Electronic Journal of Probability

Electron. J. Probab. **29** (2024), article no. 114, 1–44.
ISSN: 1083-6489 https://doi.org/10.1214/24-EJP1176

Inverting Ray-Knight identities on trees*

Xiaodan Li† Yushu Zheng‡

Abstract

In this paper, we first introduce the Ray-Knight identity and percolation Ray-Knight identity related to loop soup with intensity $\alpha(\geq 0)$ on trees. Then we present the inversions of the above identities, which are expressed in terms of repelling jump processes. The inversion in the case of $\alpha = 0$ gives the conditional law of a continuous-time Markov chain given its local time field; while the inversion in the case of $\alpha > 0$ gives the conditional law of a Markovian loop soup given its local time field. We further show that the fine mesh limits of these repelling jump processes are the self-repelling diffusions involved in the inversion of the Ray-Knight identity on the corresponding metric graph.

Keywords: Ray-Knight identities; loop soups; vertex repelling jump processes.
MSC2020 subject classifications: Primary 0K35; 60J55, Secondary 60G55; 60J27; 60J65.
Submitted to EJP on February 13, 2023, final version accepted on July 24, 2024.

1 Introduction

Imagine a Brownian crook who spent a month in a large metropolis. The number of nights he spent in hotels A, B, C, $\cdots$, etc. is known; but not the order, nor his itinerary. So the only information the police have is the total hotel bills. This vivid story is quoted from [25], which is also the paper from which the name 'Brownian burglar' comes. In [25], Warren and Yor constructed the Brownian burglar to describe the law of reflected Brownian motion conditioned on its local time field. Meanwhile, Aldous [3] used the tree structure of the Brownian excursion to show that the genealogy of the conditioned Brownian motion is a time-changed Kingman coalescent. The article [25] can be viewed as a construction of the process in time while [3] as a construction in space. Here we consider the analogous problem for continuous-time Markov chains (CTMC) on electric networks.

(Q1) How can we describe the law of a CTMC conditionally on its local time field?

*The first author is supported by NSFC, China (No.12301184) and Fundamental Research Funds for the Central Universities. The second author is supported by China Postdoctoral Science Foundation (No. 2023000117).

†Shanghai University of Finance and Economics, China. E-mail: lixiaodan@mail.shufe.edu.cn
‡Chinese Academy of Sciences, China. E-mail: yszheng666@gmail.com

This problem can actually be seen as a special case in a more general class of recovery problems that we are to explain. The local time field of a CTMC is considered in the generalized second Ray-Knight theorem, which provides an identity between the law of the sum of half of a squared Gaussian free field (GFF) with boundary condition 0 and the local time field of an independent Markovian path on the one hand, and the law of half of a squared GFF with boundary condition $\sqrt{2u}$ on the other hand (see [8]). We call this identity a Ray-Knight identity. It is well-known that the local time field of a loop soup with intensity $\alpha = 1/2$ is distributed as half of a squared GFF by Le Jan's isomorphism [14, §6.2]. Therefore the generalized second Ray-Knight identity can also be stated using the loop soup with intensity $1/2$. In the case of a loop soup with arbitrary intensity $\alpha > 0$, an analogous identity holds: adding the local time field of a CTMC to the local time field of a loop soup with 'boundary condition' 0 gives the distribution of the local time field of a loop soup with 'boundary condition' u, see Proposition 2.2 below for a precise statement. We call any such identity a Ray-Knight identity. Inverting the Ray-Knight identity refers to recovering the CTMC conditioned on the total local time field.

Vertex-reinforced jump processes (VRJP), conceived by Werner and first studied by Davis and Volkov [5, 6], are continuous-time jump processes favouring sites with higher local times. Surprisingly, Sabot and Tarrès [24] found that a time change of a variant of VRJP provides an inversion of the Ray-Knight identity on a general graph in the case $\alpha = 1/2$. It is natural to wonder whether an analogous description holds for an arbitrary intensity α.

(Q2) For any $\alpha > 0$, how can we describe the process that inverts the Ray-Knight identity related to loop soup with intensity $\alpha > 0$?

Note that (Q1) can be viewed as a special case of (Q2) with $\alpha = 0$ if we generalize (Q2). Intuitively, when $\alpha = 0$, the external interference of the loop soup disappears. Hence it reduces to extracting the CTMC from its own local time field. Another equivalent interpretation of (Q2) is to recover the loop soup with intensity α conditioned on its local time field.

In [15], Lupu gave a 'signed version' of Le Jan's isomorphism where the loop soup at intensity $\alpha = 1/2$ is naturally coupled with a signed GFF. In [17, Theorem 8], Lupu, Sabot, and Tarrès gave the corresponding version of the Ray-Knight identity (we call it a percolation Ray-Knight identity). Besides the identity of local time fields, by adding a percolation along with the Markovian path and finally sampling signs in every connected component of the percolation, one can start with a GFF with boundary condition 0 and end up getting a GFF with boundary condition $\sqrt{2u}$. The inversion of the percolation Ray-Knight identity is represented as another self-interacting process [17, §3]. This leads to the following question.

(Q3) Can we generalize the percolation Ray-Knight identity to the case of loop soup with intensity $\alpha > 0$? If so, how can we describe the process that inverts the percolation Ray-Knight identity?

