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Moderate deviations on Poisson chaos*

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Abstract

This paper deals with U-statistics of Poisson processes and multiple Wiener-Itô integrals on the Poisson space. Via sharp bounds on the cumulants for both classes of random variables, moderate deviation principles, concentration inequalities and normal approximation bounds with Cramér correction are derived. It is argued that the results obtained in this way are in a sense best possible and cannot be improved systematically. Applications in stochastic geometry and to functionals of Ornstein-Uhlenbeck-Lévy processes are investigated.

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1 Introduction

Probabilistic limit theorems for functionals of Poisson processes were intensively studied over the past decades. Starting with the seminal work [34], this field of research got a particular new drive. In that paper, Malliavin calculus for Poisson processes was combined for the first time with Stein's method for normal approximation to deduce new central limit theorems with explicit error bounds. Previously, this connection had been established and exploited for functionals of Gaussian processes and has led to a large number of exciting new developments, see e.g. the monograph [32] for an excellent introduction. As an example we mention the celebrated fourth moment theorem, which states that a sequence of random variables living inside a fixed Wiener chaos and having unit variance converges in distribution to a standard Gaussian random variable if and only if their fourth moments converge to 3, the fourth moment of the standard Gaussian distribution. For the Poisson space, which we consider throughout this paper, a fourth moment theorem in the same spirit as well as some refinements were established in

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[11, 12, 13]. Indeed, for a sequence of random variables $(F_n)_{n\in\mathbb{N}}$ living inside a fixed Poisson chaos, so-called multiple Wiener-Itô integrals, such that $\mathbb{E}[F_n^2] = 1$ for each $n \in \mathbb{N}$ one has that

$$F_n \xrightarrow{d} N \sim \mathcal{N}(0,1) \qquad \text{if} \qquad \mathbb{E}[F_n^4] \to 3,$$

as $n \to \infty$. This can equivalently be rephrased by saying that

$$F_n \xrightarrow{d} N \sim \mathcal{N}(0,1) \qquad \text{if} \qquad \mathrm{cum}_4(F_n) \to 0,$$

as $n \to \infty$, where $\mathrm{cum}_4(F_n) := \mathbb{E}[F_n^4] - 3$ stands for the fourth cumulant of F_n. In case that the sequence $(F_n^4)_{n\in\mathbb{N}}$ is uniformly integrable also the reverse direction of this implication is true.

Poisson functionals, i.e. random variables depending only on a Poisson process, play a crucial role in stochastic geometry, where one frequently studies random structures constructed from an underlying Poisson process. For such situations the Malliavin-Stein method is a very useful approach that has been extensively employed over the last years (see e.g. the volume of survey articles [33]). Although multiple Wiener-Itô integrals are very important and interesting objects, statistics of interest in stochastic geometry are usually sums of multiple Wiener-Itô integrals and not single ones as considered in the fourth moment theorem above. However, many of them are so-called U-statistics of Poisson processes. Since these Poisson U-statistics can be written as finite sums of multiple Wiener-Itô integrals, they are closely related to multiple Wiener-Itô integrals and very well suited for the Malliavin-Stein approach. This technique was used in e.g. [8, 19, 24, 25, 29, 36, 39, 44] to derive general normal approximation results for Poisson U-statistics, which were applied to different situations such as

(i) geometric random graphs [8, 24, 25, 39, 40],
(ii) intersection and flat processes [5, 20, 22, 29, 39, 45],
(iii) random simplicial complexes [1, 9],
(iv) the statistical analysis of spherical Poisson fields [6, 7].

We also point to the works [3, 4] for the study of concentration bounds for U-statistics of Poisson processes. For a survey on Poisson U-statistics we refer to [26].

The present paper is focussed on refinements of the central limit theorem for multiple Wiener-Itô integrals and U-statistics of Poisson processes. For finite sums of such random variables we study the validity of the Gaussian tail behaviour on scales beyond the one of the central limit theorem. We do this by proving moderate deviation principles (MDPs) as well as concentration inequalities and normal approximation bounds with Cramér correction.

Our proofs rely on the so-called method of cumulants, which requires fine estimates on the cumulants of all orders. It is well known that such bounds encode much information about the fine probabilistic behaviour of the involved random variables, see the monograph [41]. In particular, sharp bounds on cumulants lead to moderate deviation principles, see [14]. For more details on the method of cumulants we also refer to the recent survey [15]. In order to control the cumulants, so-called product formulas for the moments and cumulants of multiple Wiener-Itô integrals are needed. In the present article we improve existing results in this direction (see e.g. [29, 35, 47]) by deriving such bounds under weaker (and partially even optimal) integrability assumptions. These findings are of independent interest.

Our work can be regarded as a continuation of our previous article [46], where we studied similar questions for multiple stochastic integrals on the Wiener space, that is, stochastic integrals with respect to Gaussian processes. On the Wiener space it has been shown in [46] that all cumulants of a multiple stochastic integral are bounded in terms of the fourth cumulant. Roughly speaking, this can be seen as a consequence of the hypercontractivity property on the Wiener space. Since no such property is available for Poisson processes, a similar bound for the cumulants in terms of only the fourth one cannot be expected for the classes of random variables we consider. This together with the much more involved combinatorial structure of the product formulas for multiple stochastic integrals on the Poisson space makes the derivation of cumulant estimates for U-statistics and multiple stochastic integrals a challenging problem, which is tackled in the present text. It is one of the main features that our results will turn out to be best possible. In fact, we shall identify a range of scales on which general finite sums of Poisson U-statistics and general finite sums of multiple Wiener-Itô integrals satisfy a MDP and we construct examples of such functionals that cannot satisfy a similar MDP beyond this range of scales. We highlight that this is in sharp contrast to the situation studied in [46], where such an example could not be found so far. This in turn led to a range of scalings for which we could not answer in [46] whether or not a MDP is valid for a general sequence of multiple stochastic integrals.

Our general findings for multiple Wiener-Itô integrals and Poisson U-statistics will be illustrated by means of three examples. We start by specialising our estimates to U-statistics having a fixed kernel. As an application we consider the intersection process of order q generated by a Poisson process of k-dimensional totally geodesic submanifolds in a d-dimensional standard space of constant curvature $\kappa \in \{-1, 0, 1\}$. More specifically, we consider the $d - q(d - k)$-dimensional Riemannian volume associated with such an intersection process within a fixed observation window. This naturally connects to the recent line of research in non-Euclidean stochastic geometry. As a second model we investigate the random geometric graph in which two points of a homogeneous Poisson process within some convex body in $\mathbb{R}^d$ are connected by an edge, provided their Euclidean distance does not exceed some given threshold. The Poisson functional we consider is a linear combination of classical subgraph counting statistics. Finally we study the Ornstein-Uhlenbeck process generated by a Poisson process in space and time. More precisely, our focus lies on the quadratic variation functional of this stochastic process, which admits a representation as a sum of Wiener-Itô integrals of order one and two.

This paper is organised as follows. After introducing some preliminaries and notation in Section 2, we present and discuss our main results in Section 3. We consider moderate deviations for multiple Wiener-Itô integrals and Poisson U-statistics in Subsection 3.1, while Subsection 3.2 deals with product formulas, which are essential ingredients of our proofs. Section 4 is devoted to applications, before the proofs are given in Sections 5, 6 and 7.

2 Preliminaries

2.1 Poisson processes and multiple Wiener-Itô integrals

Let $(\mathbb{X}, \mathcal{X})$ be a measurable space, which is supplied with a σ-finite measure μ. A random counting measure η on $\mathbb{X}$ is called a Poisson process with intensity measure μ, provided that

i) for all $B \in \mathcal{X}$, $\eta(B)$ is a (possibly degenerate) Poisson distributed random variable with mean $\mu(B)$,

ii) the random variables $\eta(B_1), \ldots, \eta(B_n)$ are independent for all pairwise disjoint $B_1, \ldots, B_n \in \mathcal{X}$, $n \in \mathbb{N}$.

For $q, r \in \mathbb{N}$ we let $L^r(\mu^q)$ be the space of measurable functions $f : \mathbb{X}^q \to \mathbb{R}$ with the property that $|f|^r$ is integrable with respect to μ^q, the q-fold product measure of the underlying measure μ. Moreover, we shall denote by $L^r_s(\mu^q)$ the subspace of symmetric functions, that is, functions $f \in L^r(\mu^q)$ that are invariant with respect to arbitrary permutations of their arguments.

For a counting measure ξ on $\mathbb{X}$ and $m \in \mathbb{N}$ let us define the measure $\xi^{(m)}$ on $\mathbb{X}^m$ by

$$\xi^{(m)}(\,\cdot\,) := \int_{\mathbb{X}} \cdots \int_{\mathbb{X}} \mathbf{1}((x_1,\dots,x_m) \in \,\cdot\,) \Big(\xi - \sum_{i=1}^{m-1} \delta_{x_i}\Big)(\mathrm{d}x_m)\Big(\xi - \sum_{i=1}^{m-2} \delta_{x_i}\Big)(\mathrm{d}x_{m-1}) \cdots \times (\xi - \delta_{x_1})(\mathrm{d}x_2)\xi(\mathrm{d}x_1)\,,$$

where δ_x stands for the Dirac measure at $x \in \mathbb{X}$. For $f \in L^1_s(\mu^q)$ we define the pathwise multiple stochastic integral $I_q(f)$ of f with respect to the (compensated) Poisson process η by

$$I_q(f) := \sum_{J \subset [q]} (-1)^{q-|J|} \int_{\mathbb{X}^{|J|}} \int_{\mathbb{X}^{q-|J|}} f(x_1,\dots,x_q)\, \mu^{q-|J|}(\mathrm{d}x_{J^c})\, \eta^{(|J|)}(\mathrm{d}x_J)$$

with $[q] := \{1,\dots,q\}$, $x_J{:=}(x_j : j \in J)$ and where $|J|$ stands for the cardinality of J (for $J = [q]$ we interpret the inner integral as $f(x_1,\dots,x_q)$). The q-fold Wiener-Itô integral of $f \in L^2_s(\mu^q)$ is defined as the limit of $(I_q(f_n))_{n\in\mathbb{N}}$ in the space of square integrable random variables, where $(f_n)_{n\in\mathbb{N}}$ is a sequence of simple symmetric functions approximating f in $L^2_s(\mu^q)$. We recall that $\mathbb{E}[I_q(f)] = 0$ and that

$$\mathbb{E}[I_{q_1}(f_1)I_{q_2}(f_2)] = q_1!\mathbf{1}(q_1 = q_2)\,\langle f_1, f_2\rangle_{L^2(\mu^{q_1})},$$

where $q_1, q_2 \in \mathbb{N}$, $f_1 \in L^2_s(\mu^{q_1})$, $f_2 \in L^2_s(\mu^{q_2})$ and $\langle\,\cdot\,,\,\cdot\,\rangle_{L^2(\mu^{q_1})}$ denotes the usual scalar product in $L^2(\mu^{q_1})$. In particular, if a random variable has the form $F := I_{q_1}(f_1) + \ldots + I_{q_k}(f_k)$ with $k \in \mathbb{N}$, distinct $q_1,\dots,q_k \in \mathbb{N}$ and $f_i \in L^2_s(\mu^{q_i})$, $i \in \{1,\dots,k\}$, we have that $\mathbb{E}F = 0$ and that the variance of F is given by

$$\operatorname{Var} F = q_1!\|f_1\|^2_{L^2(\mu^{q_1})} + \ldots + q_k!\|f_k\|^2_{L^2(\mu^{q_k})}\,. \tag{2.1}$$

Let us also remark that the collection of all random variables of the form $I_q(f)$ with $f \in L^2_s(\mu^q)$ is called the q^{th} Poisson chaos (with respect to the measure μ).

We refer to [28] for background material concerning Poisson processes and a detailed construction of the multiple Wiener-Itô integral.

2.2 Poisson U-statistics

As in the previous section, let $(\mathbb{X}, \mathcal{X})$ be a measurable space and η be a Poisson process on $\mathbb{X}$ with σ-finite intensity measure μ. A Poisson U-statistic is a random variable of the form

$$S := \sum_{(x_1,\dots,x_q)\in\eta^q_{\neq}} f(x_1,\dots,x_q),$$

where $q \in \mathbb{N}$, $\eta^q_{\neq}$ stands for the set of all q-tuples of distinct points of η and $f \in L^1_s(\mu^q)$. Since we sum over all permutations of any combination of q distinct points of η, we can assume without loss of generality that f is symmetric. We denote q as order and refer to f as the kernel of S. The previous definition covers most of the relevant situations. However, our general framework even allows situations where the Poisson process is not given by its atoms, cf. [28, Section 12.3]. In this case, we mean by a Poisson U-statistic of order q a functional of the type

$$S := \int_{\mathbb{X}^q} f(x_1,\dots,x_q)\, \eta^{(q)}(\mathrm{d}(x_1,\dots,x_q))\,.$$

In the sequel we always use the notation with the sum since we believe that it is more intuitive and is the typical situation for most of our examples.

The Poisson U-statistic S is square integrable if and only if

$$\int_{\mathbb{X}^i}\left(\int_{\mathbb{X}^{q-i}} f(y_1,\dots,y_i,x_1,\dots,x_{q-i})\,\mu^{q-i}(\mathrm{d}(x_1,\dots,x_{q-i}))\right)^2\mu^i(\mathrm{d}(y_1,\dots,y_i))<\infty \quad (2.2)$$

for all $i\in\{0,\dots,q\}$ (see [28, Proposition 12.12] and [39, Section 3]). For $i=0$ and $i=q$ this means that $f\in L_s^1(\mu^q)$ and $f\in L_s^2(\mu^q)$, respectively. If (2.2) is satisfied, the functions $f_i:\mathbb{X}^i\to\mathbb{R}$, $i\in\{1,\dots,q\}$, given by

$$f_i(y_1,\dots,y_i):=\binom{q}{i}\int_{\mathbb{X}^{q-i}} f(y_1,\dots,y_i,x_1,\dots,x_{q-i})\,\mu^{q-i}(\mathrm{d}(x_1,\dots,x_{q-i})), \quad (2.3)$$

are square integrable. In [39, Section 3] it is shown that a square integrable Poisson U-statistic S has the representation

$$S=\mathbb{E}S+\sum_{i=1}^q I_i(f_i), \quad (2.4)$$

and that its variance is given by

$$\operatorname{Var}S=\sum_{i=1}^q i!\|f_i\|^2_{L^2(\mu^i)}\,, \quad (2.5)$$

compare with (2.1). The decomposition (2.4) is called the Wiener-Itô chaos expansion of S. We emphasise that any square-integrable Poisson functional F, i.e. any random variable depending on a Poisson process only, has a representation as a sum of its expectation and (possibly infinitely many) multiple Wiener-Itô integrals. In other words, (2.4) says that square-integrable Poisson U-statistics have a finite Wiener-Itô chaos expansion. On the other hand, in [39, Theorem 3.6] it is shown that any square-integrable Poisson functional with a finite Wiener-Itô chaos expansion having integrable kernels f_i, which, by the square-integrability of the Poisson functional, are automatically square-integrable, can be written as a sum of finitely many Poisson U-statistics and a constant.

