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Stochastic integrability of heat-kernel bounds for random walks in a balanced random environment*

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Abstract

In $\mathbb{Z}^d$, $d \geq 2$, we consider random walks in a balanced random environment with a finite range of dependence. We first obtain both positive and negative exponential moment bounds for the invariant measure of the process of the environment as viewed from the particle. We then deduce the exponential integrability for both the lower and upper bounds of the heat kernel of the RWRE which greatly improves the known L^p bounds. Using these bounds, we prove the optimal decay rate for the semigroup generated by the heat kernel for $d \geq 3$ when the environment is i.i.d. As a consequence, we deduce a functional central limit theorem for the environment viewed from the particle.

Keywords: random walk in a balanced random environment; heat-kernel estimates; Green function; environment viewed from the particle.
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1 Introduction

In the model of random walks in random environment (RWRE), the invariant measure for the process of the environment viewed from the particle plays a crucial role in the study of the limiting behavior, cf. [47, 46, 39, 23, 50] and references therein. It has been used to derive the central limit theorem (CLT) and to determine the effective equations in homogenization, cf. e.g., [42, 36, 22, 38, 32, 27]. For time-reversible random walks in random environment, Kipnis and Varadhan [37] proved a functional central limit theorem (FCLT) which states that the process of the environment as viewed from the particle (centered at the invariant measure) has a diffusive scaling limit. It is interesting to investigate whether such behavior is still shared by non-ballistic RWRE models which in general are non-reversible in time.

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The Gaussian bounds of the heat kernel, which compare the transition probability of the RWRE to that of the Brownian motion, offer finer descriptions of the diffusivity than the CLT. Such heat kernel estimates (HKE) have been established for RWRE where either the invariant measure is explicitly known (e.g., the conductance model [19, 20, 5, 1]) or models where the environment process decorrelates with a relatively fast speed (e.g., ballistic RWRE [9]). For general RWRE, the HKE, which is usually in terms of the invariant measure, is not expected to achieve deterministic Gaussian bounds. (An important class of such models is the random walk in a balanced environment, a process generated by a non-divergence form operator [29, 28, 45]). Although the HKE may have Gaussian bounds in the limit, the lack of good moments of the random HKE could significantly limit its applicability in the quantitative comparison to the Brownian motion. Therefore it is essential to establish good stochastic integrability for both the invariant measure and the HKE.

In this article we consider random walks in a uniformly elliptic, balanced random environment in $\mathbb{Z}^d$ for $d \geq 2$. We will prove the exponential integrability for both the lower and upper bounds of the invariant measure and the heat kernel, greatly improving known results of $L^{d/(d-1)}$ integrability in the ergodic setting. Of course, the $L^{d/(d-1)}$ integrability holds in the weaker stationary and ergodic setting, while we need a finite range of dependence condition to obtain the exponential integrability. Furthermore, we will obtain optimal variance decay rate for the semigroup generated by the heat kernel in dimensions $d \geq 3$. As a consequence, we deduce a FCLT for the environment viewed from the particle. Our results are crucially inspired by an argument from Armstrong and Lin [3].

1.1 Settings

Let $\mathbb{S}_{d\times d}$ be the set of $d \times d$ positive-definite diagonal matrices. A map

$$\omega : \mathbb{Z}^d \to \mathbb{S}_{d\times d}$$

is called an *environment*. Denote the set of all environments by Ω and let $\mathbb{P}$ be a probability measure on Ω so that

$$\left\{\omega(x) = \mathrm{diag}[\omega_1(x), \ldots, \omega_d(x)], x \in \mathbb{Z}^d\right\}$$

is stationary and ergodic with respect to the *spatial shifts* $\theta_x : \Omega \to \Omega$ defined by

$$(\theta_x\omega)(\cdot) = \omega(x + \cdot).$$

Expectation with respect to $\mathbb{P}$ is denoted by $\mathbb{E}$ or $E_{\mathbb{P}}$.

Let $\{e_1, \ldots, e_d\}$ be the canonical basis for $\mathbb{R}^d$. We set

$$\omega(x, x \pm e_i) := \frac{\omega_i(x)}{2\mathrm{tr}\omega(x)} \quad \text{for } i = 1, \ldots, d, \tag{1.1}$$

and $\omega(x, y) = 0$ if $|x - y| \neq 1$. Namely, we normalize ω to get a transition probability. We remark that the configuration of $\{\omega(x, y) : x, y \in \mathbb{Z}^d\}$ is called a *balanced environment* in the literature [42, 35, 11].

Definition 1.1. *For a fixed $\omega \in \Omega$, the random walk $(X_n)_{n\geq 0}$ in the environment ω started at $x \in \mathbb{Z}^d$ is a Markov chain in $\mathbb{Z}^d$ with law P^x_ω specified by $P^x_\omega(X_0 = x) = 1$ and*

$$P^x_\omega\left(X_{n+1} = z | X_n = y\right) = \omega(y, z). \tag{1.2}$$

The expectation with respect to P^x_ω is written as E^x_ω. When the starting point of the random walk is 0, we sometimes omit the superscript and simply write P^0_ω, E^0_ω as P_ω and

E_ω, respectively. Notice that under P^0_ω, $\omega \in \Omega$, the process

$$\bar{\omega}^i = \theta_{X_i}\omega \in \Omega, \quad i \in \mathbb{Z}_{\geq 0},$$

is also a Markov chain, called the *environment viewed from the particle* process. With abuse of notation, we enlarge our probability space so that P_ω still denotes the joint law of the random walks and $(\bar{\omega}^i)_{i\geq 0}$.

We also consider the *continuous-time* RWRE (Y_t) started at $x \in \mathbb{Z}^d$ on $\mathbb{Z}^d$.

Definition 1.2. *For any function $u : \mathbb{Z}^d \to \mathbb{R}$, we let $\nabla_i^2 u(x) = u(x+e_i) + u(x-e_i) - 2u(x)$ for $1 \leq i \leq d$. Define $\nabla^2 = \mathrm{diag}[\nabla_1^2, \ldots, \nabla_d^2]$, which is a diagonal matrix. Let $(Y_t)_{t\geq 0}$ be the Markov process on $\mathbb{Z}^d$ with generator*

$$L_\omega u(x) = \sum_y \omega(x,y)[u(y) - u(x)] = \frac{1}{2\mathrm{tr}\omega(x)} \mathrm{tr}(\omega(x)\nabla^2 u). \tag{1.3}$$

With abuse of notation, we still let P^x_ω denote the quenched law of (Y_t). If there is no ambiguity from the context, we also write, for $x, y \in \mathbb{Z}^d$, $n \in \mathbb{Z}, t \in \mathbb{R}$, the transition kernels of the discrete and continuous time walks as

$$p^\omega_n(x,y) = P^x_\omega(X_n = y), \quad \text{and} \quad p^\omega_t(x,y) = P^x_\omega(Y_t = y),$$

respectively. We still denote the process of the environment viewed from the particle as

$$\bar{\omega}^t = \theta_{Y_t}\omega, \quad t \in \mathbb{R}_{\geq 0}. \tag{1.4}$$

Both the discrete- and continuous-time RWRE share the same trajectory, and their behavior are very much the same. The solutions to the Dirichlet problem can be characterized using the discrete-time RWRE, whereas for the transition kernels it is easier to manipulate the continuous-time case where the derivatives in time have less cumbersome notation compared to theirs discrete counterparts. Hence we will use both (X_n) and (Y_t) in our paper for convenience.

1.2 Main assumptions

We assume the following points throughout the paper.

(A1) The measure $\mathbb{P}$ is translation-invariant under shifts $\{\theta_x : x \in \mathbb{Z}^d\}$, and $\mathbb{P}$ has a finite range $\Delta > 0$ of dependence. That is, for any subsets $A, B \subset \mathbb{Z}^d$ with $\mathrm{dist}(A,B) = \inf\{|x-y| : x \in A, y \in B\} \geq \Delta$, the collections of variables $\{\omega(x) : x \in A\}$ and $\{\omega(y) : y \in B\}$ are independent.

(A2) $\frac{\omega}{\mathrm{tr}\omega} \geq 2\kappa \mathrm{I}$ for $\mathbb{P}$-almost every ω and some constant $\kappa \in (0, \frac{1}{2d}]$.

Of course, a special case of (A1) is the i.i.d. assumption below.

(A3) $\left\{\omega(x), x \in \mathbb{Z}^d\right\}$ are i.i.d. under the probability measure $\mathbb{P}$.

In this paper, we use c, C to denote positive constants which may change from line to line but that only depend on the dimension d and the ellipticity constant κ unless otherwise stated. We write $A \lesssim B$ if $A \leq CB$, and $A \asymp B$ if both $A \lesssim B$ and $A \gtrsim B$ hold.

1.3 Earlier results in the literature

The following quenched central limit theorem (QCLT) was proved by Lawler [42], which is a discrete version of Papanicolaou, Varadhan [47].

Theorem A. Assume (A2) and that law $\mathbb{P}$ of the environment is translation-invariant and ergodic under spatial shifts $\{\theta_x : x \in \mathbb{Z}^d\}$. Then

(i) There exists a probability measure $\mathbb{Q} \sim \mathbb{P}$ such that $(\bar{\omega}^i)_{i\geq 0}$ is an ergodic (with respect to time shifts) sequence under law $\mathbb{Q} \times P_\omega$.

(ii) For $\mathbb{P}$-almost every ω, the rescaled path $X_{n^2 t}/n$ converges weakly (under law P_ω) to a Brownian motion with covariance matrix $\bar{a} = E_{\mathbb{Q}}[\omega/\mathrm{tr}\omega] > 0$.

QCLT for the balanced RWRE in static environments under weaker ellipticity assumptions can be found at [35, 11]. For dynamic balanced random environment, QCLT was established in [25] and finer results concerning the local limit theorem and heat kernel estimates were obtained at [24]. An algebraic rate of convergence for the QCLT in the balanced environment was obtained in [32]. When the RWRE is allowed to make long jumps, non-CLT stable limits of the balanced random walk are considered in [17, 18]. We refer to the lecture notes [13, 50, 12, 26, 41] for QCLT results in different models of RWRE.

Denote the Radon-Nikodym derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$ as

$$\rho(\omega) = \mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P}(\omega). \tag{1.5}$$

For any $x \in \mathbb{Z}^d$ and finite set $A \subset \mathbb{Z}^d$, we write

$$\rho_\omega(x) := \rho(\theta_x \omega) \quad \text{and} \quad \rho_\omega(A) = \sum_{x\in A} \rho_\omega(x).$$

It is known that (see, e.g., [24]) for $\mathbb{P}$-almost all ω, the measure $\rho_\omega(\cdot)$ on $\mathbb{Z}^d$ is the unique (up to a multiplicative constant) invariant measure for the RWRE $(X_n)_{n\geq 0}$. In this sense, $\mathbb{Q}$ is the *steady state* for the environmental process. To investigate the long term behavior of the RWRE and the homogenization of the corresponding diffusion equations, it is essential to characterize the invariant measure ρ_ω. For ergodic balanced random environments, the following stochastic bounds for the invariant measure ρ_ω and the heat kernel were proved in [42, 29, 24].

For $r \geq 0, t > 0$, denote by

$$\mathfrak{h}(r,t) = \frac{r^2}{r \vee t} + r \log\left(\frac{r}{t} \vee 1\right), \quad r \geq 0, t > 0. \tag{1.6}$$

Theorem B. Assume (A2) and that law $\mathbb{P}$ of the environment is translation-invariant and ergodic under spatial shifts $\{\theta_x : x \in \mathbb{Z}^d\}$. There exist constants $q > \frac{d}{d-1}$ and $p > 0$ both depending on (d, κ) such that

(i)

$$\mathbb{E}[\rho^q] < \infty, \qquad \mathbb{E}[\rho^{-p}] < \infty;$$

(ii) $\mathbb{P}$-almost surely, for any $r > 0$,

$$\rho_\omega(B_{2r}) \leq C\rho_\omega(B_r);$$

(iii) $\mathbb{P}$-almost surely, for all $x \in \mathbb{Z}^d$, $t > 0$,

$$\frac{c\rho_\omega(0)}{\rho_\omega(B_{\sqrt{t}})} e^{-C|x|^2/t} \leq p_t^\omega(x,0) \leq \frac{C\rho_\omega(0)}{\rho_\omega(B_{\sqrt{t}})} e^{-c\mathfrak{h}(|x|,t)}. \tag{1.7}$$

Moreover, for $x \in \mathbb{Z}^d$, $t > 0$,

$$\|p_t^\omega(0,x)\|_{L^q(\mathbb{P})} \leq \frac{C}{(t+1)^{d/2}} e^{-c\mathfrak{h}(|x|,t)}, \tag{1.8}$$

$$\|p_t^\omega(0,x)\|_{L^{-p}(\mathbb{P})} \geq \frac{c}{(t+1)^{d/2}} e^{-C|x|^2/t}. \tag{1.9}$$

Remark 1.3. The positive moment bound in (i) with $q = \frac{d}{d-1}$ was obtained by Lawler [42]. The negative moment bound (i) and volume-doubling property (ii) were proved by Bauman [6]. The L^q bound of the invariant measure in (i) and a heat kernel moment bound of $\|p_t^\omega(0,x)\|_{L^q(\mathbb{P})}$ were obtained by Fabes and Stroock [29]. Note that although [6, 29] are results for deterministic operators, they yield the corresponding stochastic integrability of the RWRE by Birkhoff's ergodic theorem. Deterministic heat kernel bounds of the form (1.7) were shown by Escauriaza [28] in the PDE setting, and by Mustapha [45] for discrete time balanced random walks. In the more general dynamic ergodic balanced environment setting, Theorem B was proved by Deuschel and the first named author [24]. The optimality of the L^q moment, $q > \frac{d}{d-1}$, of the heat kernel was also discussed in [29, (4.10)] with examples where q could be as close to $\frac{d}{d-1}$ as possible. Note that in the general time-dependent environment setting, the number $\frac{d}{d-1}$ in Theorem B, which is due to the integrability of ρ in the static environment should be replaced by $\frac{d+1}{d}$.

Roughly speaking, the term $\frac{\rho_\omega(0)}{\rho_\omega(B_{\sqrt{t}})}$ in the HKE is the long term ratio between the time the RWRE visits the origin and the time it spends in the ball $B_{\sqrt{t}}$. For the special deterministic environment $a \equiv I$, i.e., when the RWRE is a simple random walk, we have $\rho \equiv 1$ and this term becomes a constant $Ct^{-d/2}$.

However, as an important feature of the non-divergence form model, both ρ_ω and $\frac{\rho_\omega(0)t^{d/2}}{\rho_\omega(B_{\sqrt{t}})}$ are not expected to have deterministic upper and (nonzero) lower bounds in general. For instance, Gantert's example in [50, Page 281] shows that there exists a uniformly elliptic balanced environment with an arbitrarily strong local "blocking" property so that $\rho_\omega(0) > 0$ can be as small as we want. In other words, there are examples of balanced i.i.d. uniformly elliptic random environments with the property $\mathbb{P}(\inf_{x\in\mathbb{Z}^d} \rho_\omega(x) = 0) = 1$. The poor stochastic integrability of the heat kernel bounds could greatly limit their usefulness in the prediction of the diffusive behavior of the RWRE.