The analogous problems can also be considered for Brownian motion and Brownian loop soup. In [18], Lupu, Sabot, and Tarrès constructed a self-repelling diffusion out of a divergent Bass-Burdzy flow which inverts the Ray-Knight identity related to GFF on the line and showed that the self-repelling diffusion is the fine mesh limit of the vertex repelling jump processes involved in the case of grid graphs on the line. More generally, it was shown in [25, 2] that the self-repelling diffusion inverting the Ray-Knight identity on the positive half line can be constructed either with the burglar process or the Bass-Burdzy flow.

We want to explore the relationship between the repelling jump processes in (Q1)-(Q3) and the self-repelling diffusions in [2, 3, 25]. Our last question is the following.

(Q4) Are the fine mesh limits of the repelling jump processes involved in (Q1)-(Q3) the self-repelling diffusions?

In this paper, we will focus on the case where the graph in the electric network is a tree and give a complete answer to the above questions (Q1)-(Q4). It is shown that the percolation Ray-Knight identity has a simple form. We construct two repelling jump processes, namely the vertex repelling jump process and the percolation vertex repelling jump process, that invert the Ray-Knight identity and percolation Ray-Knight identity related to loop soup respectively, and show that the fine mesh limits of these repelling jump processes are self-repelling diffusions involved in the inversion of the Ray-Knight identity on the corresponding metric graph. Besides, the inversions in the case of general graphs have been constructed in another paper of ours in preparation, whose jump rates are expressed via the α-permanent of matrices. In particular, contrary to the case of trees, the jump rates in the general case are generally non-local. We restrict our discussion to the simple case of trees here mainly due to its own interests.

The main feature of this paper is the intuitive way of constructing vertex repelling jump processes, which is rather different from [24, 17]. It is enlightened by the recovery of the loop soup with intensity $1/2$ given its local time field in [27, §2.5] and [26, Proposition 7], where Werner involves the crossings of loop soup that greatly simplify the recovery. In our case, the introduction of crossings translates the problem into a 'discrete-time version' of inverting Ray-Knight identity, which can be stated as recovering the path of a discrete-time Markov chain conditioned on the number of crossings over each edge, see Proposition 3.4. This inversion has a surprisingly nice description, which can be seen as a 'reversed' oriented version of the edge-reinforced random walk (ERRW).

The paper is organized as follows. In §2, we introduce the Ray-Knight identity and percolation Ray-Knight identity related to loop soup and give the main results of the paper. In §3-4, the vertex repelling jump process and the percolation vertex repelling jump process are shown to invert the Ray-Knight identity and the percolation Ray-Knight identity respectively. In §5, we verify that the mesh limits of repelling jump processes are the self-repelling diffusions. In Appendix A, we give the rigorous definition and basic properties of a class of self-interacting processes called processes with terminated jump rates, which contains repelling jump processes.

2 Statements of main results

In this section, we first recall the Ray-Knight identity. Then we introduce a new Ray-Knight identity that we call percolation Ray-Knight identity. Finally, we present our results concerning the inversion of these identities and the fine mesh limit of the inversions.

2.1 Notation

We will use the following notation throughout the paper. $\mathbb{N} = \{0, 1, 2, \cdots\}$, $\mathbb{R}^+ = [0, \infty)$. For any stochastic process R on some state space $\mathcal{S}$, for $t, u \geq 0$, $x \in \mathcal{S}$, and a specified point $x_0 \in \mathcal{S}$, we denote by

- $L^R(t, x)$ the local time of R. When $\mathcal{S}$ is discrete, $L^R(t, x) := \int_0^t 1_{\{R_s = x\}} \, \mathrm{d}s$;
- $\tau_u^R := \inf\{t > 0 : L^R(t, x_0) > u\}$ the right-continuous inverse of $t \mapsto L^R(t, x_0)$;
- T^R the lifetime of R;
- $H_x^R := \inf\{t > 0 : R_t = x\}$ the hitting time of x;
- $R_{[0,t]} := (R_s : 0 \leq s \leq t)$ the path of R up to time t.

The superscripts in the above notation are omitted when $R = X$, the CTMC to be introduced immediately.

For any measurable subset D of $\mathcal{S}$, the print of R on D is defined to be the process $\left(R_{A^{-1}(t)} : 0 \le t < A(T^R)\right)$, where $A(t) = \int_0^t 1_{\{R_s \in D\}}\, \mathrm{d}s$ and A^{-1} is the right-continuous inverse of A.

2.2 Ray-Knight identity related to loop soup

Consider a tree $\mathbb{T}$, i.e. a finite or countable connected graph without cycle, with root x_0. Denote by V its set of vertices, by E, resp. $\vec{E}$, its set of undirected, resp. directed, edges. Assume that any vertex $x \in V$ has finite degree. We write $x < y$ if x is an ancestor of y and $x \sim y$ if x and y are neighbours. Denote by $\mathfrak{p}(x)$ the parent of x. For $x \sim y$, we simply write $xy := (x, y)$ for a directed edge. The tree is endowed with a killing measure $(k_x)_{x \in V}$ on V and conductances $(C_{xy})_{xy \in \vec{E}}$ on $\vec{E}$. We assume $C_{xy} > 0$ for any $xy \in \vec{E}$ and define $C_{xy} := 0$ for $xy \notin \vec{E}$. We do not assume the symmetry of the conductances at the moment. Write $C^*_{xy} = C^*_{yx} = \sqrt{C_{xy} C_{yx}}$ for $x \sim y$.

