Electronic Journal of Probability

Electron. J. Probab. **29** (2024), article no. 115, 1–26.
ISSN: 1083-6489 https://doi.org/10.1214/24-EJP1179

Power variations and limit theorems for stochastic processes controlled by fractional Brownian motions

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Abstract

In this paper we establish limit theorems for power variations of stochastic processes controlled by fractional Brownian motions with Hurst parameter $H \le 1/2$. We show that the power variations of such processes can be decomposed into the mix of several weighted random sums plus some remainder terms, and the convergences of power variations are dominated by different combinations of those weighted sums depending on whether $H < 1/4$, $H = 1/4$, or $H > 1/4$. We show that when $H \ge 1/4$ the centered power variation converges stably at the rate $n^{-1/2}$, and when $H < 1/4$ it converges in probability at the rate n^{-2H}. We determine the limit of the mixed weighted sum based on a rough path approach developed in [33].

Keywords: power variation; discrete rough integral; fractional Brownian motion; controlled rough path; limit theorems; estimation of volatility.
MSC2020 subject classifications: 60B10; 60G15.
Submitted to EJP on January 23, 2024, final version accepted on July 27, 2024.

1 Introduction

In this paper we establish limit theorems for power variations of low-regularity processes in a general rough path framework. Recall that for a stochastic process $(y_t, t \in [0,1])$ the power variation of order $p > 0$ (p-variation for short) is defined as

$$\sum_{k=0}^{n-1} \left| y_{t_{k+1}} - y_{t_k} \right|^p, \tag{1.1}$$

where $0 = t_0 < t_1 < \cdots < t_n = 1$ is a partition of the time interval $[0,1]$. The power variation has been widely used in quantitative finance for the estimation of volatility and related parameters; see [1, 5, 6, 7, 8, 14] and references therein.

When y is a semimartingale the power variation has been discussed in [4, 6, 9, 27, 30, 31, 44, 45]. The case of stationary Gaussian was treated in [26, 32]. When y is a Young

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integral (see [46]) driven by fractional Gaussian processes the power variation has been investigated in [3, 16, 36]. The study of power variations (1.1) in the non-semimartingale case is closely related to the limits of weighted random sums. For example, a key step in [3, 16, 36] is to observe that when y is a Young integral of the form: $y_t = \int_0^t z_s dx_s$ and the integrator x is Hölder continuous of order greater than $1/2$, the increment $y_{t_{k+1}} - y_{t_k}$ in (1.1) can be replaced by its first-order approximation $z_{t_k}(x_{t_{k+1}} - x_{t_k})$. We refer the reader to [12, 13, 15, 33, 34, 37, 39, 38, 41, 43] for discussions about limit theorems of weighted random sums.

Recently, empirical evidence was found that security volatility actually has much lower regularity than semimartingales (see [21]). The statement is further supported by other empirical work based on both return data (see [11, 20, 22]) and option data (see [10, 19, 35]). Motivated by these advances in quantitative finance, it is then natural to ask the following question: Is there a limit theorem for power variations when the process is “rougher” than semimartingale, and if so, under what conditions does the limit theorem hold?

A main difficulty in the low-regularity case is that the aforementioned relation between y and its first-order approximation is no longer true. In fact, we will see that the difference between the power variations of a low-regularity process y and that of its first-order approximation has nonzero contribution to the limit of power variation. A second difficulty is that the weighted sums corresponding to (1.1) involve functionals of the forms $|x|^p$ and $|x|^p \cdot \text{sign}(x)$, where x is the underlying Gaussian process for y (see Definition 2.1 for our definition of the processes of x and y). While both functionals can be expressed as sums of finite chaos functionals, these sums are infinite whenever p is non-integer. So the known results in the literatures (e.g. [38, 40]) about weighted sums do not apply, and a proper treatment for these infinite sums is required.

In this paper we show that a limit theorem of power variation does hold under the assumption that y is a process “controlled” by a fractional Brownian motion (fBm for short). The controlled process is a main concept in the theory of rough path, and it is broad enough to contain two important models of stochastic processes we have in mind: the rough integrals and the solutions to rough differential equations (see Example 2.3). Our result generalizes [16] to fBms with any Hurst parameter $H \in (0, 1)$.

Remark 1.1. It should be noted that there is no essential difference between rough differential equations and rough integrals in the study of p-variation. In fact, for a rough differential equation of the form $y_t = \int_0^t f(y_s) dx_s$ the process $f(y_t)$, $t \geq 0$ can be considered a controlled process of x. Therefore, the solution to a rough differential equation can be seen as an important example of the controlled rough integral. On the other hand, a rough integral can also be formed as the solution to a proper rough differential equation.

Our main result can be informally stated as follows. The reader is referred to Theorem 4.2 for a more precise statement.

Theorem 1.2. *Let x be a fBm with Hurst parameter $H \leq 1/2$ and $(y, y', \dots, y^{(\ell-1)})$ be a process controlled by x almost surely (see Definition 2.1) for some $\ell \in \mathbb{N}$. Define the function $\phi(x) = |x|^p$, $x \in \mathbb{R}$ for some constant $p > 0$, and denote ϕ' and ϕ'' the derivatives of ϕ, and c_p and σ are constants given in* (3.56) *and* (3.61)*, respectively. Let*

$$U_t^n = n^{pH-1} \sum_{0 \leq t_k < t} \phi(y_{t_{k+1}} - y_{t_k}) - c_p \int_0^t \phi(y'_u) du \qquad t \in [0, 1]. \tag{1.2}$$

Let W be a standard Brownian motion independent of x. Then *(i) For $1/4 < H \leq 1/2$, $p \in [3,\infty) \cup \{2\}$ and $\ell \geq 4$ we have the convergence in law:*

$$n^{1/2} U_1^n \to \sigma \int_0^1 \phi(y'_u) dW_u, \qquad \text{as } n \to \infty.$$

(ii) For $H = 1/4$, $p \in [5,\infty) \cup \{2,4\}$ and $\ell \geq 6$ the following convergence in law holds:

$$n^{1/2} U_1^n \to \sigma \int_0^1 \phi(y'_u) dW_u - \frac{c_p}{8} \int_0^1 \phi''(y'_u)(y''_u)^2 du + \frac{(p-2)c_p}{24} \int_0^1 \phi'(y'_u) y'''_u du$$

as $n \to \infty$.
(iii) For $H < 1/4$, $p \in [5,\infty) \cup \{2,4\}$ and $\ell \geq 6$ we have the convergence in probability:

$$n^{2H} U_1^n \to -\frac{c_p}{8} \int_0^1 \phi''(y'_u)(y''_u)^2 du + \frac{(p-2)c_p}{24} \int_0^1 \phi'(y'_u) y'''_u du \qquad \text{as } n \to \infty.$$

As mentioned previously, the limit of power variation in the low-regularity case is not solely determined by the first-order approximation of y. A first step of our proof is thus to consider the higher-order approximation of y and to estimate the corresponding weighted random sums and remainder terms. The convergences of mixed weighted sums and power variation are based on a rough path approach developed in [33]. In particular, we will see that the rough path approach allows us to avoid the application of Malliavin integration by parts for functionals of infinite chaos.

Remark 1.3. The p-variation for integrals involving fBm x with Hurst parameter $H < 1/2$ has already been considered in the literature; see [16, 33]. However, the article [16] focuses on the case of Young integrals only. Namely, the authors consider p-variation of the integral $\int_0^t u_s dx_s$ in which u requires a regularity of order higher than $1 - H$. In the current paper, we extend the results in [16] for an integrand process u which has the same regularity as the fBm x. In fact, we will see in Remark 4.4 that one recovers from Theorem 1.2 (i) the results obtained in [16, Theorem 4].

In [33] the authors have shown that the convergence $U_t^n \to 0$ in probability holds as $n \to \infty$ under the settings of Theorem 1.2, where U^n is given in (1.2). The current paper is thus a study of the asymptotic error distribution for this convergence. The reader is referred to Remark 4.3 for a further discussion about the relation between a result in [33] and the current contribution.

The paper is structured as follows: In Section 2 we introduce the concept of discrete rough paths and discrete rough integrals and recall some basic results of the rough paths theory. In Section 3 we derive some useful estimates and limit theorem results for weighted random sums related to fBm. In Section 4 we prove the limit theorem of power variation for processes controlled by fBm.

1.1 Notation

Let $\pi : 0 = t_0 < t_1 < \cdots < t_n = 1$ be a partition on $[0,1]$. For $s,t \in [0,1]$ such that $s < t$, we write $[\![s,t]\!]$ for the discrete interval that consists of t_k's such that $t_k \in [s,t]$ and the two endpoints s and t. Namely, $[\![s,t]\!] = \{t_k : s \leq t_k \leq t\} \cup \{s,t\}$. For $N \in \mathbb{N} = \{1,2,\dots\}$ we denote the discrete simplex $\mathcal{S}_N([\![s,t]\!]) = \{(u_1,\dots,u_N) \in [\![s,t]\!]^N : u_1 < \cdots < u_N\}$. Similarly, we denote the continuous simplex: $\mathcal{S}_N([s,t]) = \{(u_1,\dots,u_N) \in [s,t]^N : u_1 < \cdots < u_N\}$.

Throughout the paper we work on a probability space $(\Omega, \mathcal{F}, P)$. If X is a random variable, we denote by $|X|_{L_p}$ the L_p-norm of X. The letter K stands for a constant independent of any important parameters which can change from line to line. We write $A \lesssim B$ if there is a constant $K > 0$ such that $A \leq KB$. We denote $[a]$ the integer part of a.

2 Preliminary results

In this section, we introduce the concept of discrete rough paths and discrete rough integrals, and recall some basic results of the rough paths theory. In the second part of the section we recall the elements of Wiener chaos expansion and fractional Brownian motion.

2.1 Controlled rough paths and algebraic properties

This subsection is devoted to introducing the main rough paths notations which will be used in the sequel. The reader is referred to [17, 18] for an introduction to the rough path theory.

Recall that the continuous simplex $\mathcal{S}_k([0,1])$ is defined in Section 1.1. We denote by $\mathcal{C}_k$ the set of functions $g:\mathcal{S}_k([0,1])\to\mathbb{R}$ such that $g_{u_1\cdots u_k}=0$ whenever $u_i=u_{i+1}$ for some $i\le k-1$. Such a function will be called a $(k-1)$-*increment*. For $f\in\mathcal{C}_1$ and $g\in\mathcal{C}_2$ we define the operator δ as follows:

$$\delta f_{st}=f_t-f_s \quad \text{and} \quad \delta g_{sut}=g_{st}-g_{su}-g_{ut}. \tag{2.1}$$

We introduce a general notion of controlled rough process which will be used throughout the paper:

Definition 2.1. Let x and $y,y',y'',\dots,y^{(\ell-1)}$ be real-valued continuous processes on $[0,1]$. Denote the 2-increments: $x^i_{st}=(\delta x_{st})^i/i!$, $(s,t)\in\mathcal{S}_2([0,1])$, $i=0,1,\dots,\ell-1$. For convenience, we also write $y^{(0)}=y$, $y^{(1)}=y'$, $y^{(2)}=y''$,…, and $\mathbf{y}=(y^{(0)},\dots,y^{(\ell-1)})$. We define the remainder processes

$$\begin{aligned} r^{(\ell-1)}_{st} &= \delta y^{(\ell-1)}_{st} \\ r^{(k)}_{st} &= \delta y^{(k)}_{st}-y^{(k+1)}_s x^1_{st}-\cdots-y^{(\ell-1)}_s x^{\ell-k-1}_{st}, \qquad k=0,1,\dots,\ell-2, \end{aligned} \tag{2.2}$$

for $(s,t)\in\mathcal{S}_2([0,1])$. We call $\mathbf{y}$ *a rough path controlled by* (x,ℓ,α) *almost surely* for some constant $\alpha\in(0,1)$ if for any $\varepsilon>0$ there is a finite random variable $G_{\mathbf{y}}\equiv G_{\mathbf{y},\varepsilon}$ (that is, $G_{\mathbf{y}}<\infty$ almost surely) such that $|r^{(k)}_{st}|\le G_{\mathbf{y}}(t-s)^{(\ell-k)\alpha-\varepsilon}$ for all $(s,t)\in\mathcal{S}_2([0,1])$ and $k=0,1,\dots,\ell-1$. We call $\mathbf{y}$ *a rough path controlled by* (x,ℓ,α) *in* L_p for some $p>0$ if there exist constants $K>0$, $\alpha\in(0,1)$ such that $|r^{(k)}_{st}|_{L_p}\le K(t-s)^{(\ell-k)\alpha}$ for all $(s,t)\in\mathcal{S}_2([0,1])$ and $k=0,\dots,\ell-1$.

Remark 2.2. Let x be a α-Hölder continuous path with $\alpha>0$. Let z be a rough path controlled by (x,ℓ,α) almost surely (see Definition 2.1). According to the rough path theory (see e.g. [18, 24]) if we take $\ell=[1/\alpha]$ then for $t\in[0,1]$ the compensated Riemann sum:

$$\sum_{0\le t_k<t}\sum_{i=0}^{\ell-1} z^{(i)}_{t_k}x^{i+1}_{t_kt_{k+1}}$$

converges as $n\to\infty$, and its limit is defined as the controlled rough integral $\int_0^t z_s dx_s$. Nevertheless, the current paper aims to consider p-variation of a controlled process in general, and not only the controlled rough integrals. So we have left the value of ℓ unspecified. In particular, our main result (Theorem 1.2) includes the case of y being a rough integral: $y_t=\int_0^t z_s dx_s$ as a special case.