2.3 The method of cumulants

In this section we present what is called the method of cumulants. We start with the definition of cumulants and by setting up the notation. For real-valued random variables $X_1,\dots,X_m$, $m\in\mathbb{N}$, the joint characteristic function $\varphi_{X_1,\dots,X_m}:\mathbb{R}^m\to\mathbb{C}$ is given by

$$\varphi_{X_1,\dots,X_m}(t_1,\dots,t_m):=\mathbb{E}\exp\left(\mathbf{i}\sum_{i=1}^m t_iX_i\right),$$

where $\mathbf{i}$ is the imaginary unit. The joint cumulant of $X_1,\dots,X_m$ is then defined as

$$\operatorname{cum}(X_1,\dots,X_m):=(-\mathbf{i})^m\frac{\partial^m}{\partial t_1\dots\partial t_m}\log\varphi_{X_1,\dots,X_m}(t_1,\dots,t_m)\Big|_{t_1=\dots=t_m=0}.$$

In the following we consider random variables that have finite moments of all orders. This implies that all joint cumulants of these random variables are well defined. Note that the joint cumulants are linear in each coordinate. For a real-valued random variable X and $m\in\mathbb{N}$ we shall write $\operatorname{cum}_m(X):=\operatorname{cum}(X,\dots,X)$ for the m^{th} cumulant of X.

Before we can summarise the main elements of the method of cumulants, we provide the definition of a moderate deviation principle. Let us recall from [10] that a sequence

$(\mathbb{P}_n)_{n\in\mathbb{N}}$ of probability measures on a topological space $\mathcal{Z}$ with σ-field $\mathscr{Z}$ satisfies a large deviation principle with speed $s_n \to \infty$ and good rate function $\mathcal{I}$ if the level sets $\{z\in \mathcal{Z} : \mathcal{I}(z) \le a\}$ are compact for all $0 \le a < \infty$ and if for all $A \in \mathscr{Z}$,

$$-\inf_{z\in\mathrm{int}(A)} \mathcal{I}(z) \le \liminf_{n\to\infty} s_n^{-1} \log \mathbb{P}_n(A) \le \limsup_{n\to\infty} s_n^{-1} \log \mathbb{P}_n(A) \le -\inf_{z\in\mathrm{cl}(A)} \mathcal{I}(z)\,,$$

where $\mathrm{int}(A)$ and $\mathrm{cl}(A)$ stand for the interior and the closure of A, respectively. Moreover, a sequence $(X_n)_{n\in\mathbb{N}}$ of random variables satisfies a LDP if their distributions do. We will speak about a moderate deviation principle (MDP) instead of a LDP if the scaling of the involved random variables is between that of a law of large numbers and that of a central limit theorem.

Now, let $(X_n)_{n\in\mathbb{N}}$ be a sequence of square-integrable random variables, $\gamma \ge 0$ be a constant and $(\Delta_n)_{n\in\mathbb{N}}$ be a positive real-valued sequence. To keep the presentation of our results more transparent, we introduce the following shorthand notation and say that $(X_n)_{n\in\mathbb{N}}$ satisfies

- $\mathbf{MDP}(\gamma, (\Delta_n)_{n\in\mathbb{N}})$ if for any positive real-valued sequence $(a_n)_{n\in\mathbb{N}}$ with

  $$\lim_{n\to\infty} a_n = \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{a_n}{\Delta_n^{1/(1+2\gamma)}} = 0$$

  the re-scaled random variables $(a_n^{-1}(X_n - \mathbb{E}X_n)/\sqrt{\mathrm{Var}\, X_n})_{n\in\mathbb{N}}$ satisfy a moderate deviation principle (MDP) with speed a_n^2 and good rate function $\mathcal{I}(z) = z^2/2$,
- $\mathbf{CI}(\gamma, (\Delta_n)_{n\in\mathbb{N}})$ if the Bernstein-type concentration inequality

  $$\mathbb{P}(|X_n - \mathbb{E}X_n| \ge z\sqrt{\mathrm{Var}\, X_n}) \le 2\exp\left(-\frac{1}{4}\min\left\{\frac{z^2}{2^{1+\gamma}},\, (z\Delta_n)^{1/(1+\gamma)}\right\}\right)$$

  holds for all $n \in \mathbb{N}$ and $z \ge 0$,
- $\mathbf{NACC}(\gamma, (\Delta_n)_{n\in\mathbb{N}})$ if a normal approximation bound with Cramér correction holds, that is, if there exist constants $c_0, c_1, c_2 > 0$ only depending on γ such that for all $n \in \mathbb{N}$ and $z \in [0, c_0\Delta_n^{1/(1+2\gamma)}]$,

  $$\mathbb{P}(X_n - \mathbb{E}X_n \ge z\sqrt{\mathrm{Var}\, X_n}) = e^{L^+_{n,z}}(1-\Phi(z))\Big(1 + c_1\theta^+_{n,z}\frac{1+z}{\Delta_n^{1/(1+2\gamma)}}\Big)$$

  and

  $$\mathbb{P}(X_n - \mathbb{E}X_n \le -z\sqrt{\mathrm{Var}\, X_n}) = e^{L^-_{n,z}}(1-\Phi(z))\Big(1 + c_1\theta^-_{n,z}\frac{1+z}{\Delta_n^{1/(1+2\gamma)}}\Big)$$

  with $\theta^+_{n,z}, \theta^-_{n,z} \in [-1,1]$ and $L^+_{n,z}, L^-_{n,z} \in (-c_2z^3/\Delta_n^{1/(1+2\gamma)}, c_2z^3/\Delta_n^{1/(1+2\gamma)})$, where Φ is the distribution function of a standard Gaussian random variable.

The main tool for proving our results are sharp estimates for cumulants and their implications. The next proposition is our main device. It collects findings taken from the monograph [41], the paper [14] and the survey article [15] (see also [46, Lemma 11]). It summarises fine probabilistic estimates, which are available under certain natural bounds on cumulants.

Proposition 2.1 (MDP, CI and NACC under cumulant bounds)**.** *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of real-valued random variables such that $\mathbb{E}[X_n] = 0$, $\mathbb{E}[X_n^2] = 1$ and $\mathbb{E}[|X_n|^m] < \infty$ for all $m \ge 3$. Suppose that there exist a constant $\gamma \ge 0$ and a positive real-valued sequence $(\Delta_n)_{n\in\mathbb{N}}$ such that*

$$|\,\mathrm{cum}_m(X_n)| \le \frac{(m!)^{1+\gamma}}{\Delta_n^{m-2}} \tag{2.6}$$

for all $m \ge 3$ and $n \in \mathbb{N}$. Then $(X_n)_{n\in\mathbb{N}}$ satisfies $\mathbf{MDP}(\gamma, (\Delta_n)_{n\in\mathbb{N}})$, $\mathbf{CI}(\gamma, (\Delta_n)_{n\in\mathbb{N}})$ and $\mathbf{NACC}(\gamma, (\Delta_n)_{n\in\mathbb{N}})$.

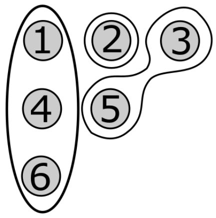

Figure 1: An element of $\Pi(3,2,1)$.

Remark 2.2. The cumulant bound (2.6) immediately implies a central limit theorem for the random variables $(X_n)_{n\in\mathbb{N}}$, as soon as $\Delta_n \to \infty$, if we let $n \to \infty$. In addition, it also delivers a bound for the speed of convergence in terms of the Kolmogorov distance defined as the sup-norm of the difference of the distribution function of X_n and that of a standard Gaussian random variable, see e.g. [41, Corollary 2.1]. However, since this leads for all the random variables we consider in this paper to rates that are weaker than those already available in the existing literature, we have decided not to pursue this direction in this text.

2.4 Partitions

Let $m \in \mathbb{N}$ and let $q_1, \dots, q_m \in \mathbb{N}$. We define $N_0 := 0$, $N_\ell := \sum_{i=1}^{\ell} q_i$, $\ell \in \{1,\dots,m\}$, and $N := N_m$, and put $J_\ell := \{N_{\ell-1}+1,\dots,N_\ell\}$, $\ell \in \{1,\dots,m\}$. A partition σ of $[N] := \{1,\dots,N\}$ is a collection $\{B_1,\dots,B_k\}$ of $1 \le k \le N$ pairwise disjoint non-empty sets, called blocks, such that $B_1 \cup \dots \cup B_k = [N]$. The number k of blocks of σ is denoted by $|\sigma|$. By $\Pi(q_1,\dots,q_m)$ we denote the set of all partitions σ such that $|B \cap J_\ell| \le 1$ for all $\ell \in \{1,\dots,m\}$ and $B \in \sigma$.

It is instructive to graphically represent a partition $\sigma \in \Pi(q_1,\dots,q_m)$ as follows. We imagine the $q_1+\dots+q_m$ elements of $\{1,\dots,q_1+\dots+q_m\}$ arranged in an array of m rows, where the numbers $1,\dots,q_1$ (i.e. the elements of J_1) form the first row, $q_1+1,\dots,q_1+q_2$ (i.e. the elements of J_2) the second row and so on. The blocks of the partition σ are then indicated by closed curves, where all elements encircled by the same curve belong to the same block of σ, see Figure 1.

Every partition $\sigma \in \Pi(q_1,\dots,q_m)$ induces a partition σ^* of $\{1,\dots,m\}$ in the following way: $i,j \in \{1,\dots,m\}$ are in the same block of σ^* whenever there is a block $B \in \sigma$ such that $|B \cap J_i| = 1$ and $|B \cap J_j| = 1$. Let $\widetilde{\Pi}(q_1,\dots,q_m)$ be the set of all partitions $\sigma \in \Pi(q_1,\dots,q_m)$ such that $|\sigma^*| = 1$. We remark that the partitions in $\Pi(q_1,\dots,q_m)$ and $\widetilde{\Pi}(q_1,\dots,q_m)$ are known as non-flat and connected non-flat, respectively, see e.g. [35, Chapter 4]. By $\Pi_{\ge 2}(q_1,\dots,q_m)$ and $\widetilde{\Pi}_{\ge 2}(q_1,\dots,q_m)$ we denote the sets of all $\sigma \in \Pi(q_1,\dots,q_m)$ and of all $\sigma \in \widetilde{\Pi}(q_1,\dots,q_m)$ such that $|B| \ge 2$ for all $B \in \sigma$. Finally, we introduce the set $\overline{\Pi}(q_1,\dots,q_m)$ of all partitions $\sigma \in \Pi(q_1,\dots,q_m)$ such that for each $\ell \in \{1,\dots,m\}$ there exists a block $B \in \sigma$ with $|B| \ge 2$ and $B \cap J_\ell \neq \varnothing$. In other words, in each row in the graphical representation of σ there exists at least one element, which belongs to some block $B \in \sigma$ with $|B| \ge 2$. In case that $q_1 = \dots = q_m = q$ we write $\Pi^m(q)$, $\widetilde{\Pi}^m(q)$, $\Pi^m_{\ge 2}(q)$, $\widetilde{\Pi}^m_{\ge 2}(q)$ and $\overline{\Pi}^m(q)$ instead of $\Pi(q_1,\dots,q_m)$, $\widetilde{\Pi}(q_1,\dots,q_m)$, $\Pi_{\ge 2}(q_1,\dots,q_m)$, $\widetilde{\Pi}_{\ge 2}(q_1,\dots,q_m)$ and $\overline{\Pi}(q_1,\dots,q_m)$, respectively.

For functions $f^{(\ell)} : \mathbb{X}^{q_\ell} \to \mathbb{R}$, $\ell \in \{1,\dots,m\}$, we define their tensor product $\otimes_{\ell=1}^m f^{(\ell)} : \mathbb{X}^N \to \mathbb{R}$ by

$$(\otimes_{\ell=1}^m f^{(\ell)})(x_1,\dots,x_N) := \prod_{\ell=1}^{m} f^{(\ell)}(x_{N_{\ell-1}+1},\dots,x_{N_\ell})\,.$$

For $\sigma \in \Pi(q_1,\dots,q_m)$ the function $(\otimes_{\ell=1}^m f^{(\ell)})_\sigma : \mathbb{X}^{|\sigma|} \to \mathbb{R}$ is obtained by replacing in $(\otimes_{\ell=1}^m f^{(\ell)})$ all variables that belong to the same block of σ by a new common variable. Note that this way $(\otimes_{\ell=1}^m f^{(\ell)})_\sigma$ is only defined up to permutations of its arguments. Since in what follows we always integrate with respect to all arguments of $(\otimes_{\ell=1}^m f^{(\ell)})_\sigma$, this does not cause problems.

3 Main results

3.1 Moderate deviation estimates

We are now prepared to present the main results of this paper. They show that finite sums of multiple stochastic integrals on the Poisson space as well as finite sums of Poisson U-statistics satisfy $\mathbf{MDP}(\gamma,(\Delta_n)_{n\in\mathbb{N}})$, $\mathbf{CI}(\gamma,(\Delta_n)_{n\in\mathbb{N}})$ as well as $\mathbf{NACC}(\gamma,(\Delta_n)_{n\in\mathbb{N}})$, and we determine the parameters γ and $(\Delta_n)_{n\in\mathbb{N}}$ in both situations. In fact, it will turn out later that the parameter γ we obtain cannot be improved systematically. In the next two theorems we implicitly assume that all occurring integrals are well defined. We start with the result for multiple Wiener-Itô integrals.

