

On convex integration solutions to the surface quasi-geostrophic equation driven by generic additive noise*

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Abstract

We study the surface quasi-geostrophic equation driven by a generic additive noise process W . By means of convex integration techniques, we establish existence of weak solutions whenever the stochastic convolution z associated with W is well defined and fulfills certain regularity constraints. Quintessentially, we show that the so constructed solutions to the non-linear equation are controlled by z in a linear fashion. This allows us to deduce further properties of the so constructed solutions, without relying on structural probabilistic properties such as Gaussianity, Markovianity or a martingale property of the underlying noise W .

Keywords: SQG equation; stochastic partial differential equations; additive noise; convex integration; stochastic fluid dynamics.

MSC2020 subject classifications: 60H15; 35R60; 35Q35.

Submitted to EJP on December 20, 2023, final version accepted on October 2, 2024.

1 Introduction

In this work, we are concerned with the construction of solutions to the surface quasi-geostrophic equation (SQG equation) on the two-dimensional torus \mathbb{T}^2

$$\begin{aligned} \partial_t \theta + u \cdot \nabla \theta &= -\nu \Lambda^\gamma \theta + \Lambda^\delta \dot{W} \\ u &= \nabla^\perp \Lambda^{-1} \theta = (-\partial_2 \Lambda^{-1} \theta, \partial_1 \Lambda^{-1} \theta) = (-\mathcal{R}_2 \theta, \mathcal{R}_1 \theta) = \mathcal{R}^\perp \theta \end{aligned} \tag{1.1}$$

where $\Lambda = (-\Delta)^{1/2}$, $\nu \geq 0$, $\gamma \in [0, 3/2)$, \mathcal{R} is the pair of Riesz-transforms, and \dot{W} is a generic noise process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ for which the associated stochastic

*Florian Bechtold and Theresa Lange have received funding from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (grant agreement No. 949981). Jörn Wichmann was partially supported by the Australian Government through the Australian Research Council’s Discovery Projects funding scheme (grant number DP220100937).

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convolution

$$z_t := \int_0^t e^{-\nu(t-s)\Lambda^\gamma} \Lambda^{\delta-1} dW_s$$

is well defined and satisfies certain integrability conditions to be made precise below. Towards this end, we adapt a convex integration scheme originating from [7] and subsequently extended in [21, 20] working on the level of the transformed unknown $g = \Lambda^{-1}\theta - z$, which satisfies

$$\nabla \cdot (-\partial_t \mathcal{R}g + \nabla^\perp(g+z)\Lambda(g+z)) = \nabla \cdot (\nu\Lambda^{\gamma-1}\nabla g) \quad \text{in } \Omega \times \mathbb{R} \times \mathbb{T}^2, \quad (1.2)$$

Definition 1.1. We call $g \in C_{\text{loc}}(\mathbb{R}; \dot{H}^{1/2}(\mathbb{T}^2))$ a weak solution to (1.2) if

$$\begin{aligned} \langle \mathcal{R}g, \nabla \xi \rangle(t) - \langle \mathcal{R}g, \nabla \xi \rangle(s) - \int_s^t \langle \Lambda(g+z), \nabla^\perp(g+z) \cdot \nabla \xi \rangle dr \\ = \nu \int_s^t \langle g, \nabla \cdot [\Lambda^{\gamma-1}\nabla \xi] \rangle dr \end{aligned} \quad (1.3)$$

for all $t \geq s$, $\xi \in C^\infty(\mathbb{T}^2)$ and \mathbb{P} -a.s., where the non-linearity is understood in the sense of Lemma A.2.

For the rigorous connection between (1.2) and (1.1), we refer to subsection A.2 in the appendix. Our main result is the following. For the notational conventions employed, we refer to Section 2.

Theorem 1.2. Let W be a stochastic process such that the expression

$$z_t = \int_0^t e^{-\nu(t-s)\Lambda^\gamma} \Lambda^{\delta-1} dW_s$$

is well defined and satisfies

$$[z]_{L_\omega^{2m} C_t^\alpha C_x^{1+}} + \|z\|_{L_\omega^{2m} C_{\text{loc}} C_x^{1+\kappa}} < \infty \quad (1.4)$$

for some $m \geq 1, \alpha > 0$ and $\kappa > 1/2, \delta \in \mathbb{R}$ and $\gamma < 3/2$. Then there exists a weak solution g to (1.2) in the sense of Definition 1.1 satisfying

$$\|g\|_{L_\omega^{2m} C_{\text{loc}} B_{\infty,1}^{1/2}} \leq C([z]_{L_\omega^{2m} C_t^\alpha C_x^{1+}} + \|z\|_{L_\omega^{2m} C_{\text{loc}} C_x^{1+\kappa}} + 1). \quad (1.5)$$

Let us comment on the main insights and novelties of the above result, to be discussed also later in the paper:

- Despite the equation (1.2) being non-linear in g , we are able to construct solutions which are controlled by the forcing in a linear fashion as expressed by (1.5).
- As a consequence, whenever quantified moment estimates on z are available, they carry over to the solution g . For example, in the case of W being a cylindrical Wiener process on $L^2(\mathbb{T}^2)$ with sufficient spatial coloring (which in particular is Gaussian), this immediately implies sub-Gaussian moment estimates of the form

$$\mathbb{E} \left[\exp \epsilon \|g\|_{C_{\text{loc}} B_{\infty,1}^{1/2}}^2 \right] < \infty$$

for some $\epsilon > 0$, unavailable in previous stochastic convex integration schemes.

- We stress that in terms of the underlying stochastic process W , our theorem merely requires (1.4). In particular, it is not necessary to assume much on the specific probabilistic structure of W such as Markovianity, Gaussianity or the martingale property. Indeed, (1.4) can be assured in rather general contexts, as we demonstrate in a dedicated examples section 4. This conveys a remarkable robustness of the convex integration scheme.

- To the best of our knowledge, we demonstrate in particular for the first time that convex integration methods can prevail in the face of temporally rough noises (i.e. noises which are rougher than “white in time”). Given our ensuing non-uniqueness results (Section 5.1), the observation “rougher noises yield more substantial regularization effects” made for stochastic differential equations e.g. (see for example [16]) does not apply in our context. This again underlines the robustness of the convex integration scheme.

1.1 Literature on the SQG equation

Let us begin by discussing results in the literature on the case of the deterministic SQG equations, i.e. (1.1) with $W = 0$. As a special case of the quasi-geostrophic equations, they form a model of potential temperature in the context of geophysical fluid dynamics, describing the evolution of boundary temperature in a fast rotating stratified fluid. For more details on its physical interpretation, derivation and application in atmospheric sciences, meteorology and oceanography we refer to e.g. [36], [31], [8], [18], where in particular [8] illustrates an interesting analogy to the 3D Euler equations underlining the interest in studying the SQG equation. In the deterministic case, though not exhaustive, one may highlight the following results: some works on well-posedness of the SQG equation show global existence of weak solutions in case $\nu \geq 0, 0 < \gamma \leq 2$ in $L_t^\infty L_x^2$ for any initial data in $L^2(\mathbb{T}^2)$ [38], as well as in the case $\nu > 0, 0 < \gamma \leq 2$ in $L_t^\infty H_x^{-1/2}$ for initial conditions in $\dot{H}_x^{1/2}(\mathbb{R}^2)$, and of regularity $L_t^\infty L_x^p$ for initial conditions in $L_x^p(\mathbb{R}^2), p \geq 4/3$ [33]. Some results and discussions on potential blow-up of solutions can be found in [34], [11], [12], [9], [5], [41]. Furthermore, global existence of regular solutions was derived in [13] via the existence of an invariant measure on \mathbb{T}^2 and for all initial conditions in the support of that measure. In terms of uniqueness, [10] showed uniqueness of SQG patches with moving boundary satisfying an arc-chord condition in the case $\nu = 0$; however, non-unique weak solutions $(-\Delta)^{-1/2}\theta \in C_t^\sigma C_x^\beta$ for $\nu \geq 0, 0 < \gamma < 2 - \beta, 1/2 < \beta < 4/5, \sigma < \frac{\beta}{2-\beta}$ were established in [3] via convex integration. Also see [29] for a direct approach. For the steady-state SQG equation, in [7] the authors derive existence of stationary weak solutions employing a similar convex integration scheme yielding non-uniqueness of the solutions.

There have been several results in the direction of “regularization by noise” obtained in connection with the SQG equations. Observe that the authors in [13] derive an invariant measure for the deterministic case via invariant measures for the SQG equation perturbed by a carefully chosen additive noise. Including random perturbations in general has shed new light on questions of well-posedness: in the cases of additive and linear multiplicative noises, the authors in [39] and [40] provide local existence and uniqueness (continuous in some H^s on a smooth bounded domain [39], and in $L^p(\mathbb{T}^2), p \in (2, \infty)$ [40]) as well as non-explosion and ergodicity results. In [51] existence of a random attractor is analysed. We would furthermore like to mention the works [14] and [28] on local well-posedness for space-time white noise via regularity structures and paracontrolled distributions, respectively, as well as a large deviation principle for small multiplicative noise and small time in [32]. Further note the regularization result in [2] via random diffusions.

1.2 Stochastic convex integration results

After the advent of deterministic convex integration results (for instance, see [4] and references therein), it became natural to ask if the techniques developed in this context carry over to corresponding stochastic problems in order to establish non-uniqueness results. With this background, Hofmanová, Zhu and Zhu were able to adapt

convex integration methods to hydrodynamic equations with noise. Their first papers [24, 22, 23] did so by working with stopping time techniques, which has the advantage of avoiding moment estimates along the convex integration iteration. For a gentle overview of these results, refer also to [19]. In this context, let us also mention the works of Yamazaki [46, 45, 48] and Schenke and Rehmeier [37] and in particular [47], which treats the case of SQG equations with additive noise deriving existence and non-uniqueness in law of solutions to the corresponding momentum equations of regularity $C_t C_x^{1/2+\iota} \cap C_t^\eta C_x, \iota \in (0, 1/2 - \gamma/3), \eta \in (0, 1/3]$. The case of linear multiplicative noise was subsequently studied in [49]. Additionally, prescribed energy solutions and non-unique Markov selection for the SQG equation with additive and linear multiplicative noise were established in [43]. Establishing moment estimates poses a problem in all these works, as convex integration schemes typically involve super-linear estimates that need to be controlled through careful parameter-tuning. In particular, as the estimate of a given moment in iteration $n + 1$ thus requires higher and higher moment estimates on the previous levels, one is unable to close moment estimates.

In [6], this issue was for the first time overcome by a moment-dependent parameter-tuning that allowed control of the blow-up of higher and higher moments along the iteration scheme, exploiting the quantified moment behaviour of the underlying Gaussian noise. While this opened the door for the establishment of stationary solutions through a Krylov-Bogoliobov type argument in [26], the drawback of the approach consists in the fact that one can not simultaneously study different moments of a specific convex integration, as the scheme itself depends on the moment one intends to investigate in the beginning. Also, as the tuning depends on the exact moment behaviour of the underlying noise, extensions to different noises are not obvious.

Inspired by [7], Hofmanová et al. study moment-based convex integration in the context of the SQG equations: for the case of only space dependent noise [21] and recently space-time white noise [20], they obtain infinitely many non-Gaussian weak solutions (of regularity $L_{\text{loc}}^p(0, \infty; B_{\infty,1}^{-1/2}) \cap C_b([0, \infty); B_{\infty,1}^{-1/2-}) \cap C_b^1([0, \infty); B_{\infty,1}^{-3/2-})$ for all $p \in [1, \infty)$ [21], and of regularity $L_{\text{loc}}^p([0, \infty); B_{\infty,1}^{-1/2}) \cap C([0, \infty), B_{\infty,1}^{-1/2-}) \cap C^{1/2-}([0, \infty), B_{\infty,1}^{-1-})$ for all $p \in [1, \infty)$ [20], respectively) as well as infinitely many stationary (ergodic) solutions. While one main focus of [20] consists in demonstrating that convex integration schemes can serve to establish solutions to supercritical SPDEs, which are inaccessible by tools such as regularity structures or paracontrolled distributions, the authors were able to also show that the convex integration scheme can be set up independently of a given moment fixed in the beginning. This allows them to show that arbitrary moments of the constructed convex integration solutions exist, provided this is the case for the initial condition. Further quantified moment bounds, however, are lacking. In particular, the existence of exponential moments is left open and the consideration of more general noises is lacking.

1.3 Structure of the paper

After introducing some basic notation and preliminaries in Section 2, we proceed to give the convex integration construction and thus the proof of our main theorem in Section 3. We then provide some concrete examples of noises to which our result is applicable, i.e. for which we explicitly verify the condition (1.4). We treat one class of general noises in the Young integration setting by means of the Sewing Lemma and another class of noises by means of more classical stochastic calculus arguments, in neither assuming Gaussianity, the Markov nor martingale property. We finally conclude by discussing some immediate consequences of our main result which can be obtained by a slight adaption of arguments in [21, 20].

2 Notation and preliminaries

We write $f \lesssim g$ for two non-negative quantities f and g if f is bounded by g up to a multiplicative universal constant. Accordingly we define \gtrsim and \asymp . We denote the maximum respectively minimum of two real numbers a and b by $a \vee b$ respectively $a \wedge b$. For two Banach spaces X and Y we write $X \hookrightarrow Y$ if the inclusion is continuous. We denote by $\langle \cdot, \cdot \rangle$ the inner product in $L^2(\mathbb{T}^2)$. The convolution of two functions in time is denoted by $*_t$, i.e., $f *_t g(s) = \int_{\mathbb{R}} f(s-r)g(r) dr$.

We denote by $C_x^{n+\kappa}$, $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\kappa \in [0, 1)$, the space of n -times continuously differentiable functions with κ -Hölder continuous n -th derivative equipped with the norm

$$\|g\|_{C_x^{n+\kappa}} = \sum_{|k| \leq n} \sup_{x \in \mathbb{T}^2} |\partial^k g(x)| + \sum_{|k|=n} \sup_{x \neq y \in \mathbb{T}^2} \frac{|\partial^k g(x) - \partial^k g(y)|}{|x - y|^\kappa},$$

where $\partial^k = \partial^{k_1} \partial^{k_2}$ for a multi index $k \in \mathbb{N}_0^2$.

Let E be a Banach space and $\alpha \in [0, 1)$. We define the Banach space of (α -Hölder) continuous E -valued functions on $[t, t + 1]$ by $C_t E$ and $C_t^\alpha E$, respectively. We equip them with the norms

$$\|g\|_{C_t E} := \sup_{s \in [t, t+1]} \|g(s)\|_E,$$

and

$$[g]_{C_t^\alpha E} := \sup_{s \neq s' \in [t, t+1]} \frac{\|g(s) - g(s')\|_E}{|s - s'|^\alpha},$$

$$\|g\|_{C_t^\alpha E} := \|g\|_{C_t E} + [g]_{C_t^\alpha E},$$

respectively. Moreover, for any $T > 0$, we define similarly $C_{[0, T]}$ and $C_{[0, T]}^\alpha$ via

$$\|g\|_{C_{[0, T]} E} := \sup_{s \in [0, T]} \|g(s)\|_E,$$

and

$$[g]_{C_{[0, T]}^\alpha E} := \sup_{s \neq s' \in [0, T]} \frac{\|g(s) - g(s')\|_E}{|s - s'|^\alpha},$$

$$\|g\|_{C_{[0, T]}^\alpha E} := \|g\|_{C_{[0, T]} E} + [g]_{C_{[0, T]}^\alpha E}.$$

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. The space of strongly measurable L^m -integrable, $m \in [1, \infty]$, E -valued random variables is denoted by $L_\omega^m E$ equipped with the usual L^m -norm. Let $L_\omega^m C_{\text{loc}} E$, respectively $L_\omega^m C_{\text{loc}}^\alpha E$, denote the space of L^m -integrable locally (α -Hölder) continuous E -valued functions equipped with the uniform time-localized (semi)-norms, $m \in [1, \infty)$, $\alpha \in (0, 1)$, defined by

$$\|g\|_{L_\omega^m C_{\text{loc}} E} := \sup_{t \in \mathbb{R}} \mathbb{E} [\|g\|_{C_t E}^m],$$

$$[g]_{L_\omega^m C_{\text{loc}}^\alpha E} := \sup_{t \in \mathbb{R}} \mathbb{E} [[g]_{C_t^\alpha E}^m],$$

$$\|g\|_{L_\omega^m C_{\text{loc}}^\alpha E} := \|g\|_{L_\omega^m C_{\text{loc}} E} + [g]_{L_\omega^m C_{\text{loc}}^\alpha E}.$$

The Fourier transform on \mathbb{T}^2 is defined by

$$\mathbb{Z}^2 \ni k \mapsto \widehat{f}(k) := \int_{\mathbb{T}^2} e^{ix \cdot k} f(x) dx.$$

For $s \in \mathbb{R}$, we define $\dot{H}^s = \{f : \int_{\mathbb{T}^2} f(x) dx = 0, \|\Lambda^s f\|_{L^2(\mathbb{T}^2)} < \infty\}$ equipped with the norm

$$\|f\|_{\dot{H}^s}^2 := \sum_{0 \neq k \in \mathbb{Z}^2} |k|^{2s} |\widehat{f}(k)|^2.$$

Let $(\Delta_j)_{j \geq -1}$ denote Littlewood-Paley blocks corresponding to a dyadic partition of unity. We define Besov spaces $B_{p,q}^s$, $s \in \mathbb{R}$, $p, q \in [1, \infty]$, as the completion of smooth functions with respect to the norm

$$\|g\|_{B_{p,q}^s} := \|\{2^{j\alpha} \|\Delta_j u\|_{L^p}\}_j\|_{\ell^q}.$$

Here $\|\cdot\|_{\ell^q}$ denotes the usual ℓ^q -sequence norm.

Let $\psi \in C_c^\infty(\mathbb{R}^2)$ satisfy $\psi(k) = 0$ for $|k| \geq 1$ and $\psi(k) = 1$ for $|k| \leq 1/2$. Define the frequency truncation $P_{\leq \lambda}$ for $\lambda > 0$ by

$$\widehat{P_{\leq \lambda} g}(k) = \psi\left(\frac{k}{\lambda}\right) \widehat{g}(k).$$

Notice that $\|P_{\leq \lambda} g\|_{L^\infty} \lesssim \|g\|_{L^\infty}$ for all $g \in L^\infty$ and $\lambda \geq 1$. For more details on Besov spaces and the stability of frequency truncation we refer to [44, Section 1.3]. We write $g \in B_{p,q}^{s-}$ if $g \in B_{p,q}^{s-\epsilon}$ for any $\epsilon > 0$ and $g \in B_{p,q}^{s+}$ if there exists $\epsilon > 0$ such that $g \in B_{p,q}^{s+\epsilon}$.