In the following we give some examples of controlled rough paths defined in Definition 2.1.

Example 2.3. Let x_t, $t\in[0,1]$ be a standard fractional Brownian motion (fBm in the sequel) with Hurst parameter $H\in(0,1)$ (see Section 2.3 for the definition of fBm). It is well-known that for any $\varepsilon>0$ the fBm is $(H-\varepsilon)$-Hölder continuous almost surely. For a

continuous function V defined on $\mathbb{R}$, we define the differential operator $\mathcal{L}_V$ such that for any differentiable function f we have $\mathcal{L}_V f = V f'$. Denote $\mathcal{L}_V^i = \mathcal{L}_V \circ \cdots \circ \mathcal{L}_V$ the ith iteration of $\mathcal{L}_V$.
(i) Let V be a sufficiently smooth function on $\mathbb{R}$. Set $z_t^{(i)} = V^{(i)}(x_t)$ for $i = 0, \dots, \ell - 1$. Then $(z, z', \dots, z^{(\ell-1)})$ is a rough path controlled by (x, ℓ, H) almost surely.
(ii) Let b and V be continuous functions defined on $\mathbb{R}$. According to [18, Theorem 12.10], given proper regularity conditions for b and V there exists a unique solution for the stochastic differential equation (SDE in the sequel): $dy_t = b(y_t)dt + V(y_t)dx_t$. (When x is a Brownian motion, that is when $H = 1/2$, the differential equation coincides the corresponding classical Stratonovich-type SDE.) Let $y_t' = V(y_t)$ and $y_t^{(i)} = \mathcal{L}_V^{i-1} V(y_t)$, $i = 2, \dots, \ell - 1$. Then $(y, y', \dots, y^{(\ell-1)})$ is a rough path controlled by (x, ℓ, H) almost surely.
(iii) Let $\ell = [1/H]$ and let $(z, z', \dots, z^{(\ell-1)})$ be a rough path controlled by (x, ℓ, H) almost surely. Let y be the rough integral $y_t := \int_0^t z_s dx_s$, $t \in [0,1]$ in the sense of [24]. An explicit example of rough integral is $y_t = \int_0^t V(x_s)dx_s$. Denoting $y' = z, \dots, y^{(\ell)} = z^{(\ell-1)}$, then $(y, y', \dots, y^{(\ell)})$ is a rough path controlled by $(x, \ell + 1, H)$ almost surely.

By Definition 2.1 it is easy to show that the partial sequence of $\mathbf{y}$ and the functions of y are both controlled rough paths. We state this fact and omit the proof for sake of conciseness. We refer the readers to e.g. [17, Lemma 4.1, Theorem 4.16, Lemma 7.3] for related results in the multidimensional case, and refer to [25] for a theory of controlled rough path at arbitrary level of roughness.

Lemma 2.4. *Let $\mathbf{y}$ be a rough path controlled by (x, ℓ, α) almost surely (resp., in L_p for some $p > 0$). Then*
(i) For any $i = 0, \dots, \ell - 1$, $(y^{(i)}, \dots, y^{(\ell-1)})$ is a rough path controlled by $(x, \ell - i, \alpha)$ almost surely (resp., in L_p).
(ii) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function which has derivatives up to order $(L-1)$ and the $(L-1)$th derivative $f^{(L-1)}$ is Lipschitz. Let $z_s^{(0)} = f(y_s)$ and

$$z_s^{(r)} = \sum_{i=1}^{r} \frac{f^{(i)}(y_s)}{i!} \sum_{\substack{1 \le j_1, \dots, j_i \le (L \wedge \ell) - 1 \\ j_1 + \cdots + j_i = r}} \frac{r!}{j_1! \cdots j_i!} y_s^{(j_1)} \cdots y_s^{(j_i)}, \qquad r = 0, \dots, (L \wedge \ell) - 1.$$

for $s \in [0,1]$. For example, we have $z^{(1)} = f'(y_s)y_s'$ and $z^{(2)} = f''(y_s)(y_s')^2 + f'(y_s)y_s''$. Then $(z^{(0)}, \dots, z^{((L \wedge \ell)-1)})$ is a rough path controlled by $(x, L \wedge \ell, H)$ almost surely (resp., in L_p).

Let us also recall an algebraic result from [33, Lemma 2.5].

Lemma 2.5. *Let x, $\mathbf{y}$ and $r^{(i)}$, $i = 0, \dots, \ell - 1$ be continuous processes satisfying* (2.2). *Then we have the following relation: $\delta r_{sut}^{(0)} = \sum_{i=1}^{\ell-1} r_{su}^{(i)} x_{ut}^i$.*

2.2 Discrete rough integrals

We introduce some "discrete" integrals defined as Riemann type sums. Namely, let f and g be functions on $\mathcal{S}_2([0,1])$ with values in vector spaces V and W, respectively. Let $\mathcal{D}_n = \{0 = t_0 < \cdots < t_n = 1\}$ be a generic partition of $[0,1]$. We define the discrete integral of f with respect to g as:

$$\mathcal{J}_s^t(f, g) := \sum_{s \le t_k < t} f_{st_k} \otimes g_{t_k t_{k+1}}, \qquad (s,t) \in \mathcal{S}_2([0,1]), \tag{2.3}$$

where we use the convention that $\mathcal{J}_s^t(f, g) = 0$ whenever $\{t_k : s \le t_k < t\} = \emptyset$. We highlight that f in (2.3) is a function of two variables. Similarly, if f is a path on $[0,1]$, then we define the discrete integral of f with respect to g as:

$$\mathcal{J}_s^t(f,g) := \sum_{s\le t_k<t} f_{t_k}\otimes g_{t_kt_{k+1}}, \qquad (s,t)\in\mathcal{S}_2([0,1]). \tag{2.4}$$

Remark 2.6. Let V and W be two vector spaces over a field F, with respective dimensions $n,m\in\mathbb{N}$ and bases $B_V=\{v_i, i=1,\dots,n\}$ and $B_W=\{w_j, j=1,\dots,m\}$. Then the tensor product space $V\otimes W$ can be identified as the vector space of F-valued $n\times m$ matrices with the canonical basis, and $v_i\otimes w_j$ is identified with the matrix with entry 1 in the ith row, jth column and 0 elsewhere. Let $v\in V$ and $w\in W$ be such that $v=\sum_i a_iv_i$ and $w=\sum_j b_jw_j\in W$ for some $a_i,b_j\in F$. Then we have $v\otimes w=\sum_{i,j}a_ib_jv_i\otimes w_j$.

2.3 Chaos expansion and fractional Brownian motions

Let $d\gamma(x)=(2\pi)^{-1/2}e^{-x^2/2}dx$ be the standard Gaussian measure on the real line, and let $f\in L_2(\gamma)$ be such that $\int_{\mathbb{R}}f(x)d\gamma(x)=0$. It is well-known that the function f can be expanded into a series of Hermite polynomials as: $f(x)=\sum_{q=d}^\infty a_qH_q(x)$, where $d\ge 1$ is some integer and $H_q(x)=(-1)^qe^{\frac{x^2}{2}}\frac{d^q}{dx^q}e^{-\frac{x^2}{2}}$ is the Hermite polynomial of order q. Recall that we have the iteration formula: $H_{q+1}(t)=xH_q(x)-H_q'(x)$. If $a_d\neq 0$, then d is called the *Hermite rank* of the function f. Since $f\in L_2(\gamma)$, we have $\|f\|^2_{L_2(\gamma)}=\sum_{q=d}^\infty|a_q|^2q!<\infty$. The reader is referred to [42, 40] for an introduction of chaos expansion.

Let x be a standard fractional Brownian motion (fBm for short) with Hurst parameter $H\in(0,1)$, that is x is a continuous Gaussian process such that $\mathbb{E}[x_sx_t]=\frac{1}{2}(|s|^{2H}+|t|^{2H}-|s-t|^{2H})$. The fBm x is almost surely γ-Hölder continuous for all $\gamma<H$. Define the covariance function ρ by

$$\rho(k)=\mathbb{E}(\delta x_{01}\delta x_{k,k+1}). \tag{2.5}$$

Let $\mathcal{H}$ be the completion of the space of indicator functions with respect to the inner product $\langle\mathbf{1}_{[u,v]},\mathbf{1}_{[s,t]}\rangle_{\mathcal{H}}=\mathbb{E}(\delta x_{uv}\delta x_{st})$.

Remark 2.7. It can be shown that $\sum_{k\in\mathbb{Z}}\rho(k)=0$ whenever $H<\frac{1}{2}$. In fact, consider the relation:

$$\mathbb{E}[\delta x_{01}(x_m-x_{-n})]=\sum_{k=-n}^{m-1}\mathbb{E}[\delta x_{01}\delta x_{k,k+1}]=\sum_{k=-n}^{m-1}\rho(k).$$

Then we have the convergence:

$$\lim_{n,m\to\infty}\mathbb{E}[\delta x_{01}(x_m-x_{-n})]=\sum_{k\in\mathbb{Z}}\rho(k).$$

On the other hand, a direct computation shows that

$$\begin{aligned}\mathbb{E}[\delta x_{01}(x_m-x_{-n})] &= \mathbb{E}[x_1x_m]-\mathbb{E}[x_1x_{-n}]\\ &= \frac{1}{2}(1+m^{2H}-(m-1)^{2H})-\frac{1}{2}(1+n^{2H}-(n+1)^{2H}).\end{aligned} \tag{2.6}$$

Note that $m^{2H}-(m-1)^{2H}\to 0$ as $m\to\infty$ when $H<1/2$. This implies that (2.6) converges to 0, and thus $\sum_{k\in\mathbb{Z}}\rho(k)=0$.

The following result shows that given two sequences of stable convergence and convergence in probability, respectively, their joint sequence is also of stable convergence. Recall that X_n is called convergent to X stably if $(X_n,Z)\to(X,Z)$ in distribution as $n\to\infty$ for any $Z\in\mathcal{F}$. The reader is referred to [2, 29] for an introduction to stable convergence.

Lemma 2.8. *Let $Y_n^{(1)}$, $Y_n^{(2)}$, $n \in \mathbb{N}$ be two sequences of random variables and denote the σ-field: $\mathcal{F}^Y = \sigma\{Y_n^{(i)}, i = 1, 2, n \in \mathbb{N}\}$. Let $Y^{(1)}$ be a random variable such that the convergence in distribution $(Y_n^{(1)}, Z) \to (Y^{(1)}, Z)$ as $n \to \infty$ holds for any $Z \in \mathcal{F}^Y$. Suppose that we have the convergence in probability $Y_n^{(2)} \to Y^{(2)}$ as $n \to \infty$ for some random variable $Y^{(2)}$. Then the convergence in distribution $(Y_n^{(1)}, Y_n^{(2)}, Z) \to (Y^{(1)}, Y^{(2)}, Z)$ as $n \to \infty$ holds for any $Z \in \mathcal{F}^Y$. In particular, we have the convergence in distribution $(Y_n^{(1)} + Y_n^{(2)}, Z) \to (Y^{(1)} + Y^{(2)}, Z)$ as $n \to \infty$ for any $Z \in \mathcal{F}^Y$.*

Proof. Since $Y_n^{(2)} - Y^{(2)} \to 0$ in probability it follows that the two sequences $(Y_n^{(1)}, Y^{(2)} + (Y_n^{(2)} - Y^{(2)}), Z)$ and $(Y_n^{(1)}, Y^{(2)}, Z)$ have the same limit. On the other hand, by the convergence in distribution of $Y_n^{(1)}$ and the fact that $Y^{(2)} \in \mathcal{F}^Y$ we have the convergence $(Y_n^{(1)}, Y^{(2)}, Z) \to (Y^{(1)}, Y^{(2)}, Z)$. We conclude that the convergence $(Y_n^{(1)}, Y_n^{(2)}, Z) \to (Y^{(1)}, Y^{(2)}, Z)$ as $n \to \infty$ holds. This completes the proof. □

3 Upper-bound estimate and limit theorem for weighted random sums

In this section we derive some useful estimates and limit theorem results for weighted random sums related to fBm.

3.1 Upper-bound estimate of weighted random sums

We prove a general upper-bound estimate result for weighted random sums. In the second part of the subsection we apply this estimate result to weighted sums involving fBms. Recall that for a continuous process x_t, $t \in [0, 1]$ and an integer $i \in \mathbb{N}$ we denote the 2-increment: $x^i_{st} = (\delta x_{st})^i / i!$, $(s, t) \in \mathcal{S}_2([0, 1])$.

