &\geq \|w_m - w_t\| - \|w_{m_i} - w_t\| - \|w_{n_k} - w_m\| \\ &> 4 \cdot 2^{h-3} - 2\sqrt{2}2^{h-4} > 2^{h-3} \geq D \text{ (see (4.59))}.\end{aligned} \tag{4.72}$$

Thus each of the vertices w_{m_i} which satisfies (4.69) automatically has distance more than D to each of the vertices n_i which satisfy (4.71). It follows that we can combine the vertices satisfying (4.68), (4.69) with those satisfying (4.70), (4.71) and that under (4.58), (4.59) and (4.67),

$$Z(\gamma, h, D) \geq Z(\gamma, h-4, D) + Z(\widetilde{\gamma}, h-4, D).$$

If we set

$$\xi = I[\widetilde{\gamma} \text{ exists}] = I[(4.67) \text{ occurs}],$$

and

$$\widetilde{Z}(h-4, D) = Z(\widetilde{\gamma}, h-4, D),$$

then this says that the stochastic lower bound (4.62) for $Z(\gamma, h, D)$ is valid when $\xi = 1$. When $\xi = 0$, then (4.62) merely says $Z(\gamma, h, D) \geq Z(\gamma, h-4, D)$, and as we saw, this always holds.

It remains to verify that $Z(\gamma, h-4, D), \widetilde{Z}(h-4, D)$ and ξ are independent and that their joint distribution has the properties (4.63)-(4.65). For fixed γ, $Z(\gamma, h-4, D)$ depends only on connections inside $\big(w_t + S(2^{h-4})\big) \cap (R(\gamma) \cup \Delta R(\gamma))$ and vertices $w_p \in J$ with $t \leq i < p \leq i + \lceil 2c_{32} \log j \rceil$ for some w_i for which $Y(w_i, \gamma, h-4)$ can occur. Such w_i have to lie in $w_t + S(2^{h-4})$ and $i \geq t$ must hold (see the lines following (4.55)). Thus, $Z(\gamma, h-4, D)$ depends only on vertices within distance $\sqrt{2} \cdot 2^{h-4} + \lceil 2c_{32} \log j \rceil < 2^{h-2}$ from w_t. ξ only depends on the vertices in the annulus G of (4.66), because $\widetilde{\delta}$ has to satisfy (4.48). Since the distance from G to w_t is at least $4 \cdot 2^{h-3} > 2^{h-3} + \lceil 2c_{32} \log j \rceil$ for the h-values under consideration (see (4.59)), $Z(\gamma, h-4, D)$ and ξ are indeed independent. For the same reason, the location of the path $\widetilde{\delta}$ when it exists, is independent of $Z(\gamma, h-4, D)$. Moreover,

$$P\{\xi = 1\} \geq P\{\exists \text{ vacant circuit in the annulus } G \text{ of } (4.66)\} \tag{4.73}$$

for the same reasons as for equation (7.72) in Kesten (1982); see also Lemma 3 in Kesten (1980)). Since the right hand side here is bounded away from 0, (4.63) holds for $c_{44} > 0$ given by the right hand side of (4.73).

Finally we need to consider the conditional distribution of distribution of $\widetilde{Z}$. The conditional distribution when $\xi = 0$ is irrelevant, because in this case $\widetilde{Z}$ does not figure in (4.62). To finish our proof it suffices to show that

$$\begin{aligned} &P\{\widetilde{Z}(h-4, D) \in B | \text{vertices within distance } 2^{h-2} \text{ of } w_t, \xi = 1, \widetilde{\delta} = \overline{\delta}\} \\ &\quad = P\{Z(\gamma^*(\overline{\delta}), h-4, D) \in B\}. \end{aligned} \tag{4.74}$$

Indeed, (4.74) implies that

$$\begin{aligned} &P\{\widetilde{Z}(h-4, D) \in B | Z(h-4, \gamma, D), \xi = 1\} \\ &= \sum_{\overline{\delta}} P\{\widetilde{\delta} = \overline{\delta} | \xi = 1\} P\{Z(\gamma^*(\overline{\delta}), h-4, D) \in B\}, \end{aligned} \tag{4.75}$$

so that (4.64) holds with

$$q(\overline{\gamma}) = \sum_{\overline{\delta}: \gamma^*(\overline{\delta}) = \overline{\gamma}} P\{\widetilde{\delta} = \overline{\delta} | \xi = 1\}.$$

This distribution also satisfies (4.65), because if (4.58) holds, then the distance of any point in the annulus G of (4.66) to $\Delta \widehat{S}$ is at least equal the the distance of $w_t = y_1$ to $\Delta \widehat{S}$ minus $5 \cdot 2^{h-3} > 2^{h-3}$. In particular this holds for the initial point of all possible $\widetilde{\delta}$.

Finally, to prove (4.74), we first check more precisely on which vertices the event $\{\xi = 1, \widetilde{\delta} = \overline{\delta}\}$ depends. For fixed γ and $\overline{\delta}$, this only depends on the vertices in $G \cap \Big(U(\overline{\delta}, \gamma) \cup \overline{\delta} \cup \{w_t, \dots, w_{\overline{m}}\} \cup (\text{interior of the piece of } \gamma \text{ from } w_t = y_1 \text{ to } \overline{z}) \Big)$, where $w_{\overline{m}}$ is the initial point of $\overline{\delta}$ on J and $\overline{z}$ its final point on γ. Indeed, we only need to check vertices in this set to check the minimality of $\overline{\delta}$, as follows from the desription of $\Delta U(\delta)$ in the lines following (4.66). We claim that

$$U(\overline{\delta}, \gamma) \cup \text{ (interior of the piece of } \gamma \text{ from } w_t = y_1 \text{ to } \overline{z}) \subset L(\gamma^*(\overline{\delta})). \tag{4.76}$$

In fact, we claim that for any δ which satisfies (4.48)-(4.50), it holds that

$$\begin{aligned} &L(\gamma) \cup U(\delta, \gamma) \cup (\text{interior of the piece of } \gamma \text{ from } w_t = y_1 \text{ to } z_u) \\ &= L(\gamma^*(\delta)). \end{aligned} \tag{4.77}$$

(4.76) follows once we have (4.77) by replacing δ by $\overline{\delta}$. (4.77) is the analogue of equation (8.70) in Kesten (1982). It follows by observing that $R(\gamma^*(\delta))$ and $V(\delta, \gamma)$ have the same boundary and therefore are the same. Hence also

$$\begin{aligned} L(\gamma^*(\delta)) &= I \setminus (R(\gamma^*(\delta)) \cup \gamma^*(\delta)) = I \setminus (V(\delta, \gamma) \cup \gamma^*(\delta)) \\ &= \Big(L(\gamma) \cup R(\gamma) \cup (\gamma \setminus \{y_1, y_q\}) \Big) \setminus (V(\delta, \gamma) \cup \gamma^*(\delta)) \\ &= L(\gamma) \cup U(\delta, \gamma) \cup (\text{interior of the piece of } \gamma \text{ from } w_t = y_1 \text{ to } z_u). \end{aligned}$$

Alternatively, we can check directly that the two sides of (4.77) have the same boundary. Thus (4.77) holds, as claimed.

Just before (4.73) we described on which vertices $Z(\gamma, h-4, D)$ depends. All these vertices have to lie within distance 2^{h-2} of w_t. For the same reasons, $Z(\gamma^*(\overline{\delta}),\ h-4, D)$ depends only on the vertices in $R(\gamma^*(\overline{\delta})) \cup \{w_i : \overline{m} < i \le \rho\}$ which are within distance 2^{h-2} of the initial point $w_{\overline{m}}$ of $\overline{\delta}$ (that is, once $\overline{\delta}$ is fixed). But

$$\|w_{\overline{m}} - w_t\| > 4 \cdot 2^{h-3},$$

and no vertex $w_i \in J$ can lie in $L(\gamma^*(\overline{\delta}) \cup R(\gamma^*(\overline{\delta}) \subset I \subset \mathring{J}$. Therefore

$$\Big(R(\gamma^*(\delta)) \cup \{w_i : \overline{m} < i \le \rho\}\Big) \cap \{\text{vertices within distance } 2^{h-2} \text{ of } w_{\overline{m}}\}$$

is disjoint from

$$L(\gamma^*(\overline{\delta})) \cup \overline{\delta} \cup \{w_{t+1}, \dots, w_{\overline{m}}\} \cup \{\text{vertices within distance } 2^{h-2} \text{ of } w_t\}.$$

Thus even under the conditioning in the left hand side of (4.74), the vertices in $R(\gamma^*(\overline{\delta})) \cup \{w_i : \overline{m} < i \le \rho\}$ at distance $\le 2^{h-2}$ from $w_{\overline{m}}$ are still independently occupied or vacant with probability 1/2. Therefore, the conditional distribution of $Z(\widetilde{\gamma}, h-4, D)$, given the conditions in the left hand side of (4.74) is just the unconditional distribution of $Z(\gamma^*(\overline{\delta}), h-4, D)$. This is precisely what (4.74) states.

Step (v) In this last step we assemble the pieces proven in Steps (i)-(iv) to complete the proof of Proposition 1. First we must generalize the estimate (4.60) for (4.47), in order to obtain an estimate for (4.38). Recall that $\zeta^{(1)}$ was the first vacant permissible path (in the ordering introduced in (4.46)). More generally, we now introduce the successive disjoint permissible vacant paths $\zeta^{(i)}$: $\zeta^{(1)}$ is the first such path, and $\zeta^{(i)}$ is the first vacant permissible path which comes after $\zeta^{(i-1)}$ and is disjoint from $\zeta^{(i)}$. At the first i for which there is no such path we say that $\zeta^{(i)}$ and $\zeta^{(i')}$ with $i' > i$ do not exist. Now it is not hard to see that if ζ' and ζ'' are two permissible paths which are either disjoint or have only their endpoint on J in common, then one must precede the other. In fact, we already stated that Lemma 1 of Kesten (1980) or Proposition 2.3 in Kesten (1982) shows that there exists a path ζ''' which precedes both ζ' and ζ''. The construction of ζ''' (or a direct study of the boundaries of $L(\zeta')$ and $L(\zeta''))$ shows that ζ''' can be taken as one of ζ' or ζ'' when these paths are disjoint or intersect only on J. It follows that if ζ is any vacant permissible path, then it must intersect one of the $\zeta^{(i)}$ in a point not on J, because otherwise ζ would be comparable to all of them, and appear as one of the $\zeta^{(i)}$ itself. But if ζ intersects $\zeta^{(i)}$ in a point not on J, then the vacant cluster of ζ in $\mathring{C}$ is the same as the vacant cluster of $\zeta^{(i)}$ in $\mathring{C}$, and if (4.3) fails for

ζ, then it fails for $\zeta^{(i)}$. Consequently, (for fixed C, J, F) the probability in (4.38) is bounded by

$$\sum_{i \geq 1} P\{\zeta^{(i)} \text{ exists, but (4.3) fails for } \zeta^{(i)} | E(p, C)\}. \tag{4.78}$$

Instead of only (4.47) we should therefore estimate this sum. Now the argument leading to (4.56) remains valid without change when we replace $\zeta^{(1)}$ by $\zeta^{(i)}$. Indeed, the only property of $\zeta^{(1)}$ which is used is that the event $\{\zeta^{(1)} = \gamma\}$ depends only on the vertices in $L(\gamma) \cup \gamma \cup \{w_1, \dots, w_t\} \cup (\Delta L(\gamma) \cap \overset{\circ}{J})$. However, this is equally true with $\zeta^{(i)}$ instead of $\zeta^{(1)}$. Therefore (see (4.60)), the sum in (4.78) is at most

$$\sum_{i \geq 1} P\{\zeta^{(i)} \text{ exists}\} \exp[\alpha \exp\{c_{31} j\}] \Lambda(\alpha, h, D).$$

But (see (4.14))

$$\begin{aligned} &P\{\zeta^{(i)} \text{ exists}\} \\ &\leq P\{\text{there exist at least } i \text{ disjoint crossings of } \widetilde{S} \setminus S\} \\ &\leq (1 - c_{41})^i. \end{aligned}$$

Combining this with our preceding estimate we find that (4.38) is bounded by

$$[c_{41}]^{-1} \exp[\alpha \exp\{c_{31} j\}] \Lambda(\alpha, h, D).$$

Finally this yields the bound

$$\frac{c_{42} j^6}{c_{41}} 2^{2(1 - c_{30}) j} \exp[\alpha \exp\{c_{31} j\}] \Lambda(\alpha, h, D) \tag{4.79}$$

for the sum in (4.15) (recall (4.39)).