For any vector fields v, w we write $v \approx w$ if there exists a smooth scalar function p such that $v = w + \nabla^\perp p$.

We define, similarly to [7, (2.7)], the Riesz-type transforms \mathcal{R}_j^o , $j = 1, 2$, by

$$\widehat{\mathcal{R}_1^o}(k) := \frac{25(k_2^2 - k_1^2)}{12|k|^2}, \quad \widehat{\mathcal{R}_2^o}(k) := \frac{7(k_2^2 - k_1^2)}{12|k|^2} + \frac{4k_1 k_2}{|k|^2}$$

and

$$\|q\|_X := \|q\|_{L^\infty} + \sum_{j=1}^2 \|\mathcal{R}_j^o q\|_{L^\infty}.$$

3 Convex integration

We aim to construct a solution as the limit of an iterative algorithm, i.e.,

$$g = \lim_{n \rightarrow \infty} g_{\leq n}, \quad g_{\leq n} = g_{\leq n-1} + \tilde{g}_n,$$

where \tilde{g}_n are constructed in an explicit manner.

At first, the iterates $g_{\leq n}$ do not need to satisfy (1.2) exactly, but are allowed to carry a residual \tilde{q}_n , i.e.,

$$-\partial_t \mathcal{R} g_{\leq n} + \nabla^\perp (g_{\leq n} + z) \Lambda (g_{\leq n} + z) - \nu \Lambda^{\gamma-1} \nabla g_{\leq n} \approx \nabla \tilde{q}_n.$$

The main difficulty (that enables convex integration) is to find an explicit update rule \tilde{g}_n that corrects $g_{\leq n}$ in a way such that the magnitude of the correction is proportional to the residual. Ultimately, we hope for concentration of $g_{\leq N}$ on g and a vanishing residual.

Crucially, Cheng, Kwon and Li found suitable building blocks to correct the iterates adapted to the SQG equation; see [7, Section 2.1]. They were modified to the stochastic setting by Hofmanová, Zhu and Zhu in [21] (see also [20]).

We split our convex integration strategy into the following steps:

1. construction and properties of explicit building blocks,

2. propagation of residual magnitudes,
3. an example that allows us to close the argumentation.

Let us highlight that it is a particular feature of this adapted convex integration scheme that it allows for a sublinear error estimate (3.32) uniformly over all moments. This is in stark contrast to for example [26, Proposition 3.3], where quantified moment explosions are derived in order to assure finiteness of one fixed moment (i.e. the convex integration scheme also depends on the moment fixed in the beginning). The latter is a manifestation of superlinear error estimates that need to be controlled already in the deterministic setting. Indeed, in the stochastic setting, superlinear error estimates mean that by taking expectation, one would need to control higher and higher moments along the iteration, necessitating the mentioned quantified moment explosion. In order to study for example sub-Gaussian moments of equations perturbed by other kinds of noises (e.g. transport [35, 25, 30] and linear multiplicative [1]) in other problems, it would thus be essential to assure that the corresponding deterministic convex integration scheme allows for sublinear error estimates, since otherwise moment explosions would occur that need to be controlled in a quantified sense as in [26, Proposition 3.3]. In this sense, an adaption of our investigations to other problems is already impeached by the superlinear error estimates typically encountered in the corresponding deterministic convex integration scheme. A possible adaption of our investigations to the SQG equation perturbed by linear multiplicative or transport noise using adaptations of the convex integration scheme in [7] is left for future work.

3.1 Building blocks

Let $(\lambda_n, r_n, \ell_n, \mu_n) \in (0, \infty)^4$ be control parameters. One can think of $\lambda_n (\nearrow \infty)$ measuring blow up and frequency truncation, $r_n (\searrow 0)$ concentration, $\ell_n (\searrow 0)$ convolution and $\mu_n (\nearrow \infty)$ frequency truncation on an intermediate scale, respectively. We will keep track on the requirements of the control parameters in order to close the argumentation. A possible choice can be found in section 3.4.

3.1.1 Ansatz

Let $z_n = P_{\leq \lambda_n} z$. We start with the Ansatz

$$g_{\leq 0} = 0, \tag{3.1a}$$

$$\nabla q_0 \stackrel{\approx}{\sim} \nabla^\perp z_0 \Lambda z_0, \tag{3.1b}$$

and update recursively, for $n \geq 1$,

$$g_{\leq n} = g_{\ell_n} + g_n, \tag{3.2a}$$

$$\nabla q_n \stackrel{\approx}{\sim} -\partial_t \mathcal{R} g_{\leq n} + \nabla^\perp (g_{\leq n} + z_n) \Lambda (g_{\leq n} + z_n) - \nu \Lambda^{\gamma-1} \nabla g_{\leq n}. \tag{3.2b}$$

Here the building blocks are defined by

$$g_n(t, x) = \sum_{j=1}^2 a_{j,n}(t, x) \cos(5\lambda_n l_j \cdot x), \tag{3.3a}$$

$$a_{j,n} = 2 \sqrt{\frac{\chi_{\ell_n}}{5\lambda_n} P_{\leq \mu_n}} \sqrt{C_0 + \mathcal{R}_j^o \frac{q_{\ell_n}}{\chi_{\ell_n}}}, \tag{3.3b}$$

with

$$g_{\ell_n} = g_{\leq n-1} *_t \psi_{\ell_n}, \tag{3.4a}$$

$$q_{\ell_n} = q_{n-1} *_t \psi_{\ell_n}, \tag{3.4b}$$

$$\chi_{\ell_n} = \|q_{n-1}\|_X *_t \psi_{\ell_n} + r_{n-1}, \tag{3.4c}$$

where $\psi_\varepsilon(t)$ denotes a one-sided time mollifier, e.g., $\psi_\varepsilon(t) = \varepsilon^{-1}\psi_0(\varepsilon^{-1}(t + 2))$ for some $\psi_0 \in C^\infty(\mathbb{R}; [0, 1])$, $\psi_0(t) = 1$ for $|t| \leq 1/4$, $\psi_0(t) = 0$ for $|t| \geq 2$ and $\int_{\mathbb{R}} \psi_0(t) dt = 1$. The free parameter $C_0 \geq 2$ can be used to derive non-uniqueness results as done e.g. in [7, 21, 20].

Next, we derive bounds of the building blocks in terms of the control parameters and the residual. Let $m \geq 1$ be the probabilistic integrability, $\alpha > 0$ be the time Hölder regularity and $\kappa > 1/2$ be the space Hölder-regularity. Set

$$C_z := \|z\|_{L^\infty_{\omega} L^{2m} C_{\text{loc}} C_x^{1+\kappa}} + \|z\|_{L^\infty_{\omega} L^{2m} C_{\text{loc}}^\alpha C_x^{1+}}. \tag{3.5}$$

It turns out that $C_z < \infty$ is sufficient for finding suitable control parameters.

3.1.2 Properties

Similarly to [20, Remark 6.1] we find that g_n is frequency localized in the annulus $\{k : 5\lambda_n - \mu_n \leq |k| \leq 5\lambda_n + \mu_n\}$. In particular, g_n is mean-free. Since the mollifier in (3.4) is one-sided, g_n is adapted to the filtration $\mathcal{F}_t = \sigma(z_s | s \leq t)$. We derive some further preparatory estimates for g_n .

3.1.3 Continuous scale

Since g_n is frequency localized

$$\|g_n(t)\|_{B_{\infty,1}^{1/2}} = \sum_{j \geq -1} 2^{j/2} \|\Delta_j g_n(t)\|_{L^\infty} = \sum_{j: 5\lambda_n - \mu_n \leq 2^j \leq 5\lambda_n + \mu_n} 2^{j/2} \|\Delta_j g_n(t)\|_{L^\infty}.$$

In order to simplify the estimate we restrict to

$$\mu_n \leq \lambda_n. \tag{R1}$$

Then $\{k : 5\lambda_n - \mu_n \leq |k| \leq 5\lambda_n + \mu_n\} \subset \{k : 4\lambda_n \leq |k| \leq 6\lambda_n\}$ and

$$\begin{aligned} \sum_{j: 5\lambda_n - \mu_n \leq 2^j \leq 5\lambda_n + \mu_n} 2^{j/2} \|\Delta_j g_n(t)\|_{L^\infty} &\leq (6\lambda_n)^{1/2} \sum_{j: 4\lambda_n \leq 2^j \leq 6\lambda_n} \|\Delta_j g_n(t)\|_{L^\infty} \\ &\lesssim \lambda_n^{1/2} \|g_n(t)\|_{L^\infty}. \end{aligned}$$

Here we used that

$$\sum_{j: 4\lambda_n \leq 2^j \leq 6\lambda_n} 1 \lesssim 1,$$

since $\#\{j : 4\lambda_n \leq 2^j \leq 6\lambda_n\} = \#\{j : 2 + \log_2(\lambda_n) \leq j \leq \log_2(6) + \log_2(\lambda_n)\}$ and $C^0 \hookrightarrow B_{\infty,\infty}^0$ (see for example [42, p.89]). Moreover, due to stability of the frequency truncation and $\|q_{\ell_n}\|_X \leq \chi_{\ell_n}$,

$$\begin{aligned} \|g_n(t)\|_{L^\infty} &\leq \sum_{j=1}^2 \|a_{j,n}(t)\|_{L^\infty} \\ &\lesssim \sqrt{\frac{\chi_{\ell_n}(t)}{5\lambda_n}} \sum_{j=1}^2 \|C_0 + \mathcal{R}_j^o \frac{q_{\ell_n}(t)}{\chi_{\ell_n}(t)}\|_{L^\infty}^{1/2} \\ &\lesssim \sqrt{\frac{\chi_{\ell_n}(t)}{\lambda_n}} \left(C_0 + \frac{\|q_{\ell_n}(t)\|_X}{\chi_{\ell_n}(t)} \right)^{1/2} \\ &\lesssim \sqrt{\frac{\chi_{\ell_n}(t)}{\lambda_n}} (C_0 + 1)^{1/2}. \end{aligned}$$

Thus,

$$\|g_n(t)\|_{B_{\infty,1}^{1/2}} \lesssim \sqrt{(C_0 + 1)\chi_{\ell_n}(t)}.$$

Take the supremum over $t \in [s, s + 1]$, L_{ω}^{2m} -norm and supremum in $s \in \mathbb{R}$,

$$\|g_n\|_{L_{\omega}^{2m} C_{\text{loc}} B_{\infty,1}^{1/2}} \lesssim (C_0 + 1) \sup_{s \in \mathbb{R}} \left(\mathbb{E} \left[\sup_{t \in [s, s+1]} \chi_{\ell_n}(t)^m \right] \right)^{1/2m}.$$

We restrict our setting to

$$\ell_n \leq 1. \tag{R2}$$

Then $[s - \ell_n, s + 1 + \ell_n] \subset [s - 1, s + 2] = \cup_{k=0}^2 [s - 1 + k, s + k]$ and thus,

$$\sup_{t \in [s, s+1]} \chi_{\ell_n}(t) \leq \sup_{t \in [s-1, s+2]} \|q_{n-1}(t)\|_X + r_{n-1}. \tag{3.6}$$

Therefore,

$$\|g_n\|_{L_{\omega}^{2m} C_{\text{loc}} B_{\infty,1}^{1/2}} \lesssim \sqrt{(C_0 + 1)(\|q_{n-1}\|_{L_{\omega}^m C_{\text{loc}} X} + r_{n-1})}. \tag{3.7}$$

Similarly,

$$\|g_n\|_{L_{\omega}^{2m} C_{\text{loc}} C_x^{\vartheta}} \lesssim \lambda_n^{\vartheta-1/2} \sqrt{(C_0 + 1)(\|q_{n-1}\|_{L_{\omega}^m C_{\text{loc}} X} + r_{n-1})}. \tag{3.8}$$

Remark 3.1. The estimate (3.7) shows that the building blocks decay as a function of the residual. Moreover, (3.8) allows for stronger spatial scales provided the residual decays sufficiently fast.

3.1.4 Hölder scale

Let $s \neq s' \in [t, t + 1]$. Analogously to previous section, one finds for increments

$$\|g_n(s) - g_n(s')\|_{B_{\infty,1}^{1/2}} \lesssim \lambda_n^{1/2} \sum_{j=1}^2 \|a_{j,n}(s) - a_{j,n}(s')\|_{L^{\infty}}.$$

Using $|\sqrt{a} - \sqrt{b}| \leq \sqrt{|a - b|}$ and the stability of frequency truncation

$$\begin{aligned} \|a_{j,n}(s) - a_{j,n}(s')\|_{L^{\infty}} &\lesssim \lambda_n^{-1/2} \sqrt{|\chi_{\ell_n}(s) - \chi_{\ell_n}(s')|} \left\| \sqrt{C_0 + \mathcal{R}_j^o \frac{q_{\ell_n}(s)}{\chi_{\ell_n}(s)}} \right\|_{L^{\infty}} \\ &\quad + \lambda_n^{-1/2} \sqrt{\chi_{\ell_n}(s')} \left\| \mathcal{R}_j^o \frac{q_{\ell_n}(s)}{\chi_{\ell_n}(s)} - \mathcal{R}_j^o \frac{q_{\ell_n}(s')}{\chi_{\ell_n}(s')} \right\|_{L^{\infty}}^{1/2}. \end{aligned} \tag{3.9}$$

Before we continue the estimation, we observe the following: Due to Jensen’s inequality,

$$|\mathcal{R}_j^o q_{\ell_n}| \leq \|\mathcal{R}_j^o q_{\ell_n}\|_{L^{\infty}} \leq \|q_{\ell_n}\|_X \leq \|q_{n-1}\|_X * \psi_{\ell_n} \leq \chi_{\ell_n}.$$

Thus, $\mathcal{R}_j^o q_{\ell_n} \chi_{\ell_n}^{-1} \in [-1, 1]$ and

$$\left\| \sqrt{C_0 + \mathcal{R}_j^o \frac{q_{\ell_n}}{\chi_{\ell_n}}} \right\|_{L^{\infty}} \leq \sqrt{C_0 + 1}, \quad \left\| \left(C_0 + \mathcal{R}_j^o \frac{q_{\ell_n}}{\chi_{\ell_n}} \right)^{-1/2} \right\|_{L^{\infty}} \leq \frac{1}{\sqrt{C_0 - 1}}. \tag{3.10}$$

Additionally,

$$\begin{aligned} \left\| \mathcal{R}_j^o \frac{q_{\ell_n}(s)}{\chi_{\ell_n}(s)} - \mathcal{R}_j^o \frac{q_{\ell_n}(s')}{\chi_{\ell_n}(s')} \right\|_{L^\infty} &= \frac{\|\chi_{\ell_n}(s')\mathcal{R}_j^o q_{\ell_n}(s) - \chi_{\ell_n}(s)\mathcal{R}_j^o q_{\ell_n}(s')\|_{L^\infty}}{\chi_{\ell_n}(s)\chi_{\ell_n}(s')} \\ &\leq \frac{|\chi_{\ell_n}(s') - \chi_{\ell_n}(s)|}{\chi_{\ell_n}(s')} \frac{\|\mathcal{R}_j^o q_{\ell_n}(s)\|_{L^\infty}}{\chi_{\ell_n}(s)} + \frac{\|\mathcal{R}_j^o q_{\ell_n}(s) - \mathcal{R}_j^o q_{\ell_n}(s')\|_{L^\infty}}{\chi_{\ell_n}(s')} \\ &\leq \frac{|\chi_{\ell_n}(s') - \chi_{\ell_n}(s)|}{\chi_{\ell_n}(s')} + \frac{\|q_{\ell_n}(s) - q_{\ell_n}(s')\|_X}{\chi_{\ell_n}(s')}, \end{aligned}$$

which implies

$$\begin{aligned} \left\| \mathcal{R}_j^o \frac{q_{\ell_n}(s)}{\chi_{\ell_n}(s)} - \mathcal{R}_j^o \frac{q_{\ell_n}(s')}{\chi_{\ell_n}(s')} \right\|_{L^\infty}^{1/2} \\ \leq \frac{1}{\sqrt{\chi_{\ell_n}(s')}} (|\chi_{\ell_n}(s') - \chi_{\ell_n}(s)| + \|q_{\ell_n}(s) - q_{\ell_n}(s')\|_X)^{1/2}. \end{aligned}$$

Using the above in (3.9) leads to

$$\begin{aligned} \|a_{j,n}(s) - a_{j,n}(s')\|_{L^\infty} \\ \lesssim \lambda_n^{-1/2} (|\chi_{\ell_n}(s) - \chi_{\ell_n}(s')|(C_0 + 1) + \|q_{\ell_n}(s) - q_{\ell_n}(s')\|_X)^{1/2}. \end{aligned}$$

Next, we estimate each term separately.