One of the goals in this paper is to show that, in a balanced environment with finite range of dependence, both ρ_ω and the heat kernel have positive and negative exponential moment bounds.

Since $\mathbb{Q}$ is the limiting ergodic measure of the environment, it is expected that, as $t \to \infty$, $\psi(\bar\omega^t) \to E_{\mathbb{Q}}[\psi]$ almost surely for $\psi \in L^1(\mathbb{P})$. (Recall the process $\bar\omega^t$ of the environment as viewed from the particle in (1.4).) When the balanced environment satisfies a finite range of dependence and ψ is an $L^\infty(\mathbb{P})$ local function, it is shown in [32, Theorem 1.2] that, with overwhelming annealed probability, the average $t^{-1}\int_0^t \psi(\bar\omega^s)\,\mathrm{d}s$ converges to $E_{\mathbb{Q}}[\psi]$ at an algebraic speed. For time-reversible RWRE, Kipnis and Varadhan [37] proved that the process of the environment as viewed from the particle has diffusive behavior. Further, in the conductance model where the walk is generated by divergence form operators, algebraic rates for the decay of $E_\omega[\psi(\bar\omega^t)]$ are obtained in [44, 31, 21].

Another goal in this paper is to establish the optimal decay rate of $E_\omega[\psi(\bar\omega^t)] - E_{\mathbb{Q}}\psi$ for the the balanced RWRE. To this end, we will adapt the strategy of Gloria, Neukamm, Otto [31] on the semigroup decay of divergence form equations into the non-divergence form setting.

In our proof of improved moment bounds for the invariant measure and heat kernels, we will employ the following quantitative homogenization result for the elliptic non-divergence form operator L_ω.

We need some notations to describe the homogenization problem. For a function f on $\mathbb{R}^d$ we let $D^2 f$ denote the Hessian matrix of f. For any function $u : \mathbb{Z}^d \to \mathbb{R}$, recall that $\nabla_i^2 u(x) = u(x+e_i) + u(x-e_i) - 2u(x)$ for $1 \le i \le d$. Also, $\nabla^2 = \mathrm{diag}[\nabla_1^2, \dots, \nabla_d^2]$ is a diagonal matrix. For $r > 0$, $y \in \mathbb{R}^d$ we let

$$\mathbb{B}_r(y) = \left\{x \in \mathbb{R}^d : |x-y| < r\right\}, \quad B_r(y) = \mathbb{B}_r(y) \cap \mathbb{Z}^d$$

denote the continuous and discrete balls with center y and radius r, respectively. When $y = 0$, we also write $\mathbb{B}_r = \mathbb{B}_r(0)$ and $B_r = B_r(0)$. For any $B \subset \mathbb{Z}^d$, its *discrete boundary* is the set

$$\partial B := \left\{z \in \mathbb{Z}^d \setminus B : \text{dist}(z,x) = 1 \text{ for some } x \in B\right\}.$$

Let $\bar{B} = B \cup \partial B$. Note that with abuse of notation, whenever confusion does not occur, we also use ∂A and $\bar{A}$ to denote the usual continuous boundary and closure of $A \subset \mathbb{R}^d$, respectively.

A function $\psi : \Omega \to \mathbb{R}$ is said to be *local* if it is measurable and depends only on the environment $\{\omega(x) : x \in S\}$ in a finite set $S \subset \mathbb{Z}^d$.

Proposition C. Assume (A1), (A2), and that the ψ is a local function. Recall the measure $\mathbb{Q}$ in Theorem A. Suppose $g \in C^\alpha(\partial \mathbb{B}_1)$, $f \in C^\alpha(\mathbb{B}_1)$ for some $\alpha \in (0,1]$, and ψ is a measurable function of $\omega(0)$ with $\|\psi/\text{tr}\omega\|_\infty := \operatorname{ess\,sup}_\omega \frac{|\psi(\omega(0))|}{\omega(0)} < \infty$. Let $\bar{u}$ be the solution of the Dirichlet problem

$$\begin{cases} \frac{1}{2}\text{tr}(\bar{a}D^2\bar{u}) = f\bar{\psi} & \text{in } \mathbb{B}_1, \\ \bar{u} = g & \text{on } \partial\mathbb{B}_1, \end{cases}$$

with $\bar{a} = E_{\mathbb{Q}}[\omega/\text{tr}\omega] > 0$ being a positive-definite matrix and $\bar{\psi} = E_{\mathbb{Q}}[\psi/\text{tr}\omega]$.

For any $q \in (0,d)$, there exist a random variable $\mathscr{X}_q = \mathscr{X}_q(\omega, d, \kappa)$ with $\mathbb{E}[\exp(c\mathscr{X}_q^d)] < \infty$, and a constant $\beta = \beta(d,\kappa,q) \in (0,1)$ such that for any $y \in B_{3R}$, the solution u of

$$\begin{cases} \frac{1}{2}\text{tr}(\omega\nabla^2 u(x)) = \frac{1}{R^2} f(\frac{x-y}{R})\psi(\theta_{x-y}\omega) & x \in B_R(y), \\ u(x) = g(\frac{x-y}{|x-y|}) & x \in \partial B_R(y) \end{cases} \tag{1.10}$$

satisfies, with $A_1 = \|f\|_{C^{0,\alpha}(\mathbb{B}_1)} \|\frac{\psi}{\text{tr}(\omega)}\|_\infty + [g]_{C^{0,\alpha}(\partial\mathbb{B}_1)}$,

$$\max_{x \in B_R(y)} \left| u(x) - \bar{u}(\tfrac{x-y}{R}) \right| \lesssim A_1(1 + \mathscr{X}_q R^{-q/d}) R^{-\alpha\beta}, \tag{1.11}$$

Remark 1.4. The proof of Proposition C, which is a small modification of [32, Theorem 1.5], can be found in the Appendix.

In terms of the quantitative homogenization of elliptic non-divergence form operators in the PDE setting, Yurinski derived a second moment estimate of the homogenization error in [49] for linear elliptic case, and Caffarelli, Souganidis [16] proved a logarithmic convergence rate for the nonlinear case. Afterwards, Armstrong, Smart [4] achieved an algebraic convergence rate for fully nonlinear elliptic equations. Armstrong, Lin [3] obtained quantitative estimates for the approximate corrector problems.

In the random walks in a balanced random environment setting, the above algebraic rate (1.11) was proved in [32] by Peterson and the authors, following ideas of Armstrong, Smart [4] in the PDE setting. When the environment is non-elliptic but genuinely d-dimensional and i.i.d., Harnack inequalities with near optimal constants were proved in [8] for elliptic operators by Berger, Cohen, Deuschel and the first named author, and in [10] for parabolic operators by Berger and Criens, respectively.

1.4 Main results

For RWRE in a balanced, uniformly elliptic environment with a finite range of dependence, we will establish natural lower and upper bounds for both the invariant measure and the heat kernel (Theorem 1.5) which possess both positive and negative exponential moments, greatly improving the stochastic integrability in Theorem B for the ergodic setting. When the environment is i.i.d., we will obtain the optimal decay rate of the semigroup generated by the heat kernel for $d \geq 3$ in Theorem 1.6.

Theorem 1.5. *Assume* (A1), (A2)*, and* $d \geq 2$*. Let* $s = s(d,\kappa) = 2 + \frac{1}{2\kappa} - d \geq 2$*. For any* $\varepsilon > 0$*, there exists a random variable* $\mathscr{H}(\omega) = \mathscr{H}(\omega, d, \kappa, \varepsilon) > 0$ *with* $\mathbb{E}[\exp(c\mathscr{H}^{d-\varepsilon})] < \infty$ *such that the following properties hold.*

(a) For $\mathbb{P}$*-almost all* ω*,*

$$c\mathscr{H}^{-s} \leq \rho(\omega) \leq C\mathscr{H}^{d-1}.$$

In particular, for any $q \in (-\frac{d}{s}, \frac{d}{d-1})$*, we have*

$$\mathbb{E}[\exp(c\rho^q)] < \infty.$$

(b) For any $r \geq 1$ *and* $\mathbb{P}$*-almost all* ω*,*

$$c\mathscr{H}^{-s} \leq \frac{r^d \rho_\omega(0)}{\rho_\omega(B_r)} \leq C\mathscr{H}^{d-1}.$$

(c) Recall the function $\mathfrak{h}$ *in* (1.6)*. For any* $x \in \mathbb{Z}^d, t > 0$*, and* $\mathbb{P}$*-almost all* ω*,*

$$p_t^\omega(x,0) \leq C\mathscr{H}^{d-1}(1+t)^{-d/2} e^{-c\mathfrak{h}(|x|,t)},$$
$$p_t^\omega(x,0) \geq c\mathscr{H}^{-s}(1+t)^{-d/2} e^{-C|x|^2/t}.$$

Recall the continuous time RWRE $(Y_t)_{t\geq 0}$ in Definition 1.2. For any measurable function $\zeta : \Omega \to \mathbb{R}$, we define its *stationary extension* $\bar\zeta : \mathbb{Z}^d \times \Omega \to \mathbb{R}$ as

$$\bar\zeta(x) = \bar\zeta(x;\omega) := \zeta(\theta_x \omega).$$

Define the semigroup P_t, $t \geq 0$, on $\mathbb{R}^\Omega$ by

$$P_t\zeta(\omega) = E_\omega^0[\zeta(\bar\omega^t)] = \sum_z p_t^\omega(0,z)\bar\zeta(z;\omega).$$

The following theorem estimates the speed of decorrelation of the environmental process $\bar\omega^t$ from the original environment under (A3). It gives a rate $t^{-d/4}$ of decay for the semigroup, which is optimal. A function $\zeta : \Omega \to \mathbb{R}$ is said to be *local* if it depends only on the environment $\{\omega(x) : x \in S\}$ in a finite set $S \subset \mathbb{Z}^d$. The smallest such a set S is called the *support* of ζ and denoted by $\mathrm{Supp}(\zeta)$.

Theorem 1.6. *Assume* (A2), (A3)*, and* $d \geq 3$*. For any local measurable function* $\zeta : \Omega \to \mathbb{R}$ *with* $\|\zeta\|_\infty \leq 1$ *and* $t \geq 0$*, we have, for* $C = C(d, \kappa, \#\mathrm{Supp}(\zeta))$*,*

$$\mathrm{Var}_{\mathbb{Q}}(P_t\zeta) \leq C(1+t)^{-d/2}; \tag{1.12}$$

$$\|P_t\zeta - E_{\mathbb{Q}}\zeta\|_{L^1(\mathbb{P})} + \|P_t\zeta - \mathbb{E}[P_t\zeta]\|_{L^p(\mathbb{P})} \leq C_p(1+t)^{-d/4} \quad \text{for all } p \in (0,2). \tag{1.13}$$

For divergence form operators, optimal diffusive decay of the semigroup generated by the heat kernel was obtained by Gloria, Neukamm, Otto [31], de Buyer, Mourrat [21]. Our proof of Theorem 1.6 is motivated by the approach of [31], which uses an Efron-Stein type inequality and the Duhamel representation formula for the vertical derivative. However, unlike the divergence form setting [31], there are no deterministic Gaussian bounds for the heat kernel, and the steady state $\mathbb{Q}$ of the environment process $(\bar\omega^t)_{t\geq 0}$ is not only different from the original measure $\mathbb{P}$ but also without an explicit formula. To overcome these difficulties, our heat kernel estimates and the (negative and positive) exponential moment bounds of the Radon-Nikodym derivative $\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}}$ in Theorem 1.5 play crucial roles. Another feature of our non-divergence setting is that there are no Caccioppoli estimates for $p > 2$. See Lemma 3.2.

As a consequence of Theorem 1.6 and the CLT of [22], we obtain a CLT for the additive functional of the environmental process. Recall that $\bar{\omega}^t = \theta_{Y_t}\omega$ is an ergodic process under the law $\mathbb{Q} \times P_\omega$ and time shifts. The following CLT says that, when $d \geq 3$, the fluctuation around the ergodic mean is approximately Gaussian under the diffusive rescaling.

Corollary 1.7. *Assume* (A2), (A3). *Let $d \geq 3$. For $\mathbb{P}$-almost all ω and any bounded measurable local function ζ of the environment with $E_{\mathbb{Q}}\zeta = 0$, the P_ω law of*

$$\frac{1}{\sqrt{t}}\int_0^t \zeta(\bar{\omega}^s)\,\mathrm{d}s$$

converges weakly to a Brownian motion with a deterministic diffusivity constant.

Another immediate consequence of Theorem 1.6 is the existence of a stationary corrector in $d \geq 5$ for the non-divergence form homogenization problem (1.10).

Corollary 1.8. *Assume* (A2), (A3). *When $d \geq 5$, for any bounded local measurable function $\zeta : \Omega \to \mathbb{R}$, there exists $\phi : \Omega \to \mathbb{R}$ such that $\phi \in L^p(\mathbb{P})$ for all $p \in (0, 2)$, and for $\mathbb{P}$-almost all ω, its stationary extension $\bar{\phi}(x) = \phi(\theta_x\omega)$ solves*

$$L_\omega\bar{\phi}(x) = \bar{\zeta}(x) - E_{\mathbb{Q}}[\zeta], \quad \text{for all } x \in \mathbb{Z}^d. \tag{1.14}$$

Remark 1.9. In the classical periodic environment setting, it is well-known that the existence of a stationary corrector implies that the optimal homogenization error of problem (1.10) is generically of scale R^{-1}. Readers may refer to the classical books [7, 36] for the derivation of the rate in the periodic setting, and [34, 48, 33] for discussions on the optimality of the rates.

We remark that our Corollary 1.8 is a weaker version of [3, Theorem 7.1] where a stretched exponential tail was obtained for the corrector. Neverthless, our Corollary 1.8 is an immediate consequence of the optimal semigroup decay rate.

To obtain the above results, we need to study properties of the Green functions. For $d \geq 2, R \geq 1$, denote the exit time from B_R of the RWRE by

$$\tau = \tau_R = \inf\{n \geq 0 : X_n \notin B_R\}. \tag{1.15}$$

Definition 1.10. *For $R \geq 1$, $\omega \in \Omega$, $x \in \mathbb{Z}^d$, $S \subset \mathbb{Z}^d$, the Green function $G_R(\cdot, \cdot)$ in the ball B_R for the balanced random walk is defined by*

$$G_R(x, S) = G_R^\omega(x, S) := E_\omega^x\left[\sum_{n=0}^{\tau_R - 1} \mathbb{1}_{X_n \in S}\right], \quad x \in \bar{B}_R.$$

We also write $G_R(x, y) := G_R^\omega(x, \{y\})$ and $G_R(x) := G(x, 0)$.