We shall now use the recurrence relation (4.61) to get an explicit estimate for $\Lambda(\alpha, h, D)$. To this end we rewrite (4.61) as

$$\Lambda(\alpha, h, D) \leq f(\Lambda(\alpha, h - 4, D)), \tag{4.80}$$

where

$$f(s) = (1 - c_{44}) s + c_{44} s^2, \quad 0 \leq s \leq 1.$$

The relation (4.80) is of the form of the recurrence relation for the generating function of a Bienaymé-Galton-Watson branching process in which each individual has one child with probability $1 - c_{44}$ and two children with probability c_{44}. If

W_n denotes the size of the n-th generation in such a process, and we start with $W_0 = 1$ individual, then

$$Es^{W_n} = f^{(n)}(s), \quad |s| \le 1,$$

where $f^{(n)}$ denotes the n-th iterate of f. Recall that (4.80) holds if $\alpha \ge 0$ and (4.59) holds. Thus, if

$$\alpha \ge 0,\ 4 \le h \le \ell, \text{ and } 2^{h-4n} \ge 8D \vee 32c_{32} \log j, \tag{4.81}$$

we can iterate (4.80) n times to obtain (for any $A > 0$)

$$\Lambda(\alpha, h, D) \le f^{(n)}(\Lambda(\alpha, h - 4n, D)) \le P\{W_n \le A\} + [\Lambda(\alpha, h - 4n, D)]^A. \tag{4.82}$$

We shall choose our parameters as follows:

$$D = 2^{c_{33}c_{30}(j+1)}, h = \ell, n = \lfloor (1 - c_{33})c_{30}j/8 \rfloor, A = (1 + c_{45})^n. \tag{4.83}$$

(4.81) will be satisfied for large j, provided we fix $c_{33} < 1$ (see (4.9) for ℓ).

The first term in the right hand side of (4.82) can now be estimated by standard branching process methods. Indeed, it is known (see Athreya and Ney (1972), Corollary 1 in Section I.11) that for our branching process, which has extinction probability 0 and $f'(0) = 1 - c_{44}$,

$$f^{(n)}(s) \le c_{46}(s)[f'(0)]^n = c_{46}(s)[1 - c_{44}]^n, \quad 0 \le s < 1.$$

Here $c_{46}(s)$ depends on s only. Therefore, if we fix $s > [1 - c_{44}]^{1/2}$, then we obtain for suitable constants $c_{47} < \infty, c_{48} < 1$,

$$P\{W_{\lfloor n/2 \rfloor} \le n\} \le s^{-n} c_{46}(s)[1 - c_{44}]^{\lfloor n/2 \rfloor} \le c_{47}[c_{48}]^n. \tag{4.84}$$

We improve this further by noting that W_n consists of the offspring of the $W_{\lfloor n/2 \rfloor}$ individuals in the $\lfloor n/2 \rfloor$-th generation. If any one of these individuals has more than $(1 + c_{45})^n$ children in the n-th generation, then $W_n > (1 + c_{45})^n$. Therefore, for any choice of $c_{45} \ge 0$,

$$P\{W_n \le (1 + c_{45})^n\} \le P\{W_{\lfloor n/2 \rfloor} \le n\} + \left[P\{W_{n - \lfloor n/2 \rfloor} \le (1 + c_{45})^n\}\right]^n. \tag{4.85}$$

Since the W-process is supercritical and has zero extinction probability, we can choose $c_{45} > 0$ such that

$$P\{W_{n - \lfloor n/2 \rfloor} \le (1 + c_{45})^n\} \to 0 \quad (n \to \infty). \tag{4.86}$$

Together with (4.84) and (4.85) this gives for large j

$$P\{W_n \le A\} = P\{W_n \le (1+c_{45})^n\} \le 2c_{47}[c_{48}]^n \le c_{49}\exp[-c_{50}(1-c_{33})c_{30}j]. \tag{4.87}$$

This is an exponential bound (in j) for the first term in the right hand side of (4.82).

To estimate the second term in the right hand side of (4.82) we note that it follows from the definition (4.57) that

$$\Lambda(\alpha, h-4n, D) \le 1-(1-e^{-\alpha})\inf_{\gamma} P\{Z(\gamma, h-4n, D) \ge 1\}, \tag{4.88}$$

where γ ranges over the paths satisfying (4.41)-(4.45) and

$$\text{distance of } y_1 \text{ to } \Delta\widehat{S} > 2^{h-4n}. \tag{4.89}$$

For such a γ,

$$\begin{aligned} P\{Z(\gamma, h-4n, D) \ge 1\} &\ge P\{Y(w_t, \gamma, h-4n) \text{ occurs}\} \\ &\ge P\{w_{t+1}, \dots, w_{t+\lceil 2c_{32}\log j\rceil} \text{ are vacant}\} \\ &\ge 2^{-2c_{32}\log j - 1} \ge j^{-c_{51}c_{32}}. \end{aligned} \tag{4.90}$$

For the second inequality here we note that if the first vertex of γ, $y_1 = w_t$ satisfies (4.89), then automatically $t \le \rho - \lceil 2c_{32}\log j\rceil$ because w_ρ is adjacent to $w_1 \in \Delta\widehat{S}$. Thus (4.89) implies $\|w_t - w_\rho\| \ge 2^{h-4n} - 2 > 2\lceil 2c_{32}\log j\rceil$. From (4.82) and (4.87)-(4.90), we obtain for $0 \le \alpha \le 1/2$,

$$\begin{aligned} \Lambda(\alpha, h, D) &\le c_{49}\exp[-c_{50}(1-c_{33})c_{30}j] + \left[1 - \frac{\alpha}{2j^{c_{51}c_{32}}}\right]^A \\ &\le c_{49}\exp[-c_{50}(1-c_{33})c_{30}j] + \exp[-\frac{(1+c_{45})^n\alpha}{2j^{c_{51}c_{32}}}]. \end{aligned} \tag{4.91}$$

Finally we take

$$c_{31} = c_{31}(c_{30}, c_{33}) < \frac{(1-c_{33})c_{30}}{16}\log(1+c_{45}) \tag{4.92}$$

and $\alpha = \alpha(j, c_{30}, c_{31})$ so that

$$\alpha e^{c_{31}j} = (1-c_{30})j$$

and $c_{30} \in (0,1)$ so close to 1 that

$$\frac{1-c_{30}}{c_{30}} \le \frac{1}{6}c_{50}(1-c_{33}).$$

Note that with these choices

$$\frac{(1+c_{45})^n\alpha}{2j^{c_{51}c_{32}}} = \frac{(1+c_{45})^{\lfloor (1-c_{33})c_{30}j/8\rfloor}(1-c_{30})}{2j^{c_{51}c_{32}-1}}e^{-c_{31}j} \ge c_{50}(1-c_{33})c_{30}j$$

for large j. Substituting these values of the parameters in the bound (4.79) for the sum in (4.15) finally gives us the exponential bound (4.5). ∎

5. Usefulness and existence of double paths.

We first demonstrate that one can use so-called 'double-paths' to write (a piece) of an arbitrary word. We define a *double path* to be a pair of selfavoiding paths π', π'' such that

$$\pi' \text{ is occupied and } \pi'' \text{ is vacant;} \tag{5.1}$$

$$\text{the initial points of } \pi' \text{ and } \pi'', u' \text{ and } u'', \text{ are neighbors;} \tag{5.2}$$

$$\text{the final points of } \pi' \text{ and } \pi'', v' \text{ and } v'', \text{ are neighbors;} \tag{5.3}$$

and such that the following minimality property holds. For any occupied selfavoiding path $\widehat{\pi}'$ from u' to v' and any selfavoiding vacant path $\widehat{\pi}''$ from u'' to v'' denote by $R(\widehat{\pi}', \widehat{\pi}'')$ the interior of the Jordan curve formed by concatenating $\widehat{\pi}', \{v', v''\}$, (the reverse of) $\widehat{\pi}'', \{u'', u'\}$. (Note that $\widehat{\pi}'$ and $\widehat{\pi}''$ are automatically disjoint, since one of them is occupied and the other is vacant.) Let $\overline{R}(\widehat{\pi}', \widehat{\pi}'')$ be the union of $R(\widehat{\pi}', \widehat{\pi}'')$ and its boundary (that is the above Jordan curve). Then we further require that

$$\overline{R}(\pi', \pi'') \text{ is minimal among all such } \overline{R}(\widehat{\pi}', \widehat{\pi}''). \tag{5.4}$$

By means of Proposition 2.2 in Kesten (1982) one can show that this implies the following property:

$$\begin{aligned}
&\text{every vertex } u \in \pi' \setminus \{u', v'\} \text{ has a neighbor } w \in (R(\pi', \pi'') \cup \pi'') \\
&\text{such that } w \text{ is connected to a vertex of } \pi'' \text{ by a vacant path in } \overline{R}(\pi', \pi'') \\
&\text{and such that the edge } \{u, w\} \subset \overline{R}(\pi', \pi''); \\
&\text{the same property holds when the single and double prime superscripts are} \\
&\text{interchanged (and 'vacant' is replaced by 'occupied').}
\end{aligned} \tag{5.5}$$

E.g., to obtain the first part interchange occupied and vacant in Proposition 2.2 of Kesten (1982) and take $A_1 = \{u\}, A_3 = \pi'', A_2 = \{u'', u'\}$ followed by the piece of π' from u' to u and $A_4 =$ piece of π' from u to v' followed by $\{v', v''\}$. By the minimality property (5.4) there does not exist an occupied path inside $\overline{R}(\pi', \pi'') \setminus (\{u\} \cup \pi'')$ from a vertex of $\mathring{A}_2$ to a vertex of $\mathring{A}_4$. Note that $J \setminus (A_1 \cup A_3)$ should be $\overline{J} \setminus (A_1 \cup A_3)$ on lines 1 and 2 f.b. of p. 30 in Kesten (1982). We point out that in the first part of (5.5) $w \in \pi''$ is possible; in this case the vacant path from w to π'' consists of w only. A similar comment applies to the second part of (5.5).

We shall repeatedly use the fact that if $\widehat{\pi}', \widehat{\pi}''$ are an occupied and vacant path, respectively, which start at the adjacent points u', u'' and end at the adjacent points v', v'', but for which $\overline{R}(\widehat{\pi}', \widehat{\pi}'')$ is not minimal, then there exist an occupied path π' from u' to v' and a vacant path π'' from u'' to v'' such that (π', π'') is a double path with $\overline{R}(\pi', \pi'') \subset \overline{R}(\widehat{\pi}', \widehat{\pi}'')$. To find such paths, first take π' as an occupied path from u' to v' in $\overline{R}(\widehat{\pi}', \widehat{\pi}'')$ which minimizes $\overline{R}(\pi', \widehat{\pi}'')$ over all such paths. Such a π' exists by the argument for Proposition 2.3 in Kesten (1982) (see also Lemma 1 in Kesten (1980)). After that take π'' as a vacant path from u'' to v'' in $\overline{R}(\pi', \widehat{\pi}'')$ which minimizes $\overline{R}(\pi', \pi'')$ over all such paths. One easily checks that (5.4) holds for these π', π''.

Now let (π', π'') be a double path starting at (u', u'') and let

$$\Theta = \min(\|v' - u'\|, \|v'' - u''\|) - 1. \tag{5.6}$$

Finally, let $\xi = (\xi_1, \dots)$ be any infinite word (here each $\xi_i \in \{0, 1\}$ with 0 (1) corresponding to vacant (respectively, occupied). The next lemma (which is purely deterministic) shows how one can 'see' an initial segment of the word ξ inside $\overline{R}(\pi', \pi'')$.