Proposition 3.1. *Let x be a continuous process on $[0, 1]$. Let $\mathbf{y} = (y^{(0)}, \dots, y^{(\ell-1)})$ be a rough path on $[0, 1]$ controlled by (x, ℓ, α) in L_2 for some $\alpha > 0$ and $\ell \in \mathbb{N}$, and let $(r^{(i)}, i = 0, \dots, \ell - 1)$ be the remainder processes of $\mathbf{y}$ defined in Definition 2.1. Let h be a 1-increment defined on $\mathcal{S}_2([\![0, 1]\!])$. Let $\beta_i \in [0, 1]$, $i = 0, 1, \dots, \ell - 1$ be some constants such that*

$$\beta := \min_{i=0,\dots,\ell-1} \{(\ell - i)\alpha + \beta_i\} > 1. \tag{3.1}$$

Suppose that there exists a constant $K > 0$ such that

$$|\mathcal{J}_s^t(x^i, h)|_{L_2} \leq K(t - s)^{\beta_i} \tag{3.2}$$

for any $(s, t) \in \mathcal{S}_2([0, 1])$ satisfying $t - s \geq 1/n$. Then we can find a constant $K > 0$ independent of n such that the following estimates hold:

$$|\mathcal{J}_s^t(r^{(0)}, h)|_{L_1} \leq K(t - s)^{\beta} \qquad \text{and} \qquad |\mathcal{J}_s^t(y, h)|_{L_1} \leq K(t - s)^{\beta_0} \tag{3.3}$$

for $(s, t) \in \mathcal{S}_2([0, 1])$ such that $t - s \geq 1/n$.

Proof. Denote $R_{st} := \mathcal{J}_s^t(r^{(0)}, h)$ for $(s, t) \in \mathcal{S}_2([0, 1])$. Recall that the operator δ for 2-increment is defined in (2.1). So, for $(s, u, t) \in \mathcal{S}_3([0, 1])$, we have

$$\delta R_{sut} = \mathcal{J}_s^t(r^{(0)}, h) - \mathcal{J}_s^u(r^{(0)}, h) - \mathcal{J}_u^t(r^{(0)}, h) = \sum_{u \leq t_k < t} (r^{(0)}_{st_k} - r^{(0)}_{ut_k}) h_{t_k t_{k+1}}. \tag{3.4}$$

Note that by definition of $\delta r^{(0)}$ we have the relation $r^{(0)}_{st_k} - r^{(0)}_{ut_k} = \delta r^{(0)}_{sut_k} + r^{(0)}_{su}$. Substituting this into (3.4) and then invoking Lemma 2.5 we obtain

$$\delta R_{sut} = r^{(0)}_{su} \mathcal{J}_u^t(1, h) + \sum_{i=1}^{\ell-1} r^{(i)}_{su} \mathcal{J}_u^t(x^i, h). \tag{3.5}$$

We can now bound δR as follows: Taking the L_1-norm on both sides of (3.5) gives

$$|\delta R_{sut}|_{L_1} \leq \sum_{i=0}^{\ell-1} |r_{su}^{(i)}|_{L_2} \cdot |\mathcal{J}_u^t(x^i, h)|_{L_2}. \tag{3.6}$$

Applying condition (3.2) to $|\mathcal{J}_u^t(x^i, h)|_{L_2}$ in (3.6) and invoking the relation $|r_{st}^{(i)}|_{L_2} \leq K(t-s)^{(\ell-i)\alpha}$ given in Definition 2.1 we get

$$|\delta R_{sut}|_{L_1} \lesssim \sum_{i=0}^{\ell-1} (u-s)^{(\ell-i)\alpha}(t-u)^{\beta_i} \lesssim (t-s)^{\beta} \tag{3.7}$$

for $(s, u, t) \in \mathcal{S}_3([0, 1])$ such that $t - u \geq 1/n$, where β is defined in (3.1).

Take $(s, t) \in \mathcal{S}_2([0, 1])$ such that $t - s \geq 1/n$. Consider the partition $[\![s, t]\!]$ of the interval $[s, t]$: $s < t_k < \cdots < t_{k'} < t$, where k and k' are such that $t_{k-1} \leq s < t_k$ and $t_{k'} < t \leq t_{k'+1}$. In the following we show that (3.7) holds for all $(u_1, u_2, u_3) \in \mathcal{S}_3([\![s, t]\!])$. In view of (3.7) it remains to show that the estimate (3.7) holds for $|\delta R_{ut_{k'}t}|_{L_1}$, $u \in [\![s, t]\!] : u \leq t_{k'}$. Indeed, by definition (2.3) we have $\mathcal{J}_{t_{k'}}^t(x^i, h) = 0$ and

$$|\mathcal{J}_{t_{k'}}^t(1, h)|_{L_2} = |h_{t_{k'}t_{k'+1}}|_{L_2} \leq (1/n)^{\beta_0} \leq (t-s)^{\beta_0}.$$

Applying these estimates to the right-hand side of (3.6) we obtain the estimate (3.7) for $|\delta R_{ut_{k'}t}|_{L_1}$.

By (2.3) it is clear that for any two consecutive partition points u, v in $[\![s, t]\!]$ and $u < v$ we have $R_{uv} = 0$. Applying the discrete sewing lemma [33, Lemma 2.5] to R on the partition $[\![s, t]\!]$ and then invoking the estimate (3.7) of δR on $\mathcal{S}_3([\![s, t]\!])$ we obtain $|R_{st}|_{L_1} \lesssim (t-s)^{\beta}$. This proves the first estimate in (3.3).

Recall that by (2.2) we have $y_t = \sum_{i=0}^{\ell-1} y_s^{(i)} x_{st}^i + r_{st}^{(0)}$. Substituting this into $\mathcal{J}_s^t(y, h)$ we get the relation

$$\mathcal{J}_s^t(y, h) = \sum_{i=0}^{\ell-1} y_s^{(i)} \mathcal{J}_s^t(x^i, h) + \mathcal{J}_s^t(r^{(0)}, h). \tag{3.8}$$

Applying (3.2) and the first estimate in (3.3) to the right-hand side of (3.8) we obtain the desired estimate of $\mathcal{J}_s^t(y, h)$ in (3.3). ☐

In the next result we apply Proposition 3.1 to weighted sums which involve fBms.

Proposition 3.2. *Let x be a one-dimensional fBm with Hurst parameter $H \leq 1/2$. Suppose that $(y, y', \dots, y^{(\ell-1)})$, $\ell \in \mathbb{N}$ is a process controlled by $(x, \ell, H-\varepsilon)$ in L_2 for some sufficiently small $\varepsilon > 0$. Let $f = \sum_{q=d}^{\infty} a_q H_q \in L_2(\mathbb{R}, \gamma)$ with Hermite rank $d > 0$ and f belongs to the Sobolev space $W^{2(\ell-1),2}(\mathbb{R}, \gamma)$, where γ denotes the standard Gaussian measure on the real line; see e.g. Page 28 in [42]. We define a family of increments $\{h^n; n \geq 1\}$ by:*

$$h_{st}^n := \sum_{s \leq t_k < t} f(n^H \delta x_{t_k t_{k+1}}), \qquad (s, t) \in \mathcal{S}_2([\![0, 1]\!]). \tag{3.9}$$

(i) Suppose that $d > \frac{1}{2H}$ and that ℓ is the least integer such that $\ell H + \frac{1}{2} > 1$, that is $\ell = [\frac{1}{2H}] + 1$. Then there is a constant K independent of n such that

$$|\mathcal{J}_s^t(y, h^n)|_{L_1} \leq K n^{1/2} (t-s)^{1/2} \tag{3.10}$$

for all $(s, t) \in \mathcal{S}_2([0, 1])$ satisfying $t - s \geq 1/n$.
(ii) Suppose that $d \leq \frac{1}{2H}$ and that $\ell = d + 1$. Then there is a constant K independent of n such that

$$|\mathcal{J}_s^t(y, h^n)|_{L_1} \leq K n^{1-dH} (t-s)^{1-dH} \tag{3.11}$$

for all $(s, t) \in \mathcal{S}_2([0, 1])$ satisfying $t - s \geq 1/n$.

Remark 3.3. The space $W^{2(\ell-1),2}(\mathbb{R},\gamma)$ is defined in the same way as the classical Sobolev space $W^{2(\ell-1),2}(\mathbb{R})$ except that we use the measure γ in place of the Lebesgue measure. One can consider it a Gaussian measure weighted Sobolev space.

Proof of Proposition 3.2. We assume that $d > \frac{1}{2H}$. In the following we prove (i) by applying Proposition 3.1. We first recall the estimate in [33, equation (4.24)]:

$$|\mathcal{J}_s^t(x^i,h^n)|_{L_2} \le Kn^{1/2}(t-s)^{iH+1/2}, \qquad (s,t)\in\mathcal{S}_2([0,1]) : t-s\ge 1/n, \tag{3.12}$$

for all $i=0,\dots,[\frac{1}{2H}]$. The estimate (3.12) implies that relation (3.2) holds for $h := h^n/\sqrt{n}$ and $\beta_i := iH+1/2$. Take $\alpha = H-\varepsilon$ and recall that ℓ is the least integer such that $\ell H + 1/2 > 1$, or $\ell H > 1/2$. It is thus readily checked that condition (3.1) is satisfied. We conclude that (3.10) holds.

We turn to the case when $d \le \frac{1}{2H}$. As before, our estimate will be an application of Proposition 3.1. We first derive an estimate of $|\mathcal{J}_s^t(x^i,h^n)|_{L_2}$. Let $f_1(x) = \sum_{q=[\frac{1}{2H}]+1}^{\infty} a_q H_q(x)$ and $f_2(x) = \sum_{q=d}^{[\frac{1}{2H}]} a_q H_q(x)$. We consider the following decomposition

$$h_{st}^n = h_{st}^{n,(1)} + h_{st}^{n,(2)}, \tag{3.13}$$

where

$$h_{st}^{n,(1)} = \sum_{s\le t_k<t} f_1(n^H \delta x_{t_k t_{k+1}}), \qquad h_{st}^{n,(2)} = \sum_{s\le t_k<t} f_2(n^H \delta x_{t_k t_{k+1}}).$$

Note that the Hermite rank of f_1 is greater than $\frac{1}{2H}$. So we can apply (3.12) to get the estimate

$$|\mathcal{J}_s^t(x^i,h^{n,(1)})|_{L_2} \le Kn^{1/2}(t-s)^{iH+1/2} \tag{3.14}$$

for $(s,t)\in\mathcal{S}_2([0,1]) : t-s\ge 1/n$ and $i=0,\dots,d$. By assumption we have $1/2-dH>0$. It follows that

$$1 \le n^{1/2-dH}(t-s)^{1/2-dH}, \tag{3.15}$$

and therefore we can enlarge the bound in (3.14) to be:

$$|\mathcal{J}_s^t(x^i,h^{n,(1)})|_{L_2} \le Kn^{1-dH}(t-s)^{1+iH-dH}. \tag{3.16}$$

Let us turn to the estimate of $|\mathcal{J}_s^t(x^i,h^{n,(2)})|_{L_2}$. We first have the bound

$$|\mathcal{J}_s^t(x^i,h^{n,2})|_{L_2} \le \sum_{q=d}^{[\frac{1}{2H}]} |a_q|\cdot|\mathcal{J}_s^t(x^i,h^{n,q})|_{L_2}, \qquad (s,t)\in\mathcal{S}_2([0,1]). \tag{3.17}$$

Recall the estimate in [33, Lemma 4.11 (ii)]:

$$|\mathcal{J}_s^t(x^i,h^{n,q})|_{L_2} \lesssim \begin{cases} n^{1-qH}(t-s)^{1+iH-qH} & \text{when } q\le i \\ n^{1/2}(t-s)^{iH+1/2} & \text{when } q>i \end{cases} \tag{3.18}$$

for $(s,t)\in\mathcal{S}_2([0,1]) : t-s\ge 1/n$, $i=0,\dots,d$ and $q<\frac{1}{2H}$. Substituting (3.18) into the right-hand side of (3.17) and then applying the relation $1\le n(t-s)$ we obtain that

$$|\mathcal{J}_s^t(x^i,h^{n,2})|_{L_2} \le Kn^{1-dH}(t-s)^{1+iH-dH}, \qquad (s,t)\in\mathcal{S}_2([0,1]) \tag{3.19}$$

for $i=0,\dots,d$.

Combining the two estimates (3.16) and (3.19) and taking into account the decomposition (3.13), we obtain that (3.2) holds for $\beta_i := 1+iH-dH$ and $h := h^n/n^{1-dH}$. It is readily checked that condition (3.1) is satisfied for $\ell = d+1$. Applying Proposition 3.1 we thus conclude the desired estimate (3.11). □

3.2 Convergence of Riemann sum

Let y be a continuous process controlled by the fBm x. This subsection is devoted to the convergence of Riemann sum for the regular integral $\int_0^t y_u du$. For convenience we will consider the uniform partition of $[0,1]$: $t_i = i/n$, $i = 0, 1, \dots, n$.

We start by proving the following weighted limit theorem result:

Lemma 3.4. *Let x be a one-dimensional fBm with Hurst parameter $H < 1/2$. Let (y, y') be a rough path controlled by $(x, 2, H)$ almost surely. Define the increment*

$$h^n_{st} = \sum_{s \le t_k < t} \int_{t_k}^{t_{k+1}} x^1_{t_k u} du \qquad \text{for } (s,t) \in \mathcal{S}_2(\llbracket 0,1 \rrbracket). \tag{3.20}$$

Then for each $(s,t) \in \mathcal{S}_2([0,1])$ we have the convergence in probability:

$$n^{2H} \mathcal{J}_s^t(y, h^n) \to -\frac{1}{4H+2} \int_s^t y'_u du \qquad \text{as } n \to \infty. \tag{3.21}$$

Proof. The proof is divided into several steps. By localization (cf. [28, Lemma 3.4.5]) we can and will assume that (y, y') is controlled by $(x, 2, H - \varepsilon)$ in L_2 for any $\varepsilon > 0$.