Note that for $d \geq 3$, by [35, Theorem 1], the RWRE is transient, and so the Green function in the whole space

$$G^\omega(x) := \lim_{R\to\infty} G_R(x) < \infty$$

is well-defined for all $x \in \mathbb{Z}^d$, $\mathbb{P}$-almost surely. Whereas, when $d = 2$, the RWRE is recurrent, and thus the Green function in the whole $\mathbb{Z}^2$ is infinity. In this case, the *potential kernel*

$$A(x) = A^\omega(x) = \sum_{n=0}^{\infty}[p_n^\omega(0, 0) - p_n^\omega(x, 0)], \quad x \in \mathbb{Z}^2, \tag{1.16}$$

is well-defined. Note that G and A are both non-negative functions, and, for $x \in \mathbb{Z}^d$,

$$L_\omega G(x) = -\mathbb{1}_{x=0}, \quad \text{if } d \geq 3,$$

and

$$L_\omega A(x) = \mathbb{1}_{x=0}, \quad \text{if } d = 2.$$

Theorem 1.11. *Assume* (A1), (A2). *For $r > 0$, let*

$$U(r) := \begin{cases} -\log r & d = 2, \\ r^{2-d} & d \geq 3. \end{cases} \tag{1.17}$$

For any $\varepsilon > 0$, there exists a random variable $\mathscr{H} = \mathscr{H}(\omega, d, \kappa, \varepsilon) > 0$ with $\mathbb{E}[\exp(c\mathscr{H}^{d-\varepsilon})] < \infty$ such that, $\mathbb{P}$-almost surely, for all $x \in B_R$,

$$\mathscr{H}^{-s}[U(|x|+1) - U(R+2)] \lesssim G_R^\omega(x) \lesssim \mathscr{H}^{d-1}[U(|x|+1) - U(R+2)],$$

where $s = s(d, \kappa) = 2 + \frac{1}{2\kappa} - d \geq 2$.

In terms of the Green function of non-divergence form operators, $L^{d/(d-1)}$ and L^q, $q > \frac{d}{d-1}$ upper bounds were proved by Bauman [6] and Fabes, Stroock [29]. For non-divergence form operators in a random environment with a finite range of dependence, an upper bound for the Green function of the *approximate corrector* with exponential integrability was obtained by Armstrong, Lin [3]. For balanced RW in a time-dependent ergodic environments, positive and negative moment bounds and scaling limits of the Green's function were obtained by Deuschel and the first named author in [24].

Our proof of the bounds of G_R^ω follows the idea of Armstrong, Lin [3, Proposition 4.1]. In Theorem 1.11, we apply their idea to obtain both upper and lower bounds for the Green function G_R in a finite region.

With the bounds in Theorem 1.11 and the heat kernel estimates Theorem 1.5, we can deduce bounds of the Green functions on the whole space.

Corollary 1.12. *Assume* (A1), (A2). *Let s be as in Theorem 1.11. For any $\varepsilon > 0$, there exists a random variable $\mathscr{H} = \mathscr{H}(\omega, d, \kappa, \varepsilon) > 0$ with $\mathbb{E}[\exp(c\mathscr{H}^{d-\varepsilon})] < \infty$ such that, $\mathbb{P}$-almost surely, for all $x \in \mathbb{Z}^d$,*

$$\mathscr{H}^{-s}\log(|x|+1) \lesssim A^\omega(x) \lesssim \mathscr{H}\log(|x|+1), \text{ when } d = 2;$$
$$\mathscr{H}^{-s}(1+|x|)^{2-d} \lesssim G^\omega(x) \lesssim \mathscr{H}^{d-1}(1+|x|)^{2-d}, \text{ when } d \geq 3.$$

The proof of Corollary 1.12 will be given in Section 3.2.

2 Bounds of the Green function in a ball

By the Markov property, $G_R(x, S)$ satisfies $G_R = 0$ on ∂B_R and

$$L_\omega G_R(x, S) = -\mathbb{1}_{x \in S}, \quad x \in B_R. \tag{2.1}$$

We will establish upper and lower bounds of the Green function G_R (Theorem 1.11) by comparing G_R to test functions, an idea we learnt from Armstrong-Lin [3, Proposition 4.1]. Note that [3] only obtained an upper bound of the Green function corresponding to the "approximate" operator defined on the whole $\mathbb{R}^d$, while our Green function corresponds to the original non-divergence form operator within a finite ball B_R. Hence, in our case, the challenge lies in finding appropriate test functions (for both lower and upper bounds) so that they have the desired boundary values and concavity near the discrete boundary. More details are explained below.

Let us call a function $u : \mathbb{Z}^d \to \mathbb{R}$ *ω-harmonic* on $A \subset \mathbb{Z}^d$ if $L_\omega u(y) = 0$ for $y \in A$. Clearly, the Green function $G_R(x,0)$ is ω-harmonic on $B_R \setminus \{0\}$.

To obtain the upper bound, we construct a function h which is almost ω-harmonic away from the origin and with (almost) zero boundary values, so that $G_R - h$ is subharmonic at places that are either close to the origin or the boundary of B_R. As a result, if $(G_R - h)(x_0) = \max_{B_R}(G_R - h)$ were positive, then the maximum principle forces the maximizer x_0 to be sufficiently far away from both the origin and the boundary. This allows enough space for the homogenization to occur around x_0, i.e., h is close to its continuous harmonic counterpart up to an algebraic error. On the other hand, by the maximal principle, the ω-harmonic counterpart of $G_R - h$ (which is an algebraic error away from $G_R - h$) cannot achieve its maximum over the ball $\bar{B}_{|x_0|/2}(x_0)$ in the center. This would contradict the assumption that the maximizer is x_0, if we can exploit the fact that h is ω-superharmonic to give it enough room to absorb the algebraic error.

The proof of the lower bound follows similar philosophy.

The following Lemmas contain properties of some deterministic functions that will be useful in our construction of the test functions in the next subsections.

Lemma 2.1. *Let $\delta = \beta/2$, where $\beta = \beta(d, \kappa, q_\varepsilon)$ is as defined in Proposition C. Define $\bar{\zeta}, \bar{\xi} : (0, \infty) \to \mathbb{R}$ as*

$$\bar{\zeta}(r) = \begin{cases} -(\log r)\exp(r^{-\delta}/\delta) & d = 2 \\ r^{2-d}\exp(-r^{-\delta}/\delta) & d \geq 3, \end{cases}$$

$$\bar{\xi}(r) = \begin{cases} -(\log r)\exp(-r^{-\delta}/\delta) & d = 2 \\ r^{2-d}\exp(r^{-\delta}/\delta) & d \geq 3. \end{cases}$$

Define two functions $\zeta, \xi : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$ as

$$\zeta(y) = \bar{\zeta}(|y|), \quad \text{and} \quad \xi(y) = \bar{\xi}(|y|), \quad y \neq 0.$$

Then, the following statements hold.

(i) $\bar{\zeta}, \bar{\xi}$ are decreasing functions on (C, ∞). Moreover, for $r \geq C$,

$$-\bar{\zeta}'(r) \asymp r^{1-d}, \quad \text{and} \quad \frac{1}{2}r^{1-d} \leq -\bar{\xi}'(r) \leq \left(d - \frac{1}{2}\right) r^{1-d}.$$

(ii) For $|y| \geq C$, we have

$$-\Delta\zeta(y) \geq |y|^{-(2+\delta)}|\zeta(y)|, \quad \text{and} \quad -\Delta\xi(y) \leq -|y|^{-(2+\delta)}|\xi(y)|.$$

(iii) For $|y| \geq 2$ and $k \in \mathbb{N}$, there exists $C = C(k, d)$ such that

$$|D^k\zeta(y)| \leq C|y|^{-k}|\zeta(y)|, \quad \text{and} \quad |D^k\xi(y)| \leq C|y|^{-k}|\xi(y)|.$$

Lemma 2.2. *There exist constants $\alpha_0 \in (0,1)$ and $A_0 \geq 1$ depending on κ such that, for any $\alpha \in (0, \alpha_0)$, $A \geq A_0$,*

$$L_\omega(e^{-2\alpha|x|/R}) \leq 0 \quad \text{in } B_R \setminus B_{R/2}, \text{ when } R \geq A_0; \tag{2.2}$$

$$L_\omega(e^{-A|x|^2}) \geq -\mathbb{1}_{x=0}, \quad x \in \mathbb{Z}^d; \tag{2.3}$$

$$L_\omega(e^{-A|x|^2/R^2}) > 0, \quad x \in B_R \setminus B_{R/2}, \text{ when } R \geq A^2. \tag{2.4}$$

The proof of Lemma 2.2 is in Section A.2 of the Appendix.

2.1 Upper bounds of Green's functions

Recall $\mathscr{X}_q$ in Proposition C. For any $\varepsilon \in (0,1)$, we write

$$R_0 = R_0(\omega, d, \kappa, \varepsilon) := \mathscr{X}_{d-\varepsilon}^{d/(d-\varepsilon)} + K,$$

where K is a sufficiently large constant depending on (d,κ), and denote the exit time from B_{R_0} as

$$s_0 = \min\{n \geq 0 : X_n \notin B_{R_0}\}. \tag{2.5}$$

Note that R_0 plays the role of a "homogenization radius" in the sense that for all $R \geq R_0$ and $y \in B_{3R}$, the upper bound in (1.11) can be replaced by the algebraic term $CA_1 R^{-\alpha\beta}$.

Let $\alpha = \alpha(d,\kappa) > 0$ be a constant to be determined in Lemma 2.4, and set

$$C_{\alpha,R} := \frac{[\bar\zeta(R/2) - \bar\zeta(R)]R^{d-2}}{e^{-\alpha+2\alpha/R} - e^{-2\alpha}} \asymp \alpha^{-1}, \quad \text{when } R \geq R_0.$$

Definition 2.3. *Let $\bar\zeta, \zeta$ be as in Lemma 2.1. For any fixed $R \geq 4R_0$, we define a function $h : \bar B_R \to [0,\infty)$ by*

$$h(x) = \begin{cases} h_1(x), & x \in B_{R_0}, \\ h_2(x), & x \in B_{R/2} \setminus B_{R_0}, \\ h_3(x), & x \in \bar B_R \setminus B_{R/2}, \end{cases}$$

where the functions h_1, h_2, h_3 are defined as below

$$\begin{aligned} h_1(y) &= E_\omega^y[h_2(X_{s_0}) + |X_{s_0}| - |y|], \quad y \in \bar B_{R_0}, \\ h_2(y) &= R_0^{d-1}\left[(\alpha^{-1} - 1)(\bar\zeta(R/2) - \bar\zeta(R)) + \zeta(y) - \bar\zeta(R)\right], \quad y \in \mathbb{R}^d \setminus \{0\}, \\ h_3(y) &= R_0^{d-1}\alpha^{-1}C_{\alpha,R}R^{2-d}[e^{-2\alpha(|y|-1)/R} - e^{-2\alpha}], \quad y \in \mathbb{R}^d. \end{aligned}$$

Note that h_2 is defined first and the definition of h_1 follows.

Lemma 2.4. *When $R \geq 4R_0$, there exists a constant $\alpha > 0$ such that the functions h_1, h_2, h_3, h given in Definition 2.3 have the following properties.*

(a) $L_\omega h_1(x) = -L_\omega(|x|) \leq -\mathbb{1}_{x=0}$ *for* $x \in B_{R_0}$*;*

(b) $h_1 = h_2$ *on* ∂B_{R_0}*, and* $h_2 = h_3$ *on* $\partial \mathbb{B}_{R/2}$*;*

(c) $h_2 \geq h_1$ *in* $B_{R_0} \setminus B_{R_0/2}$*.*

(d) $h_2 \geq h_3$ *in* $\mathbb{B}_R \setminus \mathbb{B}_{R/2}$*, and* $h_2 \leq h_3$ *in* $\mathbb{B}_{R/2} \setminus \mathbb{B}_{R/2-1}$*.*

(e) $L_\omega h_3 \leq 0$ *in* $B_R \setminus B_{R/2}$*.*

Proof. (a) and (b) are obvious. To see (c), note that $h_2(x) - h_1(x) = E_\omega^x[f(|x|) - f(|X_{s_0}|)]$, where $f(r) = r + R_0^{d-1}\bar\zeta(r)$. Since $R_0 \geq K$, by Lemma 2.1(i), taking K sufficiently large, $f(r)$ is a decreasing function for $r \in [R_0/2, R_0+1]$.

Next, we will prove (d). Indeed, we can write, for $y \in \mathbb{R}^d \setminus \{0\}$,

$$h_2(y) - h_3(y) = R_0^{d-1}a(|y|) + A(R_0, R),$$

where $A(R_0,R)$ is a constant, and $a(r) = \bar\zeta(r) - \alpha^{-1}C_{\alpha,R}R^{2-d}e^{-2\alpha(r-1)/R}$. For $r \in [R/2-1, R]$, by Lemma 2.1(i), taking $\alpha > 0$ sufficiently small, we have

$$a'(r) \geq -Cr^{1-d} + 2C_{\alpha,R}e^{-2\alpha}R^{1-d} \geq -Cr^{1-d} + c\alpha^{-1}R^{1-d} \geq 0.$$

Hence, $h_2 - h_3$ is radially increasing in $\mathbb{B}_R \setminus \mathbb{B}_{R/2-1}$. Item (d) then follows from the fact that $h_2 - h_3 = 0$ on $\partial\mathbb{B}_{R/2}$.

Item (e) is a consequence of (2.2) in Lemma 2.2. □

Proof of the upper bound in Theorem 1.11. For $0 < a < b$ and $d \geq 2$, we have an elementary inequality

$$b - a \leq b^{d-1}(U(a) - U(b)). \tag{2.6}$$

When $R \in (1, 4R_0]$, note that for $x \in B_R$ one has $L_\omega[G_R(x) - (R + 1 - |x|)] \geq 0$, and $G_R = 0 \leq R + 1 - |x|$ on ∂B_R. By the maximum principle, we have, for $x \in B_R$,

$$G_R(x) \leq R + 1 - |x| \overset{(2.6)}{\leq} (R+2)^{d-1}[U(|x|+1) - U(R+2)].$$

Hence the upper bound in Theorem 1.11 holds when $R \in (1, 4R_0]$. It remains to consider the case $R > 4R_0$.