Lemma 8. *At least the initial segment $(\xi_1, \dots, \xi_\Theta)$ of ξ can be seen inside $\overline{R}(\pi', \pi'')$ from u' as well as from u''. In fact, there exist paths $\sigma' = (\sigma'_0 = u', \sigma'_1, \dots), \sigma'' = (\sigma''_0 = u'', \sigma''_1, \dots) \subset \overline{R}(\pi', \pi'')$, which start at u' and u'', respectively, and which have as endpoint one of $\{v', v''\}$ such that one sees an initial segment of ξ from $u'(u'')$ along $\sigma'(\sigma'')$. The length of these paths is at least Θ, so that one can see at least $(\xi_1, \dots, \xi_\Theta)$ from both u' and u''. For a fixed occupancy configuration, and any i, the event $\{$length $(\sigma' \setminus \{u'\}) \le i\}$ depends on $\xi_1, \dots, \xi_i$ only and not on $\xi_j, j > i$. If length $(\sigma') \ge i$, then $\sigma'_1, \dots, \sigma'_i$ also depend on $\xi_1, \dots, \xi_i$ only. The same statement with single and double prime superscripts interchanged is also valid.*

Proof. We only prove that we can find the path σ' from u'. This means that we can find a selfavoiding path $\sigma = (\sigma_0 = u', \sigma_1, \dots, \sigma_\nu) \subset \overline{R}(\pi', \pi'')$ such that σ_1 is adjacent to $u', \sigma_\nu \in \{v', v''\}$, and such that σ_i is occupied (vacant) if $\xi_i = 1$ (respectively $\xi_i = 0$). We prove this in the following recursive way. We find a neighbor σ_1 of $u' \in \overline{R}(\pi', \pi'')$ such that σ_1 is occupied (vacant) if $\xi_1 = 1$ ($\xi_1 = 0$) and a new double path (π'_1, π''_1) with σ_1 the initial point of one of them and with endpoints (v', v''), and such that

$$\overline{R}(\pi'_1, \pi''_1) \subset \overline{R}(\pi', \pi'') \setminus \{u'\}. \tag{5.7}$$

We then repeat this step, that is we find a neighbor σ_2 of σ_1 such that σ_2 is occupied (σ_2 is vacant) if $\xi_2 = 1$ ($\xi_2 = 0$) and a further double path etc. This construction will continue until we first use a vertex σ_ν from $\{v', v''\}$. It is clear from the recursive nature of the construction that the event $\{\nu \le i\}$ is independent of $\xi_{i+1}, \dots$. Similarly, if $\nu \ge i$, then the piece $(u', \sigma_1, \dots, \sigma_i)$ of σ is independent of $\xi_{i+1}, \dots$.

Two cases have to be distinguished in our construction, depending on the value of ξ_1.

Case (i) $\xi_1 = 1$. We now take σ_1 as the neighbor of u' on π'. This uniquely determines σ_1. We take $\widehat{\pi}'$ to be the piece of π' from σ_1 to v' (this is just π' minus its first edge). We also want a vacant path $\widehat{\pi}''$. To choose this, we observe that by (5.5) there must exist a vacant path π_2 in $\overline{R}(\pi', \pi'')$ from a neigbor u_1 of σ_1 to π''. Now form the vacant path $\widehat{\pi}''$ from u_1 to v'' which consists of π_2 followed by the piece of π'' from the endpoint of π_2 to v''. Then $(\widehat{\pi}', \widehat{\pi}'')$ is a pair of paths, occupied and vacant, respectively, from (σ_1, u_1) to (v', v''). By construction $\widehat{\pi}', \widehat{\pi}''$ and $\{\sigma_1, u_1\}$ are contained in $\overline{R}(\pi', \pi'')$, so that

$$\overline{R}(\widehat{\pi}', \widehat{\pi}'') \subset \overline{R}(\pi', \pi'')$$

(compare Newman (1951), Theorem 11.1 and its proof). In fact, u' can be connected to ∞ by a path whose only point in $\overline{R}(\pi', \pi'')$ is u, and $u \notin \widehat{\pi}' \cup \widehat{\pi}''$. We therefore even have

$$\overline{R}(\widehat{\pi}', \widehat{\pi}'') \subset \overline{R}(\pi', \pi'') \setminus \{u'\}.$$

It is not clear that $(\widehat{\pi}', \widehat{\pi}'')$ itself has the minimality property corresponding to (5.4), but we take for (π_1', π_1'') the occupied and vacant pair of paths from (σ_1, u_1) to (v', v'') which makes $\overline{R}(\pi_1', \pi_1'')$ minimal. This will automatically satisfy

$$\overline{R}(\pi_1', \pi_1'') \subset \overline{R}(\widehat{\pi}', \widehat{\pi}'') \subset \overline{R}(\pi', \pi'') \setminus \{u'\}.$$

Thus (5.7) will be satisfied and we are done with our recursive step in Case (i).

Case (ii) $\xi_1 = 0$. This time we take σ_1 as a vacant neighbor of u' on π''. However, we do not necessarily take $\sigma_1 = u''$, because it is not clear that u'' has a neighbor on $\widehat{\pi}' := \pi' \setminus \{\text{first edge of } \pi'\}$. Instead, we find σ_1 as follows. Let u_1 be the neighbor of u' on π' (this was called σ_1 in Case (i)). Now consider the six neighbors of u' on $\mathcal{T}$ (see Figure 9). One of these neighbors is u_1, which is occupied. Another one of these neighbors is u'', which is vacant. Now $\{u', u_1\}$ is a side of two triangular faces of $\mathcal{T}$, one of which belongs to $\overline{R}(\pi', \pi'')$. Call the third vertex of this triangle u_2 (the others are u' and u_1). We claim that u_2 must be vacant. To see this we must examine two subcases. We assume that u_2 is occupied. In each case we shall see that this assumption contradicts the minimality property (5.4).

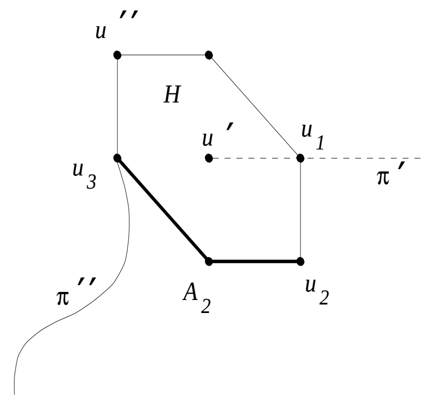

FIGURE 9. The six neighbors of u' (the vertex in the center) and the arc A_2. The dashed path is π', while π'' consists of the edge $\{u'', u_3\}$ plus the solidly drawn path. A_2 is the boldly drawn arc.

Subcase (a) u_2 is a vertex on π'. In this case, replace the piece of π' from u' to u_2 by the single edge $\{u', u_2\}$ This removes at least (the interior of) the triangle (u', u_1, u_2) from $\overline{R}(\pi', \pi'')$, and yields a strictly smaller $\overline{R}$, in contradiction to (5.4).
Subcase (b) u_2 is occupied, but is not on π'. In this subcase we can replace the edge $\{u', u_1\}$ of π' by the two edges $\{u', u_2\}$ and $\{u_2, u_1\}$. This replacement again removes (the interior of) the triangle (u', u_1, u_2) from $\overline{R}(\pi', \pi'')$, and leads to a contradiction as before. This proves that u_2 must be vacant as claimed.

We next claim that there must exist a vacant path π_2 from u_2 to π'' in $\overline{R}(\pi', \pi'')$. If $u_2 \in \pi''$, then this path consists of $\{u_2\}$ only, and the existence of π_2 is clear. If $u_2 \notin \pi''$, then u_2 must lie in $R(\pi', \pi'')$, because it is a vertex of a triangle in $R(\pi', \pi'')$ which does not lie on $\Delta R(\pi', \pi'')$ ($u_2 \notin \pi'$ because u_2 is vacant). Now to find π_2, denote the hexagon whose vertices are the neighbors of u' by H and move from u_2 towards u'' along the arc of the boundary of H which does not contain u_1. We continue till we first hit a vertex, u_3 say, of $\pi' \cup \pi''$. Since $u'' \in \pi''$, we must have $u_3 = u''$ or we must reach u_3 before u''. Let A_2 be the arc of H from u_2 to u_3. Then $\{u', u_2\}$ and A_2, minus the endpoints u' and u_3, lie in $R(\pi', \pi'')$, because $u_2 \in R(\pi', \pi'')$. This shows that $u_3 \in \pi'$ is impossible. Indeed, if $u_3 \in \pi'$, then we can replace the piece of π' from u' via u_1 to u_3 by the single edge from u' to u_3 and so decrease $\overline{R}(\pi', \pi'')$. But then $u_3 \in \pi''$, u_3 itself is vacant, and the path from u_1 to u_3 consisting of $\{u_1, u_2\}$ followed by A_2 lies in $\overline{R}(\pi', \pi'')$. If A_2 is vacant, then we can take A_2 itself for π_2. If not, then apply Proposition 2.2 of Kesten (1982) again. If there does not exist a vacant path π_2 as claimed, then there exists an occupied path π_3 from a vertex in $\mathring{A}_2$ to a vertex in $\mathring{A}_4$, where

$$A_4 = \{v'', v'\}, \text{ followed by the (reversed) piece of } \pi' \text{ from } v' \text{ to } u_1$$
$$\text{followed by the edge } \{u_1, u_2\};$$

this time take $A_1 = \{u_2\}$ and $A_3 =$ piece of π'' from u_3 to v''. But then the path consisting of the edge from u' to the initial vertex of π_3 on $\overset{\circ}{A}_2$ followed by π_3 itself is an occupied path in $\overline{R}(\pi', \pi'')$, which runs from u' to a point of π', but which does not start with the edge $\{u', u_1\}$. This again contradicts the minimality property (5.4). Thus π_3 cannot exist, but π_2 does exist.

We now take $\sigma_1 = u_2$, $\widehat{\pi}' =$ piece of π' from u_1 to v', and $\widehat{\pi}'' = \pi_2$ followed by the piece of π'' from the endpoint of π_2 on π'' to v''. From here on we continue as in Case (i).

We can continue using the procedure of Case (i) or Case (ii) as long as both π', π'' each have at least two points. Assume then that we arrive at the situation where one of them has only one point. For the sake of argument let σ_m be occupied ($\xi_m = 1$) and let the remaining path $\pi'' = \{v''\}$ be a one point path only; π' will be an occupied path from σ_m to the neighbor v' of v''. If now $\xi_{m+1} = 0$, then we simply take as the next point $\sigma_{m+1} = v''$ and we stop our procedure. If, however, $\xi_{m+1} = 1$, then we are in Case (i). The argument there still applies and σ_{m+1} has to be taken as the neighbor of σ_m on π'. Even though π'' has only one vertex, the next vacant path (π_1'' in our notation) may again have more than one vertex, but this causes no problem. At each step $\overline{R}$ becomes strictly smaller, so our process must stop at some time. ■

Corollary 1. *Let (π', π'') be a double path with initial points (u', u'') and endpoints (v', v''). Let $y \in \pi'$. Then for all $\xi = (\xi_1, \xi_2, \dots)$ there exists a λ and a path $(z_0 = y, z_1, \dots, z_\lambda) \subset \overline{R}(\pi', \pi'')$ which starts at y and ends at $z_\lambda \in \{v', v''\}$ such that $(\xi_1, \dots, \xi_\lambda)$ is seen along this path. Moreover $\lambda \geq \min(\|v' - y\|, \|v'' - y\|) - 1$. The same holds when $y \in \pi''$.*

Proof. There is nothing new to prove if $y = u'$ or $y = v'$. So assume $y \in \pi' \setminus \{u', v'\}$. Let $\widehat{\pi}'$ be the piece of π' from y to v'. This is an occupied path. We next construct a vacant path from a neighbor y'' of y to v'' inside $\overline{R}(\pi', \pi'')$. To this end, recall that there exists a neighbor y'' of y such that $\{y, y''\} \subset \overline{R}(\pi', \pi'')$ and such that y'' is connected by a vacant path in $\overline{R}(\pi', \pi'')$, π_1 say, to π'' (see (5.5)). Let the endpoint of π_1 on π'' be z''. Then take for $\widehat{\pi}''$ the concatenation of π_1 and the piece of π'' from z'' to v''. By construction $\widehat{\pi}', \widehat{\pi}'' \subset \overline{R}(\pi', \pi'')$. Hence also $\overline{R}(\widehat{\pi}', \widehat{\pi}'') \subset \overline{R}(\pi', \pi'')$. $(\widehat{\pi}', \widehat{\pi}'')$ is not necessarily a double path, because $\overline{R}(\widehat{\pi}', \widehat{\pi}'')$ may not be minimal, but we can now find a double path in $\overline{R}(\widehat{\pi}', \widehat{\pi}'')$ with the starting points (y, y'') and endpoints (v', v'') by the procedure outlined just before Lemma 3. The Corollary now follows by applying Lemma 3 to this new double path. ■

Unfortunately there are with probability 1 no infinite double paths, because we are working with critical percolation on $\mathcal{T}$. We shall now show, though, that we can find suitable double paths between two successive circuits $\mathcal{C}_k$ and $\mathcal{C}_{k+1}$.