The conductances and killing measure naturally induce an irreducible CTMC $X = (X_t)_{0 \le t < T^X}$ on V which being at x, jumps to a neighbour y with rate C_{xy} and is killed with rate k_x. T^X is the time when the process is killed or explodes. Let $\mathcal{L} = \mathcal{L}_\alpha$ be the unrooted oriented loop soup with parameter $\alpha > 0$ associated to X. (See for example [4, §4.1] for the precise definition.)

Denote by $L_\cdot(\mathcal{L})$ the local time field of $\mathcal{L}$, i.e. for all $x \in V$, $L_x(\mathcal{L})$ is the sum of the occupation time at x of each loop in $\mathcal{L}$. It is well-known (Cf. for example [4, Corollary 4.3]) that when X is transient, $L_x(\mathcal{L})$ follows a Gamma($\alpha, G(x,x)^{-1}$) distribution[1], where G is the Green function of X; when X is recurrent, $L_x(\mathcal{L}) = \infty$ for all $x \in V$ a.s.. We first suppose X is transient, which ensures that the conditional distribution of $\mathcal{L}$ given $L_{x_0}(\mathcal{L})$ exists. For $u \ge 0$, let $\mathcal{L}^{(u)}$ have the law of $\mathcal{L}$ given $L_{x_0}(\mathcal{L}) = u$. Without particular mention, we always assume that X starts from x_0. The next proposition (see [17, Proposition 3.7] or [4, Proposition 5.3]) connects the path of X with the loops in $\mathcal{L}^{(u)}$ that visit x_0.

Proposition 2.1. *For any $u > 0$, consider the path $(X_t)_{0 \le t \le \tau_u}$ conditioned on $\tau_u < T^X$. Let $D = (d_1, d_2, \cdots)$ be a Poisson-Dirichlet $(0, \alpha)$ partition, independent of X. Set $s_n := u \cdot \sum_{k=1}^n d_k$. Then the family of the unrooted loops*

$$\left\{ \pi \left(\left(X_{\tau_{s_{j-1}} + t} \right)_{0 \le t \le \tau_{s_j} - \tau_{s_{j-1}}} \right) : j \ge 1 \right\}$$

is distributed as the family of the loops in $\mathcal{L}^{(u)}$ that visit x_0, where π is the quotient map that maps a rooted loop to its corresponding unrooted loop.

By Proposition 2.1, we can take $\mathcal{L}^{(0)} = \{\gamma \in \mathcal{L} : \gamma \text{ does not visit } x_0\}$ and $\mathcal{L}^{(u)}$ to be the collection of loops in $\mathcal{L}^{(0)}$ and loops derived by partitioning $X_{[0,\tau_u]}$, where X and $\mathcal{L}^{(0)}$ are required to be independent. This special choice of $\left(\mathcal{L}^{(u)} : u \ge 0\right)$ provides a continuous version of the conditional distribution. We work on this version from now on. Note that the above definition also makes sense when $\alpha = 0$. In this case, $\mathcal{L}^{(0)} = \emptyset$ and $\mathcal{L}^{(u)}$ consists of a single loop $\pi\left(X_{[0,\tau_u]}\right)$ (Poisson-Dirichlet $(0,0)$ partition is considered as the trivial partition $D = (1, 0, 0, \cdots)$). So we also allow $\alpha = 0$ henceforth. The generalized second Ray-Knight theorem related to loop soup reads as follows, which is direct from the above definition.

Proposition 2.2 (Ray-Knight identity)**.** *Let $\mathcal{L}^{(0)}$ and X be independent. Then for $u > 0$, conditionally on $\tau_u < T^X$,*

$$\left(L_x(\mathcal{L}^{(0)}) + L(\tau_u, x) \right)_{x \in V} \text{ has the same law as } \left(L_x(\mathcal{L}^{(u)}) \right)_{x \in V}.$$

[1] The density of Gamma(a, b) distribution at x is $1_{\{x>0\}} \frac{b^a}{\Gamma(a)} x^{a-1} e^{-bx}$.

2.3 Percolation Ray-Knight identity related to loop soup

In this part, we assume the symmetry of the conductances (i.e. $C_{xy} = C_{yx}$ for any $x \sim y$) and that $0 < \alpha < 1$. An element O in $\{0,1\}^E$ is also called a configuration on E. When $O(e) = 1$ (resp. $O(e) = 0$), the edge e is thought of as being open (resp. closed). A percolation on E refers to a random configuration on E.