Recalling (R2) and $|s - s'| \leq 1$, and applying Young's inequality and properties of convolution

$$\begin{aligned} |\chi_{\ell_n}(s) - \chi_{\ell_n}(s')| &= \left| \int \|q_{n-1}(\nu)\|_X (\psi_{\ell_n}(s - \nu) - \psi_{\ell_n}(s' - \nu)) \, d\nu \right| \\ &\leq \sup_{\nu \in [s - \ell_n, s + \ell_n] \cup [s' - \ell_n, s' + \ell_n]} \|q_{n-1}(\nu)\|_X \int |\psi_{\ell_n}(s - \nu) - \psi_{\ell_n}(s' - \nu)| \, d\nu \\ &\lesssim \sup_{\nu \in [t-1, t+3]} \|q_{n-1}(\nu)\|_X |s - s'|^{2\alpha} \ell_n^{-2\alpha}. \end{aligned}$$

Similarly to the estimate for increments of χ_{ℓ_n} ,

$$\begin{aligned} \|q_{\ell_n}(s) - q_{\ell_n}(s')\|_X &\leq \left\| \int q_{n-1}(\nu) (\psi_{\ell_n}(s - \nu) - \psi_{\ell_n}(s' - \nu)) \, d\nu \right\|_X \\ &\lesssim \sup_{\nu \in [t-1, t+3]} \|q_{n-1}(\nu)\|_X |s - s'|^{2\alpha} \ell_n^{-2\alpha}. \end{aligned}$$

Collecting all estimates shows

$$\|a_{j,n}(s) - a_{j,n}(s')\|_{L^\infty} \lesssim \left((C_0 + 1) \lambda_n^{-1} \ell_n^{-2\alpha} |s - s'|^{2\alpha} \sup_{\nu \in [t-1, t+3]} \|q_{n-1}(\nu)\|_X \right)^{1/2}$$

and

$$[g_n]_{L_\omega^{2m} C_{loc}^\alpha B_{\infty,1}^{1/2}} \lesssim \ell_n^{-\alpha} \sqrt{(C_0 + 1) \|q_{n-1}\|_{L_\omega^m C_{loc} X}}. \tag{3.11}$$

Similarly,

$$[g_n]_{L_\omega^{2m} C_{loc}^\alpha C_x^\beta} \lesssim \lambda_n^{\beta-1/2} \ell_n^{-\alpha} \sqrt{(C_0 + 1) \|q_{n-1}\|_{L_\omega^m C_{loc} X}}. \tag{3.12}$$

3.1.5 C_t^1 scale

Take the time derivative in (3.3b)

$$\begin{aligned} \partial_t a_{j,n} &= \frac{1}{\sqrt{5}\lambda_n} \left(\frac{\partial_t \chi_{\ell_n}}{\sqrt{\chi_{\ell_n}}} P_{\leq \mu_n} \sqrt{C_0 + \mathcal{R}_j^o \frac{q_{\ell_n}}{\chi_{\ell_n}}} \right. \\ &\quad \left. + \sqrt{\chi_{\ell_n}} P_{\leq \mu_n} \left(C_0 + \mathcal{R}_j^o \frac{q_{\ell_n}}{\chi_{\ell_n}} \right)^{-\frac{1}{2}} \mathcal{R}_j^o \left(\frac{\partial_t q_{\ell_n}}{\chi_{\ell_n}} - \frac{q_{\ell_n} \partial_t \chi_{\ell_n}}{\chi_{\ell_n}^2} \right) \right). \end{aligned}$$

Using (3.10), we find that

$$\|\partial_t a_{j,n}\|_{L^\infty} \lesssim \lambda_n^{-\frac{1}{2}} \left(\frac{|\partial_t \chi_{\ell_n}|}{\sqrt{\chi_{\ell_n}}} \left(\sqrt{C_0 + 1} + \frac{1}{\sqrt{C_0 - 1}} \right) + \frac{1}{\sqrt{C_0 - 1}} \frac{\|\partial_t q_{\ell_n}\|_X}{\sqrt{\chi_{\ell_n}}} \right).$$

Additionally, for $t \in [s, s + 1]$

$$\begin{aligned} |\partial_t \chi_{\ell_n}(t)| \vee \|\partial_t q_{\ell_n}(t)\|_X &\leq \|q_{n-1}\|_X *_t |\partial_t \psi_{\ell_n}| \\ &\leq \sup_{t \in [s-1, s+2]} \|q_{n-1}(t)\|_X \|\partial_t \psi_{\ell_n}\|_{L_t^1} \\ &\lesssim \sup_{t \in [s-1, s+2]} \|q_{n-1}(t)\|_X \ell_n^{-1}. \end{aligned}$$

All together, using $\chi_{\ell_n} \geq r_{n-1}$,

$$\|\partial_t g_n\|_{L_w^m C_{loc} L^\infty} \lesssim \lambda_n^{-1/2} r_{n-1}^{-1/2} \ell_n^{-1} \|q_{n-1}\|_{L_w^m C_{loc} X} \left(\sqrt{C_0 + 1} + \frac{1}{\sqrt{C_0 - 1}} \right). \tag{3.13}$$

3.2 The residual in terms of control parameters

The following proposition provides a quantified bound on the magnitude of the new residual in terms of the control parameters provided that the old residuals are controlled by the concentration parameter r_n .

Proposition 3.2. *Let us assume that for $n \leq N$ the following conditions are satisfied:*

1. (condition on control parameters) (R1), (R2) and

$$\lambda_n \in \mathbb{N}, \quad 12\lambda_n \leq 4\mu_{n+1}, \quad 10 \leq \mu_n, \quad 48\lambda_n \leq \lambda_{n+1}; \tag{R3}$$

2. (control on past residuals)

$$\|q_n\|_{L_w^m C_{loc} X} \leq r_n. \tag{3.14}$$

Then there exists a constant $C_{it} \geq 1$ (independent of N) such that

$$\|q_{N+1}\|_{L_w^m C_{loc} X} \leq C_{it} (\mathfrak{E}_{N+1}^{miss} + \mathfrak{E}_{N+1}^{com} + \mathfrak{E}_{N+1}^{time} + \mathfrak{E}_{N+1}^{dis} + \mathfrak{E}_{N+1}^{trans} + \mathfrak{E}_{N+1}^{sto}), \tag{3.15}$$

where

$$\mathfrak{E}_{N+1}^{miss} = \left\{ \log \mu_{N+1} \left(\frac{\lambda_N}{\mu_{N+1}} + \frac{\mu_{N+1}}{\lambda_{N+1}} \right)^2 + \frac{\lambda_N}{\lambda_{N+1}} \right\} r_N, \tag{3.16a}$$

$$\begin{aligned} \mathfrak{E}_{N+1}^{com} &= \log(\lambda_{N+1}) \lambda_N \ell_{N+1}^\alpha \mathfrak{S}_{N,\alpha} \mathfrak{S}_{N,0} \\ &\quad + \log(\lambda_{N+1}) \lambda_N^{1/2} (\ell_{N+1}^\alpha \mathfrak{S}_{N,\alpha} + \lambda_N^{-\kappa} \mathfrak{S}_{N,0}) C_z \\ &\quad + \log(\lambda_{N+1}) (\ell_{N+1}^\alpha + \lambda_N^{-\kappa}) C_z^2, \end{aligned} \tag{3.16b}$$

$$\mathfrak{E}_{N+1}^{time} = \log(\lambda_{N+1}) \lambda_{N+1}^{-3/2} r_N^{1/2} \ell_{N+1}^{-1}, \tag{3.16c}$$

$$\mathfrak{E}_{N+1}^{dis} = \log(\lambda_{N+1}) |\nu| \lambda_{N+1}^{\gamma-3/2} r_N^{1/2}, \tag{3.16d}$$

$$\mathfrak{E}_{N+1}^{trans} = \log(\lambda_{N+1}) \lambda_N^{1/2} \lambda_{N+1}^{-1/2} r_N^{1/2} \mathfrak{S}_{N,0}, \tag{3.16e}$$

$$\mathfrak{E}_{N+1}^{sto} = \log(\lambda_{N+1}) \lambda_{N+1}^{-1/2} r_N^{1/2} C_z. \tag{3.16f}$$

Here C_z is given by (3.5) and

$$\mathfrak{S}_{N,\alpha} := \sum_{n=1}^N 4^{N-n} \ell_n^{-\alpha} r_{n-1}^{1/2}. \tag{3.17}$$

Proposition 3.2 can be used recursively provided that r_{N+1} dominates the right hand side of (3.15) and the control parameter obey (R1), (R2) and (R3) also at later times. The former is trivially the case if the right hand side defines r_{N+1} . However, only if r_N vanishes asymptotically one can hope for a limit $g = \lim_N g_{\leq N}$ that satisfies (1.3). Finding a suitable family of controls that achieves this is a difficult task. One needs to carefully balance the blow up of (λ_n, μ_n) and the convergence of (ℓ_n, r_n) . We present a possible choice in Section 3.4.

3.3 Proof of Proposition 3.2

The proof proceeds as the one of [20, Proposition 5.1]. See also [21, Proposition 3.1] for more details. However, contrary to [20, Proposition 5.1] we have not yet fixed an explicit choice for the control parameters.

3.3.1 Control of $g_{\leq N+1}$ in terms of building blocks

By definition (3.2a) we have full control on the next step $g_{\leq N+1}$ in terms of the previous step and the explicit building block g_{N+1} . This allows for the recursive estimate (recall (R2))

$$\begin{aligned} [g_{\leq N+1}]_{L_{\omega}^{2m} C_{\text{loc}}^{\alpha} B_{\infty,1}^{1/2}} &\leq [g_{\ell_{N+1}}]_{L_{\omega}^{2m} C_{\text{loc}}^{\alpha} B_{\infty,1}^{1/2}} + [g_{N+1}]_{L_{\omega}^{2m} C_{\text{loc}}^{\alpha} B_{\infty,1}^{1/2}} \\ &\leq 4[g_{\leq N}]_{L_{\omega}^{2m} C_{\text{loc}}^{\alpha} B_{\infty,1}^{1/2}} + [g_{N+1}]_{L_{\omega}^{2m} C_{\text{loc}}^{\alpha} B_{\infty,1}^{1/2}} \\ &\leq \sum_{n=1}^{N+1} 4^{N+1-n} [g_n]_{L_{\omega}^{2m} C_{\text{loc}}^{\alpha} B_{\infty,1}^{1/2}}. \end{aligned}$$

Using (3.11), respectively (3.7), and (3.14)

$$[g_{\leq N+1}]_{L_{\omega}^{2m} C_{\text{loc}}^{\alpha} B_{\infty,1}^{1/2}} \lesssim \sqrt{(C_0 + 1)} \mathfrak{S}_{N+1,\alpha}, \tag{3.18a}$$

$$\|g_{\leq N+1}\|_{L_{\omega}^{2m} C_{\text{loc}}^{\alpha} B_{\infty,1}^{1/2}} \lesssim \sqrt{(C_0 + 1)} \mathfrak{S}_{N+1,0}, \tag{3.18b}$$

where $\mathfrak{S}_{N,\alpha}$ is defined by (3.17).

3.3.2 Decomposition of new residual

A mollification of (3.2b) on the scale ℓ_{N+1} at level N yields

$$\nabla q_{\ell_{N+1}} \overset{\approx}{\approx} -\partial_t \mathcal{R} g_{\ell_{N+1}} + \nabla^{\perp} (g_{\ell_{N+1}} + z_{N+1}) \Lambda (g_{\ell_{N+1}} + z_{N+1}) - \nu \Lambda^{\gamma-1} \nabla g_{\ell_{N+1}} - \mathbf{R}_{\text{com}}, \tag{3.19}$$

where the commutator is given by

$$\begin{aligned} \mathbf{R}_{\text{com}} &= \nabla^{\perp} (g_{\ell_{N+1}} + z_{N+1}) \Lambda (g_{\ell_{N+1}} + z_{N+1}) \\ &\quad - (\nabla^{\perp} (g_{\leq N} + z_N) \Lambda (g_{\leq N} + z_N)) *_t \psi_{\ell_{N+1}}. \end{aligned}$$

Using the explicit update rule (3.2a) in (3.2b)

$$\begin{aligned}
 \nabla q_{N+1} &\overset{\approx}{\approx} -\partial_t \mathcal{R}(g_{\ell_{N+1}} + g_{N+1}) + \nabla^\perp(g_{\ell_{N+1}} + g_{N+1} + z_{N+1})\Lambda(g_{\ell_{N+1}} + g_{N+1} + z_{N+1}) \\
 &\quad - \nu\Lambda^{\gamma-1}\nabla(g_{\ell_{N+1}} + g_{N+1}) \\
 &= \underbrace{-\partial_t \mathcal{R}g_{\ell_{N+1}} + \nabla^\perp(g_{\ell_{N+1}} + z_{N+1})\Lambda(g_{\ell_{N+1}} + z_{N+1}) - \nu\Lambda^{\gamma-1}\nabla g_{\ell_{N+1}}}_{\approx \nabla q_{\ell_{N+1}} + R_{\text{com}} \text{ by (3.19)}} + \nabla^\perp g_{N+1}\Lambda g_{N+1} \\
 &\quad - \partial_t \mathcal{R}g_{N+1} + \nabla^\perp(g_{\ell_{N+1}} + g_{N+1} + z_{N+1})\Lambda(g_{\ell_{N+1}} + g_{N+1} + z_{N+1}) - \nu\Lambda^{\gamma-1}\nabla g_{N+1} \\
 &\quad - \nabla^\perp(g_{\ell_{N+1}} + z_{N+1})\Lambda(g_{\ell_{N+1}} + z_{N+1}) - \nabla^\perp g_{N+1}\Lambda g_{N+1} \\
 &\overset{\approx}{\approx} \underbrace{\nabla q_{\ell_{N+1}} + \nabla^\perp g_{N+1}\Lambda g_{N+1}}_{\approx \nabla q_M \text{ (mismatch error)}} + R_{\text{com}} - \underbrace{\partial_t \mathcal{R}g_{N+1}}_{\text{(time error)}} - \underbrace{\nu\Lambda^{\gamma-1}\nabla g_{N+1}}_{\text{(dissipation error)}} \\
 &\quad + \underbrace{\nabla^\perp g_{N+1}\Lambda g_{\ell_{N+1}} + \nabla^\perp g_{\ell_{N+1}}\Lambda g_{N+1}}_{\approx \nabla q_T \text{ (transport error)}} + \underbrace{\nabla^\perp g_{N+1}\Lambda z_{N+1} + \nabla^\perp z_{N+1}\Lambda g_{N+1}}_{\approx \nabla q_S \text{ (stochastic error)}}.
 \end{aligned}$$

Notice that ∇q_{N+1} is only defined up to a smooth scalar function $\nabla^\perp p$. Therefore, we subtract the mean-value of each summand on the right hand side without changing the notation. On mean-free functions we can invert the gradient $\Delta^{-1}\nabla \cdot = \nabla^{-1}$.

3.3.3 Mismatch error

Due to (R3) and (R1), we can follow the line of argumentation used in [7, Proposition 3.1] to derive

$$\|q_M\|_X \lesssim \left\{ \log \mu_{N+1} \left(\frac{\lambda_N}{\mu_{N+1}} + \frac{\mu_{N+1}}{\lambda_{N+1}} \right)^2 + \frac{\lambda_N}{\lambda_{N+1}} \right\} (C_0 + 1)\chi_{\ell_{N+1}}.$$

This together with (3.6) gives

$$\|q_M\|_{L^m C_{\text{loc}} X} \lesssim \left\{ \log \mu_{N+1} \left(\frac{\lambda_N}{\mu_{N+1}} + \frac{\mu_{N+1}}{\lambda_{N+1}} \right)^2 + \frac{\lambda_N}{\lambda_{N+1}} \right\} (C_0 + 1)r_N.$$

3.3.4 Commutator error

We decompose the commutator error

$$\begin{aligned}
 R_{\text{com}} &= -(\nabla^\perp(z_N - z_{N+1})\Lambda(g_{\leq N} + z_{N+1}) + \nabla^\perp(g_{\leq N} + z_N)\Lambda(z_N - z_{N+1})) *_t \psi_{\ell_{N+1}} \\
 &\quad + \nabla^\perp(g_{\ell_{N+1}} - g_{\leq N})\Lambda(g_{\leq N} + z_{N+1}) + \nabla^\perp(g_{\ell_{N+1}} + z_{N+1})\Lambda(g_{\ell_{N+1}} - g_{\leq N}) \\
 &\quad + \nabla^\perp(g_{\leq N} + z_{N+1})\Lambda(g_{\leq N} + z_{N+1}) \\
 &\quad - (\nabla^\perp(g_{\leq N} + z_{N+1})\Lambda(g_{\leq N} + z_{N+1})) *_t \psi_{\ell_{N+1}} \\
 &=: R_{\text{com}}^1 + R_{\text{com}}^2 + R_{\text{com}}^3.
 \end{aligned}$$

Notice that R_{com}^i , $i \in \{1, 2, 3\}$, is frequency localized to $\{k : |k| \leq 2\lambda_{N+1}\}$. Thus, by Lemma A.1, Poincaré's and Hölder's inequalities

$$\begin{aligned}
 \|\Delta^{-1}\nabla \cdot R_{\text{com}}^1\|_{L^m C_{\text{loc}} X} &\lesssim \log(\lambda_{N+1})\|\nabla^\perp(z_N - z_{N+1})\Lambda(g_{\leq N} + z_{N+1})\|_{L^m C_{\text{loc}} L^\infty} \\
 &\quad + \log(\lambda_{N+1})\|\nabla^\perp(g_{\leq N} + z_N)\Lambda(z_N - z_{N+1})\|_{L^m C_{\text{loc}} L^\infty} \\
 &\lesssim \log(\lambda_{N+1}) \left(\lambda_N^{1/2}\|g_{\leq N}\|_{L^m C_{\text{loc}} B_{\infty,1}^{1/2}} + \|z\|_{L^m C_{\text{loc}} C_x^{1+}} \right) \lambda_N^{-\kappa} \|z\|_{L^m C_{\text{loc}} C_x^{1+\kappa}}.
 \end{aligned}$$

Let us stress at this point that it is here, where we require the condition $\kappa > 1/2$ to hold in order to compensate for the diverging factor $\lambda_N^{1/2}$ above. This explains the

additional regularity requirement expressed in κ compared with [20], where a more detailed analysis of the noise term is performed (however, this more detailed analysis exploits the Gaussianity of the underlying noise W and thus does not extend to the setting of more general noises we are concerned with here). Similarly,

$$\begin{aligned} \|\Delta^{-1}\nabla \cdot R_{\text{com}}^2\|_{L_{\omega}^m C_{\text{loc}} X} &\lesssim \log(\lambda_{N+1})\lambda_N^{1/2}\ell_{N+1}^{\alpha}[g_{\leq N}]_{L_{\omega}^{2m} C_{\text{loc}}^{\alpha} B_{\infty,1}^{1/2}} \\ &\quad \times \left(\lambda_N^{1/2}\|g_{\leq N}\|_{L_{\omega}^{2m} C_{\text{loc}} B_{\infty,1}^{1/2}} + \|z\|_{L_{\omega}^{2m} C_{\text{loc}} C_x^{1+}} \right) \end{aligned}$$

and

$$\begin{aligned} \|\Delta^{-1}\nabla \cdot R_{\text{com}}^3\|_{L_{\omega}^m C_{\text{loc}} X} &\lesssim \log(\lambda_{N+1})\ell_{N+1}^{\alpha}[\nabla^{\perp}(g_{\leq N} + z_{N+1})\Lambda(g_{\leq N} + z_{N+1})]_{L_{\omega}^m C_{\text{loc}}^{\alpha} L^{\infty}} \\ &\leq \log(\lambda_{N+1})\ell_{N+1}^{\alpha}\|\nabla^{\perp}(g_{\leq N} + z_{N+1})\|_{L_{\omega}^{2m} C_{\text{loc}} L^{\infty}}[\Lambda(g_{\leq N} + z_{N+1})]_{L_{\omega}^{2m} C_{\text{loc}}^{\alpha} L^{\infty}} \\ &\quad + \log(\lambda_{N+1})\ell_{N+1}^{\alpha}\|\Lambda(g_{\leq N} + z_{N+1})\|_{L_{\omega}^{2m} C_{\text{loc}} L^{\infty}}[\nabla^{\perp}(g_{\leq N} + z_{N+1})]_{L_{\omega}^{2m} C_{\text{loc}}^{\alpha} L^{\infty}} \\ &\leq \log(\lambda_{N+1})\ell_{N+1}^{\alpha} \left(\lambda_N^{1/2}\|g_{\leq N}\|_{L_{\omega}^{2m} C_{\text{loc}} B_{\infty,1}^{1/2}} + \|z\|_{L_{\omega}^{2m} C_{\text{loc}} C_x^{1+}} \right) \\ &\quad \times \left(\lambda_N^{1/2}[g_{\leq N}]_{L_{\omega}^{2m} C_{\text{loc}}^{\alpha} B_{\infty,1}^{1/2}} + [z]_{L_{\omega}^{2m} C_{\text{loc}}^{\alpha} C_x^{1+}} \right). \end{aligned}$$