Let $\mathcal{C}_i$ be fixed as C_i for some sequence of circuits with $C_i \subset \overset{\circ}{C}_{i+1}$. We first consider the case that both C_k and C_{k+1} are occupied. Let $v = v^{(k)} \in C_k$. Because C_k is the outermost occupied circuit in $\overset{\circ}{C}_{k+1}$, there exists a vacant path $\mathcal{D} = \mathcal{D}^{(k)}$ from a neighbor w of v to a vertex u' adjacent to C_{k+1} (see Remark 1). Without loss of generality we assume that

$$u' \text{ is the only vertex of } \mathcal{D} \text{ adjacent to } C_{k+1}. \tag{5.8}$$

Since the vacant path $\mathcal{D}$ cannot intersect the occupied circuits C_k and C_{k+1} we may further assume that

$$w \text{ and } \mathcal{D} \subset C_k^{\text{ext}} \cap \overset{\circ}{C}_{k+1}. \tag{5.9}$$

In addition there is no vacant circuit between the successive circuits C_k and C_{k+1}. Therefore, again by Remark 1, there also exists an occupied path ρ connecting some vertex $a \in C_k$ to some vertex b of C_{k+1}. Without loss of generality we may assume that

$$\rho \setminus \{a, b\} \subset C_k^{\text{ext}} \cap \overset{\circ}{C}_{k+1}.$$

We could construct a double path from $\mathcal{D}, \rho$ and pieces of C_k, C_{k+1}. However, we need double paths with some further special properties.

To construct these special double paths we must first fix the constants $c_{30} - c_{33}$ appearing in Proposition 1. We take c_{32} so large that with probability 1

$$\frac{c_{32}}{2} \log\log(\text{diam } (\mathcal{C}_k)) > M_k := \left\lceil \frac{2 \log k}{\log(\beta \wedge (1-\beta))} \right\rceil \text{ eventually} \tag{5.10}$$

(β is the parameter in the measure μ in (1.2).) This can be done by virtue of (3.3). We shall assume in the sequel that k is so large that (5.10) holds. Next we choose two pairs c_{30}, c_{33} and $\widehat{c}_{30}, \widehat{c}_{33}$ in $(0, 1)$ with corresponding c_{31} and $\widehat{c}_{31}$ (see (4.92)) so that the estimate (4.5) holds for each of these choices of the parameters, and so that

$$\widehat{c}_{33}\widehat{c}_{30} > c_{30}. \tag{5.11}$$

Finally we take for j the unique integer with

$$2^j \leq \text{diam } (\mathcal{C}_{k+1}) < 2^{j+1}. \tag{5.12}$$

Now assume that $v = v^{(k)} \in \mathcal{C}_k$ satisfies

$$d(v, C_{k+1}) \geq 2[\text{diam}(C_{k+1})]^{\widehat{c}_{30}} \tag{5.13}$$

and that C_{k+1} is $(\widehat{c}_{30} - \widehat{c}_{33})$-good (with $\widehat{c}_{32} = c_{32}$). We shall now use Proposition 1 to construct (in a deterministic way) under assumptions (5.13), (5.10) and the further assumptions (5.23) and (5.25) below, a double path $(\pi', \pi'') = (\pi'(k,v), \pi''(k,v))$ with the following properties:

$$\text{the initial points of } \pi' \text{ and } \pi'' \text{ are } v \text{ and } w, \text{ respectively;} \tag{5.14}$$

$$\begin{aligned}&\text{the final points of } \pi' \text{ and } \pi'' \text{ are some vertices } y \in C_{k+1}\\ &\qquad\qquad \text{and } u \in \overset{\circ}{C}_{k+1}, \text{ respectively;}\end{aligned} \tag{5.15}$$

$$R(\pi', \pi'') \subset C_k^{\text{ext}} \cap \overset{\circ}{C}_{k+1} \text{ and consequently } \overline{R}(\pi', \pi'') \subset (C_k^{\text{ext}} \cup C_k) \cap \overline{C}_{k+1}; \tag{5.16}$$

$$\begin{aligned}&\text{there exists a vacant path } \theta = \theta^{(k+1)} = (z_1, z_2, \dots, z_{q-1}, u)\\ &\subset C_k^{\text{ext}} \cap \overset{\circ}{C}_{k+1} \text{ such that } \theta \text{ is adjacent to } C_{k+1},\\ &q = \text{ length } (\theta) = M_{k+1}, \text{ and such that } z_1, \dots, z_{q-1} \notin \overline{R}(\pi', \pi'');\end{aligned} \tag{5.17}$$

$$\begin{aligned}&\text{there exists a vacant path } \widehat{\tau} = \widehat{\tau}^{(k)} \text{ from } w \text{ to } \theta \text{ with only its}\\ &\text{endpoint on } \theta \text{ adjacent to } C_{k+1};\end{aligned} \tag{5.18}$$

$$\begin{aligned}&\text{if } v(z_i) \in C_{k+1} \text{ is a point adjacent to } z_i, 1 \le i \le q \text{ (with } z_q = u\\ &\text{and } v(z_q) = v(u) = y), \text{and } V^{(k+1)} = \{v(z_i) : 1 \le i \le q\}, \text{ then}\\ &d(V^{(k+1)}, C_{k+2}) \ge 2[\text{diam } (C_{k+2})]^{c_{30}}.\end{aligned} \tag{5.19}$$

To start our construction, consider the vacant path $\mathcal{D}$ from w to u'. It satisfies (for large k)

$$\begin{aligned}\text{diam } (\mathcal{D}) &\ge \|w - u'\| \ge \|u' - v\| - \|w - v\| \ge d(v, C_{k+1}) - 4\\ &\qquad\qquad\qquad\qquad \text{(because } u' \text{ is adjacent to } C_{k+1})\\ &\ge 2[\text{diam } (C_{k+1})]^{\widehat{c}_{30}} - 4 > [\text{ diam } C_{k+1}]^{\widehat{c}_{30}}.\end{aligned} \tag{5.20}$$

Therefore, the fact that C_{k+1} is good implies (see (4.4)) that there exist [diam $(C_{k+1})]^{\widehat{c}_{31}} \ge 2^{\widehat{c}_{31} j}$ vacant paths $\theta_m \subset C_k^{\text{ext}} \cap \overset{\circ}{C}_{k+1}$ in the vacant cluster of $\mathcal{D}$ in $\overset{\circ}{C}_{k+1}$ which are adjacent to C_{k+1} and satisfy

$$\text{length } (\theta_m) \ge c_{32} \log\log(\text{diam } (C_{k+1})) \ge 2M_{k+1} \tag{5.21}$$

and

$$d(\theta_p, \theta_q) \geq [\text{diam } (C_{k+1})]^{\widehat{c}_{33}\widehat{c}_{30}}, \quad p \neq q.$$

We take $\widehat{c}_{33}\widehat{c}_{30} > \widehat{c} > c_{30}$ so that this implies (for k large enough)

$$d(\theta_p, \theta_q) \geq 16[\text{diam } (C_{k+1})]^{\widehat{c}}, \quad p \neq q. \tag{5.22}$$

For each such θ_m and $z \in \theta_m$ pick a neighbor $v(z)$ of z in C_{k+1} and let $V_m^{(k+1)} = \{v(z) : z \in \theta_m\}$. We now also assume that

$$N(\overline{j}, \widehat{c}) \leq \overline{j}^2 \text{ and } N^{(3)}(\overline{j}, \widehat{c}_{30}) = 0 \text{ for all } \overline{j} \geq j. \tag{5.23}$$

We know from Lemma 6 that with probability 1 this holds for large j. Also for large j, $2^{\widehat{c}_{31}j} > j^2$, so that by Lemma 7 for at least one m

$$d(V_m^{(k+1)}, C_{k+2}) > 2^{\widehat{c}j}. \tag{5.24}$$

Pick such an m and denote θ_m by $\widehat{\theta}$. Let $\widehat{\theta} = (z_1, \dots, z_p)$, $p \geq 2M_{k+1}$. It is part of (4.3) that there exists a vacant path $\widehat{\tau}$ from some vertex $z \in \mathcal{D}$ to some vertex $z_q \in \widehat{\theta}$ such that z_q is the only vertex of $\widehat{\tau}$ adjacent to C_{k+1}. We can always extend $\widehat{\tau}$ by the piece of $\mathcal{D}$ from z to w, and at the other end we may stop at the first vertex of $\widehat{\tau}$ on $\widehat{\theta}$. Thus we may assume without loss of generality that $z = w$ and

$$\widehat{\tau} \setminus \{z_q\} \text{ is disjoint from } \widehat{\theta} \text{ and contains no vertex adjacent to } C_{k+1}.$$

Also without loss of generality we assume that the vertices of $\widehat{\theta}$ are indexed such that $q \geq M_{k+1}$.

Now consider the following two paths from C_k to C_{k+1} which lie in $C_k^{\text{ext}} \cap \overset{\circ}{C}_{k+1}$, except for their initial and final points: the path τ, which consists of the edge from v to w, followed by $\widehat{\tau}$, and finally an edge from the endpoint z_q of $\widehat{\tau}$ to $y := v(z_q) \in C_{k+1}$ (such a neighbor y of z_q exists because $\widehat{\theta}$ is adjacent to C_{k+1}); and the path ρ, which runs from $a \in C_k$ to $b \in C_{k+1}$. These two paths divide the ring $C_k^{\text{ext}} \cap \overset{\circ}{C}_{k+1}$ into two components, each of which is bounded by τ, ρ and different arcs of C_k between v and a and of C_{k+1} between b and y. One of these components contains $\{z_1, \dots, z_{q-1}\}$, because ρ does not intersect $\widehat{\theta}$ at all (recall that $\widehat{\theta}$ is vacant and ρ is occupied) and τ intersects $\widehat{\theta}$ only in z_q. We denote by W the component of $(C_k^{\text{ext}} \cap \overset{\circ}{C}_{k+1}) \setminus (\tau \cup \rho)$ which does *not* contain $z_1, \dots, z_{q-1}$ (see Figure 10).

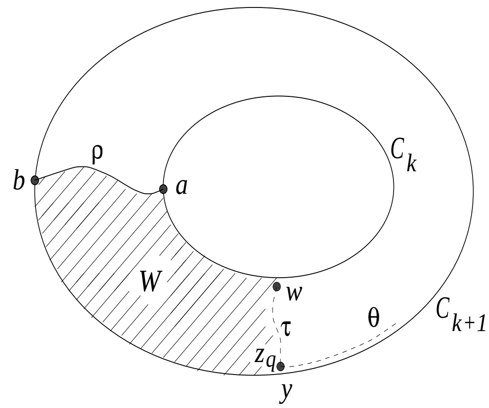

FIGURE 10. Illustration of the region W. The circuits C_k, C_{k+1} and the path ρ are occupied. τ and θ are vacant.