Theorem 3.1 (MDP, CI and NACC for multiple integrals)**.** *Let $(\eta_n)_{n\in\mathbb{N}}$ be a family of Poisson processes over σ-finite measure spaces $((\mathbb{X}_n,\mathcal{X}_n,\mu_n))_{n\in\mathbb{N}}$ and let $f_n^{(i)} \in L_s^2(\mu_n^{q_i})$, $i \in \{1,\dots,k\}$, $n\in\mathbb{N}$, with distinct $q_1,\dots,q_k\in\mathbb{N}$ and $k\in\mathbb{N}$ be such that $\sum_{i=1}^k \|f_n^{(i)}\|^2_{L^2(\mu_n^{q_i})} > 0$ for all $n\in\mathbb{N}$. Define $q := \max\{q_1,\dots,q_k\}$ and let $Y_n := \sum_{i=1}^k I_{q_i}(f_n^{(i)})$ for $n\in\mathbb{N}$. Assume that there is a positive real-valued sequence $(\alpha_n)_{n\in\mathbb{N}}$ such that, for any $n\in\mathbb{N}$,*

$$(\operatorname{Var} Y_n)^{-m/2}\left|\int_{\mathbb{X}_n^{|\sigma|}} (\otimes_{\ell=1}^m f_n^{(i_\ell)})_\sigma \, \mathrm{d}\mu_n^{|\sigma|}\right| \le \alpha_n^{m-2} \tag{3.1}$$

for all $\sigma\in\widetilde{\Pi}_{\ge 2}(q_{i_1},\dots,q_{i_m})$, $i_1,\dots,i_m\in\{1,\dots,k\}$ and $m\ge 3$. Then $(Y_n)_{n\in\mathbb{N}}$ satisfies $\mathbf{MDP}(q-1,(c_{q,k}\alpha_n^{-1})_{n\in\mathbb{N}})$, $\mathbf{CI}(q-1,(c_{q,k}\alpha_n^{-1})_{n\in\mathbb{N}})$ and $\mathbf{NACC}(q-1,(c_{q,k}\alpha_n^{-1})_{n\in\mathbb{N}})$ with $c_{q,k} := 1/(kq^q)^3$.

We can compare the assumptions of Theorem 3.1 with those of the corresponding result for multiple Wiener-Itô integrals on the Wiener space proved in [46]. In the latter case, it was sufficient to impose a condition on the *fourth* cumulant only in order to bound all higher-order cumulants and to deduce **MDP**, **CI** and **NACC**. In contrast, our condition (3.1) (and also the condition (3.2) in case of Poisson U-statistics below) impose restrictions to *all* cumulants of order $m\ge 3$ simultaneously. In view of the complicated combinatorial structure and in view of the absent hypercontractivity property on the Poisson space, this is unavoidable by our method, which is based on sharp cumulant bounds.

Next, we shall discuss a version of Theorem 3.1 for sums of Poisson U-statistics.

Theorem 3.2 (MDP, CI and NACC for Poisson U-statistics)**.** *Let $(\eta_n)_{n\in\mathbb{N}}$ be a family of Poisson processes over σ-finite measure spaces $((\mathbb{X}_n,\mathcal{X}_n,\mu_n))_{n\in\mathbb{N}}$ and let $f_n^{(i)} : \mathbb{X}_n^{q_i}\to\mathbb{R}$, $i\in\{1,\dots,k\}$, $n\in\mathbb{N}$, with distinct $q_1,\dots,q_k\in\mathbb{N}$ and $k\in\mathbb{N}$, be measurable and satisfy* (2.2) *and $\sum_{i=1}^k \|f_n^{(i)}\|^2_{L^2(\mu_n^{q_i})} > 0$ for all $n\in\mathbb{N}$. Define $q:=\max\{q_1,\dots,q_k\}$ and the random variables*

$$Z_n := \sum_{i=1}^k S_n^{(i)} \qquad \textit{with} \qquad S_n^{(i)} := \sum_{(x_1,\dots,x_{q_i})\in\eta_{n,\neq}^{q_i}} f_n^{(i)}(x_1,\dots,x_{q_i}), \quad i\in\{1,\dots,k\},$$

for $n\in\mathbb{N}$. Assume that there is a positive real-valued sequence $(\beta_n)_{n\in\mathbb{N}}$ such that, for any $n\in\mathbb{N}$,

$$(\operatorname{Var} Z_n)^{-m/2}\left|\int_{\mathbb{X}_n^{|\sigma|}} (\otimes_{\ell=1}^m f_n^{(i_\ell)})_\sigma \, \mathrm{d}\mu_n^{|\sigma|}\right| \le \beta_n^{m-2} \tag{3.2}$$

for all $\sigma \in \widetilde{\Pi}(q_{i_1}, \dots, q_{i_m})$*,* $i_1, \dots, i_m \in \{1, \dots, k\}$ *and* $m \geq 3$*. Then* $(Z_n)_{n\in\mathbb{N}}$ *satisfies* $\mathbf{MDP}(q-1, (c_{q,k}\beta_n^{-1})_{n\in\mathbb{N}})$*,* $\mathbf{CI}(q-1, (c_{q,k}\beta_n^{-1})_{n\in\mathbb{N}})$ *and* $\mathbf{NACC}(q-1, (c_{q,k}\beta_n^{-1})_{n\in\mathbb{N}})$ *with* $c_{q,k} := 1/(kq^q)^3$*.*

Remark 3.3. The assumption $\sum_{i=1}^k \|f_n^{(i)}\|^2_{L^2(\mu_n^{q_i})} > 0$ for all $n \in \mathbb{N}$ in Theorems 3.1 and 3.2 ensures that, for all $n \in \mathbb{N}$, $\operatorname{Var} Y_n > 0$ and $\operatorname{Var} Z_n > 0$, respectively. Indeed, by (2.1), we have

$$\operatorname{Var} Y_n = \sum_{i=1}^k q_i! \|f_n^{(i)}\|^2_{L^2(\mu_n^{q_i})} \geq \sum_{i=1}^k \|f_n^{(i)}\|^2_{L^2(\mu_n^{q_i})} > 0.$$

In case of Z_n there exists a unique $i_n \in \{1, \dots, k\}$ such that $q_{i_n} = \max\{q_1, \dots, q_k : \|f_n^{(i)}\|^2_{L^2(\mu_n^{q_i})} > 0\}$. From (2.4) we deduce that the q_{i_n}-th kernel of the chaos expansion of Z_n is $f_n^{(i_n)}$, whence, by (2.1), $\operatorname{Var} Z_n \geq q_{i_n}! \|f_n^{(i_n)}\|^2_{L^2(\mu_n^{q_{i_n}})} > 0$.

Remark 3.4. Moderate deviation principles and also concentration inequalities are well known in the case of 'classical' U-statistics based on fixed numbers of i.i.d. random variables, see e.g. [18] and [31]. Although the classical U-statistics and the Poisson U-statistics we consider are close in the L^2-sense by a result of Dynkin and Mandelbaum [16], this is not sufficient to push the classical results to the Poisson case since we investigate U-statistics on an exponential scale. In addition we remark that – in sharp contrast to typical assumptions imposed on classical U-statistics – the random variables Y_n and Z_n considered in Theorem 3.1 and 3.2 satisfy $\mathbb{E}[e^{sY_n}] = \mathbb{E}[e^{sZ_n}] = \infty$ for any $s > 0$, provided that $f_n^{(i)} \geq 0$ for all $i \in \{1, \dots, k\}$ and that for some $i \in \{1, \dots, k\}$, $q_i \geq 2$ and $\|f_n^{(i)}\|_{L^2(\mu_n^{q_i})} > 0$ (see [29, Corollary 2]).

Theorem 3.1 and Theorem 3.2 imply a moderate deviation principle for a range of scales $(a_n)_{n\in\mathbb{N}}$ 'close' to that in the related central limit theorem, which in turn would correspond to the choice $a_n = 1$ for all $n \in \mathbb{N}$. We shall now discuss whether or not it is possible in general to enlarge this range of scales. To this end we use a simple example based on a sum of independent and identically distributed random variables. Let us denote for $q \in \{0, 1, 2, \dots\}$ by H_q the q^{th} Poisson-Charlier polynomial for the Poisson distribution with parameter 1. This family of orthogonal polynomials is recursively defined by

$$H_0(x) := 1 \qquad \text{and} \qquad H_{q+1}(x) := xH_q(x-1) - H_q(x), \qquad x \in \mathbb{R},$$

for integers $q \geq 0$, see e.g. [35, Equation (10.0.2)]. For example,

$$\begin{aligned} H_1(x) &= x - 1, \\ H_2(x) &= x^2 - 3x + 1, \\ H_3(x) &= x^3 - 6x^2 + 8x - 1, \\ H_4(x) &= x^4 - 10x^3 + 29x^2 - 24x + 1. \end{aligned}$$

In particular, H_q is a polynomial of degree q with leading coefficient equal to 1.

Theorem 3.5. *Fix* $q \in \mathbb{N}$ *and let* $(Z_k)_{k\in\mathbb{N}}$ *be a sequence of independent and Poisson distributed random variables with parameter* 1*. For each* $n \in \mathbb{N}$ *define* $S_n := \sum_{k=1}^n H_q(Z_k)$ *and let* $(a_n)_{n\in\mathbb{N}}$ *be a sequence of positive real numbers such that* $\lim_{n\to\infty} a_n = \infty$ *and* $\lim_{n\to\infty} a_n/\sqrt{n} = 0$*.*

a) Assume that

$$\lim_{n\to\infty} a_n^{-2+1/q} n^{1/(2q)} \log(a_n\sqrt{n}) = \infty.$$

Then the sequence of random variables $\left(\frac{S_n}{a_n\sqrt{nq!}}\right)_{n\in\mathbb{N}}$ *satisfies a MDP with speed* a_n^2 *and good rate function* $\mathcal{I}(z) = z^2/2$*.*

b) Assume that

$$\lim_{n\to\infty} a_n^{-2+1/q} n^{1/(2q)} \log(a_n\sqrt{n}) < \infty.$$

Then, the sequence of random variables $\big(\frac{S_n}{a_n\sqrt{nq!}}\big)_{n\in\mathbb{N}}$ *cannot satisfy a MDP with a good rate function* $\mathcal{I}(z)$ *satisfying* $\mathcal{I}(z) > 0$ *for* $z \neq 0$ *and with* $\mathcal{I}(z) \to \infty$*, as* $z \to \pm\infty$*.*

To discuss the relation between Theorem 3.5, which is not derived by the method of cumulants but by findings from [2, 17] for sums of i.i.d. random variables, and Theorem 3.1, we let η be a Poisson process over a σ-finite measure space $(\mathbb{X}, \mathcal{X}, \mu)$ and Z_1 be a Poisson random variable with parameter 1. Then, for every $q \in \mathbb{N}$ and every $B \in \mathcal{X}$ with $\mu(B) = 1$ we have that the random variable $H_q(Z_1)$ and the multiple Wiener-Itô integral $I_q(g_B)$ with $g_B(z_1,\dots,z_q) := \mathbf{1}_B^{\otimes q}(z_1,\dots,z_q) := \mathbf{1}_B(z_1)\cdots\mathbf{1}_B(z_q)$ are identically distributed, see [35, Proposition 10.0.2]. Thus, for each $n \in \mathbb{N}$, S_n has the same distribution as $I_q(f_n)$ with $f_n := \sum_{i=1}^n g_{B_i}$ with pairwise disjoint measurable subsets $(B_i)_{i\in\mathbb{N}}$ of $\mathbb{X}$ with $\mu(B_i) = 1$ for all $i \in \mathbb{N}$. For $m \in \mathbb{N}$ with $m \geq 3$ and $\sigma \in \widetilde{\Pi}^m_{\geq 2}(q)$, we obtain

$$\int_{\mathbb{X}^{|\sigma|}} (\otimes_{\ell=1}^m f_n)_\sigma \,\mathrm{d}\mu^{|\sigma|} = \sum_{i_1,\dots,i_m=1}^n \int_{\mathbb{X}^{|\sigma|}} (\otimes_{\ell=1}^m g_{B_{i_\ell}})_\sigma \,\mathrm{d}\mu^{|\sigma|} = \sum_{i=1}^n \int_{\mathbb{X}^{|\sigma|}} (\otimes_{\ell=1}^m g_{B_i})_\sigma \,\mathrm{d}\mu^{|\sigma|} = n,$$

where we used the construction of the $(g_{B_i})_{i\in\mathbb{N}}$, the pairwise disjointness of $(B_i)_{i\in\mathbb{N}}$ and $\mu(B_i) = 1$ for $i \in \mathbb{N}$. Consequently, the random variables $(S_n/\sqrt{nq!})_{n\in\mathbb{N}}$ satisfy (3.1) with $\alpha_n := \frac{1}{\sqrt{n}}$. So, Theorem 3.1 implies that if $(a_n)_{n\in\mathbb{N}}$ is a sequence of positive real numbers such that

$$\lim_{n\to\infty} a_n = \infty \qquad \text{and} \qquad \lim_{n\to\infty} \frac{a_n}{\alpha_n^{-1/(2q-1)}} = \lim_{n\to\infty} \frac{a_n}{n^{1/(4q-2)}} = 0, \tag{3.3}$$

then the sequence of random variables $(S_n/(a_n\sqrt{nq!}))_{n\in\mathbb{N}}$ satisfies a MDP with speed a_n^2 and good rate function $\mathcal{I}(z) = z^2/2$. However, the growth condition on a_n we just obtained by means of the method of cumulants coincides up to subpolynomial factors with the optimal one from Theorem 3.5, which in turn is based on a different method. Indeed, it is easy to verify that the condition of a) in Theorem 3.5 is satisfied in case of (3.3), while the condition of b) holds if

$$\lim_{n\to\infty} \frac{a_n}{n^{\varepsilon+1/(4q-2)}} = \infty$$

for some $\varepsilon > 0$. In other words this means that the polynomial order of the range of scalings in Theorem 3.1 and, thus, presumably also in Theorem 3.2 cannot be improved systematically. On the other hand, this does not necessarily exclude the possibility that for special choices of functions f_n the MDP might hold beyond this range of scales.

We also note in this context that such an optimality result is not available for sequences of multiple stochastic integrals on a Wiener space. In fact, it has been argued in [46] that there exists a non-trivial interval of scales a_n for which it is not clear whether or not a moderate deviation principle is satisfied in general. We find it rather remarkable that despite the much more involved combinatorial nature of stochastic integrals on Poisson spaces, such a gap does not exist in this set-up.

3.2 Product formulas

One of the main devices for deriving sharp bounds on cumulants of multiple stochastic integrals and Poisson U-statistics are explicit combinatorial formulas for the (joint) cumulants of such random variables. The moment and cumulant formulas provided in this section are known, but we were able to derive them under weaker – sometimes even minimal – integrability assumptions in comparison with the existing literature. We consider the same framework as in Section 2.