All together, using (3.18),

$$\begin{aligned} \|\Delta^{-1}\nabla \cdot R_{\text{com}}\|_{L_{\omega}^m C_{\text{loc}} X} &\lesssim \log(\lambda_{N+1})\lambda_N\ell_{N+1}^{\alpha}(C_0 + 1)\mathfrak{S}_{N,\alpha}\mathfrak{S}_{N,0} \\ &\quad + \log(\lambda_{N+1})\lambda_N^{1/2}\sqrt{(C_0 + 1)}(\ell_{N+1}^{\alpha}\mathfrak{S}_{N,\alpha} + \lambda_N^{-\kappa}\mathfrak{S}_{N,0})C_z \\ &\quad + \log(\lambda_{N+1})(\ell_{N+1}^{\alpha} + \lambda_N^{-\kappa})C_z^2. \end{aligned}$$

3.3.5 Time error

Since g_{N+1} is frequency localized to an annulus of size λ_{N+1}

$$\|\Delta^{-1}\nabla \cdot \partial_t \mathcal{R}g_{N+1}\|_{L_{\omega}^m C_{\text{loc}} X} \lesssim \log(\lambda_{N+1})\lambda_{N+1}^{-1}\|\partial_t g_{N+1}\|_{L_{\omega}^m C_{\text{loc}} L^{\infty}}.$$

Using (3.13) and (3.14)

$$\|\Delta^{-1}\nabla \cdot \partial_t \mathcal{R}g_{N+1}\|_{L_{\omega}^m C_{\text{loc}} X} \lesssim \log(\lambda_{N+1})\lambda_{N+1}^{-3/2}r_N^{1/2}\ell_{N+1}^{-1} \left(\sqrt{C_0 + 1} + \frac{1}{\sqrt{C_0 - 1}} \right).$$

3.3.6 Dissipation error

As g_{N+1} is frequency localized to an annulus of size λ_{N+1}

$$\|\Delta^{-1}\nabla \cdot \nu\Lambda^{\gamma-1}\nabla g_{N+1}\|_{L^m C_{\text{loc}} X} \lesssim \log(\lambda_{N+1})|\nu|\lambda_{N+1}^{\gamma-3/2}\|g_{N+1}\|_{L_{\omega}^m C_{\text{loc}} B_{\infty,1}^{1/2}}.$$

Using Hölder's inequality, (3.7) and (3.14)

$$\|\Delta^{-1}\nabla \cdot \nu\Lambda^{\gamma-1}\nabla g_{N+1}\|_{L^m C_{\text{loc}} X} \lesssim \log(\lambda_{N+1})|\nu|\lambda_{N+1}^{\gamma-3/2}r_N^{1/2}\sqrt{C_0+1}.$$

3.3.7 Transport error

Notice that $\nabla^\perp g_{N+1} \Lambda g_{\ell_{N+1}}$ and $\nabla^\perp g_{\ell_{N+1}} \Lambda g_{N+1}$ are frequency localized to an annulus of size λ_{N+1} . Hölder's inequality, (3.7) and (3.18b) imply

$$\begin{aligned} & \|\Delta^{-1} \nabla \cdot [\nabla^\perp g_{N+1} \Lambda g_{\ell_{N+1}} + \nabla^\perp g_{\ell_{N+1}} \Lambda g_{N+1}]\|_{L_\omega^m C_{\text{loc}} X} \\ & \lesssim \log(\lambda_{N+1}) \lambda_{N+1}^{-1} \|\nabla^\perp g_{N+1} \Lambda g_{\ell_{N+1}} + \nabla^\perp g_{\ell_{N+1}} \Lambda g_{N+1}\|_{L_\omega^m C_{\text{loc}} L^\infty} \\ & \leq \log(\lambda_{N+1}) \lambda_N^{1/2} \lambda_{N+1}^{-1/2} \|g_{N+1}\|_{L_\omega^{2m} C_{\text{loc}} B_{\infty,1}^{1/2}} \|g_{\ell_{N+1}}\|_{L_\omega^{2m} C_{\text{loc}} B_{\infty,1}^{1/2}} \\ & \lesssim \log(\lambda_{N+1}) \sqrt{C_0 + 1} \lambda_N^{1/2} \lambda_{N+1}^{-1/2} r_N^{1/2} \|g_{\leq N}\|_{L_\omega^{2m} C_{\text{loc}} B_{\infty,1}^{1/2}} \\ & \lesssim \log(\lambda_{N+1}) (C_0 + 1) \lambda_N^{1/2} \lambda_{N+1}^{-1/2} r_N^{1/2} \mathfrak{E}_{N,0}. \end{aligned}$$

3.3.8 Stochastic error

Since $\nabla^\perp g_{N+1} \Lambda z_{N+1}$ and $\nabla^\perp z_{N+1} \Lambda g_{N+1}$ are frequency localized to an annulus of size λ_{N+1} , Hölder's inequality and (3.7)

$$\begin{aligned} & \|\Delta^{-1} \nabla \cdot [\nabla^\perp g_{N+1} \Lambda z_{N+1} + \nabla^\perp z_{N+1} \Lambda g_{N+1}]\|_{L_\omega^m C_{\text{loc}} X} \\ & \lesssim \log(\lambda_{N+1}) \lambda_{N+1}^{-1} \|\nabla^\perp g_{N+1} \Lambda z_{N+1} + \nabla^\perp z_{N+1} \Lambda g_{N+1}\|_{L_\omega^m C_{\text{loc}} L^\infty} \\ & \lesssim \log(\lambda_{N+1}) \lambda_{N+1}^{-1/2} \|g_{N+1}\|_{L_\omega^{2m} C_{\text{loc}} B_{\infty,1}^{1/2}} \|z\|_{L_\omega^{2m} C_{\text{loc}} C_x^{1+}} \\ & \lesssim \log(\lambda_{N+1}) \sqrt{C_0 + 1} \lambda_{N+1}^{-1/2} r_N^{1/2} C_z. \end{aligned}$$

Collecting all estimates and choosing C_{it} as the maximum of all implicit constants and the terms depending on C_0 verify the claim. From now on, we will no longer indicate the dependence of C_{it} on C_0 .

3.4 An explicit choice of controls

Let $a, b \in \mathbb{N}$, $\beta \in (0, 1)$, and define

$$\forall n \geq 0: \quad \lambda_n = a^{(b^n)}, \quad r_n = C_{\text{start}} (C_z^2 + 1) \lambda_0^\beta \lambda_n^{-\beta}, \quad (3.20a)$$

$$\forall n \geq 1: \quad \ell_n = \lambda_n^{-1}, \quad \mu_n = \sqrt{\lambda_{n-1} \lambda_n}, \quad (3.20b)$$

where C_z is given by (3.5) and $C_{\text{start}} \geq 1$ will be appropriately chosen later (see (3.35)), depending on the parameter a . C_{start} can be used to ensure that the initial residual is dominated by r_0 .

The next lemma allows us to iterate Proposition 3.2 for specific choices of (a, b, β) .

Lemma 3.3. *Let $m \geq 1$, $\alpha > 0$, $\kappa > 1/2$, $\gamma < 3/2$ and $z \in L_\omega^{2m} (C_{\text{loc}} C_x^{1+\kappa} \cap C_{\text{loc}}^\alpha C_x^{1+})$.*

Then for all $C > 1$ there exist $a, b \in \mathbb{N}$ and $\beta \in (0, 1)$ such that the control choice (3.20) satisfies for all $n \geq 1$ the conditions (R1), (R2), (R3) and

$$C (\mathfrak{E}_n^{\text{miss}} + \mathfrak{E}_n^{\text{com}} + \mathfrak{E}_n^{\text{time}} + \mathfrak{E}_n^{\text{dis}} + \mathfrak{E}_n^{\text{trans}} + \mathfrak{E}_n^{\text{sto}}) \leq r_n, \quad (3.21)$$

where \mathfrak{E}_n^\bullet is defined by (3.16).

Proof. We aim to trace the restrictions on a, b and β such that (R1), (R2), (R3) and (3.21) are realized individually.

By assumption C_z is finite. Notice that λ_n and μ_n are increasing and, thus, r_n and ℓ_n are decreasing. Moreover (R1) and (R2) are satisfied for all $a, b > 1$. We find that (R3)

imposes the restrictions

$$12\lambda_n \leq 4\mu_{n+1} \Leftrightarrow 9 \leq a^b, \tag{3.22a}$$

$$10 \leq \mu_n \Leftrightarrow 100 \leq a^{b+1}, \tag{3.22b}$$

$$48\lambda_n \leq \lambda_{n+1} \Leftrightarrow 48 \leq a^{b-1}. \tag{3.22c}$$

Observe that $r_m = r_n \lambda_m^{-\beta} \lambda_n^\beta$ for each $m, n \in \mathbb{N}$. Next, we will find sufficient conditions such that each term on the left hand side of (3.21) is bounded by r_n .

ad (3.16a). As $(1 - b)(1 - \beta) < 0$ the factor in front of r_n ,

$$C\mathfrak{E}_n^{\text{miss}} = C (b^{n-1}(1 + b) \log(a) + 1) a^{b^{n-1}(1-b)(1-\beta)} r_n,$$

can be made arbitrarily small.

Before we proceed let us estimate the sums (3.17). Neglecting the decaying factor and estimating the geometric series show

$$\mathfrak{S}_{n-1,\alpha} = r_n^{1/2} a^{b^n \beta/2} \sum_{k=1}^{n-1} 4^{n-1-k} a^{b^{k-1}(\alpha b - \beta/2)} \leq r_n^{1/2} a^{b^{n-1}(b\beta/2 + \alpha)} 4^n. \tag{3.23}$$

ad (3.16b). Using (3.23),

$$\begin{aligned} C\mathfrak{E}_n^{\text{com}} &\leq C b^n \log(a) 4^{2n} a^{b^{n-1}(1-\alpha(b-1)+\beta b)} r_n \\ &\quad + C b^n \log(a) 4^n \left(a^{b^{n-1}(1/2-\alpha b+\beta b)} + a^{b^{n-1}(1/2-\kappa+\beta b)} \right) r_n \\ &\quad + C b^n \log(a) \left(a^{b^n(\beta-\alpha)} + a^{b^{n-1}(b\beta-\kappa)} \right) r_n. \end{aligned}$$

We need to restrict to

$$1 - \alpha(b - 1) + \beta b < 0 \Leftrightarrow \beta < \frac{\alpha(b - 1) - 1}{b}, \tag{3.24}$$

$$\frac{\alpha(b - 1) - 1}{b} > 0 \Leftrightarrow b > 1 + \frac{1}{\alpha}, \tag{3.25}$$

$$1/2 - \kappa + \beta b < 0 \Leftrightarrow \beta < \frac{1}{b} \left(\kappa - \frac{1}{2} \right). \tag{3.26}$$

ad (3.16c). Notice that $r_n^{1/2} \leq a^{b^n \beta/2} r_n$. Therefore,

$$\begin{aligned} C\mathfrak{E}_n^{\text{time}} &= C b^n \log(a) a^{b^{n-1}(-b/2+b\beta/2-\beta/2)} r_n^{1/2} \\ &\leq C b^n \log(a) a^{b^{n-1}(-b/2+b\beta-\beta/2)} r_n. \end{aligned}$$

We need to impose

$$-b/2 + b\beta - \beta/2 < 0 \Leftrightarrow \beta < \frac{b}{2b - 1}. \tag{3.27}$$

ad (3.16d). Notice that $r_{n-1}^{1/2} \leq a^{b^{n-1}(b\beta-\beta/2)} r_n$. This implies

$$C\mathfrak{E}_n^{\text{dis}} \leq C |\nu| b^n \log(a) a^{b^{n-1}(b(\gamma-3/2+\beta)-\beta/2)} r_n.$$

The condition reads

$$b(\gamma - 3/2 + \beta) - \beta/2 < 0 \Leftrightarrow \beta < \frac{2b}{2b - 1} \left(\frac{3}{2} - \gamma \right). \tag{3.28}$$

ad (3.16e). Using (3.23)

$$C\mathfrak{E}_n^{\text{trans}} \leq Cb^n \log(a)4^n a^{b^{n-1}(1/2-b/2+b\beta-\beta/2)}r_n.$$

We impose

$$1/2 - b/2 + b\beta - \beta/2 < 0 \iff \beta < \frac{b-1}{2b-1}. \tag{3.29}$$

ad (3.16f). Notice that $r_{n-1}^{1/2}C_z \leq a^{b^{n-1}(b\beta-\beta/2)}r_n$. Therefore,

$$C\mathfrak{E}_n^{\text{sto}} \leq Cb^n \log(a)a^{b^{n-1}(-b/2+b\beta-\beta/2)}r_n.$$

This calls for the condition

$$-b/2 + b\beta - \beta/2 < 0 \iff \beta < \frac{b}{2b-1}. \tag{3.30}$$

Lastly, we need to verify that all conditions can be satisfied simultaneously. First the value of b is determined by (3.25). This allows us to find $\beta > 0$ such that (3.24), (3.26) - (3.30) hold. Now, b and β are fixed. Let $\varsigma > 0$ quantify the slowest convergence, i.e.,

$$-\varsigma = \max \left\{ (1-b)(1-\beta), 1-\alpha(b-1)+\beta b, 1/2-\kappa+\beta b, -b/2+b\beta-\beta/2, b(\gamma-3/2+\beta)-\beta/2, 1/2-b/2+b\beta-\beta/2 \right\}.$$

As $\mathbb{N} \ni n \mapsto (4b)^{2n}a^{-b^{n-1}\varsigma}$ is decreasing for $a \geq a_0(b, \varsigma)$, we estimate

$$\begin{aligned} C(\mathfrak{E}_n^{\text{miss}} + \mathfrak{E}_n^{\text{com}} + \mathfrak{E}_n^{\text{time}} + \mathfrak{E}_n^{\text{dis}} + \mathfrak{E}_n^{\text{trans}} + \mathfrak{E}_n^{\text{sto}}) \\ \leq C \log(a)(10+|\nu|)(4b)^{2n}a^{-b^{n-1}\varsigma}r_n \\ \leq C \log(a)(10+|\nu|)(4b)^2a^{-\varsigma}r_n. \end{aligned}$$

The value of a is chosen such that (3.22), $a \geq a_0(b, \varsigma)$ and

$$C \log(a)(10+|\nu|)(4b)^2a^{-\varsigma} \leq 1 \tag{3.31}$$

hold. □

Remark 3.4. We want to stress that Lemma 3.3 does not depend on the value C_{start} nor the explicit value C_z . In particular, it is independent of $m \geq 1$ as long as $C_z < \infty$.

Notice that the exponent b is purely determined by the α -Hölder regularity in time of z , cf. (3.25), as was the case already in [24, p.193]. In other words, a time irregular stochastic term (α small) forces us to introduce higher frequencies at a faster rate, i.e., $n \mapsto \lambda_n \sim a^{\alpha^{-n}}$ increases rapidly.

We are ready to proof convergence of the iteration introduced in (3.2).

Theorem 3.5. *Let $m \geq 1$, $\alpha > 0$, $\kappa > 1/2$, $\gamma < 3/2$ and $z \in L_\omega^{2m}(C_{\text{loc}}C_x^{1+\kappa} \cap C_{\text{loc}}^\alpha C_x^{1+})$.*

Then there exist $a, b \in \mathbb{N}$, $\beta \in (0, 1)$ and $C_{\text{start}} \geq 1$ such that for all $n' \geq n \geq 0$

$$\|q_n\|_{L_\omega^m C_{\text{loc}}X} \leq r_n, \tag{3.32}$$

$$\|g_{\leq n'} - g_{\leq n}\|_{L_\omega^{2m} C_{\text{loc}}B_{\infty,1}^{1/2}} \lesssim \sqrt{r_n}. \tag{3.33}$$

In particular, $g = \lim_{n \rightarrow \infty} g_{\leq n} \in L_\omega^{2m} C_{\text{loc}}B_{\infty,1}^{1/2}$ exists, solves (1.2) and satisfies

$$\|g\|_{L_\omega^{2m} C_{\text{loc}}B_{\infty,1}^{1/2}} \lesssim \sqrt{C_{\text{start}}} \left(\|z\|_{L_\omega^{2m} C_{\text{loc}}C_x^{1+\kappa}} + [z]_{L_\omega^{2m} C_{\text{loc}}^\alpha C_x^{1+}} + 1 \right). \tag{3.34}$$

Proof. We apply Lemma 3.3 for $C = C_{\text{it}}$ from Proposition 3.2 and obtain a parameter configuration (a, b, β) such that (R1), (R2), (R3) and (3.21) hold for all $n \geq 1$. We can choose (a, b, β) such that (3.24) holds.

Now, we proceed via induction. First, we argue that it is possible to start the iteration. By (3.1b), the frequency locality of z_0 and Lemma A.1, Poincaré’s and Hölder’s inequalities

$$\begin{aligned} \|q_0\|_{L_\omega^m C_{\text{loc}} X} &= \|\Delta^{-1} \nabla \cdot [\nabla^\perp z_0 \Lambda z_0]\|_{L_\omega^m C_{\text{loc}} X} \\ &\lesssim \log(\lambda_0) \|\Delta^{-1} \nabla \cdot [\nabla^\perp z_0 \Lambda z_0]\|_{L_\omega^m C_{\text{loc}} L_x^\infty} \\ &\lesssim \log(\lambda_0) \|\nabla^\perp z_0 \Lambda z_0\|_{L_\omega^m C_{\text{loc}} L_x^\infty} \\ &\lesssim \log(\lambda_0) \|z_0\|_{L_\omega^{2m} C_{\text{loc}} C_x^{1+}}^2 \\ &\lesssim \log(\lambda_0) (C_z^2 + 1). \end{aligned}$$

Therefore, choosing

$$C_{\text{start}} \approx \log(\lambda_0) = \log(a) \tag{3.35}$$

ensures then $\|q_0\|_{L_\omega^m C_{\text{loc}} X} \leq r_0$.

Since Lemma 3.3 guarantees that the assumptions of Proposition 3.2 are satisfied, the induction step follows and we have shown (3.32).

