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Large deviations principle for the inviscid limit of fluid dynamic systems in 2D bounded domains

Federico Butori* Eliseo Luongo†

Abstract

Using a weak convergence approach, we establish a Large Deviation Principle (LDP) for the solutions of fluid dynamic systems in two-dimensional bounded domains subjected to no-slip boundary conditions and perturbed by additive noise. Our analysis considers the convergence of both viscosity and noise intensity to zero. Specifically, we focus on three important scenarios: Navier-Stokes equations in a Kato-type regime, Navier-Stokes equations for fluids with circularly symmetric flows and Second-Grade Fluid equations. In all three cases, we demonstrate the validity of the LDP, taking into account the critical topology $C([0,T];L^2)$.

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1 Introduction

1.1 The problem of the inviscid limit

An important role in the understanding of the behavior of turbulent fluid is given by the analysis of the so-called inviscid limit. In a naive way, given $u^{NS,\varepsilon}$ and $\bar{u}$ solutions, in a suitable sense, of the systems below

$$\begin{cases} \partial_t u^{NS,\varepsilon} - \varepsilon\Delta u^{NS,\varepsilon} + \nabla p^{NS,\varepsilon} + u^{NS,\varepsilon}\cdot\nabla u^{NS,\varepsilon} &= f^\varepsilon \\ \operatorname{div} u^{NS,\varepsilon} &= 0 \\ u^{NS,\varepsilon}(0) &= u_0^\varepsilon \end{cases} \tag{NS-E}$$

$$\begin{cases} \partial_t \bar{u} + \nabla\bar{p} + \bar{u}\cdot\nabla\bar{u} &= \bar{f} \\ \operatorname{div}\bar{u} &= 0 \\ \bar{u}(0) &= \bar{u}_0, \end{cases} \tag{E-E}$$

*Scuola Normale Superiore, Pisa, Italy. E-mail: federico.butori@sns.it
†Scuola Normale Superiore, Pisa, Italy. E-mail: eliseo.luongo@sns.it

the problem of the inviscid limit consists in showing that $u^{NS,\varepsilon}$, the solution of the Navier-Stokes equations, converges to $\bar{u}$, the solution of the corresponding Euler equations, as $\varepsilon \to 0$ in the topology $L^\infty(0,T;L^2(D))$, D being the domain where the equations evolve.

The difficulty of answering to this problem changes drastically considering different boundary conditions. Some frameworks where the problem above has a positive answer have been presented in [23], [44]. We sum up their results.

1. If previous equations evolve in a two-dimensional domain D without boundary, assuming

$$f^\varepsilon = \bar{f} \equiv 0, \;\; u_0^\varepsilon, \; \bar{u}_0 \in H^k, \;\; \mathrm{div}(u_0^\varepsilon) = \mathrm{div}(\bar{u}_0) = 0, \;\; \|u_0^\varepsilon - \bar{u}_0\|_{H^k} \overset{\varepsilon \to 0}{\to} 0$$

implies the convergence

$$\|u^{NS,\varepsilon} - \bar{u}\|_{C([0,T];H^k)} \overset{\varepsilon \to 0}{\to} 0.$$

2. If $u^{NS,\varepsilon}$ is a solution of the Navier-Stokes equations with Navier-Boundary conditions, i.e. $u^{NS,\varepsilon} \cdot n|_{\partial D} = 0, \; \partial_1 u_2^{NS,\varepsilon} - \partial_2 u_1^{NS,\varepsilon}|_{\partial D} = 0$,

$$u_0^\varepsilon \overset{L^2(D)}{\to} \bar{u}_0, \quad f^\varepsilon \overset{L^2((0,T)\times D)}{\to} \bar{f},$$

then each sequence, u^{ε_k}, has a subsequence, $u^{\varepsilon_{h_k}}$, converging to a weak solution of the Euler equations in $C([0,T];L^2(D))$. Moreover, if the solution of the Euler equation is unique, then $u^{NS,\varepsilon} \to \bar{u}$.

In the case of no-slip boundary conditions, i.e. $u^{NS,\varepsilon}|_{\partial D} = 0$, the convergence of $u^{NS,\varepsilon}$ to $\bar{u}$ in the topology $L^\infty(0,T;L^2(D))$ is an open problem with few results available:

1. Unconditioned results. They are based on strong assumptions about the symmetry of the domain and of the data [45], or real analytic data [57], [2].
2. Conditioned results. They are based on stating some criteria about the behavior of the solutions of the Navier-Stokes equations in the boundary layer in order to prove that the inviscid limit holds. This line of research started with the seminal work by Kato [36], see also [18], [61], [63], [38] and the references therein for other results in this direction.

Beside its mathematical interest, the analysis of the inviscid limit in the case of no-slip boundary conditions is a relevant problem also from a physical prospective of the understanding of turbulence. The no-slip condition $u^{NS,\varepsilon}|_{\partial D} = 0$ provokes large stress near the boundary, if $u^{NS,\varepsilon}$ is large nearby and this stress, when the viscosity is small enough, may lead to instabilities and generate vortices. This is the so-called phenomenon of the emergence of a boundary layer: close to the boundary the fluid presents a turbulent behavior for $\varepsilon \to 0$. The thickness of the boundary layer and some control on the behavior of the fluid in this region are very challenging and mostly open questions, see [3] for a review on the topic and [20], [21], [31] for some attempts of describing the generation of vorticity at the boundary.

Contrary to Navier-Stokes equations, the problem of the inviscid limit in bounded domains and no-slip boundary conditions has been solved for the second grade fluid equations, at least in a suitable regime of the parameters, see [46]. The second-grade fluid equations are a model for viscoelastic fluids, with two parameters: $\varepsilon > 0$, corresponding to the elastic response, and $\nu > 0$, corresponding to viscosity. For such fluids, assuming their density, ρ, being constant and equal to 1, the stress tensor is given by

$$T = -p^{SG,\varepsilon} I + \nu A_1 + \varepsilon A_2 - \varepsilon A_1^2,$$

where

$$
\begin{aligned}
A_1 &= \frac{\nabla u^{SG,\varepsilon} + \nabla(u^{SG,\varepsilon})^T}{2}, \\
A_2 &= \partial_t A_1 + A_1 \nabla u^{SG,\varepsilon} + (\nabla u^{SG,\varepsilon})^T A_1,
\end{aligned}
$$

being $p^{SG,\varepsilon}$ the pressure and $u^{SG,\varepsilon}$ the velocity field. Given this stress tensor, the equations of motion for an incompressible homogeneous fluid of grade 2 are given by

$$
\begin{cases}
\partial_t v^{SG,\varepsilon} &= \nu\Delta u^{SG,\varepsilon} - \operatorname{curl}(v^{SG,\varepsilon}) \times u^{SG,\varepsilon} + \nabla p^{SG,\varepsilon} + f^{SG,\varepsilon} \\
\operatorname{div} u^{SG,\varepsilon} &= 0 \\
v^{SG,\varepsilon} &= u^{SG,\varepsilon} - \varepsilon\Delta u^{SG,\varepsilon} \\
u^{SG,\varepsilon}|_{\partial D} &= 0 \\
u^{SG,\varepsilon}(0) &= u_0^{SG,\varepsilon},
\end{cases}
\tag{1.1}
$$

where $f^{SG,\varepsilon}$ describes some external forces acting on the fluid, see [24], [56] for further details on the physics behind this system. The analysis of the second-grade fluid equations started with [16], where some results, not restricted to the two-dimensional case, for global existence and uniqueness of solutions of the problem above have been shown. Setting, formally, $\varepsilon = 0$ the system above reduces to the Navier-Stokes system. Thus it can be seen as a generalization of the Navier-Stokes equations. Moreover, the convergence of the solution of the second-grade fluid equations to the solution of the Navier-Stokes equations has been shown rigorously in [32].

1.2 The inviscid limit in the stochastic framework

In the last decades, stochastic forcing have been added to the fluid dynamic systems of the previous section. We refer to [27, Chapter 5] for some justifications for the introduction of stochastic forcing terms in fluid dynamic models. After establishing the well-posedness of Navier-Stokes equations, Second-Grade fluid equations and Euler equations with Gaussian additive noise, see for example [26], [55], [5] for some results in this direction, a natural question is trying to understand the validity of the inviscid limit in such stochastic models. According to [43, Chapter 10], the relevant scaling of the parameters in order to study the inviscid limit is the following one

$$
\begin{cases}
du^{NS,\varepsilon} &= (\varepsilon\Delta u^{NS,\varepsilon} - u^{NS,\varepsilon}\cdot\nabla u^{NS,\varepsilon} + \nabla p^{NS,\varepsilon})dt + \sqrt{\varepsilon}dW_t \\
\operatorname{div} u^{NS,\varepsilon} &= 0 \\
u^{NS,\varepsilon}|_{\partial D} &= 0 \\
u^{NS,\varepsilon}(0) &= u_0,
\end{cases}
\tag{1.2}
$$

$$
\begin{cases}
dv^{SG,\varepsilon} &= (\nu\Delta u^{SG,\varepsilon} - \operatorname{curl}(v^{SG,\varepsilon}) \times u^{SG,\varepsilon} + \nabla p^{SG,\varepsilon})dt + \sqrt{\varepsilon}dW_t \\
\operatorname{div} u^{SG,\varepsilon} &= 0 \\
v^{SG,\varepsilon} &= u^{SG,\varepsilon} - \varepsilon\Delta u^{SG,\varepsilon} \\
u^{SG,\varepsilon}|_{\partial D} &= 0 \\
u^{SG,\varepsilon}(0) &= u_0.
\end{cases}
\tag{1.3}
$$

Difficulties analogous to those described in subsection 1.1 appear also in the stochastic framework, even considering different scaling of the parameters:

1. The validity of the inviscid limit in the case of Navier-Boundary conditions has been shown in [17].
2. The validity of some conditioned results for the stochastic Navier-Stokes with no-slip boundary conditions has been shown in [48], [64].

3. The validity of the inviscid limit for the stochastic Second-Grade fluid equations with no-slip boundary conditions under suitable assumptions between ν and ε has been shown in [49].

These results can be seen as a sort of law of large numbers for the stochastic systems above. It is natural then to investigate Large Deviations principles for the aforementioned systems focusing, in case, on their relation with some form of Kato-type condition.

In this paper we will study the validity of a Large Deviation Principle for the inviscid limit of Navier-Stokes equations and Second-Grade Fluid Equations with additive Gaussian noise in two-dimensional bounded domains and no-slip boundary conditions, presenting, for the first system, a natural Kato-type condition that closely resembles the ones from classical conditioned results [36], [48]. According to the discussion in subsection 1.1 this is the critical case to analyze. Regarding the Large Deviation Principle for the inviscid limit of the Navier-Stokes equations, one technical issue that need to be addressed is the interplay between Kato-type conditions, i.e. some controls on the dissipation of the energy in the solutions of the stochastic Navier-Stokes equations within the boundary layer, and the large fluctuations away from the zero-noise and zero-viscosity limit.

Large Deviation for fluid dynamic models in 2D have been established in the case of Navier-Stokes with additive noise, see [15], and multiplicative noise, see [59]. While the first result is based on a technique developed by Freidlin and Wentzell, based on a discretization of the equation and the application of the so-called contraction principle, the second one resorts to the the Weak Convergence Approach developed in [11, 12]. While the Freidlin-Wentzell technique is best suited for equations with additive noise, the Weak Convergence Approach has proved to be much more flexible in many other situations. As an example, in [29], the authors proved a LDP for the convergence of the Euler equation with transport noise on the 2D torus to a deterministic Navier-Stokes system using the weak convergence approach. We adopt this approach as well, even if our equations have additive noise, as the vanishing of the viscosity together with the noise constitutes a technical issue that cannot be addressed via a classical contraction argument.

The validity of a Large Deviation Principle for the inviscid limit of the Navier-Stokes equations with periodic or free boundary conditions has been shown in [6] using the weak convergence approach. Similarly to other results with these kind of boundary conditions, the result of [6] is based on the validity of the enstrophy equality, which allows to obtain stable estimates in the limit $\varepsilon \to 0$ stronger than the one guaranteed by the energy equality. These relations are not available in the case of no-slip boundary conditions, due to the generation of vorticity close the boundary. Therefore, the introduction of some Kato-type hypothesis, see [36], is required in order to show the validity of the Large Deviation Principle, similarly to the validity of the inviscid limit. On the contrary, as described in subsection 1.1, there are fluid dynamic frameworks where the inviscid limit holds in the bounded domain without any assumption on the behavior of the solution in the boundary layer. This is the case of the second grade fluid equations with the scaling of the parameters introduced in [46] and the case of the Navier-Stokes equations in the open ball with forcing and initial conditions with radial symmetry.

1.3 Plan of the paper

The goal of this paper is to show the validity of the large deviations principle for the inviscid limit of relevant fluid dynamical systems in 2D bounded domain via the weak convergence approach introduced in [11, 12]. In section 2 we will introduce some facts used repeatedly among the paper. In particular, in subsection 2.1, we will recall several notions about Large Deviations, presenting their classical formulation and some equivalent ones. In subsection 2.2 we will present some well-known facts about the systems under study that will play a role in the rest of the paper. In subsection 2.3 we will state our main theorems. The analysis of the validity of a Large Deviation Principle for the inviscid limit of the Navier-Stokes equations is the object of section 3. In particular we will start proving the validity of the Large Deviation Principle assuming initial conditions and forcing terms with radial symmetry in subsection 3.1. Secondly, we will prove Theorem 2.23 in subsection 3.2, namely the validity of the Large Deviation Principle assuming a Kato condition for the Navier-Stokes equations, but without imposing any symmetry in the flow of the fluid. In section 4 we will prove the validity of the Large Deviation Principle for the Second-Grade fluid equations under the scaling of the parameter introduced in [46] and provide a toy model where we can compute the asymptotic behaviour of our rate function in terms of some parameters modeling the oscillations of the flow of the fluid. Lastly we add some comments on the Kato condition assumed in this paper in section 5.

2 Preliminaries and main results

2.1 Large deviations principle

We recall here the abstract framework of the weak convergence approach to Large Deviations developed in [11, 12]. We begin with an usual filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, $t \in [0,T]$. Let $\mathcal{H}$ be an Hilbert space and $\mathcal{Q}$ a trace-class operator on $\mathcal{H}$. We can endow the space $\mathcal{H}_0 := \mathcal{Q}^{1/2}\mathcal{H}$ with the metric induced by $\mathcal{Q}$, that is

$$\langle g, h\rangle_0 = \langle \mathcal{Q}^{-1/2} g, \mathcal{Q}^{-1/2} h\rangle$$

which makes $\mathcal{H}_0$ a Hilbert space. The norm induced by this inner product will be denoted $\|\cdot\|_{\mathcal{H}_0}$. When $\mathcal{Q}$ is the covariance operator of a Wiener process $\{W_t\}_{t\in[0,T]}$ on $\mathcal{H}$, we call $\mathcal{H}_0$ the reproducing kernel Hilbert space of W_t, or simply RKHS. We also define the space

$$S^N := S^N(\mathcal{H}_0) := \Big\{ v \in L^2(0,T;\mathcal{H}_0) : \quad \int_0^T \|v_s\|^2_{\mathcal{H}_0} ds \leqslant N \Big\},$$

which makes a Polish space when endowed with the weak topology. We denote by $\mathcal{P}_2 := \mathcal{P}_2(\mathcal{H}_0)$ the space of $\mathcal{H}_0$-valued, $\mathcal{F}_t$-predictable and $\mathbb{P}$-a.s. square integrable processes. Next we define

$$\mathcal{P}_2^N := \{\phi \in \mathcal{P}_2 : \ \phi(\omega) \in S^N \quad \mathbb{P} - a.s.\}.$$

Let $\mathcal{E}$ and $\mathcal{E}_0$ be Polish spaces.

Definition 2.1. *A function $I : \mathcal{E} \to [0,\infty]$ is called a rate function if for any $M < \infty$, the level set $\{f \in \mathcal{E} : I(f) \leqslant M\}$ is a compact subset of $\mathcal{E}$. A family of rate functions I_x on $\mathcal{E}$, parametrized by $x \in \mathcal{E}_0$, is said to have compact level sets on compacts if for all compact subsets K of $\mathcal{E}_0$ and each $M < \infty$, $\cup_{x\in K}\{f \in \mathcal{E} : I_x(f) \leqslant M\}$ is a compact subset of $\mathcal{E}$.*

Let us give now the definition of LDP in the original formulation by Varadhan (see [62])

Definition 2.2. *We say that a Large Deviation Principle holds for a family μ^ε of probability measures on a metric space $(\mathcal{E}, d)$ with rate function I and speed ε if for every Borel set Γ of $\mathcal{E}$*

$$I(\mathring{\Gamma}) \leqslant \liminf_{\varepsilon\to 0} \varepsilon \log(\mu^\varepsilon(\Gamma)) \leqslant \limsup_{\varepsilon\to 0} \varepsilon \log(\mu^\varepsilon(\Gamma)) \leqslant I(\bar{\Gamma}) \tag{2.1}$$

where $I(A) := -\inf_{f\in A} I(f)$.

This condition has been proved equivalent by Bryc in [8] to the so called Laplace principle. Here, we state a uniform version of this principle, that is, we let I and μ^ε depend also on some parameter $x \in \mathcal{E}_0$.

Definition 2.3. ***(Uniform Laplace Principle)*** *Let I_x be a family of rate functions on E parameterized by $x \in \mathcal{E}_0$ and assume that this family has compact level sets on compacts. The family of random variables $\{X^{x,\varepsilon}\}$ distributed according to $\mu^{x,\varepsilon}$ are said to satisfy the Laplace principle on $\mathcal{E}$ with rate function I_x, uniformly on compacts, if for all compact subsets $K \subset \mathcal{E}_0$ and all bounded continuous functions h mapping $\mathcal{E}$ into $\mathbb{R}$,*

$$\lim_{\varepsilon \to 0} \sup_{x \in K} \Big| \varepsilon \log \mathbb{E}_x \Big[\exp\big(-\varepsilon^{-1} h(X^{x,\varepsilon})\big)\Big] + \inf_{f \in \mathcal{E}} \{h(f) - I_x(f)\} \Big| = 0 \tag{2.2}$$

We are interested in the case when the family of measures μ^ε is given by the laws of some stochastic process $X^{x,\varepsilon}$ solving some SPDE and driven by $\varepsilon^{1/2} W_t$. In this case, we can often represent $X^{x,\varepsilon} = \mathcal{G}^\varepsilon(x, \varepsilon^{1/2} W)$ for some measurable map $\mathcal{G}^\varepsilon : \mathcal{E}_0 \times C([0,T];\mathcal{H}) \to \mathcal{E}$. In this setting, in [12] the authors provided a handy criterion that allows to deduce the uniform Laplace principle. This is known as the *weak convergence approach* to Large deviations. The criterion goes as follows.

Hypothesis 2.4. *There exists a measurable map $\mathcal{G}^0 : \mathcal{E}_0 \times C([0,T];\mathcal{H}) \to \mathcal{E}$ such that:*

1. *For any $N < \infty$ and compact set $K \subset \mathcal{E}_0$, $\Gamma_{K,N} := \{\mathcal{G}^0(x, \int_0^\cdot v_s ds) : v \in S^N,\ x \in K\}$ is a compact subset of $\mathcal{E}$.*
2. *Consider $N < \infty$ and families $\{x^\varepsilon\} \subset \mathcal{E}_0,\ \{u^\varepsilon\} \subset \mathcal{P}_2^N$ such that, as $\varepsilon \to 0$, $x^\varepsilon \to x$ and u^ε converge in law to u as S^N-valued random element, then $\mathcal{G}^\varepsilon\big(x^\varepsilon, \varepsilon^{1/2} W + \int_0^\cdot u_s^\varepsilon ds\big)$ converges in law to $\mathcal{G}^0(x, \int_0^\cdot u_s ds)$ in the topology of $\mathcal{E}$.*

Theorem 2.5. *Let $X^{\varepsilon,x} = \mathcal{G}^\varepsilon(x, \varepsilon^{1/2} W)$ and suppose Hypothesis 2.4 holds. Define, for $x \in \mathcal{E}_0$ and $f \in \mathcal{E}$*

$$I_x(f) := \inf_{\{v \in L_t^2 \mathcal{H}_0 :\ f = \mathcal{G}^0(x, \int_0^\cdot v_s ds)\}} \int_0^T \|v_s\|_{\mathcal{H}_0}^2 ds$$

with the convention that $\inf \varnothing = +\infty$. Assume that for all $f \in \mathcal{E}$, $x \to I_x(f)$ is a lower semicontinuous map from $\mathcal{E}_0$ to $[0,\infty]$. Then for all $x \in \mathcal{E}_0$, $f \to I_x(f)$ is a rate function on $\mathcal{E}$ and the family I_x, $x \in \mathcal{E}_0$ of rate functions has compact level sets on compacts. Furthermore, the family $\{X^{\varepsilon,x}\}$ satisfies the Laplace principle on $\mathcal{E}$, with the rate functions $\{I_x\}$, uniformly on compact subsets of $\mathcal{E}_0$.

2.2 Well-Known facts on fluid dynamic models

Let us start this section introducing some general assumptions which will be always adopted under our analysis even if not recalled.

Hypothesis 2.6.

- $0 < T < +\infty$.
- *D is a bounded, smooth, simply connected domain.*
- *$(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ is a filtered probability space such that $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space, $(\mathcal{F}_t)_{t \in [0,T]}$ is a right continuous filtration and $\mathcal{F}_0$ contains every $\mathbb{P}$ null subset of Ω.*

For square integrable semimartingales taking value in separable Hilbert spaces $U_1,\ U_2$ we will denote by $[M,N]_t$ the quadratic covariation process. If M, N take values in the same separable Hilbert space U with orthonormal basis u_i, we will denote by $\langle\langle M, N\rangle\rangle_t = \sum_{i \in \mathbb{N}} [\langle M, u_i\rangle_U, \langle N, u_i\rangle_U]_t$. For each $k \in \mathbb{N},\ 1 \leqslant p \leqslant \infty$ we will denote by

$L^p(D)$ and $W^{k,p}(D)$ the well-known Lebesgue and Sobolev spaces. We will denote by $C_c^\infty(D)$ the space of smooth functions with compact support and by $W_0^{k,p}(D)$ their closure with respect to the $W^{k,p}(D)$ topology. If $p=2$, we will write $H^k(D)$ (resp. $H_0^k(D)$) instead of $W^{k,2}(D)$ (resp. $W_0^{k,2}(D)$). Let X be a separable Hilbert space, denote by $L^p(\mathcal{F}_{t_0}, X)$ the space of p integrable random variables with values in X, measurable with respect to $\mathcal{F}_{t_0}$. We will denote by $L^p(0,T;X)$ the space of measurable functions from $[0,T]$ to X such that

$$\|u\|_{L^p(0,T;X)} := \left(\int_0^T \|u_t\|_X^p \, dt\right)^{1/p} < +\infty, \quad 1 \leqslant p < \infty$$

and obvious generalization for $p=\infty$. For any $r,\ p \geqslant 1$, we will denote by

$$L^p(\Omega, \mathcal{F}, \mathbb{P}; L^r(0,T;X))$$

the space of processes with values in X such that

1. $u(\cdot,t)$ is progressively measurable.
2. $u(\omega,t) \in X$ for almost all (ω,t) and

$$\mathbb{E}\left[\|u(\omega,\cdot)\|_{L^r(0,T;X)}^p\right] < +\infty.$$

Obvious generalizations for $p=\infty$ or $r=\infty$.