Theorem 3.6 (Moment and cumulant formulas for stochastic integrals)**.** *Let* $m \in \mathbb{N}$*, let* $\widetilde{m} \in \mathbb{N}$ *with* $\widetilde{m} \geq m$ *be even and let* $f^{(\ell)} \in L^2_s(\mu^{q_\ell})$ *with* $q_\ell \in \mathbb{N}$*,* $\ell \in \{1, \ldots, m\}$*, be such that*

$$\int_{\mathbb{X}^{|\sigma|}} |(\otimes_{\ell=1}^{\widetilde{m}} f^{(i)})_\sigma| \, \mathrm{d}\mu^{|\sigma|} < \infty, \qquad \sigma \in \Pi^{\widetilde{m}}_{\geq 2}(q_i),\, i \in \{1, \ldots, m\}, \tag{3.4}$$

$$\int_{\mathbb{X}^{|\sigma|}} \big|(\otimes_{\ell=1}^{m} f^{(\ell)})_\sigma\big| \, \mathrm{d}\mu^{|\sigma|} < \infty, \qquad \sigma \in \Pi_{\geq 2}(q_1, \ldots, q_m)\,. \tag{3.5}$$

Then,

$$\mathbb{E}\Big[\prod_{\ell=1}^{m} I_{q_\ell}(f^{(\ell)})\Big] = \sum_{\sigma \in \Pi_{\geq 2}(q_1,\ldots,q_m)} \int_{\mathbb{X}^{|\sigma|}} (\otimes_{\ell=1}^{m} f^{(\ell)})_\sigma \, \mathrm{d}\mu^{|\sigma|}\,,$$

$$\mathrm{cum}(I_{q_1}(f^{(1)}), \ldots, I_{q_m}(f^{(m)})) = \sum_{\sigma \in \widetilde{\Pi}_{\geq 2}(q_1,\ldots,q_m)} \int_{\mathbb{X}^{|\sigma|}} (\otimes_{\ell=1}^{m} f^{(\ell)})_\sigma \, \mathrm{d}\mu^{|\sigma|}\,.$$

Formulas as those for the moments and cumulants in the previous theorem are known as product or diagram formulas in the literature. This line of research for the Poisson case goes back to the work [47]. There as well as in [28, 29, 35, 43] such formulas were derived under stronger integrability assumptions. For m even and $f^{(1)} = \ldots = f^{(m)}$, (3.4) and (3.5) only require that the integrals appearing on the right-hand side of the moment formula exist as finite numbers, whence one can regard the integrability assumptions as minimal. The related but different problem of not only computing the expectation but the whole chaos expansion of a product of multiple Wiener-Itô integrals is studied, for example, in [12, 27, 35, 47].

Next, we present corresponding moment and a cumulant formulas for Poisson U-statistics.

Theorem 3.7 (Moment and cumulant formulas for Poisson U-statistics)**.** *Let* $m \in \mathbb{N}$*, let* $\widetilde{m} \in \mathbb{N}$ *with* $\widetilde{m} \geq m$ *be even and let* $f^{(\ell)} \in L^1_s(\mu^{q_\ell})$ *with* $q_\ell \in \mathbb{N}$*,* $\ell \in \{1, \ldots, m\}$*, be such that*

$$\int_{\mathbb{X}^{|\sigma|}} |(\otimes_{\ell=1}^{\widetilde{m}} f^{(i)})_\sigma| \, \mathrm{d}\mu^{|\sigma|} < \infty, \qquad \sigma \in \overline{\Pi}^{\widetilde{m}}(q_i),\, i \in \{1, \ldots, m\}, \tag{3.6}$$

$$\int_{\mathbb{X}^{|\sigma|}} \big|(\otimes_{\ell=1}^{m} f^{(\ell)})_\sigma\big| \, \mathrm{d}\mu^{|\sigma|} < \infty, \qquad \sigma \in \overline{\Pi}(q_1, \ldots, q_m)\,. \tag{3.7}$$

Then, for $S_\ell := \sum_{(x_1,\ldots,x_{q_\ell}) \in \eta^{q_\ell}_{\neq}} f^{(\ell)}(x_1, \ldots, x_{q_\ell})$*,* $\ell \in \{1, \ldots, m\}$*,*

$$\mathbb{E}\Big[\prod_{\ell=1}^{m} (S_\ell - \mathbb{E}S_\ell)\Big] = \sum_{\sigma \in \overline{\Pi}(q_1,\ldots,q_m)} \int_{\mathbb{X}^{|\sigma|}} (\otimes_{\ell=1}^{m} f^{(\ell)})_\sigma \, \mathrm{d}\mu^{|\sigma|}\,,$$

$$\mathrm{cum}(S_1, \ldots, S_m) = \sum_{\sigma \in \widetilde{\Pi}(q_1,\ldots,q_m)} \int_{\mathbb{X}^{|\sigma|}} (\otimes_{\ell=1}^{m} f^{(\ell)})_\sigma \, \mathrm{d}\mu^{|\sigma|}\,.$$

Note that the difference between the formulas for the joint moments and cumulants of multiple Wiener-Itô integrals in Theorem 3.6 and those in Theorem 3.7 for Poisson U-statistics is the appearance of different sets of partitions one has to sum over. We remark that the formulas in Theorem 3.7 generalise [28, Proposition 12.13], [29, Corollary 1] and [43, Corollary 3.5].

4 Applications

4.1 U-statistics with fixed kernel and k-geodesic processes

We start our collection of applications by considering Poisson U-statistics whose kernel function does not depend on the intensity parameter of the underlying Poisson process.

Corollary 4.1. *Let $(\mathbb{X}, \mathcal{X}, \mu)$ be a probability space, let $(t_n)_{n\in\mathbb{N}}$ be a sequence of real numbers satisfying $t_n \geq 1$ and $t_n \to \infty$, as $n \to \infty$, and let $f : \mathbb{X}^q \to \mathbb{R}$ for some $q \in \mathbb{N}$ be measurable and such that $\|f\|_\infty := \sup\{|f(x)| : x \in \mathbb{X}^q\} < \infty$ and*

$$v_f := q^2 \int_{\mathbb{X}} \left(\int_{\mathbb{X}^{q-1}} f(x, x_1, \dots, x_{q-1}) \, \mu^{q-1}(\mathrm{d}(x_1, \dots, x_{q-1})) \right)^2 \mu(\mathrm{d}x) > 0 \,. \tag{4.1}$$

For each $n \in \mathbb{N}$, let η_n be a Poisson process on $\mathbb{X}$ with intensity measure $t_n\mu$. Then the sequence $(S_n)_{n\in\mathbb{N}}$ given by

$$S_n := \sum_{(x_1,\dots,x_q)\in\eta^q_{n,\neq}} f(x_1, \dots, x_q)$$

satisfies $\mathbf{MDP}(q-1, (\tau_n)_{n\in\mathbb{N}})$, $\mathbf{CI}(q-1, (\tau_n)_{n\in\mathbb{N}})$ and $\mathbf{NACC}(q-1, (\tau_n)_{n\in\mathbb{N}})$ with $\tau_n := \sqrt{t_n}/(q^{3q} \max\{(\|f\|_\infty/\sqrt{v_f})^3, \|f\|_\infty/\sqrt{v_f}\})$.

The previous corollary can be applied, for example, to Poisson hyperplane or Poisson k-flat processes in $\mathbb{R}^d$, one of the principal models considered in stochastic geometry, cf. [42]. Following [5, 20, 21, 23] we treat this model in greater generality and denote for $d \geq 2$ and $\kappa \in \{-1, 0, 1\}$ by $\mathbb{M}^d_\kappa$ the d-dimensional standard space of constant curvature κ. As a model for $\mathbb{M}^d_1$ we can take the d-dimensional unit sphere $\mathbb{S}^d \subset \mathbb{R}^{d+1}$, for $\mathbb{M}^d_0$ the d-dimensional Euclidean space $\mathbb{R}^d$ and for $\mathbb{M}^d_{-1}$ the Beltrami-Klein model in the interior $B^d := \{x = (x_1, \dots, x_d) \in \mathbb{R}^d : x_1^2 + \dots + x_d^2 < 1\}$ of the d-dimensional unit ball, see [38, Chapter 6]. For $k \in \{1, \dots, d-1\}$ a k-geodesic of $\mathbb{M}^d_\kappa$ is a totally geodesic k-dimensional submanifold of $\mathbb{M}^d_\kappa$ and by $A_\kappa(d, k)$ we indicate the space of k-geodesics in $\mathbb{M}^d_\kappa$. In the language of our model spaces, the elements of $A_1(d, k)$ arise as intersections of $\mathbb{S}^d$ with $(k+1)$-dimensional linear subspaces of $\mathbb{R}^{d+1}$, $A_0(d, k)$ is the space of k-dimensional affine subspaces of $\mathbb{R}^d$ and each k-geodesic in the Beltrami-Klein model for $\mathbb{M}^d_{-1}$ is the non-empty intersection of B^d with an element from $A_0(d, k)$. The spaces $A_\kappa(d, k)$ carry natural measures $\mu^d_{k,\kappa}$ which are invariant under the action of the full isometry group of $\mathbb{M}^d_\kappa$ and are unique up to a multiplicative factor. Since $\mathbb{M}^d_1$ is a compact space, we can take for $\mu^d_{k,1}$ the invariant probability measure, whereas for $\mu^d_{k,0}$ we choose the same normalisation as in [42] and for $\mu^d_{k,-1}$ the normalisation as in [20].

Next, we fix a Borel set $W \subset \mathbb{M}^d_\kappa$ such that $c_{k,\kappa,W} := \mu_{k,\kappa}(\{E \in A_\kappa(d, k) : E \cap W \neq \emptyset\}) \in (0, \infty)$ and denote by $\mu^d_{k,\kappa,W}$ the restriction of $c^{-1}_{k,\kappa,W}\mu^d_{k,\kappa}$ to $\{E \in A_\kappa(d, k) : E \cap W \neq \emptyset\}$. Now, let $(t_n)_{n\in\mathbb{N}}$ be a sequence such that $t_n \geq 1$ for any $n \in \mathbb{N}$ and $t_n \to \infty$, as $n \to \infty$, and for each $n \in \mathbb{N}$, let $\eta_n = \eta_{k,\kappa,W,n}$ be a Poisson process on $A_\kappa(d, k)$ with intensity measure $t_n\mu^d_{k,\kappa,W}$. In the classical Euclidean case $\kappa = 0$, η_n is called a Poisson k-flat process and we refer to [42] for further details and background material concerning such processes and their most fundamental properties. Spherical and hyperbolic processes of k-geodesics were only recently studied in detail in [20, 21, 23] for $k = d-1$ and in [5] for general k, respectively.

Assume now that $2k \geq d$ and let $q \in \mathbb{N}$ be such that $d - q(d-k) \geq 0$. The intersection process of order q induced by η_n arises by considering the intersections of any q pairwise different k-flats from η_n. This intersection is almost surely either empty or an element of

$A_\kappa(d, d-q(d-k))$, see [5, Lemma 2.2]. Now, for each $n \in \mathbb{N}$ define the Poisson U-statistic

$$S_n := \frac{1}{q!} \sum_{(E_1,\dots,E_q)\in\eta^q_{n,\neq}} \mathcal{H}^{d-q(d-k)}_\kappa(E_1 \cap \ldots \cap E_q \cap W)$$

of order q with fixed kernel $\mathcal{H}^{d-q(d-k)}_\kappa(\,\cdot\, \cap W)/q!$, where $\mathcal{H}^{d-q(d-k)}_\kappa$ denotes the $(d-q(d-k))$-dimensional Hausdorff measure with respect to the Riemannian metric in $\mathbb{M}^d_\kappa$. In other words, S_n is the total $(d-q(d-k))$-volume of the trace of the intersection process of order q induced by η_n within W.

The U-statistics $(S_n)_{n\in\mathbb{N}}$ satisfy the assumptions of Corollary 4.1 with explicitly known constants in (4.1), see [29, Section 6] for the Euclidean case $\kappa = 0$ and [5] for general κ (in fact, the constant v_f is implicit in [5, Proposition 3.1]) and $\|f\|_\infty = \max\{\mathcal{H}^{d-q(d-k)}(F\cap W) : F \in A_\kappa(d, d-q(d-k))\} < \infty$. Thus, Corollary 4.1 yields moderate deviation principles, concentration inequalities as well as a normal approximation bound with Cramér correction in this situation.

4.2 Subgraph counts in random geometric graphs

Let $(t_n)_{n\in\mathbb{N}}$ be a positive real-valued sequence such that $t_n \to \infty$, as $n \to \infty$, and let $(r_n)_{n\in\mathbb{N}}$ be a positive real-valued sequence. Further, we fix a compact convex set $W \subset \mathbb{R}^d$ with interior points and let, for each $n \in \mathbb{N}$, η_n be a Poisson process whose intensity measure μ_n is t_n times the restriction of the Lebesgue measure to W. Based on this data, we construct the random geometric graph $\mathrm{RGG}(\eta_n, r_n)$ by taking the points of η_n as the vertices of the graph and by connecting two distinct points by an edge whenever their (Euclidean) distance is strictly positive and does not exceed the given threshold r_n.

We are interested in the subgraph counting statistics associated with the random geometric graph. To define them, let G be a fixed connected graph with q vertices, and for $x_1, \dots, x_q \in W$ we let $f_=(x_1, \dots, x_q; G, r_n)$ be $1/q!$ times the indicator that the random geometric graph $\mathrm{RGG}(\{x_1, \dots, x_q\}, r_n)$ is isomorphic to G, while $f_\subset(x_1, \dots, x_q; G, r_n)$ is $1/q!$ times the number of subgraphs of $\mathrm{RGG}(\{x_1, \dots, x_q\}, r_n)$ that are isomorphic to G. The subgraph counting statistics are given by

$$S^\diamondsuit_n(G) := \sum_{(x_1,\dots,x_q)\in\eta^q_{n,\neq}} f_\diamondsuit(x_1, \dots, x_q; G, r_n)\,, \quad \diamondsuit \in \{=, \subset\}.$$

Here, $S^=_n(G)$ and $S^\subset_n(G)$ are the numbers of induced and non-induced copies of G in $\mathrm{RGG}(\eta_n, r_n)$. For example, if G is the graph with three vertices and two edges, three vertices connected by three edges in $\mathrm{RGG}(\eta_n, r_n)$ are counted thrice in $S^\subset_n(G)$ but do not to contribute to $S^=_n(G)$. From Theorem 3.2 we can deduce the following result.