Next, we verify (3.33). Notice that, due to (3.2a) for arbitrary $k \geq 0$,

$$\|g_{\leq k+1} - g_{\leq k}\|_{L_\omega^{2m} C_{\text{loc}} B_{\infty,1}^{1/2}} \lesssim \ell_{k+1}^\alpha [g_{\leq k}]_{L_\omega^{2m} C_{\text{loc}} B_{\infty,1}^{1/2}} + \|g_{k+1}\|_{L_\omega^{2m} C_{\text{loc}} B_{\infty,1}^{1/2}}.$$

Therefore, using (3.7) and (3.18a),

$$\|g_{\leq k+1} - g_{\leq k}\|_{L_\omega^{2m} C_{\text{loc}} B_{\infty,1}^{1/2}} \lesssim \ell_{k+1}^\alpha \mathfrak{S}_{k,\alpha} + \sqrt{r_k}.$$

Recall (3.23) and (3.24). This allows us to bound

$$\ell_{k+1}^\alpha \mathfrak{S}_{k,\alpha} \leq \sqrt{r_{k+1}} a^{b^k(b\beta/2 - \alpha(b-1))} 4^{k+1} \leq \sqrt{r_{k+1}} \leq \sqrt{r_k}.$$

Notice that $r_m = r_n \lambda_m^{-\beta} \lambda_n^\beta$ for all $m, n \in \mathbb{N}$. Therefore,

$$\begin{aligned} \|g_{\leq n'} - g_{\leq n}\|_{L_\omega^{2m} C_{\text{loc}} B_{\infty,1}^{1/2}} &\leq \sum_{k=n}^{n'-1} \|g_{\leq k+1} - g_{\leq k}\|_{L_\omega^{2m} C_{\text{loc}} B_{\infty,1}^{1/2}} \\ &\lesssim \sqrt{r_n} \sum_{k=n}^{n'-1} a^{-b^n \beta(b^{k-n} - 1)} \lesssim \sqrt{r_n}, \end{aligned}$$

where we used the estimate

$$\sum_{k=n}^{n'-1} a^{-\beta b^n (b^{k-n} - 1)} \leq \sum_{k=0}^{n'-n-1} a^{-\beta(b-1)k} \leq \frac{1}{1 - a^{-\beta(b-1)}}.$$

Lastly, we need to check that g solves (1.2) and satisfies (3.34). First notice that, since $z_n = P_{\lambda_n} z$ and $C^{1+\kappa} \hookrightarrow \dot{H}^{3/2}$,

$$\|z_n - z\|_{\dot{H}^{1/2}}^2 \leq \sum_{|k| \geq \lambda_n} |k| |\widehat{z}(k)|^2 \leq \lambda_n^{-2} \|z\|_{\dot{H}^{3/2}}^2 \lesssim \lambda_n^{-2} \|z\|_{C_x^{1+\kappa}}^2.$$

This implies $z_n \rightarrow z \in L_\omega^{2m} C_{\text{loc}} \dot{H}^{1/2}$. Moreover, as $B_{\infty,1}^{1/2} \hookrightarrow \dot{H}^{1/2}$ the strong convergence of $g_{\leq n} \in L_\omega^{2m} C_{\text{loc}} B_{\infty,1}^{1/2}$ implies $g_{\leq n} + z_n \rightarrow g + z \in L_\omega^{2m} C_{\text{loc}} \dot{H}^{1/2}$.

Let $\xi \in C^\infty(\mathbb{T}^2)$. Due to (3.2b),

$$\langle -\nabla q_n - \partial_t \mathcal{R}g_{\leq n} + \nabla^\perp(g_{\leq n} + z_n)\Lambda(g_{\leq n} + z_n) - \nu\Lambda^{\gamma-1}\nabla g_{\leq n}, \nabla\xi \rangle = 0.$$

Multiply by $\zeta \in L^\infty_\omega C^\infty_c(\mathbb{R})$, integrate in time, take expectation and split up the sum,

$$\begin{aligned} \mathbb{E} \left[\int_{\mathbb{R}} \langle \nabla q_n, \nabla\xi \rangle \zeta \, dr \right] + \mathbb{E} \left[\int_{\mathbb{R}} \langle \partial_t \mathcal{R}g_{\leq n}, \nabla\xi \rangle \zeta \, dr \right] + \mathbb{E} \left[\int_{\mathbb{R}} \langle \nu\Lambda^{\gamma-1}\nabla g_{\leq n}, \nabla\xi \rangle \zeta \, dr \right] \\ = \mathbb{E} \left[\int_{\mathbb{R}} \langle \nabla^\perp(g_{\leq n} + z_n)\Lambda(g_{\leq n} + z_n), \nabla\xi \rangle \zeta \, dr \right]. \end{aligned} \tag{3.36}$$

Next, we will establish the convergence of each term.

The residual vanishes. Indeed, using integration by parts, Hölder’s inequality and (3.32),

$$\begin{aligned} \mathbb{E} \left[\int_{\mathbb{R}} \langle \nabla q_n, \nabla\xi \rangle \zeta \, dr \right] &= \mathbb{E} \left[\int_{\mathbb{R}} \langle q_n, -\Delta\xi \rangle \zeta \, dr \right] \\ &\leq \|q_n\|_{L^m_\omega C_{loc}L^\infty} \|\xi\|_{L^1} \|\zeta\|_{L^1_\omega L^1_t} \lesssim r_n \xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

Similarly, one establishes

$$\mathbb{E} \left[\int_{\mathbb{R}} \langle \partial_t \mathcal{R}g_{\leq n}, \nabla\xi \rangle \zeta \, dr \right] \xrightarrow{n \rightarrow \infty} -\mathbb{E} \left[\int_{\mathbb{R}} \langle \mathcal{R}g, \nabla\xi \rangle \partial_t \zeta \, dr \right],$$

and

$$\mathbb{E} \left[\int_{\mathbb{R}} \langle \nu\Lambda^{\gamma-1}\nabla g_{\leq n}, \nabla\xi \rangle \zeta \, dr \right] \xrightarrow{n \rightarrow \infty} \mathbb{E} \left[\int_{\mathbb{R}} \langle \nu g, \nabla \cdot [\Lambda^{\gamma-1}\nabla\xi] \rangle \zeta \, dr \right].$$

Using Lemma A.2, we find for the non-linear term

$$\begin{aligned} \mathbb{E} \left[\int_{\mathbb{R}} \langle \nabla^\perp(g_{\leq n} + z_n)\Lambda(g_{\leq n} + z_n) - \nabla^\perp(g + z)\Lambda(g + z), \nabla\xi \rangle \zeta \, dr \right] \\ \leq \mathbb{E} \left[\|(g_{\leq n} + z_n) - (g + z)\|_{C_{loc}\dot{H}^{1/2}} \right. \\ \quad \times \left. \left(\|g_{\leq n} + z_n\|_{C_{loc}\dot{H}^{1/2}} + \|g + z\|_{C_{loc}\dot{H}^{1/2}} \right) \|\xi\|_{\dot{H}^4} \|\zeta\|_{L^1_t} \right] \\ \leq \|(g_{\leq n} + z_n) - (g + z)\|_{L^{2m}_\omega C_{loc}\dot{H}^{1/2}} \\ \quad \times \left(\|g_{\leq n} + z_n\|_{L^{2m}_\omega C_{loc}\dot{H}^{1/2}} + \|g + z\|_{L^{2m}_\omega C_{loc}\dot{H}^{1/2}} \right) \|\xi\|_{\dot{H}^4} \|\zeta\|_{L^\infty_\omega L^1_t}. \end{aligned}$$

It remains to localise the time-global integrals and to rewrite the time-derivative of ζ in terms of point-values. Let $t > s$. Since $\zeta \in L^\infty_\omega C^\infty_c(\mathbb{R})$ is to our disposal, we choose a specific structure: For $|t - s|/2 > \varepsilon > 0$, we construct $\zeta_\varepsilon(\omega, t) = \zeta_\omega(\omega)\tau_\varepsilon(t)$ by prescribing arbitrary $\zeta_\omega \in L^\infty_\omega$ and specific $\tau_\varepsilon \in C^\infty_c(\mathbb{R})$. We choose τ_ε such that

- it takes values in $[0, 1]$;
- it equals 1 on $[s + \varepsilon, t - \varepsilon]$;
- it vanishes on $\mathbb{R} \setminus [s - \varepsilon, s + \varepsilon]$;
- it is monotone increasing on $[s - \varepsilon, s + \varepsilon]$;
- it is monotone decreasing on $[t - \varepsilon, t + \varepsilon]$;
- its derivative is normalized in both transition regions, i.e.,

$$\int_{s-\varepsilon}^{s+\varepsilon} |\tau'_\varepsilon| \, dr = 1 = \int_{t-\varepsilon}^{t+\varepsilon} |\tau'_\varepsilon| \, dr;$$

- it is smooth.

In other words, τ_ε is a smooth approximation of the indicator function on $[s, t]$. Using this specific test function and dominated convergence, we find

$$\begin{aligned} & \mathbb{E} \left[\int_{\mathbb{R}} \langle \mathcal{R}g, \nabla \xi \rangle \tau'_\varepsilon \, dr \, \zeta_\omega \right] \\ &= \mathbb{E} \left[\int_{s-\varepsilon}^{s+\varepsilon} \langle \mathcal{R}g, \nabla \xi \rangle \tau'_\varepsilon \, dr \, \zeta_\omega \right] + \mathbb{E} \left[\int_{t-\varepsilon}^{t+\varepsilon} \langle \mathcal{R}g, \nabla \xi \rangle \tau'_\varepsilon \, dr \, \zeta_\omega \right] \\ &\xrightarrow{\varepsilon \rightarrow 0} \mathbb{E} [\langle \mathcal{R}g, \nabla \xi \rangle (s) \zeta_\omega] - \mathbb{E} [\langle \mathcal{R}g, \nabla \xi \rangle (t) \zeta_\omega]. \end{aligned}$$

The other terms in (3.36) are handled similarly. Overall, since $\zeta_\omega \in L^\infty_\omega$ and $\xi \in C^\infty(\mathbb{T})$ are arbitrary, we have verified that g satisfies (1.3).

The estimate (3.34) follows from (3.33) by noticing that $g_{\leq 0} = 0$. □

In Theorem 3.5 we traced the concentration speed of $g_{\leq n}$ in $L^\infty_\omega C_{\text{loc}} B_{\infty,1}^{1/2}$ as it is sufficient for the identification of the non-linearity. However, the explicit structure of the solution allows us to read of improved regularity based on the parameter choice.

Corollary 3.6 (Regularity). *Let the framework of Theorem 3.5 be satisfied. Moreover, let $(a, b, \beta, C_{\text{start}})$ be given by Theorem 3.5 and g denote the corresponding solution. Let $\vartheta_t, \vartheta_x \in [0, 1)$ satisfy $\vartheta_t + \vartheta_x < 1/2 + \beta/(2b)$ and either*

1. (high spatial regularity)

$$\vartheta_x \geq \frac{1}{2} - \alpha + \frac{\beta}{2b} \quad \text{and} \quad \vartheta_t < \alpha - \frac{1}{b} \left(\vartheta_x - \left(\frac{1}{2} - \alpha + \frac{\beta}{2b} \right) \right), \quad (3.37)$$

2. (high temporal regularity) or

$$\vartheta_x < \frac{1}{2} - \alpha + \frac{\beta}{2b} \quad \text{and} \quad \vartheta_t < \alpha. \quad (3.38)$$

Then $g \in L^\infty_\omega C_{\text{loc}}^{\vartheta_t} C^{\vartheta_x}$ with

$$[g]_{L^\infty_\omega C_{\text{loc}}^{\vartheta_t} C^{\vartheta_x}} \lesssim \sqrt{C_{\text{start}}} \left(\|z\|_{L^\infty_\omega C_{\text{loc}} C_x^{1+\kappa}} + [z]_{L^\infty_\omega C_{\text{loc}}^\alpha C_x^{1+}} + 1 \right). \quad (3.39)$$

Proof. Let $(a, b, \beta, C_{\text{start}})$ be a parameter configuration given by Theorem 3.5 and g its corresponding convex integration solution. In particular, $\|q_n\|_{L^\infty_\omega C_{\text{loc}} X} \leq r_n$ for all $n \in \mathbb{N}_0$. Let $\vartheta_t, \vartheta_x \in [0, 1)$ satisfy $\vartheta_t + \vartheta_x < 1/2 + \beta/(2b)$ and $N \in \mathbb{N}$.

Then, using (3.2a) and (3.12),

$$\begin{aligned} [g_{\leq N+1} - g_{\leq N}]_{L^\infty_\omega C_{\text{loc}}^{\vartheta_t} C^{\vartheta_x}} &\lesssim \ell_{N+1}^{\alpha-\vartheta_t} [g_{\leq N}]_{L^\infty_\omega C_{\text{loc}}^\alpha C^{\vartheta_x}} + [g_{N+1}]_{L^\infty_\omega C_{\text{loc}}^{\vartheta_t} C^{\vartheta_x}} \\ &\lesssim \ell_{N+1}^{\alpha-\vartheta_t} \sum_{n=1}^N 4^{N-n} [g_n]_{L^\infty_\omega C_{\text{loc}}^\alpha C^{\vartheta_x}} + [g_{N+1}]_{L^\infty_\omega C_{\text{loc}}^{\vartheta_t} C^{\vartheta_x}} \\ &\lesssim \ell_{N+1}^{\alpha-\vartheta_t} \sum_{n=1}^N 4^{N-n} \lambda_n^{\vartheta_x-1/2} \ell_n^{-\alpha} r_{n-1}^{1/2} + \lambda_{N+1}^{\vartheta_x-1/2} \ell_{N+1}^{-\vartheta_t} r_N^{1/2}. \end{aligned} \quad (3.40)$$

Notice the following identity of the second term,

$$\lambda_{N+1}^{\vartheta_x-1/2} \ell_{N+1}^{-\vartheta_t} r_N^{1/2} = a^{b^N (b(\vartheta_t + \vartheta_x - 1/2) - \beta/2)} a^{\beta/2} r_0^{1/2} = r_0^{1/2} a^{\beta/2} a^{-s_0} b^{N-1},$$

where $-s_0 = b^2 (\vartheta_t + \vartheta_x - 1/2) - b\beta/2 < 0$ by assumption.

We proceed by investigating the first term. Our analysis is split into two cases: high spatial regularity and high temporal regularity.

Case 1: High spatial regularity. Let us assume that ϑ_t and ϑ_x additionally satisfy (3.37), which is equivalent to

$$b(\vartheta_x + \alpha - 1/2) - \beta/2 \geq 0 \quad \text{and} \quad b^2(\vartheta_t - \alpha) + b(\vartheta_x + \alpha - 1/2) - \beta/2 < 0.$$

Estimating the super exponential by the last summand and bounding the geometric series imply

$$\begin{aligned} & \ell_{N+1}^{\alpha-\vartheta_t} \sum_{n=1}^N 4^{N-n} \lambda_n^{\vartheta_x-1/2} \ell_n^{-\alpha} r_{n-1}^{1/2} \\ &= r_0^{1/2} a^{\beta/2} a^{b^{N+1}(\vartheta_t-\alpha)} \sum_{n=1}^N 4^{N-n} a^{b^{n-1}(b(\vartheta_x+\alpha-1/2)-\beta/2)} \\ &\leq r_0^{1/2} a^{\beta/2} a^{-\varsigma_{\text{space}}} b^{N-1} 4^N, \end{aligned}$$

where $-\varsigma_{\text{space}} = b^2(\vartheta_t - \alpha) + b(\vartheta_x + \alpha - 1/2) - \beta/2 < 0$.

Case 2: High temporal regularity. In this case, we assume that ϑ_t and ϑ_x additionally satisfy (3.38). Neglecting the super exponential in the sum shows

$$\begin{aligned} & \ell_{N+1}^{\alpha-\vartheta_t} \sum_{n=1}^N 4^{N-n} \lambda_n^{\vartheta_x-1/2} \ell_n^{-\alpha} r_{n-1}^{1/2} \\ &= r_0^{1/2} a^{\beta/2} a^{b^{N+1}(\vartheta_t-\alpha)} \sum_{n=1}^N 4^{N-n} a^{b^{n-1}(b(\vartheta_x+\alpha-1/2)-\beta/2)} \\ &\leq r_0^{1/2} a^{\beta/2} a^{-\varsigma_{\text{time}}} b^{N-1} 4^N, \end{aligned}$$

where $-\varsigma_{\text{time}} = b^2(\vartheta_t - \alpha) < 0$.

In both cases, there exists $\varsigma_1 > 0$ such that

$$\ell_{N+1}^{\alpha-\vartheta_t} \sum_{n=1}^N 4^{N-n} \lambda_n^{\vartheta_x-1/2} \ell_n^{-\alpha} r_{n-1}^{1/2} \lesssim r_0^{1/2} a^{-\varsigma_1} b^{N-1} 4^N.$$

We now fix $\varsigma_1 > 0$ with this property. Notice that, for each fixed choice of parameters $a > 1$, $b > 1$ and $\varsigma > 0$, there exists a constant $C > 0$ such that for all $N \in \mathbb{N}$,

$$a^{-\varsigma b^{N-1}} (4^N + 1) \leq C a^{-N\varsigma}. \tag{3.41}$$

Next, we revisit (3.40). Using the estimates for the first and second terms, and applying (3.41) yield

$$[g_{\leq N+1} - g_{\leq N}]_{L_\omega^{2m} C_{\text{loc}}^{\vartheta_t} C^{\vartheta_x}} \lesssim r_0^{1/2} a^{-N\varsigma_2},$$

where $\varsigma_2 = \varsigma_1 \wedge \varsigma_0$.

Finally, Assertion (3.39) follows from $g_{\leq 0} = 0$, (3.12) and the summability of $a^{-n\varsigma_2}$,

$$\begin{aligned} [g]_{L_\omega^{2m} C_{\text{loc}}^{\vartheta_t} C^{\vartheta_x}} &\leq [g_{\leq 1}]_{L_\omega^{2m} C_{\text{loc}}^{\vartheta_t} C^{\vartheta_x}} + \sum_{n=1}^{\infty} [g_{\leq n+1} - g_{\leq n}]_{L_\omega^{2m} C_{\text{loc}}^{\vartheta_t} C^{\vartheta_x}} \\ &\lesssim r_0^{1/2} + r_0^{1/2} \sum_{n=1}^{\infty} a^{-n\varsigma_2} \lesssim r_0^{1/2}. \end{aligned}$$

□

Remark 3.7. If we choose $\vartheta_t = 0$, we can pick $\vartheta_x \approx 1/2 + \beta/(2b) > 1/2$. On the other hand, if we choose $\vartheta_x = 1/2$, we can take $\vartheta_t \approx \min\{\alpha, \beta/(2b)\} > 0$. Therefore, g enjoys improved regularity in the sense that

$$g \in L_\omega^{2m} \left(C_{\text{loc}} C_x^{1/2+\beta/(2b)-} \cap C_{\text{loc}}^{\min\{\alpha, \beta/(2b)\}-} B_{\infty,1}^{1/2} \right). \tag{3.42}$$

4 Examples of noises W

As seen in Theorem 3.5 and more precisely in (3.34), all our convex integration scheme requires is the finiteness of the term

$$\|z\|_{L_\omega^{2m} C_{\text{loc}} C_x^{1+\kappa}} + [z]_{L_\omega^{2m} C_{\text{loc}} C_x^{1+\alpha}} \tag{4.1}$$

for some $\kappa > 1/2$ and $\gamma < 3/2$ and $\alpha > 0$. Recalling that

$$z_t = \int_0^t e^{-\nu(t-s)\Lambda^\gamma} \Lambda^{\delta-1} dW_s$$

we provide some examples of noises W and according constraints on the parameter δ which assure that (4.1) is finite. For the sake of conciseness, we set the viscosity $\nu = 1$ in this section.