Lastly we will denote by $C_{\mathcal{F}}([0,T];X)$ the space of continuous adapted processes with values in X such that

$$\mathbb{E}\left[\sup_{t\in[0,T]}\|u_t\|_X^2\right] < +\infty.$$

Set

$$H = \{f \in L^2(D;\mathbb{R}^2),\ \operatorname{div} f = 0,\ f\cdot n|_{\partial D} = 0\},\ V = H_0^1(D;\mathbb{R}^2) \cap H,$$
$$D(A) = H^2(D;\mathbb{R}^2) \cap V.$$

Moreover we introduce the vector space

$$W_\varepsilon = \{u \in V:\ \operatorname{curl}(u - \varepsilon\Delta u) \in L^2(D;\mathbb{R}^2)\}$$

with norm $\|u\|_{W_\varepsilon}^2 = \|u\|^2 + \varepsilon\|\nabla u\|_{L^2(D;\mathbb{R}^2)}^2 + \|\operatorname{curl}(u-\varepsilon\Delta u)\|_{L^2(D)}^2$. We simply write W in place of W_1. It is well-known, see for example [16], that we can identify W_ε with the space

$$\hat{W} = \{u \in H^3(D;\mathbb{R}^2) \cap V\}.$$

Moreover there exists a constant such that

$$\|u\|_{H^3}^2 \leqslant C\left(\|u\|^2 + \|\nabla u\|_{L^2(D;\mathbb{R}^2)}^2 + \|\operatorname{curl}(u-\Delta u)\|_{L^2(D)}^2\right). \tag{2.3}$$

We denote by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ the inner product and the norm in H respectively. With some abuse of notation, sometimes, we will denote also the inner product and the norm in L^2 with the same symbols. Other norms and scalar products will be denoted with the proper subscript. On V we introduce the family of norms $\|u\|_{V_\varepsilon}^2 = \|u\|^2 + \varepsilon\|\nabla u\|_{L^2(D;\mathbb{R}^2)}^2$. Again, in case of $\varepsilon = 1$ we continue to write V in place of V_1. We will shortly denote by

$\|u\|_{\varepsilon,*} = \|\text{curl}(u - \varepsilon\Delta u)\|_{L^2(D)}$. Obviously the following inequality holds for $u \in V$, where C_p is the Poincaré constant associated to D,

$$\frac{\|u\|^2_{V_\varepsilon}}{\varepsilon + C_p^2} \leqslant \|\nabla u\|^2_{L^2(D;\mathbb{R}^2)} \leqslant \frac{\|u\|^2_{V_\varepsilon}}{\varepsilon} \tag{2.4}$$

Denote by P the linear projector of $L^2\left(D;\mathbb{R}^2\right)$ on H and define the unbounded linear operator $A : D(A) \subseteq H \to H$ by the identity

$$\langle Av, w\rangle = \langle \Delta v, w\rangle_{L^2(D;\mathbb{R}^2)} \tag{2.5}$$

for all $v \in D(A),\ w \in H$. A will be called the Stokes operator. It is well-known (see for example [60]) that A is self-adjoint, generates an analytic semigroup of negative type on H and moreover $V \sim D\left((-A)^{1/2}\right)$. We will denote by $e^{\varepsilon At}$ the strongly continuous semigroup on H generated by εA. Denote by $\mathbb{L}^4$ the space $L^4\left(D, \mathbb{R}^2\right) \cap H$, with the usual topology of $L^4\left(D, \mathbb{R}^2\right)$. Define the trilinear, continuous form $b : \mathbb{L}^4 \times V \times \mathbb{L}^4 \to \mathbb{R}$ as

$$b\left(u, v, w\right) = \langle u, P(\nabla v w)\rangle. \tag{2.6}$$

We will use also some properties of the projection operator P and the solution map of the Stokes operator. We refer to [60] for the proof of the lemmas below.

Lemma 2.7. *Let $r > 0$, the restriction of the projection operator $P : L^2(D;\mathbb{R}^2) \to H$ to $H^r(D;\mathbb{R}^2)$ is a continuous and linear map between $H^r(D;\mathbb{R}^2)$ and itself, i.e. if $u \in H^r(D;\mathbb{R}^2)$ than also $Pu \in H^r(D;\mathbb{R}^2)$ and*

$$\|Pu\|_{H^r(D;\mathbb{R}^2)} \leqslant C(D, r)\, \|u\|_{H^r(D;\mathbb{R}^2)}\,.$$

Lemma 2.8. *The injection of V in H is compact. Thus there exists a sequence e_i of elements of H which forms an orthonormal basis in H and an orthogonal basis in V. This sequence verifies*

$$-Ae_i = \lambda_i e_i$$

where $\lambda_{i+1} > \lambda_i > 0,\ i = 1, 2, \ldots$. Moreover $\lambda_i \to +\infty$. Lastly $e_i \in C^\infty(\overline{D};\mathbb{R}^2)$ under our assumptions on D.

The tools introduced above are the standard ingredients in order to deal with the Navier-Stokes equations. We need to recall some other facts in order to treat Second-Grade fluid equations. We refer to [16], [54], [55], [28] for the proof of the various statements.

Lemma 2.9. *For any smooth, divergence free $\phi,\ v,\ w$ the following relation holds*

$$\langle \text{curl}\,\phi \times v, w\rangle_{L^2} = b(v, \phi, w) - b(w, \phi, v). \tag{2.7}$$

Moreover for $u,\ v,\ w$ the following inequalities hold

$$|\langle \text{curl}(u - \varepsilon\Delta u) \times v, w\rangle_{L^2}| \leqslant C_\varepsilon \|u\|_{H^3}\|v\|_{V_\varepsilon}\|w\|_{W_\varepsilon}, \tag{2.8}$$

$$|\langle \text{curl}(u - \varepsilon\Delta u) \times u, w\rangle_{L^2}| \leqslant C_\varepsilon \|u\|^2_{V_\varepsilon}\|w\|_{W_\varepsilon} \tag{2.9}$$

Therefore there exists a bilinear operator $\hat{B}_\varepsilon : W_\varepsilon \times V_\varepsilon \to W_\varepsilon^*$ such that

$$\langle \hat{B}_\varepsilon(u, v), w\rangle_{W_\varepsilon^*, W_\varepsilon} = \langle P(\text{curl}(u - \varepsilon\Delta u) \times v), w\rangle \tag{2.10}$$

which satisfies for $u \in V,\ v \in W$

$$\|\hat{B}_\varepsilon(v, u)\|_{W_\varepsilon^*} \leqslant C_\varepsilon \|u\|_{V_\varepsilon}\|v\|_{W_\varepsilon}, \tag{2.11}$$

$$\|\hat{B}_\varepsilon(u, u)\|_{W_\varepsilon^*} \leqslant C_\varepsilon \|u\|^2_{V_\varepsilon}. \tag{2.12}$$

Lastly, for $u \in W,\ v \in V,\ w \in W$

$$\langle \hat{B}_\varepsilon(u, v), w\rangle_{W_\varepsilon^*, W_\varepsilon} = -\langle \hat{B}_\varepsilon(u, w), v\rangle_{W_\varepsilon^*, W_\varepsilon}. \tag{2.13}$$

Theorem 2.10. *Each function $f \in H^2(D)$ satisfies the following inequality:*

$$\|f\|_{H^1} \leqslant C\|f\|_{L^2}^{1/2}\|f\|_{H^2}^{1/2}. \tag{2.14}$$

Now we are ready to introduce some assumptions on the stochastic part of systems (1.2), (1.3).

Hypothesis 2.11. $W_t = \sum_{k\in K} \sigma_k W_t^k$ *where*

- K *is a (possibly countable) set of indexes,* $\gamma \geqslant 2$.
- $\sigma_k \in D((-A)^\gamma)$ *satisfying*

$$\sum_{k\in K} \|\sigma_k\|^2_{D((-A)^\gamma)} < +\infty.$$

- $\{W_t^k\}_{k\in K}$ *is a sequence of real, independent Brownian motions adapted to* $\mathcal{F}_t$.

We denote by H_0 the RKHS associated to W_t.

Remark 2.12. Previous assumptions on the noise implies in particular that $H_0 \hookrightarrow D((-A)^\gamma)$ and that W is a process with continuous paths with values in $D((-A)^\gamma)$. Since $\lambda_i \sim Ci$, see [33], a simple example of noise satisfying Hypothesis 2.11 is $W_t = (-A)^{-\gamma-1/2-\delta}W_t^H$, $\delta > 0$ and W_t^H being the cylindrical Wiener process on H. With this particular choice of the coefficients σ_k, $H_0 = D((-A)^{\gamma+1/2+\delta})$.

Since we are going to prove the validity of the Large Deviation Principle via the weak convergence approach, we will need to analyze the well-posedness of some partial differential equations, slightly more general than (1.2), (1.3). Therefore, let $\beta \geqslant 0$ and $f \in \mathcal{P}_2^N,\ N \geqslant 0$ arbitrary we consider the stochastic partial differential equations below

$$\begin{cases} du^{NS,\varepsilon} &= (\varepsilon\Delta u^{NS,\varepsilon} - u^{NS,\varepsilon}\cdot\nabla u^{NS,\varepsilon} + \nabla p^{NS,\varepsilon} + f)dt + \sqrt{\beta}dW_t \\ \operatorname{div} u^{NS,\varepsilon} &= 0 \\ u^{NS,\varepsilon}|_{\partial D} &= 0 \\ u^{NS,\varepsilon}(0) &= u_0, \end{cases} \tag{2.15}$$

$$\begin{cases} dv^{SG,\varepsilon} &= (\nu\Delta u^{SG,\varepsilon} - \operatorname{curl}(v^{SG,\varepsilon}) \times u^{SG,\varepsilon} + \nabla p^{SG,\varepsilon} + f)dt + \sqrt{\beta}dW_t \\ \operatorname{div} u^{SG,\varepsilon} &= 0 \\ v^{SG,\varepsilon} &= u^{SG,\varepsilon} - \varepsilon\Delta u^{SG,\varepsilon} \\ u^{SG,\varepsilon}|_{\partial D} &= 0 \\ u^{SG,\varepsilon}(0) &= u_0. \end{cases} \tag{2.16}$$

Definition 2.13. *A stochastic process with continuous trajectories with values in H is a weak solution of equation* (2.15) *if*

$$u^{NS,\varepsilon} \in C_{\mathcal{F}}([0,T];H) \cap L^2(\Omega,\mathcal{F},\mathbb{P};L^2(0,T;V))$$

and $\mathbb{P} - a.s.$ for every $t \in [0,T]$ and $\phi \in D(A)$ we have

$$\langle u_t^{NS,\varepsilon} - u_0, \phi\rangle + \int_0^t \varepsilon\langle\nabla u_s^{NS,\varepsilon}, \nabla\phi\rangle_{L^2(D;\mathbb{R}^2)} = \int_0^t b(u_s^{NS,\varepsilon}, \phi, u_s^{NS,\varepsilon})ds + \int_0^t \langle f_s, \phi\rangle ds + \sqrt{\beta}\langle W_t, \phi\rangle.$$

Definition 2.14. *A stochastic process with weakly continuous trajectories with values in W is a weak solution of equation* (2.16) *if*

$$u^{SG,\varepsilon} \in L^2(\Omega,\mathcal{F},\mathbb{P};L^\infty(0,T;W))$$

and $\mathbb{P}-a.s.$ *for every* $t\in[0,T]$ *and* $\phi\in W$ *we have*

$$\langle u_t^{SG,\varepsilon}-u_0,\phi\rangle_{V_\varepsilon}+\int_0^t \nu\langle\nabla u_s^{SG,\varepsilon},\nabla\phi\rangle+\langle \mathrm{curl}(u_s^{SG,\varepsilon}-\varepsilon\Delta u_s^{SG,\varepsilon})\times u_s^{SG,\varepsilon},\phi\rangle ds$$
$$=\int_0^t\langle f_s,\phi\rangle ds+\sqrt{\beta}\langle W_t,\phi\rangle.$$

The well-posedness of (2.15) (resp. (2.16)) in the sense of Definition 2.13 (resp. Definition 2.14) is guaranteed by Theorem 2.15 below, see [27] (resp. Theorem 2.16, see [55], [49, Section 6]).

Theorem 2.15. *For each* $u_0\in H^3(D;\mathbb{R}^2)\cap H$ *there exists a unique weak solution of* (2.15) *in the sense of Definition 2.13. Moreover the following relation holds true*

$$\|u_t^{NS,\varepsilon}\|^2+2\varepsilon\int_0^t\|\nabla u_s^{NS,\varepsilon}\|_{L^2}^2ds=\|u_0\|^2+t\beta\sum_{k\in K}\|\sigma_k\|^2+2\sqrt{\beta}\int_0^t\langle u^{NS,\varepsilon},dW_s\rangle$$
$$+2\int_0^t\langle f_s,u^{NS,\varepsilon}\rangle ds\quad \mathbb{P}-a.s. \tag{2.17}$$

Theorem 2.16. *For each* $u_0\in W$ *there exists a unique weak solution of* (2.16) *in the sense of Definition 2.14. Moreover* $u^{SG,\varepsilon}$ *has continuous paths with values in* V *and it holds*

$$\|u_t^{SG,\varepsilon}\|_{V_\varepsilon}^2+2\nu\int_0^t\|\nabla u_s^{SG,\varepsilon}\|_{L^2}^2ds=\|u_0\|_{V_\varepsilon}^2+t\beta\sum_{k\in K}\|(I-\varepsilon A)^{-1/2}\sigma_k\|^2$$
$$+2\sqrt{\beta}\int_0^t\langle u^{SG,\varepsilon},dW_s\rangle+2\int_0^t\langle f_s,u^{SG,\varepsilon}\rangle ds\quad \mathbb{P}-a.s. \tag{2.18}$$

Calling $q^{SG,\varepsilon}=\mathrm{curl}(u^{SG,\varepsilon}-\varepsilon\Delta u^{SG,\varepsilon}),\ s_k=\mathrm{curl}\,\sigma_k$ *it holds*

$$\|q_t^{SG,\varepsilon}\|_{L^2}^2=\|u_0\|_{\varepsilon,*}^2-\frac{2\nu}{\varepsilon}\int_0^t\langle q_s^{SG,\varepsilon}-\mathrm{curl}\,u_s^{SG,\varepsilon},q_s^{SG,\varepsilon}\rangle ds+t\beta\sum_{k\in K}\|s_k\|^2$$
$$+2\sqrt{\beta}\sum_{k\in K}\int_0^t\langle s_k,q_s\rangle dW_s^k+2\int_0^t\langle \mathrm{curl}\,f_s,q_s\rangle ds\quad \mathbb{P}-a.s. \tag{2.19}$$

Lastly we need to recall some results about the well-posedness of Euler equations with forcing term $f\in\mathcal{P}_2^N$, namely solutions of

$$\begin{cases}\partial_t\overline{u}+\overline{u}\cdot\nabla\overline{u} &=\nabla\overline{p}+f\\ \mathrm{div}\,\overline{u} &=0\\ \overline{u}\cdot n|_{\partial D} &=0\\ \overline{u}(0) &=u_0.\end{cases} \tag{2.20}$$

Definition 2.17. *A stochastic process with continuous trajectories with values in* H *is a weak solution of equation* (2.20) *if* $\mathbb{P}-a.s.$ *for every* $t\in[0,T]$ *and* $\phi\in C_c^\infty(D;\mathbb{R}^2)$ *we have*

$$\langle\overline{u}_t-u_0,\phi\rangle=\int_0^t b(\overline{u}_s,\phi,\overline{u}_s)ds-\int_0^t\langle f_s,\phi\rangle ds.$$

The well-posedness of (2.20) in regular spaces is a classical result, see for example [5], [4], [30].

Theorem 2.18. *For each* $u_0 \in H^3(D;\mathbb{R}^2) \cap H$ *there exists a unique weak solution of* (2.20) *with trajectories in* $C([0,T];W^{2,4}(D;\mathbb{R}^2))$*. Moreover*

$$\|\overline{u}_t\|^2 = \|u_0\|^2 + 2\int_0^t \langle f_s, \overline{u}_s\rangle ds \quad \mathbb{P}-a.s. \tag{2.21}$$

$$\sup_{t\in[0,T]}\|\overline{u}_t\|_{W^{2,4}} \leqslant C(\|u_0\|_{W^{2,4}}, N) \quad \mathbb{P}-a.s. \tag{2.22}$$

Remark 2.19. The well-posedness of equation (2.15), equation (2.16) holds under weaker assumptions on the noise than Hypothesis 2.11. We need to assume a noise so regular in order to guarantee that there exists a unique solution of (2.20) which belongs to $C([0,T];W^{2,4}(D;\mathbb{R}^2)) \cap C([0,T];H)$.

2.3 Main results

As stated in section 1, our goal is to prove a Large Deviation Principle via the weak convergence approach introduced in subsection 2.1. Therefore we need to introduce some maps $\mathcal{G}^{NS,\varepsilon},\ \mathcal{G}^{SG,\varepsilon},\ \mathcal{G}^0$. Following the notation of subsection 2.1, let

$$\mathcal{E}_0^{NS} := H^3(D;\mathbb{R}^2) \cap H, \quad \mathcal{E}_0^{SG} := W, \quad \mathcal{E} := C([0,T];H).$$

According to Theorem 2.18 we can introduce the measurable map

$$\mathcal{G}^{NS,0} : \mathcal{E}_0^{NS} \times C([0,T];H) \to \mathcal{E} \quad \big(\textit{resp. } \mathcal{G}^{SG,0} : \mathcal{E}_0^{SG} \times C([0,T];H) \to \mathcal{E}\big)$$

which associates to each $u_0 \in \mathcal{E}_0^{NS}$ (resp. $u_0 \in \mathcal{E}_0^{SG}$) and $\int_0^\cdot f_s ds,\ f \in L^2(0,T;H_0)$ the unique regular solution of (2.20) with initial condition u_0 and forcing term f guaranteed by Theorem 2.18, 0 otherwise. Analogously thanks to Theorem 2.15 (resp. Theorem 2.16) we can introduce the measurable map

$$\mathcal{G}^{NS,\varepsilon} : \mathcal{E}_0^{NS} \times C([0,T];H) \to \mathcal{E} \quad \big(\textit{resp. } \mathcal{G}^{SG,\varepsilon} : \mathcal{E}_0^{SG} \times C([0,T];H) \to \mathcal{E}\big)$$

such that for each $u_0 \in \mathcal{E}_0^{NS}$ (resp. $u_0 \in \mathcal{E}_0^{SG}$), $\mathcal{G}^{NS,\varepsilon}(u_0, \sqrt{\varepsilon}W_\cdot)$ (resp. $\mathcal{G}^{SG,\varepsilon}(u_0, \sqrt{\varepsilon}W_\cdot)$ is the unique weak solution of (2.15) (resp. (2.16)) with $\beta = \varepsilon$, initial condition u_0 and null forcing term guaranteed by Theorem 2.15 (resp. Theorem 2.16). More generally, it follows that, if $f \in \mathcal{P}_2^N$, $\mathcal{G}^{NS,\varepsilon}(u_0, \sqrt{\varepsilon}W_\cdot + \int_0^\cdot f_s ds)$ (resp. $\mathcal{G}^{SG,\varepsilon}(u_0, \sqrt{\varepsilon}W_\cdot + \int_0^\cdot f_s ds)$) is the unique solution of (2.15) (resp. (2.16)) $\beta = \varepsilon$, initial condition u_0 and forcing term f. When dealing with the inviscid limit for Navier-Stokes equations and no-slip boundary conditions one can choose either to assume a Kato-type hypothesis or to require strong assumptions on the regularity of the domain, initial conditions and forcing term. We will follow both these lines. In the following, given $c > 0$, we will denote $\Gamma_{c\varepsilon} = \{x \in D : \quad d(x,\partial D) \leqslant c\varepsilon\}$.

Hypothesis 2.20 (Strong Kato Hypothesis)**.** *For each* $N \in \mathbb{N}$, $u_0^\varepsilon,\ u_0 \in \mathcal{E}_0^{NS}$ *and* $f^\varepsilon,\ f \in \mathcal{P}_2^N$ *such that* $u_0^\varepsilon \to u_0$ *in* $\mathcal{E}_0^{NS}$ *and* $f^\varepsilon \to_{\mathcal{L}} f$ *in* S^N*, if* $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ *is a filtered probability space where all* f^ε*,* f *are defined together and* $f^\varepsilon \to f$ $\mathbb{P}-a.s.$ *in* S^N*, then, it exists* $c > 0$ *such that for every* $\delta > 0$

$$\mathbb{P}\left(\varepsilon\int_0^T \left\|\nabla\mathcal{G}^{NS,\varepsilon}\left(u_0^\varepsilon, \sqrt{\varepsilon}W_\cdot + \int_0^\cdot f_s^\varepsilon ds\right)\right\|_{L^2(\Gamma_{c\varepsilon})}^2 ds > \delta\right) \to 0.$$

Remark 2.21. In the previous condition, the set $\mathcal{E}_0^{NS}$ need not to be the full space $H^3(D;\mathbb{R}^2) \cap H$, but it can be a closed subset of it, even consisting of a singleton. Of course, the LDP that we will be able to prove will then only be uniform with respect to initial conditions belonging to such subset (cf. Definition 2.3). In contrast, the space in which the forcing f varies cannot be restricted as easily, without substantially weakening the strength of the LDP (see the discussion in Section 5).

Remark 2.22. By Skorokhod's representation theorem, given $f^\varepsilon,\ f \in \mathcal{P}_2^N$ such that $f^\varepsilon \to_{\mathcal{L}} f$ in S^N there exists at least a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ where all f^ε, f are defined together and $f^\varepsilon \to f\ \mathbb{P} - a.s.$ in S^N.

Now we are ready to state our main result on the validity of a Large Deviation Principle under Hypothesis 2.20.

Theorem 2.23. *Assuming Hypothesis 2.20, the family* $\{u^{NS,\varepsilon} = \mathcal{G}^{NS,\varepsilon}(u_0, \sqrt{\varepsilon}W_\cdot)\}_{u_0 \in \mathcal{E}_0^{NS}}$ *satisfies the uniform Laplace principle with the rate function*

$$\begin{aligned} I_{u_0}^{NS}(v) &= \inf_{\{f \in L^2(0,T;H_0):\ v = \mathcal{G}^{NS,0}(u_0, \int_0^\cdot f_s ds)\}} \frac{1}{2}\int_0^T \|f_s\|_{H_0}^2 ds \\ &= \frac{1}{2}\int_0^T \|\partial_t v_s + P(v_s \cdot \nabla v_s)\|_{H_0}^2 ds. \end{aligned}$$

where $u_0 \in \mathcal{E}_0^{NS},\ v \in C([0,T];H)$*, with the convention that* $I_{u_0}^{NS}(v) = +\infty$ *anytime* v *is not in the range of* $\mathcal{G}^{NS,0}(u_0, \cdot)$.