Corollary 4.2. *For $k \in \mathbb{N}$, $a_1, \dots, a_k \in \mathbb{R}\setminus\{0\}$, $\diamondsuit_1, \dots, \diamondsuit_k \in \{=, \subset\}$ and connected graphs $G_1, \dots, G_k$ such that G_i has q_i vertices for $i \in \{1, \dots, k\}$, let $S_n := \sum_{i=1}^k a_i S^{\diamondsuit_i}_n(G_i)$, and define $p := \min\{q_1, \dots, q_k\}$, $q := \max\{q_1, \dots, q_k\}$ and $a := \max\{|a_1|, \dots, |a_k|\}$. Assume that $t_n \to \infty$, as $n \to \infty$, and that there is a constant $v > 0$ such that*

$$\operatorname{Var} S_n \geq v \max\{t_n^{2q-1}(\kappa_d r_n^d)^{2q-2}, t_n^p(\kappa_d r_n^d)^{p-1}\}\,, \qquad n \in \mathbb{N}\,. \tag{4.2}$$

Then the sequence $(S_n)_{n\in\mathbb{N}}$ satisfies $\mathbf{MDP}(q-1, (\tau_n)_{n\in\mathbb{N}})$, $\mathbf{CI}(q-1, (\tau_n)_{n\in\mathbb{N}})$ and $\mathbf{NACC}(q-1, (\tau_n)_{n\in\mathbb{N}})$ with τ_n given by

$$\tau_n := \frac{\sqrt{v t_n \min\{1, t_n \kappa_d r_n^d\}^{p-1}}}{(kq^q)^3 \max\{a, a^3 \operatorname{Vol}(W)/v\}}\,, \qquad n \in \mathbb{N}\,.$$

Our concentration inequality for the subgraph counting statistic is not the first one in this direction that appeared in the literature. More precisely, for a fixed connected graph

G with $q \geq 2$ vertices, it was shown in [4, Theorem 1.1] that, essentially, the subgraph counting statistic $S_n^{\subset}(G)$ satisfies a concentration inequality of the form

$$\mathbb{P}(|S_n^{\subset}(G) - \mathbb{E}S_n^{\subset}(G)| \geq z) \leq 2\exp(-c_{d,q}z^{1/q}), \qquad z \geq 0,$$

where $c_{d,q} \in (0,\infty)$ is a constant that only depends on the space dimension d as well as on the vertex number q of G. We emphasise that our Corollary 4.2 yields a result of a similar nature and especially shows the same exponent $1/q$ for z (for the special case of the edge counting statistic corresponding to the choice $q = 2$ we refer to [3, 40] and the discussions therein). On the other hand, our framework allows to deal simultaneously with a finite number of graphs and also leads to moderate deviation principles and normal approximation bounds with Cramér correction, which have no counterparts in the existing literature.

Remark 4.3. The assumption on the existence of a strictly positive constant $v > 0$ in the lower variance bound (4.2) for S_n is always satisfied if we choose $k = 1$ and $\diamondsuit_1 =$ '$\subset$'. For $\diamondsuit_1 =$ '$=$' we additionally need to assume that G_1 is feasible in the sense that $\mathbb{P}(\mathrm{RGG}(\{X_1,\dots,X_{q_1}\},r)$ is isomorphic to $G_1) > 0$ for some $r > 0$ and independent random points $X_1,\dots,X_{q_1}$ uniformly distributed on W, cf. [37, Chapter 3] for a discussion of the latter concept. This follows, for example, from the results in [37, Chapter 3.3] and especially from the remark after Proposition 3.7 there. On the other hand, if $k \geq 2$ the assumption seems unavoidable and does not need to be satisfied in general. For example, if $k = 2$ and if we take $G_1 = G_2 = G$ for some connected graph G (on $q \geq 2$ vertices) as well as $a_1 = -a_2$, then S_n is identically zero with probability one and so (4.2) is satisfied only with $v = 0$.

Remark 4.4. A generalisation of random geometric graphs is the random connection model, where similarly to the Erdős-Rényi random graph it is decided independently for each pair of points of the underlying Poisson process if they are connected by an edge, but the probability of an edge depends on the relative spatial position of the vertices. In the recent preprint [30], for subgraph counts of random connection models cumulant bounds are derived in order to establish rates of convergence for the normal approximation in Kolmogorov distance. Although the random geometric graph is a special case of the random connection model, the cumulant estimates required for our Corollary 4.2 do not follow directly from the findings of [30], since this paper works with rescalings of a fixed connection function, which does not cover the situation of Corollary 4.2 with the two parameters t and r_t. However, the moment and cumulant formulas for subgraph counts in [30] are very similar to our general formulas for Poisson U-statistics, since subgraph counts can be seen as U-statistics of a Poisson process with an additional randomisation coming from drawing edges randomly. In particular, the same classes of partitions are used. The lower bound in our Proposition 6.1 implies that the exponent r in Lemma 2.6 of [30] is in fact optimal.

4.3 Quadratic functionals of Ornstein-Uhlenbeck Lévy processes

Let η be a Poisson process on $\mathbb{R} \times \mathbb{R}$ with intensity measure $\mu := \lambda \otimes \nu$, where λ is the Lebesgue measure on $\mathbb{R}$ and ν is a σ-finite measure on $\mathbb{R}$ with $\int_{\mathbb{R}} u^2\,\nu(\mathrm{d}u) = 1$. We denote by $\hat{\eta}$ the compensated Poisson process $\eta - \mu$. For a fixed parameter $\varrho > 0$ the Ornstein-Uhlenbeck Lévy process $(U_t)_{t\in\mathbb{R}}$ is given as the stochastic integral

$$U_t := \sqrt{2\varrho}\int_{(-\infty,t]\times\mathbb{R}} ue^{-\varrho(t-x)}\,\hat{\eta}(\mathrm{d}(x,u))$$

with respect to $\hat{\eta}$. We are interested in the behaviour of the quadratic functionals

$$Q(T) := \int_0^T U_t^2\,\mathrm{d}t, \quad T \geq 0.$$

A central limit theorem for $Q(T)$ with a rate of convergence has been obtained in [34] in terms of the Wasserstein distance and in terms of the Kolmogorov distance in [19]. Our next result adds a moderate deviation principle, a concentration inequality as well as a normal approximation bound with Cramér correction. To formulate it, define the constant $c_\nu := \int_{-\infty}^{\infty} u^4 \, \nu(\mathrm{d}u)$.

Corollary 4.5. *Let $(T_n)_{n\in\mathbb{N}}$ be a positive real-valued sequence such that $T_n > 1/\varrho$ and $T_n \to \infty$, as $n \to \infty$, and assume that there exists a constant $M \geq 1$ such that $\int_{-\infty}^{\infty} |u|^m \, \nu(\mathrm{d}u) \leq M^m$ for all $m \in \mathbb{N}$. Then the sequence $(Q(T_n))_{n\in\mathbb{N}}$ satisfies $\mathbf{MDP}(1, (\tau_n)_{n\in\mathbb{N}})$, $\mathbf{CI}(1, (\tau_n)_{n\in\mathbb{N}})$ and $\mathbf{NACC}(1, (\tau_n)_{n\in\mathbb{N}})$ with $(\tau_n)_{n\in\mathbb{N}}$ given by*

$$\tau_n := \frac{1}{512} \min\left\{1, \frac{c_\nu\big(T_n - 1/\varrho\big)}{(4M^4)^2 \max\{1, 2/\varrho\}^2 (T_n + 1/(2\varrho))}\right\} \frac{\sqrt{c_\nu\big(T_n - 1/\varrho\big)}}{4M^4 \max\{1, 2/\varrho\}}.$$

5 Proofs: product formulas

The proof of Theorem 3.6 is prepared by the following lemma. Since the measure μ was assumed to be σ-finite, there exists a sequence $(A_n)_{n\in\mathbb{N}}$ of measurable subsets of $\mathbb{X}$ with $A_n \subseteq A_{n+1}$ and $\mu(A_n) < \infty$ for $n \in \mathbb{N}$ satisfying $\bigcup_{n\in\mathbb{N}} A_n = \mathbb{X}$. Now, for a function $f : \mathbb{X}^q \to \mathbb{R}$ with $q \in \mathbb{N}$ and $n \in \mathbb{N}$ define its truncation $f_n : \mathbb{X}^q \to \mathbb{R}$ by

$$f_n(x) := \mathbf{1}_{A_n^q}(x) \mathbf{1}_{[-n,n]}(f(x)) f(x), \qquad x \in \mathbb{X}^q. \tag{5.1}$$

Lemma 5.1. *Fix an even number $m \in \mathbb{N}$ and $f \in L^2_s(\mu^q)$ for some $q \in \mathbb{N}$. Assume that*

$$\int_{\mathbb{X}^{|\sigma|}} |(\otimes_{\ell=1}^m f)_\sigma| \, \mathrm{d}\mu^{|\sigma|} < \infty \tag{5.2}$$

for all $\sigma \in \Pi^m_{\geq 2}(q)$. Then $\lim_{n\to\infty} \mathbb{E} I_q(f - f_n)^m = 0$ and

$$\lim_{n\to\infty} \mathbb{E} I_q(f_n)^m = \mathbb{E} I_q(f)^m = \sum_{\sigma \in \Pi^m_{\geq 2}(q)} \int_{\mathbb{X}^{|\sigma|}} (\otimes_{\ell=1}^m f)_\sigma \, \mathrm{d}\mu^{|\sigma|} < \infty.$$

Proof. We note that the sequence of functions $(f_n)_{n\in\mathbb{N}}$ satisfies the following properties:

- each f_n is bounded and has support of finite μ-measure,
- f_n converges to f as $n \to \infty$ μ-almost everywhere,
- $|f_n| \leq |f|$.

By the dominated convergence theorem, it follows that $\|f - f_n\|^2_{L^2(\mu^q)} \to 0$, as $n \to \infty$. Now, let $\varepsilon > 0$ and use Chebychev's inequality to see that

$$\mathbb{P}(|I_q(f) - I_q(f_n)| \geq \varepsilon) = \mathbb{P}(|I_q(f - f_n)| \geq \varepsilon) \leq \frac{\mathbb{E} I_q(f - f_n)^2}{\varepsilon^2} = \frac{q! \|f - f_n\|^2_{L^2(\mu^q)}}{\varepsilon^2},$$

where the last expression tends to 0, as $n \to \infty$. Hence, $I_q(f_n)$ converges to $I_q(f)$ in probability and in distribution, as $n \to \infty$. Thus,

$$\mathbb{E} I_q(f)^m \leq \liminf_{n\to\infty} \mathbb{E} I_q(f_n)^m$$

by the Portmanteau theorem. Note that the expression $\mathbb{E} I_q(f)^m$ on the left-hand side is in fact well defined since we have assumed m to be even. To $\mathbb{E} I_q(f_n)^m$ we can now apply [29, Theorem 1] which yields

$$\mathbb{E} I_q(f)^m \leq \liminf_{n\to\infty} \sum_{\sigma \in \Pi^m_{\geq 2}(q)} \int_{\mathbb{X}^{|\sigma|}} (\otimes_{\ell=1}^m f_n)_\sigma \, \mathrm{d}\mu^{|\sigma|} = \sum_{\sigma \in \Pi^m_{\geq 2}(q)} \int_{\mathbb{X}^{|\sigma|}} (\otimes_{\ell=1}^m f)_\sigma \, \mathrm{d}\mu^{|\sigma|},$$

where the last step is verified by the dominated convergence theorem, which is applicable due to (5.2). Consequently, replacing f by $f - f_n$, we obtain

$$\mathbb{E}I_q(f-f_n)^m \le \sum_{\sigma\in\Pi^m_{\ge 2}(q)} \int_{\mathbb{X}^{|\sigma|}} (\otimes_{\ell=1}^m (f-f_n))_\sigma \,\mathrm{d}\mu^{|\sigma|}$$

for all $n \in \mathbb{N}$. Once again by the dominated convergence theorem and (5.2) the right-hand side tends to 0, as $n \to \infty$. This proves the first assertion.

On the other hand, using the reverse triangle inequality we see that

$$|(\mathbb{E}I_q(f)^m)^{1/m} - (\mathbb{E}I_q(f_n)^m)^{1/m}| \le (\mathbb{E}I_q(f-f_n)^m)^{1/m}.$$

Thus, by the convergence of the right-hand side to zero and by [29, Theorem 1],

$$\begin{aligned}\mathbb{E}I_q(f)^m &= \lim_{n\to\infty} \mathbb{E}I_q(f_n)^m \\ &= \lim_{n\to\infty} \sum_{\sigma\in\Pi^m_{\ge 2}(q)} \int_{\mathbb{X}^{|\sigma|}} (\otimes_{\ell=1}^m f_n)_\sigma \,\mathrm{d}\mu^{|\sigma|} \\ &= \sum_{\sigma\in\Pi^m_{\ge 2}(q)} \int_{\mathbb{X}^{|\sigma|}} (\otimes_{\ell=1}^m f)_\sigma \,\mathrm{d}\mu^{|\sigma|},\end{aligned}$$

where the last step follows once more by the dominated convergence theorem. This completes the argument. □

Proof of Theorem 3.6. For each $\ell \in \{1,\dots,m\}$ and $n \in \mathbb{N}$ define the function $f_n^{(\ell)}$ as in (5.1). Then, using [29, Theorem 1] in the first and the dominated convergence theorem, which is applicable due to (3.5), in the second step, we have that

$$\begin{aligned}\lim_{n\to\infty} \mathbb{E}\Big[\prod_{\ell=1}^m I_{q_\ell}(f_n^{(\ell)})\Big] &= \lim_{n\to\infty} \sum_{\sigma\in\Pi_{\ge 2}(q_1,\dots,q_m)} \int_{\mathbb{X}^{|\sigma|}} (\otimes_{\ell=1}^m f_n^{(\ell)})_\sigma \,\mathrm{d}\mu^{|\sigma|} \\ &= \sum_{\sigma\in\Pi_{\ge 2}(q_1,\dots,q_m)} \int_{\mathbb{X}^{|\sigma|}} (\otimes_{\ell=1}^m f^{(\ell)})_\sigma \,\mathrm{d}\mu^{|\sigma|}.\end{aligned}$$

From Lemma 5.1, whose assumption is satisfied by (3.4), one deduces that, for $\ell \in \{1,\dots,m\}$,

$$\lim_{n\to\infty} \mathbb{E}I_{q_\ell}(f^{(\ell)} - f_n^{(\ell)})^{\widetilde{m}} = 0 \quad \text{and} \quad \lim_{n\to\infty} \mathbb{E}I_{q_\ell}(f_n^{(\ell)})^{\widetilde{m}} = \mathbb{E}I_{q_\ell}(f^{(\ell)})^{\widetilde{m}} < \infty. \tag{5.3}$$

Thus, using Hölder's inequality, we find that

$$\begin{aligned}&\lim_{n\to\infty} \Big|\mathbb{E}\Big[\prod_{\ell=1}^m I_{q_\ell}(f^{(\ell)})\Big] - \mathbb{E}\Big[\prod_{\ell=1}^m I_{q_\ell}(f_n^{(\ell)})\Big]\Big| \\ &\le \lim_{n\to\infty} \sum_{\ell=1}^m \mathbb{E}\Big|\prod_{j=1}^{\ell-1} I_{q_j}(f^{(j)}) \times I_{q_\ell}(f^{(\ell)} - f_n^{(\ell)}) \times \prod_{j=\ell+1}^m I_{q_j}(f_n^{(j)})\Big| \\ &\le \lim_{n\to\infty} \sum_{\ell=1}^m \prod_{j=1}^{\ell-1} (\mathbb{E}|I_{q_j}(f^{(j)})^m|)^{1/m} \times (\mathbb{E}|I_{q_\ell}(f^{(\ell)} - f_n^{(\ell)})^m|)^{1/m} \times \prod_{j=\ell+1}^m (\mathbb{E}|I_{q_j}(f_n^{(j)})^m|)^{1/m} \\ &\le \lim_{n\to\infty} \sum_{\ell=1}^m \prod_{j=1}^{\ell-1} (\mathbb{E}I_{q_j}(f^{(j)})^{\widetilde{m}})^{1/\widetilde{m}} \times (\mathbb{E}I_{q_\ell}(f^{(\ell)} - f_n^{(\ell)})^{\widetilde{m}})^{1/\widetilde{m}} \times \prod_{j=\ell+1}^m (\mathbb{E}I_{q_j}(f_n^{(j)})^{\widetilde{m}})^{1/\widetilde{m}},\end{aligned}$$

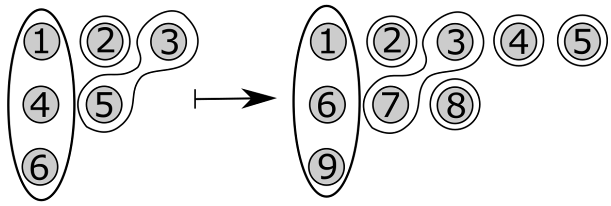

Figure 2: Illustration of the mapping $\Theta_{3,2,1}^{5,3,1} : \Pi(3,2,1) \to \Pi(5,3,1)$.

where in the last step we used that $\widetilde{m} \geq m$ is even. By (5.3), the two products ranging over $j \in \{1,\dots,\ell-1\}$ and $j \in \{\ell+1,\dots,m\}$, respectively, converge to finite constants, while the middle term $(\mathbb{E}I_{q_\ell}(f^{(\ell)} - f_n^{(\ell)})^{\widetilde{m}})^{1/\widetilde{m}}$ tends to zero, as $n \to \infty$. This completes the proof of the first part of Theorem 3.6.