Definition 2.3. *For $u \geq 0$, let $\mathcal{O}^{(u)}$ be the percolation on E such that, conditionally on $L_\cdot(\mathcal{L}^{(u)}) = \ell$:*

(i) edges are open independently;

(ii) the edge $\{x, y\}$ is open with probability

$$\begin{cases} \dfrac{I_{1-\alpha}\left(2C_{xy}\sqrt{\ell_x \ell_y}\right)}{I_{\alpha-1}\left(2C_{xy}\sqrt{\ell_x \ell_y}\right)}, & \text{if } \ell_x, \ell_y > 0; \\ 0, & \text{if } \ell_x \wedge \ell_y = 0, \end{cases} \tag{2.1}$$

where I_ν is the modified Bessel function of the first kind: for $\nu \geq -1$ and $z > 0$,

$$I_\nu(z) = (z/2)^\nu \sum_{n=0}^{\infty} \frac{(z/2)^{2n}}{n!\,\Gamma(n+\nu+1)}.$$

Remark 2.4. Let $K_\nu(z) := \frac{\pi}{2} \frac{I_{-\nu}(z) - I_\nu(z)}{\sin(\nu\pi)}$, $\nu \geq -1$ and $z > 0$ (when ν is an integer, the right-hand side is replaced by its limiting value), which is the modified Bessel function of the second kind. It holds that $K_\nu(z) > 0$ for $\nu \geq 0$ and $z > 0$ (Cf. [20, Theorem 8.1]). This implies that the quantity in (2.1) is less than 1.

Consider the loop soup $\mathcal{L}^{(0)}$ and the percolation $\mathcal{O}^{(0)}$. For $t \geq 0$, define the aggregated local times

$$\phi_t(x) := L_x(\mathcal{L}^{(0)}) + L(t, x),\ x \in V. \tag{2.2}$$

The process $\mathcal{X} := (X_t, \mathcal{O}_t)_{0 \leq t \leq \tau_u}$ is defined as follows: $(X_0, \mathcal{O}_0) := \left(x_0, \mathcal{O}^{(0)}\right)$. Conditionally on $L_\cdot(\mathcal{L}^{(0)})$ and $(\mathcal{X}_s : 0 \leq s \leq t)$, if $X_t = x$, then for any $y \sim x$,

- X_t jumps to y with rate C_{xy} and $\mathcal{O}_t(\{x, y\})$ is set to 1 after jumping (if it was not already).
- In case $\mathcal{O}_t(\{x, y\}) = 0$, $\mathcal{O}_t(\{x, y\})$ is set to 1 without X_t jumping with rate

$$\begin{cases} C_{xy}\sqrt{\dfrac{\phi_t(y)}{\phi_t(x)}} \cdot \dfrac{K_\alpha\left(2C_{xy}\sqrt{\phi_t(x)\phi_t(y)}\right)}{K_{1-\alpha}\left(2C_{xy}\sqrt{\phi_t(x)\phi_t(y)}\right)}, & \text{if } \phi_t(x), \phi_t(y) > 0; \\ 0, & \text{if } \phi_t(y) = 0; \\ \infty, & \text{if } \phi_t(x) = 0 \text{ and } \phi_t(y) > 0. \end{cases} \tag{2.3}$$

  where $K_\nu(z)$ is defined in Remark 2.4.

Be careful that although the rate is ∞ in the case $\phi_t(x) = 0$ and $\phi_t(y) > 0$, $\{x, y\}$ will not be opened immediately. Indeed, conditionally on X_t having no jump during $[t, t + \Delta t]$, the probability that $\{x, y\}$ keeps closed until $t + \Delta t$ is

$$\exp\left(-\int_0^{\Delta t} C_{xy}\sqrt{\frac{\phi_t(y)}{s}} \cdot \frac{K_\alpha\left(2C_{xy}\sqrt{s\phi_t(y)}\right)}{K_{1-\alpha}\left(2C_{xy}\sqrt{s\phi_t(y)}\right)}\,\mathrm{d}s\right). \tag{2.4}$$

Using the asymptotic $K_\nu(z) \sim \frac{1}{2}\Gamma(\nu)\left(\frac{1}{2}z\right)^{-\nu}$ as $z \to 0+$, we have the above integrand $\asymp s^{-\alpha}$ as $s \to 0+$. Hence, the probability in (2.4) goes to 1 as $\Delta t \to 0+$. This implies a.s. $\{x, y\}$ will not be opened immediately.

Theorem 2.5 (Percolation Ray-Knight identity)**.** *With the notation above, conditionally on $\tau_u < T^X$, $(\phi_{\tau_u}, \mathcal{O}_{\tau_u})$ has the same law as $(L_\cdot(\mathcal{L}^{(u)}), \mathcal{O}^{(u)})$.*

Theorem 2.5 will be proved in §4. The process $(\mathcal{O}_t)_{0\le t\le\tau_u}$ has a natural interpretation in terms of loop soup on a metric graph. Specifically, let $\widetilde{\mathbb{T}}$ be the metric graph associated to $\mathbb{T}$, where the edges are considered as intervals, so that one can construct a Brownian motion B which moves continuously on $\widetilde{\mathbb{T}}$, and whose print on the vertices is distributed as X (see §4 for details). Let $\widetilde{\mathcal{L}}^{(0)}$ be the unrooted oriented loop soup with intensity α associated to B with 'boundary condition' 0 at x_0. Starting with $\widetilde{\mathcal{L}}^{(0)}$ and an independent Brownian motion B starting at x_0, one can consider the field $(\widetilde{\phi}_t(x),\, x \in \widetilde{\mathbb{T}})$ which is the aggregated local time at x of the loop soup $\widetilde{\mathcal{L}}^{(0)}$ and the Brownian motion B up to time t. Then one can construct $\mathcal{O}_t$ as the configuration where an edge e is open if the field $\widetilde{\phi}_t$ does not have any zero on the edge e. See Proposition 4.1.