The last equality is guaranteed by the injectivity of the map $\mathcal{G}^{NS,0}$ in the second component, which in turn is a consequence of the uniqueness for the Euler system in our setting, see Theorem 2.18.

The validity of Hypothesis 2.20 is in contrast to the so-called *Kolmogorov's zeroth law of turbulence*, see [40], [41], [39]. The latter describing the physical evidence that the anomalous dissipation of the kinetic energy holds for three dimensional fluids at high Reynolds number. While, nowadays, the *Kolmogorov's zeroth law of turbulence* is a well-accepted assumption for three dimensional fluids where counterexamples to Hypothesis 2.20 have been shown in the case of deterministic forcing and domains without boundaries, see for example [7] for an explicit counterexample and [35], [53] for some numerical discussions, the situation is less clear in the two dimensional case. This is due the fact that either Navier-Stokes and Euler's flows preserve smooth solutions. We refer to [19] and the references therein for further discussions on this topic. Indeed, even if Hypothesis 2.20 may look too restrictive, we will provide, thanks to Theorem 2.27, an explicit example where it is satisfied, see also Remark 3.9 and Remark 3.10 below. Moreover, we will come back on the meaning of Hypothesis 2.20 and its relation with a more classical version of the Kato Hypothesis, i.e. non depending on f^ε, in section 5.

As pointed out in [46], [49] in order to obtain an unconditioned result for the Second-Grade fluid equations, we cannot take any scaling of $\nu \to 0$ but it is necessary to assume:

Hypothesis 2.24. $\nu = O(\varepsilon)$.

Now we can state our main result on the Second Grade Fluid equations.

Theorem 2.25. *Assuming Hypothesis 2.24, the family* $\{u^{SG,\varepsilon} = \mathcal{G}^{SG,\varepsilon}(u_0, \sqrt{\varepsilon}W_\cdot)\}_{u_0 \in \mathcal{E}_0^{SG}}$ *satisfies the uniform Laplace principle with the rate function*

$$\begin{aligned} I_{u_0}^{SG}(v) &= \frac{1}{2}\inf_{f \in L^2(0,T;H_0):\ v = \mathcal{G}^{SG,0}(u_0, \int_0^\cdot f_s ds)} \int_0^T \|f_s\|_{H_0}^2 ds \\ &= \frac{1}{2}\int_0^T \|\partial_t v_s + P(v_s \cdot \nabla v_s)\|_{H_0}^2 ds \end{aligned}$$

where $u_0 \in \mathcal{E}_0^{SG},\ v \in C([0,T];H)$*, with the convention that* $I_{u_0}^{SG}(v) = +\infty$ *anytime* v *is not in the range of* $\mathcal{G}^{SG,0}(u_0, \cdot)$.

Lastly we want to consider the case of fluids with radial symmetry. In such case the inviscid limit in general holds without any assumptions on the behavior of the fluid in the boundary layer as observed in [45]. Therefore, calling B the open ball in $\mathbb{R}^2$, centered in

0 with radius 1, we introduce

$$\mathcal{H}^{RS} = \overline{\left\{ \frac{x^{\perp}}{|x|}\bar{u}(|x|), \quad \bar{u} \in C_c^{\infty}(0,1) \right\}}^{D((-A)^{\gamma})}$$

endowed with the $D((-A)^{\gamma})$ norm and

$$\mathcal{E}_0^{RS} = \mathcal{E}_0^{NS} \cap \left\{ u = \frac{x^{\perp}}{|x|}\bar{u}(|x|), \quad \bar{u} \in L^2(0,1) \right\}$$

endowed with the H^3 norm. As above we need to introduce a particular forced Navier-Stokes systems:

$$\begin{cases} du^{RS,\varepsilon} & = (\varepsilon \Delta u^{RS,\varepsilon} - u^{RS,\varepsilon} \cdot \nabla u^{RS,\varepsilon} + \nabla p^{RS,\varepsilon} + f)dt + \sqrt{\varepsilon} dW_t^{RS} \\ \operatorname{div} u^{RS,\varepsilon} & = 0 \\ u^{RS,\varepsilon}|_{\partial D} & = 0 \\ u^{RS,\varepsilon}(0) & = u_0. \end{cases} \tag{2.23}$$

Now can introduce the assumptions in order to deal the case with radial symmetry and study the Large Deviation Principle in this framework:

Hypothesis 2.26. *$D = B$, $W_t^{RS} = \sum_{k \in K} \sigma_k W_t^k$ where*

- *K is a (possibly countable) set of indexes, $\gamma \geqslant 2$.*
- *$\sigma_k \in \mathcal{H}^{RS}$ satisfying*

$$\sum_{k \in K} \|\sigma_k\|^2_{D((-A)^{\gamma})} < +\infty.$$

- *$\{W_t^k\}_{k \in K}$ is a sequence of real, independent Brownian motions adapted to $\mathcal{F}_t$.*

We denote by H_0^{RS} the RKHS associated to W_t^{RS}.
Since Theorem 2.15, Theorem 2.18 continue to hold considering

$$u_0 \in \mathcal{E}_0^{RS}, \quad f \in L^2(0,T; H_0^{RS})$$

and assuming Hypothesis 2.26, we can define the measurable maps $\mathcal{G}^{RS,\varepsilon}$ and $\mathcal{G}^{RS,0}$ as above for $\mathcal{G}^{NS,\varepsilon}$ and $\mathcal{G}^{NS,0}$ considering $\mathcal{E}_0^{RS}$ instead of $\mathcal{E}_0^{NS}$.

Theorem 2.27. *Assuming Hypothesis 2.26, the family $\{u^{RS,\varepsilon} = \mathcal{G}^{RS,\varepsilon}(u_0, \sqrt{\varepsilon} W_\cdot^{RS})\}_{u_0 \in \mathcal{E}_0^{RS}}$ satisfies the uniform Laplace principle with the rate function*

$$\begin{aligned} I_{u_0}^{RS}(v) &= \frac{1}{2} \inf_{f \in L^2(0,T;H_0^{RS}):\ v = \mathcal{G}^{RS,0}(u_0, \int_0^{\cdot} f_s ds)} \int_0^T \|f_s\|^2_{H_0^{RS}} ds \\ &= \frac{1}{2} \int_0^T \|\partial_t v_s + P(v_s \cdot \nabla v_s)\|^2_{H_0^{RS}} ds \end{aligned}$$

where $u_0 \in \mathcal{E}_0^{RS},\ v \in C([0,T]; H)$, with the convention that $I_{u_0}^{SG}(v) = +\infty$ anytime v is not in the range of $\mathcal{G}^{RS,0}(u_0, \cdot)$.

Let us observe that all of the three functionals have the same integral representation, the only difference being the space where u_0 and the noise take values. This phenomenon roughly means that the LDPs we are able to establish are not strong enough to distinguish between the two models at the level of the asymptotics of their fluctuations. In principle, a sharper LDP could display such differences. This kind of results would be of utmost interest also in the periodic framework, acting as selection principles for solutions coming from convex integration schemes. Secondly, let us discuss about possible consequences of our main results. The problems of the exit from a domain in the state space and the large deviations for the invariant measures are very common applications in literature of Large Deviation Principles to the description of rare events. The first problem is related to estimating the typical (exponential) timescale on which the solution of the system starting from a domain in the state space $\mathcal{O}$ that contains one of its equilibrium point, leave this set, and possibly identify also the escaping set (the subset of $\partial\mathcal{O}$ where the escape is most likely to occur) and the most typical trajectory of this escape. The second problem is related to the (exponential) rate of convergence of the invariant measure of the system. In the context of stochastic Navier-Stokes, the exit time problem was studied, in the small noise limit, by [9], then their results were used to study the LDP for the invariant measure in [10]. Let us mention also the works [13] and [14] which proved the same results but for a noise that becomes both small and white (in space) in the limit $\varepsilon \to 0$. Concerning these problems in our framework, the study of the convergence of the invariant measures in the small noise-small viscosity limit, is central in the study of the statistical properties of a turbulent fluid, see for example [51], [42], [43, Chapter 10] and the references therein. In the case of the zero-noise, zero viscosity limit, this kind of results are, however, much harder to obtain from the large deviation principle due to the conservative nature of the limit object and the presence of multiple stable configurations of the deterministic system. Indeed, no results are available in this direction even in the periodic setting and we think our results, as well as the ones of [6], as a first step in order to address such questions.

Remark 2.28. Theorem 2.23, Theorem 2.25 and Theorem 2.27 continue to hold also if we add a deterministic forcing term g in $L^2(0,T;H_0)$ or $L^2(0,T;H_0^{RS})$ in equations (1.2), (1.3), up to re-defining the maps $\mathcal{G}^\varepsilon$ and $\mathcal{G}^0$ accordingly. Indeed the computations below can be easily adapted to this framework. Moreover, it is enough to assume the validity of Hypothesis 2.20 for equation (1.2) without any forcing g in order to prove the validity of Theorem 2.23 also if we add the forcing term g.

Remark 2.29. With particular choices of the noise coefficients, the Large deviations functional can be made more explicit, as a relevant example, in the framework of Remark 2.12, $I_{u_0}^{NS}(v)$ reduces to

$$I_{u_0}^{NS}(v) = \frac{1}{2}\int_0^T \|(-A)^{\gamma+1/2+\delta}\left(\partial_s v_s + P(v_s\cdot\nabla v_s)\right)\|^2 ds,$$

similarly for the second grade fluid equations.

We conclude this section with few notation that will be adopted: by C we will denote several constant independent from ε, ν, perhaps changing value line by line. If we want to keep track of the dependence of C from some parameter ρ we will use the symbol $C(\rho)$. Sometimes we will use the notation $a \lesssim b$, if it exists a constant independent from ν and ε such that $a \leqslant Cb$. In order to simplify the notation we will denote Sobolev spaces by $W^{s,p}$, H^s, forgetting domain and range and use Einstein summation convention.

3 Navier-Stokes

Let us start discussing Condition 1 in Hypothesis 2.4. Since $\mathcal{E}_0^{RS} \hookrightarrow \mathcal{E}_0^{NS}$ and $\mathcal{H}^{RS} \hookrightarrow D((-A)^\gamma)$, by definition of the maps $\mathcal{G}^{NS,0}, \mathcal{G}^{RS,0}$ we have

$$\mathcal{G}^{NS,0}(u_0, \int_0^\cdot f_s ds) = \mathcal{G}^{RS,0}(u_0, \int_0^\cdot f_s ds)$$

if D is the open ball centered in 0 with radius 1, $u_0 \in \mathcal{E}_0^{RS}$ and $f_s \in L^2(0,T;H_0^{RS})$. Moreover, since $\mathcal{E}_0^{RS}$ and $\mathcal{E}_0^{NS}$ (resp. $\mathcal{H}^{RS}$ and $D((-A)^\gamma)$) are Banach spaces endowed with the same norm, it is enough to show the validity of Condition 1 in Hypothesis 2.4 for $\mathcal{G}^{NS,0}$ in order to have that the same holds also for $\mathcal{G}^{RS,0}$. Therefore we limit ourselves to show the validity of Condition 1 in Hypothesis 2.4 for $\mathcal{G}^{NS,0}$.

Let us fix $N>0,\ K$ a compact subset of $\mathcal{E}_0^{NS}$, we want to show that the set

$$K_N=\{\mathcal{G}^{NS,0}(x_0,\int_0^{\cdot} f_s ds)\quad v\in S^N, x_0\in K\}\overset{c}{\hookrightarrow}\mathcal{E}.$$

Therefore let us fix two sequences $\{x_0^n\}_{n\in\mathbb{N}}\subset K,\ \{f^n\}_{n\in\mathbb{N}}\subset S^N$. Since K is compact subset of $\mathcal{E}_0^{NS},\ \|f^n\|^2_{L^2(0,T;H_0)}\leqslant N$ we can find a subsequence $\{n_k\}_{k\in\mathbb{N}},\ x\in K,\ f\in S^N$ such that $x_0^{n_k}\to x_0$ in $\mathcal{E}_0^{NS},\ f^{n_k}\rightharpoonup f$ in $L^2(0,T;H_0)$. Let $u^{n_k}:=\mathcal{G}^{NS,0}(x_0^{n_k},\int_0^{\cdot} f_s^{n_k}ds)$ (resp. $u:=\mathcal{G}^{NS,0}(x_0,\int_0^{\cdot} f_s ds)$). According to Theorem 2.18, u^{n_k} (resp. u) is the unique regular weak solution of (2.20) with initial condition $x_0^{n_k}$ (resp. x_0) and forcing term f^{n_k} (resp. f). Our goal is to show that $u^{n_k}\to u$ in $\mathcal{E}$. We emphasize, once for all, that the weak convergence of f^{n_k} to f is not directly sufficient in order to show the validity of *Condition 1* in Hypothesis 2.4. Therefore, we will need to move to some integrated-in-time version of f^{n_k} and f in order to gain strong convergence, uniform in time, in weaker Sobolev spaces. We will adopt similar arguments also in the forthcoming sections in order to establish the validity of *Condition 2* in Hypothesis 2.4. Fix $\theta>0$ arbitrarily small and define $F^{n_k}=\int_0^{\cdot} f_s^{n_k}ds,\ F=\int_0^{\cdot} f_s ds$. By Hypothesis 2.11, $H_0\hookrightarrow D((-A)^2)$. This implies, see for example [27, Proposition 2.10] that

$$F^{n_k}\to F\quad \text{in } C([0,T];D((-A)^{2-\theta})). \tag{3.1}$$

Obviously

$$\sup_{k\geqslant 1}\|F^{n_k}\|_{C([0,T];D((-A)^2))}+\|F\|_{C([0,T];D((-A)^2))}\leqslant C(N). \tag{3.2}$$

Lastly, since $x_0^{n_k}\to x_0$ in $\mathcal{E}_0^{NS}$, from (2.22) we can find a constant $C=C(N,x_0)$ only depending on N and $\|x\|_{\mathcal{E}_0^{NS}}$ such that

$$\sup_{k\geqslant 1}\|u^{n_k}\|_{C([0,T];W^{2,4})}+\|u\|_{C([0,T];W^{2,4})}\leqslant C(\|x_0\|_{W^{2,4}},N). \tag{3.3}$$

We introduce

$$v_t^{n_k}=u_t^{n_k}-F_t^{n_k},\quad v_t=u_t-F_t.$$

By triangle inequality $v^{n_k},\ v$ satisfy relation (3.3), too. Since $F^{n_k}\to F$ in $\mathcal{E}$, it is enough to show that $v^{n_k}\to v$ in $\mathcal{E}$ in order to prove the validity of Condition 1 in Hypothesis 2.4. This is the aim of Lemma 3.1 below. We will follow the idea introduced in [65] to show uniqueness of the solutions with bounded vorticity of the Euler equations. However, in order to prove the continuous dependence from the data we exploit the higher regularity and the uniform bounds guaranteed by relation (3.3).

Lemma 3.1. $v^{n_k}\to v$ *in* $C([0,T];H)$.

Proof. Let

$$\begin{aligned}
\zeta^{n_k}&=\operatorname{curl} u_t^{n_k},\quad \zeta_t=\operatorname{curl} u_t,\\
\phi_t^{n_k}&=\operatorname{curl} F_t^{n_k},\quad \phi_t=\operatorname{curl} F_t,\\
h_t^{n_k}&=\operatorname{curl} v_t^{n_k}=\zeta_t^{n_k}-\phi_t^{n_k},\quad h_t=\operatorname{curl} v_t=\zeta_t-\phi_t.
\end{aligned}$$

$h_t^{n_k}$ (resp. h_t) is a weak solution of the vorticity equation

$$\begin{cases}\partial_t h^{n_k}+u^{n_k}\cdot\nabla(h^{n_k}+\phi^{n_k})=0\\ h_0^{n_k}=\operatorname{curl} x_0^{n_k}\end{cases}\qquad\left(\text{resp. }\begin{cases}\partial_t h+u\cdot\nabla(h+\phi)=0\\ h_0=\operatorname{curl} x_0\end{cases}\right). \tag{3.4}$$

Thanks to (3.1), (3.2), (3.3) $h^{n_k},\ h,\ \phi^{n_k},\ \phi$ satisfy

$$\phi^{n_k} \to \phi \quad in\ C([0,T];H^{3-\theta}), \tag{3.5}$$
$$\sup_{k\geqslant 1}\|\phi^{n_k}\|_{C([0,T];H^3)}+\|\phi\|_{C([0,T];H^3)} \leqslant C(N), \tag{3.6}$$
$$\sup_{k\geqslant 1}\|h^{n_k}\|_{C([0,T];W^{1,4})}+\|h\|_{C([0,T];W^{1,4})} \leqslant C(\|x_0\|_{W^{2,4}},N). \tag{3.7}$$

We need to introduce the stream function $\psi_t^{n_k}$ (resp. ψ_t) which is the weak solution of

$$\begin{cases}-\Delta\psi_t^{n_k}=h_t^{n_k}\\ \psi_t^{n_k}|_{\partial D}=0\end{cases} \qquad \left(\text{resp. } \begin{cases}-\Delta\psi_t=h_t\\ \psi_t|_{\partial D}=0\end{cases}\right). \tag{3.8}$$

By standard elliptic regularity theory, see for example [1], and the uniform bound (3.7), it holds

$$\sup_{k\geqslant 1}\|\psi^{n_k}\|_{C([0,T];W^{3,4})}+\|\psi\|_{C([0,T];W^{3,4})} \leqslant C(\|x_0\|_{W^{2,4}},N). \tag{3.9}$$

Lastly, introducing

$$\alpha_t^{n_k}=\psi_t^{n_k}-\psi_t,\quad g_t^{n_k}=\phi_t^{n_k}-\phi_t,\quad G_t^{n_k}=F_t^{n_k}-F_t,$$

It is well-known that $v^{n_k}=-\nabla^{\perp}\psi^{n_k},\ v=-\nabla^{\perp}\psi$, see for example [50]. With this notation in mind, thanks to equations (3.4) and (3.8), $-\Delta\alpha^{n_k}$ solves in a weak sense

$$\begin{aligned}\partial_t(-\Delta\alpha^{n_k})+&\Big[(-\nabla^{\perp}\psi^{n_k}+F^{n_k})\cdot\nabla\Big]\Big(-\Delta\alpha^{n_k}+g^{n_k}\Big)=\\ &-\Big[-\nabla^{\perp}\alpha^{n_k}+G^{n_k}\cdot\nabla\Big](-\Delta\psi+\phi).\end{aligned} \tag{3.10}$$

Therefore, arguing as in [65, Theorem 3.1], we use α^{n_k} itself as a test function in (3.10), obtaining

$$\begin{aligned}\frac{1}{2}\|\nabla\alpha_t^{n_k}\|^2&=\frac{1}{2}\|\nabla\alpha_0^{n_k}\|^2+\int_0^t\int_D\Big((-\nabla^{\perp}\psi_s^{n_k}\cdot\nabla\alpha_s^{n_k})\big(-\Delta\alpha_s^{n_k}+g_s^{n_k}\big)\\ &\quad+F_s^{n_k}\cdot\nabla\alpha_s^{n_k}(-\Delta\alpha_s^{n_k}+g_s^{n_k})+\big(-\Delta\psi_s+\phi_s\big)\big(G_s^{n_k}\cdot\nabla\alpha_s^{n_k}\big)\Big)dxds\\ &=\frac{1}{2}\|\nabla\alpha_0^{n_k}\|^2+I_1(t)+I_2(t)+I_3(t)+I_4(t)+I_5(t),\end{aligned} \tag{3.11}$$

where

$$\begin{aligned}I_1(t)&=\int_0^t\int_D\nabla^{\perp}\psi_s^{n_k}\cdot\nabla\alpha_s^{n_k}\Delta\alpha_s^{n_k}dxds,\\ I_2(t)&=-\int_0^t\int_D\nabla^{\perp}\psi_s^{n_k}\cdot\nabla\alpha_s^{n_k}g_s^{n_k}dxds,\\ I_3(t)&=-\int_0^t\int_D F_s^{n_k}\cdot\nabla\alpha_s^{n_k}\Delta\alpha_s^{n_k}dxds,\\ I_4(t)&=\int_0^t\int_D F_s^{n_k}\cdot\nabla\alpha_s^{n_k}g_s^{n_k}dxds,\\ I_5(t)&=\int_0^t\int_D(-\Delta\psi_s+\phi_s)\big(G_s^{n_k}\cdot\nabla\alpha_s^{n_k}\big)dxds.\end{aligned}$$

Therefore we need to understand the behavior of $I_1(t)$, $I_2(t)$, $I_3(t)$, $I_4(t)$, $I_5(t)$ in equation (3.11). $I_1(t)$ can be estimated easily integrating by parts, thanks to the uniform bound (3.3) and Hölder's inequality. Indeed it holds:

$$\begin{aligned}-\int_D v_i^{n_k}\partial_i\alpha^{n_k}\partial_{j,j}\alpha^{n_k}dx &= -\int_D \partial_j v_i^{n_k}\partial_i\alpha^{n_k}\partial_j\alpha^{n_k}dx - \int_D v_i^{n_k}\partial_{i,j}\alpha^{n_k}\partial_j\alpha^{n_k}dx\\ &\leqslant \|v^{n_k}\|_{W^{1,\infty}}\|\nabla\alpha^{n_k}\|^2 - \frac{1}{2}\int_D v_i^{n_k}\partial_i|\partial_j\alpha^{n_k}|^2dx\\ &\leqslant C(\|x_0\|_{W^{2,4}},N)\|\nabla\alpha^{n_k}\|^2.\end{aligned}$$

Therefore

$$I_1(t) \leqslant C(\|x_0\|_{W^{2,4}},N)\int_0^t \|\nabla\alpha_s^{n_k}\|^2ds. \tag{3.12}$$

I_3 can be estimated similarly integrating by parts, thanks to the uniform bound (3.2) and Hölder's inequality:

$$\begin{aligned}\int_D F_i^{n_k}\partial_i\alpha^{n_k}\partial_{j,j}\alpha^{n_k}dx &= -\int_D \partial_j F_i^{n_k}\partial_i\alpha^{n_k}\partial_j\alpha^{n_k}dx - \int_D F_i^{n_k}\partial_{i,j}\alpha^{n_k}\partial_j\alpha^{n_k}dx\\ &\leqslant \|F^{n_k}\|_{W^{1,\infty}}\|\nabla\alpha^{n_k}\|^2 - \frac{1}{2}\int_D F_i^{n_k}\partial_i|\partial_j\alpha^{n_k}|^2dx\\ &\leqslant C(N)\|\nabla\alpha^{n_k}\|^2.\end{aligned}$$

Therefore

$$I_3(t) \leqslant C(N)\int_0^t \|\nabla\alpha_s^{n_k}\|^2ds. \tag{3.13}$$

$I_2(t)$, $I_4(t)$ and $I_5(t)$ can be bounded easily by Hölder and Young inequalities and the uniform bounds (3.2), (3.3), (3.6), (3.7), (3.9). Indeed it holds:

$$\begin{aligned}I_2(t)+I_4(t)+I_5(t) &\leqslant \int_0^t \left(\|v_s^{n_k}\|_{L^\infty}+\|F_s^{n_k}\|_{L^\infty}\right)\|\nabla\alpha_s^{n_k}\|\|g_s^{n_k}\|dsds\\ &\quad+\int_0^t \left(\|\phi_s\|_{L^\infty}\|h_s\|_{L^\infty}\right)\|\nabla\alpha_s^{n_k}\|\|G_s^{n_k}\|\\ &\leqslant \int_0^t \|\nabla\alpha_s^{n_k}\|^2ds\\ &\quad+C\left(\|v^{n_k}\|_{C([0,T];W^{1,4})}+\|F^{n_k}\|_{C([0,T];W^{1,4})}\right)^2\|g^{n_k}\|^2_{C([0,T];L^2)}\\ &\quad+C\left(\|\phi\|_{C([0,T];W^{1,4})}+\|h\|_{C([0,T];W^{1,4})}\right)^2\|G^{n_k}\|^2_{C([0,T];L^2)}\\ &\leqslant \int_0^t \|\nabla\alpha_s^{n_k}\|^2ds\\ &\quad+C(\|x_0\|_{W^{2,4}},N)\left(\|g^{n_k}\|^2_{C([0,T];L^2)}+\|G^{n_k}\|^2_{C([0,T];L^2)}\right).\end{aligned} \tag{3.14}$$

Combining relations (3.12), (3.13) and (3.14) we get

$$\begin{aligned}\frac{1}{2}\|\nabla\alpha_t^{n_k}\|^2 &\leqslant \frac{1}{2}\|\nabla\alpha_0^{n_k}\|^2 + C(\|x_0\|_{W^{2,4}},N)\int_0^t\|\nabla\alpha_s^{n_k}\|^2ds\\ &\quad+C(\|x_0\|_{W^{2,4}},N)\left(\|g^{n_k}\|^2_{C([0,T];L^2)}+\|G^{n_k}\|^2_{C([0,T];L^2)}\right),\end{aligned}$$

which implies, by Grönwall's Lemma,

$$\begin{aligned}\|v^{n_k}-v\|_{C([0,T];H)} &\leqslant C(\|x_0\|_{W^{2,4}},N)\big(\|x_0^{n_k}-x_0\|+\|g^{n_k}\|_{C([0,T];L^2)}\\ &\quad+\|G^{n_k}\|_{C([0,T];L^2)}\big).\end{aligned} \tag{3.15}$$

Thanks to our assumptions the thesis follows. □

3.1 The Case of fluids with radial symmetry

The goal of this Section is to prove the validity of the Large Deviation Principle for the zero noise-zero viscosity limit of the Navier-Stokes equations in presence of strong assumptions on the domain and data, see Hypothesis 2.26. Therefore in this section we assume that D is the open ball centered in 0 with radius 1.