The formula for the joint cumulants eventually follows from this exactly as in the proof of Theorem 1 in [29]. □

As a preparation for the proof of Theorem 3.7 we need to introduce an operation on partitions. Fix integers $m \geq 1$, $q_1,\dots,q_m, j_1,\dots,j_m \geq 1$ with $j_1 \leq q_1,\dots,j_m \leq q_m$. First, define $\tau_{j_1,\dots,j_m}^{q_1,\dots,q_m} : \{1,\dots,j_1+\dots+j_m\} \to \{1,\dots,q_1+\dots+q_m\}$ by putting

$$\tau_{j_1,\dots,j_m}^{q_1,\dots,q_m}(u) := \sum_{i=1}^{k-1} q_i + r_k \qquad \text{if } u = \sum_{i=1}^{k-1} j_i + r_k,\ r_k \leq j_k \text{ for some } k \in \{1,\dots,m\}.$$

Then we define the mapping $\Theta_{j_1,\dots,j_m}^{q_1,\dots,q_m} : \Pi(j_1,\dots,j_m) \to \Pi(q_1,\dots,q_m)$, which assigns to a partition $\pi = \{B_1,\dots,B_v\} \in \Pi(j_1,\dots,j_m)$ with $v \geq 1$ blocks the partition

$$\Theta_{j_1,\dots,j_m}^{q_1,\dots,q_m}(\pi) := \{\tau(B_1),\dots,\tau(B_v)\} \cup \bigcup_{\substack{z\in\{1,\dots,q_1+\dots+q_m\} \\ z \notin \cup_{w=1}^{v} \tau(B_w)}} \{\{z\}\}, \qquad \tau := \tau_{j_1,\dots,j_m}^{q_1,\dots,q_m}.$$

Let us explain the mechanism behind $\Theta_{j_1,\dots,j_m}^{q_1,\dots,q_m}$. The partition $\sigma := \Theta_{j_1,\dots,j_m}^{q_1,\dots,q_m}(\pi) \in \Pi(q_1,\dots,q_m)$ arises from $\pi \in \Pi(j_1,\dots,j_m)$ by first adding to the first row corresponding to the graphical representation of π the elements $j_1+1,\dots,q_1$, each of which becomes an individual block of σ. In the second row corresponding to π we first shift the labels of the elements there by $q_1 - j_1$ and then add the elements $q_1+j_1+1,\dots,q_1+q_2$, which again become new individual blocks of σ. In the third row the elements are shifted by $q_1 - j_1 + q_2 - j_2$ and then new elements $q_1+q_2+j_3+1,\dots,q_1+q_2+q_3$ are added as individual blocks etc. The procedure is illustrated in Figure 2 by an example.

Proof of Theorem 3.7. Fix $i \in \{1,\dots,m\}$. Applying (2.4) to the Poisson U-statistic S_i we have that

$$S_i - \mathbb{E}S_i = \sum_{j=1}^{q_i} I_j(f_j^{(i)}) \tag{5.4}$$

with functions $f_j^{(i)} : \mathbb{X}^j \to \mathbb{R}$ given by

$$f_j^{(i)}(x_1,\dots,x_j) := \binom{q_i}{j} \int_{\mathbb{X}^{q_i-j}} f^{(i)}(x_1,\dots,x_j,x_{j+1},\dots,x_{q_i})\, \mu^{q_i-j}(\mathrm{d}(x_{j+1},\dots,x_{q_i}))$$

for $j \in \{1,\dots,q_i\}$. By (3.6) and since $\widetilde{m}$ is even, we have that $f_j^{(i)} \in L_s^2(\mu^j)$ for each $j \in \{1,\dots,q_i\}$ and that $f_j^{(i)}$ satisfies (3.4). Indeed, for every $\pi \in \Pi_{\geq 2}^{\widetilde{m}}(j)$ one has that

$$\int_{\mathbb{X}^{|\pi|}} |(\otimes_{\ell=1}^{\widetilde{m}} f_j^{(i)})_\pi|\, \mathrm{d}\mu^{|\pi|} \leq \binom{q_i}{j}^{\widetilde{m}} \int_{\mathbb{X}^{|\sigma|}} |(\otimes_{\ell=1}^{\widetilde{m}} f^{(i)})_\sigma|\, \mathrm{d}\mu^{|\sigma|},$$

where we applied the definition of $f_j^{(i)}$, used the triangle inequality and put $\sigma := \Theta_{j,\dots,j}^{q_i,\dots,q_i}(\pi) \in \overline{\Pi}^{\widetilde{m}}(q_i)$. Similarly, we have for $j_\ell \in \{1,\dots,q_\ell\}$ with $\ell \in \{1,\dots,m\}$ and $\pi \in \Pi_{\geq 2}(j_1,\dots,j_m)$ that

$$\int_{\mathbb{X}^{|\pi|}} |(\otimes_{\ell=1}^m f_{j_\ell}^{(\ell)})_\pi| \,\mathrm{d}\mu^{|\pi|} \leq \prod_{\ell=1}^m \binom{q_\ell}{j_\ell} \int_{\mathbb{X}^{|\sigma|}} |(\otimes_{\ell=1}^m f^{(\ell)})_\sigma| \,\mathrm{d}\mu^{|\sigma|}$$

for $\sigma := \Theta_{j_1,\dots,j_m}^{q_1,\dots,q_m}(\pi) \in \overline{\Pi}(q_1,\dots,q_m)$. Thus, the functions $f_{j_\ell}^{(\ell)}$, $\ell \in \{1,\dots,m\}$, also satisfy (3.5) due to (3.7).

We can now repeatedly apply Theorem 3.6 to (5.4) to conclude that

$$\begin{aligned}
&\mathbb{E}\Big[\prod_{\ell=1}^m (S_\ell - \mathbb{E}S_\ell)\Big] \\
&= \sum_{1\leq j_1\leq q_1} \cdots \sum_{1\leq j_m\leq q_m} \mathbb{E}\Big[\prod_{\ell=1}^m I_{j_\ell}(f_{j_\ell}^{(\ell)})\Big] \\
&= \sum_{1\leq j_1\leq q_1} \cdots \sum_{1\leq j_m\leq q_m} \sum_{\pi\in\Pi_{\geq 2}(j_1,\dots,j_m)} \int_{\mathbb{X}^{|\pi|}} (\otimes_{\ell=1}^m f_{j_\ell}^{(\ell)})_\pi \,\mathrm{d}\mu^{|\pi|} \\
&= \sum_{1\leq j_1\leq q_1} \cdots \sum_{1\leq j_m\leq q_m} \Big(\prod_{\ell=1}^m \binom{q_\ell}{j_\ell}\Big) \\
&\qquad \times \sum_{\pi\in\Pi_{\geq 2}(j_1,\dots,j_m)} \int_{\mathbb{X}^{|\Theta_{j_1,\dots,j_m}^{q_1,\dots,q_m}(\pi)|}} (\otimes_{\ell=1}^m f^{(\ell)})_{\Theta_{j_1,\dots,j_m}^{q_1,\dots,q_m}(\pi)} \,\mathrm{d}\mu^{|\Theta_{j_1,\dots,j_m}^{q_1,\dots,q_m}(\pi)|}.
\end{aligned}$$

Next, we note that $\Theta_{j_1,\dots,j_m}^{q_1,\dots,q_m}(\pi)$ has $q_\ell - j_\ell$ singleton blocks at the end of row $\ell \in \{1,\dots,m\}$ in the representing diagram. Moreover, the prefactor $\prod_{\ell=1}^m \binom{q_\ell}{j_\ell}$ is precisely the number of possibilities to choose $q_\ell - j_\ell$ singletons in each row $\ell \in \{1,\dots,m\}$ of a diagram corresponding to a partition in $\overline{\Pi}(q_1,\dots,q_m)$. This together with the symmetry of the functions $f^{(1)},\dots,f^{(m)}$ implies that

$$\begin{aligned}
&\Big(\prod_{\ell=1}^m \binom{q_\ell}{j_\ell}\Big) \sum_{\pi\in\Pi_{\geq 2}(j_1,\dots,j_m)} \int_{\mathbb{X}^{|\Theta_{j_1,\dots,j_m}^{q_1,\dots,q_m}(\pi)|}} (\otimes_{\ell=1}^m f^{(\ell)})_{\Theta_{j_1,\dots,j_m}^{q_1,\dots,q_m}(\pi)} \,\mathrm{d}\mu^{|\Theta_{j_1,\dots,j_m}^{q_1,\dots,q_m}(\pi)|} \\
&\quad = \sum_{\substack{\sigma\in\overline{\Pi}(q_1,\dots,q_m) \\ \sigma \text{ has } q_1 - j_1 \text{ singletons in row } 1 \\ \cdots \\ \sigma \text{ has } q_m - j_m \text{ singletons in row } m}} \int_{\mathbb{X}^{|\sigma|}} (\otimes_{\ell=1}^m f^{(\ell)})_\sigma \,\mathrm{d}\mu^{|\sigma|}.
\end{aligned}$$

Thus,

$$\mathbb{E}\Big[\prod_{\ell=1}^m (S_\ell - \mathbb{E}S_\ell)\Big] = \sum_{\sigma\in\overline{\Pi}(q_1,\dots,q_m)} \int_{\mathbb{X}^{|\sigma|}} (\otimes_{\ell=1}^m f^{(\ell)})_\sigma \,\mathrm{d}\mu^{|\sigma|},$$

proving the first part of the theorem.

For the second part we use (5.4), multilinearity of joint cumulants and apply Theorem 3.6 to see that

$$\begin{aligned}
&\mathrm{cum}(S_1,\dots,S_m) \\
&= \sum_{1\leq j_1\leq q_1} \cdots \sum_{1\leq j_m\leq q_m} \mathrm{cum}(I_{j_1}(f_{j_1}^{(1)}),\dots,I_{j_m}(f_{j_m}^{(m)})) \\
&= \sum_{1\leq j_1\leq q_1} \cdots \sum_{1\leq j_m\leq q_m} \sum_{\pi\in\widetilde{\Pi}_{\geq 2}(j_1,\dots,j_m)} \int_{\mathbb{X}^{|\pi|}} (\otimes_{\ell=1}^m f_{j_\ell}^{(\ell)})_\pi \,\mathrm{d}\mu^{|\pi|}
\end{aligned}$$

$$= \sum_{1\le j_1\le q_1} \cdots \sum_{1\le j_m\le q_m} \Big(\prod_{\ell=1}^m \binom{q_\ell}{j_\ell}\Big)$$
$$\times \sum_{\pi\in\widetilde{\Pi}_{\ge 2}(j_1,\dots,j_m)} \int_{\mathbb{X}^{|\Theta^{q_1,\dots,q_m}_{j_1,\dots,j_m}(\pi)|}} (\otimes_{\ell=1}^m f^{(\ell)})_{\Theta^{q_1,\dots,q_m}_{j_1,\dots,j_m}(\pi)} \,\mathrm{d}\mu^{|\Theta^{q_1,\dots,q_m}_{j_1,\dots,j_m}(\pi)|}$$
$$= \sum_{1\le j_1\le q_1} \cdots \sum_{1\le j_m\le q_m} \sum_{\substack{\sigma\in\widetilde{\Pi}(q_1,\dots,q_m)\\ \sigma \text{ has } q_1-j_1 \text{ singletons in row } 1\\ \cdots \\ \sigma \text{ has } q_m-j_m \text{ singletons in row } m}} \int_{\mathbb{X}^{|\sigma|}} (\otimes_{\ell=1}^m f^{(\ell)})_\sigma \,\mathrm{d}\mu^{|\sigma|}$$
$$= \sum_{\sigma\in\widetilde{\Pi}(q_1,\dots,q_m)} \int_{\mathbb{X}^{|\sigma|}} (\otimes_{\ell=1}^m f^{(\ell)})_\sigma \,\mathrm{d}\mu^{|\sigma|}\,,$$

where we used the same arguments as above. This completes the proof. □

6 Proofs: moderate deviation estimates

The first aim of this section is to upper bound the right-hand sides of the cumulant formulas in Theorems 3.6 and 3.7 in such a way that Proposition 2.1 can be applied. For that purpose, the following estimates are going to be important. Recall the definitions of the classes $\Pi^m(q)$ and $\widetilde{\Pi}^m_{\ge 2}(q)$ of partitions from Subsection 2.4.