Step 1: Estimate of h^n. By the self-similarity of the fBm we have $\mathbb{E}[x^1_{t_k u} x^1_{t_{k'} u'}] = n^{-2H} \mathbb{E}[x^1_{k, nu} x^1_{k', nu'}]$. Here we have written $x^1_{k,nu}$ instead of x^1_{knu} to avoid confusion. Applying this relation and then the change of variable $nu' \to u'$ and $nu \to u$ we get

$$\mathbb{E}[|h^n_{st}|^2] = n^{-2H-2} \sum_{ns \le k, k' < nt} \int_k^{k+1} \int_{k'}^{k'+1} \mathbb{E}[x^1_{ku} x^1_{k'u'}] du' du, \qquad (s,t) \in \mathcal{S}_2(\llbracket 0,1 \rrbracket).$$

Applying the estimate $|\mathbb{E}[x^1_{ku} x^1_{k'u'}]| \lesssim |k - k'|^{2H-2}$ for $k \ne k'$ we obtain

$$\mathbb{E}[|h^n_{st}|^2] \lesssim n^{-2H-2} \sum_{\substack{ns \le k, k' < nt \\ k \ne k'}} |k - k'|^{2H-2} \lesssim n^{-2H-1}(t-s), \qquad (s,t) \in \mathcal{S}_2(\llbracket 0,1 \rrbracket). \tag{3.22}$$

Step 2: A decomposition of $|\mathcal{J}_s^t(x^1, h^n)|^2_{L_2}$. Let $(s,t) \in \mathcal{S}_2([0,1])$ such that $t - s > 1/n$. By definition (2.4) we can express $|\mathcal{J}_s^t(x^1, h^n)|^2_{L_2}$ as

$$|\mathcal{J}_s^t(x^1, h^n)|^2_{L_2} = \sum_{s \le t_k, t_{k'} < t} \mathbb{E} \int_{t_k}^{t_{k+1}} \int_{t_{k'}}^{t_{k'+1}} x^1_{st_k} x^1_{st_{k'}} x^1_{t_k u} x^1_{t_{k'} u'} du' du. \tag{3.23}$$

Applying the moment formula for multivariate Gaussian random variables to the product $x^1_{st_k} x^1_{st_{k'}} x^1_{t_k u} x^1_{t_{k'} u'}$ in (3.23) we obtain

$$\mathbb{E}\left(x^1_{st_k} x^1_{st_{k'}}, x^1_{t_k u} x^1_{t_{k'} u'} \right) = A_1 + A_2 + A_3\,, \tag{3.24}$$

where

$$A_1 = \mathbb{E}(x^1_{st_k} x^1_{st_{k'}}) \mathbb{E}(x^1_{t_k u} x^1_{t_{k'} u'}) \tag{3.25}$$

$$A_2 = \mathbb{E}(x^1_{st_k} x^1_{t_{k'} u'}) \mathbb{E}(x^1_{t_k u} x^1_{st_{k'}}) \tag{3.26}$$

$$A_3 = \mathbb{E}(x^1_{st_k} x^1_{t_k u}) \mathbb{E}(x^1_{t_{k'} u'} x^1_{st_{k'}}) \tag{3.27}$$

Step 3: Estimate of A_1. Recall that $\left|\mathbb{E}\big(x^1_{st_k} x^1_{st_{k'}}\big)\right| \le (t-s)^{2H}$; see e.g. [38, Lemma 5.1]. On the other hand, similar to the estimate of $\mathbb{E}[\delta x_{t_k u} \delta x_{t_{k'} u'}]$ in Step 1 we have $|\mathbb{E}(x^1_{t_k u} x^1_{t_{k'} u'})| \lesssim n^{-2H} |k - k'|^{2H-2}$. Substituting these two estimates into (3.25) we obtain

$|A_1| \lesssim (t-s)^{2H}|k-k'|^{2H-2}n^{-2H}$. The above estimate for $|A_1|$ together with the relation $\sum_{s\le t_k, t_{k'}<t}|k-k'|^{2H-2} \lesssim n(t-s)$ shows that

$$\sum_{s\le t_k, t_{k'}<t} \int_{t_k}^{t_{k+1}} \int_{t_{k'}}^{t_{k'+1}} A_1 du' du \lesssim (t-s)^{2H+1} n^{-1-2H}. \tag{3.28}$$

Step 4: Estimate of A_2 and A_3. Recall that A_2 and A_3 are defined in (3.26)-(3.27). It follows that we have

$$\sum_{s\le t_k, t_{k'}<t} \int_{t_k}^{t_{k+1}} \int_{t_{k'}}^{t_{k'+1}} (A_2 + A_3) du' du = \tilde{A}_2 + \tilde{A}_3\,, \tag{3.29}$$

where

$$\begin{aligned} \tilde{A}_2 &= \sum_{s\le t_k, t_{k'}<t} \int_{t_k}^{t_{k+1}} \int_{t_{k'}}^{t_{k'+1}} \langle \mathbf{1}_{[s,t_k]}, \mathbf{1}_{[t_{k'},u']}\rangle_{\mathcal{H}} \langle \mathbf{1}_{[t_k,u]}, \mathbf{1}_{[s,t_{k'}]}\rangle_{\mathcal{H}} du' du \\ \tilde{A}_3 &= \sum_{s\le t_k, t_{k'}<t} \int_{t_k}^{t_{k+1}} \int_{t_{k'}}^{t_{k'+1}} \langle \mathbf{1}_{[s,t_k]}, \mathbf{1}_{[t_k,u]}\rangle_{\mathcal{H}} \langle \mathbf{1}_{[t_{k'},u']}, \mathbf{1}_{[s,t_{k'}]}\rangle_{\mathcal{H}} du' du. \end{aligned} \tag{3.30}$$

In the following we bound $\tilde{A}_2$ and $\tilde{A}_3$. We first have

$$|\tilde{A}_2| \le 2 \sum_{s\le t_k \le t_{k'}<t} \int_{t_k}^{t_{k+1}} \int_{t_{k'}}^{t_{k'+1}} |\langle \mathbf{1}_{[s,t_k]}, \mathbf{1}_{[t_{k'},u']}\rangle_{\mathcal{H}}| \cdot |\langle \mathbf{1}_{[t_k,u]}, \mathbf{1}_{[s,t_{k'}]}\rangle_{\mathcal{H}}| du' du.$$

Invoking the elementary estimates

$$|\langle \mathbf{1}_{[t_k,u]}, \mathbf{1}_{[s,t_{k'}]}\rangle_{\mathcal{H}}| \le n^{-2H} \qquad \text{and} \qquad |\langle \mathbf{1}_{[s,t_k]}, \mathbf{1}_{[t_{k'},u']}\rangle_{\mathcal{H}}| \lesssim n^{-2H}|k-k'|^{2H-1}$$

for $t_k \le t_{k'}$ we obtain

$$\begin{aligned} |\tilde{A}_2| \lesssim \sum_{s\le t_k \le t_{k'}<t} \int_{t_k}^{t_{k+1}} \int_{t_{k'}}^{t_{k'+1}} n^{-2H} \cdot n^{-2H} \cdot (k'-k)^{2H-1} du' du \\ \lesssim (t-s)^{2H+1} n^{-2H-1}. \end{aligned} \tag{3.31}$$

We turn to the estimate of $\tilde{A}_3$. A change of variables in (3.30) gives

$$\tilde{A}_3 = n^{-4H-2} \sum_{ns\le k,k'<nt} \tilde{A}_{3,kk'}, \tag{3.32}$$

where

$$\tilde{A}_{3,kk'} = \int_k^{k+1} \int_{k'}^{k'+1} \langle \mathbf{1}_{[ns,k]} \otimes \mathbf{1}_{[ns,k']}, \mathbf{1}_{[k,u]} \otimes \mathbf{1}_{[k',u']}\rangle_{\mathcal{H}^{\otimes 2}} du' du. \tag{3.33}$$

It is clear that $|\tilde{A}_{3,kk'}|$ is uniformly bounded in (k,k'). Therefore, from (3.32) we obtain the estimate

$$|\tilde{A}_3| \lesssim (t-s)^2 n^{-4H}. \tag{3.34}$$

Step 5: Estimate of $\mathcal{J}_s^t(x^1, h^n)$. Putting together the estimates (3.28), (3.31), and (3.34) and taking into account the decompositions (3.23)-(3.24) and (3.29) we obtain the estimate

$$|\mathcal{J}_s^t(x^1, h^n)|_{L_2} \lesssim (t-s) n^{-2H} \tag{3.35}$$

for $(s,t) \in \mathcal{S}_2([0,1])$ such that $t-s > 1/n$.

Step 6: Convergence of the second moment of $\mathcal{J}_s^t(x^1, h^n)$. Let $(s,t) \in \mathcal{S}_2([0,1])$. In this step we show the convergence:

$$n^{2H}|\mathcal{J}_s^t(x^1, h^n)|_{L_2} \to \frac{1}{4H+2}(t-s) \qquad \text{as } n \to \infty. \tag{3.36}$$

Recall our decomposition of $|\mathcal{J}_s^t(x^1, h^n)|_{L_2}^2$ in (3.23)-(3.24) and of A_2 in (3.29). So the estimates in (3.28), (3.31) and (3.34) together shows that the convergence of $|\mathcal{J}_s^t(x^1, h^n)|_{L_2}^2$ is dominated by that of A_{22}. Namely, we have

$$\lim_{n\to\infty} n^{4H}|\mathcal{J}_s^t(x^1, h^n)|_{L_2}^2 = \lim_{n\to\infty} n^{4H} A_{22}. \tag{3.37}$$

In the following we focus on the computation of $\lim_{n\to\infty} n^{4H} A_{22}$.

Recall the expression of A_{22} in (3.32)-(3.33). We first note that since $|A_{22,kk'}| \sim O(1)$ we can replace the summation $\sum_{ns\le k,k'<nt}$ in (3.32) by $\sum_{ns+n^\varepsilon\le k,k'<nt}$ for $0<\varepsilon<1$ without changing the limit of A_{22}. Next, by stationary increment and self-similarity of the fBm we have

$$\begin{aligned}\langle \mathbf{1}_{[ns,k]} \otimes \mathbf{1}_{[ns,k']}, \mathbf{1}_{[k,u]} \otimes \mathbf{1}_{[k',u']}\rangle_{\mathcal{H}^{\otimes 2}} &= \langle \mathbf{1}_{[ns-k,0]} \otimes \mathbf{1}_{[ns-k',0]}, \mathbf{1}_{[0,u-k]} \otimes \mathbf{1}_{[0,u'-k']}\rangle_{\mathcal{H}^{\otimes 2}} \\ &= \langle \mathbf{1}_{[\frac{ns-k}{u-k},0]} \otimes \mathbf{1}_{[\frac{ns-k'}{u'-k'},0]}, \mathbf{1}_{[0,1]} \otimes \mathbf{1}_{[0,1]}\rangle_{\mathcal{H}^{\otimes 2}} (u-k)^{2H}(u'-k')^{2H} \\ &= \langle \mathbf{1}_{(-\infty,0]} \otimes \mathbf{1}_{(-\infty,0]}, \mathbf{1}_{[0,1]} \otimes \mathbf{1}_{[0,1]}\rangle_{\mathcal{H}^{\otimes 2}} (u-k)^{2H}(u'-k')^{2H} + o(1),\end{aligned}$$

where the last equation holds for k and k' such that $k-ns \ge n^\varepsilon$ and $k'-ns \ge n^\varepsilon$. Using the relation $\langle \mathbf{1}_{(-\infty,0]}, \mathbf{1}_{[0,1]}\rangle_{\mathcal{H}} = -1/2$ we obtain

$$\langle \mathbf{1}_{[ns,k]} \otimes \mathbf{1}_{[ns,k']}, \mathbf{1}_{[k,u]} \otimes \mathbf{1}_{[k',u']}\rangle_{\mathcal{H}^{\otimes 2}} = \frac{1}{4}(u-k)^{2H}(u'-k')^{2H} + o(1). \tag{3.38}$$

Substituting (3.38) into (3.32) we obtain

$$\begin{aligned} A_{22} &= n^{-4H-2}\frac{1}{4} \sum_{ns+n^\varepsilon \le k,k'<nt} \int_k^{k+1}\int_{k'}^{k'+1} (u-k)^{2H}(u'-k')^{2H} du' du + n^{-4H} o(1) \\ &= n^{-4H}(t-s)^2 \cdot \frac{1}{4}(2H+1)^{-2} + n^{-4H} o(1). \end{aligned}$$

It follows that $\lim_{n\to\infty} n^{4H} A_{22} = (t-s)^2 \cdot \frac{1}{4}(2H+1)^{-2}$. Recalling relation (3.37), we thus obtain the convergence in (3.36).