4.1 Cylindrical Wiener noise

The canonical example is to take W as a cylindrical Wiener process. Let us stress that in our setting, the parameter δ will always be negative. The process z should be accordingly thought of as the stochastic convolution of the colored noise $\tilde{W} := \Lambda^{\delta-1} W$. We cite the following result from [20] on the regularity of the stochastic convolution in this setting.

Lemma 4.1 (Regularity of stochastic convolution). *Let W be a cylindrical Wiener noise on $L^2(\mathbb{T}^2)$. Let $\gamma \in (0, 3/2)$, $\kappa > 1/2$ and $\delta < (-1 - \kappa + \gamma/2) \wedge -1$. Then for all $m \geq 1$ and $\alpha < 1/2$ there exists $C > 1$ such that*

$$[z]_{L_\omega^{2m} C_t^\alpha C_x^{1+\alpha}} + \|z\|_{L_\omega^{2m} C_t C_x^{1+\kappa}} \leq C < \infty. \tag{4.2}$$

Proof. Refer to [20, Proposition 4.1]. □

Note that in this case, z is a Gaussian random variable. As a direct consequence, from (3.34) we conclude that the moments of our convex integration solution g are sub-Gaussian. Indeed, by (3.34) we obtain

$$\|g\|_{L_\omega^{2m} C_{\text{loc}} B_{\infty,1}^{1/2}} \lesssim [z]_{L_\omega^{2m} C_t^\alpha C_x^{1+\alpha}} + \|z\|_{L_\omega^{2m} C_t C_x^{1+\kappa}} \lesssim C(2m)^{1/2}$$

where in the last step, we exploited that also the Hölder modulus of continuity of a Gaussian process admits sub-Gaussian moments (see for example [27]). In particular, this quantified moment bound in m implies that g admits square exponential moments, i.e. there exists an $\epsilon > 0$ such that

$$\mathbb{E} \left[\exp \epsilon \|g\|_{C_{\text{loc}} B_{\infty,1}^{1/2}}^2 \right] < \infty,$$

which is a result not available in the setting of [20].

4.2 Noises constructed via Young integrals

In this section, we show how to construct the stochastic convolution z for a given noise W via sewing techniques. The latter constrains us to work on a compact time interval $[0, T]$, fixed throughout this subsection. We are able to cover the class of processes which admit some temporal Hölder regularity, provided the spatial coloring is strong enough, i.e. δ is sufficiently negative. Let $(\beta^k)_k$ be a sequence of independent copies of a stochastic process β such that

$$\mathbb{E} \left[\left(\sup_{s \neq t \in [0, T]} \frac{|\beta_t - \beta_s|}{|t - s|^H} \right)^{2m} \right] < \infty. \tag{4.3}$$

For $(e_k)_k$ an orthonormal basis of $L^2(\mathbb{T}^2)$, let us consider a noise of the form

$$W_t = \sum_{|k|=0}^{\infty} e_k \beta_t^k.$$

We then have the following.

Lemma 4.2. *Let W be as above. Suppose $\gamma < 3/2$, $\kappa > 1/2$ and $H \in (0, 1)$. Suppose moreover $\delta < -2 - (\gamma \vee \kappa + \gamma(1 - H))$. Then*

$$\|z\|_{L_{\omega}^2 C_{[0, T]}^H C_x^{1+\kappa}} + \|z\|_{L_{\omega}^2 C_{[0, T]} C_x^{1+\kappa}} \lesssim \mathbb{E} \left[[\beta]_{C^H}^{2m} \right]^{\frac{1}{2m}}$$

Example 4.3. A canonical example for β in the above would be a $H + \epsilon$ -fractional Brownian motion for $H \in (0, 1)$ and $\epsilon > 0$ arbitrarily small. Indeed, in this case (4.3) is finite and moreover also enjoys sub-Gaussian moments. Similar to the above case of cylindrical Wiener noise, this implies that our constructed convex integration solution g enjoys square exponential moments in the sense that for some $\epsilon' > 0$, we have

$$\mathbb{E} \left[\exp \epsilon' \|g\|_{C_{[0, T]}^{1/2} B_{\infty, 1}^{1/2}}^2 \right] < \infty.$$

We stress that the construction is feasible for all $H \in (0, 1)$, as the integrand of the stochastic convolution is smooth in time.

Remark 4.4. Note that in the case $H = 1/2$, which recovers the case of cylindrical Wiener noise, the condition in the above lemma is worse than that given in Lemma 4.1. Besides the fact that we loose optimality due to Sobolev embeddings, this would still be natural to expect, as the sewing arguments are entirely pathwise and thus do not capture any stochastic cancellations.

Proof of Lemma 4.2. Let $(e_k)_k$ be the sequence of eigenvectors of Λ and $(\lambda_k)_k$ be the corresponding sequence of eigenvalues. Note that $\lambda_k \simeq |k|$. We begin by showing that the expression

$$\int_0^t e^{-\lambda_k^\gamma(t-r)} d\beta_r \tag{4.4}$$

is well defined as the sewing of the germ $A_{s,r}^t = e^{-\lambda_k^\gamma(t-s)}(\beta_r - \beta_s)$. For the reader unfamiliar with sewing techniques and the corresponding terminologies, we refer to section A.3 in the appendix. Let us point out that $A^t : \Delta_2([0, t]) \rightarrow \mathbb{R}$ is indeed a well defined object for any $t \in [0, T]$, since – contrary to Volterra settings – no singularities in $t \in [0, T]$ are present. Using the Schauder estimate Lemma A.5, we note that $0 < \epsilon < H$

$$|(\delta A^t)_{s,u,r}| = |(\beta_r - \beta_u) e^{-\lambda_k^\gamma t} (e^{\lambda_k^\gamma s} - e^{\lambda_k^\gamma u})| \lesssim [\beta]_{C^H} |r - u|^H (\lambda_k^\gamma)^{1-H+\epsilon} |u - s|^{1-H+\epsilon}.$$

By the Sewing Lemma A.4, we therefore obtain the existence of an associated integral $(\mathcal{I}A^t)$, which we use to define (4.4)

$$(\mathcal{I}A^t)_t =: \int_0^t e^{-\lambda_k^\gamma(t-r)} d\beta_r.$$

Using the a priori bound (A.4) that comes with the Sewing Lemma A.4, we have

$$|(\mathcal{I}A^t)_t - (\mathcal{I}A^t)_s| \lesssim |t - s|^H [\beta]_{C^H} + |t - s|^{1+\epsilon} [\beta]_{C^H} (\lambda_k^\gamma)^{1-H+\epsilon}.$$

With the definition of the one dimensional stochastic convolution at hand, we define

$$z_t := \sum_{|k|=1}^\infty \lambda_k^{\delta-1} \int_0^t e^{-\lambda_k^\gamma(t-s)} d\beta_s^k e_k$$

We note that

$$\begin{aligned} \|z_t - z_s\|_{H^\sigma}^2 &= \sum_{|k|=1}^\infty \lambda_k^{2(\delta-1)} |k|^{2\sigma} \left(\int_0^t e^{-\lambda_k^\gamma(t-r)} d\beta_r^k - \int_0^s e^{-\lambda_k^\gamma(s-r)} d\beta_r^k \right)^2 \\ &\lesssim \sum_{|k|=1}^\infty \lambda_k^{2(\delta-1)} |k|^{2\sigma} \left(\int_s^t e^{-\lambda_k^\gamma(t-r)} d\beta_r^k \right)^2 \\ &\quad + \sum_{|k|=1}^\infty \lambda_k^{2(\delta-1)} |k|^{2\sigma} (e^{-\lambda_k^\gamma(t-s)} - 1)^2 \left(\int_0^s e^{-\lambda_k^\gamma(s-r)} d\beta_r^k \right)^2 \end{aligned}$$

We can now use the above estimate to deduce that

$$\left| \int_s^t e^{-\lambda_k^\gamma(t-r)} d\beta_r^k \right| = |(\mathcal{I}A^t)_t - (\mathcal{I}A^t)_s| \lesssim |t - s|^H [\beta^k]_{C^H} + |t - s|^{1+\epsilon} [\beta^k]_{C^H} (\lambda_k^\gamma)^{1-H+\epsilon}$$

and

$$\left| \int_0^s e^{-\lambda_k^\gamma(s-r)} d\beta_r^k \right| = |(\mathcal{I}A^s)_s - (\mathcal{I}A^s)_0| \lesssim T^H [\beta^k]_{C^H} + T^{1+\epsilon} [\beta^k]_{C^H} (\lambda_k^\gamma)^{1-H+\epsilon}.$$

We therefore conclude that

$$\begin{aligned} &\mathbb{E} \left[\sum_{|k|=1}^\infty \lambda_k^{2(\delta-1)} |k|^{2\sigma} \left(\int_s^t e^{-\lambda_k^\gamma(t-r)} d\beta_r^k \right)^2 \right] \\ &\lesssim \mathbb{E} [[\beta]_{C^H}^2] |t - s|^{2H} \sum_{|k|=1}^\infty \lambda_k^{2(\delta-1)} |k|^{2\sigma} (\lambda_k^\gamma)^{2(1-H+\epsilon)}, \end{aligned}$$

where the above sum converges provided

$$2(\delta - 1) + 2\sigma + 2\gamma(1 - H + \epsilon) < -2. \tag{4.5}$$

Concerning the second sum, we obtain again thanks to Schauder estimates $|e^{-\lambda_k^\gamma(t-s)} - 1| \lesssim \lambda_k^{\gamma H} |t - s|^H$ (Lemma A.5) and thus

$$\begin{aligned} &\mathbb{E} \left[\sum_{|k|=1}^\infty \lambda_k^{2(\delta-1)} |k|^{2\sigma} (e^{-\lambda_k^\gamma(t-s)} - 1)^2 \left(\int_0^s e^{-\lambda_k^\gamma(s-r)} d\beta_r^k \right)^2 \right] \\ &\lesssim \mathbb{E} [[\beta]_{C^H}^2] \sum_{|k|=1}^\infty \lambda_k^{2(\delta-1)} |k|^{2\sigma} (\lambda_k^\gamma)^{2H+2(1-H+\epsilon)} |t - s|^{2H} \end{aligned}$$

where the above sum converges provided

$$2(\delta - 1) + 2\sigma + 2\gamma(1 + \epsilon) < -2. \tag{4.6}$$

Noting that (4.6) implies (4.5), we can conclude that $z \in L^2_\omega C_t^H H_x^\sigma$ provided (4.6) holds. Using the Sobolev embedding $H^\sigma \hookrightarrow C^{1+\epsilon}$ for $\sigma > 2$ and $\epsilon > 0$ sufficiently small, we have $z \in L^2_\omega C_t^H C_x^{1+\epsilon}$. Concerning the other regularity claim on z , note that it suffices to ask for arbitrarily small Hölder regularity, meaning that in this case we may use the Schauder estimate $|e^{-\lambda_k^\gamma(t-s)} - 1| \lesssim \lambda_k^{\gamma\epsilon} |t - s|^\epsilon$, which lets (4.6) collapse to (4.5). We therefore obtain that $z \in L^2_\omega C_t^\epsilon H_x^\sigma$ for some $\epsilon > 0$, provided (4.5) holds. Using again the Sobolev embedding $H^\sigma \hookrightarrow C^{1+\kappa+\epsilon}$ for $\sigma > 2 + \kappa$, we obtain the claim. \square

4.3 Another class of general stochastic processes

We give another example of a class of stochastic processes W that assures the existence of a stochastic convolution z and that can assure the finiteness of (4.1). Contrary to the previous subsection, we will provide a construction in spirit of that of the Itô integral (i.e. in particular not a pathwise one), yet going beyond the martingale and Gaussian setting.

Let $\beta : \Omega \times [0, \infty) \rightarrow \mathbb{R}$ be a stochastic process which starts in zero, is continuous and centered and has independent increments. We moreover assume that

$$\text{Var}(\beta_t - \beta_s) = (t - s), \quad \mathbb{E}[|\beta_t - \beta_s|^4] \leq C|t - s|^{1+\nu} \tag{4.7}$$

for some $\nu \in (0, 1]$. Note that the above are obviously satisfied for β being a Brownian motion with $\nu = 1$, however β need not be a Brownian motion or even a Gaussian process (if it was a Gaussian process, the above conditions would imply that β is a Brownian motion). We consider again

$$W_t = \sum_k e_k \beta_t^k$$

where $(e_k)_k$ is an orthonormal basis of $L^2(\mathbb{T}^2)$ and $(\beta^k)_k$ are independent copies of β , which we describe below. In the case of deterministic integrands which we are concerned with, the above conditions on β suffice to establish an analogue of the Paley-Wiener integral by means of an isometry. Time regularity of the so obtained integral can then be recovered by means of the Kolmogorov continuity theorem (which is why we require a fourth moment assumption).

Lemma 4.5. *Let $f : [0, t] \rightarrow \mathbb{R}$ be a deterministic elementary function meaning for some $N \in \mathbb{N}$ and $0 \leq t_1 < \dots < t_N \leq t$ and $c_k \in \mathbb{R}$, we have*

$$f(t) = \sum_{k=0}^{N-1} c_k \mathbf{1}_{[t_k, t_{k+1}]}$$

Suppose β as above, i.e. a continuous centered process starting in zero satisfying (4.7). Then we have

$$\mathbb{E} \left(\int_0^t f_s d\beta_s \right)^2 = \int_0^t f_s^2 ds$$

Proof. The proof follows from a direct computation as in the case of the Itô isometry, exploiting that the coefficients $(c_k)_{k \leq N}$ are deterministic. \square

As the elementary functions are dense in $L^2([0, t])$, we can use the above isometry to define the Paley-Wiener integral for $f \in L^2([0, t])$. Indeed, if $f \in L^2([0, t])$, we find a sequence of elementary function $(f^n)_n$ such that $\|f^n - f\|_{L^2} \rightarrow 0$. Thus

$$\mathbb{E} \left(\int_0^t f_s^n - f_s^m d\beta_s \right)^2 = \int_0^t (f_s^n - f_s^m)^2 ds$$

i.e. the $\left(\int_0^t f_s^n d\beta_s\right)$ is Cauchy in $L^2(\Omega)$, meaning we obtain a limit object for which the above isometry carries over. In particular, we have

$$\mathbb{E} \left(\int_0^t e^{-\lambda(t-s)} d\beta_s \right)^2 = \int_0^t e^{-2\lambda(t-s)} ds = \frac{1}{2\lambda}(1 - e^{-2\lambda t}).$$

Moreover, we can use the estimate on the fourth moment in (4.7) to deduce the existence of a Hölder continuous modification of the integral process via Kolmogorov’s continuity theorem.

Lemma 4.6. *Let $f : [0, T] \rightarrow \mathbb{R}$ be bounded. We have*

$$\mathbb{E} \left(\int_s^t f_r d\beta_r \right)^4 \leq 6 \left(\int_s^t f_r^2 dr \right)^2 + C \|f\|_\infty^{4-2(1+v)} \left(\int_s^t f_r^2 dr \right)^{1+v}.$$

Consequently, the process $X_t = \int_0^t f_s d\beta_s$ admits a $\frac{v}{4}$ -Hölder continuous modification.

Proof. By density, we may argue again through approximation via elementary functions: let

$$f_t^N = \sum_{k=1}^N c_k \mathbf{1}_{[t_k, t_{k+1}]}(t).$$

We have

$$\begin{aligned} & \mathbb{E} \left(\int_s^t f_r^N d\beta_r \right)^4 \\ &= \sum_{k,j,n,m=1}^N c_k c_j c_n c_m \mathbb{E}(\beta_{t_{k+1}} - \beta_{t_k})(\beta_{t_{j+1}} - \beta_{t_j})(\beta_{t_{n+1}} - \beta_{t_n})(\beta_{t_{m+1}} - \beta_{t_m}). \end{aligned}$$

Observe that due to the independent increments and centeredness, the above expectation vanishes whenever all four running indices k, j, n, m are distinct and whenever three of them coincide but one is distinct. We are therefore left with

$$\begin{aligned} \mathbb{E} \left(\int_s^t f_r^N d\beta_r \right)^4 &= 6 \sum_{k \neq j}^N c_k^2 c_j^2 \mathbb{E}(\beta_{t_{k+1}} - \beta_{t_k})^2 (\beta_{t_{j+1}} - \beta_{t_j})^2 + \sum_{k=1}^N c_k^4 \mathbb{E}(\beta_{t_{k+1}} - \beta_{t_k})^4 \\ &\leq 6 \sum_{k=1}^N c_k^2 (t_{k+1} - t_k) \sum_{j=1}^N c_j^2 (t_{j+1} - t_j) + C \sum_{k=1}^N c_k^4 (t_{k+1} - t_k)^{1+v} \\ &\leq 6 \left(\int_s^t (f_r^N)^2 dr \right)^2 + C \|f^N\|_\infty^{4-2(1+v)} \sum_{k=1}^N (c_k^2 (t_{k+1} - t_k))^{1+v} \\ &\leq 6 \left(\int_s^t (f_r^N)^2 dr \right)^2 + C \|f^N\|_\infty^{4-2(1+v)} \left(\int_s^t (f_r^N)^2 dr \right)^{1+v}. \end{aligned}$$

We then conclude by density and Kolmogorov’s continuity theorem. □

From the above proof, it is clear that the statement extends to deterministic bounded functions depending also on the upper integration variable, that is for $f : \Delta_2 \rightarrow \mathbb{R}$ bounded deterministic, we have

$$\mathbb{E} \left(\int_s^t f(t, r) d\beta_r \right)^4 \leq 6 \left(\int_s^t f^2(t, r) dr \right)^2 + C \|f\|_\infty^{4-2(1+v)} \left(\int_s^t f^2(t, r) dr \right)^{1+v}.$$

Consequently, $|\int_s^t f(t, r) d\beta_r| \lesssim |t - s|^{v/4}$ \mathbb{P} -almost surely. This essentially gives us Hölder regularity of the stochastic integral. To conclude that the stochastic convolution is Hölder

continuous, we need to also obtain some Hölder regularity from the semigroup, for which we use again the Schauder estimate of Lemma A.5. Indeed, for $z_t = \int_0^t e^{-a(t-s)} d\beta_s$ we have

$$z_t - z_s = \int_s^t e^{-a(t-r)} d\beta_r + \int_0^s (e^{-a(t-r)} - e^{-a(s-r)}) d\beta_s$$

and therefore using Lemma 4.6 and the Schauder estimate A.5 (using it once with $\theta = (1 + \nu)/4$ and once with $\theta = 1/2$), we obtain

$$\begin{aligned} \mathbb{E}|z_t - z_s|^4 &\lesssim (t - s)^2 + (t - s)^{1+\nu} \\ &\quad + \left(\int_0^s (e^{-a(t-r)} - e^{-a(s-r)})^2 dr \right)^2 + \left(\int_0^s (e^{-a(t-r)} - e^{-a(s-r)})^2 dr \right)^{1+\nu} \\ &\lesssim (t - s)^2 + (t - s)^{1+\nu} + a^{(1+\nu)}(t - s)^{(1+\nu)}. \end{aligned} \tag{4.8}$$

Hence, we can apply the Kolmogorov continuity theorem to obtain the existence of a continuous modification. In the infinite dimensional setting, we need to again make sure that we stay summable along Fourier modes but the calculation is essentially similar.