Finally we define

$$\widehat{\pi}' = \text{the path consisting of the arc of } C_k \text{ from } v \text{ to } a \text{ in the boundary of } W, \text{ followed by } \rho \text{ and finally the arc of } C_{k+1} \text{ from } b \text{ to } y \text{ in the boundary of } W,$$

$$\widehat{\pi}'' = \widehat{\tau}.$$

These paths are occupied and vacant, respectively, and their initial points, v, w are adjacent, and so are their final points y, z_q. Moreover the construction of our paths is such that $\overline{R}(\widehat{\pi}', \widehat{\pi}'') = W \cup \Delta W$ and therefore $z_1, \dots, z_{q-1}$ lie outside $\overline{R}(\widehat{\pi}', \widehat{\pi}'')$. Finally take for π' and π'' an occupied and vacant path from v to y and from w to z_q, respectively, which lie in $\overline{R}(\widehat{\pi}', \widehat{\pi}'')$ and form a double path. These paths have the properties (5.14)-(5.16) by construction, with the choice z_q for u. (5.17) holds if we take $\theta = (z_{q-M_{k+1}+1}, z_{q-M_{k+1}+2}, \dots, z_q = u)$ and renumber its vertices. Condition (5.18) was already verified right after (5.24). Lastly, let $V^{(k+1)} = V_m^{(k+1)}$ for the m for which $\widehat{\theta} = \theta_m$. To obtain (5.19) we shall further assume that

$$\text{diam } (C_{k+i+1}) \le (k+i)^{c_{10}+2} \text{diam } (C_{k+i}), \quad i \ge 0. \tag{5.25}$$

We shall show in the next section that this is justified because this holds with probability 1 for all large k. (5.24), (5.12), (5.25) and $\widehat{c} > c_{30}$ show that for large j

$$d(V^{(k+1)}, C_{k+2}) \ge 2[\text{diam } (C_{k+2})]^{c_{30}},$$

which gives (5.19). This completes the construction of our special double path when C_k and C_{k+1} are both occupied (or both vacant) and when (5.13), (5.23) and (5.25) hold.

If C_{k+2} is also occupied, then we can repeat the above construction with k replaced by $k+1$ and for any $v^{(k+1)} \in V^{(k+1)}$. (5.19) takes the place of (5.13). However, the exponent $\widehat{c}_{30}$ in (5.13) has been replaced by the smaller c_{30}, and hence we are only able to construct a double path and sets $\theta^{(k+2)}, V^{(k+2)}$ which satisfy (5.14)-(5.17) with k replaced by $k+1$, but with (compare (5.24))

$$d(V^{(k+2)}, C_{k+3}) \geq 2[\text{diam } (C_{k+3})]^{c}$$

for any $c < c_{33}c_{30}$, instead of (5.19). Thus with each repetion of the construction our estimate seems to deteriorate. We shall now prove that we can actually obtain

$$d(V^{(k+2)}, C_{k+3}) \geq 2[\text{diam } (C_{k+3})]^{c_{30}}, \tag{5.26}$$

so that our estimate does not deteriorate after the first repetion, and we can continue to find double paths from $v^{(k+i)} \in V^{(k+i)} \subset C_{k+i}$ and can satisfy (5.26) with $k+2$ replaced by $k+i$.

To see (5.26), assume we picked $v^{(k+1)} \in V^{(k+1)}$ and then a neighbor $w^{(k+1)} \in C_{k+1}^{\text{ext}} \cap \overset{\circ}{C}_{k+2}$ of $v^{(k+1)}$ so that $w^{(k+1)}$ has a vacant connection to a neighbor of C_{k+2}. Then find a tentative $\widehat{\theta}^{(k+2)}$ and $\widehat{V}^{(k+2)} \subset C_{k+2}$ as in the preceding construction with k replaced by $k+1$. By (5.18) this comes with a vacant path $\widehat{\tau}^{(k+1)}$ from $w^{(k+1)}$ to some vertex in $\widehat{\theta}^{(k+2)}$, such that this vertex is the only vertex of $\widehat{\tau}^{(k+1)}$ adjacent to C_{k+2}. We identify $\mathcal{D}^{(k+1)}$ with $\widehat{\tau}^{(k+1)}$. Now our construction only guarantees that

$$\text{diam } (\mathcal{D}^{(k+1)}) \geq d(v^{(k+1)}, C_{k+2}) - 4 \geq 2[\text{diam } (C_{k+2})]^{c_{30}} - 4.$$

However, if actually the better estimate

$$\text{diam } (\mathcal{D}^{(k+1)}) \geq 2[\text{diam } (C_{k+2})]^{\widehat{c}_{30}} - M_{k+2} - 4 > [\text{diam } (C_{k+2})]^{\widehat{c}_{30}}$$

holds, then we can simply go through the construction with k replaced by $k+1$ to find a $V^{(k+2)}$ which satisfies (5.26). (Note that our construction did not use (5.13) itself, but only its consequence (5.20).) We therefore only have to consider the case when

$$d(V^{(k+1)}, C_{k+2}) - 4 \leq \text{diam } (\mathcal{D}^{(k+1)}) < 2[\text{diam } (C_{k+2})]^{\widehat{c}_{30}} - M_{k+2} - 4.$$

In this case we have for any $v^{(k+2)} \in \widehat{V}^{(k+2)}$ some $z \in \widehat{\theta}^{(k+2)}$ adjacent to $v^{(k+2)}$ and

$$\begin{aligned}
&\|v^{(k+1)} - v^{(k+2)}\| \leq \|v^{(k+1)} - z\| + 2 \\
&\leq \text{diam } (\mathcal{D}^{(k+1)}) + \text{ length}(\widehat{\theta}^{(k+2)}) + 2 < 2[\text{diam } (C_{k+2})]^{\widehat{c}_{30}}.
\end{aligned}$$

We claim that in this case we must even have

$$d(v^{(k+2)}, C_{k+3}) \geq 2[\text{diam } (C_{k+2})]^{\widehat{c}_{30}}. \tag{5.27}$$

Indeed, if this fails, then each of C_{k+1}, C_{k+2} and C_{k+3} must have a point in

$$v^{(k+2)} + S(2[\text{diam } (C_{k+2})]^{\widehat{c}_{30}}).$$

Therefore, if

$$2^{\overline{j}} \leq \text{diam } (C_{k+2}) < 2^{\overline{j}+1},$$

then there would exist some r, s so that

$$v^{(k+2)} \in [r2^{\widehat{c}_{30}(\overline{j}+1)}, (r+1)2^{\widehat{c}_{30}(\overline{j}+1)}] \times [s2^{\widehat{c}_{30}(\overline{j}+1)}, (s+1)2^{\widehat{c}_{30}(\overline{j}+1)}].$$

Therefore, if $\widehat{c}_{30}q \geq 4$,

$$\begin{aligned} &T(\lfloor 2^{-\widehat{c}_{30}q}(r-2)\rfloor, \lfloor 2^{-\widehat{c}_{30}q}(s-2)\rfloor; \widehat{c}_{30}, \overline{j}+q) \\ &= [\lfloor 2^{-\widehat{c}_{30}q}(r-2)\rfloor 2^{\widehat{c}_{30}(\overline{j}+q+1)}, (\lfloor 2^{-\widehat{c}_{30}q}(r-2)\rfloor + 3)2^{\widehat{c}_{30}(\overline{j}+q+1)}] \\ &\qquad\qquad \times [\lfloor 2^{-\widehat{c}_{30}q}(s-2)\rfloor 2^{\widehat{c}_{30}(\overline{j}+q+1)}, (\lfloor 2^{-\widehat{c}_{30}q}(s-2)\rfloor + 3)2^{\widehat{c}_{30}(\overline{j}+q+1)}] \\ &\supset [r2^{\widehat{c}_{30}(\overline{j}+1)}, (r+1)2^{\widehat{c}_{30}(\overline{j}+1)}] \times [s2^{\widehat{c}_{30}(\overline{j}+1)}, (s+1)2^{\widehat{c}_{30}(\overline{j}+1)}] + S(2 \cdot 2^{\widehat{c}_{30}(\overline{j}+1)}) \end{aligned}$$

would intersect $S(2^{\overline{j}+1}) \subset S(2^{\overline{j}+q+1})$, as well as $C_{k+1}, C_{k+2}, C_{k+3}$. Since diam$(C_{k+3})$ $\geq$ diam $(C_{k+2}) \geq 2^{\overline{j}}$ and diam$(C_{k+1}) \geq (k+1)^{-c_{10}-2}$ diam$(C_{k+2}) \geq 2^{\overline{j}-(\log \overline{j})^2}$ for large $\overline{j}$, we would have $N^{(3)}(\overline{j}+q, \widehat{c}_{30}) \neq 0$. This is ruled out by (5.23), so that (5.27) must hold. Since this holds for any $v^{(k+2)} \in \widehat{V}^{(k+2)}$ this implies (5.26) with $\theta^{(k+2)}, V^{(k+2)}$ taken equal to $\widehat{\theta}^{(k+2)}, \widehat{V}^{(k+2)}$.

We have described a construction of double paths between two successive circuits which are occupied. Of course this works equally well (after an interchange of 'occupied' and 'vacant') between two successive circuits which are vacant. We now briefly describe how to find a double path with the properties (5.14) -(5.17) and (5.19) when C_k is vacant and C_{k+1} is occupied. In fact we obtain (5.19) for any fixed $c_{30} < 1$ for large k even without the assumption (5.13). We merely need that the simpler (3.28) holds. The treatment of vertices adjacent to C_{k+1} will be somewhat different from the preceding construction and this is why (3.28) comes in.

In the present situation we start again with a vertex $v = v^{(k)} \in C_k$. Since C_k is the outermost vacant circuit in $\overset{\circ}{C}_{k+1}$, there is an occupied path τ from a neighbor w of v to a vertex $x \in C_{k+1}$. The occupied τ cannot intersect the vacant circuit C_k and we may stop τ at its first vertex on C_{k+1}. We then have

$$w \text{ and } \tau \setminus \{x\} \subset C_k^{\text{ext}} \cap \overset{\circ}{C}_{k+1}.$$

In addition there is no occupied circuit between C_k and C_{k+1}. Therefore, again by Remark 1, there also exists a vacant path ρ' from some vertex in C_k to a vertex adjacent to C_{k+1}. Let ρ be the piece of ρ' from its last intersection with C_k to its first point adjacent to C_{k+1}. Let ρ run from $a \in C_k$ to z' adjacent to C_{k+1}. Then

$$\begin{aligned}&\rho \setminus \{z'\} \text{ does not contain any vertex adjacent to } C_{k+1}\\ &\text{and } \rho \setminus \{a\} \subset C_k^{\text{ext}} \cap \overset{\circ}{C}_{k+1}.\end{aligned}$$

Now define $\widehat{\mathcal{D}}$ as the path consisting of (the reverse of) ρ plus an arc of C_k starting at a and of diameter $\geq \frac{1}{2}$ diam (C_k). This path lies in $\overset{\circ}{C}_{k+1}$ but it may contain vertices adjacent to C_{k+1} other than z'. However, the number of vertices on $\widehat{\mathcal{D}}$ adjacent to C_{k+1} cannot exceed

$$1+ \text{ (the number of vertices of } C_k \text{ adjacent to } C_{k+1}) \leq 1 + k^2,$$

provided (3.28) holds. Thus $\widehat{\mathcal{D}}$ contains a subpath $\mathcal{D}$ which contains exactly one vertex adjacent to C_{k+1} and satisfies

$$\begin{aligned}\text{diam } (\mathcal{D}) &\geq \frac{1}{2k^2} \text{diam } (\widehat{\mathcal{D}})\\ &\geq \frac{1}{4k^2} \text{diam } (C_k) \geq 2[\text{diam } (C_{k+1})]^{\widehat{c}_{30}} \text{ (by (5.25))}.\end{aligned}$$

Indeed, we can take for $\mathcal{D}$ a piece of $\widehat{\mathcal{D}}$ between two successive points adjacent to C_{k+1} (and including one of these points) of maximal diameter. As before, we can now apply Proposition 1 to $\mathcal{D}$ and then use Lemma 7 to find in the vacant cluster of $\mathcal{D}$ in $\overset{\circ}{C}_{k+1}$ a path $\widehat{\theta}^{(k+1)}$ and a set $V^{(k+1)} \in C_{k+1}$ such that

$$\widehat{\theta}^{(k+1)} \text{ is adjacent to } V^{(k+1)} \subset C_{k+1}$$

and such that (5.24) holds. However, $\mathcal{D}$ is a piece of $\widehat{\mathcal{D}}$, which is part of the vacant cluster of v. Therefore $\widehat{\theta}$ is also in the vacant cluster of v in $\overset{\circ}{C}_{k+1}$. From here on we continue essentially as from (5.24). We merely point out that for $\widehat{\pi}''$ we shall take a path from v to a vertex z_q of $\widehat{\theta}^{(k+1)}$ in $(C_k^{\text{ext}} \cup C_k) \cap \overset{\circ}{C}_{k+1}$ in the vacant cluster of v (or equivalently of $\mathcal{D}$); we can take this path in $C_k^{\text{ext}} \cup C_k$ because we can always replace the piece from v to the last intersection of $\widehat{\pi}''$ with C_k by an arc of C_k. (We are not claiming this time that z_q is the only point of $\widehat{\pi}''$ which is adjacent to C_{k+1}.) For $\widehat{\pi}'$ we shall take a path consisting of the path τ from w to $x \in C_{k+1}$ followed by a suitable arc of C_{k+1} from x to a neighbor y of z_q.