Step 7: Convergence of $\mathcal{J}_s^t(x^1, h^n)$. In this step, we show the L_2-convergence of $\mathcal{J}_s^t(x^1, h^n)$:

$$n^{2H}\mathcal{J}_s^t(x^1, h^n) \to -\frac{1}{4H+2}(t-s). \tag{3.39}$$

In view of the convergence (3.36), it suffices to show that:

$$n^{2H}\mathbb{E}\mathcal{J}_s^t(x^1, h^n) \to -\frac{1}{4H+2}(t-s) \qquad \text{as } n \to \infty. \tag{3.40}$$

The convergence (3.40) can be proved in the similar way as in Step 5. Indeed, we have:

$$\begin{aligned} \mathbb{E}[\mathcal{J}_s^t(x^1, h^n)] &= \mathbb{E} \sum_{s\le t_k<t} x^1_{st_k} \int_{t_k}^{t_{k+1}} x^1_{t_k u} du = \sum_{s\le t_k<t} \int_{t_k}^{t_{k+1}} \langle \mathbf{1}_{[s,t_k]}, \mathbf{1}_{[t_k,u]}\rangle_{\mathcal{H}} du \\ &= n^{-2H}(t-s)(-1/2)(2H+1)^{-1} + n^{-2H} o(1). \end{aligned}$$

The convergence (3.40) then follows. The two convergences (3.36) and (3.40) together implies that the convergence (3.39) holds.

Step 8: Convergence of $\mathcal{J}_s^t(y,h^n)$. Let $(s,t) \in \mathcal{S}_2([0,1])$. We start by taking a partition of $[s,t]$: $s = s_0 < s_1 < \cdots < s_m = t$ such that $\max_{j=0,\dots,m-1} |s_{j+1} - s_j| \le 1/m$ for some $m < n$. Then we can write

$$\mathcal{J}_s^t(y,h^n) = \sum_{j=0}^{m-1} \mathcal{J}_{s_j}^{s_{j+1}}(y,h^n). \tag{3.41}$$

Since (y,y') is controlled by $(x,2,H-\varepsilon)$ in L_2 we have the expansion $y_{t_k} = y_{s_j} + y'_{s_j} x^1_{s_j t_k} + r^{(0)}_{s_j t_k}$. Substituting this into $\mathcal{J}_{s_j}^{s_{j+1}}(y,h^n)$ in (3.41) we obtain

$$\mathcal{J}_s^t(y,h^n) = \sum_{j=0}^{m-1} y_{s_j} \mathcal{J}_{s_j}^{s_{j+1}}(1,h^n) + \sum_{j=0}^{m-1} y'_{s_j} \mathcal{J}_{s_j}^{s_{j+1}}(x^1,h^n) + \sum_{j=0}^{m-1} \mathcal{J}_{s_j}^{s_{j+1}}(r^{(0)},h^n). \tag{3.42}$$

In the following we consider the convergence of the three terms on the right-hand side of (3.42).

We note that it follows from relations (3.22) and (3.35) that conditions (3.1)-(3.2) hold for $h := n^{2H} h^n$, $\alpha = H - \varepsilon$, $\beta_0 := 1 - H$ and $\beta_1 := 1$. Therefore, applying Proposition 3.1 we have

$$n^{2H} |\mathcal{J}_s^t(r^{(0)},h^n)|_{L_1} \lesssim (t-s)^{1+H-\varepsilon}.$$

This implies that

$$\lim_{m\to\infty} \limsup_{n\to\infty} n^{2H} \Big| \sum_{j=0}^{m-1} \mathcal{J}_{s_j}^{s_{j+1}}(r^{(0)},h^n) \Big|_{L_1} \lesssim \lim_{m\to\infty} \sum_{j=0}^{m-1} (s_{j+1}-s_j)^{1+H-\varepsilon} = 0. \tag{3.43}$$

We turn to the other two terms in the right-hand side of (3.42). Applying (3.22) we have

$$\begin{aligned} n^{2H} \Big| \sum_{j=0}^{m-1} y_{s_j} \mathcal{J}_{s_j}^{s_{j+1}}(1,h^n) \Big|_{L_1} &= n^{2H} \sum_{j=0}^{m-1} \Big| y_{s_j} h^n_{s_j s_{j+1}} \Big|_{L_1} \\ &\lesssim \sum_{j=0}^{m-1} |y_{s_j}|_{L_2} (s_{j+1}-s_j)^{1/2} (1/n)^{1/2-H} \to 0 \qquad \text{as } n \to \infty. \end{aligned} \tag{3.44}$$

Finally, according to (3.39) we have the convergence in probability:

$$\begin{aligned} \lim_{m\to\infty} \lim_{n\to\infty} n^{2H} \sum_{j=0}^{m-1} y'_{s_j} \mathcal{J}_{s_j}^{s_{j+1}}(x^1,h^n) &= -\frac{1}{4H+2} \lim_{m\to\infty} \sum_{j=0}^{m-1} y'_{s_j} (s_{j+1}-s_j) \\ &= -\frac{1}{4H+2} \int_0^t y'_u du. \end{aligned} \tag{3.45}$$

Putting together the convergences (3.43)-(3.45) and recalling the relation (3.42) we conclude the convergence (3.21). □

With Lemma 3.4 in hand, in the following we consider the convergence of a Riemann sum.

Proposition 3.5. *Let x be a one-dimensional fBm with Hurst parameter $H < 1/2$. Let (y,y',y'') be a rough path controlled by $(x,3,H)$ almost surely.Then we have the convergence in probability:*

$$n^{2H} \left(\frac{1}{n} \sum_{0 \le t_k < t} y_{t_k} - \int_0^t y_u du \right) \to 0 \qquad \textit{as } n \to \infty. \tag{3.46}$$

Proof. The proof is divided into several steps.
Step 1: A decomposition of the error of Riemann sum. We first have

$$\int_0^t y_u du - \frac{1}{n} \sum_{0 \le t_k < t} y_{t_k} = \sum_{0 \le t_k < t} \int_{t_k}^{t_{k+1}} \delta y_{t_k u} du. \tag{3.47}$$

Substituting the expansion $\delta y_{t_k u} = y'_{t_k} x^1_{t_k u} + y''_{t_k} x^2_{t_k u} + r^{(0)}_{t_k u}$ into (3.47) we get the expansion:

$$\int_0^t y_u du - \frac{1}{n} \sum_{0 \le t_k < t} y_{t_k} = I_1 + I_2 + I_3, \tag{3.48}$$

where

$$I_1 = \sum_{0 \le t_k < t} y'_{t_k} \int_{t_k}^{t_{k+1}} x^1_{t_k u} du, \qquad I_2 = \sum_{0 \le t_k < t} y''_{t_k} \int_{t_k}^{t_{k+1}} x^2_{t_k u} du, \qquad I_3 = \sum_{0 \le t_k < t} \int_{t_k}^{t_{k+1}} r^0_{t_k u} du.$$

In the following we consider the convergence of I_1, I_2 and I_3 which together will give the desired convergence in (3.46).
Step 2: Convergence of I_1 and I_3. Since $|r^{(0)}_{t_k u}|_{L_1} \lesssim n^{-3H}$ it follows that

$$n^{2H} I_3 \to 0 \tag{3.49}$$

in probability as $n \to \infty$. On the other hand, a direct application of Lemma 3.4 yields the convergence

$$n^{2H} I_1 \to -\frac{1}{4H+2} \int_s^t y''_u du \qquad \text{as } n \to \infty. \tag{3.50}$$

Step 3: Convergence of I_2. We consider the following decomposition of I_2:

$$I_2 = \sum_{0 \le t_k < t} y''_{t_k} \int_{t_k}^{t_{k+1}} x^2_{t_k u} du = I_{21} + I_{22},$$

where

$$\begin{aligned} I_{21} &= \sum_{0 \le t_k < t} y''_{t_k} \int_{t_k}^{t_{k+1}} (x^2_{t_k u} - \frac{1}{2}(u - t_k)^{2H}) du \qquad (3.51) \\ I_{22} &= \frac{1}{2} \sum_{0 \le t_k < t} y''_{t_k} \int_{t_k}^{t_{k+1}} (u - t_k)^{2H} du = \frac{1}{2} \sum_{0 \le t_k < t} y''_{t_k} \cdot (1/n)^{2H+1} (2H+1)^{-1}. \end{aligned}$$

Note that I_{22} is a Riemann sum and we thus have the convergence:

$$n^{2H} I_{22} \to \frac{1}{4H+2} \int_0^t y''_u du \qquad \text{in probability as } n \to \infty.$$

We turn to the convergence of I_{21}. We first note that a direct computation shows that

$$\begin{aligned} \Big| \sum_{0 \le t_k < t} \int_{t_k}^{t_{k+1}} \Big(x^2_{t_k u} - \frac{1}{2}(u - t_k)^{2H} \Big) du \Big|^2_{L_2} &\lesssim \sum_{0 \le t_k, t_{k'} < t} \int_{t_k}^{t_{k+1}} \int_{t_{k'}}^{t_{k'+1}} n^{-4H} |\rho(k - k')|^2 du' du \\ &\lesssim n^{-4H-2} n(t-s) = n^{-4H-1}(t-s). \qquad (3.52) \end{aligned}$$

Applying Proposition 3.1 to I_{21} in (3.51) with $\ell = 1$ and $\beta_0 = 1 - H + \varepsilon$ and invoking the estimate (3.52) we obtain $n^{2H} |I_{21}|_{L_1} \lesssim (t-s)^{1-H+\varepsilon} (1/n)^{H-\varepsilon}$ for any $\varepsilon > 0$. In particular, we have $n^{2H} |I_{21}|_{L_1} \to 0$ as $n \to \infty$. Combining the convergence of I_{21} and I_{22} we obtain

$$n^{2H} I_2 \to \frac{1}{4H+2} \int_0^t y''_u du \qquad \text{in probability as } n \to \infty. \tag{3.53}$$

Step 4: Conclusion. Substituting the convergences of I_i, $i = 1, 2, 3$ in (3.49), (3.50) and (3.53) into (3.48) we obtain the convergence (3.46). □

Remark 3.6. We conjecture that the exact rate of convergence in (3.46) is $O(n^{-H-1/2})$ given that y satisfies proper regularity conditions. We will explore this problem in a future paper. Note that the rate $o(n^{-2H})$ we have obtained in (3.46) is sufficient for our purpose in this paper, and it requires a weaker condition of y.

In the next result we consider the convergence rate of the Riemann sum under a weaker condition. We will also include the case when $H = 1/2$.

Proposition 3.7. *Let x be a one-dimensional fBm with Hurst parameter $H \leq 1/2$. Let (y, y') be a rough path controlled by $(x, 2, H - \varepsilon)$ in L_2 for some $\varepsilon > 0$. Then there is a constant K independent of n such that:*

$$\Big|\frac{1}{n}\sum_{0\leq t_k<t} y_{t_k} - \int_0^t y_u du\Big|_{L_1} \leq K n^{-2H+2\varepsilon} \qquad \textit{for all } t \in [0,1]. \tag{3.54}$$

Proof. Because y is controlled by $(x, 2, H)$ we have the relation $\delta y_{t_k u} = y'_{t_k} x^1_{t_k u} + r^{(0)}_{t_k u}$. So, similar to (3.48), we have the decomposition

$$\int_0^t y_u du - \frac{1}{n}\sum_{0\leq t_k<t} y_{t_k} = I_1 + I_2, \tag{3.55}$$

where

$$I_1 = \sum_{0\leq t_k<t} y'_{t_k}\int_{t_k}^{t_{k+1}} x^1_{t_k u} du, \qquad I_2 = \sum_{0\leq t_k<t}\int_{t_k}^{t_{k+1}} r^0_{t_k u} du.$$

It is readily checked that $|I_2|_{L_1} \lesssim n^{-2H+2\varepsilon}$. Let h^n be defined in (3.20). Applying Proposition 3.1 with $h = n^{2H-2\varepsilon}h^n$, $\ell = 1$ and $\beta_0 = 1 - H + 2\varepsilon$ we obtain that $n^{2H-2\varepsilon}|I_1|_{L_1} \lesssim 1$. Combining the estimate of I_1 and I_2 in (3.55) we obtain the desired estimate (3.54). □

3.3 Weighted p-variations

In this subsection we consider limit theorems for weighted random sums of some fBms functionals. For $p > -1$, we denote

$$c_p = \mathbb{E}(|N|^p) = \frac{2^{p/2}}{\sqrt{\pi}}\Gamma\left(\frac{p+1}{2}\right). \tag{3.56}$$

It is easy to see that $c_{p+2} = (p+1)c_p$, and when p is an even integer we have $c_p = \mathbb{E}(N^p) = (p-1)(p-3)\cdots 1$. We define the sign function:

$$\text{sign}(x) = 1, -1, 0 \text{ for } x > 0,\ x < 0 \text{ and } x = 0, \text{ respectively.} \tag{3.57}$$

Lemma 3.8. *Let x be a fBm with Hurst parameter $H < 1/2$. Let (y, y') be a process controlled by $(x, 2, H)$ almost surely. Take $p > 1/2$ and let*

$$f(x) = |x|^{p+1}\cdot \text{sign}(x), \qquad x \in \mathbb{R}. \tag{3.58}$$

Then we have the following convergence in probability:

$$n^{H-1}\sum_{0\leq t_k<t} y_{t_k} f(n^H x^1_{t_k t_{k+1}}) \to -\frac{1}{2}c_{p+2}\int_0^t y'_u du \qquad \textit{as } n \to \infty. \tag{3.59}$$

Proof. We prove the convergence (3.59) by applying [33, Theorem 4.14 (ii)]. It is easy to see that the function f in (3.58) belongs to $L_2(\gamma)$ with Hermite rank $d = 1$ as long as $p > -3/2$. Take $\ell = d + 1 = 2$. By assumption (y, y') is a rough path controlled by (x, ℓ, H) almost surely.