The reason why this particular geometry can be treated lies in the fact the we can show that the solution of the Navier Stokes equations given by Theorem 2.15 posses radial symmetry, and in turn show that the nonlinear term in the equation vanishes. We will be able to represent the solution $u^\varepsilon_t(x) = v^\varepsilon_t(x)\frac{x^\perp}{|x|}$ where v^ε is a radial function satisfying an appropriate auxiliary equation. Then we will exploit this particular representation formula in order to prove the validity of Theorem 2.27.

By radial functions, we mean functions g such that $g(R_\theta x) = g(x)$ for a.e. $x \in D$, for each $\theta \in [0, 2\pi]$, $R_\theta : D \to D$ being the counterclockwise rotation of the disk about its center by the angle θ. Any function $u(x)$ that can be written as $\bar{u}(|x|)\frac{x^\perp}{|x|}$ will be called circularly symmetric and the radial function $\bar{u}$ will be called its radial part.

We want to consider the following equation for the scalar function v^ε:

$$dv^\varepsilon = \Big[\varepsilon(\Delta v^\varepsilon - \frac{v^\varepsilon}{|x|^2}) + \bar{f}_t\Big]dt + \sqrt{\varepsilon}\sum_{k\in K}\sigma_k(|x|)dW^k_t \tag{3.16}$$

where the forcing $\bar{f}_t$ and the initial datum χ^ε are radial functions in $L^2(D)$. In order to study problem (3.16) we need to introduce some space of functions and operators, we refer to [45, Section 3] and the references therein for the proof of this statement and some discussions on this topic.

Let $\mathcal{H}^1 := H^1_0(D) \cap L^2(D, \frac{dx}{|x|^2})$ endowed with the following scalar product

$$\langle u, v\rangle_{\mathcal{H}^1} = \langle \nabla u, \nabla v\rangle + \langle \frac{u}{|x|}, \frac{v}{|x|}\rangle.$$

Define the operator $-\tilde{A} : D(\tilde{A}) \to L^2(D)$ as $-\tilde{A}u = g$ whenever there exist $g \in L^2(D)$ such that

$$\langle u, v\rangle_{\mathcal{H}^1} = \langle g, v\rangle.$$

Then the following statement holds.

Lemma 3.2. *The operator $\tilde{A}$ generates a self-adjoint analytic semigroup of negative type $e^{\tilde{A}t}$ over $L^2(D)$, $D((-\tilde{A})^{1/2}) = \mathcal{H}^1$. Moreover, if x_0 has radial symmetry then the same holds also for $e^{\tilde{A}t}x_0$ $\forall t \geqslant 0$.*

Therefore, according to [22], problem (3.16) can be interpreted in mild form as

$$v^\varepsilon_t := e^{\varepsilon\tilde{A}t}\chi^\varepsilon + \int_0^t e^{\varepsilon\tilde{A}(t-s)}\bar{f}_s ds + \sqrt{\varepsilon}\sum_{k\in K}\int_0^t e^{\varepsilon\tilde{A}(t-s)}\sigma_k(|\cdot|)dW^k_s \tag{3.17}$$

We introduce the notion of weak solution of problem (3.16).

Definition 3.3. *We say that v^ε is a weak solution of* (3.16) *if*

$$v^\varepsilon \in C_{\mathcal{F}}\left([0,T]; L^2(D)\right) \cap L^2(\Omega, \mathcal{F}, \mathbb{P}; L^2(0,T;\mathcal{H}^1))$$

and for every $\phi \in D(\tilde{A})$, we have

$$\langle v^\varepsilon_t, \phi\rangle = \langle \chi^\varepsilon, \phi\rangle + \varepsilon\int_0^t \left\langle v^\varepsilon_s, \tilde{A}\phi\right\rangle ds + \int_0^t \langle \bar{f}_s, \phi\rangle ds + \sqrt{\varepsilon}\sum_{k\in K}\langle s_k, \phi\rangle W^k_t$$

for every $t \in [0,T], \mathbb{P}-$ a.s., where $s_k(x) = \sigma_k(|x|)$.

Thanks to [22, Theorem 5.4] the mild formula (3.17) gives the unique weak solution of (3.16). Indeed the following hold

Lemma 3.4. *$v^\varepsilon \in C([0,T];L^2(D)) \cap L^2([0,T];\mathcal{H}^1)$ and has radial symmetry. Moreover, it is a weak solution in the sense of Definition 3.3 of equation* (3.16).

We introduced the problem (3.16), because of the representation formula for the unique solution of (2.23) guaranteed by the following proposition.

Proposition 3.5. *The unique weak solution u^ε of the Navier-Stokes system* (2.23) *for $f_t(x) = \bar{f}_t(|x|)\frac{x^\perp}{|x|} \in H_0^{RS}$, $u_0(x) = \bar{u}_0(|x|)\frac{x^\perp}{|x|} \in \mathcal{E}_0^{RS}$ and with noise W^{RS} is given by $\bar{u}_t^\varepsilon(|x|)\frac{x^\perp}{|x|}$ where $\bar{u}^\varepsilon$ solves equation* (3.16) *with forcing $\bar{f}$ and initial datum $\bar{u}_0$*

Proof. We start showing that for every $\phi \in C_c^\infty(D;\mathbb{R}^2)$ divergence free

$$b\left(\bar{u}_s^\varepsilon(x)\frac{x^\perp}{|x|},\phi,\bar{u}_s^\varepsilon(x)\frac{x^\perp}{|x|}\right) = 0.$$

Indeed

$$\begin{aligned}\int_D |\bar{u}_s^\varepsilon(|x|)|^2\frac{x^\perp}{|x|^2}\cdot(x^\perp\cdot\nabla\,\phi)dx &= \int_D |\bar{u}_s^\varepsilon(|x|)|^2\frac{x^\perp}{|x|^2}\cdot(\nabla(x^\perp\cdot\phi)+\phi^\perp)dx\\ &= \int_D \frac{|\bar{u}_s^\varepsilon(|x|)|^2}{|x|^2}(x^\perp\cdot(\nabla(x^\perp\cdot\phi))+x\cdot\phi)dx = I_1+I_2.\end{aligned}$$

Now we have

$$I_1 = \int_D \frac{|\bar{u}_s^\varepsilon(|x|)|^2}{|x|^2}\operatorname{div}(x^\perp(x^\perp\cdot\phi))dx = -\int_D \nabla\Big[\frac{|\bar{u}_s^\varepsilon(|x|)|^2}{|x|^2}\Big]\cdot x^\perp(x^\perp\cdot\phi)dx = 0$$

since the gradient of a radial function is always parallel to x. While, if we define $V(\rho) = \int_0^\rho \frac{|\bar{u}_s^\varepsilon(r)|^2}{r}dr$, we have

$$I_2 = \int_D \frac{|\bar{u}_s^\varepsilon(|x|)|^2}{|x|^2}x\cdot\phi dx = \int_D \nabla V(|x|)\cdot\phi dx = -\int_D V(|x|)\operatorname{div}\phi dx = 0.$$

Therefore we are left to show that for each $\phi \in D(A)$, neglecting the non-linear term which is zero for divergence free test functions,

$$\langle u_t^\varepsilon - u_0,\phi\rangle + \int_0^t \varepsilon\langle\nabla u_s^\varepsilon,\nabla\phi\rangle ds = \int_0^t\langle f_s,\phi\rangle ds + \sqrt{\varepsilon}\sum_{k\in K}\langle\sigma_k\frac{x^\perp}{|x|},\phi\rangle W_t^k. \tag{3.18}$$

We rewrite this as

$$\begin{aligned}\int_D[\bar{u}_t^\varepsilon(|x|)-\bar{u}_0(|x|)]\phi(x)\cdot\frac{x^\perp}{|x|}dx + &\int_0^t\int_D\varepsilon\nabla[\bar{u}_s^\varepsilon(|x|)\frac{x^\perp}{|x|}]\cdot\nabla\phi(x)dxds\\ &= \int_0^t\int_D\bar{f}_s(|x|)\phi(x)\cdot\frac{x^\perp}{|x|}dxds\\ &+\sqrt{\varepsilon}\Big(\sum_{k\in K}\int_D\sigma_k(|x|)\phi(x)\cdot\frac{x^\perp}{|x|}dx\Big)W_t^k,\end{aligned}$$

which, comparing with the Definition 3.3, holds true if we prove that

$$\int_0^t\int_D\nabla[\bar{u}_s^\varepsilon(|x|)\frac{x^\perp}{|x|}]\cdot\nabla\phi(x)dxds = \int_0^t\int_D(-\tilde{A})^{1/2}\bar{u}_s^\varepsilon(|x|)(-\tilde{A})^{1/2}(\phi(x)\cdot\frac{x^\perp}{|x|})dxds \tag{3.19}$$

for each $\phi \in D(A)$. In order to prove the claim, we observe that if $\phi \in C_c^\infty(D \setminus \{0\}; \mathbb{R}^2)$, then $\phi \cdot \frac{x^\perp}{|x|} \in D(\tilde{A})$. Therefore (3.19) holds for each $\phi \cdot \frac{x^\perp}{|x|} \in D(\tilde{A})$, by simple calculuations upon noticing that

$$\Delta(\phi(x) \cdot \frac{x^\perp}{|x|}) = \Delta(\phi(x)) \cdot \frac{x^\perp}{|x|} + \phi(x) \cdot \frac{x^\perp}{|x|^3} - 2\operatorname{div}\left(\frac{x \cdot \phi}{|x|^3} x^\perp\right). \tag{3.20}$$

In particular (3.18) is satisfied for each $\phi \in C_c^\infty(D \setminus \{0\}; \mathbb{R}^2)$. Finally, we obtain that u^ε is a weak solution of (2.23) by observing that the closure of $C_c^\infty(D \setminus \{0\}; \mathbb{R}^2)$ vectors field in the H^1 norm, is exactly $H_0^1(D; \mathbb{R}^2)$, which implies that (3.18) holds in particular for every $\phi \in D(A)$. □

In the same manner we can prove the analogous result for the Euler system, that is

Proposition 3.6. *Given $u_0(|x|)\frac{x^\perp}{|x|} \in \mathcal{E}_0^{RS}$ the unique solution of the system* (2.20) *in*

$$C([0,T]; W^{2,4}(D; \mathbb{R}^2)) \cap C([0,T]; H)$$

is given by $\bar{u}_t(|x|)\frac{x^\perp}{|x|}$ where the radial function u_t is given by

$$\bar{u}_t(|x|) := u_0(|x|) + \int_0^t f_s(|x|) ds.$$

3.1.1 Condition 2

In this section we prove that the second condition in the weak convergence approach is easily fulfilled in the case of fluids with radial symmetries. Let $u_0^\varepsilon(x) := \bar{u}_0^\varepsilon(|x|)\frac{x^\perp}{|x|} \to u_0(x) := \bar{u}_0(|x|)\frac{x^\perp}{|x|}$ in $\mathcal{E}_0^{RS}$, which we recall being endowed with the H^3 norm and $f^\varepsilon(t,x) := \bar{f}^\varepsilon(t,|x|)\frac{x^\perp}{|x|} \to f(t,x) := \bar{f}(t,|x|)\frac{x^\perp}{|x|}$ in law as S^N-random variables. We will show that for each sequence $\varepsilon_n \to 0$, $\mathcal{G}^{\varepsilon_n,RS}\big(u_0^{\varepsilon_n}, {\varepsilon_n}^{1/2} W + \int_0^\cdot f_s^{\varepsilon_n} ds\big)$ converges in law to $\mathcal{G}^{0,RS}(u_0, \int_0^\cdot f_s ds)$ in the topology of $\mathcal{E}$. This implies the validity of the second condition in Hypothesis 2.4. In order to simplify the notation, we will consider $\varepsilon > 0$ in the following dropping the subscript ε_n, having in mind it is a countable family. Since S^N is a Polish space, by Skorokhod's representation theorem, see [25, Chapter 3] and [34], we can introduce a further filtered probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathcal{F}}_t, \tilde{\mathbb{P}})$ and random variables $\tilde{f}^\varepsilon, \tilde{W}^\varepsilon, \tilde{f}$ such that $(\tilde{f}^\varepsilon, \tilde{W}^\varepsilon)$ has the same joint law of (f^ε, W), $\tilde{f}$ has the same law of f and $\tilde{f}^\varepsilon \to_{\tilde{\mathbb{P}}-a.s.} \tilde{f}$ in $L^2(0,T; H_0^{RS}) \hookrightarrow L^2(0,T; V)$, see for example [27] for details. In the following, with some abuse of notation, we will drop the tilde in our notation and simply use $\mathbb{P},\ \mathbb{E},\ f^\varepsilon,\ f, W^\varepsilon$ instead of $\tilde{\mathbb{P}},\ \tilde{\mathbb{E}},\ \tilde{f}^\varepsilon,\ \tilde{f}, \tilde{W}^\varepsilon$. Thanks to Theorem 2.15 for each ε we can define u^ε as the unique solution of (2.15) with forcing term f^ε, initial condition u_0^ε and Brownian forcing term W^ε, i.e. $u^\varepsilon = \mathcal{G}^{\varepsilon,RS}(u_0^\varepsilon, \int_0^\cdot f_s^\varepsilon ds + \sqrt{\varepsilon} W^\varepsilon)$. Moreover, by Theorem 2.18 we can define u^E as the unique regular solution of (2.20) with forcing term f and initial condition u_0, i.e. $u^E = \mathcal{G}^{0,RS}(u_0, \int_0^\cdot f_s ds)$. From the results in the previous section, we have that both u^ε and u have circular symmetry, and their radial parts are given by

$$\begin{aligned}
\bar{u}_t^\varepsilon &:= e^{\varepsilon \tilde{A} t} \bar{u}_0^\varepsilon + \int_0^t e^{\varepsilon \tilde{A}(t-s)} \bar{f}_s^\varepsilon ds + \sqrt{\varepsilon} \sum_{k \in K} \int_0^t e^{\varepsilon \tilde{A}(t-s)} \sigma_k(|\cdot|) dW_s^{\varepsilon,k}, \\
\bar{u}_t &:= \bar{u}_0 + \int_0^t \bar{f}_s ds.
\end{aligned}$$

Actually we will prove the stronger result:

$$\mathbb{E}\left[\sup_{t \in [0,T]} \|u_t^\varepsilon - u_t^E\|^2\right] \to 0 \tag{3.21}$$

in the probability space defined via Skorokhod's representation theorem. From our representation formula, it is sufficient to show

$$\mathbb{E}\left[\sup_{t\in[0,T]}||\bar{u}_t^\varepsilon - \bar{u}_t||^2\right] \to 0.$$

To do so we write

$$\begin{aligned}\bar{u}_t^\varepsilon - \bar{u}_t =& e^{\varepsilon\tilde{A}t}(\bar{u}_0^\varepsilon - \bar{u}_0) + (e^{\varepsilon\tilde{A}t} - I)\bar{u}_0 + \int_0^t e^{\varepsilon\tilde{A}(t-s)}(\bar{f}_s^\varepsilon - \bar{f}_s)ds \\ &+ \int_0^t (e^{\varepsilon\tilde{A}(t-s)} - I)\bar{f}_s + \sqrt{\varepsilon}\sum_{k\in K}\int_0^t e^{\varepsilon\tilde{A}(t-s)}\sigma_k(|\cdot|)dW_s^{\varepsilon,k}. \end{aligned} \tag{3.22}$$

Preliminarily we observe that in the proof of Proposition 3.5, we showed equality (3.19) for every vector field $\phi(x) \in H_0^1$. If we disregard the time integration and let $\bar{u}_s^\varepsilon$ in (3.19) be any generic radial function in $\mathcal{H}^1$ we obtain that the map $J : \mathcal{H}_R^1 \to D((-A)^{1/2})$ that sends any radial function $v(|x|)$ in $\mathcal{H}^1$ to the circular symmetric vector field $v(|x|)\frac{x^\perp}{|x|}$ is an isometry (where we indicated with $\mathcal{H}_R^1$ the set of radial function of $\mathcal{H}^1$). We then obtain

$$\|f^\varepsilon\|_{D((-A)^{1/2})} = \|\bar{f}_s^\varepsilon(|x|)\frac{x^\perp}{|x|}\|_{D((-A)^{1/2})} = \|\bar{f}_s^\varepsilon(|x|)\|_{\mathcal{H}^1},$$
$$\|f\|_{D((-A)^{1/2})} = \|\bar{f}_s(|x|)\frac{x^\perp}{|x|}\|_{D((-A)^{1/2})} = \|\bar{f}_s(|x|)\|_{\mathcal{H}^1},$$

which gives exactly

$$\int_0^T \|\bar{f}_s\|_{\mathcal{H}^1}^2 + \sup_{\varepsilon>0}\int_0^T \|\bar{f}_s^\varepsilon\|_{\mathcal{H}^1}^2 \leqslant C(N) \quad \mathbb{P}-a.s., \tag{3.23}$$

since $f^\varepsilon,\ f \in S^N$. Moreover $\bar{f}^\varepsilon \rightharpoonup \bar{f}$ in $L^2(0,T;\mathcal{H}^1) \quad \mathbb{P}-a.s.$ Now we can treat $\overline{u}^\varepsilon - \overline{u}$. The first and second terms in (3.22) go to zero in L^2 norm thanks to the strong convergence of $\bar{u}_0^\varepsilon$ to $\bar{u}_0$ and the continuity of the semigroup. The convergence is uniform in time since, for the first term $\sup_{t\leqslant T}\|e^{\varepsilon\tilde{A}t}(\bar{u}_0^\varepsilon - \bar{u}_0)\| \leqslant \|\bar{u}_0^\varepsilon - \bar{u}_0\|$ while for the second, we choose for every ε, t_ε for which $\|(e^{\varepsilon\tilde{A}t_\varepsilon} - I)\bar{u}_0\|$ achieves its maximum over $[0,T]$. Then by observing that $t_\varepsilon \leqslant T$, we get that as $\varepsilon \to 0$, $\varepsilon t_\varepsilon \to 0$, and we conclude by the continuity of the semigroup. The stochastic integral term in (3.22) can be easily controlled using the Itô formula and Burkholder-Davis-Gundy inequality for Stochastic Convolutions, see for example [58], obtaining

$$\mathbb{E}\left[\sup_{t\leqslant T}\|\sqrt{\varepsilon}\sum_{k\in K}\int_0^t e^{\varepsilon\tilde{A}(t-s)}\sigma_k(|\cdot|)dW_s^{\varepsilon,k}\|^2\right] \leqslant 2T\sqrt{\varepsilon}\mathbb{E}\left[\sup_{t\leqslant T}\|\bar{u}_s^\varepsilon\|^2\right]\left(\sum_{k\in K}\|\sigma_k\|^2\right)$$

which converges to zero as all the quantities are bounded. In order to study the third term in (3.22), call $\bar{F}_t^\varepsilon = \int_0^t \bar{f}_s^\varepsilon ds$ and $\bar{F}_t = \int_0^t \bar{f}_s ds$. By [27, Proposition 2.10],

$$\bar{F}^\varepsilon \to \bar{F} \quad in\ C([0,T];L^2) \quad \mathbb{P}-a.s. \tag{3.24}$$

The third term in (3.22) can can be rewritten as

$$\int_0^t e^{\varepsilon\tilde{A}(t-s)}(\bar{f}_s^\varepsilon - \bar{f}_s)ds = \varepsilon\int_0^t \tilde{A}e^{\varepsilon\tilde{A}(t-s)}(\bar{F}_s^\varepsilon - \bar{F}_s)ds + (\bar{F}_t^\varepsilon - \bar{F}_t).$$

The second term above converges to 0 $\mathbb{P}-a.s.$ and in $L^2(\Omega,\mathbb{P})$ thanks to (3.23) and (3.24). Concerning the other we use standard properties of analytic semigroup, see for example

[52, Chapter 2, Theorem 6.13], and (3.23) obtaining

$$\mathbb{E}\left[\sup_{t\in[0,T]}\|\varepsilon\int_0^t \tilde{A}e^{\varepsilon\tilde{A}(t-s)}(\bar{F}^{\varepsilon}_s-\bar{F}_s)ds\|^2\right]$$
$$\leqslant C\varepsilon\mathbb{E}\left[\sup_{t\in[0,T]}\left(\int_0^t\frac{\|\bar{F}^{\varepsilon}-\bar{F}\|_{C([0,T];\mathcal{H}^1)}}{(t-s)^{1/2}}ds\right)^2\right]\leqslant C(N,T)\varepsilon\to 0.$$

Finally, for the fourth term in (3.22), since $\mathcal{H}^1=D((-\tilde{A})^{1/2})$ according to Lemma 3.2, by standard properties of analytic semigroups, see for example [52, Chapter 2, Theorem 6.13], and our assumption on f, see (3.23), we obtain

$$\begin{aligned}\sup_{t\in[0,T]}\|\int_0^t(e^{\varepsilon\tilde{A}(t-s)}-I)\bar{f}_sds\|^2 &\leqslant\left(\sup_{t\in[0,T]}\int_0^t\|(e^{\varepsilon\tilde{A}(t-s)}-I)\bar{f}_s\|ds\right)^2\\ &\leqslant C\varepsilon\left(\sup_{t\in[0,T]}\int_0^t(t-s)^{1/2}\|\bar{f}_s\|_{\mathcal{H}^1}ds\right)^2\\ &\leqslant C(N,T)\varepsilon\quad\mathbb{P}-a.s.\end{aligned}$$

Therefore we get

$$\mathbb{E}\Big[\sup_{[0,T]}\|\int_0^t(e^{\varepsilon\tilde{A}(t-s)}-I)\bar{f}_sds\|^2\Big]\to 0.$$

This proves the validity of relation (3.21). Now we are ready to prove Theorem 2.27.