This arc can be chosen so that $\overline{R}(\widehat{\pi}', \widehat{\pi}'') \subset (C_k^{\text{ext}} \cup C_k) \cap \overline{C}_{k+1}$. (5.16) will be a consequence of this.

This completes our construction of special double paths. It shows that if (5.13) holds for some k_0 and some v_{k_0} and (5.25), (3.28) hold for all $k \geq k_0$ and if (5.23) holds for the corresponding j (see (5.12)), then one can successively choose $\theta^{(k+i)}, V^{(k+i)}$ and for $v \in V^{(k+i)}$ a special double path $(\pi'(k+i, v), \pi''(k+i, v)$ so that (5.14)-(5.17) with k replaced by $k+i$ hold.

6. Synthesis.

We now put the various lemmas about circuits and double paths together to show that the probability of seeing a random word on $\mathcal{T}$ is bounded away from 0. Our strategy is to first choose all the $\mathcal{C}_k$ and to decide whether they are occupied or vacant. In other words, we shall condition on

$$\{\mathcal{C}_k = C_k, \kappa(C_k) = \varepsilon_k, k \geq 1\}. \tag{6.1}$$

Here $\{C_k\}_{k\geq 1}$ is a sequence of circuits for which $C_k \subset \overset{\circ}{C}_{k+1}$ and $\varepsilon_k \in \{0, 1\}$. After having chosen $\mathcal{C}_k$ and $\kappa(\mathcal{C}_k)$ we choose the vertices not on $\cup_1^\infty C_k$. We shall show that by ignoring a P-null set we may assume that the following conditions hold for $k \geq$ some (random) k_0, respectively $j \geq$ some (random) j_0 and fixed $\overline{c} > \widehat{c}_{30}$, $\widehat{c} \in (c_{30}, \widehat{c}_{33}\widehat{c}_{30})$:

$$C_k \text{ is good;} \tag{6.2}$$

$$\text{diam } (C_{k+1}) \leq \text{diam } (C_k)[\log(\text{diam } (C_k)]^{c_{10}+1}; \tag{6.3}$$

$$N(j, \widehat{c}) \leq j^2 \text{ and } N(j, \overline{c}) \leq j^2; \tag{6.4}$$

$$N^{(3)}(j, \widehat{c}_{30}) = 0; \tag{6.5}$$

and finally (3.28) and (5.10). For an occupancy configuration which satisfies (6.3)-(6.5), (3.28) and (5.10) we now make a number of choices recursively. At the k-th stage we will have singled out a vertex $v^k \in C_k$ and a double path

$$(\pi'(k), \pi''(k)) = (\pi'(k, v^{(k)}), \pi''(k, v^{(k)})) \subset (C_k^{\text{ext}} \cup C_k) \cap \overline{C}_{k+1}$$

such that

$$v^{(k)} \in \Delta\overline{R}(\pi'(k, v^{(k)}), \pi''(k, v^{(k)})).$$

The endpoints of $\pi'(k)$ and $\pi''(k)$ are on C_{k+1} or adjacent to C_{k+1}, depending on whether C_{k+1} is occupied or vacant. For the sake of argument let us describe the situation when C_{k+1} is occupied. Then the endpoint $y^{(k+1)}$ of $\pi'(k)$ will lie on C_{k+1} and the endpoint $u^{(k+1)}$ of $\pi''(k)$ will be in $C_k^{\text{ext}} \cap \overset{\circ}{C}_{k+1}$, adjacent to C_{k+1}. Associated with $(\pi'(k), \pi''(k))$ will be a vacant path $\theta^{(k+1)} = (z_1^{(k+1)}, \dots, z_{M_{k+1}}^{(k+1)})$

which is adjacent to C_{k+1}, ends at $z^{(k+1)}_{M_{k+1}} = u^{(k+1)}$ and is such that $\theta^{(k+1)} \setminus \{u^{(k+1)}\}$ lies outside $\overline{R}(\pi'(k), \pi''(k))$. Moreover (5.19) will hold.

We now concentrate on the case where ξ is random with the distribution μ of (1.2). A few words to handle the case of a deterministic ξ with bounded $r_m(\xi)$ will be said at the end. Let

$$\mathcal{F}_n = \sigma\text{-field in } \Xi \text{ generated by } \xi_1, \dots, \xi_n.$$

To avoid double subscripts we shall write $\xi(i)$ instead of ξ_i in the remainder of this section. At stage k we will also have found an $\{\mathcal{F}_n\}$-stopping time λ_k and we will have chosen $\xi(1), \dots, \xi(\lambda_k)$ and a path $(w(0), \dots, w(\lambda_k))$ such that

$$(w(0), \dots, w(\lambda_k)) \text{ is a selfavoiding path;} \tag{6.6}$$

$$w(i) \text{ occupied or vacant according as } \xi_i = 1 \text{ or } \xi_i = 0, \quad 1 \le i \le \lambda_k; \tag{6.7}$$

$$w(\lambda_k) = v^{(k)} \in \Delta\overline{R}(\pi'(k), \pi''(k)) \tag{6.8}$$

but

$$w(i) \notin \overline{R}(\pi'(k), \pi''(k)) \cup C_k^{\text{ext}} \text{ for } i < \lambda_k. \tag{6.9}$$

We will have carried out the construction in such a way that

$$\begin{aligned}&\text{the construction of } (w(0), w(1), \dots, w(\lambda_k)) \text{ depends only on}\\ &\xi(1), \xi(2), \dots, \xi(\lambda_k), \text{ and conditionally on this information}\\ &\xi(\lambda_k + 1), \xi(\lambda_k + 2), \dots \text{ still have the distribution } \mu.\end{aligned} \tag{6.10}$$

The $(k+1)$-th stage will consist of first choosing the set $V^{k+1} \subset C_{k+1}$ of neighbors to the points in $\theta^{(k+1)}$ as described in (5.19) and then for each $v \in V^{(k+1)}$ construct the special double path $(\pi'(k+1, v), \pi''(k+1, v))$ described in the preceding section. We next need to choose for the $(k+1)$-th stage an $\{\mathcal{F}_n\}$-stopping time λ_{k+1} and the path

$$\gamma_{k+1} := (w(\lambda_k + 1), w(\lambda_k + 2), \dots, w(\lambda_{k+1}))$$

so that (6.6) and (6.7) with $k+1$ instead of k hold. At the same time we will have to find the new $v^{(k+1)}, (\pi'(k+1, v^{(k+1)}), \pi''(k+1, v^{(k+1)})) \subset (C^{\text{ext}}_{k+1} \cup C_{k+1}) \cap \overline{C}_{k+2}$ so that also (6.8) and (6.9) with $k+1$ instead of k hold. Together with this we must also find $\theta^{(k+2)}$ so that (5.17) holds with k replaced by $k+1$. We shall start this all at some k_0 with $\lambda(k_0) = 0$ and some vertex $w(0) \in C_{k_0}$ and with

$$d(w(0), C_{k_0+1}) \ge 2[\text{diam } (C_{k_0+1})]^{\widehat{c}_{30}}. \tag{6.11}$$

We shall make the choice for the starting parameters explicit when we fill in the details. First we note that if we are successful with our construction at every stage, then we see $\xi = (\xi_1, \xi_2, \dots)$ along the infinite selfavoiding path $(w(0), w(1), \dots)$ which is the concatenation of the γ_k. The principal estimate will be that the conditional probability of successfully choosing $(\xi(\lambda_k+1), \dots, \xi(\lambda_{k+1}))$ and the path γ_{k+1} in the above manner, given the occupancy configuration, λ_k and $\xi(1), \dots, \xi(\lambda_k)$, is at least $1 - 1/k^2$. This will then guarantee that the probability of seeing the random word $(\xi(1), \xi(2), \dots)$ from some point $w(0)$ is at least

$$\prod_{k_0}^{\infty}(1 - \frac{1}{k^2}) \geq \frac{1}{2} \tag{6.12}$$

(if k_0 is chosen sufficiently large). By the zero-one law (1.3) this will then show that

$$\mu\{\xi : \rho(\xi) = 1\} = 1$$

and complete the proof of our Theorem.

Let us fill in the remaining details. Most of the work has been done in Sections 2-5 already. We fix $\widehat{c} \in (c_{30}, \widehat{c}_{33}\widehat{c}_{30})$ and $\overline{c} \in (\widehat{c}_{30}, 1)$. First we prove that we may assume (6.2)-(6.5), (3.28) and (5.10). Of these, (6.4), (6.5) and (3.28) are explicitly stated in Lemma 6, and (6.2) follows from Proposition 1. As for (6.3), this is almost immediate from Lemma 4. Indeed, for large k,

$$\begin{aligned}\text{diam }(\mathcal{C}_{k+1}) &\leq 2 \max_{x \in \mathcal{C}_{k+1}} \|x\| \leq 2\frac{(k+1)^{c_{10}}}{c_9} \min_{x \in \mathcal{C}_k} \|x\| \text{ (by (3.5))} \\ &\leq 2\frac{(k+1)^{c_{10}}}{c_9} \text{ diam }(\mathcal{C}_k) \text{ (recall that } \mathcal{C}_k \text{ surrounds } \mathbf{0}) \\ &\leq c_{52}[\log(\text{diam }(\mathcal{C}_k))]^{c_{10}} \text{ diam }(\mathcal{C}_k) \text{ (by (3.3)).}\end{aligned}$$

Finally, (5.10) follows from (3.3) when c_{32} is sufficiently large. From now on we assume that (6.2)-(6.5), (3.28) and (5.10) hold for $k \geq k_0, j \geq j_0$, respectively, where we assume that j_0 satisfies (5.12) with k_0 for k. If necessary we raise k_0 so that

$$\prod_{k_0}^{\infty}(1 - \frac{1}{k^2}) \geq \frac{1}{2}, \quad 2^{j_0}/(8 \cdot 2^{\overline{c}j_0}\sqrt{2} + 1) > j_0^2 \text{ and } 2^{\overline{c}j_0} \geq 8 \cdot 2^{\widehat{c}_{30}j_0}.$$

We now begin finding $(w(0), w(1), \dots)$ and λ_k satisfying (6.6)-(6.10) by an application of Lemma 7 with $V_m^{(k_0)} = \{v_m\}$ for a collection of vertices $v_m \in C_{k_0}$ with

$$\|v_p - v_q\| \geq 8 \cdot 2^{\overline{c}j_0} \text{ for } p \neq q.$$

Since diam $(C_{k_0}) \geq 2^{j_0}$, we can find at least $2^{j_0}/(8 \cdot 2^{\overline{c} j_0}\sqrt{2}+1) > j_0^2$ such vertices. Then by virtue of (6.4) and Lemma 7 there exists at least one v_m with

$$d(v_m, C_{k_0+1}) \geq 2^{\overline{c} j_0} \geq 8 \cdot 2^{\widehat{c}_{30} j_0}.$$

We take $w(0) \in C_{k_0}$ as one of those v_m.