To study the link between the loop soup on the discrete graph and that on the metric graph is the motivation we introduce the percolation Ray-Knight identity. In [15], Lupu uses this link to provide new insights into the isomorphism theorem. We restrict to the case $0 < \alpha < 1$ because the other cases are trivial in the metric graph setting: when $\alpha \ge 1$, a.s. $\widetilde{\phi}_t > 0$ everywhere for any $t > 0$; when $\alpha = 0$, $\widetilde{\phi}_t$ has no zero on e if and only if e is crossed by $B_{[0,t]}$.

Remark 2.6. Since the laws involved in the Ray-Knight and percolation Ray-Knight do not depend on the killing rate k_{x_0} (due to the conditioning $\tau_u < T^X$), we can generalize the above results to the case where X is recurrent.

2.4 Inversion of Ray-Knight identities

To ease the presentation, write $\phi^{(u)}$ for $L_\cdot(\mathcal{L}^{(u)})$ from now on. Theorem 2.5 allows us to identify $(\phi^{(u)}, \mathcal{O}^{(u)})$ with $(\phi_{\tau_u}, \mathcal{O}_{\tau_u})$, which we will do.

Definition 2.7. *We call the triple*

$$\left(\phi^{(0)}, X_{[0,\tau_u]}, \phi^{(u)}\right)$$

a Ray-Knight triple (with parameter α) associated to X. Similarly, recalling the notation $\mathcal{X}_t = (X_t, \mathcal{O}_t)$, the triple

$$\left((\phi^{(0)}, \mathcal{O}^{(0)}), \mathcal{X}_{[0,\tau_u]}, (\phi^{(u)}, \mathcal{O}^{(u)})\right)$$

will be called a percolation Ray-Knight triple.

Inverting the Ray-Knight, resp. percolation Ray-Knight identity is to deduce the conditional law of $X_{[0,\tau_u]}$, resp. $\mathcal{X}_{[0,\tau_u]}$, given $\phi^{(u)}$, resp. $(\phi^{(u)}, \mathcal{O}^{(u)})$.

We introduce an adjacency relation on $V \times \{0,1\}^E$: For $(x, O_1), (y, O_2) \in V \times \{0,1\}^E$, (x, O_1) and (y, O_2) are neighboured if they satisfy either one of the followings: (1) $O_1 = O_2$ and $x \sim y$; (2) O_1 and O_2 differ by exactly one edge e and $x, y \in e$. This defines a graph $\mathbb{T}'$ with finite degree and $\mathcal{X}_t$ is a nearest-neighbour jump process on $\mathbb{T}'$.

The inversion of Ray-Knight identities is expressed in terms of several processes defined by jump rates. Readers are referred to Appendix A for the rigorous definition and basic properties of such processes. All the continuous-time processes defined below are assumed to be right-continuous, minimal, nearest-neighbour jump processes with a finite or infinite lifetime. The collection of all such sample paths on $\mathbb{T}$ (resp. $\mathbb{T}'$) is denoted by Ω (resp. Ω').

Given $\lambda \in (\mathbb{R}^+)^V$, set for $t \ge 0$, $x, y \in V$ with $x \sim y$, and $\omega \in \Omega$ with $T^\omega > t$,

$$\begin{aligned} \Lambda_t(x) &= \Lambda_t(\lambda,\omega)(x) = \Lambda_t(\lambda,\omega_{[0,t]})(x) := \lambda(x) - \int_0^t 1_{\{\omega_s = x\}}\,\mathrm{d}s, \\ \varphi_t(xy) &= \varphi_t(\lambda,\omega)(xy) = \varphi_t(\lambda,\omega_{[0,t]})(xy) := 2C^*_{xy}\sqrt{\Lambda_t(x,\omega)\Lambda_t(y,\omega)}. \end{aligned} \tag{2.5}$$

Intuitively, λ is viewed as the initial local time field and Λ_t stands for the remaining local time field while running the process ω until time t. Although these quantities depend on ω, we will systematically drop the notation ω for the sake of concision whenever the process is clear from the context.

2.4.1 Inversion of Ray-Knight identity

Keep in mind that most of the following definitions have a parameter $\alpha \geq 0$, which is always omitted in the notation for simplicity.

Set

$$\mathfrak{R} = \begin{cases} \left\{\lambda \in (\mathbb{R}^+)^V : \lambda(x) > 0 \ \forall x \in V\right\}, & \text{if } \alpha > 0; \\ \left\{\lambda \in (\mathbb{R}^+)^V : \lambda(x_0) > 0, \text{ supp}(\lambda) \text{ is connected and finite}\right\}, & \text{if } \alpha = 0. \end{cases} \tag{2.6}$$

Note that with probability 1, $\phi^{(u)} \in \mathfrak{R}$. We will take $\mathfrak{R}$ as the range of $\phi^{(u)}$ and consider only the law of $X_{[0,\tau_u]}$ given $\phi^{(u)} = \lambda \in \mathfrak{R}$.