Lemma 4.7. *Let $\beta : \Omega \times [0, \infty) \rightarrow \mathbb{R}$ be a centered, continuous stochastic process starting in zero satisfying (4.7). Consider as noise*

$$W_t = \sum_k e_k \beta_t^k$$

where (e_k) is an orthonormal basis of $L^2(\mathbb{T}^2)$ of eigenvectors of Λ and $(\beta^k)_k$ are independent copies of β . Then the process

$$z_t := \int_0^t e^{\Lambda^\gamma(t-s)} \Lambda^{\delta-1} dW_s = \sum_{|k|=1}^\infty \int_0^t e^{-\lambda_k^\gamma(t-s)} \lambda_k^{\delta-1} d\beta_s^k e_k \tag{4.9}$$

is well defined and we have for any $\epsilon > 0$

$$z \in L_\omega^4 C_{[0,T]}^{v/4-\epsilon} H_x^\sigma$$

provided the parameters satisfy

$$4(\delta - 1) + 4\sigma + (1 + \nu)\gamma < -2.$$

Using again Sobolev embeddings, the above implies:

Corollary 4.8. *Let $\gamma \in (0, 3/2)$ and $\kappa > 1/2$. Suppose that ν as above and*

$$\delta < -3/2 - (1 + \nu)\gamma/4 - \kappa.$$

Then the process z in (4.9) satisfies $z \in L_\omega^4 C_{[0,T]}^{v/4-\epsilon} C_x^{1+\kappa}$.

Proof of Lemma 4.7. Note that

$$\|z_t - z_s\|_{H^\sigma}^2 = \sum_{|k|=1}^\infty \lambda_k^{2(\delta-1)} |k|^{2\sigma} \left(\int_0^t e^{-\lambda_k^\gamma(t-r)} d\beta_r^k - \int_0^s e^{-\lambda_k^\gamma(s-r)} d\beta_r^k \right)^2$$

and thus

$$\begin{aligned} & \mathbb{E} \|z_t - z_s\|_{H^\sigma}^4 \\ &= \sum_{k \neq j}^{\infty} \lambda_k^{2(\delta-1)} |k|^{2\sigma} \mathbb{E} \left(\int_0^t e^{-\lambda_k^\gamma(t-r)} d\beta_r^k - \int_0^s e^{-\lambda_k^\gamma(s-r)} d\beta_r^k \right)^2 \\ & \quad \times \lambda_j^{2(\delta-1)} |j|^{2\sigma} \mathbb{E} \left(\int_0^t e^{-\lambda_j^\gamma(t-r)} d\beta_r^j - \int_0^s e^{-\lambda_j^\gamma(s-r)} d\beta_r^j \right)^2 \\ & + \sum_{|k|=1}^{\infty} \lambda_k^{4(\delta-1)} |k|^{4\sigma} \mathbb{E} \left(\int_0^t e^{-\lambda_k^\gamma(t-r)} d\beta_r^k - \int_0^s e^{-\lambda_k^\gamma(s-r)} d\beta_r^k \right)^4 \\ & \lesssim \left(\sum_{|k|=1}^{\infty} \lambda_k^{2(\delta-1)} |k|^{2\sigma} \mathbb{E} \left(\int_0^t e^{-\lambda_k^\gamma(t-r)} d\beta_r^k - \int_0^s e^{-\lambda_k^\gamma(s-r)} d\beta_r^k \right)^2 \right)^2 \\ & + \sum_{|k|=1}^{\infty} \lambda_k^{4(\delta-1)} |k|^{4\sigma} \mathbb{E} \left(\int_0^t e^{-\lambda_k^\gamma(t-r)} d\beta_r^k - \int_0^s e^{-\lambda_k^\gamma(s-r)} d\beta_r^k \right)^4 \end{aligned}$$

Using (4.8), we have

$$\begin{aligned} & \sum_{|k|=1}^{\infty} \lambda_k^{4(\delta-1)} |k|^{4\sigma} \mathbb{E} \left(\int_0^t e^{-\lambda_k^\gamma(t-r)} d\beta_r^k - \int_0^s e^{-\lambda_k^\gamma(s-r)} d\beta_r^k \right)^4 \\ & \lesssim \sum_{|k|=1}^{\infty} \lambda_k^{4(\delta-1)} |k|^{4\sigma} \left(|t-s|^2 + \lambda_k^{\gamma(1+v)} (t-s)^{1+v} \right). \end{aligned}$$

The above sum converges if

$$4(\delta - 1) + 4\sigma + (1 + v)\gamma < -2 \tag{4.10}$$

in which case the above sum is bounded by

$$\left| \sum_{|k|=1}^{\infty} \lambda_k^{4(\delta-1)} |k|^{4\sigma} \mathbb{E} \left(\int_0^t e^{-\lambda_k^\gamma(t-r)} d\beta_r^k - \int_0^s e^{-\lambda_k^\gamma(s-r)} d\beta_r^k \right)^4 \right| \lesssim |t-s|^{1+v}.$$

Concerning the second sum, note that we can bound similarly using again (4.8)

$$\begin{aligned} & \sum_{|k|=1}^{\infty} \lambda_k^{2(\delta-1)} |k|^{2\sigma} \mathbb{E} \left(\int_0^t e^{-\lambda_k^\gamma(t-r)} d\beta_r^k - \int_0^s e^{-\lambda_k^\gamma(s-r)} d\beta_r^k \right)^2 \\ & \lesssim \sum_{|k|=1}^{\infty} \lambda_k^{2(\delta-1)} |k|^{2\sigma} \left(\mathbb{E} \left(\int_0^t e^{-\lambda_k^\gamma(t-r)} d\beta_r^k - \int_0^s e^{-\lambda_k^\gamma(s-r)} d\beta_r^k \right)^4 \right)^{1/2} \\ & \lesssim \sum_{|k|=1}^{\infty} \lambda_k^{2(\delta-1)} |k|^{2\sigma} \left(|t-s|^2 + \lambda_k^{\gamma(1+v)} (t-s)^{1+v} \right)^{1/2}. \end{aligned}$$

Under condition (4.10), the above sum converges in which case we obtain

$$\left(\sum_{|k|=1}^{\infty} \lambda_k^{2(\delta-1)} |k|^{2\sigma} \mathbb{E} \left(\int_0^t e^{-\lambda_k^\gamma(t-r)} d\beta_r^k - \int_0^s e^{-\lambda_k^\gamma(s-r)} d\beta_r^k \right)^2 \right)^2 \lesssim |t-s|^{1+v}.$$

Overall, this shows

$$\mathbb{E} \|z_t - z_s\|_{H^\sigma}^4 \lesssim |t-s|^{1+v}$$

and thus, we may conclude by Kolmogorov's continuity theorem. \square

5 Consequences

In this section, we collect a selection of consequences of Section 3. By nature of the presented estimates, the following results directly carry over from the analysis in [21] and [20], hence we will content ourselves with merely giving the statements and for the detailed proofs refer the reader to the corresponding arguments in those papers. We would like to stress here that we do not require Gaussianity of the noise, hence the results of these sections hold for any noise W such that the associated stochastic convolution z is a well defined object satisfying (4.1). However, in order to define stationary (ergodic) solutions, Section 5.5 may only deal with noises allowing for an extension of the corresponding solution to the full time horizon.

5.1 Non-uniqueness, infinitely many solutions and continuum of solutions

Theorem 5.1. *The solution constructed in Theorem 3.5 is not unique; in fact, there exist infinitely many such solutions and the set of solutions forms a continuum.*

These statements follow analogously to the analysis in Sections 3.2.2, 3.2.3, and 3.2.4, respectively, in [21].

5.2 Initial value problem

Let θ_0 be an \mathcal{F}_0 -measurable initial condition independent of the noise with zero mean and such that for all $m \geq 1$ and some $\eta > \frac{1}{2}$ it holds

$$\mathbb{E} [\|\theta_0\|_{C^\eta}^{2m}] < \infty. \tag{5.1}$$

Similar to [20] we may incorporate this initial condition into the stochastic convolution giving

$$z_t = e^{-t\Lambda^\gamma} \Lambda^{-1} \theta_0 + \Lambda^{-1} \int_0^t e^{-(t-s)\nu\Lambda^\gamma} \Lambda^{\delta-1} dW_s$$

where the first summand is in $C([0, \infty); C^{\eta+1}) \cap C_t^{1/2} C_x^{\eta+1-\gamma/2}$, cf. [50, Lemma 2.8]. In the convex integration construction of Section 3, adjust by introducing the cut-off

$$\chi(t) = \begin{cases} 0, & t \leq 2^{-n-1} \\ \in (0, 1), & t \in (2^{-n-1}, 2^{-n}) \\ 1, & t \geq 2^{-n} \end{cases}$$

and redefining the perturbations as

$$g_{n+1} := \chi g_{n+1}^{\text{old}}.$$

This way, the scheme does not manipulate the initial condition and though we merely obtain a bounded error

$$\left(\mathbb{E} \left[\sup_{t \in [0, 2^{-n}]} \|q_{n+1}(t)\|_X^r \right] \right)^{\frac{1}{r}} \lesssim C_{\text{start}} (C_z^2 + 1)$$

in that region, this is compensated by the decreasing size of the time interval. In contrast to Section 3, the constant C_{start} will now depend on the initial condition; more precisely it depends on the moment bound 5.1. Therefore in this setting, one first needs to fix m to proceed with the analysis; nevertheless, the result holds for all $m \geq 1$. Thus in analogy to [20] we obtain:

Theorem 5.2. *For the above choice of initial condition θ_0 there exist infinitely many (\mathcal{F}_t) -adapted non-Gaussian analytically weak solutions θ to (1.1) in the sense of Lemma A.3 with initial value θ_0 , belonging to*

$$L^p_{\text{loc}}([0, \infty); B_{\infty,1}^{-1/2}) \cap C([0, \infty), B_{\infty,1}^{-1/2-}) \cap C^{1/2-}([0, \infty), B_{\infty,1}^{-1-})$$

\mathbb{P} -a.s. for all $p \in [1, \infty)$. Furthermore there exists $\sigma > \frac{1}{2}$ such that for all $m \geq 1$

$$\begin{aligned} \sup_{t \geq 4} \left(\mathbb{E} \left[\sup_{s \in [t, t+1]} \|\theta\|_{B_{\infty,1}^{\sigma-1}}^{2m} \right] \right)^{\frac{1}{2m}} + \|\theta\|_{L^2_{\omega} C^{1/2-}_{\text{loc}} B_{\infty,1}^{-1-}} \\ \lesssim \sqrt{C_{\text{start}}} (\|z\|_{L^2_{\omega} C_{\text{loc}} C_x^{1+\kappa}} + [z]_{L^2_{\omega} C_{\text{loc}} C_x^{1+}} + 1). \end{aligned}$$

In particular, for any $\epsilon > 0$ there exists such θ such that

$$\|\theta\|_{L^2_{\omega} C_{\text{loc}} B_{\infty,1}^{-1/2-}} \leq \sqrt{C_{\text{start}}} (\|z\|_{L^2_{\omega} C_{\text{loc}} C_x^{1+\kappa}} + [z]_{L^2_{\omega} C_{\text{loc}} C_x^{1+}} + 1) (\|\theta_0\|_{L^2_{\omega} B_{\infty,1}^{-1/2-}} + \epsilon). \tag{5.2}$$

Proof. Let $\sigma \in (\frac{1}{2}, \frac{b}{2}(\alpha - \beta) \wedge \frac{1}{2} + \frac{\beta}{4b})$.

First observe that analogous to [20, (6.2)] it holds

$$\begin{aligned} \|g_{\ell_{n+1}} - g_{\leq n}\|_{L^2_{\omega} C_{\text{loc}} C_x^{\sigma}} &\lesssim \ell_{n+1}^{\alpha} \lambda_n^{\sigma - \frac{1}{2}} \|g_{\leq n}\|_{L^2_{\omega} C_{\text{loc}} B_{\infty,1}^{1/2}} \\ &\lesssim \sqrt{C_{\text{start}}(1 + C_z^2)} \lambda_{n+1}^{-\alpha} \lambda_n^{\sigma} \leq \frac{1}{4} r_{n+1}^{\frac{1}{2}}, \end{aligned}$$

since $\sigma < \frac{b}{2}(\alpha - \beta)$. Hence

$$\begin{aligned} \sup_{t \geq 2^{-n+2}} \left(\mathbb{E} \left[\sup_{s \in [t, t+1]} \|g_{\leq n+1}(s) - g_{\leq n}(s)\|_{C^{\sigma}}^{2m} \right] \right)^{\frac{1}{2m}} \\ \leq \|g_{\ell_{n+1}} - g_{\leq n}\|_{L^2_{\omega} C_{\text{loc}} C_x^{\sigma}} + \sup_{t \geq 2^{-n+2}} \left(\mathbb{E} \left[\sup_{s \in [t, t+1]} \|g_{n+1}\|_{C^{\sigma}}^{2m} \right] \right)^{\frac{1}{2m}} \\ \lesssim r_{n+1}^{\frac{1}{2}} + \lambda_{n+1}^{\sigma - \frac{1}{2}} \sqrt{(C_0 + 1) (\|q_n\|_{L^2_{\omega} C_{\text{loc}} X} + r_n)} \\ \lesssim r_{n+1}^{\frac{1}{2}} + \lambda_{n+1}^{\sigma - \frac{1}{2}} r_n^{\frac{1}{2}}, \end{aligned}$$

where we used that

$$\lambda_{n+1}^{\sigma - \frac{1}{2}} r_n^{\frac{1}{2}} = \sqrt{C_{\text{start}}(C_z^2 + 1)} \lambda_0^{\frac{\beta}{2}} \lambda_{n+1}^{\sigma - \frac{1}{2} - \frac{\beta}{2b}} \leq \sqrt{C_{\text{start}}(C_z^2 + 1)} \lambda_0^{\frac{\beta}{2}} \lambda_{n+1}^{-\frac{\beta}{4b}},$$

since $\sigma < \frac{1}{2} + \frac{\beta}{4b}$. In total this gives

$$\left(\sup_{t \geq 2^{-n+2}} \mathbb{E} \left[\sup_{s \in [t, t+1]} \|g_{\leq n+1}(s) - g_{\leq n}(s)\|_{C^{\sigma}}^{2m} \right] \right)^{\frac{1}{2m}} \leq \sqrt{C_{\text{start}}(C_z^2 + 1)} \lambda_0^{\frac{\beta}{2}} \lambda_{n+1}^{-\frac{\beta}{4b}} + r_{n+1}^{\frac{1}{2}}.$$

Conclude as in [20, Theorem 7.1], and [21, Theorem 3.2]. In particular, non-Gaussianity follows analogously: for non-Gaussian θ_0 the parameter a can be chosen large enough so as to ensure that $P_{\leq \lambda_0} \theta_0$ and $P_{\lambda_0} z$ are non-Gaussian and conclude via the construction. For Gaussian θ_0 , start the construction from $g_{\leq 0} = \hat{\chi} Z$ where Z is a smooth bounded non-Gaussian \mathcal{F}_0 -measurable random variable with finitely many non-zero Fourier modes, independent of θ_0 and the noise, and $\hat{\chi}$ a smooth cut-off in time equal to 0 if $t \leq \frac{1}{2}$ and 1 if $t > 1$. Choosing λ_0 large enough such that $P_{\leq \lambda_0} Z = Z$, we hence ensure that the first Fourier modes of the constructed solution correspond to that of a non-Gaussian and is thus non-Gaussian itself. \square

5.3 Prescribing terminal values

As it turns out, a similar procedure as in Section 5.2 can be used to prescribe terminal values at arbitrary times $T \geq 4$:

Theorem 5.3. *Let $T \geq 4$ be arbitrary. For all $m \geq 1$, let $v \in L^m_\omega C^1([0, T], C^\eta)$, $\eta > \frac{1}{2}$ be given such that v has zero mean \mathbb{P} -almost surely, $v(t) = v(T)$ for all $t \in [T - 1, T]$ and $\partial_t v(0) = 0$. Then there exists an analytically weak solution θ to (1.1) in the sense of Lemma A.3 such that $\theta(0) = v(0)$ and $\theta(T) = v(T)$ and*

$$\theta \in L^p(0, T; B_{\infty,1}^{-1/2}) \cap C([0, T], B_{\infty,1}^{-1/2-}) \cap C^{1/2-}([0, T], B_{\infty,1}^{-1-})$$

\mathbb{P} -a.s. for all $p \in [1, \infty)$. Furthermore for any $\epsilon > 0$ there exists such θ such that for all $m \geq 1$

$$\|\theta\|_{L^m_\omega C_T B_{\infty,1}^{-1/2-}} \leq \|v\|_{L^m_\omega C_T B_{\infty,1}^{-1/2-}} + \epsilon.$$

Proof. Analogously to [21, Section 3.3] we may in this case set up a convex integration procedure of the form: let $z(0) = 0$ and let

$$\begin{aligned} & -\partial_t \mathcal{R}g_{\leq n} + \nabla^\perp(g_{\leq n} + z_n)\Lambda(g_{\leq n} + z_n) - \nu\Lambda^{\gamma-1}\nabla g_{\leq n} \approx \nabla q_n, \\ & g_{\leq n}(0) = P_{\lambda_n/3}\Lambda^{-1}v(0), \\ & g_{\leq n}(t) = P_{\lambda_n/3}\Lambda^{-1}v(T), \quad \forall t \in [T - 2^{-n}, T]. \end{aligned}$$