Proposition 6.1. *For any $q\ge 2$ one has*

$$|\Pi^m(q)| \le q^{qm}(m!)^q\,, \qquad m\in\mathbb{N}\,. \tag{6.1}$$

Moreover, there are no constants $c>0$ and $\gamma\in(0,q)$ such that

$$|\widetilde{\Pi}^m_{\ge 2}(q)| \le c^m(m!)^\gamma\,, \qquad m\in\mathbb{N}\,. \tag{6.2}$$

Proof. We can construct each partition $\sigma\in\Pi^{m+1}(q)$ by taking a partition $\hat\sigma\in\Pi^m(q)$ and deciding for each $i\in\{qm+1,\dots,q(m+1)\}$ whether it joins an existing block or forms a new block on its own. Since $\hat\sigma$ has at most qm blocks, we obtain from a partition $\hat\sigma\in\Pi_m(q)$ at most $(qm+1)^q$ new partitions. This implies the inequality

$$|\Pi^{m+1}(q)| \le (qm+1)^q|\Pi^m(q)| \le q^q(m+1)^q|\Pi^m(q)|\,.$$

Putting $C:=\max\{|\Pi^1(q)|,q^q\}$, this yields $|\Pi^m(q)|\le C^m(m!)^q$ for $m\in\mathbb{N}$ by iteration. Because of $|\Pi^1(q)|=1$, this is (6.1).

In order to show the second part, we prove that for any $\tilde\gamma\in(0,q)$ there exists a constant $\hat c>0$ such that

$$\limsup_{m\to\infty} \frac{|\widetilde{\Pi}^m_{\ge 2}(q)|}{\hat c^m(m!)^{\tilde\gamma}} > 1\,. \tag{6.3}$$

This in turn implies (6.2) by the following consideration. Assume that $\gamma\in(0,q)$ and $c>0$ satisfy (6.2). For $\tilde\gamma:=(\gamma+q)/2$ there is a constant $\hat c>0$ such that

$$1 < \limsup_{m\to\infty} \frac{|\widetilde{\Pi}^m_{\ge 2}(q)|}{\hat c^m(m!)^{\tilde\gamma}} = \limsup_{m\to\infty} \frac{|\widetilde{\Pi}^m_{\ge 2}(q)|}{c^m(m!)^{\gamma}(\hat c/c)^m(m!)^{\tilde\gamma-\gamma}}\,.$$

Since, by construction, $(\hat c/c)^m(m!)^{\tilde\gamma-\gamma}\to\infty$, as $m\to\infty$, this means that

$$\limsup_{m\to\infty} \frac{|\widetilde{\Pi}^m_{\ge 2}(q)|}{c^m(m!)^{\gamma}} > 1\,,$$

which contradicts (6.2).

So, let us prove (6.3) for a given $\tilde\gamma\in(0,q)$. Let $k,u\in\mathbb{N}$ with k even and put $m=uk$. In this situation we can construct partitions $\sigma\in\widetilde{\Pi}^{uk}_{\ge 2}(q)$ in the following way:

1) We group $\{1 + (\ell - 1)q : \ell \in \{1, \dots, uk\}\}$ into u blocks of size k.
2) Now, we order these blocks as follows:
$\{qi_1^{(1)} + 1, \dots, qi_k^{(1)} + 1\}, \dots, \{qi_1^{(u)} + 1, \dots, qi_k^{(u)} + 1\}$ with $i_1^{(1)} < i_1^{(2)} < \dots < i_1^{(u)}$ and $i_1^{(\ell)} < i_2^{(\ell)} < \dots < i_k^{(\ell)}$ for $\ell \in \{1, \dots, u\}$. Next, we add the blocks $\{qi_1^{(\ell)}+2, qi_k^{(\ell+1)}+2\}$, $\ell \in \{1, \dots, u-1\}$, and $\{qi_1^{(u)} + 2, qi_k^{(1)} + 2\}$. Then we form blocks of size $k-2$ from the remaining elements of $\{2 + (\ell - 1)q : \ell \in \{1, \dots, uk\}\}$ in an arbitrary way.
3) For any $j \in \{3, \dots, q\}$ we group $\{j + (\ell - 1)q : \ell \in \{1, \dots, uk\}\}$ into u blocks of size k.

Note that $\sigma \in \Pi_{\geq 2}^{uk}(q)$ since, by construction, σ is non-flat and all its blocks contain at least two elements. In addition, steps 1) and 2) are sufficient to ensure that σ is also connected and, thus, belongs to $\widetilde{\Pi}_{\geq 2}^{uk}(q)$.

According to steps 1)–3) there are

$$\frac{(uk)!}{u!(k!)^u} \cdot \frac{((k-2)u)!}{u!((k-2)!)^u} \cdot \left(\frac{(uk)!}{u!(k!)^u}\right)^{q-2}$$

possibilities to form such partitions $\sigma \in \widetilde{\Pi}_{\geq 2}^{uk}(q)$. It is easy to verify that

$$\frac{((k-2)u)!}{u!((k-2)!)^u} \geq \frac{((k-2)u)!}{u!(k!)^u} \geq \frac{(uk)!}{u!(k!)^u(2u)!\binom{uk}{(k-2)u}} \geq \frac{(uk)!}{u!(k!)^u(2u)!2^{uk}} \,,$$

whence

$$|\widetilde{\Pi}_{\geq 2}^{uk}(q)| \geq \frac{((uk)!)^q}{(k!)^{uq}(u!)^q(2u)!2^{uk}} \geq \frac{((uk)!)^{\tilde{\gamma}}}{(2(k!)^{q/k})^{uk}} \frac{((uk)!)^{q-\tilde{\gamma}}}{(u!(2u)!)^q}.$$

Since $(uk)! \geq ((2u)!)^{k/2}$ for all $u \in \mathbb{N}$ and all even $k \in \mathbb{N}$, we can choose an even $k_0 \in \mathbb{N}$ such that

$$\frac{((uk_0)!)^{q-\tilde{\gamma}}}{(u!(2u)!)^q} > 1 \qquad \text{for all} \qquad u \in \mathbb{N}\,.$$

Consequently, we have that

$$\limsup_{u\to\infty} \frac{|\widetilde{\Pi}_{\geq 2}^{uk_0}(q)|}{((uk_0)!)^{\tilde{\gamma}}(2(k_0!)^{q/k_0})^{-uk_0}} > 1\,,$$

which yields (6.3) and completes the proof. □

Proof of Theorem 3.1. We let Y_n be as in the statement of the theorem and fix $m \geq 3$. Then,

$$\begin{aligned}\Big|\operatorname{cum}_m\Big(\frac{Y_n}{\sqrt{\operatorname{Var} Y_n}}\Big)\Big| &= \frac{1}{(\operatorname{Var} Y_n)^{m/2}} |\operatorname{cum}(Y_n, \dots, Y_n)| \\ &\leq \frac{1}{(\operatorname{Var} Y_n)^{m/2}} \sum_{(i_1,\dots,i_m)\in[k]^m} |\operatorname{cum}(I_{q_{i_1}}(f_n^{(i_1)}), \dots, I_{q_{i_m}}(f_n^{(i_m)}))|\,,\end{aligned}$$

where we used the definition of $\operatorname{cum}_m(\,\cdot\,)$ in terms of a joint cumulant and then applied the multilinearity of the latter together with the triangle inequality. Next, we apply the cumulant formula in Theorem 3.6 to deduce that

$$\begin{aligned}|\operatorname{cum}(I_{q_{i_1}}(f_n^{(i_1)}), \dots, I_{q_{i_m}}(f_n^{(i_m)}))| &\leq |\widetilde{\Pi}_{\geq 2}(q_{i_1}, \dots, q_{i_m})| \\ &\quad\times \sup_{\sigma\in\widetilde{\Pi}_{\geq 2}(q_{i_1},\dots,q_{i_m})} \Big|\int_{\mathbb{X}^{|\sigma|}} (\otimes_{\ell=1}^m f_n^{(i_\ell)})_\sigma \,\mathrm{d}\mu^{|\sigma|}\Big|.\end{aligned}$$

Recalling $q = \max\{q_1, \dots, q_k\}$, we see that according to the first part of Proposition 6.1, we have

$$|\widetilde{\Pi}_{\geq 2}(q_{i_1}, \dots, q_{i_m})| \leq |\Pi^m(q)| \leq q^{qm}(m!)^q\,, \tag{6.4}$$

while the assumption (3.1) of the theorem ensures that

$$\sup_{\sigma \in \widetilde{\Pi}_{\geq 2}(q_{i_1}, \dots, q_{i_m})} \frac{1}{(\operatorname{Var} Y_n)^{m/2}} \Big| \int_{\mathbb{X}^{|\sigma|}} (\otimes_{\ell=1}^m f_n^{(i_\ell)})_\sigma \, \mathrm{d}\mu^{|\sigma|} \Big| \leq \alpha_n^{m-2}\,.$$

Putting the previous estimates together leads to the bound

$$\Big| \operatorname{cum}_m \Big(\frac{Y_n}{\sqrt{\operatorname{Var} Y_n}} \Big) \Big| \leq q^{qm}(m!)^q \alpha_n^{m-2}\, k^m \leq (m!)^q\, (q^{3q} k^3 \alpha_n)^{m-2}$$

for all $n \in \mathbb{N}$ and $m \geq 3$. Thus, the statements of the theorem follow from Proposition 2.1. □

Proof of Theorem 3.2. The argument is very similar to the one for Theorem 3.1. Basically, the only change is that now one uses the cumulant formula in Theorem 3.7 instead of the one in Theorem 3.6 and assumption (3.2) instead of (3.1). We leave the details to the reader. □

Remark 6.2. Note that the second part of Proposition 6.1 implies that one cannot achieve a smaller exponent than q at $m!$ in (6.4). This means that the choice $\gamma = q - 1$ in the cumulant bound (2.6) cannot be improved systematically. Thus, Theorem 3.1 is the best result one can obtain by the method of cumulants. This agrees with the discussion next to Theorem 3.5, which shows that Theorem 3.1 yields for the situation considered in Theorem 3.5 up to subpolynomial factors the optimal range of scales for the MDP.

Also the monograph [41] discusses cumulant bounds for multiple stochastic integrals with respect to compensated Poisson processes in Chapter 5.2. However, it appears that the estimate for the number of the involved partitions there is not correct. As discussed above the exponent at $m!$ cannot be smaller than q, while the exponent $q/2$ was used in [41].

Proof of Theorem 3.5. We start by noting that $\mathbb{E} H_q(Z_1) = 0$ and $\operatorname{Var} H_q(Z_1) = q!$, and hence $\mathbb{E} S_n = 0$ and $\operatorname{Var} S_n = nq!$. So, according to [2, Theorem 2.11] (see also [17, Theorem 2.2]) the sequence of random variables $(S_n/(a_n\sqrt{nq!}))_{n\in\mathbb{N}}$ satisfies a MDP with speed a_n^2 and a good rate function $\mathcal{I}$ with $\mathcal{I}(z) > 0$ for $z \neq 0$ and $\mathcal{I}(z) \to \infty$, as $z \to \pm\infty$, if and only if

$$\limsup_{n\to\infty} \frac{1}{a_n^2} \Big(\log n + \log \mathbb{P}\big(|H_q(Z_1)| > a_n \sqrt{nq!}\big) \Big) = -\infty. \tag{6.5}$$

In addition, if condition (6.5) is satisfied, then the MDP holds with the good rate function $\mathcal{I}(z) = z^2/2$.

To analyse the probability in (6.5), we start by observing that, for sufficiently large x, $\frac{1}{2}x^q \leq |H_q(x)| \leq 2x^q$ and hence, for sufficiently large n, we have that

$$\mathbb{P}\big(Z_1^q > 2a_n\sqrt{nq!}\big) \leq \mathbb{P}\big(|H_q(Z_1)| > a_n\sqrt{nq!}\big) \leq \mathbb{P}\Big(Z_1^q > \frac{1}{2}a_n\sqrt{nq!}\Big).$$

Moreover, for sufficiently large $m \in \mathbb{N}$, one has that the Poisson random variable Z_1 satisfies $\mathbb{P}(Z_1 \geq m) \leq 2\mathbb{P}(Z_1 = m)$. Thus, we obtain

$$\mathbb{P}\big(Z_1 = \big\lceil \big(2a_n\sqrt{nq!}\big)^{1/q} \big\rceil + 1\big) \leq \mathbb{P}\big(|H_q(Z_1)| > a_n\sqrt{nq!}\big) \leq 2\mathbb{P}\Big(Z_1 = \Big\lfloor \Big(\frac{1}{2}a_n\sqrt{nq!}\Big)^{1/q} \Big\rfloor\Big)$$

for sufficiently large n. Next, the elementary inequality $(m/2)^{m/2} \le m! \le m^m$ and $\mathbb{P}(Z_1 = m) = \frac{e^{-1}}{m!}$ imply that

$$-2m\log m \le -m\log m - 1 \le \log \mathbb{P}(Z_1 = m) \le -\frac{m}{2}(\log m - \log 2) - 1 \le -\frac{m}{4}\log m$$

for large enough $m \in \mathbb{N}$. It follows that

$$\begin{aligned}
-2\Big(\big\lceil \big(2a_n\sqrt{nq!}\big)^{1/q}\big\rceil + 1\Big)\log\Big(\big\lceil \big(2a_n\sqrt{nq!}\big)^{1/q}\big\rceil + 1\Big)& \\
\le \log \mathbb{P}\big(|H_q(Z_1)| > a_n\sqrt{nq!}\big)& \\
\le \log 2 - \frac{1}{4}\Big\lfloor \Big(\frac{1}{2}a_n\sqrt{nq!}\Big)^{1/q}\Big\rfloor \log \Big\lfloor \Big(\frac{1}{2}a_n\sqrt{nq!}\Big)^{1/q}\Big\rfloor&
\end{aligned}$$

for large enough $n \in \mathbb{N}$. The left- and the right-hand side in the previous chain of inequalities asymptotically behave like constant multiples of

$$-a_n^{1/q} n^{1/(2q)} \log(a_n\sqrt{n}),$$

as $n \to \infty$. Clearly, this expression asymptotically dominates the summand $\log n$ in (6.5) by our assumption that $a_n \to \infty$, as $n \to \infty$. This means that there exist constants $c_q, C_q \in (0,\infty)$ depending only on q such that

$$\begin{aligned}
&- c_q \limsup_{n\to\infty} a_n^{-2+1/q} n^{1/(2q)} \log(a_n\sqrt{n}) \\
&\quad\le \limsup_{n\to\infty} \frac{1}{a_n^2}\Big(\log n + \log \mathbb{P}\big(|H_q(Z_1)| > a_n\sqrt{nq!}\big)\Big) \\
&\quad\quad\le -C_q \limsup_{n\to\infty} a_n^{-2+1/q} n^{1/(2q)} \log(a_n\sqrt{n}).
\end{aligned}$$

Especially, if the sequence $(a_n)_{n\in\mathbb{N}}$ satisfies the conditions in part a), both the left-hand and the right-hand side are $-\infty$. On the other hand, the left- and the right-hand side tend to constants under the condition of part b). So, the necessary and sufficient condition (6.5) for a MDP for a sequence of independent and identically distributed random variables completes the argument. □