Furthermore, one can show that $f \in W^{2,2}(\mathbb{R},\gamma)$ when $p > 1/2$ (please refer to Remark 3.3 for the definition of the Sobolev space $W^{2,2}(\mathbb{R},\gamma)$). To see this, it suffices to show that the variables $f(N)$, $f'(N)$, $f''(N)$ have finite second moments. A direct computation gives:

$$\mathbb{E}[f''(N)^2] = \mathbb{E}[|N|^{2p-2}],$$

and so the moment is finite for $p > 1/2$. The moments of $f(N)$ and $f'(N)$ can be estimated in the similar way.

In summary, we have shown that the conditions in [33, Theorem 4.14 (ii)] hold for the weighted sum in (3.59). Since f is an odd function it has the decomposition $f(x) = \sum_{q=0}^{\infty} a_{2q+1}H_{2q+1}(x)$. We compute the first coefficient: $a_1 = \mathbb{E}[|N|^{p+1} \cdot \text{sign}(N)N] = \mathbb{E}[|N|^{p+2}] = c_{p+2}$. Applying [33, Theorem 4.14] we thus obtain the convergence (3.59). □

Let $f(x) = |x|^p - c_p$, $x \in \mathbb{R}$. It is easily seen that $f \in L_2(\gamma)$ with Hermite rank $d = 2$ when $p > -\frac{1}{2}$. Furthermore, we have the decomposition $f(x) = \sum_{q=1}^{\infty} a_{2q}H_{2q}(x)$, where the constants a_{2q} are given by:

$$a_{2q} = \sum_{r=0}^{q} \frac{(-1)^r}{2^r r!(2q-2r)!} c_{2q-2r+p}, \qquad q = 1, 2, \dots. \tag{3.60}$$

We also set the constant σ:

$$\sigma^2 = \sum_{q=1}^{\infty} (2q)! a_{2q}^2 \sum_{k\in\mathbb{Z}} \rho(k)^{2q}, \tag{3.61}$$

where ρ is defined in (2.5). Note that when $H = 1/2$ we have $\rho(0) = 1$ and $\rho(k) = 0$ for $k \neq 0$, and so (3.61) gives $\sigma = \|f\|_{L_2(\gamma)} = (c_{2p} - c_p^2)^{1/2}$.

The next limit theorem result is an application of [33, Theorem 4.7 and Theorem 4.14]. The proof is similar to Lemma 3.8 and is omitted for sake of conciseness. In the following $\xrightarrow{stable\ f.d.d.}$ stands for the stable convergence of finite dimensional distributions. That is, we say $X_t^n \xrightarrow{stable\ f.d.d.} X_t$, $t \in [0,1]$ if the finite dimensional distribution of the process X_t^n, $t \in [0,1]$ converges stably to that of the process X_t, $t \in [0,1]$ as $n \to \infty$.

Proposition 3.9. *Let x be a fBm with Hurst parameter $H \le 1/2$. Let $(y^{(0)}, \dots, y^{(\ell-1)})$ be a process controlled by (x, ℓ, H) almost surely for some $\ell \in \mathbb{N}$. Let a_{2q} and σ be constants given in* (3.60)-(3.61)*. Then:*
(i) For $\frac{1}{2} \ge H > \frac{1}{4}$, $\ell = 2$ and $p \in (3/2, \infty)$ we have the convergence:

$$\frac{1}{\sqrt{n}} \sum_{0 \le t_k < t} y_{t_k}(|n^H x^1_{t_k t_{k+1}}|^p - c_p) \xrightarrow{stable\ f.d.d.} \sigma \int_0^t y_t dW_t \qquad \text{for } t \in [0,1], \tag{3.62}$$

where W is a Wiener process independent of x.
(ii) For $H = \frac{1}{4}$, $\ell = 3$ and $p \in (7/2, \infty) \cup \{2\}$ we have the convergence:

$$\frac{1}{\sqrt{n}} \sum_{0 \le t_k < t} y_{t_k}(|n^H x^1_{t_k t_{k+1}}|^p - c_p) \xrightarrow{stable\ f.d.d.} \sigma \int_0^t y_u dW_u + \frac{pc_p}{8} \int_0^t y''_u du \tag{3.63}$$

for $t \in [0,1]$.
(iii) For $H < \frac{1}{4}$, $\ell = 3$ and $p \in (7/2, \infty) \cup \{2\}$ we have the convergence in probability:

$$n^{2H-1} \sum_{0 \le t_k < t} y_{t_k}(|n^H x^1_{t_k t_{k+1}}|^p - c_p) \longrightarrow \frac{pc_p}{8} \int_0^t y''_u du \tag{3.64}$$

for $t \in [0,1]$.

Remark 3.10. Let us apply Propositions 3.8–3.9 to verify some classical results of weighted sums.

For $p>0$ an even integer, we recall the elementary relation $c_p=(p-1)c_{p-2}$. Then we have the relation $\frac{pc_p}{8}=\frac{1}{4}\binom{p}{2}c_{p-2}$. Substituting this relation into (3.64) and taking $y_t=f(x_t)$, we thus recover the convergence (1.23) in [38, Corollary 3].

Recall that a_{2q}, $q=1,\dots,p/2$ are coefficients of the chaos expansion for x^p-c_p. In other words, we have the relation $\mathbb{E}[(N^p-c_p)H_{2q}(N)]=(2q)!a_{2q}$. On the other hand, applying integration by parts yields

$$\mathbb{E}[(N^p-c_p)H_{2q}(N)]=p(p-1)\cdots(p-2q+1)\mathbb{E}[N^{p-2q}]=p(p-1)\cdots(p-2q+1)c_{p-2q}.$$

It follows that we have the relation

$$a_{2q}=\binom{p}{2q}c_{p-2q},\quad q=1,\dots,p/2. \tag{3.65}$$

Substituting relation (3.65) into (3.61) and taking into account that $a_i=0$ for $i\neq 2,\dots,p$, we thus recover from (3.62)-(3.63) the convergences in (1.24)–(1.25) in [38, Corollary 3], respectively.

Take p an even integer as before. It is clear that in this case we have $|x|^{p+1}\cdot sign(x)=x^{p+1}$. So we recover from (3.59) the weighted limit theorem obtained in [23].

4 Limit theorem for p-variation of processes controlled by fBm

In this section we consider the convergence of p-variation for processes controlled by fBm. Throughout the section we let $\phi(x)=|x|^p$, $x\in\mathbb{R}$ for $p\geq 1$. We first state the following elementary result.

Lemma 4.1. *Denote by $\phi^{(j)}$ the jth derivative of ϕ. For convenience we will also write $\phi(x)=\phi^{(0)}(x)$, $\phi'(x)=\phi^{(1)}(x)$ and $\phi''(x)=\phi^{(2)}(x)$. For $j=0,1,\dots,[p]$ we set*

$$\phi_j(x)=|x|^{p-j}\cdot\operatorname{sign}(x)^j\qquad\textit{and}\qquad K_j=p\cdots(p-j+1)=\prod_{i=0}^{j-1}(p-i), \tag{4.1}$$

where recall that $\operatorname{sign}(x)$ is defined in (3.57) and we use the convention that $\prod_{i=0}^{-1}(p-i)=1$. For example, we have $K_0=1$, $K_1=p$, $K_2=p(p-1)$. Then
(i) When p is odd, ϕ has derivative up to order $[p]-1$, and $\phi^{([p]-1)}$ is Lipschitz. When p is even, ϕ has derivative of all orders. When p is non-integer, ϕ has derivative up to order $[p]$.
(ii) For $x\in\mathbb{R}$ we have

$$\phi^{(j)}(x)=K_j\cdot\phi_j(x),\qquad j=0,1,\dots,[p], \tag{4.2}$$

with the exception that $\phi^{(p)}(0)$ is undefined when p is an odd number. In particular, when p is an odd number we have $\phi^{(j)}(x)=K_jx^{p-j}\operatorname{sign}(x)$, while when p is even we get $\phi^{(j)}(x)=K_jx^{p-j}$.

Following is our main result. Recall that we define $\phi(x)=|x|^p$, $x\in\mathbb{R}$, and for a continuous process y the p-variation of y over the time interval $[0,t]$ is defined as $\sum_{0\leq t_k<t}\phi(\delta y_{t_kt_{k+1}})=\sum_{0\leq t_k<t}|\delta y_{t_kt_{k+1}}|^p$, where $t_k=k/n$, $k=0,1,\dots,n$ is a uniform partition of $[0,1]$.

Theorem 4.2. *Let x be a fBm with Hurst parameter $H\leq 1/2$ and $(y^{(0)},\dots,y^{(\ell-1)})$ be a process controlled by (x,ℓ,H) almost surely (see Definition 2.1) for some $\ell\in\mathbb{N}$. Let ϕ'*

and ϕ'' *be derivatives of* ϕ *defined in* (4.2)*, and* c_p *and* σ *are constants given in* (3.56) *and* (3.61)*, respectively. Let*

$$U_t^n = n^{pH-1} \sum_{0 \le t_k < t} \phi(\delta y_{t_k t_{k+1}}) - c_p \int_0^t \phi(y'_u) du \qquad t \in [0,1].$$

Then
(i) When $1/4 < H \le 1/2$*,* $p \in [3,\infty) \cup \{2\}$ *and* $\ell \ge 4$ *we have the stable f.d.d. convergence*

$$\left(n^{1/2} U^n, x\right) \to (U, x), \qquad \text{as } n \to \infty, \tag{4.3}$$

where $U_t = \sigma \int_0^t \phi(y'_u) dW_u$*,* $t \in [0,1]$*, and* W *is a standard Brownian motion independent of* x*.*
(ii) When $H = 1/4$*,* $p \in [5,\infty) \cup \{2,4\}$ *and* $\ell \ge 6$ *we have the stable f.d.d. convergence*

$$\left(n^{1/2} U^n, x\right) \to (U, x), \qquad \text{as } n \to \infty, \tag{4.4}$$

where

$$U_t = \sigma \int_0^t \phi(y'_u) dW_u - \frac{c_p}{8} \int_0^t \phi''(y'_u)(y''_u)^2 du + \frac{(p-2)c_p}{24} \int_0^t \phi'(y'_u) y'''_u du.$$

(iii) When $H < 1/4$*,* $p \in [5,\infty) \cup \{2,4\}$ *and* $\ell \ge 6$ *we have the convergence in probability*

$$n^{2H} U_t^n \to U_t \qquad \text{as } n \to \infty \tag{4.5}$$

for $t \in [0,1]$*, where*

$$U_t = -\frac{c_p}{8} \int_0^t \phi''(y'_u)(y''_u)^2 du + \frac{(p-2)c_p}{24} \int_0^t \phi'(y'_u) y'''_u du.$$

Remark 4.3. Let us point out the cruicial difference between Proposition 3.9 and Theorem 4.2. Proposition 3.9 is about the weighted sum $\sum_{t_k} z_{t_k} |\delta x_{t_k t_{k+1}}|^p$ for a fBm x with Hurst parameter $H \le 1/2$ and a process z controlled by x (see Definition 2.1), while Theorem 4.2 is about the p-variation $\sum_{t_k} |\delta y_{t_k t_{k+1}}|^p$ of a process y controlled by x (e.g. $y_t = f(x_t)$ and $y'_t = f'(x_t)$; see again Definition 2.1). One might get an impression that these are very similar results, and we could get one from the other. This is indeed true when $H > 1/2$, as is shown in e.g. [16]. More precisely, by an elementary application of mean value theorem one can prove that the difference between the p-variation and the corresponding weighted sum

$$\sum_{t_k} |\delta y_{t_k t_{k+1}}|^p - \sum_{t_k} |y'_{t_k}|^p \cdot |\delta x_{t_k t_{k+1}}|^p \tag{4.6}$$

is negligible comparing to the p-variation $\sum_{t_k} |\delta y_{t_k t_{k+1}}|^p$ or the weighted sum $\sum_{t_k} |y'_{t_k}|^p \cdot |\delta x_{t_k t_{k+1}}|^p$.

However, the convergence of p-variation for the case $H \le 1/2$ is much more involved. Specifically, when $1/4 < H \le 1/2$ the previous argument no longer gives the neglibility of the difference (4.6). Instead, a careful upper-bound estimate of the L_p-norm of (4.6) is required. On the other hand, when $H \le 1/4$, the difference (4.6) is no longer negligible. A main effort of the proof of Theorem 4.2 will be to study the limit theorem for (4.6) and its joint distribution with the weighted sum $\sum_{t_k} |y'_{t_k}|^p \cdot |\delta x_{t_k t_{k+1}}|^p$.

Proof of Theorem 4.2. Take $\varepsilon > 0$ sufficiently small. Recall that $y^{(i)}$, $r^{(i)}$, $i = 0, \ldots, \ell - 1$ and $G_{\mathbf{y}} = G_{\mathbf{y},\varepsilon}$ are defined in Definition 2.1. Let $G_x = G_{x,\varepsilon}$ be a finite random variable such that $|x^1_{st}| \leq G_x(t-s)^{H-\varepsilon}$. By localization (cf. [28, Lemma 3.4.5]) we can and will assume that there exists some constant $C_0 > 0$ such that

$$\sum_{i=0}^{\ell-1} \Big(\sup_{t\in[0,1]} |y^{(i)}_t| + \sup_{s,t\in[0,1]} |r^{(i)}_{st}| \Big) + G_{\mathbf{y}} + G_x < C_0 \qquad \text{almost surely.} \tag{4.7}$$

Note that under this assumption it is clear that $(y^{(0)}, \ldots, y^{(\ell-1)})$ is controlled by $(x, \ell, H - \varepsilon)$ in L_p for any $p > 0$ (see Definition 2.1).