Proof of Theorem 2.27. Since we already checked the validity of Condition 1 and Condition 2 in Hypothesis 2.4, it remains to show that for each $v\in\mathcal{E}$ the map $u_0\to I^{RS}_{u_0}(v)$ is a lower continuous map from $\mathcal{E}^{RS}_0$ to $[0,+\infty]$ in order to apply Theorem 2.5 and complete the proof. The arguments goes in this way. Fix $u_0\in\mathcal{E}^{RS}_0$ and a family $\{u^n_0\}_{n\in\mathbb{N}}\subseteq\mathcal{E}^{RS}_0$ converging to u_0. Without loss of generality we may assume $\liminf_{n\to+\infty}I^{RS}_{u^n_0}(v)=M<+\infty$ otherwise we have nothing to prove. Therefore, thanks to the well-posedness of the Euler equations guaranteed by Theorem 2.18, there exists a subsequence n_k and family $\{f^{n_k}\}_{n_k\in\mathbb{N}}\subseteq S^{2M}$ such that $\mathcal{G}^{RS,0}(u^{n_k}_0,\int_0^{\cdot} f^{n_k}_s ds)=v$. Moreover $f^{n_k}\in S^{2M}$ for all k. Up to passing to a further subsequence, which we continue to denote by f^{n_k} for simplicity of notation, there exists $f\in S^{2M}$ such that $f^{n_k}\rightharpoonup f$ in $L^2(0,T;H^{RS}_0)$. Thanks to (3.1) and (3.15) it follows that $\mathcal{G}^{RS}_0:\mathcal{E}^{RS}_0\times L^2([0,T];H^{RS}_0)\to\mathcal{E}$ is a continuous map endowing $L^2([0,T];H^{RS}_0)$ with the weak topology. Therefore $\mathcal{G}^{RS,0}(u_0,f)=v$ and from the lower semicontinuity of the norm with respect to the weak convergence the thesis follow immediately:

$$I^{RS}_{u_0}(v)\leqslant\frac{1}{2}\int_0^T\|f_s\|^2_{H^{RS}_0}ds\leqslant\liminf_{k\to+\infty}\frac{1}{2}\int_0^T\|f^{n_k}_s\|^2_{H^{RS}_0}ds\leqslant M=\liminf_{n\to+\infty}I^{RS}_{u^n_0}(v).$$

□

3.2 Proof of Theorem 2.23

We already provided the validity of Condition 1 of Hypothesis 2.4 at the beginning of Section 3. Moreover, the argument for showing the lower continuity of the map $I^{RS}_{u_0}(v)$ for $v\in\mathcal{E}$ fixed repeats verbatim for $I^{NS}_{u_0}(v)$. Therefore, it is enough to show the validity of Condition 2 of Hypothesis 2.4 in order to prove Theorem 2.23. This is the aim of the next subsection.

3.2.1 Condition 2

Fix $N > 0$, let $\tilde{f}^\varepsilon, \tilde{f} \in \mathcal{P}_2^N,\ u_0^\varepsilon, u_0 \in \mathcal{E}_0^{NS}$ such that $\tilde{f}^\varepsilon \to_{\mathcal{L}} \tilde{f}$ weakly in $L^2(0,T;H_0),\ u_0^\varepsilon \to u_0$ in $\mathcal{E}_0^{NS}$. We will show that for each sequence $\varepsilon_n \to 0$, $\mathcal{G}^{\varepsilon_n,NS}\big(u_0^{\varepsilon_n}, {\varepsilon_n}^{1/2}W + \int_0^\cdot \tilde{f}_s^{\varepsilon_n} ds\big)$ converges in law to $\mathcal{G}^{0,NS}(u_0, \int_0^\cdot \tilde{f}_s ds)$ in the topology of $\mathcal{E}$. This implies the validity of the second condition in Hypothesis 2.4. In order to simplify the notation, we will consider $\varepsilon > 0$ in the following dropping the subscript ε_n, having in mind it is a countable family. Since S^N is a Polish space, by Skorokhod's representation theorem, see [25, Chapter 3] and [34], we can introduce a further filtered probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathcal{F}}_t, \tilde{\mathbb{P}})$ and random variables $f^\varepsilon, W^\varepsilon, f$ such that $(f^\varepsilon, W^\varepsilon)$ has the same joint law of $(\tilde{f}^\varepsilon, W)$, f has the same law of $\tilde{f}$ and $f^\varepsilon \to_{\tilde{\mathbb{P}}-a.s.} f$ in $L^2(0,T;H_0)$, see for example [27] for details. Thanks to Theorem 2.15 for each ε we can define u^ε as the unique solution of (2.15) with forcing term f^ε, initial condition u_0^ε and Brownian forcing term W^ε. The family $\{u^\varepsilon\}_{\varepsilon>0}$ satisfies Hypothesis 2.20. Moreover, by Theorem 2.18 we can define u^E as the unique regular solution of (2.20) with forcing term f and initial condition u_0. We will show that u_0^ε converges to u^E in probability in $C([0,T];H)$. This implies the validity of Condition 2.

Before starting with the computation we recall some facts. In the following, with some abuse of notation, we will simply use $\mathbb{P},\ \mathbb{E}$ instead of $\tilde{\mathbb{P}},\ \tilde{\mathbb{E}}$. Fix $\theta > 0$ arbitrarily small and define $F_t^\varepsilon = \int_0^t f_s^\varepsilon ds,\ F_t = \int_0^t f_s ds$. By Hypothesis 2.11, $H_0 \hookrightarrow D((-A)^\gamma)$. This implies, see for example [27, Proposition 2.10] that

$$F^\varepsilon \to_{\mathbb{P}-a.s.} F \quad \textit{in } C([0,T];D((-A)^{\gamma-\theta})). \tag{3.25}$$

Obviously

$$\sup_{\varepsilon>0}\|F^\varepsilon\|_{C([0,T];D((-A)^\gamma))} + \|F\|_{C([0,T];D((-A)^\gamma))} \leqslant C(N) \quad \mathbb{P}-a.s. \tag{3.26}$$

Starting from (2.17), Burkholder-Davis-Gundy inequality, Grönwall's lemma and the convergence of u_0^ε to u_0 imply

$$\sup_{\varepsilon>0}\left\{\mathbb{E}\left[\sup_{t\in[0,T]}\|u_t^\varepsilon\|^2\right] + \varepsilon\mathbb{E}\left[\int_0^T \|\nabla u_s^\varepsilon\|^2 ds\right]\right\} \leqslant C(N, \|u_0\|). \tag{3.27}$$

In order to show the convergence of u^ε to u^E we will introduce $z^\varepsilon = \int_0^t e^{\varepsilon A(t-s)} f_s^\varepsilon ds,\ v^\varepsilon = u^\varepsilon - z^\varepsilon,\ v^E = u^E - F$ and show separately the convergence of z^ε to F and of v^ε to v^E. While the convergence of z^ε to F will be established by exploiting the properties of the Stokes semigroup, the convergence of v^ε to v^E will be the more demanding part of the argument relying on the Strong Kato Condition Hypothesis 2.20 and the introduction of a corrector of the boundary layer for v^E satisfying suitable estimates. This way to proceed is typical in the case of the analysis of the inviscid limit for the Navier-Stokes equations with no-slip boundary conditions, see also [36], [18], [61], [63].

We start with the convergence of z^ε to F.

Lemma 3.7. *For each $\theta > 0$, $z_t^\varepsilon \to F$ in $C([0,T];D((-A)^{\gamma-1/2-\theta}))\quad \mathbb{P}-a.s.$ and in $L^2(\Omega,\mathbb{P})$.*

Proof. z^ε can be rewritten as

$$z_t^\varepsilon = \int_0^t f_s^\varepsilon ds + \varepsilon\int_0^t Ae^{\varepsilon(t-s)A} F_s^\varepsilon ds = I_1 + I_2.$$

$I_1 \to F \in C([0,T];D((-A)^{\gamma-1/2-\theta})\ \mathbb{P}-a.s.$ thanks to (3.25). Moreover, since (3.26) holds, previous convergence holds also in $L^2(\Omega,\mathbb{P})$ by Lebesgue theorem. It remains to show that $I_2 \to 0$ properly. The $\mathbb{P}-a.s.$ convergence can be obtained as follows

$$\begin{aligned}\varepsilon \sup_{t\in[0,T]} \left\|\int_0^t Ae^{\varepsilon(t-s)A}F_s^\varepsilon ds\right\|_{D((-A)^{\gamma-1/2-\theta}} &\leqslant \varepsilon \sup_{t\in[0,T]} \int_0^t \left\|(-A)^{1/2+\gamma-\theta} e^{\varepsilon(t-s)A}F_s^\varepsilon\right\| ds \\ &\leqslant \varepsilon^{1/2+\theta} \sup_{t\in[0,T]} \int_0^t \frac{\|F_s^\varepsilon\|_{D(-A)^\gamma}}{(t-s)^{1/2-\theta}} ds \\ &\leqslant C(N)\varepsilon^{1/2+\theta} \to 0 \quad \mathbb{P}-a.s.\end{aligned}$$

Since previous bound is uniform in $\omega \in \Omega$ the convergence holds also in $L^2(\Omega, \mathbb{P})$ and the thesis follows. □

In order to prove the convergence of v^ε to v^E we observe that they solve in a sense analogous to Definition 2.13, Definition 2.17

$$dv^\varepsilon + P\left((v^\varepsilon + z^\varepsilon)\cdot\nabla(v^\varepsilon+z^\varepsilon)\right)dt + \varepsilon Av^\varepsilon dt = \sqrt{\varepsilon}dW_t, \tag{3.28}$$

$$\partial_t v^E + P(u^E\cdot\nabla u^E) = 0. \tag{3.29}$$

By triangle inequality and the uniform bound guaranteed by Lemma 3.7, estimates analogous to (3.27), (2.22) hold for v^ε and v^E, too. We observe that, thanks to the regularity of u^E guaranteed by Theorem 2.18

$$\begin{aligned}\|\partial_t v^E\|_{C([0,T];L^\infty(D)))} \lesssim \|\partial_t v^E\|_{C([0,T];W^{1,4}(D))} &\lesssim \|P(u^E\cdot\nabla u^E)\|_{C([0,T];W^{1,4}(D))} \\ &\lesssim \|u^E\|^2_{C([0,T];W^{2,4}(D))} \leqslant C(N,u_0) \quad \mathbb{P}-a.s.\end{aligned} \tag{3.30}$$

Following the idea of [36], let v the corrector of the boundary layer of width $\delta=\delta(\varepsilon)$, i.e. a divergence free vector field with support in a strip of the boundary of width δ such that $v^E - v \in V$ and $\mathbb{P}-a.s.$ uniformly in $t\in[0,T],\ \omega\in\Omega$

$$\begin{aligned}&\|v_t\|_{L^\infty(D)} \leqslant C(N,u_0),\ \|v_t\| \leqslant C(N,u_0)\delta^{\frac12},\ \|\partial_t v_t\| \leqslant C(N,u_0)\delta^{\frac12},\\ &\|\nabla v_t\|_{L^\infty(D)} \leqslant C(N,u_0)\delta^{-1},\ \|\nabla v_t\| \leqslant C(N,u_0)\delta^{-1/2},\ \|\rho\nabla v_t\|_{L^\infty(D)} \leqslant C(N,u_0),\\ &\|\rho^2\nabla v_t\|_{L^\infty(D)} \leqslant C(N,u_0)\delta,\ \|\rho\nabla v_t\| \leqslant C(N,u_0)\delta^{\frac12},\end{aligned} \tag{3.31}$$

ρ being the distance function to ∂D. Now we are ready to show the convergence of v^ε to v^E.

Lemma 3.8. *$v^\varepsilon \to v^E$ in $C([0,T];H)$ in probability.*

Proof. Arguing as in [49, Theorem 9] one can show that the following relations hold true.

$$\begin{aligned}\|v_t^\varepsilon\|^2 + 2\varepsilon\int_0^t \|\nabla v^\varepsilon\|^2 ds = \|u_0^\varepsilon\|^2 - 2\int_0^t b(v^\varepsilon+z^\varepsilon, z^\varepsilon, v^\varepsilon)ds + 2\varepsilon^{1/2}\int_0^t \langle dW_s^\varepsilon, v^\varepsilon\rangle \\ + \varepsilon t Tr(Q) \quad \mathbb{P}-a.s.,\end{aligned} \tag{3.32}$$

$$\|v_t^E\|^2 = \|u_0\|^2 - 2\int_0^t b(v^E+F, F, v^E)ds \quad \mathbb{P}-a.s. \tag{3.33}$$

Exploiting relations (3.32), (3.33) we can study $\|v_t^\varepsilon - v_t^E\|^2$. Indeed, it holds

$$\begin{aligned}\|v_t^\varepsilon - v_t^E\|^2 &= \|v_t^\varepsilon\|^2 + \|v_t^E\|^2 - 2\langle v_t^\varepsilon, v_t^E\rangle \\ &\leqslant \|u_0^\varepsilon\|^2 - 2\int_0^t b(v_s^\varepsilon+z_s^\varepsilon, z_s^\varepsilon, v_s^\varepsilon)ds + 2\varepsilon^{1/2}\int_0^t \langle dW_s^\varepsilon, v_s^\varepsilon\rangle + \varepsilon t Tr(Q) \\ &+ \|u_0\|^2 - 2\int_0^t b(v_s^E+F_s, F_s, v_s^E)ds - 2\langle v_t^\varepsilon, v_t^E - v_t\rangle - 2\langle v_t^\varepsilon, v_t\rangle.\end{aligned} \tag{3.34}$$

Thanks to the fact that $v^E - v \in C^1([0,T];V)$ we can rewrite $\langle v_t^\varepsilon, v_t^E - v_t\rangle$ via Itô formula: $\langle v_t^\varepsilon, v_t^E - v_t\rangle = \langle u_0^\varepsilon, u_0 - v_0\rangle + \int_0^t \langle v_s^\varepsilon, \partial_s(v_s^E - v_s)\rangle ds + \int_0^t \langle dv_s^\varepsilon, v_s^E - v_s\rangle$. Therefore

$$\begin{aligned}\langle v_t^\varepsilon, v_t^E - v_t\rangle &= \langle u_0^\varepsilon, u_0 - v_0\rangle + \int_0^t \langle v_s^\varepsilon, \partial_s(v_s^E - v_s)\rangle ds - \varepsilon \int_0^t \langle \nabla v_s^\varepsilon, \nabla(v_s^E - v_s)\rangle ds \\ &+ \int_0^t b(v_s^\varepsilon + z_s^\varepsilon, v_s^E - v_s, v_s^\varepsilon + z_s^\varepsilon) ds + \varepsilon^{1/2}\langle W_t^\varepsilon, v_t^E - v_t\rangle \\ &- \varepsilon^{1/2}\int_0^t \langle W_s^\varepsilon, \partial_s(v_s^E - v_s)\rangle ds. \end{aligned} \tag{3.35}$$

Let us observe that by assumptions

$$\|u_0^\varepsilon\|^2 + \|u_0\|^2 - 2\langle u_0^\varepsilon, u_0\rangle = \|u_0^\varepsilon - u_0\|^2 = o(1), \quad \varepsilon t Tr(Q) \leqslant \varepsilon T Tr(Q) = o(1).$$

Moreover, thanks to the properties of the boundary layer corrector (3.31), $\mathbb{P}-a.s.$ it holds

$$\begin{aligned}&\langle u_0^\varepsilon, v_0\rangle \leqslant C(u_0, N)\delta^{1/2} = o(1),\\ &\langle v_t^\varepsilon, v_t\rangle \leqslant C(N, u_0)\delta^{1/2}\sup_{t\in[0,T]}\|v_t^\varepsilon\|,\\ &\varepsilon^{1/2}\langle W_t^\varepsilon, v_t^E - v_t\rangle - \varepsilon^{1/2}\int_0^t \langle W_s^\varepsilon, \partial_s(v_s^E - v_s)\rangle ds \leqslant \varepsilon^{1/2} C(N, u_0)\sup_{t\in[0,T]}\|W_t^\varepsilon\|.\end{aligned}$$

Exploiting these facts, inserting relation (3.35) in (3.34) we obtain

$$\begin{aligned}\|v_t^\varepsilon - v_t^E\|^2 &\leqslant o(1) + \delta^{1/2} C(N, u_0)\sup_{t\in[0,T]}\|v_t^\varepsilon\| \\ &+ \varepsilon^{1/2} C(N, u_0)\sup_{t\in[0,T]}\|W_t^\varepsilon\| - 2\int_0^t b(v_s^\varepsilon + z_s^\varepsilon, z_s^\varepsilon, v_s^\varepsilon) ds \\ &+ 2\varepsilon^{1/2}\int_0^t \langle dW_s^\varepsilon, v_s^\varepsilon\rangle - 2\int_0^t b(v_s^E + F_s, F_s, v_s^E) ds \\ &- 2\int_0^t \langle v_s^\varepsilon, \partial_s(v_s^E - v_s)\rangle ds + 2\varepsilon\int_0^t \langle \nabla v_s^\varepsilon, \nabla(v_s^E - v_s)\rangle ds \\ &- 2\int_0^t b(v_s^\varepsilon + z_s^\varepsilon, v_s^E - v_s, v_s^\varepsilon + z_s^\varepsilon) ds \quad \mathbb{P}-a.s.\end{aligned} \tag{3.36}$$

In order to understand the behavior of $\int_0^t \langle v_s^\varepsilon, \partial_s(v_s^E - v_s)\rangle ds$, we observe that, thanks to (3.31),

$$\int_0^t \langle v_s^\varepsilon, \partial_s v_s\rangle ds \leqslant \delta^{1/2} C(N, u_0)\sup_{t\in[0,T]}\|v_t^\varepsilon\| \quad \mathbb{P}-a.s.$$

Moreover, since v^E satisfies (3.29), we have $\int_0^t \langle v_s^\varepsilon, \partial_s v_s^E\rangle ds = -\int_0^t b(v_s^E + F_s, v_s^E + F_s, v_s^\varepsilon) ds$. Let us rewrite the trilinear forms appearing (3.36):

$$\begin{aligned}&b(v^E + F, v^E + F, v^\varepsilon) - b(v^\varepsilon + z^\varepsilon, z^\varepsilon, v^\varepsilon) - b(v^E + F, F, v^E)\\ &- b(v^\varepsilon + z^\varepsilon, v^E - v, v^\varepsilon + z^\varepsilon)\\ &= b(v^E, v^E, v^\varepsilon - v^E) + b(v^E + F, F, v^\varepsilon) + b(F, v^E + F, v^\varepsilon)\\ &- b(v^\varepsilon + z^\varepsilon, z^\varepsilon, v^\varepsilon) - b(v^E + F, F, v^E) - b(v^\varepsilon, v^E, v^\varepsilon - v^E)\\ &- b(z^\varepsilon, v^E - v, v^\varepsilon + z^\varepsilon) - b(v^\varepsilon, v^E, z^\varepsilon) + b(v^\varepsilon, v, v^\varepsilon + z^\varepsilon).\end{aligned} \tag{3.37}$$

By simple computations the terms in (3.37) can be rewritten as:

$$|b(v^E, v^E, v^\varepsilon - v^E) - b(v^\varepsilon, v^E, v^\varepsilon - v^E)| \leqslant \|\nabla v^E\|_{L^\infty}\|v^\varepsilon - v^E\|^2. \tag{3.38}$$

$$b(v^\varepsilon, v, v^\varepsilon + z^\varepsilon) - b(z^\varepsilon, v^E - v, v^\varepsilon + z^\varepsilon) = b(u^\varepsilon, v, u^\varepsilon) - b(z^\varepsilon, v^E, u^\varepsilon). \tag{3.39}$$

$$-b(v^\varepsilon + z^\varepsilon, z^\varepsilon, v^\varepsilon) + b(v^\varepsilon, v^E, z^\varepsilon) = b(v^\varepsilon, z^\varepsilon, v^E - v^\varepsilon) - b(z^\varepsilon, z^\varepsilon, v^\varepsilon). \tag{3.40}$$

$$\begin{aligned} &b(F, v^E + F, v^\varepsilon) + b(v^E + F, F, v^\varepsilon) - b(v^E + F, F, v^E) \\ &= b(v^E, F, v^\varepsilon - v^E) + b(F, F, v^\varepsilon - v^E) + b(F, v^E, v^\varepsilon). \end{aligned} \tag{3.41}$$

Preliminarily, let us rewrite the last terms in each of (3.39), (3.40) and (3.41) obtaining

$$- b(z^\varepsilon, v^E, u^\varepsilon) - b(z^\varepsilon, z^\varepsilon, v^\varepsilon) + b(F, v^E, v^\varepsilon) = b(F - z^\varepsilon, v^E, v^\varepsilon) + b(z^\varepsilon, v^\varepsilon - v^E, z^\varepsilon). \tag{3.42}$$

Let us leave out $b(u^\varepsilon, v, u^\varepsilon)$ from our analysis for a moment. Indeed, it well be treated differently. Then considering the other terms appearing in (3.39), (3.40) and (3.41) and exploiting (3.42), we have

$$\begin{aligned} &b(v^\varepsilon, z^\varepsilon, v^E - v^\varepsilon) + b(v^E, F, v^\varepsilon - v^E) + b(F, F, v^\varepsilon - v^E) \\ &+ b(F - z^\varepsilon, v^E, v^\varepsilon) - b(z^\varepsilon, z^\varepsilon, v^\varepsilon - v^E) \pm b(z^\varepsilon, F, v^\varepsilon - v^E) \\ &\pm b(v^E, z^\varepsilon, v^E - v^\varepsilon) \\ &= b(v^E + z^\varepsilon, F - z^\varepsilon, v^\varepsilon - v^E) + b(v^\varepsilon - v^E, z^\varepsilon, v^E - v^\varepsilon) \\ &+ b(F - z^\varepsilon, v^E, v^\varepsilon) + b(F - z^\varepsilon, F, v^\varepsilon - v^E). \end{aligned} \tag{3.43}$$