First, we will prove via contradiction that

$$G_R \leq h \quad \text{in } \bar{B}_R. \tag{2.7}$$

Assume by contradiction that (2.7) fails, i.e., $\max_{\bar{B}_R}(G_R - h) > 0$. By Lemma 2.4(a), $L_\omega(G_R - h) \geq 0$ in B_{R_0} and so $\max_{\bar{B}_R}(G_R - h)$ is achieved outside of B_{R_0}. Further, note that $(G_R - h)|_{\partial B_R} = (-h_3)|_{\partial B_R} \leq 0$. By Lemma 2.4(e), $L_\omega(G_R - h_3) \geq 0$ in $B_R \setminus B_{R/2}$, and so, by the maximum principle and Lemma 2.4(d),

$$\max_{\bar{B}_R \setminus B_{R/2}} (G_R - h) \leq \max_{\partial(B_R \setminus B_{R/2})} (G_R - h_3) \leq 0 \vee \max_{B_{R/2} \setminus B_{R_0}} (G_R - h).$$

Hence, if $\max_{\bar{B}_R}(G_R - h) > 0$, then there exists $x_0 \in B_{R/2} \setminus B_{R_0}$ so that

$$(G_R - h)(x_0) = \max_{\bar{B}_R}(G_R - h) > 0.$$

Since $x_0 \in B_{R/2} \setminus B_{R_0}$, by Lemma 2.4(c)(d),

$$(G_R - h_2)(x_0) \geq \max_{\bar{B}_{|x_0|/2}(x_0)} (G_R - h_2),$$

which is equivalent to

$$(G_R - R_0^{d-1}\zeta)(x_0) \geq \max_{\bar{B}_{|x_0|/2}(x_0)} (G_R - R_0^{d-1}\zeta). \tag{2.8}$$

Recall $\bar{a} = E_{\mathbb{Q}}[\omega/\mathrm{tr}\omega] > 0$. Without loss of generality, assume that $\bar{a} = I$, and set

$$\tilde{\zeta}(y) := \zeta(y) + c|x_0|^{-(2+\delta)}|\zeta(x_0)||y - x_0|^2, \quad y \in \mathbb{R}^d \setminus \{0\},$$

where $c > 0$ is chosen so that (by Lemma 2.1(ii))

$$\Delta\tilde{\zeta}(y) \leq -|y|^{-(2+\delta)}|\zeta(y)| + 2nc|x_0|^{-(2+\delta)}|\zeta(x_0)| \leq 0 \quad \text{for } y \in \mathbb{B}_{|x_0|/2}(x_0).$$

Then (by Lemma 2.1(iii)) $|D\tilde{\zeta}| \leq C|x_0|^{-1}|\zeta(x_0)|$ in $\mathbb{B}_{1+|x_0|/2}(x_0)$, and

$$(G_R - R_0^{d-1}\tilde{\zeta})(x_0) \overset{(2.8)}{\geq} \max_{\partial B_{|x_0|/2}(x_0)} (G_R - R_0^{d-1}\tilde{\zeta}) + CR_0^{d-1}|x_0|^{-\delta}|\zeta(x_0)|. \tag{2.9}$$

Let $\bar{v} : \bar{\mathbb{B}}_1 \to \mathbb{R}$ and $v : \bar{B}_{|x_0|/2}(x_0) \to \mathbb{R}$ be the solutions of (Here $\bar{a} = I$.)

$$\begin{cases} \mathrm{tr}(\bar{a}D^2\bar{v}) = \Delta\bar{v} = 0 & x \in \mathbb{B}_1 \\ \bar{v}(x) = R_0^{d-1}\tilde{\zeta}(x_0 + \frac{|x_0|}{2}x) & x \in \partial\mathbb{B}_1, \end{cases}$$

and

$$\begin{cases} L_\omega v(x) = 0 & x \in B_{|x_0|/2}(x_0) \\ v(x) = \bar{v}(\frac{x - x_0}{|x - x_0|}) & x \in \partial B_{|x_0|/2}(x_0). \end{cases}$$

We will show that v can be controlled by $R_0^{d-1}\tilde{\zeta}$ both on the boundary and inside of $B_{|x_0|/2}(x_0)$. Indeed, for $x \in \partial B_{|x_0|/2}(x_0)$,

$$\begin{aligned}|v(x) - R_0^{d-1}\tilde{\zeta}(x)| &= R_0^{d-1}\Big|\tilde{\zeta}(x_0 + \tfrac{|x_0|(x-x_0)}{2|x-x_0|}) - \tilde{\zeta}(x)\Big| \\ &\le R_0^{d-1} \sup_{\bar{\mathbb{B}}_{1+|x_0|/2}(x_0)} |D\tilde{\zeta}| \\ &\le CR_0^{d-1}|x_0|^{-1}|\zeta(x_0)|. \end{aligned} \tag{2.10}$$

For $x \in B_{|x_0|/2}(x_0)$, applying Proposition C with $q = d - \varepsilon$ to the case $\alpha = 1$, there exists $\beta = \beta(d, \kappa, \varepsilon) \in (0, 1)$ such that

$$\begin{aligned}v(x) &\le \bar{v}(\tfrac{x-x_0}{|x_0|/2}) + C|x_0|^{-\beta}R_0^{d-1} \sup_{y\in\partial\mathbb{B}_1}\Big|D\tilde{\zeta}(x_0 + \tfrac{|x_0|}{2}y)\Big| \\ &\le \bar{v}(\tfrac{x-x_0}{|x_0|/2}) + CR_0^{d-1}|x_0|^{-\beta}|\zeta(x_0)|. \end{aligned} \tag{2.11}$$

Furthermore, using the fact that $\Delta\tilde{\zeta}(x_0 + \frac{|x_0|}{2}x) \le 0$ for $x \in \mathbb{B}_1$, we get $\bar{v}(x) \le R_0^{d-1}\tilde{\zeta}(x_0 + \frac{|x_0|}{2}x)$ in $\mathbb{B}_1$. This, together with (2.11), yields, for $x \in B_{|x_0|/2}(x_0)$,

$$v(x) \le R_0^{d-1}\tilde{\zeta}(x) + CR_0^{d-1}|x_0|^{-\beta}|\zeta(x_0)|. \tag{2.12}$$

Notice that $(G_R - v)$ is an ω-harmonic function on $B_{|x_0|/2}(x_0)$, and so

$$\max_{B_{|x_0|/2}(x_0)}(G_R - v) \le \max_{\partial B_{|x_0|/2}(x_0)}(G_R - v).$$

Therefore, combining this inequality and (2.10), (2.12), we get

$$\max_{B_{|x_0|/2}(x_0)}(G_R - R_0^{d-1}\tilde{\zeta}) \le \max_{\partial B_{|x_0|/2}(x_0)}(G_R - R_0^{d-1}\tilde{\zeta}) + CR_0^{d-1}|x_0|^{-\beta}|\zeta(x_0)|$$

which contradicts (2.9), since $|x_0| \in (R_0, R)$, $\delta = \beta/2$ by definition in Lemma 2.1, and $R_0 \ge K$ is chosen to be sufficiently large. Inequality (2.7) is proved.

Finally, when $x \in B_{R/2} \setminus B_{R_0}$, we have $G_R(x) \overset{(2.7)}{\le} h_2(x)$, and, by Lemma 2.1(i),

$$\begin{aligned}h_2(x) &\le R_0^{d-1}\alpha^{-1}[\bar{\zeta}(|x|) - \bar{\zeta}(R)] \\ &\lesssim R_0^{d-1}\int_{|x|}^{R}(-\bar{\zeta})'(r)\,\mathrm{d}r \\ &\lesssim R_0^{d-1}\int_{|x|}^{R} r^{1-d}\,\mathrm{d}r \lesssim R_0^{d-1}[U(|x|) - U(R)]. \end{aligned} \tag{2.13}$$

When $x \in B_{R_0} \setminus \{0\}$,

$$\begin{aligned}G_R(x) &\overset{(2.7)}{\le} h_1(x) = E_\omega^x[h_2(X_{s_0}) + |X_{s_0}| - |x|] \\ &\overset{(2.13),(2.6)}{\lesssim} R_0^{d-1}E_\omega^x[U(|X_{s_0}|) - U(R) + U(|x|) - U(|X_{s_0}|)] \\ &= R_0^{d-1}[U(|x|) - U(R)]. \end{aligned}$$

Note that, for $|x| \ge 1$, $U(|x|) - U(R) \lesssim U(|x| + 1) - U(R + 2)$.

When $x \in B_R \setminus B_{R/2}$,

$$\begin{aligned}G_R &\le h_3 \le CR_0^{d-1}R^{2-d}(e^{-2\alpha(|x|-1)/R} - e^{-2\alpha}) \\ &\lesssim R_0^{d-1}R^{2-d}(1 - \tfrac{|x|-1}{R}) = R_0^{d-1}R^{1-d}(R + 1 - |x|) \\ &\overset{(2.6)}{\lesssim} R_0^{d-1}[U(|x| + 1) - U(R + 2)]. \end{aligned}$$

The upper bound in Theorem 1.11 is proved by putting $\mathscr{H} = R_0$. □

2.2 Lower bounds of Green's functions

The proof of the lower bound of Theorem 1.11, which is similar to that of the upper bound, is via comparing G_R to appropriate test functions. However, unlike the Green function in the whole space, G_R is defined only in bounded region, and so the test functions should be carefully designed to capture the behavior of G_R near the boundary.

Lemma 2.5. *Define $\bar\eta : \mathbb{R} \to (0,\infty)$ as $\bar\eta(r) = (1+r^2)^{-\theta}$, where*

$$\theta := 1/(4\kappa) \geq d/2. \tag{2.14}$$

Define $\eta : \mathbb{R}^d \to \mathbb{R}$ as

$$\eta(y) = \bar\eta(|y|).$$

There exists a constant $C_0 = C_0(d,\kappa) > 0$ such that, for $x \in \mathbb{Z}^d$,

$$L_\omega \eta(x) \geq -\mathbb{1}_{x\in B_{C_0\theta^2}}.$$

The proof of Lemma 2.5 is in Section A.2 of the Appendix.

Let $\gamma = \gamma(\kappa) > 0$ be a large constant to be determined, and set

$$C_{\gamma,R} := \frac{[\bar\xi(R/2) - \bar\xi(R)]R^{d-2}}{e^{-\gamma/4} - e^{-\gamma}} \asymp e^{\gamma/4} \quad \text{when } R \geq R_0.$$

Definition 2.6. *Recall $\xi, \bar\xi, \eta, \bar\eta, \theta$ from Lemma 2.1 and Lemma 2.5. For any fixed $R \geq 4R_0$, we define three functions ℓ_i, $i = 1,2,3$, as*

$$\begin{aligned}
\ell_1(y) &= E^y_\omega[\ell_2(X_{s_0}) + \eta(y) - \eta(X_{s_0})], \quad y \in \bar B_{R_0};\\
\ell_2(y) &= R_0^{d-2-2\theta}\left[(\gamma^{-2}-1)(\bar\xi(R/2) - \bar\xi(R)) + \xi(y) - \bar\xi(R)\right], \quad y \in \mathbb{R}^d \setminus \{0\};\\
\ell_3(y) &= R_0^{d-2-2\theta}\gamma^{-2} C_{\gamma,R} R^{2-d}(e^{-\gamma|y|^2/R^2} - e^{-\gamma}), \quad y \in \mathbb{R}^d,
\end{aligned}$$

Note that ℓ_2 is defined first and the definition of ℓ_1 follows. Also, for $R \geq 4R_0$, we define a function $\ell : \bar B_R \to \mathbb{R}$ by

$$\ell(x) = \begin{cases} \ell_1(x), & x \in B_{R_0},\\ \ell_2(x), & x \in B_{R/2} \setminus B_{R_0},\\ \ell_3(x), & x \in \bar B_R \setminus B_{R/2}.\end{cases}$$

Lemma 2.7. *When $R \geq 4R_0$, there exists a constant $\gamma > 0$ such that the functions $\ell_1, \ell_2, \ell_3, \ell$ given in Definition 2.6 have the following properties.*

(a) $L_\omega \ell_1 = L_\omega \eta \geq -\mathbb{1}_{x\in B_{C_0\theta^2}}$ *for* $x \in B_{R_0}$*;*

(b) $\ell_1 = \ell_2$ *on* ∂B_{R_0}*, and* $\ell_2 = \ell_3$ *on* $\partial \mathbb{B}_{R/2}$*;*

(c) $\ell_2 \leq \ell_1$ *in* $B_{R_0} \setminus B_{R_0/2}$*.*

(d) $\ell_2 \leq \ell_3$ *in* $\mathbb{B}_R \setminus \mathbb{B}_{R/2}$*, and* $\ell_2 \geq \ell_3$ *in* $\mathbb{B}_{R/2} \setminus \mathbb{B}_{R/2-2}$*.*

(e) $L_\omega \ell_3 \geq 0$ *in* $B_R \setminus B_{R/2}$*.*

Proof. (a) follows from Lemma 2.5, and (b) follows from definition. To see (c), note that $\ell_1 - \ell_2 = E^x_\omega[f(|X_{s_0}|) - f(|x|)]$, where $f(r) = R_0^{d-2-2\theta}\bar\xi(r) - \bar\eta(r)$. Since $R_0 \geq K$, by taking K sufficiently large and by Lemma 2.1(i), we have

$$\begin{aligned}
f'(r) &\geq -(d-1/2)R_0^{d-2-2\theta} r^{1-d} + 2\theta(1+\tfrac{1}{r^2})^{-\theta-1} r^{d-2-2\theta}\\
&\overset{(2.14)}{\geq} (d-1/2)(r^{d-2-2\theta} - R_0^{d-2-2\theta}) \geq 0
\end{aligned}$$

and so $f(r)$ is decreasing for $r \in [R_0/2, R_0]$. Item (c) is proved.

Next, we will show (d). Indeed, we can write, for $y \neq 0$,

$$\ell_2(y) - \ell_3(y) = R_0^{d-2-2\theta} a(|y|) + A(R_0, R),$$

where $A(R_0, R)$ is a constant, and $a(r) = \bar{\xi}(r) - \gamma^{-2} C_{\gamma,R} R^{2-d} e^{-\gamma r^2/R^2}$. By Lemma 2.1(i),

$$\begin{aligned} a'(r) &\le -cr^{1-d} + C\gamma^{-1} C_\gamma R^{-d} e^{-\gamma r^2/R^2} \\ &\le cr^{1-d}\left(-1 + C\gamma^{-1}(\tfrac{r}{R})^d\right) < 0 \end{aligned}$$

for $r \in [R/2 - 2, R]$ if γ is chosen to be sufficiently large. Hence, $\ell_2 - \ell_3$ is radially decreasing in $\mathbb{B}_R \setminus \mathbb{B}_{R/2-2}$. Item (d) then follows from the fact that $\ell_2 = \ell_3$ on $\partial \mathbb{B}_{R/2}$. Item (e) is a consequence of (2.4) in Lemma 2.2. □

Proof of the lower bound in Theorem 1.11: It suffices to show that, for $x \in B_R$,

$$G_R^\omega(x) \gtrsim R_0^{d-2-2\theta}(U(|x|+1) - U(R+1)). \tag{2.15}$$

Recall $U(r)$ in (1.17). Indeed, (2.15) is equivalent to the lower bound of Theorem 1.11 when $x \in B_{R-1}$. When $x \in B_R \setminus B_{R-1}$, $R \ge 2$, taking $y \in B_R$ with $|y| \le |x| - 1$ and $|x-y|_1 \le 2d$, by the assumption (A2), inequality (2.15) yields

$$\begin{aligned} G_R(x) &\ge P_x^\omega(X_{|x-y|_1} = y) G_R(y) \\ &\gtrsim R_0^{d-2-2\theta}(U(|x|) - U(R+1)) \\ &\gtrsim R_0^{d-2-2\theta}(U(|x|+1) - U(R+2)). \end{aligned}$$

Our proof of (2.15) consists of several steps. Let C_0 be as in Lemma 2.5 and recall $\tau = \tau_R$ in (1.15).

When $R \in (1, 2C_0\theta^2)$, by (2.3), taking $A = A(\kappa) \ge 1$ sufficiently large,

$$L_\omega[G_R - (e^{-A|x|^2} - e^{-AR^2})] \le 0 \quad \text{in } B_R.$$

Since $G_R = 0 \ge (e^{-A|x|^2} - e^{-AR^2})$ on ∂B_R, by the maximum principle, we have $G_R \ge e^{-A|x|^2} - e^{-AR^2}$ in B_R. Thus, using the inequality $e^a \ge 1 + a$ for $a \ge 0$, we get, for $x \in B_R$, (Note that $R \asymp 1$ in this case.)

$$G_R \ge e^{-AR^2}\left(e^{A(R^2-|x|^2)} - 1\right) \ge e^{-AR^2}(R^2 - |x|^2) \gtrsim R - |x|$$

By the fact $a \ge \log(1+a), a \ge 0$, we have $R - |x| \gtrsim \log \frac{R+1}{|x|+1}$. Moreover, since $R \asymp 1$, for $d \ge 3$, we also have $R - |x| \gtrsim (|x|+1)^{2-d} - (R+1)^{2-d}$. Thus (2.15) holds for this case.