We divide the proof into several steps.

Step 1: Taylor's expansion of the function ϕ. For convenience let us denote

$$q = \begin{cases} p & \text{when } p \text{ is an even number.} \\ [p] - 1 & \text{otherwise.} \end{cases}$$

Applying the Taylor expansion to $\phi(\delta y_{t_k t_{k+1}})$ at the value $y^{(1)}_{t_k} \delta x_{t_k t_{k+1}}$ we get

$$\phi(\delta y_{t_k t_{k+1}}) = I_1 + I_2, \tag{4.8}$$

where

$$I_1 = \sum_{j=0}^{q} \frac{\phi^{(j)}(y^{(1)}_{t_k} x^1_{t_k t_{k+1}})}{j!} \cdot (\delta y_{t_k t_{k+1}} - y^{(1)}_{t_k} \delta x_{t_k t_{k+1}})^j \tag{4.9}$$

$$I_2 = \frac{\phi^{(q+1)}(\xi_k)}{(q+1)!} \cdot (\delta y_{t_k t_{k+1}} - y^{(1)}_{t_k} \delta x_{t_k t_{k+1}})^{q+1}, \tag{4.10}$$

and ξ_k is some value between $\delta y_{t_k t_{k+1}}$ and $y^{(1)}_{t_k} \delta x_{t_k t_{k+1}}$.

Step 2: Estimate of I_2. We first note that when p is an even number $\phi^{(q+1)}(\xi_k) = \phi^{(p+1)}(\xi_k) = 0$, and so $I_2 = 0$. In the following we assume that p is not even and by definition of q we have $q + 1 = [p]$. It is clear that

$$|\xi_k| \leq |\delta y_{t_k t_{k+1}}| + |y^{(1)}_{t_k} \delta x_{t_k t_{k+1}}|. \tag{4.11}$$

The relation (4.11) together with the definition of $\phi^{(q+1)}$ in (4.1) yields

$$|\phi^{(q+1)}(\xi_k)| = |\phi^{([p])}(\xi_k)| \lesssim |\delta y_{t_k t_{k+1}}|^{p-[p]} + |y^{(1)}_{t_k} \delta x_{t_k t_{k+1}}|^{p-[p]}. \tag{4.12}$$

Since y is controlled by x, Definition 2.1 and the assumption (4.7) gives

$$|\delta y_{t_k t_{k+1}}| \leq G_{\mathbf{y}}(1/n)^{H-\varepsilon} \leq C_0(1/n)^{H-\varepsilon}.$$

Similarly, we have $|y^{(1)}_{t_k} \delta x_{t_k t_{k+1}}| \lesssim (1/n)^{H-\varepsilon}$. Substituting these two estimates into (4.12) we get

$$|\phi^{([p])}(\xi_k)| \lesssim (1/n)^{(p-[p])H-\varepsilon} \wedge 1, \tag{4.13}$$

where we added $\wedge 1$ to include the case when p is odd.

By Definition 2.1 of controlled processes again we have the estimate $|\delta y_{t_k t_{k+1}} - y^{(1)}_{t_k} \delta x_{t_k t_{k+1}}|_{L_2} \lesssim (1/n)^{2H-\varepsilon}$. Applying this estimate and the estimate (4.13) to (4.10) we obtain

$$\Big| \sum_{0\leq t_k<t} I_2 \Big| \leq \sum_{0\leq t_k<t} |I_2| \lesssim n \cdot (1/n)^{(p-[p])H-\varepsilon} \cdot (1/n)^{2[p]H-\varepsilon} = (1/n)^{pH+[p]H-1-2\varepsilon}. \tag{4.14}$$

It follows from (4.14) that when $1/2 \geq H > 1/4$ and $p \geq 2$ we have

$$n^{pH-1/2} \sum_{0 \leq t_k < t} I_2 \to 0 \qquad \text{in probability as } n \to \infty \tag{4.15}$$

and when $H \leq 1/4$ and $p \geq 3$ we have

$$n^{(p+2)H-1} \sum_{0 \leq t_k < t} I_2 \to 0 \qquad \text{in probability as } n \to \infty. \tag{4.16}$$

This shows that I_2 does not have contribution in the limits of U^n in (4.3)-(4.5) under the given conditions in Theorem 4.2.

Step 3: Decomposition of I_1. Recall that $\phi^{(j)}(x)$ and $\phi_j(x)$ are defined in (4.1)-(4.2). It is clear that

$$\phi^{(j)}(a \cdot b) = K_j \phi_j(a \cdot b) = K_j \phi_j(a)\phi_j(b) \qquad \text{for any } a \text{ and } b \in \mathbb{R}. \tag{4.17}$$

On the other hand, by (2.2) we have:

$$\delta y_{t_k t_{k+1}} - y^{(1)}_{t_k} \delta x_{t_k t_{k+1}} = \sum_{i=2}^{\ell-1} y^{(i)}_{t_k} x^i_{t_k t_{k+1}} + r^{(0)}_{t_k t_{k+1}}. \tag{4.18}$$

Substituting (4.17)-(4.18) into (4.9) we obtain

$$I_1 = \sum_{j=0}^{q} \frac{K_j \phi_j(y^{(1)}_{t_k}) \phi_j(x^1_{t_k t_{k+1}})}{j!} \cdot \left(\sum_{i=2}^{\ell-1} y^{(i)}_{t_k} x^i_{t_k t_{k+1}} + r^{(0)}_{t_k t_{k+1}} \right)^j. \tag{4.19}$$

In the following we consider two different decompositions of I_1 in (4.19) according to the value of H.

When $H \leq 1/4$ we consider the decomposition:

$$I_1 = J_1 + J_2 + J_3 + J_4 + J_5 + J_6, \tag{4.20}$$

where

$$\begin{aligned}
J_1 &= K_0 \phi_0(y^{(1)}_{t_k}) \phi_0(x^1_{t_k t_{k+1}}) = |y^{(1)}_{t_k}|^p \cdot |x^1_{t_k t_{k+1}}|^p \\
J_2 &= K_1 \phi_1(y^{(1)}_{t_k}) \phi_1(x^1_{t_k t_{k+1}}) \cdot y^{(2)}_{t_k} x^2_{t_k t_{k+1}} \qquad (4.21)\\
J_3 &= K_1 \phi_1(y^{(1)}_{t_k}) \phi_1(x^1_{t_k t_{k+1}}) \cdot y^{(3)}_{t_k} x^3_{t_k t_{k+1}} \\
J_4 &= \frac{K_2 \phi_2(y^{(1)}_{t_k}) \phi_2(x^1_{t_k t_{k+1}})}{2!} \cdot \left(y^{(2)}_{t_k} x^2_{t_k t_{k+1}} \right)^2 \\
J_5 &= \sum_{j=0}^{q} \frac{K_j \phi_j(y^{(1)}_{t_k}) \phi_j(x^1_{t_k t_{k+1}})}{j!} \cdot \left(\sum_{i=2}^{\ell-1} y^{(i)}_{t_k} x^i_{t_k t_{k+1}} \right)^j - \sum_{e=1}^{4} J_e. \qquad (4.22)\\
J_6 &= I_1 - \sum_{j=0}^{q} \frac{K_j \phi_j(y^{(1)}_{t_k}) \phi_j(x^1_{t_k t_{k+1}})}{j!} \cdot \left(\sum_{i=2}^{\ell-1} y^{(i)}_{t_k} x^i_{t_k t_{k+1}} \right)^j. \qquad (4.23)
\end{aligned}$$

When $H > 1/4$ we consider the decomposition

$$I_1 = J_1 + J_6 + (I_1 - J_1 - J_6). \tag{4.24}$$

Step 4: Estimate of J_6. Recall that J_6 is defined in (4.23). Note that J_6 consists of the terms in (4.19) which contain $r^{(0)}_{t_k t_{k+1}}$. Similar to the estimate in (4.13), invoking the definition of controlled processes (see Definition 2.1) and the assumption (4.7) we have

$$|r^{(0)}_{t_k t_{k+1}}| \lesssim (1/n)^{\ell H - \varepsilon} \qquad \text{and} \qquad |y^{(i)}_{t_k} x^i_{t_k t_{k+1}}| \lesssim (1/n)^{2H-\varepsilon}, \quad i = 2, \ldots, \ell - 1.$$

It follows that

$$\left|\left(\sum_{i=2}^{\ell-1} y_{t_k}^{(i)} x_{t_k t_{k+1}}^i + r_{t_k t_{k+1}}^{(0)}\right)^j - \left(\sum_{i=2}^{\ell-1} y_{t_k}^{(i)} x_{t_k t_{k+1}}^i\right)^j\right| \lesssim (1/n)^{\ell H-\varepsilon}(1/n)^{2H\cdot(j-1)-\varepsilon}. \quad (4.25)$$

On the other hand, by the definition of ϕ_j in (4.1) we have the estimate

$$\left|\frac{K_j \phi_j(y_{t_k}^{(1)})\phi_j(x_{t_k t_{k+1}}^1)}{j!}\right| \lesssim (1/n)^{(p-j)H-\varepsilon}. \quad (4.26)$$

Substituting the two estimates (4.25)-(4.26) into (4.23) we obtain

$$|J_6| \lesssim \sum_{j=1}^{q} (1/n)^{(\ell+2(j-1)+(p-j))H-\varepsilon} \lesssim (1/n)^{(\ell-1+p)H-\varepsilon}.$$

It follows that

$$|\sum_{0\leq t_k<t} J_6| \leq \sum_{0\leq t_k<t} |J_6| \lesssim (1/n)^{(\ell-1+p)H-1-\varepsilon}.$$

It is readily checked that

$$n^{pH-1/2} \sum_{0\leq t_k<t} J_6 \to 0 \qquad \text{when } \ell\geq 3 \text{ and } H>1/4 \quad (4.27)$$

and

$$n^{(p+2)H-1} \sum_{0\leq t_k<t} J_6 \to 0 \qquad \text{when } \ell\geq 4 \text{ and } H\leq 1/4. \quad (4.28)$$

Note that this shows that J_6 does not have contribution in any of the limits of U^n in (4.3)-(4.5).

Step 5: Convergence of J_1. We first consider the case when $1/2 \geq H > 1/4$. According to Proposition 3.9 given that $p > 3/2$ and that $|y'|^p$ is controlled by $(x, 2, H)$ we have the stable f.d.d. convergence:

$$\frac{1}{\sqrt{n}} \sum_{0\leq t_k<t} (n^{pH} J_1 - |y'_{t_k}|^p c_p) \to \sigma \int_0^t |y'_t|^p dW_t \qquad \text{as } n\to\infty. \quad (4.29)$$

Note that by Lemma 2.4 (ii) and Lemma 4.1 (i) for $|y'|^p$ to be controlled by $(x, 2, H)$ it requires $p \geq 2$ and that y is controlled by (x, ℓ, H) for $\ell \geq 3$.

On the other hand, given that $|y'|^p$ is controlled by $(x, 2, H)$ Proposition 3.7 implies that

$$\frac{1}{\sqrt{n}} \sum_{0\leq t_k<t} |y'_{t_k}|^p c_p - \sqrt{n}\cdot c_p \int_0^t |y'_u|^p du \to 0 \qquad \text{as } n\to\infty. \quad (4.30)$$

Combining (4.30) with the convergence in (4.29) we obtain the stable f.d.d. convergence

$$\sqrt{n}\left(\sum_{0\leq t_k<t} n^{pH-1} J_1 - c_p \int_0^t |y'_u|^p du\right) \to \sigma \int_0^t |y'_u|^p dW_u \qquad \text{as } n\to\infty. \quad (4.31)$$

We turn to the case when $H = 1/4$. According to Proposition 3.9 given that $|y'|^p$ is controlled by $(x, 3, H)$, $p \in (7/2, \infty) \cup \{2\}$ we have the stable f.d.d. convergence:

$$\frac{1}{\sqrt{n}} \sum_{0\leq t_k<t} (n^{pH} J_1 - |y'_{t_k}|^p c_p) \to \sigma \int_0^t |y'_u|^p dW_u + \frac{pc_p}{8}\int_0^t (|y'_u|^p)'' du \qquad \text{as } n\to\infty,$$

where $(|y'_u|^p)'' = (\phi(y'_u))'' = \phi''(y'_u)(y''_u)^2 + \phi'(y'_u)y'''_u$. By Lemma 2.4 (ii) and Lemma 4.1 (i) again this requires $p \in [3,\infty) \cup \{2\}$ and $\ell \geq 4$. Similar to (4.31), we can apply Proposition 3.5 to obtain the stable f.d.d. convergence:

$$\sqrt{n}\left(\sum_{0\leq t_k<t} n^{pH-1}J_1 - c_p\int_0^t |y'_u|^p du\right) \to \sigma\int_0^t |y'_u|^p dW_u + \frac{pc_p}{8}\int_0^t (|y'_u|^p)'' du\,. \tag{4.32}$$

When $H < 1/4$, given that $p \in (7/2,\infty) \cup \{2\}$ and $|y'|^p$ is controlled by $(x, 3, H)$ we have the convergence in probability:

$$n^{2H-1}\sum_{0\leq t_k<t}(n^{pH}J_1 - |y'_{t_k}|^p c_p) \to \frac{pc_p}{8}\int_0^t (|y'_u|^p)'' du\,. \tag{4.33}$$

Then it follows from Proposition 3.5 again that

$$n^{2H}\left(\sum_{0\leq t_k<t} n^{pH-1}J_1 - c_p\int_0^t |y'_u|^p du\right) \to \frac{pc_p}{8}\int_0^t (|y'_u|^p)'' du \qquad \text{in probability.}$$

Step 6: Proof of (4.3) *and the convergence of* $(I_1 - J_1 - J_6)$. In this step we show the convergence:

$$n^{pH-1/2}\sum_{0\leq t_k<t}(I_1 - J_1 - J_6) \to 0. \tag{4.34}$$

Combining (4.34) with the convergences of I_2, J_6 and J_1 respectively in (4.15), (4.27) and (4.31), and invoking the relations (4.8) and (4.24) we then obtain the convergence in (4.3).