Therefore we can simplify (3.36) and it holds

$$\begin{aligned} \|v_t^\varepsilon - v_t^E\|^2 &\leqslant o(1) + \delta^{1/2} C(N, u_0) \sup\nolimits_{t\in[0,T]} \|v_t^\varepsilon\| + \varepsilon^{1/2} C(N, u_0) \sup\nolimits_{t\in[0,T]} \|W_t^\varepsilon\| \\ &+ 2\varepsilon^{1/2} \int_0^t \langle dW_s^\varepsilon, v_s^\varepsilon \rangle + 2\varepsilon \int_0^t \langle \nabla v_s^\varepsilon, \nabla(v_s^E - v_s)\rangle ds \\ &+ 2\|\nabla v^E\|_{L^\infty(0,T;L^\infty)} \int_0^t \|v_s^\varepsilon - v_s^E\|^2 ds + 2\int_0^t b(u_s^\varepsilon, v_s, u_s^\varepsilon) ds \\ &+ 2\int_0^t \|v_s^\varepsilon - v_s^E\| \|\nabla F_s\|_{L^4} \|F_s - z_s^\varepsilon\|_{L^4} ds \\ &+ 2\int_0^t \|v_s^\varepsilon - v_s^E\| \|\nabla(F_s - z_s^\varepsilon)\| \|v_s^E + z_s^\varepsilon\|_{L^\infty} ds \\ &+ 2\|\nabla z^\varepsilon\|_{L^\infty(0,T;L^\infty)} \int_0^t \|v_s^\varepsilon - v_s^E\|^2 ds + 2\int_0^t \|\nabla v_s^E\| \|v_s^\varepsilon\| \|F_s - z_s^\varepsilon\|_{L^\infty} ds. \end{aligned} \tag{3.44}$$

Now we can treat the term $\varepsilon \int_0^t \langle \nabla v_s^\varepsilon, \nabla(v_s^E - v_s)\rangle ds$ exploiting the properties of the boundary layer corrector (3.31) and the convergence $z^\varepsilon \to F$ in $C([0,T]; D((-A)^{\gamma - \frac{1}{2} - \theta}))$ $\mathbb{P}$-a.s.

$$\begin{aligned} &2\varepsilon \int_0^t \langle \nabla v_s^\varepsilon, \nabla(v_s^E - v_s)\rangle ds \\ &= 2\varepsilon \int_0^t \langle \nabla u^\varepsilon, \nabla(v_s^E - v_s)\rangle ds - 2\varepsilon \int_0^t \langle \nabla z_s^\varepsilon, \nabla(v_s^E - v_s)\rangle ds \\ &\leqslant 2\varepsilon \int_0^t \|\nabla u_s^\varepsilon\| \|\nabla v_s^E\| ds + 2\varepsilon \int_0^t \|\nabla u_s^\varepsilon\|_{L^2(\Gamma_\delta)} \|\nabla v_s\| ds \\ &+ 2\varepsilon \int_0^t \|(-A)^{1/2} z_s^\varepsilon\| \|\nabla(v_s^E - v_s)\| ds \\ &\leqslant 2\varepsilon \int_0^t \|\nabla u_s^\varepsilon\| \|\nabla v_s^E\| ds + \delta^{-1/2} \varepsilon C(N, u_0) \int_0^t \|\nabla u_s^\varepsilon\|_{L^2(\Gamma_\delta)} \\ &+ \delta^{-1/2} \varepsilon C(N, u_0) \int_0^t \|(-A)^{1/2} z_s^\varepsilon\| ds. \end{aligned} \tag{3.45}$$

The term $\int_0^t b(u^\varepsilon, v, u^\varepsilon)ds$ is the classical term in the analysis of the inviscid limit in the Kato's regime, it can be estimated by

$$\left|\int_0^t b(u_s^\varepsilon, v_s, u_s^\varepsilon)ds\right| \leqslant \delta C(N,u_0)\int_0^t \|\nabla u_s^\varepsilon\|^2_{L^2(\Gamma_\delta)}ds \tag{3.46}$$

see [36, Equation 5.8]. Combining estimates (3.44), (3.45) and (3.46), choosing $\delta = c\varepsilon$, where c is the constant appearing in Hypothesis 2.20, it holds

$$\begin{aligned}\|v_t^\varepsilon - v_t^E\|^2 &\leqslant o(1) + \varepsilon^{1/2}C(N,u_0)\sup_{t\in[0,T]}\|v_t^\varepsilon\| + 2\varepsilon^{1/2}\int_0^t \langle dW_s^\varepsilon, v_s^\varepsilon\rangle \\ &+ \varepsilon C(N,u_0)\int_0^t \|\nabla u_s^\varepsilon\|ds + \varepsilon^{1/2}C(N,u_0)\sup_{t\in[0,T]}\|W_t^\varepsilon\| \\ &+ \varepsilon^{1/2}C(N,u_0)\int_0^t \|\nabla u_s^\varepsilon\|_{L^2(\Gamma_{c\varepsilon})}ds + \varepsilon^{1/2}C(N,u_0)\int_0^t \|\nabla u_s^\varepsilon\|_{L^2(\Gamma_{c\varepsilon})} \\ &+ 2\|\nabla v^E\|_{L^\infty(0,T;L^\infty)}\int_0^t \|v^\varepsilon - v^E\|^2 ds + \varepsilon C(N,u_0)\int_0^t \|\nabla u_s^\varepsilon\|^2_{L^2(\Gamma_{c\varepsilon})}ds \\ &+ 2(1+\|\nabla z^\varepsilon\|_{L^\infty(0,T;L^\infty)})\int_0^t \|v_s^\varepsilon - v_s^E\|^2 ds + 2\int_0^t \|v_s^\varepsilon\|\|\nabla v_s^E\|\|F_s - z_s^\varepsilon\|_{L^\infty}ds.\end{aligned} \tag{3.47}$$

Therefore, by Grönwall's inequality, equation (3.47) implies

$$\begin{aligned}\sup_{t\in[0,T]}\|v_t^\varepsilon - v_t^E\|^2 \leqslant\ & e^{2T(1+\|\nabla v^E\|_{L^\infty(0,T;L^\infty)}+\|\nabla z^\varepsilon\|_{L^\infty(0,T;L^\infty)})}\times \\ & \Bigg(o(1) + \varepsilon^{1/2}C(N,u_0)\sup_{t\in[0,T]}\|v_t^\varepsilon\| \\ &+ 2\varepsilon^{1/2}\sup_{t\in[0,T]}\left|\int_0^t \langle dW_s^\varepsilon, v_s^\varepsilon\rangle\right| \\ &+ \varepsilon^{1/2}C(N,u_0)\sup_{t\in[0,T]}\|W_t^\varepsilon\| \\ &+ \varepsilon C(N,u_0)\int_0^T \|\nabla u_s^\varepsilon\|ds \\ &+ \varepsilon^{1/2}C(N,u_0)\int_0^T \|\nabla u_s^\varepsilon\|_{L^2(\Gamma_{c\varepsilon})}ds \\ &+ \varepsilon C(N,u_0)\int_0^T \|\nabla u_s^\varepsilon\|^2_{L^2(\Gamma_{c\varepsilon})}ds \\ &+ 2\int_0^T \|v_s^\varepsilon\|\|\nabla v_s^E\|\|F_s - z_s^\varepsilon\|_{L^\infty}ds\Bigg).\end{aligned} \tag{3.48}$$

Since $\gamma \geqslant 2$ we can find $\theta > 0$ small enough such that $D((-A)^{\gamma-1/2-\theta}) \hookrightarrow W^{1,\infty}$. Therefore, $\|\nabla z^\varepsilon\|_{L^\infty(0,T;L^\infty)}$ is $\mathbb{P}-a.s.$ bounded by $C(N,u_0)$ from Lemma 3.7. Similarly, from Theorem 2.18, $\|\nabla v^E\|_{L^\infty(0,T;L^\infty)} \leqslant C(N,u_0) \quad \mathbb{P}-a.s.$

Therefore $e^{2T(1+\|\nabla v^E\|_{L_t^\infty L_x^\infty}+\|\nabla z^\varepsilon\|_{L_t^\infty L_x^\infty})} \leqslant C(N,u_0) \quad \mathbb{P}-a.s.$ This means that, in order to show that Lemma 3.8 holds, it is enough to prove that

$$\begin{aligned}&\varepsilon^{1/2}C(N,u_0)\left(\sup_{t\in[0,T]}\|W_t^\varepsilon\| + \sup_{t\in[0,T]}\left|\int_0^t \langle dW_s^\varepsilon, v_s^\varepsilon\rangle\right|\right) + \varepsilon C(N,u_0)\int_0^T \|\nabla u_s^\varepsilon\|ds \\ &+ \varepsilon^{1/2}C(N,u_0)\int_0^T \|\nabla u_s^\varepsilon\|_{L^2(\Gamma_{c\varepsilon})} + \varepsilon C(N,u_0)\int_0^T \|\nabla u_s^\varepsilon\|^2_{L^2(\Gamma_{c\varepsilon})}ds \\ &+ 2\int_0^T \|v_s^\varepsilon\|\|\nabla v_s^E\|\|F_s - z_s^\varepsilon\|_{L^\infty}ds \to 0 \quad \text{in Probability.}\end{aligned} \tag{3.49}$$

The terms

$$\varepsilon^{1/2}C(N,u_0)\int_0^T \|\nabla u_s^\varepsilon\|_{L^2(\Gamma_{c\varepsilon})}+\varepsilon C(N,u_0)\int_0^T \|\nabla u_s^\varepsilon\|_{L^2(\Gamma_{c\varepsilon})}^2 ds\to 0 \quad \text{in Probability}$$

thanks to Hypothesis 2.20. The terms

$$\begin{aligned}&\varepsilon^{1/2}C(N,u_0)\left(\sup_{t\in[0,T]}\|W_t^\varepsilon\|\right)+\varepsilon^{1/2}\sup_{t\in[0,T]}\left|\int_0^t\langle dW_s^\varepsilon,v_s^\varepsilon\rangle\right|\\&+\varepsilon C(N,u_0)\int_0^T\|\nabla u_s^\varepsilon\|ds\to 0 \quad \text{in Probability}\end{aligned}$$

since it holds by Burkholder-Davis-Gundy inequality, Hölder inequality and (3.27)

$$\begin{aligned}&\mathbb{E}\left[\sup_{t\in[0,T]}\|W_t^\varepsilon\|\right]+\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t\langle dW_s^\varepsilon,v_s^\varepsilon\rangle\right|\right]+\varepsilon^{1/2}\mathbb{E}\left[\int_0^T\|\nabla u_s^\varepsilon\|ds\right]\\&\leqslant C+C\mathbb{E}\left[\sup_{t\in[0,T]}\|v_t^\varepsilon\|^2\right]^{1/2}+C(T)\mathbb{E}\left[\varepsilon\int_0^T\|\nabla u_s^\varepsilon\|^2ds\right]\\&\leqslant C(N,u_0,T).\end{aligned}$$

Lastly

$$\int_0^T\|v_s^\varepsilon\|\|\nabla v_s^E\|\|F_s-z_s^\varepsilon\|_{L^\infty}ds\to 0 \quad \text{in Probability}$$

thanks to Lemma 3.7, (3.27) and (2.22). Indeed it holds:

$$\begin{aligned}\mathbb{E}\left[\int_0^T\|v_s^\varepsilon\|\|\nabla v_s^E\|\|F_s-z_s^\varepsilon\|_{L^\infty}ds\right]&\leqslant C(N,u_0)\mathbb{E}\left[\int_0^T\|v_s^\varepsilon\|\|F_s-z_s^\varepsilon\|_{L^\infty}ds\right]\\&\leqslant C(N,u_0)\mathbb{E}\left[\int_0^T\|v_s^\varepsilon\|^2\right]^{1/2}\mathbb{E}\left[\int_0^T\|F_s-z_s^\varepsilon\|_{L^\infty}^2ds\right]^{1/2}\\&\to 0.\end{aligned}$$

Therefore (3.49) holds and the thesis follows. □

Combining Lemma 3.7 and Lemma 3.8 the second condition in Hypothesis 2.4 holds.

Remark 3.9. As it is classical in the analysis of the inviscid limit in bounded domains, Hypothesis 2.20 for the forcing terms f^ε, f is implied by the convergence in probability of u^ε to u^E in the probability space introduced by Skorokhod's representation theorem. Let us consider (2.17) for $t=T$ and take the limsup of this expression for $\varepsilon\to 0$. It holds

$$2\,\mathrm{limsup}_{\varepsilon\to 0}\,\varepsilon\int_0^T\|\nabla u_s^\varepsilon\|_{L^2}^2ds\leqslant \limsup_{\varepsilon\to 0}\|u_0^\varepsilon\|^2+\limsup_{\varepsilon\to 0}\varepsilon T\sum_{k\in K}\|\sigma_k\|^2 \tag{3.50}$$

$$+2\limsup_{\varepsilon\to 0}\sqrt{\varepsilon}\int_0^T\langle u_s^\varepsilon,dW_s^\varepsilon\rangle$$

$$+2\limsup_{\varepsilon\to 0}\int_0^T\langle f_s^\varepsilon,u_s^\varepsilon\rangle ds+\limsup_{\varepsilon\to 0}\{-\|u_T^\varepsilon\|^2\}. \tag{3.51}$$

Under our assumptions it follows immediately that

$$\limsup_{\varepsilon\to 0}\|u_0^\varepsilon\|^2+\limsup_{\varepsilon\to 0}\varepsilon T\sum_{k\in K}\|\sigma_k\|^2+\limsup_{\varepsilon\to 0}\{-\|u_T^\varepsilon\|^2\}=\|u_0\|^2-\|u_T^E\|^2$$

in Probability. Moreover

$$\limsup_{\varepsilon\to 0}\sqrt{\varepsilon}\left|\int_0^T\langle u^\varepsilon, dW_s^\varepsilon\rangle\right| = 0 \quad \text{in Probability}$$

since by (3.27)

$$\mathbb{E}\left[\left|\int_0^T\langle dW_s^\varepsilon, u_s^\varepsilon\rangle\right|\right] \leqslant C\mathbb{E}\left[\sup_{t\in[0,T]}\|u_t^\varepsilon\|^2\right]^{1/2} \leqslant C(N,u_0,T).$$

Lastly

$$\lim_{\varepsilon\to 0}\int_0^T\langle f_s^\varepsilon, u_s^\varepsilon\rangle ds = \int_0^T\langle f_s, u_s^E\rangle ds \quad \text{in Probability.}$$

Indeed

$$\begin{aligned}\left|\int_0^T\langle f_s^\varepsilon, u_s^\varepsilon\rangle - \langle f_s, u_s^E\rangle ds\right| &\leqslant \left|\int_0^T\langle f_s^\varepsilon, u_s^\varepsilon - u_s^E\rangle ds\right| + \left|\int_0^T\langle f_s^\varepsilon - f_s, u_s^E\rangle ds\right| \\ &= I_1 + I_2.\end{aligned}$$

$I_1 \to 0$ in Probability since we assumed that $u^\varepsilon \to u^E$ in $C([0,T];H)$ in Probability. $I_1 \to 0$ $\mathbb{P}-a.s.$ since in the space introduced by Skorokhod's representation theorem $f^\varepsilon \to f$ weakly in $L^2(0,T;H_0)$ $\mathbb{P}-a.s.$ and $u^E \in \left(L^2(0,T;H_0)\right)^*$ $\mathbb{P}-a.s.$ Therefore we proved that

$$\operatorname{limsup}_{\varepsilon\to 0}\varepsilon\int_0^T\|\nabla u_s^\varepsilon\|_{L^2}^2 ds = \|u_0\|^2 - \|u_T^E\|^2 - 2\int_0^T\langle f_s, u_s^E\rangle ds \quad \text{in Probability}$$

which implies Hypothesis 2.20 by (2.22).

Remark 3.10. Combining Remark 3.9 and the results of Subsection 3.1.1 we obtain that Hypothesis 2.20 is satisfied in the case of fluids with radial symmetry.

4 Second-Grade fluids

Since $\mathcal{E}_0^{SG} \hookrightarrow \mathcal{E}_0^{NS}$, by definition of the maps $\mathcal{G}^{NS,0}, \mathcal{G}^{SG,0}$ we have

$$\mathcal{G}^{NS,0}(u_0, \int_0^\cdot f_s ds) = \mathcal{G}^{SG,0}(u_0, \int_0^\cdot f_s ds)$$

if $u_0 \in \mathcal{E}_0^{SG}$. Moreover, since $\mathcal{E}_0^{SG}$ and $\mathcal{E}_0^{NS}$ are Banach spaces endowed with the same norm, the validity of Condition 1 in Hypothesis 2.4 for $\mathcal{G}^{NS,0}$ implies the validity of the same condition for $\mathcal{G}^{SG,0}$. Moreover the argument for showing the lower continuity of the map $I_{u_0}^{RS}(v)$ for $v \in \mathcal{E}$ fixed repeats verbatim for $I_{u_0}^{SG}(v)$. Therefore, in order to prove Theorem 2.25, it is enough to show the validity of Condition 2 of Hypothesis 2.4. This is the aim of the next subsection.

4.1 Condition 2

We argue similarly to the proof of the validity Condition 2 in the case of Navier-Stokes equations. Fix $N > 0$, let $\tilde{f}^\varepsilon, \tilde{f} \in \mathcal{P}_2^N,\ u_0^\varepsilon,\ u_0 \in \mathcal{E}_0^{SG}$ such that $\tilde{f}^\varepsilon \to_{\mathcal{L}} \tilde{f}$ weakly in $L^2(0,T;H_0),\ u_0^\varepsilon \to u_0$ in $\mathcal{E}_0^{SG}$. We will show that for each sequence $\varepsilon_n \to 0, \nu_n \to 0$ s.t. $\nu_n = O(\varepsilon_n)$, $\mathcal{G}^{\varepsilon_n,SG}\big(u_0^{\varepsilon_n}, {\varepsilon_n}^{1/2}W + \int_0^\cdot f_s^{\varepsilon_n} ds\big)$ converges in law to $\mathcal{G}^{0,SG}(u_0, \int_0^\cdot u_s ds)$ in the topology of $\mathcal{E}$. This implies the validity of the second condition in Hypothesis 2.4. In order to simplify the notation, we will consider $\varepsilon > 0,\ \nu > 0$ in the following dropping the subscript ε_n, ν_n, having in mind they are countable families. Since S^N is a Polish space, by Skorokhod's representation theorem, see [25, Chapter 3] and [34], we can introduce a further filtered probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathcal{F}}_t, \tilde{\mathbb{P}})$ and random variables $f^\varepsilon, W^\varepsilon, f$ such that $(f^\varepsilon, W^\varepsilon)$ has the same joint law of $(\tilde{f}^\varepsilon, W)$, f has the same law of $\tilde{f}$ and $f^\varepsilon \rightarrow_{\tilde{\mathbb{P}}-a.s.} f$ in $L^2(0,T;H_0)$, see for example [27] for details. Thanks to Theorem 2.16 for each ε we can define u^ε as the unique solution of (2.16) with forcing term f^ε, initial condition u_0^ε and Brownian forcing term W^ε. Moreover, by Theorem 2.18 we can define u^E as the unique regular solution of (2.20) with forcing term f and initial condition u_0. We will show that u_0^ε converges to u^E in probability in $C([0,T];H)$. This implies the validity of Condition 2.

Before starting with the computations we recall some facts. In the following, with some abuse of notation, we will simply use $\mathbb{P},\ \mathbb{E}$ instead of $\tilde{\mathbb{P}},\ \tilde{\mathbb{E}}$. Fix $\theta > 0$ arbitrarily small and define $F^\varepsilon(t) = \int_0^t f_s^\varepsilon ds,\ F(t) = \int_0^t f_s ds$. By Hypothesis 2.11, $H_0 \hookrightarrow D((-A)^\gamma)$. This implies, see for example [27, Proposition 2.10] that

$$F^\varepsilon \rightarrow_{\mathbb{P}-a.s.} F \quad \text{in } C([0,T];D((-A)^{\gamma-\theta})). \tag{4.1}$$

Obviously

$$\sup_{\varepsilon>0}\|F^\varepsilon\|_{C([0,T];D((-A)^\gamma))} + \|F\|_{C([0,T];D((-A)^\gamma))} \leqslant C(N) \quad \mathbb{P}-a.s. \tag{4.2}$$

Starting from (2.18) and (2.19), under Hypothesis 2.24, Burkholder-Davis-Gundy inequality, Grönwall's lemma and the convergence of u_0^ε to u_0 imply

$$\mathbb{E}\left[\sup_{t\in[0,T]}\|u_t^\varepsilon\|^2\right] + \varepsilon\mathbb{E}\left[\sup_{t\in[0,T]}\|\nabla u_t^\varepsilon\|^2\right] + 2\nu\mathbb{E}\left[\int_0^T\|\nabla u_s^\varepsilon\|^2 ds\right] \leqslant C(N,u_0), \tag{4.3}$$

$$\varepsilon^3\mathbb{E}\left[\sup_{t\in[0,T]}\|u_t^\varepsilon\|_{H^3}^2\right] \leqslant C(N,u_0), \tag{4.4}$$

see [49, Section 6] for the details. In order to show the convergence of u^ε to u^E we will introduce

$$z^\varepsilon = \int_0^t e^{\nu(I-\varepsilon A)^{-1}A(t-s)}(I-\varepsilon A)^{-1} f_s^\varepsilon ds$$

which is the mild solution of

$$d(I-\varepsilon A)z^\varepsilon = \nu A z^\varepsilon + f^\varepsilon,$$

$v^\varepsilon = u^\varepsilon - z^\varepsilon,\ v^E = u^E - F$ and show separately the convergence of z^ε to F and of v^ε to v^E. Once again, the convergence of z^ε to F will be the easiest part of the argument, while the convergence of v^ε to v^E will be more demanding and its proof will be based on the introduction of a corrector of the boundary layer for v^E satisfying suitable properties. Before starting showing the convergence of z^ε to F, we recall that the the operators A and $(I-\varepsilon A)^{-1}$ commute on $D((-A)^\alpha)$ for each $\alpha \in \mathbb{R}$. Moreover

$$(I-\varepsilon A)^{-1}A : D((-A)^\alpha) \rightarrow D((-A)^\alpha)$$

is a linear, bounded operator for each $\alpha \in \mathbb{R}$ with operatorial norm equal to $\frac{1}{\varepsilon}$.