Now define the *vertex repelling jump process* we are interested in. Given $\lambda \in \mathfrak{R}$, its distribution $\mathbb{P}^\lambda_{x_0}$ on Ω verifies that the process $\omega = (\omega_t, 0 \leq t < T^\omega)$ starts at $\omega_0 = x_0$, behaves such that

- conditionally on $t < T^\omega$ and $(\omega_s : 0 \leq s \leq t)$ with $\omega_t = x$, it jumps to a neighbour y of x with rate

$$r^\lambda_t(x, y, \omega_{[0,t]}) := \begin{cases} C^*_{xy}\sqrt{\dfrac{\Lambda_t(y)}{\Lambda_t(x)}} \cdot \dfrac{I_{\alpha-1}\left(\varphi_t(xy)\right)}{I_\alpha\left(\varphi_t(xy)\right)}, & \text{if } y = \mathfrak{p}(x), \\ C^*_{xy}\sqrt{\dfrac{\Lambda_t(y)}{\Lambda_t(x)}} \cdot \dfrac{I_\alpha\left(\varphi_t(xy)\right)}{I_{\alpha-1}\left(\varphi_t(xy)\right)}, & \text{if } x = \mathfrak{p}(y); \end{cases} \tag{2.7}$$

- (resurrect mechanism). every time $\lim_{s \to t-} \Lambda_s(\omega_s) = 0$ and $\omega_{t-} \neq x_0$, it jumps to $\mathfrak{p}(\omega_{t-})$ at time t.

It finally stops at time $T^\omega = T^\lambda(\omega) = T^\lambda_0(\omega) \wedge T^\lambda_\infty(\omega)$ with

$$T^\lambda_0(\omega) := \sup\left\{t \geq 0 : \Lambda_t(x_0) > 0\right\};$$
$$T^\lambda_\infty(\omega) := \sup\left\{t \geq 0 : \omega_{[0,t]} \text{ has finitely many jumps}\right\}.$$

Here T^λ_0 represents the time when the local time at x_0 is exhausted. The process can be roughly described as follows: the total local time available at each vertex is given at the beginning. As the process runs, it eats the local time. The jump rates are given in terms of the remaining local time. It finally stops whenever the available local time at x_0 is used up or an explosion occurs.

Remark 2.8. It holds that $I_1 = I_{-1}$ and as $z \downarrow 0$,

$$I_\nu(z) \sim \begin{cases} \Gamma(\nu+1)^{-1}(\frac{1}{2}z)^\nu, & \text{if } \nu > -1; \\ \frac{1}{2}z, & \text{if } \nu = -1. \end{cases}$$

Hence,

$$\text{for } \alpha > 0, \ I_{\alpha-1}(z)/I_\alpha(z) \sim \alpha z^{-1}; \quad I_{-1}(z)/I_0(z) \sim z. \tag{2.8}$$

The different behaviors in (2.8) for $\alpha > 0$ and $\alpha = 0$ indicate different behaviors of the vertex repelling jump process in the two cases. Intuitively speaking, when $\alpha > 0$, as the process continues to stay at some $x \neq x_0$, the jump rate to $\mathfrak{p}(x)$ goes to infinity. So it actually does not need the resurrect mechanism in this case, and there is still some positive local time left at any vertex other than x_0 at the end. While in the case $\alpha = 0$, the process exhibits a different picture. As it continues to stay, the jump rate to the children goes to infinity when $\alpha = 0$. So the process is 'pushed' to the boundary of supp(λ) and exhausts the available local time at one of the boundaries. To guide the process back to x_0, we resurrect it by letting it jump to the parent of the vertex. In this way, the process finally ends up exhausting all the available local time at each vertex.

By Remark 2.8, we can see that the vertex repelling jump process is in line with our intuition about the inversion of Ray-Knight identity. Since in the case $\alpha > 0$, the given time field includes the external time field $\phi^{(0)}$, the inversion process will only use part of the local time at any vertex other than x_0 and end up exhausting the local time at x_0. While in the case $\alpha = 0$, the given time field is exactly the local time field of the CTMC itself, the inversion process will certainly use up the local time at each vertex.

Theorem 2.9. *Suppose $\left(\phi^{(0)}, X_{[0,\tau_u]}, \phi^{(u)}\right)$ is a Ray-Knight triple associated to X. For any $\lambda \in \mathfrak{R}$, the conditional distribution of $X_{[0,\tau_u]}$ given $\tau_u < T^X$ and $\phi^{(u)} = \lambda$ is $\mathbb{P}^{\lambda}_{x_0}$.*

2.4.2 Inversion of percolation Ray-Knight identity

Assume the symmetry of the conductances and that $0 < \alpha < 1$. Our goal is to introduce the *percolation vertex repelling jump process* that inverts the percolation Ray-Knight identity. Given $\lambda \in \mathfrak{R}$ and a configuration O on E, the distribution $\mathbb{P}^{\lambda}_{(x_0,O)}$ on Ω' verifies that the process $\omega = (\omega_t, 0 \le t < T^{\omega})$ (the first coordinate being a vertex in V and the second coordinate a configuration on E) starts at (x_0, O), moves such that conditionally on $t < T^{\omega}$ and $(\omega_s : 0 \le s \le t)$ with $\omega_t = (x, O_1)$, it jumps from (x, O_1) to (y, O_2) with rate