When $R \ge 2C_0\theta^2$, by assumption (A2), for $x \in B_R$ and any $y \in B_{C_0\theta^2}$,

$$\begin{aligned} G_R(x) &\ge \sum_{i=0}^\infty P_\omega^x(X_i = y, i < \tau_R) P_\omega^y(y \notin \{X_1, \ldots, X_{|y|_1}\}, X_{|y|_1} = 0) \\ &\ge G_R(x,y)\kappa^{|y|_1} \gtrsim G_R(x,y), \end{aligned}$$

and so (Recall $G_R(\cdot,\cdot)$ in Definition 1.10.)

$$G_R(x) \gtrsim G_R(x, B_{C_0\theta^2}) =: H_R(x), \quad x \in B_R. \tag{2.16}$$

Thus it suffices to obtain the corresponding lower bound for H_R defined above.

When $R \in (2C_0\theta^2, 4R_0]$, since (by Lemma 2.5) $L_\omega(H_R - \eta) \le 0$ in B_R, by the maximum principle, we have $H_R \ge \eta - \bar{\eta}(R)$ in B_R. Notice that

$$\bar{\eta}(r_1) - \bar{\eta}(r_2) \gtrsim R_0^{d-2-2\theta}(U(r_1+1) - U(r_2+1)), \quad \forall r_1 < r_2 \le 4R_0. \tag{2.17}$$

Indeed, for $d=2$,

$$\bar\eta(r_1)-\bar\eta(r_2)\ge\left[\left(\tfrac{1+r_1^2}{1+r_2^2}\right)^{-\theta}-1\right](1+r_2^2)^{-\theta}$$
$$\ge CR_0^{-2\theta}\log\tfrac{1+r_2^2}{1+r_1^2}\ge CR_0^{-2\theta}\log\tfrac{1+r_2}{1+r_1},$$

where we used the fact $a\ge\log(1+a)$ for $a\ge 0$ in the second inequality. For $d\ge 3$, recalling that $\theta\ge d/2$ in (2.14),

$$\begin{aligned}\bar\eta(r_1)-\bar\eta(r_2)&=(1+r_1^2)^{-\theta}-(1+r_2^2)^{-\theta}\\&\ge(1+r_2^2)^{d/2-1-\theta}[(1+r_1^2)^{1-d/2}-(1+r_2^2)^{1-d/2}]\\&\gtrsim R_0^{d-2-2\theta}[(1+r_1)^{2-d}-(1+r_2)^{2-d}]\end{aligned}$$

Hence, we obtain $H_R\gtrsim R_0^{d-2-2\theta}(U(|x|+1)-U(R+1))$ for this case.

It remains to consider the case $R\ge 4R_0$. To this end, we will prove

$$(G_R\overset{(2.16)}{\gtrsim})H_R\ge\ell\quad\text{in }\bar B_R. \tag{2.18}$$

Assume by contradiction that (2.18) fails, i.e., $\max_{\bar B_R}(\ell-H_R)>0$. By Lemma 2.7(a), $\max_{\bar B_R}(\ell-H_R)$ is achieved outside of B_{R_0}. Further, note that $(\ell-H_R)|_{\partial B_R}\le 0$. By Lemma 2.7 (e), (d), and the maximum principle,

$$\max_{\bar B_R\setminus B_{R/2}}(\ell-H_R)\le\max_{\partial(B_R\setminus B_{R/2})}(\ell_3-H_R)\le 0\vee\max_{B_{R/2}\setminus B_{R_0}}(\ell-H_R).$$

Hence, if $\max_{\bar B_R}(\ell-H_R)>0$, then there exists $x_0\in B_{R/2}\setminus B_{R_0}$ such that

$$(\ell-H_R)(x_0)=\max_{\bar B_R}(\ell-H_R).$$

Since $x_0\in B_{R/2}\setminus B_{R_0}$, by Lemma 2.7(c), (d),

$$(\ell_2-H_R)(x_0)\ge\max_{\bar B_{|x_0|/2}(x_0)}(\ell_2-H_R),$$

which is equivalent to

$$(R_0^{d-2-2\theta}\xi-H_R)(x_0)\ge\max_{\bar B_{|x_0|/2}(x_0)}(R_0^{d-2-2\theta}\xi-H_R).$$

Without loss of generality, assume $\bar a=I$, and set, for $y\in\mathbb{R}^d\setminus\{0\}$,

$$\tilde\xi(y):=\xi(y)-c|x_0|^{-(2+\delta)}|\xi(x_0)||y-x_0|^2,$$

where $c>0$ is chosen so that (Lemma 2.1(ii)) $-\Delta\tilde\xi\le 0$ for $|y|\ge C$. Then

$$(R_0^{d-2-2\theta}\tilde\xi-H_R)(x_0)\ge\max_{\partial B_{|x_0|/2}(x_0)}(R_0^{d-2-2\theta}\tilde\xi-H_R)+cR_0^{d-2-2\theta}|x_0|^{-\delta}|\xi(x_0)|. \tag{2.19}$$

Next, let $g(x)=\tilde\xi(x_0+\frac{|x_0|}{2}x)$, and let v be the solution of

$$\begin{cases}L_\omega v(x)=0 & x\in B_{|x_0|/2}(x_0)\\ v(x)=R_0^{d-2-2\theta}g(\frac{x-x_0}{|x-x_0|}) & x\in\partial B_{|x_0|/2}(x_0).\end{cases}$$

Note that v is close to $R_0^{d-2-2\theta}\tilde\xi$ on $\partial B_{|x_0|/2}(x_0)$ in the sense that $\left|g(\frac{x-x_0}{|x-x_0|})-\tilde\xi(x)\right|\le C|x_0|^{-1}|\xi(x_0)|$ for $x\in\partial B_{|x_0|/2}(x_0)$. Comparing the L_ω-harmonic functions v and H_R in $\bar B_{|x_0|/2}(x_0)$ via the maximum principle, we have, for $x\in B_{|x_0|/2}(x_0)$,

$$H_R(x)+\max_{\partial B_{|x_0|/2}(x_0)}(R_0^{d-2-2\theta}\tilde\xi-H_R)\ge v(x)-cR_0^{d-2-2\theta}|x_0|^{-1}|\xi(x_0)|. \tag{2.20}$$

By Proposition C, for $x \in B_{|x_0|/2}(x_0)$,

$$v(x) \geq \bar{v}(\tfrac{x-x_0}{|x_0|/2}) - CR_0^{d-2-2\theta}|x_0|^{-\beta}[g]_{C^{0,1}(\partial\mathbb{B}_1)} \geq \bar{v}(\tfrac{x-x_0}{|x_0|/2}) - CR_0^{d-2-2\theta}|x_0|^{-\beta}|\xi(x_0)|, \quad (2.21)$$

where $\bar{v}$ solves

$$\begin{cases} \operatorname{tr}(\bar{a}D^2\bar{v}) = \Delta\bar{v} = 0 & x \in \mathbb{B}_1 \\ \bar{v}(x) = R_0^{d-2-2\theta}g(x) & x \in \partial\mathbb{B}_1. \end{cases}$$

Furthermore, using the fact that $\Delta g(x) = |x_0|^2\Delta\tilde{\xi}(x_0 + \frac{|x_0|}{2}x) \geq 0$ for $x \in \mathbb{B}_1$, we get $\bar{v} \geq R_0^{d-2-2\theta}g$ in $\mathbb{B}_1$. Therefore, by (2.20), (2.21), for $x \in B_{|x_0|/2}(x_0)$,

$$R_0^{d-2-2\theta}g(\tfrac{x-x_0}{|x_0|/2}) - H_R(x) \leq \max_{\partial B_{|x_0|/2}(x_0)}(R_0^{d-2-2\theta}\tilde{\xi} - H_R) + CR_0^{d-2-2\theta}|x_0|^{-\beta}|\xi(x_0)|.$$

Noting that $g(\frac{x-x_0}{|x_0|/2}) = \tilde{\xi}(x)$, the above inequality contradicts (2.19) since $|x_0| > R_0 \geq K$, where K is chosen to be sufficiently large. Inequality (2.18) is proved.

When $x \in B_{R/2} \setminus B_{R_0}$, we have $G_R \overset{(2.18)}{\gtrsim} \ell_2$, and, by Lemma 2.1(i),

$$\begin{aligned} \ell_2 &\geq R_0^{d-2-2\theta}[\bar{\xi}(|x|) - \bar{\xi}(R)] \\ &\gtrsim R_0^{d-2-2\theta}\int_{|x|}^{R} r^{1-d}\,\mathrm{d}r \\ &\gtrsim R_0^{d-2-2\theta}(U(|x|+1) - U(R+1)). \end{aligned} \quad (2.22)$$

When $x \in B_{R_0}$,

$$\begin{aligned} G_R &\overset{(2.18)}{\gtrsim} \ell_1 = E_\omega^x[\ell_2(X_{s_0}) + \eta(x) - \eta(X_{s_0})] \\ &\overset{(2.22),(2.17)}{\gtrsim} R_0^{d-2-2\theta}E_\omega^x\left[U(|X_{s_0}|+1) - U(R+1) + U(|x|+1) - U(|X_{s_0}|+1)\right] \\ &= R_0^{d-2-2\theta}[U(|x|+1) - U(R+1)]. \end{aligned}$$

Finally, when $x \in B_R \setminus B_{R/2}$, using the inequality $e^a \geq 1 + a$ for $a \geq 0$,

$$\begin{aligned} G_R &\overset{(2.18)}{\gtrsim} \ell_3 \gtrsim R_0^{d-2-2\theta}R^{2-d}\left(e^{\gamma(1-|x|^2/R^2)} - 1\right) \\ &\gtrsim R_0^{d-2-2\theta}R^{2-d}(1 - \tfrac{|x|^2}{R^2}) \gtrsim R_0^{d-2-2\theta}R^{2-d}(1 - \tfrac{|x|}{R}). \end{aligned}$$

For $d = 2$, note that $1 - \frac{|x|}{R} \asymp \frac{R}{|x|} - 1 \geq \log\frac{R}{|x|}$. For $d = 3$, clearly

$$R^{2-d}(1 - \tfrac{|x|}{R}) \gtrsim |x|^{2-d}(1 - (\tfrac{|x|}{R})^{d-2}).$$

Our proof is complete. □

3 Heat kernel bounds and consequences

3.1 Integrability of ρ and the heat kernel bounds

Using bounds of the Green functions, we will obtain the exponential integrability (under $\mathbb{P}$) of the Radon-Nikodym derivative $\rho(\omega)$ (defined in (1.5)) and of the heat kernel of the RWRE.

The goal of this section is to prove Theorem 1.5. Recall the continuous-time RWRE in Definition 1.2 and its transition kernel $p_t(x,y)$. We remark that for the time continuous random walk (Y_t), setting

$$\tau^Y = \tau_R^Y := \inf\{t \geq 0 : Y_t \notin B_R\}, \quad (3.1)$$

the corresponding Green functions of (Y_t) can be defined similarly as

$$\int_0^\infty p_t^\omega(x,S)\,\mathrm{d}t,\ d\ge 3,\quad \text{and}\quad E_\omega^x\left[\int_0^{\tau^Y} \mathbb{1}_{Y_t\in S}\,\mathrm{d}t\right],\ d\ge 2,$$

and they have the same values as $G(x,S)$ and $G_R(x,S)$, respectively. Thus we do not need to distinguish notations in discrete and continuous time cases and use $G(x,S)$ and $G_R(x,S)$ to denote Green's functions in both settings.

Corollary 3.1. *Assume* (A1), (A2) *and* $d=2$. *For any* $\varepsilon>0$, *there exists a random variable* $\mathscr{H}=\mathscr{H}(\omega,d,\kappa,\varepsilon)>0$ *with* $\mathbb{E}[\exp(c\mathscr{H}^{d-\varepsilon})]<\infty$ *such that* $\mathbb{P}$*-almost surely, for* $R>0$,

$$\int_0^{R^2} p_t^\omega(x,0)\,\mathrm{d}t \lesssim \mathscr{H}(1+\log\tfrac{R+1}{|x|+1}),\quad \forall x\in B_R.$$

Proof. We only consider $R\ge 4$, because for $R<4$ and $x\in B_R$ we can simply use the statement for $R=4$ to get an upper bound $\mathscr{H}(1+\log\frac{5}{|x|+1})\lesssim\mathscr{H}$.

Let $R_k:=2^{k-1}R$, and define recursively $T_0=0$,

$$T_k:=\min\{t\ge T_{k-1}: Y_t\notin B_{R_k}\},\quad k\in\mathbb{N}.$$

Set $N_R=\max\{n\ge 0: T_n\le R^2\}$. Then, using the strong Markov property, a.s.,

$$\begin{aligned}
E_\omega^x\left[\int_0^{R^2}\mathbb{1}_{Y_t=0}\,\mathrm{d}t\right] &\le E_\omega^x\left[\sum_{k=1}^\infty\int_{T_{k-1}}^{T_k}\mathbb{1}_{\{Y_t=0,T_{k-1}\le R^2\}}\,\mathrm{d}t\right]\\
&\le \sum_{k=1}^\infty E_\omega^x\left[G_{R_k}^\omega(Y_{T_{k-1}})\mathbb{1}_{\{T_{k-1}\le R^2\}}\right]\\
&\lesssim G_R^\omega(x)+\mathscr{H}\sum_{k=2}^\infty P_\omega^x(T_{k-1}\le R^2),
\end{aligned}$$

where we used (by Theorem 1.11) the fact $G_{R_k}^\omega(Y_{T_{k-1}})\lesssim\mathscr{H}$, $k\ge 2$, in the last inequality. Note that (Y_t) is a martingale. By Hoeffding's inequality, for $k\ge 1$,

$$P_\omega^x(T_k\le R^2)\le P_\omega^x\left(\sup_{t\le R^2}|Y_t|\ge R_k\right)\le Ce^{-cR_k^2/R^2}\le C\exp(-c4^k).$$

The conclusion follows. □

Proof of Theorem 1.5: First, we will show the upper bounds in (a) and (b). To this end, for $r\ge 1$, we take $x_0\in\partial B_r$. We claim that

$$|x_0|^{2-d}\mathscr{H}^{d-1}\gtrsim\int_{r^2/2}^{r^2}p_t^\omega(x_0,0)\,\mathrm{d}t\gtrsim r^2\frac{\rho_\omega(0)}{\rho_\omega(B_r)}. \tag{3.2}$$

Indeed, the lower bound of (3.2) follows from the volume doubling property Theorem B(ii) and integrating the lower bound of (1.7). For $d=2$, the upper bound in (3.2) is a consequence of Corollary 3.1. When $d\ge 3$, by the upper bound in Theorem 1.11,

$$\int_{r^2/2}^{r^2}p_t^\omega(x_0,0)\,\mathrm{d}t\le G^\omega(x_0)=\lim_{R\to\infty}G_R^\omega(x_0)\lesssim|x_0|^{2-d}\mathscr{H}^{d-1},$$

which gives the upper bound in (3.2).

Note that $|x_0|\asymp r$. The upper bound in (b) is proved. The upper bound in (a) then follows from taking $r\to\infty$ and the ergodic theorem.