Lemma 4.1. *For each $\theta > 0$, $z^\varepsilon(t) \rightarrow F$ in $C([0,T];D((-A)^{\gamma-2\theta}))$ $\mathbb{P}-a.s.$ and in $L^2(\Omega,\mathbb{P})$.*

Proof. z^ε can be rewritten as

$$z_t^\varepsilon = \nu\int_0^t (I-\varepsilon A)^{-2} A e^{\varepsilon(I-\varepsilon A)^{-1}A(t-s)} F_s^\varepsilon ds + (I-\varepsilon A)^{-1}F_t^\varepsilon = I_1 + I_2$$

Let us show that $I_2 \to F$ and $I_1 \to 0$ properly. We start from I_2:

$$I_2 = ((I - \varepsilon A)^{-1} F_t^\varepsilon - F_t^\varepsilon) + F_t^\varepsilon = I_{2,1} + I_{2,2}.$$

Now $I_{2,2} \to F_t$ in $C([0,T]; D((-A)^{\gamma-\theta}))\ \mathbb{P} - a.s.$ and $L^2(\Omega, \mathbb{P})$ by (4.1) and (4.2). For what concerns $I_{2,1}$ we have

$$\begin{aligned}
\sup_{t\in[0,T]} \|(-A)^{\gamma-2\theta} I_{2,1}\| &= \sup_{t\in[0,T]} \|(-A)^{\gamma-2\theta} (F_t^\varepsilon - (I-\varepsilon A)^{-1} F_t^\varepsilon)\| \\
&= \sup_{t\in[0,T]} \|(-A)^{\gamma-2\theta} \left(\frac{I}{\varepsilon} - A\right)^{-1} A F_t^\varepsilon\| \\
&= \sup_{t\in[0,T]} \|(-A)^{1-\theta} \left(\frac{I}{\varepsilon} - A\right)^{-1} \| \|(-A)^{\gamma-\theta} F_t^\varepsilon\| \\
&\leqslant \varepsilon^\theta \frac{\left(\frac{1-\theta}{\theta}\right)^{1-\theta}}{1 + \frac{1-\theta}{\theta}} C(N,T) \to 0\ \mathbb{P} - a.s.
\end{aligned}$$

Since previous bound is uniform in $\omega \in \Omega$, previous inequalities imply also the convergence in $L^2(\Omega, \mathbb{P})$. For what concerns I_1 we have

$$\begin{aligned}
&\sup_{t\in[0,T]} \|(-A)^{\gamma-\theta} I_1\| \\
&\leqslant \nu \sup_{t\in[0,T]} \int_0^t \|(-A)^{1-\theta} (I - \varepsilon A)^{-2} e^{\varepsilon (I-\varepsilon A)^{-1} A(t-s)} (-A)^\gamma F_s^\varepsilon\| ds \\
&\leqslant \varepsilon^\theta \frac{\left(\frac{1-\theta}{\theta+1}\right)^{\frac{1-\theta}{2}}}{1 + \frac{1-\theta}{\theta+1}} C(N,T) \to 0\ \mathbb{P} - a.s.
\end{aligned}$$

Since previous bound is uniform in $\omega \in \Omega$, previous inequalities imply also the convergence in $L^2(\Omega, \mathbb{P})$. Combining the convergence of I_1, $I_{2,1}$ and $I_{2,2}$ the thesis follows. □

Remark 4.2. Since $H_0 \hookrightarrow D((-A)^{3/2}) \hookrightarrow W$, z^ε satisfies relations (4.3), (4.4). We show a stronger relation. Indeed it holds:

$$\begin{aligned}
\sup_{t\in[0,T]} \|(-A)^{3/2} z_t^\varepsilon\| &\leqslant \sup_{t\in[0,T]} \int_0^t \|(-A)^{3/2} e^{\nu(I-\varepsilon A)^{-1} A(t-s)} (I - \varepsilon A)^{-1} f_s^\varepsilon\| ds \\
&\leqslant \sup_{t\in[0,T]} \int_0^t \|(-A)^{3/2} f_s^\varepsilon\| ds \\
&\leqslant C(N,T) \quad \mathbb{P} - a.s.
\end{aligned} \tag{4.5}$$

Since previous bound is uniform in $\omega \in \Omega$, we have also

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|(-A)^{3/2} z_t^\varepsilon\|^2\right] \leqslant C(N,T). \tag{4.6}$$

In order to prove the convergence of v^ε to v^E we observe that they solve in a sense analogous to Definition 2.14, Definition 2.17

$$d(v^\varepsilon - \varepsilon \Delta v^\varepsilon) = (\nu \Delta v^\varepsilon - \operatorname{curl}(v^\varepsilon + z^\varepsilon - \varepsilon \Delta(v^\varepsilon + z^\varepsilon)) \times (v^\varepsilon + z^\varepsilon) + \nabla q^\varepsilon) dt + \sqrt{\varepsilon} dW_t \tag{4.7}$$

and (3.29). By triangle inequality and the uniform bound guaranteed by Remark 4.2, estimates analogous to (4.3), (4.4), (2.22) hold for v^ε and v^E, too. Moreover v^E satisfies (3.30). Again, we introduce the corrector of the boundary layer v of width $\delta = \delta(\varepsilon)$, i.e. a divergence free vector field with support in a strip of the boundary of width δ such that $v^E - v \in V$ and $\mathbb{P} - a.s.$ uniformly in $t \in [0,T],\ \omega \in \Omega$

$$\begin{aligned}
&\|v_t\|_{L^\infty(D)} \leqslant C(N,u_0),\ \|v_t\| \leqslant C(N,u_0)\delta^{\frac{1}{2}},\ \|\partial_t v_t\| \leqslant C(N,u_0)\delta^{\frac{1}{2}}, \\
&\|\nabla v_t\|_{L^\infty(D)} \leqslant C(N,u_0)\delta^{-1},\ \|\nabla v_t\|_{L^2(D)} \leqslant C(N,u_0)\delta^{-1/2},\ \|\partial_t \nabla v_t\| \leqslant C(N,u_0)\delta^{-\frac{1}{2}}.
\end{aligned} \tag{4.8}$$

We choose δ such that

$$\lim_{\varepsilon\to 0}\delta = 0, \quad \lim_{\varepsilon\to 0}\frac{\varepsilon}{\delta} = 0. \tag{4.9}$$

Now we are ready to show the convergence of v^ε to v^E.

Lemma 4.3. *$v^\varepsilon \to v^E$ in $C([0,T];H)$ in probability.*

Proof. Let $w^\varepsilon = v^\varepsilon - v^E$. Arguing as in [49, Theorem 9] one can show that the following relations hold true:

$$\begin{aligned} d\|w^\varepsilon\|^2 &= \varepsilon Tr(Q)dt + \varepsilon^3 Tr(A^2(I-\varepsilon A)^{-2}Q)dt + \varepsilon^2 Tr(A(I-\varepsilon A)^{-1}Q)dt \\ &\quad + 2\nu\langle w^\varepsilon, Av^\varepsilon\rangle dt + 2\sqrt{\varepsilon}\langle w^\varepsilon, dW_t^\varepsilon\rangle + b(v^E+F, v^E+F, w^\varepsilon)dt \\ &\quad - 2b(v^\varepsilon+z^\varepsilon, (I-\varepsilon\Delta)(v^\varepsilon+z^\varepsilon), w^\varepsilon)dt - 2b(w^\varepsilon, (I-\varepsilon\Delta)(v^\varepsilon+z^\varepsilon), v^\varepsilon+z^\varepsilon)dt \\ &\quad + 2\varepsilon\langle w^\varepsilon, dAv^\varepsilon\rangle \end{aligned} \tag{4.10}$$

where

$$\begin{aligned} \langle w^\varepsilon, dAv^\varepsilon\rangle = &-\frac{d\|(-A)^{1/2}v^\varepsilon\|^2}{2} + \frac{d\langle\langle(-A)^{1/2}v^\varepsilon, (-A)^{1/2}v^\varepsilon\rangle\rangle}{2} \\ &+ d\langle(-A)^{1/2}(v^E - v), (-A)^{1/2}v^\varepsilon\rangle - \langle\partial_t(-A)^{1/2}(v^E-v), (-A)^{1/2}v^\varepsilon\rangle \\ &- d\langle v, Av^\varepsilon\rangle + \langle\partial_t v, Av^\varepsilon\rangle. \end{aligned}$$

First we observe that

$$\varepsilon Tr(Q) + \varepsilon^3 Tr(A^2(I-\varepsilon A)^{-2}Q) + \varepsilon^2 Tr(A(I-\varepsilon A)^{-1}Q) = o(1). \tag{4.11}$$

Secondly, we can rewrite the trilinear forms as

$$\begin{aligned} &b(v^E+F, v^E+F, w^\varepsilon) - b(v^\varepsilon+z^\varepsilon, v^\varepsilon+z^\varepsilon, w^\varepsilon) + \varepsilon b(v^\varepsilon+z^\varepsilon, \Delta(v^\varepsilon+z^\varepsilon), w^\varepsilon) \\ &- \varepsilon b(w^\varepsilon, \Delta(v^\varepsilon+z^\varepsilon), v^\varepsilon+z^\varepsilon) \pm b(v^\varepsilon, v^E, w^\varepsilon) = \\ &b(F, v^E+F, w^\varepsilon) + b(v^E, F, w^\varepsilon) - b(z^\varepsilon, v^\varepsilon+z^\varepsilon, w^\varepsilon) - b(v^\varepsilon, z^\varepsilon, w^\varepsilon) - b(w^\varepsilon, v^E, w^\varepsilon) \\ &+ \varepsilon b(u^\varepsilon, \Delta u^\varepsilon, w^\varepsilon) - \varepsilon b(w^\varepsilon, \Delta u^\varepsilon, u^\varepsilon). \end{aligned} \tag{4.12}$$

Integrating in time between 0 and t equation (4.10) and exploiting (4.11), (4.12), we get

$$\begin{aligned} \|w_t^\varepsilon\|^2 + \varepsilon\|\nabla v_t^\varepsilon\|^2 &= o(1) + \|u_0^\varepsilon - u_0\|^2 + \varepsilon\|\nabla u_0^\varepsilon\|^2 + 2\varepsilon\langle\nabla(v_t^E - v_t), \nabla v_t^\varepsilon\rangle \\ &\quad - 2\varepsilon\langle\nabla(u_0 - v_0), \nabla u_0^\varepsilon\rangle - 2\varepsilon\int_0^t \langle\partial_s\nabla(v_s^E - v_s), \nabla v_s^\varepsilon\rangle ds \\ &\quad - 2\varepsilon\langle v_t, \Delta v_t^\varepsilon\rangle + 2\varepsilon\langle v_0, \Delta u_0^\varepsilon\rangle + 2\varepsilon\int_0^t\langle\partial_s v_s, \Delta v_s^\varepsilon\rangle ds \\ &\quad + 2\nu\int_0^t\langle w_s^\varepsilon, Av_s^\varepsilon\rangle ds + 2\sqrt{\varepsilon}\int_0^t\langle w_s^\varepsilon, dW_s^\varepsilon\rangle \\ &\quad + 2\int_0^t b(F_s, u_s^E, w_s^\varepsilon) - b(z_s^\varepsilon, u_s^\varepsilon, w_s^\varepsilon)ds \\ &\quad + 2\int_0^t b(v_s^E, F_s, w_s^\varepsilon) - b(v_s^\varepsilon, z_s^\varepsilon, w_s^\varepsilon) - b(w_s^\varepsilon, v_s^E, w_s^\varepsilon)ds \\ &\quad + 2\varepsilon\int_0^t b(u_s^\varepsilon, \Delta u_s^\varepsilon, w_s^\varepsilon) - b(w_s^\varepsilon, \Delta u_s^\varepsilon, u_s^\varepsilon)ds. \end{aligned} \tag{4.13}$$

In order to reach our final expression for the evolution of $\|w_t^\varepsilon\|^2$ we rewrite better the terms related to the forcing terms $f^\varepsilon,\ f$ in equation (4.12)

$$\begin{aligned}
&b(F,u^E,w^\varepsilon)+b(v^E,F,w^\varepsilon)-b(v^\varepsilon,z^\varepsilon,w^\varepsilon)-b(z^\varepsilon,u^\varepsilon,w^\varepsilon)\\
&=b(F,v^E,v^\varepsilon)+b(F,F,w^\varepsilon)+b(v^E,F,w^\varepsilon)-b(z^\varepsilon,z^\varepsilon,w^\varepsilon)\\
&+b(z^\varepsilon,v^\varepsilon,v^E)-b(v^\varepsilon,z^\varepsilon,w^\varepsilon)\pm b(F,z^\varepsilon,w^\varepsilon)\pm b(v^E,z^\varepsilon,w^\varepsilon)\\
&=b(F-z^\varepsilon,v^E,v^\varepsilon)+b(F,F-z^\varepsilon,w^\varepsilon)\\
&+b(v^E,F-z^\varepsilon,w^\varepsilon)+b(F-z^\varepsilon,z^\varepsilon,w^\varepsilon)+b(w^\varepsilon,z^\varepsilon,w^\varepsilon).
\end{aligned}\tag{4.14}$$

Since $u_0^\varepsilon\to u_0$ in $\mathcal{E}_0^{SG}$, our choice of δ, see (4.9), and the properties of the boundary layer corrector (4.8) we have easily, see [49, Equation (99)] for details,

$$\varepsilon\|\nabla u_0^\varepsilon\|^2-2\varepsilon\langle\nabla(u_0-v_0),\nabla u_0^\varepsilon\rangle+\varepsilon\langle v_0,\Delta u_0^\varepsilon\rangle=o(1).$$

Inserting (4.14) in (4.13) we get

$$\begin{aligned}
\|w_t^\varepsilon\|^2+\varepsilon\|\nabla v_t^\varepsilon\|^2&=o(1)+2\varepsilon\langle\nabla(v_t^E-v_t),\nabla v_t^\varepsilon\rangle-2\varepsilon\int_0^t\langle\partial_s\nabla(v_s^E-v_s),\nabla v_s^\varepsilon\rangle ds\\
&-2\varepsilon\langle v_t,\Delta v_t^\varepsilon\rangle+2\varepsilon\int_0^t\langle\partial_s v_s,\Delta v_s^\varepsilon\rangle ds+2\nu\int_0^t\langle w_s^\varepsilon,Av_s^\varepsilon\rangle ds\\
&+2\int_0^t b(F_s+v_s^E,F_s-z_s^\varepsilon,w_s^\varepsilon)-b(F_s-z_s^\varepsilon,v_s^E,v_s^\varepsilon)ds\\
&+2\int_0^t b(F_s-z_s^\varepsilon,z_s^\varepsilon,w_s^\varepsilon)ds\\
&-2\int_0^t b(w_s^\varepsilon,z_s^\varepsilon,w_s^\varepsilon)-b(w_s^\varepsilon,v_s^E,w_s^\varepsilon)ds\\
&+2\varepsilon\int_0^t b(u_s^\varepsilon,\Delta u_s^\varepsilon,w_s^\varepsilon)-b(w_s^\varepsilon,\Delta u_s^\varepsilon,u^\varepsilon)ds\\
&+2\sqrt{\varepsilon}\int_0^t\langle w_s^\varepsilon,dW_s^\varepsilon\rangle\\
&=I_1(t)+I_2(t)+I_3(t)+I_4(t)+I_5(t)+I_6(t)+M(t),
\end{aligned}\tag{4.15}$$

where

$$\begin{aligned}
I_1(t)&=2\varepsilon\langle\nabla(v_t^E-v_t),\nabla v_t^\varepsilon\rangle-2\varepsilon\langle v_t,\Delta v_t^\varepsilon\rangle,\\
I_2(t)&=-2\varepsilon\int_0^t\langle\partial_s\nabla(v_s^E-v_s),\nabla v_s^\varepsilon\rangle ds+2\varepsilon\int_0^t\langle\partial_s v_s,\Delta v_s^\varepsilon\rangle ds,\\
I_3(t)&=2\nu\int_0^t\langle w_s^\varepsilon,Av_s^\varepsilon\rangle ds,\\
I_4(t)&=-2\int_0^t b(w_s^\varepsilon,z_s^\varepsilon,w_s^\varepsilon)-b(w_s^\varepsilon,v_s^E,w_s^\varepsilon)ds,\\
I_5(t)&=2\int_0^t b(F_s+v_s^E,F_s-z_s^\varepsilon,w_s^\varepsilon)-b(F_s-z_s^\varepsilon,v_s^E,v_s^\varepsilon)ds+b(F_s-z_s^\varepsilon,z_s^\varepsilon,w_s^\varepsilon)ds,\\
I_6(t)&=2\varepsilon\int_0^t b(u_s^\varepsilon,\Delta u_s^\varepsilon,w_s^\varepsilon)-b(w_s^\varepsilon,\Delta u_s^\varepsilon,u^\varepsilon)ds,\\
M(t)&=2\sqrt{\varepsilon}\int_0^t\langle w_s^\varepsilon,dW_s^\varepsilon\rangle.
\end{aligned}$$

Equation (4.15) is the final expression that we will use in order to estimate the various terms and apply Grönwall's lemma. The analysis of $I_1(t)$ follows by the properties of the boundary layer corrector (4.8), our choice of δ (4.9) and the interpolation inequality (2.14). Therefore it holds:

$$\begin{aligned}
I_1(t) &\leqslant \varepsilon C(N,u_0)(1+\delta^{-1/2})\|\nabla v_t^\varepsilon\| + \varepsilon\delta^{1/2}C(N,u_0)\|v_t^\varepsilon\|_{H^2} \\
&\leqslant \frac{\varepsilon}{10}\|\nabla v_t^\varepsilon\|^2 + \varepsilon C(N,u_0)(1+\delta^{-1}) + \varepsilon\delta^{1/2}C(N,u_0)\|\nabla v^\varepsilon\|^{1/2}\|v^\varepsilon\|_{H^3}^{1/2} \\
&\leqslant o(1) + \frac{\varepsilon}{5}\|\nabla v_t^\varepsilon\|^2 + \varepsilon^3\delta\|v_t^\varepsilon\|_{H^3}^2 + \delta^{1/2}C(N,u_0) \\
&= o(1) + \frac{\varepsilon}{5}\|\nabla v_t^\varepsilon\|^2 + \varepsilon^3\delta\|v_t^\varepsilon\|_{H^3}^2. \qquad (4.16)
\end{aligned}$$

The analysis of $I_2(t)$ is analogous to the (4.16) and leads us to

$$I_2(t) \leqslant o(1) + \varepsilon\int_0^t \|\nabla v_s^\varepsilon\|^2 ds + \varepsilon^3\delta\int_0^T \|v_s^\varepsilon\|_{H^3}^2 ds. \qquad (4.17)$$

In order to treat $I_3(t)$, we split w^ε in v^ε, $v^E - v$ and v. Then the first two terms are integrated by parts. Exploiting the properties of the boundary layer corrector (4.8), our choice of δ (4.9), ν (2.24) and the interpolation inequality (2.14) it holds

$$\begin{aligned}
I_3(t) &= -\nu\int_0^t \|\nabla v_s^\varepsilon\|^2 ds - \nu\int_0^t \langle v_s^E - v_s, Av_s^\varepsilon\rangle ds - \nu\int_0^t \langle v_s, Av_s^\varepsilon\rangle ds \\
&= -\nu\int_0^t \|\nabla v_s^\varepsilon\|^2 ds + \nu\int_0^t \|\nabla(v_s^E - v_s)\|\|\nabla v_s^\varepsilon\| ds + \nu\int_0^t \|v_s\|\|v_s^\varepsilon\|_{H^2} ds \\
&\leqslant -\nu\int_0^t \|\nabla v_s^\varepsilon\|^2 ds + \nu(1+\delta^{-1/2})C(N,u_0)\int_0^t \|\nabla v_s^\varepsilon\| ds \\
&+ \nu\delta^{1/2}C(N,u_0)\int_0^t \|\nabla v_s^\varepsilon\|^{1/2}\|v_s^\varepsilon\|_{H^3}^{1/2} ds \\
&\leqslant \varepsilon\int_0^t \|\nabla v_s^\varepsilon\|^2 ds + \varepsilon^3\delta\int_0^T \|v_s^\varepsilon\|_{H^3}^2 ds + \varepsilon(1+\delta^{-1})C(N,u_0) + \delta^{1/2}C(N,u_0) \\
&= o(1) + \varepsilon\int_0^t \|\nabla v_s^\varepsilon\|^2 ds + \varepsilon^3\delta\int_0^T \|v_s^\varepsilon\|_{H^3}^2 ds. \qquad (4.18)
\end{aligned}$$

$I_4(t)$ can be bounded easily by Hölder's inequality, obtaining

$$I_4(t) \leqslant \left(\|z^\varepsilon\|_{L^\infty(0,T;W^{1,\infty})} + \|v^E\|_{L^\infty(0,T;W^{1,\infty})}\right)\int_0^t \|w_s^\varepsilon\|^2 ds. \qquad (4.19)$$

$I_5(t)$ can be handle via Hölder's inequality, exploiting the bounds available on F and v^E, see (3.26) and (2.22):

$$\begin{aligned}
I_5(t) &\leqslant \int_0^t |b(F_s + v_s^E, F_s - z_s^\varepsilon, w_s^\varepsilon)| + |b(F_s - z_s^\varepsilon, v_s^E, v_s^\varepsilon)| + |b(F_s - z_s^\varepsilon, z_s^\varepsilon, w_s^\varepsilon)| ds \\
&\leqslant \int_0^t \|w_s^\varepsilon\|\|F_s - z_s^\varepsilon\|_{H^1}\|F_s + v_s^E\|_{L^\infty} ds + \int_0^t \|v_s^\varepsilon\|\|F_s - z_s^\varepsilon\|\|v_s^E\|_{W^{1,\infty}} ds \\
&+ \int_0^t \|w_s^\varepsilon\|\|F_s - z_s^\varepsilon\|\|z_s^\varepsilon\|_{W^{1,\infty}} ds \\
&\leqslant C(N,u_0)\|F - z^\varepsilon\|_{C([0,T];H^1)}\left(1 + \|F - z^\varepsilon\|_{C([0,T];H^1)}\right) + \int_0^t \|w_s^\varepsilon\|^2 ds \\
&+ \int_0^t \|w_s^\varepsilon\|\|F_s - z_s^\varepsilon\|\|z_s^\varepsilon\|_{W^{1,\infty}} ds. \qquad (4.20)
\end{aligned}$$

Now we can move to $I_6(t)$ which is the most difficult term. Preliminarily we observe that

$$\begin{aligned}\varepsilon b(u^\varepsilon,\Delta u^\varepsilon,w^\varepsilon)-\varepsilon b(w^\varepsilon,\Delta u^\varepsilon,u^\varepsilon) &= \varepsilon b(v^\varepsilon,\Delta u^\varepsilon,v^\varepsilon)+\varepsilon b(z^\varepsilon,\Delta u^\varepsilon,w^\varepsilon)\\ &\quad-\varepsilon b(v^\varepsilon,\Delta u^\varepsilon,v^E)-\varepsilon b(v^\varepsilon,\Delta u^\varepsilon,v^\varepsilon)\\ &\quad-\varepsilon b(w^\varepsilon,\Delta u^\varepsilon,z^\varepsilon)+\varepsilon b(v^E,\Delta u^\varepsilon,v^\varepsilon)\\ &=\varepsilon b(z^\varepsilon,\Delta u^\varepsilon,w^\varepsilon)-\varepsilon b(v^\varepsilon,\Delta u^\varepsilon,v^E)\\ &\quad-\varepsilon b(w^\varepsilon,\Delta u^\varepsilon,z^\varepsilon)+\varepsilon b(v^E,\Delta u^\varepsilon,v^\varepsilon).\end{aligned}$$