Indeed, starting of with

$$\begin{aligned} g_{\leq 0} &= P_{\lambda_0/3}\Lambda^{-1}v, \\ \nabla q_0 &\approx -\partial_t \mathcal{R}g_{\leq 0} + \nabla^\perp(g_{\leq 0} + z_0)\Lambda(g_{\leq 0} + z_0) - \nu\Lambda^{\gamma-1}\nabla g_{\leq 0} \end{aligned}$$

gives the estimates

$$\begin{aligned} \|g_{\leq 0}\|_{L^m_\omega C_T^1 B_{\infty,1}^{1/2}} &\leq \|v\|_{L^m_\omega C_T^1 C^\eta} \\ \|q_0\|_{L^m_\omega C_T X} &\lesssim \|v\|_{L^m_\omega C_T C^\eta}^2 + \|v\|_{L^m_\omega C_T^1 C^\eta} + C_z. \end{aligned}$$

Thus choose $C_{\text{start}} = C\|v\|_{L^m_\omega C_T^1 C^\eta}$. In the induction step we define g_{n+1} as before and let

$$g_{\leq n+1} := g_{\ell_{n+1}} + \chi g_{n+1} + g_{n+1}^{\text{pres}}$$

where

$$\chi(t) = \begin{cases} 0, & t \in [0, 2^{-n-1}] \cup [T - 2^{-n-1}, T], \\ (0, 1), & t \in (2^{-n-1}, 2^{-n}) \cup (T - 2^{-n}, T - 2^{-n-1}), \\ 1, & t \in [2^{-n}, T - 2^{-n}] \end{cases}$$

and

$$g_{n+1}^{\text{pres}} := (P_{\lambda_{n+1}/3} - P_{\lambda_n/3})\Lambda^{-1}v.$$

Observe that

$$\|g_{n+1}^{\text{pres}}\|_{L^m_\omega C_T B_{\infty,1}^{1/2}} \lesssim \lambda_n^{-1}\|v\|_{L^m_\omega C_T^1 C^\eta} \leq \frac{1}{2}r_{n+1}^{\frac{1}{2}}.$$

Furthermore

$$\begin{aligned} g_{\leq n+1}(0) &= P_{\lambda_{n+1}/3}\Lambda^{-1}v(0), \\ g_{\leq n+1}(t) &= P_{\lambda_{n+1}/3}\Lambda^{-1}v(T), \quad \forall t \in [T - 2^{-n-1}, T] \end{aligned}$$

and a similar analysis as in [21, Section 3.3] applies. □

5.4 Coming down from infinity

Theorem 5.2 and Theorem 5.3 imply the following coming down from infinity result:

Theorem 5.4. *Let $T \geq 4, \epsilon > 0$ and θ_0 as in Theorem 5.2, then there exist infinitely measurable solutions to (1.1) in the sense of Lemma A.3 such that for all $m \geq 1$*

$$\|\theta\|_{L^m_\omega C([T, \infty), B_{\infty,1}^{-1/2-\epsilon})} \leq \epsilon.$$

Proof. Choose v as in Theorem 5.3 such that $v(0) = \theta_0$ and $v(T) = 0$, then from Theorem 5.3 we deduce the existence of θ such that

$$\begin{aligned} \theta(0) &= \theta_0, \\ \theta(T) &= 0. \end{aligned}$$

Conclude with Theorem 5.2. □

5.5 Stationary (ergodic) solutions

Stationary solutions satisfy the equation on the full time horizon \mathbb{R} hence in order to study the existence of such solutions in the above setting, we must be able to extend the convex integration results to \mathbb{R} as well. Hence in this subsection we can only treat the case of Section 4.1 which allows for the same extension procedure as described in [20, Section 8]. Analogously we use Theorem 5.2 to start in $\theta_0 = 0$ and apply the convex integration mechanism of Section 3 to

$$z_t = \int_{-\infty}^t e^{-(t-s)\Lambda^\gamma} \Lambda^{\delta-1} dW_s$$

to construct infinitely many (ergodic) stationary solutions. Indeed: due to lack of uniqueness, we understand stationarity as invariance with respect to time shifts

$$S_t(\theta, W)(\cdot) := (\theta(\cdot + t), W(\cdot + t) - W(t)), \quad t \in \mathbb{R}, (\theta, W) \in \mathcal{T},$$

on the trajectory space $\mathcal{T} := C(\mathbb{R}, B_{p,1}^{-1/2}) \times C(\mathbb{R}, B_{p,p}^{-1-})$ for some $p \in [2, \infty)$.

In analogy to [20, Definition 8.1] we hence define stationary solutions $((\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}), \theta, W)$ as analytically weak solutions of (1.1) in the sense of Lemma A.3 on $(-\infty, \infty)$ such that

$$\text{Law}(S_t(\theta, W)) = \text{Law}(\theta, W) \quad \text{for all } t \in \mathbb{R}.$$

Similarly we may define a stationary solution $((\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}), \theta, W)$ as ergodic (cf. [20, Definition 8.4]) if

$$\text{Law}(\theta, W)(A) \in \{0, 1\} \quad \text{for all } A \subset \mathcal{T} \text{ Borel and shift invariant.}$$

In analogy to [20, Theorem 8.2, Corollary 8.3, Theorem 8.5] and [21, Theorem 4.2]) we may therefore deduce

Theorem 5.5. *1. Each convex integration solution θ obtained in Theorem 5.2 with $\theta_0 = 0$ and extended to \mathbb{R} gives rise to a stationary solution $((\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t), \tilde{\mathbb{P}}), \tilde{\theta}, \tilde{W})$ such that time averages of the law of time-shifts of θ converge to the law of $(\tilde{\theta}, \tilde{W})$ and for some $\epsilon \in (0, 1), \sigma > 0$ it holds for all $m \geq 1$ $\tilde{\theta} \in L^m_\omega C_t B_{\infty,1}^{-1/2+\epsilon} \cap L^m_\omega C_t^{1/2-\sigma} B_{\infty,1}^{-1-\epsilon}$.*

2. There exist infinitely many non-Gaussian stationary solutions.

3. There exist infinitely many non-Gaussian ergodic stationary solutions such that for some $\epsilon \in (0, 1), \sigma > 0$ and for all $m \geq 1$ they lie in $\tilde{\theta} \in L^m_\omega C_t B_{\infty,1}^{-1/2+\epsilon} \cap L^m_\omega C_t^{1/2-\sigma} B_{\infty,1}^{-1-\epsilon}$.

The proofs follow a tightness argument for the ergodic averages to deduce stationarity as well as a Krein-Milman argument to deduce ergodicity. Non-Gaussianity follows as described in the proof of Theorem 5.2.

A Auxiliary results

A.1 A multiplier and commutator Lemma

Lemma A.1 ([7, Lemma 3.2]). *Suppose $a \in L^\infty(\mathbb{T}^2)$ such that a is mean-free with $\text{supp}(\widehat{a}) \subset \{|k| \leq \mu\}$ and $\mu \geq 10$. Let $m \in C^\infty(\mathbb{R}^2 \setminus \{0\})$ be a homogeneous function of degree 0 and T_m is the Fourier multiplier defined by $\widehat{T_m f}(k) = m(k)\widehat{f}(k)$. Then there exists a constant $C_m \geq 1$ such that*

$$\|T_m a\|_{L^\infty(\mathbb{T}^2)} \leq C_m \log(\mu) \|a\|_{L^\infty(\mathbb{T}^2)}. \tag{A.1}$$

The next lemma extends the definition of the non-linearity, defined in (1.3), for $g \in H^{1/2}$. Notice that, for smooth g and ξ ,

$$\langle \nabla \cdot [\nabla^\perp g \Lambda g], \xi \rangle = -\langle \Lambda g, \nabla^\perp g \cdot \nabla \xi \rangle.$$

Moreover, $(g, g') \mapsto \langle \Lambda g, \nabla^\perp g' \cdot \nabla \xi \rangle$ for fixed $\xi \in H^4(\mathbb{T}^2)$ extends to a bi-linear and bounded map from $H^{1/2}(\mathbb{T}^2) \times H^{1/2}(\mathbb{T}^2) \rightarrow \mathbb{R}$. This is closely related to the commutator estimate of Cheng, Kwon and Li [7, Proposition 5.1]. However, they work on the level $\theta = \Lambda g$, which is why for the sake of completeness, we provide the proof for our setting.

Lemma A.2. *There exists a constant $C \geq 1$ such that for all $g, g' \in H^{1/2}$ and $\xi \in H^4$,*

$$|\langle \Lambda g, \nabla^\perp g' \cdot \nabla \xi \rangle| \leq C \|g\|_{\dot{H}^{1/2}} \|g'\|_{\dot{H}^{1/2}} \|\xi\|_{\dot{H}^4}. \tag{A.2}$$

Proof. Notice that

$$\begin{aligned} \langle \Lambda g, \nabla^\perp g' \cdot \nabla \xi \rangle &= \sum_{k,l} |k|l \cdot k^\perp \widehat{g}(-k) \widehat{g}'(l) \widehat{\xi}(k-l) \\ &= \sum_{k \neq 0} \sum_{l \neq 0, k} |k|l \cdot k^\perp \widehat{g}(-k) \widehat{g}'(l) \widehat{\xi}(k-l). \end{aligned}$$

Hölder's inequality shows

$$\begin{aligned} &\sum_{k \neq 0} \sum_{l \neq 0, k} |k|l \cdot k^\perp \widehat{g}(-k) \widehat{g}'(l) \widehat{\xi}(k-l) \\ &\leq \left(\sum_{k \neq 0} |k| |\widehat{g}(k)|^2 \right)^{1/2} \left(\sum_{k \neq 0} \left| \sum_{l \neq 0, k} |k|^{1/2} l \cdot k^\perp \widehat{g}'(l) \widehat{\xi}(k-l) \right|^2 \right)^{1/2}. \end{aligned}$$

Therefore it remains to verify

$$\left(\sum_{k \neq 0} \left| \sum_{l \neq 0, k} |k|^{1/2} l \cdot k^\perp \widehat{g}'(l) \widehat{\xi}(k-l) \right|^2 \right)^{1/2} \lesssim \|g'\|_{\dot{H}^{1/2}} \|\xi\|_{\dot{H}^4}. \tag{A.3}$$

We distinguish two cases.

1. If $|l| \lesssim |k-l|$, then $|k| \lesssim |k-l|$ and

$$|k|^{1/2} l \cdot k^\perp \leq |k|^{3/2} |l| \lesssim |k-l|^2 |l|^{1/2}.$$

2. If $|k - l| \ll |l|$, then $|k| \approx |l|$ and

$$|k|^{1/2}l \cdot k^\perp = |k|^{1/2}(k - l) \cdot k^\perp \leq |k|^{3/2}|k - l| \leq |k|^{3/2}|k - l|^{1/2}|l|^{1/2}.$$

Split up the interior sum of the left hand side of (A.3) into these two cases and estimate separately. For fixed k by Hölder's inequality

$$\begin{aligned} & \sum_{l \neq 0, k \ \& \ |l| \lesssim |k-l|} |k|^{1/2}l \cdot k^\perp \widehat{g}'(l) \widehat{\xi}(k-l) \\ & \leq \sum_{l \neq 0, k \ \& \ |l| \lesssim |k-l|} |l|^{1/2} |\widehat{g}'(l)| |k-l|^2 |\widehat{\xi}(k-l)| \\ & \leq \left(\sum_{l \neq 0, k \ \& \ |l| \lesssim |k-l|} |l| |\widehat{g}'(l)|^2 \right)^{1/2} \left(\sum_{l \neq 0, k \ \& \ |l| \lesssim |k-l|} |k-l|^4 |\widehat{\xi}(k-l)|^2 \right)^{1/2}. \end{aligned}$$

The second case can be treated analogously,

$$\begin{aligned} & \sum_{l \neq 0, k \ \& \ |k-l| \ll |l|} |k|^{1/2}l \cdot k^\perp \widehat{g}'(l) \widehat{\xi}(k-l) \\ & \leq \sum_{l \neq 0, k \ \& \ |k-l| \ll |l|} |l|^{1/2} |\widehat{g}'(l)| |k|^{3/2} |k-l|^{1/2} |\widehat{\xi}(k-l)| \\ & \leq \left(\sum_{l \neq 0, k \ \& \ |k-l| \ll |l|} |l| |\widehat{g}'(l)|^2 \right)^{1/2} \left(\sum_{l \neq 0, k \ \& \ |k-l| \ll |l|} |k|^3 |k-l| |\widehat{\xi}(k-l)|^2 \right)^{1/2}. \end{aligned}$$

Lastly, we need to bound

$$\sum_{k \neq 0} \sum_{l \neq 0, k \ \& \ |l| \lesssim |k-l|} |k-l|^4 |\widehat{\xi}(k-l)|^2 \lesssim \sum_{k \neq 0} |k|^8 |\widehat{\xi}(k)|^2 = \|\xi\|_{H^4}^2.$$

A change of variables $(k, l) \mapsto (k', l') = (k, k-l)$ reveals a similar bound for the second case. Indeed, $|l'| = |k-l| \ll |l| \sim |k| = |k'|$ and, therefore,

$$\begin{aligned} & \sum_{k \neq 0} \sum_{l \neq 0, k \ \& \ |k-l| \ll |l|} |k|^3 |k-l| |\widehat{\xi}(k-l)|^2 \\ & = \sum_{k' \neq 0} \sum_{l' \neq 0, k' \ \& \ |l'| \ll |k'|} |k'|^3 |l'| |\widehat{\xi}(l')|^2 \\ & \lesssim \sum_{k' \neq 0} \sum_{l' \neq 0, k' \ \& \ |l'| \ll |k'|} |k'|^{-4} |l'|^8 |\widehat{\xi}(l')|^2 \lesssim \|\xi\|_{H^4}^2. \end{aligned}$$

□

A.2 Transformation of SQG equations

We have the following

Lemma A.3. *Let $g \in L^{2m} C_{loc} B_{\infty,1}^{1/2}$ be a weak solution to (1.3). Let*

$$z_t = \int_0^t e^{-\nu(t-s)\Lambda^\gamma} \Lambda^{\delta-1} dW_s$$

be well defined and enjoy the regularity (1.4). Then $\theta := \Lambda(g + z)$ is a weak solution to (1.1) in the sense that

$$\langle \theta_t, \xi \rangle - \langle \theta_0, \xi \rangle + \int_0^t \langle \mathcal{R}^\perp \theta_s \cdot \nabla \theta_s, \xi \rangle ds = -\nu \int_0^t \langle \Lambda^\gamma \theta_s, \xi \rangle ds + \langle \Lambda^\delta (W_t - W_0), \xi \rangle$$

for some $\theta_0 \in H^{1/2}$ and any $\xi \in C_c^\infty(\mathbb{T}^2)$ where the non-linearity is understood by means of the identity

$$\langle \mathcal{R}^\perp \theta \cdot \nabla \theta, \xi \rangle = \frac{1}{2} \langle \theta, [\mathcal{R}^\perp \cdot, \nabla \xi] \theta \rangle$$

Proof. As already mentioned prior to Lemma A.2 it was already shown in [7, Proposition 5.1] that the commutator $[\mathcal{R}^\perp \cdot, \nabla \xi]$ extends to a linear map $[\mathcal{R}^\perp \cdot, \nabla \xi] : \dot{H}^{-1/2} \rightarrow \dot{H}^{1/2}$. Hence, $\theta := \Lambda(g + z)$ enjoys sufficient regularity for the non-linearity to be well-defined. The claim thus follows by a direct calculation. \square

A.3 The Sewing Lemma

We recall the Sewing Lemma due to [17] (see also [15, Lemma 4.2]). Let E be a Banach space, $[0, T]$ a given interval. Let Δ_n denote the n -th simplex of $[0, T]$, i.e. $\Delta_n : \{(t_1, \dots, t_n) \mid 0 \leq t_1 \leq \dots \leq t_n \leq T\}$. For a function $A : \Delta_2 \rightarrow E$ define the mapping $\delta A : \Delta_3 \rightarrow E$ via

$$(\delta A)_{s,u,t} := A_{s,t} - A_{s,u} - A_{u,t}$$

Provided $A_{t,t} = 0$ we say that for $\alpha, \beta > 0$ we have $A \in C_2^{\alpha,\beta}(E)$ if $\|A\|_{\alpha,\beta} < \infty$ where

$$\begin{aligned} \|A\|_\alpha &:= \sup_{(s,t) \in \Delta_2} \frac{\|A_{s,t}\|_E}{|t-s|^\alpha}, & \|\delta A\|_\beta &:= \sup_{(s,u,t) \in \Delta_3} \frac{\|(\delta A)_{s,u,t}\|_E}{|t-s|^\beta}, \\ \|A\|_{\alpha,\beta} &:= \|A\|_\alpha + \|\delta A\|_\beta. \end{aligned}$$

For a function $f : [0, T] \rightarrow E$, we note $f_{s,t} := f_t - f_s$

Moreover, if for any sequence $(\mathcal{P}^n([s, t]))_n$ of partitions of $[s, t]$ whose mesh size goes to zero, the quantity

$$\lim_{n \rightarrow \infty} \sum_{[u,v] \in \mathcal{P}^n([s,t])} A_{u,v}$$

converges to the same limit, we note

$$(\mathcal{I}A)_{s,t} := \lim_{n \rightarrow \infty} \sum_{[u,v] \in \mathcal{P}^n([s,t])} A_{u,v}.$$

Lemma A.4 (Sewing). *Let $0 < \alpha \leq 1 < \beta$. Then for any $A \in C_2^{\alpha,\beta}(E)$, $(\mathcal{I}A)$ is well defined. Moreover, denoting $(\mathcal{I}A)_t := (\mathcal{I}A)_{0,t}$, we have $(\mathcal{I}A) \in C^\alpha([0, T], E)$ and $(\mathcal{I}A)_0 = 0$ and for some constant $c > 0$ depending only on β we have*

$$\|(\mathcal{I}A)_t - (\mathcal{I}A)_s - A_{s,t}\|_E \leq c \|\delta A\|_\beta |t-s|^\beta. \tag{A.4}$$

We say the germ A admits a sewing $(\mathcal{I}A)$ and call \mathcal{I} the sewing operator.

Lemma A.5 ('Schauder estimate'). *We have for $a \geq 0$ and $\theta \in [0, 1]$ and $0 \leq s \leq t$*

$$(e^{-at} - e^{-as}) \lesssim a^\theta (t-s)^\theta.$$

Proof. The proof follows from interpolating the bounds

$$|e^{-at} - e^{-as}| \leq 2, \quad |e^{-at} - e^{-as}| = \left| \int_{as}^{at} -e^{-x} dx \right| \leq a(t-s).$$

\square

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Acknowledgments. The authors thank Martina Hofmanová for several helpful discussions.