$$\begin{cases} 1_{\{O_1(\{x,y\})=1\}} C_{xy} \sqrt{\dfrac{\Lambda_t(y)}{\Lambda_t(x)}} \cdot \dfrac{I_{1-\alpha}\left(\varphi_t(xy)\right)}{I_{\alpha}\left(\varphi_t(xy)\right)}, & \text{if } y = \mathfrak{p}(x), O_1 = O_2, \\ 1_{\{O_1(\{x,y\})=1\}} C_{xy} \sqrt{\dfrac{\Lambda_t(y)}{\Lambda_t(x)}} \cdot \dfrac{I_{\alpha}\left(\varphi_t(xy)\right)}{I_{1-\alpha}\left(\varphi_t(xy)\right)}, & \text{if } x = \mathfrak{p}(y), O_1 = O_2, \\ C_{xy} \sqrt{\dfrac{\Lambda_t(y)}{\Lambda_t(x)}} \cdot \dfrac{I_{\alpha-1}\left(\varphi_t(xy)\right) - I_{1-\alpha}\left(\varphi_t(xy)\right)}{I_{\alpha}\left(\varphi_t(xy)\right)}, & \text{if } y = \mathfrak{p}(x), O_2 = O_1 \setminus \{x,y\}, \\ C_{xz} \sqrt{\dfrac{\Lambda_t(z)}{\Lambda_t(x)}} \cdot \dfrac{I_{-\alpha}\left(\varphi_t(xz)\right) - I_{\alpha}\left(\varphi_t(xz)\right)}{I_{1-\alpha}\left(\varphi_t(xz)\right)}, & \begin{array}{c}\text{if } x = y, O_2 = O_1 \setminus \{x,z\} \\ \text{for } z \text{ with } x = \mathfrak{p}(z),\end{array} \end{cases} \tag{2.9}$$

and stops at time $T^{\omega} = T^{\lambda,O}(\omega)$ when the process explodes or uses up the local time at x_0. Here for $\omega = (\omega^1, \omega^2) \in \Omega'$, $\Lambda_t(x, \omega)$, $\varphi_t(xy, \omega)$ are defined as $\Lambda_t(x, \omega^1)$, $\varphi_t(xy, \omega^1)$ respectively.

Theorem 2.10. *Suppose $((\phi^{(0)}, \mathcal{O}^{(0)}), \mathcal{X}_{[0,\tau_u]}, (\phi^{(u)}, \mathcal{O}^{(u)}))$ is a percolation Ray-Knight triple associated to X. For any $\lambda \in \mathfrak{R}$ and configuration O on E, the conditional distribution of $(\mathcal{X}_{\tau_u - t})_{0 \le t \le \tau_u}$ given $\left(\phi^{(u)}, \mathcal{O}^{(u)}\right) = (\lambda, O)$ and $\tau_u < T^X$ is $\mathbb{P}^{\lambda}_{(x_0,O)}$.*

2.5 Mesh limit of vertex repelling jump processes

We only state the result in the case of simple random walks on dyadic grids, which can be generalized to general CTMCs on trees (see Remark 5.8). Let B be a reflected Brownian motion on $\mathbb{R}^+$. View 0 as the 'root' and let $(\widetilde{\phi}^{(0)}, B_{[0,\tau_u^B]}, \widetilde{\phi}^{(u)})$ be a Ray-Knight triple associated to B (defined in a similar way to that for CTMCs). The conditional law of $B_{[0,\tau_u^B]}$ given $\widetilde{\phi}^{(u)} = \lambda$ is the self-repelling diffusion B^{λ}, which can be constructed with the burglar process. See §5 for details.

Denote $\mathbb{N}_k := 2^{-k}\mathbb{N}$. Consider $\mathbb{T}_k = (\mathbb{N}_k, E_k)$, where $E_k := \big\{\{x,y\} : x, y \in \mathbb{N}_k,\ |x-y| = 2^{-k}\big\}$, endowed with conductances $C^k_e = 2^{k-1}$ on each edge and no killing. The induced

CTMC $X^{(k)}$ is the print of B on $\mathbb{N}_k$. Let $\big(\phi^{(0),k}, X^{(k)}_{[0,\tau_u^{X^{(k)}}]}, \phi^{(u),k}\big)$ be a Ray-Knight triple associated to $X^{(k)}$. It holds that

$$\Big(X^{(k)}_{2^k t}, L^{X^{(k)}}(2^k t, x), \phi^{(0),k}(x)\Big)_{t\geq 0, x\in\mathbb{R}^+} \xrightarrow{d} \Big(B_t, L^B(t,x), \widetilde{\phi}^{(0)}(x)\Big)_{t\geq 0, x\in\mathbb{R}^+},$$

where $L^{X^{(k)}}(2^k t,\cdot)$ and $\phi^{(0),k}(\cdot)$ are considered to be linearly interpolated outside $\mathbb{N}_k$.

In view of this, for a sequence of $\lambda_k \in \mathfrak{R}_k$ ($\mathfrak{R}_k$ is $\mathfrak{R}$ defined in (2.6) with $V=\mathbb{N}_k$ in the definition) and a non-negative, continuous function λ on $\mathbb{R}^+$ such that λ_k converges to λ (in the sense below), we naturally consider the vertex repelling jump processes $X^{\lambda_k,(k)}$, which is obtained by speeding up 2^k times the process with the conditional law of $X^{(k)}_{[0,\tau_u^{X^{(k)}}]}$ given $\phi^{(u),k}=\lambda_k$. The jump rates of the process are given in (5.8). Denote $\mathfrak{d}_{\lambda_k} := \inf\big\{x\in\mathbb{N}_k : \lambda_k(x)=0\big\}$ and $\mathfrak{d}_\lambda := \inf\big\{x\in\mathbb{R}^+ : \lambda(x)=0\big\}$.