We next use the method described after (5.13) with $w(0)$ taking the role of $v^{(k)}$ there, and $k = k_0$. We then find a double path $(\pi'(k_0), \pi''(k_0))$, a path $\theta^{(k_0+1)}$ and a set $V^{(k_0+1)} \in C_{k_0+1}$ with the properties (5.14)-(5.17) and (5.19) in case C_{k_0+1} is occupied, and with these properties with 'occupied' and 'vacant' interchanged in case C_{k_0+1} is vacant. Now we pick for each $v \in V^{(k_0+1)}$ a corresponding special double path $(\pi'(k+1, v), \pi''(k+1, v))$, again following the recipe at the end of the last section, so that these paths satisfy (5.14)-(5.17) and (5.19) with k replaced by $k+1$. Now define

$$\mathcal{R}^{(k+1)} = \bigcup_{v \in V^{(k+1)}} \overline{R}(\pi'(k+1, v), \pi''(k+1, v)). \tag{6.13}$$

At the moment we only know what this means for $k = k_0$, but the same definition will apply later for $k > k_0$. We also take $\lambda_{k_0} = 0$. (We don't need λ_i for $i < k_0$, but if desired one can set those $\lambda_i = 0$.) Now we choose $\xi(1), \xi(2), \dots$ one at a time and a corresponding path $(w(0), w(1), \dots)$ in $\overline{R}(\pi'(k_0), \pi''(k_0))$ such that (6.6) and (6.7) hold, following the algorithm of Lemma 8. This lemma tells us that we can continue with this until the w-path reaches one of the final points of $\pi'(k_0, w(0))$ or $\pi''(k_0, w(0))$. However, for the present purposes we may want to stop earlier. We define the $\{\mathcal{F}_n\}$-stopping time

$$\nu_{k+1} = \min\{n : w(n) = \text{ final point of } \pi'(k, v^{(k)}) \text{ or } \pi''(k, v^{(k)}) \text{ or } w(n) \in \mathcal{R}^{(k+1)}\}. \tag{6.14}$$

Again we only do this for $k = k_0$ with $v^{(k_0)} = w(0)$ at the moment, but we will use the same definition for general k later. If

$$w(\nu_{k_0+1}) \in \mathcal{R}^{(k_0+1)}, \tag{6.15}$$

then we take $\lambda_{k_0+1} = \nu_{k_0+1}$. By the algorithm of Lemma 8, we still have

$$w(i) \in \overline{R}(\pi'(k_0, v^{(k_0)}), \pi''(k_0, v^{(k_0)})), \quad i \leq \nu_{k_0+1}. \tag{6.16}$$

Moreover, (6.15) says that there exists some $v^{(k_0+1)} \in V^{(k_0+1)}$ so that

$$w(\nu_{k_0+1}) \in \overline{R}(\pi'(k_0+1, v^{(k_0+1)}), \pi''(k_0+1, v^{(k_0+1)})).$$

Also by definition of ν

$$w(i) \notin \overline{R}(\pi'(k_0+1, v^{(k_0+1)}), \pi''(k_0+1, v^{(k_0+1)})) \text{ for } i < \nu_{k_0+1},$$

so that (6.9) holds for $k = k_0 + 1$ (recall (5.16) for $k = k_0$) and also

$$w(\nu_{k_0+1}) \in \Delta\overline{R}(\pi'(k_0+1, v^{(k_0+1)}), \pi''(k_0+1, v^{(k_0+1)})). \tag{6.17}$$

In turn, we get from this and (6.16) that

$$\begin{aligned} &w(\nu_{k_0+1}) \\ &\in \overline{R}(\pi'(k_0, v^{(k_0)}), \pi''(k_0, v^{(k_0)})) \cap \Delta\overline{R}(\pi'(k_0+1, v^{(k_0+1)}), \pi''(k_0+1, v^{(k_0+1)})) \\ &\subset C_{k_0+1} \text{ (again use (5.16))}. \end{aligned}$$

We shall choose future points of $w(\cdot)$ in $C^{\text{ext}}_{k_0+1} \cup C_{k_0+1}; w(\cdot)$ will not return to $\overset{\circ}{C}_{k_0+1}$ after time ν_{k_0+1}.

The second possibility in (6.14) is that $w(\nu_{k_0+1})$ is a final point of $\pi'(k_0, v^{(k_0)})$ or of $\pi''(k_0, v^{(k_0)})$. To discuss this situation more explicitly let us assume C_{k_0+1} is occupied. Then the properties (5.14)-(5.17) and (5.19) tell us that the occupied path $\pi'(k_0, v^{(k_0)})$ has its final point at $y = v(z_q)$ in $V^{(k_0+1)}$ for some $z_q \in \theta^{(k_0+1)}$ (with $q = M_{k_0+1}$), and this y is a point of $\pi'(k_0+1, v(z_q)) \subset \overline{R}(\pi'(k_0+1, v(z_q)), \pi''(k_0+1, v(z_q)))$. Thus, if $w(\nu_{k_0+1}) = y$, then we still are in the situation (6.15). The only case left to consider is when

$$w(\nu_{k_0+1}) = \text{ the final point } z_q \text{ of the vacant path } \pi''(k_0, v^{(k_0)}). \tag{6.18}$$

Since we chose our path so that (6.7) holds for $i \le \nu_{k_0+1}$, this can happen only when $\xi(\nu_{k_0+1}) = 0$. We now successively examine $\xi(i)$ for $i > \nu_{k_0+1}$ until we come to the smallest $r \ge 1$ with $\xi(\nu_{k_0} + r) = 1$. We say that we have *success at the* (k_0+1)*-th stage* if either (6.15) holds or (6.18) holds and the r there is $\le M_{k_0+1}$. If we are not successful at the (k_0+1)-th stage, then we stop our construction and we give up on seeing ξ from $w(0)$. If (6.18) holds and $r \le M_{k_0+1}$, then we define $\lambda_{k_0+1} = \nu_{k_0+1} + r$ and take

$$\begin{aligned} &w(\nu_{k_0+i}) = z_{q-i}, \quad 1 \le i < r, \\ &w(\nu_{k_0+r}) = v(z_q - r + 1) \in V^{(k_0+1)}. \end{aligned} \tag{6.19}$$

Let us check that (6.6)-(6.10) hold for $k = k_0 + 1$ with this choice. This is easy when (6.15) applies and will be left to the reader in this case. Assume then that (6.15) does not hold, but (6.18) prevails. Then (6.6) holds because $\theta^{(k_0+1)} = (z_1, \dots, z_q)$, and $\theta^{(k_0+1)} \setminus \{z_q\}$ lies outside $\overline{R}(\pi'(k_0, v^{(k_0)}), \pi''(k_0, v^{(k_0)}))$

so that $(w(0), \dots, w(\lambda_{k_0+1} - 1))$ is a selfavoiding path. Even $w(\lambda_{k_0+1})$ has not been visited before, because $w(\lambda_{k_0+1}) = w(\nu_{k_0+1} + r) \in V^{(k_0+1)} \subset \mathcal{R}^{(k_0+1)}$ and $w(i) \notin \mathcal{R}^{(k_0+1)}$ for $i < \lambda_{k_0+1}$ (recall that (6.15) does not hold and $\theta^{(k_0+1)} \subset \mathring{C}_{k_0+1}$, while $\mathcal{R}^{(k_0+1)} \subset C_{k_0+1} \cup C^{\text{ext}}_{k_0+1}$). Next, (6.7) holds by construction, since C_{k_0+1} was occupied and $\theta^{(k_0+1)}$ was vacant. (6.8) is automatic if we take $v^{(k_0+1)} = w(\lambda_{k_0+1})$ and $\pi'(k_0+1) = \pi'(k_0+1, v^{(k_0+1)})$ and similarly for π''. In this case also (6.9) holds because we assumed that (6.15) does not occur and $\theta^{(k_0+1)} \subset \mathring{C}_{k_0+1}$, and therefore is disjoint from

$$\overline{R}(\pi'(k_0+1, v^{(k_0+1)}), \pi''(k_0+1, v^{(k_0+1)})) \cup C^{\text{ext}}_{k_0+1} \subset C_{k_0+1} \cup C^{\text{ext}}_{k_0+1}.$$

(6.10) is clear from the description of the selections which we made.

It is also clear that

$$\begin{aligned} &\mu\{\text{no success at the } (k_0+1)\text{-th stage}|\text{full occupancy configuration and } \mathcal{F}_{\lambda_{k_0}}\} \\ &\le \mu\{\xi(\nu_{k_0+1} + i) = 0,\ 1 \le i \le M_{k_0+1}|\text{full occupancy configuration and } \mathcal{F}_{\lambda_{k_0}}\} \\ &\le [\beta \wedge (1-\beta)]^{-M_{k_0+1}} \le \frac{1}{(k+1)^2} \text{ (see (5.10)).} \end{aligned} \tag{6.20}$$

If we were successful at stage $k_0 + 1$, then we can now go on and repeat the above steps with k_0 replaced by $k_0+1, k_0+2, \dots$ as long as we have success. Only one step deserves further comment. In analogy with (6.17) we will have (6.8) for $k > k_0$, but there is no reason to believe that $w(\lambda_k)$ will always be an initial point of $\pi'(k)$ or $\pi''(k)$. If this is not the case for a certain k, then for choosing γ_{k+1} we should apply Corollary 1 instead of Lemma 8. See Figure 11 for a typical step.

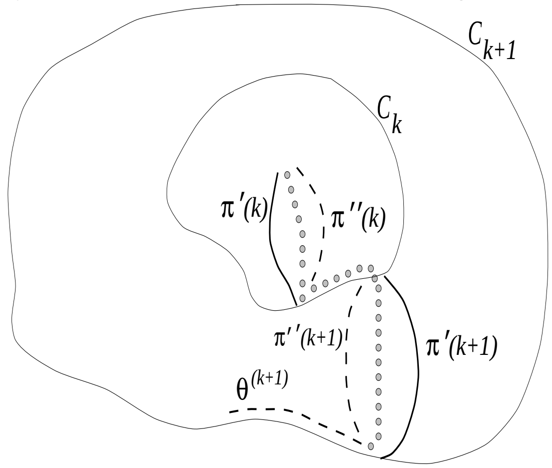

FIGURE 11. Construction of the path $\circ \circ \circ \circ$ along which a given word ξ is seen.

If we are successful at the (k_0+i)-stage for all i, then we see ξ along the path $(w(0), w(1), \dots)$ and we therefore proved (6.12) and our theorem for almost all $\xi[\mu]$ follows.

Finally, when ξ is deterministic with $r_m(\xi) \le M$ for some $M < \infty$ and all m, it is clear that the same method as for the random word will work. We now will be certain of success at the k-th stage, as soon as $M_k > M$. Thus $\rho(\xi) = 1$ for such a ξ. ∎

7. Appendix to Lemma 5.

The proof of (3.12) generally follows Kesten (1987), but since a nontrivial change is needed we outline here what this change is. Roughly speaking, we have to show that if $F(w,n)$ occurs, then we can with a probability bounded away from 0 choose the 5 paths to $\Delta S(n)$ so that their endpoints are not too close together and so that they can be extended to $\Delta S(2n)$. This will give

$$P\{F(w,2n)\} \ge c_{53} P\{F(w,n)\}, \quad w \in S(n(1-\delta)). \tag{7.1}$$

Simple monotonicity arguments then give (3.12).

We only consider $n = 2^k$ for some k. This will be enough, because $P\{F(w,n)\}$ is decreasing in n. We write $(1-\delta)2^k$ instead of $\lfloor (1-\delta)2^k \rfloor$ to simplify notation. Analogously to Kesten (1987) we introduce the following four strips, whose union is $S(2^k) \setminus \overset{\circ}{S}((1-\delta)2^k)$:

$$\begin{aligned}
\mathcal{S}_R &= [(1-\delta)2^k, 2^k] \times [-2^k, 2^k];\\
\mathcal{S}_L &= [-2^k, -(1-\delta)2^k] \times [-2^k, 2^k];\\
\mathcal{S}_T &= [-2^k, 2^k] \times [(1-\delta)2^k, 2^k];\\
\mathcal{S}_B &= [-2^k, 2^k] \times [-2^k, -(1-\delta)2^k].
\end{aligned}$$

For any occupied left-right crossing r of $\mathcal{S}_R$ we denote its endpoint on the right edge of $\Delta S(2^k)$ by $a(r) = (a_1(r), a_2(r)) = (2^k, a_2(r))$. For a vacant crossing r^* of the same rectangle we denote its endpoint on the right edge of $\Delta S(2^k)$ by $a^*(r^*)$.