We first note that by the definition of I_1, J_1 and J_6 we have

$$\sum_{0\leq t_k<t}(I_1 - J_1 - J_6) = \sum_{0\leq t_k<t}\sum_{j=1}^{q}\frac{K_j\phi_j(y^{(1)}_{t_k})\phi_j(x^1_{t_kt_{k+1}})}{j!}\cdot\left(\sum_{i=2}^{\ell-1} y^{(i)}_{t_k}x^i_{t_kt_{k+1}}\right)^j. \tag{4.35}$$

It is easy to see that (4.35) consists of weighted sums of the form $\mathcal{J}_0^t(z, h^n)$ for

$$z = \frac{K_j\phi_j(y^{(1)}_{t_k})}{j!}\cdot\sum_{\substack{2\leq i_1,\dots,i_j\leq \ell-1\\ i_1+\cdots+i_j=r}} y^{(i_1)}_{t_k}\cdots y^{(i_j)}_{t_k} \tag{4.36}$$

and

$$\begin{aligned} h^n_{st} = \sum_{s\leq t_k<t}\phi_j(x^1_{t_kt_{k+1}})\cdot(x^1_{t_kt_{k+1}})^r &= \sum_{s\leq t_k<t}(x^1_{t_kt_{k+1}})^r|x^1_{t_kt_{k+1}}|^{p-j}\text{sign}(x^1_{t_kt_{k+1}})^j \\ &= \sum_{s\leq t_k<t}|x^1_{t_kt_{k+1}}|^{p-j+r}\text{sign}(x^1_{t_kt_{k+1}})^{j+r}. \end{aligned} \tag{4.37}$$

for $j = 1,\dots,q$ and $r \geq 2j$. When $p - j + r \geq p + 2$ we can bound $|h^n_{t_kt_{k+1}}| \lesssim (1/n)^{(p+2)H-\varepsilon}$ and thus we have the convergence

$$n^{pH-1/2}\mathcal{J}_0^t(z, h^n) \to 0 \qquad \text{in probability as } n\to\infty. \tag{4.38}$$

In the following we consider the estimate of (4.37) when $p - j + r < p + 2$, that is when $r - j < 2$. Note that this implies $r - j = 1$ and thus $j = 1$ and $r = 2$. So we have

$$\mathcal{J}_0^t(z, h^n) = \sum_{0\leq t_k<t} K_1\phi_1(y'_{t_k})\phi_1(x^1_{t_kt_{k+1}})y''_{t_k}x^2_{t_kt_{k+1}}.$$

Let $f(x) = \phi_1(x)x^2$, $x \in \mathbb{R}$. It is clear that f has Hermite rank $d = 1$. According to Proposition 3.2 (ii), given that (a) $f \in W^{2,2}(\mathbb{R}, \gamma)$ and (b) $z = \phi_1(y')y''$ defined in (4.36) is controlled by $(x, 2, H)$ we have $\mathcal{J}(z, h^n) \sim (1/n)^{H-1+H(p+1)}$, which implies the convergence (4.38). By Lemma 2.4(ii) and Lemma 4.1(i) Condition (a) requires either $2(p+1-2) > -1$ or that p is even, which means we must have $p > 1/2$. By Lemma 2.4(ii) and Lemma 4.1(i) Condition (b) holds when $p \in [3, \infty) \cup \{2\}$ and $\ell \geq 4$.

In summary, we have shown that (4.38) holds for all j and r. Invoking the decomposition of $I_1 - J_1 - J_6$ in (4.35)-(4.37), we conclude (4.34) and thus (4.3).

Step 7: Convergence of J_2 when $H \leq 1/4$. Recall the definition of ϕ_j and J_2 in (4.1) and (4.21), respectively. So we have

$$\sum_{0 \leq t_k < t} J_2 = \frac{p}{2} \sum_{0 \leq t_k < t} \phi_1(y'_{t_k}) y''_{t_k} |x^1_{t_k t_{k+1}}|^{p+1} \text{sign}(x^1_{t_k t_{k+1}}).$$

Suppose that $\phi_1(y'_t)y''_t$ is controlled by $(x, 2, H)$. According to Lemma 2.4 (ii) this requires $p \in [3, \infty) \cup \{2\}$ and $\ell \geq 4$, and in this case we have

$$(\phi_1(y'_t)y''_t)' = \phi'_1(y'_t)(y''_t)^2 + \phi_1(y'_t)y'''_t.$$

Applying Lemma 3.8 we obtain the convergence in probability

$$\begin{aligned} n^{(p+2)H-1} \sum_{0 \leq t_k < t} J_2 &\to \frac{p}{2}(-\frac{1}{2}c_{p+2}) \int_0^t (\phi'_1(y'_u)(y''_u)^2 + \phi_1(y'_u)y'''_u) du \\ &= -\frac{1}{4}(p+1)c_p \int_0^t (\phi''(y'_u)(y''_u)^2 + \phi'(y'_u)y'''_u) du. \end{aligned} \tag{4.39}$$

Step 8: Convergence of J_3 when $H \leq 1/4$. We first rewrite J_3 as

$$J_3 = K_1 \phi_1(y^{(1)}_{t_k}) \phi_1(x^1_{t_k t_{k+1}}) \cdot y^{(3)}_{t_k} x^3_{t_k t_{k+1}} = \frac{p}{6} \phi_1(y'_{t_k}) y'''_{t_k} |x^1_{t_k t_{k+1}}|^{p+2}.$$

It is easy to see that we have the bound $|\sum_{0 \leq t_k < t} J_3|_{L_p} \lesssim (1/n)^{(p+2)H}$. In the following we show that $\sum_{0 \leq t_k < t} J_3$ is also convergence under proper conditions of p and ℓ.

We consider the following decomposition

$$\begin{aligned} J_3 \quad &= \quad \frac{p}{6} \phi_1(y'_{t_k}) y'''_{t_k} \left(|x^1_{t_k t_{k+1}}|^{p+2} - c_{p+2}(1/n)^{(p+2)H} \right) + \frac{p}{6} c_{p+2} \phi_1(y'_{t_k}) y'''_{t_k} (1/n)^{(p+2)H} \\ &=: \quad J_{31} + J_{32}. \end{aligned} \tag{4.40}$$

Applying Proposition 3.2 (ii) to J_{31} with $d = 2$ we obtain that $n^{(p+2)H-1} \sum_{0 \leq t_k < t} J_{31} \to 0$ in probability. Note that the application of Proposition 3.2 (ii) requires $p \in [4, \infty) \cup \{2\}$ and $\ell \geq 6$. On the other hand, by continuity of $\phi_1(y')y'''$ we have the convergence $n^{(p+2)H-1} \sum_{0 \leq t_k < t} J_{32} \to \frac{p}{6} c_{p+2} \int_0^t \phi_1(y'_u) y'''_u du$. Substituting these two convergence into (4.40) we obtain

$$n^{(p+2)H-1} \sum_{0 \leq t_k < t} J_3 \to \frac{p}{6} c_{p+2} \int_0^t \phi_1(y'_u) y'''_u du = \frac{p+1}{6} c_p \int_0^t \phi'(y'_u) y'''_u du. \tag{4.41}$$

Step 9: Convergence of J_4 when $H \leq 1/4$. We first rewrite J_4 as

$$J_4 = \frac{K_2 \phi_2(y^{(1)}_{t_k}) \phi_2(x^1_{t_k t_{k+1}})}{2!} \cdot \left(y^{(2)}_{t_k} x^2_{t_k t_{k+1}} \right)^2 = \frac{p(p-1)}{8} \phi_2(y'_{t_k})(y''_{t_k})^2 \cdot |x^1_{t_k t_{k+1}}|^{p+2}.$$

Similar to J_3 by applying Proposition 3.9 (ii)–(iii) with $d = 2$ we obtain the convergence

$$\begin{aligned} n^{(p+2)H-1} \sum_{0 \leq t_k < t} J_4 &\to \frac{p(p-1)}{8} c_{p+2} \int_0^t \phi_2(y'_u)(y''_u)^2 du \\ &= \frac{p+1}{8} c_p \int_0^t \phi''(y'_u)(y''_u)^2 du. \end{aligned} \tag{4.42}$$

Note that the application of Proposition 3.9 (ii)–(iii) requires $p \in [5,\infty) \cup \{2,4\}$ and $\ell \geq 5$.
Step 10: Convergence of J_5. Recall that J_5 is defined in (4.22). It is easy to see that we have

$$\begin{aligned} J_5 \quad &= \quad \frac{K_1 \phi_1(y_{t_k}^{(1)}) \phi_1(x^1_{t_k t_{k+1}})}{1!} \cdot \left(\sum_{i=4}^{\ell-1} y_{t_k}^{(i)} x^i_{t_k t_{k+1}} \right) \\ &\quad + \frac{K_2 \phi_2(y_{t_k}^{(1)}) \phi_2(x^1_{t_k t_{k+1}})}{2!} \cdot \left(\sum_{i=3}^{\ell-1} y_{t_k}^{(i)} x^i_{t_k t_{k+1}} \right)^2 \\ &\quad + \sum_{j=3}^{q} \frac{K_j \phi_j(y_{t_k}^{(1)}) \phi_j(x^1_{t_k t_{k+1}})}{j!} \cdot \left(\sum_{i=2}^{\ell-1} y_{t_k}^{(i)} x^i_{t_k t_{k+1}} \right)^j \\ &=: \quad J_{51} + J_{52} + J_{53}, \end{aligned}$$

where we use the convention that $\sum_{j=3}^{q} = 0$ when $q < 3$ and that $\sum_{i=4}^{\ell-1} y_{t_k}^{(i)} x^i_{t_k t_{k+1}} = 0$ when $\ell - 1 < 4$. In the following we bound each J_{5i}, $i = 1,2,3$.

For J_{51} a direct estimate shows that

$$|J_{51}| \lesssim |\phi_1(y_{t_k}^{(1)})| \cdot |\phi_1(x^1_{t_k t_{k+1}})| \cdot \sum_{i=4}^{\ell-1} |y_{t_k}^{(i)}| \cdot |x^i_{t_k t_{k+1}}| \lesssim (1/n)^{(p-1)H+4H} = (1/n)^{(p+3)H}.$$

So we have $n^{(p+2)H-1} \sum_{0 \leq t_k < t} J_{51} \to 0$. Similarly, we can bound J_{52} and J_{53} by

$$|J_{52}| \lesssim (1/n)^{(p-2)H+6H} \qquad \text{and} \qquad |J_{53}| \lesssim \sum_{j=3}^{\lfloor p \rfloor - 1} (1/n)^{(p-j)H+2jH} \lesssim (1/n)^{(p+3)H}.$$

We conclude that $n^{(p+2)H-1} \sum_{0 \leq t_k < t} (J_{52} + J_{53}) \to 0$ in probability as $n \to \infty$. We conclude that the convergence in probability

$$n^{(p+2)H-1} \sum_{0 \leq t_k < t} J_5 \to 0 \qquad \text{as } n \to \infty. \tag{4.43}$$

Step 11: Conclusion. In Step 5 we have shown that the convergence in (4.3) holds. Putting together the convergences (4.16), (4.28), (4.32), (4.39), (4.41), (4.42), (4.43) for I_2, J_i, $i = 1, \dots, 6$ and invoking Lemma 2.8, and then taking into account the decompositions (4.8) and (4.20), we obtain the convergence in (4.4). Finally, replacing (4.32) by (4.33) in the argument we obtain the convergence (4.5). □

Remark 4.4. Let us compare Theorem 4.2 to the classical results for p-variations. Consider $y_t = \int_0^t z_u dx_u$. Then we have $y'_t = z_t$. When $1/4 < H \leq 1/2$, the relation (4.3) gives the f.d.d. convergence

$$n^{1/2} \Big[n^{pH-1} \sum_{0 \leq t_k < t} |y_{t_{k+1}} - y_{t_k}|^p - c_p \int_0^t |z_u|^p du \Big] \to \sigma \int_0^t |z_u|^p dW_u$$

as $n \to \infty$. We note that this recovers the results obtained in [16, Theorem 4].

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