7 Proofs: applications

Proof of Corollary 4.1. It follows from (2.3) and (2.5) that $\operatorname{Var} S_n \ge v_f t_n^{2q-1}$ (see also [39, Theorem 5.2]). Together with the assumptions on μ_n and f this yields that, for all $m \in \mathbb{N}$ with $m \ge 3$ and $\sigma \in \widetilde{\Pi}^m(q)$,

$$\frac{1}{(\operatorname{Var} S_n)^{m/2}}\bigg|\int_{\mathbb{X}^{|\sigma|}} (\otimes_{\ell=1}^m f)_\sigma \,\mathrm{d}\mu_n^{|\sigma|}\bigg| \le \frac{t_n^{|\sigma|}\|f\|_\infty^m}{(v_f t_n^{2q-1})^{m/2}}\,.$$

Since $|\sigma| \le (q-1)m + 1$ and $t_n \ge 1$, the right-hand side is bounded by

$$t_n^{1-m/2}\bigg(\frac{\|f\|_\infty}{\sqrt{v_f}}\bigg)^m \le t_n^{-(m-2)/2} \max\{(\|f\|_\infty/\sqrt{v_f})^3, \|f\|_\infty/\sqrt{v_f}\}^{m-2}\,.$$

Now the statement follows from Theorem 3.2. □

Proof of Corollary 4.2. For a connected graph G with vertices $1,\dots,\tilde{q}$ and $\diamondsuit \in \{=,\subset\}$ we have that

$$\begin{aligned}
&f_\diamondsuit(x_1,\dots,x_{\tilde{q}}; G, r_n) &(7.1)\\
&\le \frac{1}{\tilde{q}!}\sum_{\varrho\in\operatorname{Per}(\tilde{q})} \mathbf{1}\{x_{\varrho(i)} \leftrightarrow x_{\varrho(j)} \text{ in } \operatorname{RGG}(\{x_1,\dots,x_{\tilde{q}}\}, r_n) \text{ if } i \leftrightarrow j \text{ in } G \text{ for all } i,j\in\{1,\dots,\tilde{q}\}\}
\end{aligned}$$

for $x_1, \dots, x_{\tilde{q}} \in W$, where $\leftrightarrow$ denotes an edge between two vertices and $\mathrm{Per}(\tilde{q})$ is the set of permutations of $1, \dots, \tilde{q}$. In the following let $m \in \mathbb{N}$ with $m \geq 3$, $i_1, \dots, i_m \in \{1, \dots, k\}$ and $\sigma \in \widetilde{\Pi}(i_1, \dots, i_m)$. Then, by applying (7.1) to the factors in the product of functions, we have that

$$\left| \int_W (\otimes_{\ell=1}^m f_{\diamondsuit_{i_\ell}}(\,\cdot\,; G_{i_\ell}, r_n))_\sigma \, \mathrm{d}\mu_n^{|\sigma|} \right| \leq t_n^{|\sigma|} \operatorname{Vol}(W)(\kappa_d r_n^d)^{|\sigma|-1}.$$

Here, we have used that since $\sigma \in \widetilde{\Pi}(i_1, \dots, i_m)$, for each choice of permutations $\varrho_1, \dots, \varrho_m$ coming from (7.1) the points $x_1, \dots, x_{|\sigma|}$ form a particular connected random geometric graph with distance threshold r_n, where it is completely determined which points are connected by edges. Hence, successive integration leads to the bound above. Thereby, the factorials in the denominator and the numbers of permutations cancel out.

Together with our variance assumption (4.2) we obtain that

$$\frac{\left| \int_W (\otimes_{\ell=1}^m a_{i_\ell} f_{\diamondsuit_{i_\ell}}(\,\cdot\,; G_{i_\ell}, r_n))_\sigma \, \mathrm{d}\mu_n^{|\sigma|} \right|}{(\operatorname{Var} S_n)^{m/2}} \leq \frac{t_n^{|\sigma|} a^m \operatorname{Vol}(W)(\kappa_d r_n^d)^{|\sigma|-1}}{v^{m/2} \max\{t_n^{2q-1}(\kappa_d r_n^d)^{2q-2}, t_n^p(\kappa_d r_n^d)^{p-1}\}^{m/2}}.$$

Next we consider the cases $\kappa_d t_n r_n^d \geq 1$ and $\kappa_d t_n r_n^d < 1$ separately. If $\kappa_d t_n r_n^d \geq 1$, because of $|\sigma| \leq m(q-1)+1$ we have that

$$\begin{aligned} \frac{\left| \int_W (\otimes_{\ell=1}^m a_{i_\ell} f_{\diamondsuit_{i_\ell}}(\,\cdot\,; G_{i_\ell}, r_n))_\sigma \, \mathrm{d}\mu_n^{|\sigma|} \right|}{(\operatorname{Var} S_n)^{m/2}} &\leq \frac{t_n^{m(q-1)+1} a^m \operatorname{Vol}(W)(\kappa_d r_n^d)^{m(q-1)}}{v^{m/2} t_n^{mq-m/2}(\kappa_d r_n^d)^{m(q-1)}} \\ &= \frac{t_n^{1-m/2} a^m \operatorname{Vol}(W)}{v^{m/2}} \\ &\leq \max\{1, a^2 \operatorname{Vol}(W)/v\}^{m-2} (a/\sqrt{v t_n})^{m-2}. \end{aligned}$$

Similarly, if $\kappa_d t_n r_n^d < 1$, we obtain that

$$\begin{aligned} \frac{\left| \int_W (\otimes_{\ell=1}^m a_{i_\ell} f_{\diamondsuit_{i_\ell}}(\,\cdot\,; G_{i_\ell}, r_n))_\sigma \, \mathrm{d}\mu_n^{|\sigma|} \right|}{(\operatorname{Var} S_n)^{m/2}} &\leq \frac{t_n^p a^m \operatorname{Vol}(W)(\kappa_d r_n^d)^{p-1}}{v^{m/2} t_n^{mp/2}(\kappa_d r_n^d)^{m(p-1)/2}} \\ &= \frac{a^m \operatorname{Vol}(W)}{v^{m/2} t_n^{(m-2)p/2}(\kappa_d r_n^d)^{(m-2)(p-1)/2}} \\ &\leq \max\{1, a^2 \operatorname{Vol}(W)/v\}^{m-2} \left(a/\sqrt{v t_n^p (\kappa_d r_n^d)^{p-1}}\right)^{m-2}, \end{aligned}$$

where we used that $|\sigma| \geq p$. Now, the result follows from Theorem 3.2 with $f_n^{(i)} := a_i f_{\diamondsuit_i}(\cdot; G_i, r_n)$ for $i \in \{1, \dots, k\}$. □

Proof of Corollary 4.5. It follows from the proof of the central limit theorem for $Q(T_n)$, [34, Theorem 7.2], that $Q(T_n)$ can be re-written as a sum of a first- and a second-order Wiener-Itô integral. More precisely, we have that

$$Q(T_n) = \mathbb{E}[Q(T_n)] + I_1(f_n^{(1)}) + I_2(f_n^{(2)})$$

with the functions $f_n^{(1)} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $f_n^{(2)} : (\mathbb{R} \times \mathbb{R})^2 \to \mathbb{R}$ given by

$$f_n^{(1)}((x, u)) := u^2 \tilde{f}_n^{(1)}(x) \quad \text{and} \quad f_n^{(2)}((x_1, u_1), (x_2, u_2)) := u_1 u_2 \tilde{f}_n^{(2)}(x_1, x_2),$$

where

$$\begin{aligned}\tilde f_n^{(1)}(x) &:= \mathbf{1}\{x\in(-\infty,T_n]\}\big(e^{2\varrho x}(1-e^{-2\varrho T_n})\mathbf{1}\{x\le 0\}\\ &\qquad + e^{2\varrho x}(e^{-2\varrho x}-e^{-2\varrho T_n})\mathbf{1}\{x>0\}\big)\,,\\ \tilde f_n^{(2)}(x_1,x_2) &:= \mathbf{1}\{x_1,x_2\in(-\infty,T_n]\}\big(e^{\varrho(x_1+x_2)}(1-e^{-2\varrho T_n})\mathbf{1}\{\max\{x_1,x_2\}\le 0\}\\ &\qquad + e^{\varrho(x_1+x_2)}(e^{-2\varrho\max\{x_1,x_2\}}-e^{-2\varrho T_n})\mathbf{1}\{\max\{x_1,x_2\}>0\}\big)\,.\end{aligned}$$

Note that $\tilde f_n^{(1)}$ and $\tilde f_n^{(2)}$ are both non-negative and satisfy the estimates

$$\tilde f_n^{(1)}(x)\le \mathbf{1}\{x\in(-\infty,T_n]\}\min\{e^{2\varrho x},1\} \tag{7.2}$$

and

$$\tilde f_n^{(2)}(x_1,x_2)\le \mathbf{1}\{x_1,x_2\in(-\infty,T_n]\}\,e^{-\varrho|x_1-x_2|}\,e^{2\varrho\min\{\max\{x_1,x_2\},0\}}\,. \tag{7.3}$$

In what follows, we let $m\ge 3$, $i_1,\dots,i_m\in\{1,2\}$ and $\sigma\in\widetilde\Pi_{\ge 2}(i_1,\dots,i_m)$. Then, we obtain

$$\begin{aligned}&\Big|\int_{((-\infty,T_n]\times\mathbb{R})^{|\sigma|}}(\otimes_{\ell=1}^m f_n^{(i_\ell)})_\sigma\,\mathrm{d}\mu^{|\sigma|}\Big|\\ &\le \int_{\mathbb{R}^{|\sigma|}}\prod_{j=1}^{|\sigma|}(|u_j|^{m_j}+u_j^{2m_j})\,\nu^{|\sigma|}(\mathrm{d}(u_1,\dots,u_{|\sigma|}))\int_{(-\infty,T_n]^{|\sigma|}}(\otimes_{\ell=1}^m\tilde f_n^{(i_\ell)})_\sigma\,\mathrm{d}\lambda^{|\sigma|}\,,\end{aligned}$$

where we denote by m_j the cardinality of the jth block of σ. Since $\sum_{j=1}^{|\sigma|}m_j=\sum_{\ell=1}^m i_\ell\le 2m$ and $|\sigma|\le\frac12\sum_{\ell=1}^m i_\ell\le m$, it follows from the assumptions on ν that

$$\int_{\mathbb{R}^{|\sigma|}}\prod_{j=1}^{|\sigma|}(|u_j|^{m_j}+u_j^{2m_j})\,\nu^{|\sigma|}(\mathrm{d}(u_1,\dots,u_{|\sigma|}))\le\prod_{j=1}^{|\sigma|}(M^{m_j}+M^{2m_j})\le 2^mM^{4m}\,.$$

For $j\in\{1,\dots,|\sigma|\}$ the estimates (7.2) and (7.3) lead to

$$\begin{aligned}&\int_{(-\infty,T_n]^{|\sigma|}}\mathbf{1}\{x_i\le x_j,i\in\{1,\dots,|\sigma|\}\}(\otimes_{\ell=1}^m\tilde f_n^{(i_\ell)})_\sigma(x_1,\dots,x_{|\sigma|})\,\lambda^{|\sigma|}(\mathrm{d}(x_1,\dots,x_{|\sigma|}))\\ &\le\Big(2\int_0^\infty e^{-\varrho s}\,\mathrm{d}s\Big)^{|\sigma|-1}\int_{-\infty}^{T_n}e^{\min\{2\varrho t,0\}}\,\mathrm{d}t=(\varrho/2)^{1-|\sigma|}(T_n+1/(2\varrho))\,.\end{aligned}$$

Moreover, according to (2.1) we have that

$$\operatorname{Var}Q(T_n)=\|f_n^{(1)}\|^2_{L^2(\mu^1)}+2\|f_n^{(2)}\|^2_{L^2(\mu^2)}\ge\|f_n^{(1)}\|^2_{L^2(\mu^1)}\,,$$

which leads to the lower variance bound

$$\begin{aligned}\operatorname{Var}Q(T_n)&\ge\int_{(-\infty,T_n]\times\mathbb{R}}f_n^{(1)}(x,u)^2\,\mu(\mathrm{d}(x,u))\ge\int_{-\infty}^\infty u^4\,\nu(\mathrm{d}u)\int_0^{T_n}\big(1-e^{-2\varrho(T_n-x)}\big)^2\,\mathrm{d}x\\ &\ge\int_{-\infty}^\infty u^4\,\nu(\mathrm{d}u)\int_0^{T_n}1-2e^{-2\varrho(T_n-x)}\,\mathrm{d}x\\ &=c_\nu\big(T_n-(1-e^{-2\varrho T_n})/\varrho\big)\ge c_\nu\big(T_n-1/\varrho\big)\,.\end{aligned}$$

Note that $1\le|\sigma|\le m+1$. Altogether we see that

$$\begin{aligned}&\frac{1}{(\operatorname{Var}Q(T_n))^{m/2}}\Big|\int_{((-\infty,T_n]\times\mathbb{R})^{|\sigma|}}(\otimes_{\ell=1}^m f_n^{(i_\ell)})_\sigma\,\mathrm{d}\mu^{|\sigma|}\Big|\\ &\le(2M^4)^m(\varrho/2)^{1-|\sigma|}|\sigma|\frac{T_n+1/(2\varrho)}{(c_\nu\big(T_n-1/\varrho\big))^{m/2}}\end{aligned}$$

$$
\begin{aligned}
&\le (2M^4)^m \max\{1,2/\varrho\}^m 2^m \frac{T_n + 1/(2\varrho)}{(c_\nu(T_n - 1/\varrho))^{m/2}} \\
&\le (4M^4)^m \max\{1,2/\varrho\}^m \frac{T_n + 1/(2\varrho)}{(c_\nu(T_n - 1/\varrho))^{m/2}} \\
&\le (4M^4)^2 \max\{1,2/\varrho\}^2 \frac{T_n + 1/(2\varrho)}{c_\nu(T_n - 1/\varrho)} \left(\frac{4M^4\max\{1,2/\varrho\}}{\sqrt{c_\nu(T_n - 1/\varrho)}}\right)^{m-2} \\
&\le \max\left\{1, (4M^4)^2 \max\{1,2/\varrho\}^2 \frac{T_n + 1/(2\varrho)}{c_\nu(T_n - 1/\varrho)}\right\}^{m-2} \left(\frac{4M^4\max\{1,2/\varrho\}}{\sqrt{c_\nu(T_n - 1/\varrho)}}\right)^{m-2}.
\end{aligned}
$$

Thus, Theorem 3.1 can be applied to complete the proof. □

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