If r is a left-right crossing of $\mathcal{S}_R$, then $\overset{\circ}{\mathcal{S}}_R \setminus \{r\}$ has two components, the component below r, denoted by $\mathcal{S}_R^-(r)$, and the component above r, denoted by $\mathcal{S}_R^+(r)$ (compare Kesten (1987)). We say that a left-right crossing r' of $\mathcal{S}_R$ *lies below* another left-right crossing r'' of $\mathcal{S}_R$ if $\mathcal{S}_R^-(r') \subset \mathcal{S}_R^-(r'')$. Now define r_1 as the lowest occupied left-right crossing of $\mathcal{S}_R$. When r_i exists, define r_{i+1} as the lowest occupied left-right crossing of $\mathcal{S}_R$ in $\mathcal{S}_R^+(r_i)$. Note that this requires in particular that r_{i+1} is disjoint from $r_i, r_{i-1}, \dots, r_1$. We continue choosing such r_i as long as they exist. Let $r_1, r_2, \dots, r_\nu$ be all the occupied left-right crossings of $\mathcal{S}_R$ which can be found in this way. We use the same procedure to find a maximal sequence of disjoint vacant left-right crossings $r_1^*, \dots, r_{\nu^*}^*$ of $\mathcal{S}_R$, beginning with the lowest vacant left-right crossing r_1^*.

Next we define an (η, k)*−fence* for one of the r_i. We say that an occupied left-right crossing r_i in our sequence $r_1, \dots, r_\nu$ has an (η, k)-fence if the following four analogues of (2.26)-(2.28) in Kesten (1987) hold.

$$\begin{aligned}&\text{if } r_j, j \neq i, \text{ is any other occupied left-right crossing of } \mathcal{S}_R\\&\text{in } \{r_1, \dots, r_\nu\}, \text{ then } |a_2(r_j) - a_2(r_i)| > 2\sqrt{\eta}2^k;\end{aligned} \tag{7.2}$$

$$\begin{aligned}&\text{if } r_j^* \text{ is any of the vacant left-right crossings of } \mathcal{S}_R \text{ in } \{r_1^*, \dots, r_{\nu^*}^*\},\\&\text{then } |a_2^*(r_j^*) - a_2(r_i)| > 2\sqrt{\eta}2^k;\end{aligned} \tag{7.3}$$

$$\begin{aligned}&\text{there exists an occupied vertical crossing } s \text{ of the rectangle}\\&[a_1(r_i), a_1(r_i) + \sqrt{\eta}2^k] \times [a_2(r_i) - \eta 2^k, a_2(r_i) + \eta 2^k]\\&\ = [2^k, 2^k + \sqrt{\eta}2^k] \times [a_2(r_i) - \eta 2^k, a_2(r_i) + \eta 2^k]\\&\text{which is connected to } r_i \text{ by an occupied path in } a(r_i) + S(\sqrt{\eta}2^k);\end{aligned} \tag{7.4}$$

$$\|a(r_i) - (2^k, 2^k)\| > 2\sqrt{\eta}2^k \text{ and } \|a(r_i) - (2^k, -2^k)\| > 2\sqrt{\eta}2^k. \tag{7.5}$$

The last requirement is just that $a(r_i)$ should not lie too close to the corners of $S(2^k)$ and is only needed to keep $a(r_i)$ away from the endpoints of top-bottom crossings of $\mathcal{S}_T$ and of $\mathcal{S}_B$.

The main step is to prove an analogue of Lemma 2 in Kesten (1987), namely that for all $\delta > 0, \varepsilon > 0$ there exists an $\eta = \eta(\delta, \varepsilon) > 0$ such that

$$P\{\text{there exists an occupied } r_i \text{ which does not have an } (\eta, k)\text{-fence}\} \leq \varepsilon. \tag{7.6}$$

Analogously to the reference we guarantee the existence of an (η, k)-fence for all r_i by the occurrence for each $r_i, 1 \leq i \leq \nu$, of an event $E_j(\eta, k, r_i)$ for some $\eta^{-3/8} \leq 2^j \leq \eta^{-1/2}$, as well as of an $E_{j(1)}(\eta, k, r_i)$ and an $E_{j(2)}^*(\eta, k, r_i)$, for some $2\eta^{-1/2} < 2^{j(1)}, 2^{j(2)} < \eta^{-5/8}$, and finally the existence of vacant circuits surrounding $(2^k, 2^k)$ and $(2^k, -2^k)$ in the annuli

$$(2^k, 2^k) + [S(\eta^{1/4}2^k) \setminus S(2\sqrt{\eta}2^k)] \text{ and } (2^k, -2^k) + [S(\eta^{1/4}2^k) \setminus S(2\sqrt{\eta}2^k)], \tag{7.7}$$

respectively. Here,

$$
\begin{aligned}
E_j(\eta,k,r) := \big\{&\text{there exist top-bottom crossings of the strips}\\
&[2^k-\eta 2^{k+j}, 2^k-\eta 2^{k+j-1}]\times[a_2(r)-\eta 2^{k+j}, a_2(r)+\eta 2^{k+j}]\\
&\text{and } [2^k+\eta 2^{k+j-1}, 2^k+\eta 2^{k+j}]\times[a_2(r)-\eta 2^{k+j}, a_2(r)+\eta 2^{k+j}],\\
&\text{and left-right crossings of the strips}\\
&[2^k-\eta 2^{k+j}, 2^k+\eta 2^{k+j}]\times[a_2(r)-\eta 2^{k+j}, a_2(r)-\eta 2^{k+j-1}]\\
&\text{and } [2^k-\eta 2^{k+j}, 2^k+\eta 2^{k+j}]\times[a_2(r)+\eta 2^{k+j-1}, a_2(r)+\eta 2^{k+j}];\\
&\text{all vertices on these crossings which}\\
&\text{are outside the closure of } \mathcal{S}_R^-(r) \text{ are occupied}\big\}.
\end{aligned}
\tag{7.8}
$$

$E_j^*(\eta,k,r)$ is defined in the same way as $E_j(\eta,k,r)$ except that now all vertices outside $\mathcal{S}_R^-(r)$ on the relevant crossings have to be vacant instead of occupied. The occurrence of an E_j^* was not required in Kesten (1987). We add this requirement here because occurrence of an E_j^* with j in the prescribed range will prevent another occupied left-right crossing t of $\mathcal{S}_R$ which lies above r to have $|a_2(t)-a_2(r)|\le 2\sqrt{\eta}2^k$. The necessary $E_j(\eta,k,r_i)$ and $E_j^*(\eta,k,r_i)$ for all $1\le i\le \nu$ occur with high probability when η is small. Also the vacant circuits surrounding $(2^k,\pm 2^k)$ in the annuli of (7.7) occur with high probability when η is small. In this way we obtain (7.6) for sufficiently small η, entirely as in Kesten (1987).

Finally we must show how to use (7.6) to show that if $F(w,2^k)$ occurs, then we can keep the endpoints of the five paths from w to $\Delta S(2^k)$ a distance $2\sqrt{\eta}2^k$ apart. This rests on the following deterministic facts:

$$\text{each occupied left-right crossing } r \text{ of } \mathcal{S}_R \text{ intersects some } r_i; \tag{7.9}$$

$$
\begin{aligned}
&\text{if } s_1,\dots,s_p \text{ are disjoint occupied left-right crossings of } \mathcal{S}_R,\\
&\text{then there exist } r_{i_1},\dots,r_{i_p} \text{ with } i_1,\dots,i_p \text{ distinct and}\\
&p \text{ disjoint occupied left-right crossings of } \mathcal{S}_R,\ t_1,\dots,t_p,\\
&\text{such that } t_q \text{ has the same initial point as } s_q \text{ on the left}\\
&\text{edge of } \mathcal{S}_R, \text{ and such that } t_q \text{ has the same endpoint as } r_{i_q}\\
&\text{on the right edge of } \mathcal{S}_R,\ 1\le q\le p.
\end{aligned}
\tag{7.10}
$$

(7.9) is easy, because for any two disjoint left-right crossings of $\mathcal{S}_R$, one has to lie above the other. Thus any collection of disjoint left-right crossings can be ordered so that if r' and r'' belong to this collection and r' precedes r'', then r' lies below r''. Therefore, if r is an occupied left-right crossing of $\mathcal{S}_R$ which is disjoint from all the r_i, then it lies between two successive r_i and should have been counted as one of the r_i.

(7.10) follows from the fact that there exists a path τ which connects the top and bottom edge of $\mathcal{S}_R$ in $\mathcal{S}_R$ such that τ intersects each r_i in exactly one vertex v_i, and except for these vertices v_i, τ is vacant. To see that such a τ can be found, recall that $r_1, \dots, r_\nu$ are all the disjoint r_i. Then by Proposition 2.2 of Kesten (1982) there exists a vacant path in $\mathcal{S}_R$ from some vertex w_ν in the top edge of $\mathcal{S}_R$ to a vertex u_ν which is adjacent to some $v_\nu \in r_\nu$. If we have found $v_j \in r_j$ for $j \geq i$, then again by Proposition 2.2 of Kesten (1982) there exists a vacant path from some neighbor w_i of v_i to a neighbor u_{i-1} of r_{i-1}. The vacant piece from w_i to u_{i-1} lies below r_i and above r_{i-1}. The concatenation of the pieces from w_i to u_{i-1} and the edges $\{u_{i-1}, v_{i-1}\}, \{v_{i-1}, w_{i-1}\}$ give the desired path τ.

Once we have τ with the properties of the preceding paragraph and disjoint occupied left-right crossings $s_1, \dots, s_p$, then we construct the desired left-right crossings $t_1, \dots, t_p$ by the following method: τ separates the left edge of $\mathcal{S}_R$ from its right edge, so that each s_q must intersect τ in some point. Since s_q is occupied this intersection must be at one of the v_i. Let the first intersection of s_q with τ be at v_{i_q}. These v_{i_q} must be distinct for different q, because the s_q are disjoint. Now take for t_q the first part of s_q from its initial point on the left edge of $\mathcal{S}_R$ till v_{i_q}, followed by the part of r_{i_q} from v_{i_q} till its endpoint $a(r_{i_q})$ on the right edge of $\mathcal{S}_R$. Note that the t_q for different q are disjoint, because the piece of s_q till it reaches v_{i_q} lies to the 'left of τ' and cannot intersect the piece of some r_j with $j \neq i_q$ from v_j to $a(r_j)$, because this lies to the 'right of τ'.

From here on we can basically follow Kesten (1987). If $F(w, 2^k)$ occurs, or even just the event in the middle member of (3.7), and all occupied and vacant crossings of the strips $\mathcal{S}_R, \mathcal{S}_L, \mathcal{S}_T$ and $\mathcal{S}_B$ which are analogues of the r_i, have (η, k)-fences, then by (7.9) and (7.10) we can modify the paths from w to $\Delta S(2^k)$ so that their last piece coincides with one of the r_i or r_j^* or one of the analogues of these for the strips $\mathcal{S}_L, \mathcal{S}_T$ or $\mathcal{S}_B$. Now one argues as in Lemma 4 of Kesten (1987) to show that the conditional probability given $F(w, 2^k)$, or given the event in (3.7), that there are five paths from w to $\Delta S(2^k)$ (appropriately occupied or vacant) which end in crossings which have an (η, k)-fence, is bounded away from zero (bounded in k, that is). From there one obtains (7.1) and (3.12) as in Corollary 3 of Kesten (1987).

It also follows by the same argument that the probability in (3.7) is at most $c_{54} P\{G(\mathbf{0}, n)\}$.

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