We start considering $-\varepsilon\int_0^t b(v_s^\varepsilon,\Delta u_s^\varepsilon,v_s^E)ds+\varepsilon\int_0^t b(v_s^E,\Delta u_s^\varepsilon,v_s^\varepsilon)ds$. It can treated similarly to [47, Equations (4.18)-(4.19)]. $-\varepsilon\int_0^t b(v_s^\varepsilon,\Delta u_s^\varepsilon,v_s^E)ds$ can be integrated by parts, then it holds:

$$\begin{aligned}-\varepsilon b(v^\varepsilon,\Delta u^\varepsilon,v^E) &= \varepsilon\int_D v_i^\varepsilon\partial_i v_j^E\partial_{kk}u_j^\varepsilon\,dx\\ &=-\varepsilon\int_D\partial_k v_i^\varepsilon\partial_i v_j^E\partial_k(v_j^\varepsilon+z_j^\varepsilon)dx-\varepsilon\int_D v_i^\varepsilon\partial_{i,k}v_j^E\partial_k u_j^\varepsilon dx\\ &\leqslant o(1)+2\varepsilon\|v^E\|_{W^{2,4}}\|\nabla v^\varepsilon\|^2.\end{aligned}\tag{4.21}$$

In the last step we use the fact that $\varepsilon\|v^E\|_{W^{2,4}}\|\nabla z^\varepsilon\|^2=o(1)\quad\mathbb{P}-a.s.$ by Lemma 4.1. For what concerns $\varepsilon\int_0^t b(v_s^E,\Delta u_s^\varepsilon,v_s^\varepsilon)ds$, we split it in three terms:

$$\begin{aligned}\varepsilon b(v^E,\Delta u^\varepsilon,v^\varepsilon) &= -\varepsilon\int_D v^E\cdot\nabla v^\varepsilon\Delta u^\varepsilon dx\\ &=-\varepsilon\int_D\left((v^E-v)\cdot\nabla v^\varepsilon\Delta v^\varepsilon+v\cdot\nabla v^\varepsilon\Delta v^\varepsilon+v^E\cdot\nabla v^\varepsilon\Delta z^\varepsilon\right)dx\\ &=J_1+J_2+J_3.\end{aligned}\tag{4.22}$$

J_3 is the easiest term and can be bounded by the right hand side of (4.21) arguing as above. Since $v^E-v|_{\partial D},\ v^\varepsilon|_{\partial D}=0$, we can integrate by part J_1 repeatedly, obtaining via Hölder's inequality the following estimate:

$$\begin{aligned}-\varepsilon\int_D(v^E-v)\cdot\nabla v^\varepsilon\Delta v^\varepsilon dx &= \varepsilon\int_D\partial_k(v_i^E-v_i)\partial_i v_j^\varepsilon\partial_k v_j^\varepsilon dx+\frac{\varepsilon}{2}\int_D(v_i^E-v_i)\partial_i|\partial_k v_j^\varepsilon|^2dx\\ &=\varepsilon\int_D\partial_k v_i^E\partial_i v_j^\varepsilon\partial_k v_j^\varepsilon dx-\varepsilon\int_D\partial_k v_i\partial_i v_j^\varepsilon\partial_k v_j^\varepsilon dx\\ &\leqslant\varepsilon\|v^E\|_{W^{2,4}}\|\nabla v^\varepsilon\|^2-\varepsilon\int_D\partial_k v_i\partial_i v_j^\varepsilon\partial_k v_j^\varepsilon dx\\ &=\varepsilon\|v^E\|_{W^{2,4}}\|\nabla v^\varepsilon\|^2+\varepsilon\int_D\partial_k v_i v_j^\varepsilon\partial_{i,k}v_j^\varepsilon dx\\ &=\varepsilon\|v^E\|_{W^{2,4}}\|\nabla v^\varepsilon\|^2-\frac{\varepsilon}{2}\int_D v_i\partial_i|\partial_k v_j^\varepsilon|^2dx\\ &\quad-\varepsilon\int_D v_i v_j^\varepsilon\partial_{i,k,k}v_j^\varepsilon dx\\ &=\varepsilon\|v^E\|_{W^{2,4}}\|\nabla v^\varepsilon\|^2+\varepsilon\int_D v_i\partial_i v_j^\varepsilon\partial_{k,k}v_j^\varepsilon dx\\ &=\varepsilon\|v^E\|_{W^{2,4}}\|\nabla v^\varepsilon\|^2-J_2.\end{aligned}\tag{4.23}$$

Combining (4.21), (4.22), (4.23) we get

$$\begin{aligned}&-\varepsilon\int_0^t b(v_s^\varepsilon,\Delta u_s^\varepsilon,v_s^E)ds+\varepsilon\int_0^t b(v_s^E,\Delta u_s^\varepsilon,v_s^\varepsilon)ds\\ &\leqslant o(1)+\varepsilon C(N,u_0)\|v^E\|_{L^\infty(0,T;W^{2,4})}\int_0^t\|\nabla v_s^\varepsilon\|^2ds.\end{aligned}\tag{4.24}$$

We left to estimate $\varepsilon\int_0^t b(z^\varepsilon_s,\Delta u^\varepsilon_s,w^\varepsilon_s)ds-\varepsilon\int_0^t b(w^\varepsilon_s,\Delta u^\varepsilon_s,z^\varepsilon_s)ds$.

We start considering $\varepsilon\int_0^t b(z^\varepsilon_s,\Delta u^\varepsilon_s,w^\varepsilon_s)ds$ integrating by parts repeatedly since $z^\varepsilon|_{\partial D}=0$, we obtain by Hölder's inequality and the $\mathbb{P}-a.s.$ estimates on z^ε and v^E guaranteed by equation (4.1) and (2.22)

$$\begin{aligned}
\varepsilon b(z^\varepsilon,\Delta u^\varepsilon,w^\varepsilon)&=-\varepsilon\int_D z^\varepsilon_i\partial_i w^\varepsilon_j\partial_{k,k}u^\varepsilon_j dx\\
&=\varepsilon\int_D \partial_k z^\varepsilon_i\partial_i w^\varepsilon_j\partial_k u^\varepsilon_j dx+\varepsilon\int_D z^\varepsilon_i\partial_{i,k} w^\varepsilon_j\partial_k u^\varepsilon_j dx\\
&\leqslant \varepsilon\|z^\varepsilon\|_{W^{2,4}}(\|\nabla v^\varepsilon\|+\|\nabla z^\varepsilon\|)(\|\nabla v^\varepsilon\|+\|\nabla v^E\|)\\
&+\varepsilon\int_D z^\varepsilon_i\partial_{i,k}(u^\varepsilon_j-z^\varepsilon_j-v^E_j)\partial_k u^\varepsilon_j dx\\
&\leqslant o(1)+\varepsilon\|z^\varepsilon\|_{W^{2,4}}\|\nabla v^\varepsilon\|^2\\
&+\frac{\varepsilon}{2}\int_D z^\varepsilon_i\partial_i|\partial_k u^\varepsilon_k|^2dx+\varepsilon\|z^\varepsilon\|_{L^4}(\|z^\varepsilon\|_{W^{2,4}}+\|v^E\|_{W^{2,4}})(\|\nabla v^\varepsilon\|+\|\nabla z^\varepsilon\|)\\
&\leqslant o(1)+\varepsilon\left(\|z^\varepsilon\|_{W^{2,4}}+\|v^E\|_{W^{2,4}}\right)\|\nabla v^\varepsilon\|^2. \qquad (4.25)
\end{aligned}$$

Lastly we consider $-\varepsilon\int_0^t b(w^\varepsilon_s,\Delta u^\varepsilon_s,z^\varepsilon_s)ds$. Here we want again integrate by parts repeatedly, for this reason we add and subtract $\varepsilon\int_0^t b(v_s,\Delta u^\varepsilon_s,z^\varepsilon_s)ds$ exploiting the fact that $w^\varepsilon+v|_{\partial D}=0$. Therefore, thanks to the properties of the boundary layer corrector (4.8) and computations already performed we obtain:

$$\begin{aligned}
-\varepsilon b(w^\varepsilon,\Delta u^\varepsilon,z^\varepsilon)&=\varepsilon\int_D(w^\varepsilon_i+v_i)\partial_i z^\varepsilon_k\partial_{j,j}u^\varepsilon_k dx-\varepsilon\int_D v_i\partial_i z^\varepsilon_k\partial_{j,j}u^\varepsilon_k dx\\
&=-\varepsilon\int_D\partial_j(w^\varepsilon_i+v_i)\partial_i z^\varepsilon_k\partial_j u^\varepsilon_k dx-\varepsilon\int_D(w^\varepsilon_i+v_i)\partial_{i,j}z^\varepsilon_k\partial_j u^\varepsilon_j dx\\
&+\varepsilon\|v\|\|z^\varepsilon\|_{W^{2,4}}(\|v^\varepsilon\|_{H^2}+\|z^\varepsilon\|_{H^2})\\
&\leqslant o(1)+\varepsilon C(N,u_0)\delta^{1/2}\|z^\varepsilon\|_{W^{2,4}}\|\nabla v^\varepsilon\|^{1/2}\|v^\varepsilon\|_{H^3}^{1/2}\\
&+2\varepsilon\|z^\varepsilon\|_{W^{2,4}}(\|\nabla v^\varepsilon\|+\|\nabla v^E\|+\|\nabla v\|)(\|\nabla v^\varepsilon\|+\|\nabla z^\varepsilon\|)\\
&\leqslant o(1)+\varepsilon C\|z^\varepsilon\|_{W^{2,4}}\|\nabla v^\varepsilon\|^2+\varepsilon^3\delta\|v^\varepsilon\|^2_{H^3}+\delta^{1/2}C(N,u_0)\|z^\varepsilon\|^{3/2}_{W^{2,4}}\\
&+C(N,u_0)\varepsilon\delta^{-1}\|z^\varepsilon\|^4_{W^{2,4}}. \qquad (4.26)
\end{aligned}$$

In conclusion, combining (4.24), (4.25), (4.26) we obtain

$$\begin{aligned}
I_6(t)&\leqslant o(1)+\varepsilon C\left(\|z^\varepsilon\|_{L^\infty(0,T;W^{2,4})}+\|v^E\|_{L^\infty(0,T;W^{2,4})}\right)\int_0^t\|\nabla v^\varepsilon_s\|^2ds\\
&+\varepsilon^3\delta\sup_{t\in[0,T]}\|v^\varepsilon_t\|^2_{H^3}+\delta^{1/2}C(N,u_0)\|z^\varepsilon\|^{3/2}_{L^\infty(0,T;W^{2,4})}\\
&+C(N,u_0)\varepsilon\delta^{-1}\|z^\varepsilon\|^4_{L^\infty(0,T;W^{2,4})}. \qquad (4.27)
\end{aligned}$$

Combining the various estimates on the $I_i(t),\ i\in\{1,\dots,6\}$ we get

$$\begin{aligned}
\|w^\varepsilon_t\|^2+\frac{4}{5}\varepsilon\|\nabla v^\varepsilon_t\|^2&\leqslant o(1)+C\varepsilon(1+\|v^E\|_{L^\infty(0,T;W^{2,4})}+\|z^\varepsilon\|_{L^\infty(0,T;W^{2,4})})\int_0^t\|\nabla v^\varepsilon_s\|^2ds\\
&+C(T)\varepsilon^3\delta\sup_{t\in[0,T]}\|v^\varepsilon_t\|^2_{H^3}+C\sqrt{\varepsilon}\sup_{t\in[0,T]}\left|\int_0^t\langle w^\varepsilon_s,dW_s\rangle\right|\\
&+C(1+\|z^\varepsilon\|_{L^\infty(0,T;W^{1,\infty})}+\|v^E\|_{L^\infty(0,T;W^{1,\infty})})\int_0^t\|w^\varepsilon_s\|^2ds\\
&+C(N,u_0)\|F-z^\varepsilon\|_{C([0,T];H^1)}\left(1+\|F-z^\varepsilon\|_{C([0,T];H^1)}\right)\\
&+\int_0^t\|w^\varepsilon_s\|\|F_s-z^\varepsilon_s\|\|z^\varepsilon_s\|_{W^{1,\infty}}ds. \qquad (4.28)
\end{aligned}$$

Applying Grönwall's Lemma to (4.28) we obtain

$$\begin{aligned}\sup_{t\in[0,T]}\|w_t^\varepsilon\|^2+\varepsilon\sup_{t\in[0,T]}\|\nabla v_t^\varepsilon\|^2 \leqslant e^{C(T)(1+\|v^E\|_{L^\infty(0,T;W^{2,4})}+\|z^\varepsilon\|_{L^\infty(0,T;W^{2,4})})}\times \\ \Big(o(1)+T\varepsilon^3\delta\sup_{t\in[0,T]}\|v_t^\varepsilon\|_{H^3}^2 \\ +C(N,u_0)\|F-z^\varepsilon\|_{C([0,T];H^1)} \\ +C(N,u_0)\|F-z^\varepsilon\|_{C([0,T];H^1)}^2 \\ +\int_0^T\|w_s^\varepsilon\|\|F_s-z_s^\varepsilon\|\|z_s^\varepsilon\|_{W^{1,\infty}}ds \\ +C\sqrt{\varepsilon}\sup_{t\in[0,T]}\left|\int_0^t\langle w_s^\varepsilon,dW_s\rangle\right|\Big). \qquad (4.29)\end{aligned}$$

Under our assumptions we have $e^{C(T)(1+\|v^E\|_{L^\infty(0,T;W^{2,4})}+\|z^\varepsilon\|_{L^\infty(0,T;W^{2,4})})}\leqslant C(N,u_0)\quad \mathbb{P}-a.s.$, see (2.22), (4.5). This means that, in order to show that Lemma 4.3 holds, it is enough to prove that

$$\begin{aligned}&T\varepsilon^3\delta\sup_{t\in[0,T]}\|v_t^\varepsilon\|_{H^3}^2+C(N,u_0)\|F-z^\varepsilon\|_{C([0,T];H^1)}\left(1+\|F-z^\varepsilon\|_{C([0,T];H^1)}\right)\\ &+\int_0^T\|w_s^\varepsilon\|\|F_s-z_s^\varepsilon\|\|z_s^\varepsilon\|_{W^{1,\infty}}ds+C\sqrt{\varepsilon}\sup_{t\in[0,T]}\left|\int_0^t\langle w_s^\varepsilon,dW_s\rangle\right|\to 0 \quad \text{in Probability.}\end{aligned}$$

Thanks to (4.4), we have $T\varepsilon^3\delta\sup_{t\in[0,T]}\|v_t^\varepsilon\|_{H^3}^2\to 0$ in probability. $C(N,u_0)\|F-z^\varepsilon\|_{C_tH_x^1}^2\to 0$ in probability by Lemma 4.1. Lastly, by Lemma 4.1 we have also

$$\begin{aligned}&\mathbb{E}\left[\sqrt{\varepsilon}\sup_{t\in[0,T]}\left|\int_0^t\langle w_s^\varepsilon,dW_s\rangle\right|+\int_0^T\|w_s^\varepsilon\|\|F_s-z_s^\varepsilon\|\|z_s^\varepsilon\|_{W^{1,\infty}}ds\right]\\ &\leqslant C\mathbb{E}\left[\int_0^T\|w_s^\varepsilon\|^2\right]^{1/2}\left(\sqrt{\varepsilon}+\mathbb{E}\left[\int_0^T\|F_s-z_s^\varepsilon\|^2ds\right]^{1/2}\right)\to 0.\end{aligned}$$

Now the proof is complete. □

Combining Lemma 4.1 and Lemma 4.3 the second condition in Hypothesis 2.4 holds. Therefore we can apply Theorem 2.5 and complete the proof of Theorem 2.25.

Example 4.4. Let us consider as a domain D the unit disk centered in 0 and $W_t=(-A)^{-3}W_t^H$. According to [37, equation 26] our noise in not radially symmetric. Due to the choice of the covariance of our noise we are in the framework of Remark 2.29 and

$$I_{u_0}^{SG}(v)=\frac{1}{2}\int_0^T\|(-A)^3\left(\partial_s v_s+P(v_s\cdot\nabla v_s)\right)\|^2ds.$$

According to [27, Chapter 5.5], we are interested to study $I_{u_0}^{SG}(v)$ in case of fluid flows which have faster and faster oscillations close to the boundary. Moreover, we would like that these oscillations are developed in time. Therefore as a paradigmatic example, we study the asymptotic behaviour $I_{u_0}^{SG}(v^\rho)$ as $\rho\to+\infty$, where

$$v^\rho(x,t)=(1-|x|)^7|x|^6\cos\left(\frac{\rho t}{1-|x|}\right)x^\perp,\ u_0=(1-|x|)^7|x|^6x^\perp,$$

the powers on the monomials have been introduced to make everything smooth enough for being in our framework. It is well known that for each $\theta\geqslant 0$ there exists c_θ,C_θ such that for each $f\in D((-A)^\theta))$

$$c_\theta\left\|f\right\|_{H^{2\theta}(D;\mathbb{R}^2)}^2\leqslant\left\|(-A)^\theta f\right\|^2\leqslant C_\theta\left\|f\right\|_{H^{2\theta}(D;\mathbb{R}^2)}^2.$$

Therefore

$$I_{u_0}^{SG}(v^\rho) \sim \int_0^T \|\partial_s v_s^\rho - P(v_s^\rho \cdot \nabla v_s^\rho)\|_{H^6(D;\mathbb{R}^2)}^2 ds.$$

Since for each $t \in [0,T]$ $P(v^\rho \cdot \nabla v^\rho) \equiv 0$ as discussed in the proof of Proposition 3.5, we are left to study the asymptotic behaviour of

$$\rho^2 \int_0^T \left\| (1-|x|)^6 |x|^6 \sin\left(\frac{\rho t}{1-|x|}\right) x^\perp \right\|_{H^6}^2 dt = O(\rho^{14}).$$

In particular

$$I_{u_0}^{SG}(v^\rho) = O(\rho^{14}).$$

5 Some remarks on the Kato condition

We end this work with a discussion on the Kato-type condition that we assumed in order to prove one of our main results, Theorem 2.23. Recall that the condition Hypothesis 2.20 was the following

Hypothesis 5.1 (Strong Kato Hypothesis)**.** *For each $N \in \mathbb{N}$, $u_0^\varepsilon,\ u_0 \in \mathcal{E}_0^{NS}$ and $f^\varepsilon,\ f \in \mathcal{P}_2^N$ such that $u_0^\varepsilon \to u_0$ in $\mathcal{E}_0^{NS}$ and $f^\varepsilon \to_{\mathcal{L}} f$ in S^N, if $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ is a filtered probability space where all f^ε, f are defined together and $f^\varepsilon \to f$ $\mathbb{P}-a.s.$ in S^N, then, it exists $c>0$ such that for every $\delta > 0$*

$$\mathbb{P}\left(\varepsilon \int_0^T \left\|\nabla \mathcal{G}^{NS,\varepsilon}\left(u_0^\varepsilon, \sqrt{\varepsilon} W_\cdot + \int_0^\cdot f_s^\varepsilon ds\right)\right\|_{L^2(\Gamma_{c\varepsilon})}^2 ds > \delta\right) \to 0.$$

Loosely speaking, this condition requires a control on the behaviour in the boundary layer of the solutions of the stochastic Navier-Stokes system with respect to all kind of forcings and initial data. In the course of the proofs we have assumed this condition to verify the Condition 2 in Hypothesis 2.4. As pointed out in Remark 2.21, the uniformity in the initial data is crucial only for the set of initial condition for which we wish to establish a (uniform) LDP, as we can restrict the definition of the set $\mathcal{E}_0^{NS}$ without changing the strength (topology) of the LDP. Thus we can weaken this assumption just by redefining the objects on which we apply the weak convergence approach scheme. On the contrary, if one wishes to use the weak convergence approach, condition 2 in Hypothesis 2.4 (and the definition of the space S^N) does not allow to restrict the space of forcings, without increasing the regularity of the noise W, and thus severely limiting the strength of our result. For this reason, the request on the uniformity with respect to all possible forcing cannot be weakened a priori, as for the initial data. Observe that the Strong Kato Hypotesis (SKH) is much stronger than what we ask to ensure the validity of the inviscid limit, namely

$$\mathbb{E}\left(\varepsilon \int_0^T \left\|\nabla \mathcal{G}^{NS,\varepsilon}\left(u_0^\varepsilon, \sqrt{\varepsilon} W_\cdot\right)\right\|_{L^2(\Gamma_{c\varepsilon})}^2\right) \to 0, \tag{5.1}$$

(see [48]). In the following, we will call the property described by equation (5.1) as Weak Kato Condition (WKC). In particular, this condition does not involve any control of the system for non-zero forcing. In order to weaken the SKH, one can ask if the WKC is enough to ensure the validity of the LDP. Let us notice first that what we have proved, under the SKH is not only a large deviation result for the Navier-Stokes system with zero forcing, $u^\varepsilon := \mathcal{G}^{NS,\varepsilon}(u_0, \sqrt{\varepsilon} W_\cdot)$, but actually we got, as a byproduct, a LDP result for solutions with any forcing in $L^2(0,T;H_0)$, as pointed out in Remark 2.28. Indeed if we want to include in the system a forcing $g_t \in L^2(0,T;H_0)$ we can just redefine the maps $\mathcal{G}_g^0 := \mathcal{G}^0(\cdot, \int_0^{\cdot}(\cdot + g_s)ds)$ and $\mathcal{G}_g^\varepsilon := \mathcal{G}^\varepsilon(\cdot, \int_0^{\cdot}(\cdot + g_s)ds)$ and the result that we have proved immediately imply a LDP for the solution with forcing, under the same Hypothesis 2.20. In this sense our condition is optimal for our setting: we ask controls for every forcing and we get a LDP for every forcing term.

Therefore, one way of improving our result could be to prove that the SKH can be deduced from the WKC. In some sense this would requires to be able to pass information between systems with different forcings. We shall notice that the forcing that we are working with all live in the reproducing kernel of W, therefore we might switch from one system to another just by a Girsanov transformation; however this correction explodes exponentially fast in the limit $\varepsilon \to 0$. A posteriori, if one is able to prove that the LDP holds for some forcing, one expects that the explosion of the Girsanov correction gets compensated by the exponential decay of the law of the solutions. A different approach would be to prove that one does not in fact need to ask the strong Kato condition in order to prove only a LDP for equation (1.2) (that is, only for the system with zero forcing). We formulate then the following:

Problem 5.2. (LDP under Weak Kato Hypotesys) Prove that the statement of Theorem 2.23 holds true if we replace Hypothesis 2.20 with the Kato Condition (5.1).

If the answer to this problem was positive, then we would expect to retrieve also the 'full' LDP, that is, a family of LDP for the system with any forcing $f \in L^2(0,T;\mathcal{H}_0)$. This requires to be able to pass a LDP between systems with different forcings. To see why this seems so natural, observe that every time one is able to write a family of solution X_f^ε to some S(P)DE depending by some forcing f as a continuous transformation of a Brownian motion $J(\sqrt{\varepsilon}W_\cdot)$, then by an application of the contraction principle one immediately obtains a LDP for every other forcing. In our setting however, we are not able of proving such a property of the LDP without requiring the strong Kato Condition, that is without having information about the convergence of the systems with different forcings when $\varepsilon \to 0$. Note that, by the uniqueness of the solution to the Euler system in our setting, this convergence is also a necessary condition for the 'full' LDP.

In the end, we believe that the following should be true:

Conjecture 5.3. *The Large Deviations of our system hold independently of the choice of the forcing $f \in L^2(0,T;H_0)$, that is, if a LDP holds for at least one such forcing, then it holds for every other.*

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