To obtain the lower bound in (b), for $r \geq 5$. Recall τ_r in (1.15) and $G_r(\cdot,\cdot)$ in Definition 1.10. For any fixed $y_0 \in \partial B_{r/2}$, the function $v(x) = G_r(y_0, x)/\rho_\omega(x)$ solves the *adjoint equation*

$$L_\omega^* v(x) := \sum_y \omega^*(x,y)[v(y) - v(x)] = 0, \quad x \in B_{r/2}, \tag{3.3}$$

where

$$\omega^*(x,y) := \rho_\omega(y)\omega(y,x)/\rho_\omega(x). \tag{3.4}$$

Here, we used the facts that $\sum_y \rho_\omega(y)\omega(y,x) = \rho_\omega(x)$ and $\sum_y G_r(y_0,y)\omega(y,x) = G_r(y_0,x)$. By the Harnack inequality for the adjoint operator [24, Theorem 6], we have $v(0) \asymp v(x)$ for all $x \in B_{r/4}$. Hence

$$G_r(y_0, 0)\frac{\rho_\omega(B_{r/4})}{\rho_\omega(0)} \asymp G_r(y_0, B_{r/4}). \tag{3.5}$$

Moreover, since $(|X_n|^2 - n)$ is a martingale under P_ω, by the optional stopping lemma we get $E_\omega^{y_0}[|X_{\tau_r}|^2 - \tau_r] = |y_0|^2 \geq 0$, and so

$$G_r(y_0, B_{r/4}) \leq E_\omega^{y_0}[\tau_r] \leq E_\omega^{y_0}[|X_{\tau_r}|^2] \leq Cr^2.$$

The above inequality, together with (3.5) and Theorem 1.11, yields

$$\frac{\rho_\omega(0)}{\rho_\omega(B_{r/4})} \gtrsim \frac{G_r(y_0,0)}{G_r(y_0,B_{r/4})} \gtrsim \frac{\mathscr{H}^{-s} r^{2-d}}{r^2} \gtrsim \mathscr{H}^{-s} r^{-d}.$$

The lower bound in Theorem 1.5(b) follows. Letting $r \to \infty$, we also get the lower bound in (a). □

3.2 The Green functions on the whole space: proof of Corollary 1.12

Proof of Corollary 1.12. The statement for $d \geq 3$ is an immediate consequence of Theorem 1.11. Thus it remains to prove the statement for $d = 2$ and $|x| \geq 1$.

To show the bounds for $d = 2$, recall that $A^\omega(x)$ is ω-harmonic on $\mathbb{Z}^d \setminus \{0\}$. Applying the Harnack inequality Theorem A.1 (Actually we only need the elliptic version, e.g., [40, 43]) to a constant number of balls centered on $\partial B_{|x|}$, we get

$$A^\omega(x) \asymp A^\omega(y) \quad \text{for all} \quad y \in \partial B_{|x|}.$$

In particular, letting $\tau_r = \inf\{t : Y_t \notin B_r\}$ be the exit time from B_r, we have

$$A^\omega(x) \asymp E_\omega^0[A(Y_{\tau_{|x|}})].$$

We claim that $E_\omega^0[A(Y_{\tau_R})] = G_R^\omega(0)$ for $R > 0$. Indeed, the function

$$v(y) := E_\omega^y\Big[\int_0^\infty p_t^\omega(Y_0, 0) - p_t^\omega(Y_{\tau_R}, 0)\,\mathrm{d}t\Big]$$

satisfies $L_\omega v(y) = \mathbb{1}_{y=0}$ and $v|_{\partial B_R} = 0$. Hence, by (2.1), we have

$$v(y) = G_R^\omega(y) \quad \text{for} \quad y \in B_R.$$

In particular, $v(0) = G_R^\omega(0)$, and the claim is proved. Therefore,

$$A^\omega(x) \asymp E_\omega^0[A(Y_{\tau_{|x|}})] = G_{|x|}^\omega(0)$$

Therefore, the upper and lower bounds of $A^\omega(x)$ follow from the inequality

$$\mathscr{H}^{-s}\log(1+|x|) \lesssim G_{|x|}^\omega(0) \lesssim \mathscr{H}\log(1+|x|)$$

due to Theorem 1.11. Our proof is complete. □

3.3 Optimal semigroup decay for $d \geq 3$ in i.i.d. environments: proof of Theorem 1.6

The Efron-Stein inequality (3.8) of Boucheron, Bousquet, and Massart [14] for i.i.d. ensembles will be used in our derivation of the variance decay for the semi-group. Notice that we assume (A3) throughout this subsection.

Let $\omega'(x), x \in \mathbb{Z}^d$, be independent copies of $\omega(x), x \in \mathbb{Z}^d$. For any $y \in \mathbb{Z}^d$, let $\omega'_y \in \Omega$ be the environment such that

$$\omega'_y(x) = \begin{cases} \omega(x) & \text{if } x \neq y, \\ \omega'(y) & \text{if } x = y. \end{cases}$$

That is, ω'_y is a modification of ω only at location y. For any measurable function Z of the environment ω, we write, for $y \in \mathbb{Z}^d$,

$$Z'_y = Z(\omega'_y), \quad \partial'_y Z(\omega) = Z'_y - Z, \tag{3.6}$$

and set

$$V(Z) = \sum_{y \in \mathbb{Z}^d} (\partial'_y Z)^2. \tag{3.7}$$

By an L_p version of Efron-Stein inequality [14, Theorem 3], for $q \geq 2$,

$$\mathbb{E}[|Z - \mathbb{E}Z|^q] \leq C q^{q/2} \mathbb{E}[V^{q/2}]. \tag{3.8}$$

Following the strategy of [31], our proof of the diffusive decay of the semi-group $\{P_t\}$ will make use of the Efron-Stein type inequality (3.8) and the Duhamel representation formula (3.11) for the vertical derivative. Let us reemphasize that, in the non-divergence form setting, there is no deterministic Gaussian bounds for the heat kernel, and the steady state $\mathbb{Q}$ of the environment process $(\bar{\omega}^t)_{t \geq 0}$ is not the same as the original measure $\mathbb{P}$. To overcome these difficulties, we employ crucially the heat kernel estimates and the (negative and positive) moment bounds of the Radon-Nikodym derivative $\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}}$ in Theorem 1.5.

For any $\zeta \in L^1(\Omega)$, we write

$$v(t) := P_t \zeta(\omega).$$

Then, its stationary extension $\bar{v}(t,x)$ solves the parabolic equation

$$\begin{cases} \partial_t \bar{v}(t,x) - L_\omega \bar{v}(t,x) = g(t,x) & t \geq 0, x \in \mathbb{Z}^d, \\ \bar{v}(0,x) = g_0(x) & x \in \mathbb{Z}^d, \end{cases} \tag{3.9}$$

with $g(t,x) = 0$ and $g_0(x) = \bar{\zeta}(x;\omega)$. In general, the solution of (3.9) can be represented by Duhamel's formula

$$\bar{v}(t,x) = \sum_z p_t^\omega(x,z) g_0(z) + \int_0^t p_{t-s}^\omega(x,z) g(s,z) \, \mathrm{d}s. \tag{3.10}$$

To apply (3.8), recall notations Z'_y and $\partial'_y Z$ in (3.6). By enlarging the probability space, we still use $\mathbb{P}$ to denote the joint law of (ω, ω'). For $y \in \mathbb{Z}^d$, applying ∂'_y to (3.9), we get that $\partial'_y \bar{v}$ satisfies

$$\begin{cases} \partial_t (\partial'_y \bar{v})(t,x) - L_\omega (\partial'_y \bar{v})(t,x) = (\partial'_y \omega_i(x)) \nabla_i^2 \bar{v}'_y(t,x) & t \geq 0, x \in \mathbb{Z}^d, \\ (\partial'_y \bar{v})(0,x) = \partial'_y \bar{\zeta}(x), & x \in \mathbb{Z}^d. \end{cases}$$

Here we used the convention of summation over repeated integer indices. Hence, by formula (3.10), $\partial'_y \bar{v}$ has the representation

$$\begin{aligned}\partial'_y \bar{v}(t,x) &= \sum_z \left[p_t^\omega(x,z)\partial'_y \bar{\zeta}(z) + \sum_{i=1}^d \int_0^t p_{t-s}^\omega(x,z)(\partial'_y \omega_i(z))\nabla_i^2 \bar{v}'_y(s,z)\,\mathrm{d}s \right] \\ &= \sum_{z\in y+\mathrm{Supp}(\zeta)} p_t^\omega(x,z)\partial'_y\bar{\zeta}(z) + \sum_{i=1}^d \int_0^t p_{t-s}^\omega(x,y)(\partial'_y\omega_i(y))\nabla_i^2\bar{v}'_y(s,y)\,\mathrm{d}s, \quad (3.11)\end{aligned}$$

where in the last equality we used the fact $\partial'_y \omega_i(z) = 0$ for $y \neq z$, and that $\partial'_y \bar{\zeta}(z) = 0$ for $z \notin y + \mathrm{Supp}(\zeta)$.

Proof of Theorem 1.6: Recall the notation $V(\cdot)$ in (3.7) and that we assume (A3). We write

$$K(y,s) = K(y,s;\omega,\omega') := \sum_{i=1}^d (\partial'_y \omega_i(y))\nabla_i^2 \bar{v}'_y(s,y).$$

We also write $\mathscr{H}_z := \mathscr{H}(\theta_z\omega)$ and $S := \mathrm{Supp}(\zeta)$. Without loss of generality, assume $E_{\mathbb{Q}}\zeta = 0$. Using (3.11), for any $p > 1$,

$$\begin{aligned}\|V(v(t))\|_p &= \|\sum_y (\partial'_y \bar{v}(t,0))^2\|_p \\ &\lesssim \|\sum_y \left(\sum_{z\in S} p_t^\omega(0,z+y)\right)^2\|_p + \|\sum_y \left(\int_0^t p_{t-s}^\omega(0,y)K(y,s)\,\mathrm{d}s\right)^2\|_p \\ &\lesssim (\#S)^2 \|\sum_y p_t^\omega(0,y)^2\|_p + \int_0^t \|\sum_y p_{t-s}^\omega(0,y)^2 K(y,s)^2\|_p^{1/2}\,\mathrm{d}s \\ &=: (\#S)^2 \mathrm{I} + \mathrm{II}^2, \end{aligned} \tag{3.12}$$

where $\#S$ denote the cardinality of S, and in the last inequality we applied the Cauchy-Schwarz inequality and Minkowski's integral inequality to the two norms respectively. Then, by Theorem 1.5(c),

$$\begin{aligned}\mathrm{I} = \|\sum_y p_t^\omega(0,y)^2\|_p &\lesssim (1+t)^{-d}\|\sum_z \mathscr{H}_z^{2(d-1)} e^{-c\mathfrak{h}(|z|,t)}\|_p \\ &\lesssim (1+t)^{-d}\sum_z e^{-c\mathfrak{h}(|z|,t)}\|\mathscr{H}_z^{2(d-1)}\|_p \\ &\lesssim (1+t)^{-d/2}\|\mathscr{H}^{2(d-1)p}\|_1^{1/p}, \end{aligned} \tag{3.13}$$

where in the last inequality we used the translation invariance of $\mathbb{P}$ to get $\|\mathscr{H}_z^{2(d-1)}\|_p = \|\mathscr{H}^{2(d-1)}\|_p = \|\mathscr{H}^{2(d-1)p}\|_1^{1/p}$.

Further, using Theorem 1.5(c) again,

$$\begin{aligned}\mathrm{II} &\lesssim \int_0^t (1+t-s)^{-d/2}\left(\sum_y e^{-c\mathfrak{h}(|y|,t-s)}\|\mathscr{H}_y^{2(d-1)}K(y,s)^2\|_p\right)^{1/2}\,\mathrm{d}s \\ &\lesssim \int_0^t (1+t-s)^{-d/4}\|\mathscr{H}^{2(d-1)p}K(0,s)^2\|_1^{1/(2p)}\,\mathrm{d}s \end{aligned} \tag{3.14}$$

where in the last inequality we used the translation-invariance of $\mathbb{P}$ and the fact that $\|\bar{v}\|_\infty \lesssim 1$.

Let $p' > 1$ denote the Hölder conjugate of $p > 1$. Since $\mathbb{E}[\exp(c\mathscr{H}^{d-\varepsilon})] < \infty$ for all $\varepsilon > 0$, we know that $\mathbb{E}[\mathscr{H}^k] < \infty$ for all $k > 0$. In particular, $\|\mathscr{H}^{2(d-1)p}\|_{p'} < \infty$. Hence

$$\begin{aligned}\|\mathscr{H}^{2(d-1)p}K(0,s)^2\|_1 &\le \|\mathscr{H}^{2(d-1)p}\|_{p'}\|K(0,s)^2\|_p\\ &\le C_p\sum_{i=1}^{d}\|\nabla_i^2\bar{v}(s,0)\|_{2p}^2\\ &\le C_p\sum_{e:|e|=1}\||\bar{v}(s,e)-\bar{v}(s,0)|^{2/p}\|_1^{1/p}\\ &\overset{\text{Hölder}}{\le} C_p\|\rho_\omega^{-1/p}\|_{p'}^{1/p}\sum_{e:|e|=1}\|\rho_\omega|\bar{v}(s,e)-\bar{v}(s,0)|^2\|_1^{1/p^2}\end{aligned} \tag{3.15}$$

where in the third inequality we used the fact $\|\bar{v}\|_\infty \le 1$. Then, setting

$$u(t) := \mathrm{Var}_{\mathbb{Q}}(v(t)),$$

by (3.15), Theorem 1.5(b) and Lemma 3.2, we obtain

$$\|\mathscr{H}^{2(d-1)p}K(0,s)^2\|_1 \le C_p\sum_e E_{\mathbb{Q}}[(\bar{v}(s,e)-\bar{v}(s,0))^2]^{1/p^2} \le C_p(-\frac{\mathrm{d}}{\mathrm{d}t}u)^{1/p^2}. \tag{3.16}$$

This inequality, together with (3.12),(3.13), (3.14), implies

$$\|V(v(t))\|_p^{1/2} \lesssim_p (1+t)^{-d/4} + \int_0^t (1+t-s)^{-d/4}(-\frac{\mathrm{d}}{\mathrm{d}t}u)^{1/(2p^3)}\,\mathrm{d}s,$$

where $\lesssim_p$ means that the multiplicative constant depends on (p,d,κ).

Furthermore, by Hölder's inequality and Theorem 1.5(a),

$$u(t) \le E_{\mathbb{Q}}[(v(t)-\mathbb{E}v(t))^2] \le \|\rho_\omega\|_{p'}\|(v(t)-\mathbb{E}v(t))^2\|_p \lesssim_p \|V(v(t))\|_p$$

where we applied (3.8) in the last inequality.

Therefore, we conclude that, for any $p > 1$,

$$u(t)^{1/2} \lesssim_p (1+t)^{-d/4} + \int_0^t (1+t-s)^{-d/4}(-\tfrac{\mathrm{d}}{\mathrm{d}t}u(s))^{1/(2p^3)}\,\mathrm{d}s.$$

When $d \ge 3$, we can take $p > 1$ sufficiently close to 1 and apply [2, Lemma 3.5]) (Under the notation of [2], we apply it to the case $\gamma = d/4$ and $\delta = 1/p^3$.) to obtain

$$\mathrm{Var}_{\mathbb{Q}}(v(t)) = u(t) \lesssim (1+t)^{-d/2}.$$

Thus, (1.12) is proved. By Hölder's inequality,

$$\|v(t)\|_1 \le \|\rho_\omega^{-1}\|_1^{1/2}\|\rho_\omega v(t)^2\|_1^{1/2} \le Cu(t)^{1/2} \le C(1+t)^{-d/4}.$$

Then, by the triangle inequality, we get $E_{\mathbb{Q}}[(v(t)-\mathbb{E}v(t))^2] \lesssim (1+t)^{-d/2}$. By Hölder's inequality, for any $q > 1$,

$$\||v(t)-\mathbb{E}v(t)|^{2/q}\|_1 \le \|\rho_\omega^{-1/q}\|_{q'}\|\rho^{1/q}|v(t)-\mathbb{E}v(t)|^{2/q}\|_q \lesssim_q E_{\mathbb{Q}}[(v(t)-\mathbb{E}v(t))^2]^{1/q}.$$

Display (1.13) is proved. □

Lemma 3.2. *For any bounded measurable function* $\zeta \in \mathbb{R}^\Omega$*,*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Var}_{\mathbb{Q}}(v(t)) \lesssim -\sum_{e:|e|=1} E_{\mathbb{Q}}\left[(\bar{v}(t,e)-\bar{v}(t,0))^2\right].$$

Proof. Without loss of generality, assume $\mathbb{E}[v(t)] = 0$. In the following, L_ω only acts on the spatial variables. Then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} E_{\mathbb{Q}}[v(t)^2] &= 2E_{\mathbb{Q}}[v(t)L_\omega \bar{v}(t,0)] \\ &= E_{\mathbb{Q}}\left[L_\omega(\bar{v}(t,0)^2) - \sum_{e:|e|=1} \omega(0,e)(\bar{v}(t,e)-\bar{v}(t,0))^2\right] \\ &= -\sum_{e:|e|=1} E_{\mathbb{Q}}\left[\omega(0,e)(\bar{v}(t,e)-\bar{v}(t,0))^2\right]\end{aligned}$$

where in the last equality we used the fact that (since $(\bar{\omega}^t)_{t\ge 0}$ is stationary under $\mathbb{Q}\times P_\omega$) for any $f \in L^1(\mathbb{Q})$, $E_{\mathbb{Q}}[L_\omega \bar{f}(0;\omega)] = 0$. The lemma follows by the uniform ellipticity assumption. □

3.4 Proof of Corollary 1.8 in i.i.d. environments: Existence of a stationary corrector for $d \ge 5$

Proof of Corollary 1.8: Without loss of generality, assume $E_{\mathbb{Q}}[\zeta] = 0$. By Theorem 1.6, since $\int_0^\infty (1+t)^{-d/4}\,\mathrm{d}t < \infty$ when $d \ge 5$, the limit

$$\phi(\omega) = \lim_{t\to\infty} \phi_t(\omega) := -\lim_{t\to\infty} \int_0^t (P_s\zeta)\,\mathrm{d}s \tag{3.17}$$

exists in $L^p(\mathbb{P})$ for any $p \in (0,2)$. By (3.9), $\bar{\phi}_t(x)$ satisfies

$$L_\omega \bar{\phi}_t(x) = \zeta(\theta_x\omega) - P_t\zeta(\theta_x\omega), \quad x \in \mathbb{Z}^d.$$

Note that by Theorem 1.6, $\lim_{t\to\infty} P_t\zeta = 0$ in $L^p(\mathbb{P})$, $\forall p \in (0,2)$. Therefore, taking $L^p(\mathbb{P})$ limits as $t \to \infty$, we conclude that $\bar{\phi}$ satisfies $L_\omega\bar{\phi}(x) = \bar{\zeta}(x)$ $\mathbb{P}$-a.s., $x \in \mathbb{Z}^d$. □

A Appendix

Define the parabolic operator $\mathscr{L}_\omega$ as

$$\mathscr{L}_\omega u(x,t) = \sum_{y:y\sim x} \omega(x,y)[u(y,t)-u(x,t)] - \partial_t u(x,t)$$

for every function $u : \mathbb{Z}^d \times \mathbb{R} \to \mathbb{R}$ which is differentiable in t.

Theorem A.1. ([24, Proposition 5]) Assume $\frac{\omega}{\mathrm{tr}\omega} > 2\kappa I$ for some $\kappa > 0$ and $R > 0$. Any non-negative function u with $\mathscr{L}_\omega u = 0$ in $B_{2R} \times (0, 4R^2)$ satisfies

$$\sup_{B_R\times(R^2,2R^2)} u \le C \inf_{B_R\times(3R^2,4R^2)} u.$$

The following Hölder estimate is a standard consequence of the Harnack inequality Theorem A.1.

Corollary A.2. Assume $\frac{\omega}{\mathrm{tr}\omega} > 2\kappa I$ for some $\kappa > 0$ and $(x_0,t_0) \in \mathbb{Z}^d \times \mathbb{R}$. There exists $\gamma = \gamma(d,\kappa) \in (0,1)$ such that any non-negative function u with $\mathscr{L}_\omega u = 0$ in $B_R(x_0)\times(t_0-R^2,t_0)$, $R > 0$, satisfies

$$|u(\hat{x}) - u(\hat{y})| \le C\left(\frac{r}{R}\right)^\gamma \sup_{B_R(x_0)\times(t_0-R^2,t_0)} u$$

for all $\hat{x}, \hat{y} \in B_r(x_0) \times (t_0 - r^2, t_0)$ and $r \in (0,R)$.

A.1 Proof of Proposition C

Our proof of Proposition C follows similar lines as the proof of [32, Theorem 1.5], with necessary modifications to address the fact that g and f have less regularity than the functions in [32, Theorem 1.5].

Before giving a proof, recall that by [32, Proposition 2.1], for any $p \in (0,d)$, there exists δ_p depending on (d,κ,p) such that for any $R>0$, the solution $\phi : \bar{B}_R \to \mathbb{R}$ of

$$\begin{cases} L_\omega \phi = \bar{a} - a & \text{in } B_R, \\ \phi = 0 & \text{on } \partial B_R \end{cases} \tag{A.1}$$

satisfies

$$\mathbb{P}(\max_{B_R} |\phi| \geq C R^{2-\delta_p}) \leq C \exp(-cR^p). \tag{A.2}$$

Set $\delta := \delta_1$. For $q \in (0,d)$, let $\gamma = \gamma(d,\kappa,q)$ be the constant

$$\gamma = \min\left\{ \frac{d-q}{d(1+\delta)}, \frac{1}{2} \right\} \tag{A.3}$$

and set

$$R_0 := R^\gamma, \quad \sigma = \min\{n \geq 0 : X_n - X_0 \notin B_{R_0}\}.$$

Let

$$\omega_0 = \tfrac{\omega}{\mathrm{tr}(\omega)}, \quad \psi_0 = \psi_0(\omega) = \tfrac{\psi}{\mathrm{tr}(\omega)}.$$

Following [32, Definition 4.1], we define bad points.

Definition A.1. *Let $\delta = \delta(d,\kappa)$ be as above. We say that a point x is good (and otherwise bad) if for any $\zeta(\omega) \in \{\psi_0, \omega_0\}$,*

$$\Big| E_\omega^x \Big[\sum_{i=0}^{\sigma-1} (E_{\mathbb{Q}}[\zeta] - \zeta(\bar{\omega}_i)) \Big] \Big| \leq C \|\zeta\|_\infty R_0^{2-\delta}.$$

Note that by (A.2) *and Chebyshev's inequality,* $\mathbb{P}(x \text{ is bad}) \lesssim e^{-CR_0} R_0^{2-\delta} \lesssim e^{-cR_0}$*.*

We will give first the proof for the special case $g \in C^{2,\alpha}(\partial \mathbb{B}_1)$. It is a small modification of the proof of [32, Theorem 1.5].

Proof of Proposition C for the case $f \in C^\alpha(\mathbb{B}_1), g \in C^{2,\alpha}(\partial\mathbb{B}_1)$ and $y=0$: Note that if $g \in C^{2,\alpha}(\partial \mathbb{B}_1)$, it can be extended to be a function $\tilde{g} \in C^{2,\alpha}(\mathbb{B}_2)$ such that

$$|\tilde{g}|_{2,\alpha;\mathbb{B}_2} \leq C |g|_{2,\alpha;\partial\mathbb{B}_1}.$$

By [30, Theorem 6.6],

$$|\bar{u}|_{2,\alpha,\mathbb{B}_1} \lesssim |f|_{0,\alpha;\mathbb{B}_1} \|\tfrac{\psi}{\mathrm{tr}(\omega)}\|_\infty + |g|_{2,\alpha;\partial\mathbb{B}_1} =: A \tag{A.4}$$

Step 1. Set $\bar{u}_R(x) = \bar{u}(x/R)$ for $x \in \bar{\mathbb{B}}_R$. We will show that in B_R, $\bar{u}_R$ is very close to the solution $\hat{u} : \bar{B}_R \to \mathbb{R}$ of

$$\begin{cases} L_\omega \hat{u} = \frac{1}{2}\mathrm{tr}[\omega_0 D^2 \bar{u}_R] & \text{in } B_R \\ \hat{u} = g(\frac{x}{|x|}) & \text{on } \partial B_R. \end{cases}$$

To this end, define u_+, u_- by $u_\pm(x) = \tilde{g}(\frac{x}{R+1}) \pm CA\frac{(R+1)^2 - |x|^2}{R^2}$, $x \in \bar{\mathbb{B}}_{R+1}$. Here A is as defined in (A.4). Then, for $x \in \mathbb{B}_{R+1}$, taking C large enough,

$$\mathrm{tr}[\bar{a} D^2(u_+ - \bar{u}_{R+1})] \leq \frac{c}{R^2}(|g|_{2,\alpha;\mathbb{B}_1} + |f|_{0;\mathbb{B}_1} - CA) \leq 0,$$

and similarly $\mathrm{tr}[\bar{a}(u_- - \bar{u}_{R+1})] \geq 0$. The comparison principle then yields

$$u_- \leq \bar{u}_{R+1} \leq u_+ \quad \text{in } \mathbb{B}_{R+1}.$$

In particular, for $x \in \partial B_R$, $|\bar{u}_{R+1}(x) - \tilde{g}(\frac{x}{R+1})| \lesssim A\frac{(R+1)^2-|x|^2}{R^2} \lesssim \frac{A}{R}$ and so

$$\max_{\partial B_R} |\hat{u} - \bar{u}_{R+1}| = \max_{x\in\partial B_R} |g(\tfrac{x}{|x|}) - \bar{u}_{R+1}(x)| \lesssim \frac{A}{R}. \tag{A.5}$$

Moreover, noting that $D^2\bar{u}_R(x) = R^{-2}D^2\bar{u}(\frac{x}{R})$, in B_R,

$$\begin{aligned} |L_\omega(\hat{u} - \bar{u}_{R+1})| &= |\mathrm{tr}[\omega_0(D^2\bar{u}_R - \nabla^2\bar{u}_{R+1})]| \\ &\leq |\mathrm{tr}[\omega_0(D^2\bar{u}_R - D^2\bar{u}_{R+1})]| + |\mathrm{tr}[\omega_0(D^2\bar{u}_{R+1} - \nabla^2\bar{u}_{R+1})]| \\ &\lesssim R^{-2-\alpha}|\bar{u}|_{2,\alpha;\bar{\mathbb{B}}_1} \overset{\text{(A.4)}}{\lesssim} AR^{-2-\alpha}. \end{aligned} \tag{A.6}$$

Hence, by the ABP maximum principle [32, Lemmas 2.3 and 2.4], (A.5) and (A.6) imply $\max_{B_R} |\hat{u} - \bar{u}_{R+1}| \lesssim AR^{-\alpha}$, and so, by (A.4),

$$\max_{B_R} |\hat{u} - \bar{u}_R| \lesssim AR^{-\alpha}.$$

Step 2. Let $v = u - \hat{u}$. Then v solves

$$\begin{cases} L_\omega v = \frac{1}{2}\mathrm{tr}[(\bar{a} - \omega_0)D^2\bar{u}_R] + \frac{1}{R^2}f(\frac{x}{R})(\psi_0 - \bar{\psi}) & \text{in } B_R \\ v = 0 & \text{on } \partial B_R. \end{cases}$$

Note that, for $x \in B_{R-R_0}$ and $y \in B_{R_0}(x)$,

$$\begin{aligned} &|D^2\bar{u}_R(x) - D^2\bar{u}_R(y)| \leq R^{-2}(\tfrac{R_0}{R})^\alpha[\bar{u}]_{2,\alpha;\mathbb{B}_1} \overset{\text{(A.4)}}{\lesssim} AR^{-2}(\tfrac{R_0}{R})^\alpha, \\ &|f(\tfrac{x}{R}) - f(\tfrac{y}{R})| \leq \left(\tfrac{R_0}{R}\right)^\alpha [f]_{\alpha;\mathbb{B}_1}. \end{aligned}$$

Hence, if $x \in B_{R-R_0}$ is a good point, setting

$$\bar{\omega}_0^i := \omega_0(X_i) \quad \text{and} \quad \psi_0^i := \psi_0(\theta_{X_i}\omega),$$

and noting that $E_\omega^x[\sigma] \leq (R_0+1)^2$, we have

$$\begin{aligned} &E_\omega^x[v(X_\sigma) - v(x)] \\ &= E_\omega^x\left[\sum_{i=0}^{\sigma-1} \tfrac{1}{2}\mathrm{tr}[(\bar{\omega}_0^i - \bar{a})D^2\bar{u}_R(X_i)] + \tfrac{1}{R^2}f(\tfrac{X_i}{R})(\bar{\psi} - \psi_0^i)\right] \\ &\lesssim \mathrm{tr}\left[E_\omega^x[\sum_{i=0}^{\sigma-1}(\bar{\omega}_0^i - \bar{a})]D^2\bar{u}_R(x)\right] + \tfrac{1}{R^2}f(\tfrac{x}{R})E_\omega^x[\sum_{i=0}^{\sigma-1}(\bar{\psi} - \psi_0^i)] + A\tfrac{R_0^\alpha}{R^{2+\alpha}}E_\omega^x[\sigma] \\ &\lesssim \tfrac{1}{R^2}\left|E_\omega^x[\sum_{i=0}^{\sigma-1}(\bar{\omega}_0^i - \bar{a})]\right||\bar{u}|_{2;\mathbb{B}_1} + \tfrac{1}{R^2}|f|_{0;\mathbb{B}_1}\left|E_\omega^x[\sum_{i=0}^{\sigma-1}(\bar{\psi} - \psi_0^i)]\right| + A\left(\tfrac{R_0}{R}\right)^{2+\alpha} \\ &\lesssim AR^{-2}R_0^{2-\delta} + A\left(\tfrac{R_0}{R}\right)^{2+\alpha} \lesssim AR^{-2-\alpha\delta\gamma}R_0^2. \end{aligned}$$

Let $\tau_R = \min\{n \geq 0 : X_n \notin B_R\}$ and set

$$w(x) = v(x) + C_1AR^{-2-\alpha\delta\gamma}E_\omega^x[\tau_R].$$

Then, for any good point $x \in B_{R-R_0}$, by choosing C_1 big enough,

$$E_\omega^x[w(X_\sigma) - w(x)] = E_\omega^x[v(X_\sigma) - v(x)] - C_1AR^{-2-\alpha\delta\gamma}E_\omega^x[\sigma] < 0,$$

where we used the fact that $E_\omega^x[\sigma] \geq R_0^2$. This implies

$$\partial w(x; B_R) = \emptyset \quad \text{for any good point } x \in B_{R-R_0} \tag{A.7}$$

where $\partial w(x; B_R)$ denote the sub-differential set of w at x with respect to B_R. For the definition of the sub-differential set and the ABP inequality, we refer to [32, Definition 2.2, Lemmas 2.3 and 2.4]. Next, we will apply the ABP inequality to bound $|v|$ from the above.

By [32, Lemma 2.4], since

$$L_\omega w = L_\omega v - C_1 A R^{-2-\alpha\delta\gamma} \lesssim A R^{-2},$$

we know that $|\partial w(x; B_R)| \lesssim A^d R^{-2d}$ for $x \in B_R$. Let

$$\mathscr{B}_R = \mathscr{B}_R(\omega, \gamma) := \#\text{bad points in } B_{R-R_0},$$

where $\#S$ denotes the cardinality of a set S. Display (A.7) then yields

$$|\partial w(B_R)| \lesssim [\mathscr{B}_R + \#(B_R \setminus B_{R-R_0})] A^d R^{-2d} \lesssim (\mathscr{B}_R + R^{d-1+\gamma}) A^d R^{-2d}.$$

Hence, by the ABP inequality [32, Lemma 2.4],

$$\min_{B_R} w \geq -CR|\partial w(B_R)|^{1/d} \gtrsim -A(R^{-1}\mathscr{B}_R^{1/d} + R^{-(1-\gamma)/d}).$$

Therefore, noting that $\max_{x \in B_R} E_\omega^x[\tau_R] \leq (R+1)^2$ and choosing $\delta < 1/d$,

$$\min_{B_R} v \geq \min_{B_R} w - CAR^{-\alpha\delta\gamma} \gtrsim -A(R^{-1}\mathscr{B}_R^{1/d} + R^{-\alpha\gamma\delta}).$$

Similar bound for $\min_{B_R}(-v)$ can be obtained by substituting f, g by $-f, -g$ in the problem. Therefore

$$\max_{B_R} |v| \lesssim A(R^{-1}\mathscr{B}_R^{1/d} + R^{-\alpha\gamma\delta}).$$

Step 3. Combining results in Steps 1 and 2, we get

$$\max_{B_R} |u - \bar{u}_R| \lesssim A(R^{-1}\mathscr{B}_R^{1/d} + R^{-\alpha\gamma\delta}).$$

It is shown in Step 6 of [32, Page 24, Proof of Theorem 1.5] that, with

$$\mathscr{X} = \mathscr{X}(\omega) := \max_{R \geq 1} R^{-\gamma}\mathscr{B}_R^{1/d}, \tag{A.8}$$

we have $\mathbb{E}[\exp(c\mathscr{X}^d)] < \infty$. Therefore, recalling the values of γ, A in (A.3), (A.4), we conclude that

$$\max_{B_R} |u - \bar{u}_R| \lesssim R^{-\alpha\gamma\delta}(1 + R^{-q/d}\mathscr{X})(|f|_{0,\alpha;\mathbb{B}_1}\|\psi_0\|_\infty + |g|_{2,\alpha;\partial\mathbb{B}_1}). \tag{A.9}$$

□

In what follows, we will relax the regularity of g to be $C^{0,\alpha}(\partial\mathbb{B}_1)$.

Proof of Proposition C. First, we consider the case $y = 0$. The function $g \in C^{0,\alpha}(\partial\mathbb{B}_1)$ can be extended into $\mathbb{R}^d$ so that $g \in C^{0,\alpha}(\mathbb{R}^d)$ and

$$|g|_{0,\alpha;\mathbb{B}_2} \leq C|g|_{0,\alpha;\partial\mathbb{B}_1}.$$

We can further obtain a smooth perturbation of it. To this end, let $\rho \in C^\infty(\mathbb{R}^d)$ be a *mollifier* supported on $\mathbb{B}_1$ with $\int_{\mathbb{B}_1} \rho \,\mathrm{d}x = 1$. For $h \in (0,1)$, set $\rho_h(x) := h^{-d}\rho(\frac{x}{h})$, and let $g_h = \rho_h * g$. That is, $g_h(x) = \int_{\mathbb{R}^d} \rho_h(x-z)g(z)\,\mathrm{d}z$. Then g_h satisfies

$$\begin{cases} |g - g_h|_{0;\partial\mathbb{B}_1} \le Ch^\alpha |g|_{0,\alpha;\partial\mathbb{B}_1} \\ |g_h|_{2,\alpha;\partial\mathbb{B}_1} \le Ch^{-2}|g|_{0,\alpha;\partial\mathbb{B}_1}. \end{cases}$$

Next, for $h \in (0,1)$, let $v : \bar{B}_R \to \mathbb{R}$ and $\bar{v} : \bar{\mathbb{B}}_1 \to \mathbb{R}$ be solutions of

$$\begin{cases} \frac{1}{2}\mathrm{tr}(\omega\nabla^2 v) = \frac{1}{R^2} f(\frac{x}{R})\psi(\theta_x\omega) & \text{in } B_R \\ v(x) = g_h(\frac{x}{R}) & \text{for } x \in \partial B_R \end{cases}$$

and

$$\begin{cases} \mathrm{tr}(\bar{a}D^2\bar{v}) = f\bar{\psi} & \text{in } \mathbb{B}_1 \\ \bar{v} = g_h & \text{on } \partial\mathbb{B}_1. \end{cases}$$

Then, $\max_{\mathbb{B}_1}|\bar{u} - \bar{v}| \le \max_{\partial\mathbb{B}_1}|g - g_h| \le h^\alpha |g|_{0,\alpha;\partial\mathbb{B}_1}$, and

$$\max_{B_R}|u - v| \le \max_{\partial B_R}|g(\tfrac{x}{R}) - g_h(\tfrac{x}{R})| \le Ch^\alpha|g|_{0,\alpha;\partial\mathbb{B}_1}.$$

Moreover, by (A.9), with $A_1 = \|f\|_{C^{0,\alpha}(\mathbb{B}_1)}\|\frac{\psi}{\mathrm{tr}(\omega)}\|_\infty + [g]_{C^{0,\alpha}(\partial\mathbb{B}_1)}$ as in Proposition C,

$$\begin{aligned} \max_{x\in B_R}|v(x) - \bar{v}(\frac{x}{R})| &\lesssim R^{-\alpha\gamma\delta}(1 + R^{-q/d}\mathscr{X})(|f|_{0,\alpha;\mathbb{B}_1}\|\tfrac{\psi}{\mathrm{tr}(\omega)}\|_\infty + |g_h|_{2,\alpha;\partial\mathbb{B}_1}) \\ &\lesssim A_1 h^{-2}R^{-\alpha\gamma\delta}(1 + R^{-q/d}\mathscr{X}). \end{aligned}$$

Notice that up to an additive constant, we may assume that $\inf_{\partial B_1} g = 0$, so that $|g|_{0,\alpha;\partial\mathbb{B}_1} \le C[g]_{0,\alpha;\partial\mathbb{B}_1}$. Therefore, putting $h = R^{-\alpha\gamma\delta/3}$, by the triangle inequality,

$$\max_{x\in B_R}|u - \bar{u}(\tfrac{x}{R})| \lesssim A_1 R^{-\alpha\gamma\delta/3}(1 + R^{-q/d}\mathscr{X}). \tag{A.10}$$

We proved Proposition C for the case $y = 0$ with $\beta = \gamma\delta/3$.

Finally, for any $y \in B_{3R}$, it follows from (A.10) that

$$\max_{x\in B_R(y)}\left|\bar{u}(\tfrac{x-y}{R}) - u(x)\right| \lesssim A_1 R^{-\alpha\beta}(1 + \mathscr{X}(\theta_y\omega)R^{-q/d}).$$

Let

$$\mathscr{B}_R(y) = \mathscr{B}_R(\theta_y\omega, \gamma) := \#\text{bad points in } B_{R-R_0}(y).$$

Observe that $\mathscr{B}_R(y) \le \mathscr{B}_{4R}(0)$. Thus, recalling the definition of $\mathscr{X}$ in (A.8),

$$\mathscr{X}(\theta_y\omega) \le \max_{R\ge 1} R^{-\gamma}\mathscr{B}_{4R}^{1/d} \le 4^\gamma\mathscr{X}.$$

Our proof of Proposition C is complete. □

A.2 Proofs of Lemmas 2.5 and 2.2

Proof of Lemma 2.5: By direct computation, for any $i = 1, \ldots, d$, $x \in \mathbb{Z}^d$,

$$\begin{aligned} \partial_i^2\eta(x) &= \theta(1 + |x|^2)^{-\theta-2}[4(\theta+1)x_i^2 - 2|x|^2 - 2], \\ |\partial_i^3\eta(x)| &\le C\theta^3|x|(1 + |x|^2)^{-\theta-2}. \end{aligned} \tag{A.11}$$

Moreover, noting that for any $y \in \mathbb{R}^d$ and $x \notin B_\theta$ with $|y - x| \le 1$,

$$|\partial_i^3\eta(y)| \le C\theta^3(\tfrac{1+\theta}{\theta})^{\theta+2}|x|(1 + |x|^2)^{-\theta-2} \le C\theta^3|x|(1 + |x|^2)^{-\theta-2}$$

and $|\nabla_i^2\eta(x)-\partial_i^2\eta(x)|\le C\sup_{y:|y-x|\le 1}|\partial_i^3\eta(y)|$, we have, for a sufficiently large constant $C_0(\kappa)>1$ and $x\notin B_{C_0\theta^2}$,

$$\begin{aligned}L_\omega\eta(x)&\ge\sum_{i=1}^d\frac{\omega_i(x)}{2\mathrm{tr}\omega(x)}\left[\partial_i^2\eta(x)-C\theta^3|x|(1+|x|^2)^{-\theta-2}\right]\\&\overset{\text{(A.11)}}{\ge}\theta(1+|x|^2)^{-\theta-2}[4\kappa(\theta+1)|x|^2-|x|^2-1-C\theta^2|x|]>0.\end{aligned}$$

On the other hand, for $x\in B_{C_0\theta^2}$, clearly $L_\omega\eta(x)\ge-\eta(x)\ge-1$.

The lemma is proved. □

Proof of Lemma 2.2. Computations show that, for $x\neq 0$, $i=1,\dots,d$,

$$\begin{aligned}\partial_i^2e^{-2\alpha|x|/R}&=\left(-\tfrac{2\alpha}{|x|}+\tfrac{2\alpha x_i^2}{|x|^3}+\tfrac{4\alpha^2x_i^2}{R|x|^2}\right)R^{-1}e^{-2\alpha|x|/R},\\\partial_i^3e^{-2\alpha|x|/R}&=\left(\frac{6\alpha}{R}+\frac{3}{|x|}-\frac{4\alpha x_i^2}{R|x|^2}-\frac{4\alpha^2x_i^2}{R^2|x|}-\frac{3x_i^2}{|x|^3}\right)\frac{2\alpha x_i}{R|x|^2}e^{-2\alpha|x|/R}.\end{aligned}$$

Note that, for $i=1,\dots,d$, $x\in B_R\setminus B_{R/2}$,

$$\left|(\partial_i^2-\tfrac12\nabla_i^2)e^{-2\alpha|x|/R}\right|\le C\sup_{y:|y-x|\le1}\left|\partial_i^3e^{-2\alpha|y|/R}\right|\le\frac{C\alpha}{R^3}e^{-2\alpha|x|/R},$$

and so

$$\Big|\sum_{i=1}^d\omega(x,x+e_i)(\partial_i^2-\tfrac12\nabla_i^2)e^{-2\alpha|x|/R}\Big|\le\frac{C\alpha}{R^3}e^{-2\alpha|x|/R}.\tag{A.12}$$

Further, by taking $K>0$ sufficiently large (note $R\ge K$), and choosing $\alpha>0$ to be sufficiently small, we have, for $x\in B_R\setminus B_{R/2}$,

$$\begin{aligned}&\sum_{i=1}^d\omega(x,x+e_i)\partial_i^2e^{-2\alpha|x|/R}\\&=\sum_{i=1}^d\omega(x,x+e_i)\left(-\tfrac{2\alpha}{|x|}+\tfrac{2\alpha x_i^2}{|x|^3}+\tfrac{4\alpha^2x_i^2}{R|x|^2}\right)R^{-1}e^{-2\alpha|x|/R}\\&\le\left(-\tfrac{\alpha}{|x|}+\tfrac{(1-2\kappa)\alpha|x|^2}{|x|^3}+\tfrac{2(1-2\kappa)\alpha^2|x|^2}{R|x|^2}\right)R^{-1}e^{-2\alpha|x|/R}\\&\le C(-1+C\alpha)\tfrac{\alpha}{R^2}e^{-2\alpha|x|/R}\le-\tfrac{C\alpha}{R^2}e^{-2\alpha|x|/R}.\end{aligned}$$

This, together with (A.12), implies,

$$L_\omega e^{-2\alpha|x|/R}\le-\tfrac{C\alpha}{R}e^{-2\alpha|x|/R},\quad\text{for }x\in B_R\setminus B_{R/2}.$$

Display (2.2) is proved.

To prove (2.3), note that when $x=0$, $L_\omega(e^{-A|x|^2})=e^{-A}-1>-1$. When $x\in\mathbb{Z}^2\setminus\{0\}$, choosing $A>0$ sufficiently large,

$$\begin{aligned}L_\omega(e^{-A|x|^2})&=e^{-A|x|^2}\left[\sum_{i=1}^d\omega(x,x+e_i)(e^{2Ax_i-A}+e^{-2Ax_i-A})-1\right]\\&\ge e^{-A|x|^2}[\kappa e^{2A-A}-1]>0.\end{aligned}$$

Display (2.3) is proved.

It remains to prove (2.4). Using the inequalities $e^a + e^{-a} \geq 2 + a^2, e^a \geq 1 + a$, we get, by taking A sufficiently large, for $x \in B_R \setminus B_{R/2}$,

$$\begin{aligned} L_\omega(e^{-A|x|^2/R^2}) &= e^{-A|x|^2/R^2} \sum_{i=1}^d \omega(x, x+e_i) \left[e^{-A(1+2x_i)/R^2} + e^{-A(1-2x_i)/R^2} - 2\right] \\ &\geq e^{-A|x|^2/R^2} \sum_{i=1}^d \omega(x, x+e_i) \left[e^{-A/R^2}(2 + 4A^2x_i^2/R^4) - 2\right] \\ &\geq \tfrac{A}{R^2} e^{-A|x|^2/R^2} \sum_{i=1}^d \omega(x, x+e_i) \left[\tfrac{4Ax_i^2}{R^2}(1 - \tfrac{A}{R^2}) - 1\right] \\ &\geq \tfrac{A}{R^2} e^{-A|x|^2/R^2} \left[\frac{2\kappa A|x|^2}{R^2} - 1\right] > 0. \end{aligned}$$

Our proof is complete. □

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