Theorem 2.11. *Let λ_k be a sequence in $\mathfrak{R}_k$ and $\lambda\in\widetilde{\mathfrak{R}}$ (defined in §5) such that λ_k (linearly interpolated outside $\mathbb{N}_k$) converges to λ for the local uniform topology. When $\alpha=0$, further assume that $\mathfrak{d}_{\lambda_k}\to\mathfrak{d}_\lambda$. Then the family of vertex repelling jump processes $X^{\lambda_k,(k)}$ converges weakly as $k\to\infty$ to the self-repelling diffusion B^λ for the uniform topology, where the processes are assumed to stay at 0 after the lifetime.*

We close this section with two more remarks.

Remark 2.12. We analyze detailedly in Remark 3.12 and Remark 4.5 that the results in [24] and [17] are special cases of Theorem 2.9 and Theorem 2.10 respectively when $\alpha=1/2$ and the graph is the tree.

Remark 2.13. Let us explain here why the tree structure is important for the results in the paper. The statements and proofs of most results in the paper (e.g. Theorem 2.9, Theorem 2.10, Proposition 3.2, Proposition 3.4, Lemma 4.3) rely on the tree structure. They would be much more complicated when generalized to general graphs. For example, the jump rates in (2.7) are local meaning that they depend only on the remaining local times at the current position and the positions nearby. This is due to the local property of X (resp. $\mathcal{L}$) on the tree graph: for any subtree $\mathbb{T}_0$ of $\mathbb{T}$, the print of X (resp. $\mathcal{L}$) on $\mathbb{T}_0$ is independent of its local times outside $\mathbb{T}_0$. This property does not hold on general graphs. So in general, the jump rates on the general graph are non-local which depend on the remaining local times at all positions.

3 Inverting Ray-Knight identity

In this section, we will obtain the inversion of Ray-Knight identity. The main idea is to first introduce the information on crossings and explore the law of CTMC conditioned on both local times and crossings. Then by 'averaging over crossings', we get the representation of the inversion as the vertex repelling jump process shown in Theorem 2.9.

To begin with, observe that it suffices to only consider the case when X is recurrent. In fact, for $x\in V$, we set $h(x)=\mathbb{P}^x(H_{x_0}<T^X)$, where $\mathbb{P}^x$ is the law of X starting from x. Let Y be the CTMC on V starting from x_0 induced by conductances $C^h_{xy}=\frac{h(y)}{h(x)}C_{xy}$ and no killing. It follows from Proposition A.15 that Y is recurrent and $Y_{[0,\tau_u^Y]}$ has the law of $X_{[0,\tau_u]}$ conditioned on $\tau_u<T^X$. Note that Y can also be obtained by removing the killing rate at x_0 from the h-transform of X (the latter process is killed at x_0 with rate $C_{x_0}-C^h_{x_0}$, where $C_x=k_x+\sum_{y:y\sim x}C_{xy}$ and $C^h_x=\sum_{y:y\sim x}C^h_{xy}$). Combine the two facts: (1) the law of the loop soup is invariant under h-transforms (Cf. [4, Proposition 3.2]); (2) the law of the Ray-Knight triple does not depend on the killing rate at x_0. We have the Ray-Knight triple associated to X has the same law as that associated to Y. Then it is easy to deduce Theorem 2.9 in the transient case from that in the recurrent case.

Throughout this section, we will assume X is recurrent, which particularly implies that $\tau_u < T^X$ a.s.

3.1 The representation of the inversion as a vertex-edge repelling process

Definition 3.1. *An element $n = (n(xy))_{xy\in\vec{E}} \in \mathbb{N}^{\vec{E}}$ is called a network (on $\mathbb{T}$). For any network n, set*

$$\check{n}(xy) = n(xy) + (\alpha - 1)\cdot 1_{\{y=\mathfrak{p}(x)\}},$$

and for $x \in V$, $n(x) := \sum_{y:y\sim x} n(xy)$ and $\check{n}(x) := \sum_{y:y\sim x} \check{n}(xy)$. If $n(xy) = n(yx)$ for any $\{x, y\}$ in E, we say that the network n is sourceless, denoted by $\partial n = \emptyset$. Given $\lambda \in \mathfrak{R}$, we call $\mathcal{N}$ a sourceless α-random network associated to λ if $\mathcal{N}$ is a sourceless network, and $\{\mathcal{N}(xy) : x = \mathfrak{p}(y)\}$ are independent with $\mathcal{N}(xy)$ following the Bessel $(\alpha - 1$, $2C^*_{xy}\sqrt{\lambda(x)\lambda(y)})$ distribution[2].

More generally, let $\mathfrak{p}(x_0, x)$ be the unique self-avoiding path from x_0 to x, also seen as a collection of unoriented edges. For $i \in V \setminus \{x_0\}$, we say that n has sources (x_0, i), denoted by $\partial n = (x_0, i)$, if for any $xy \in \vec{E}$ with $x = \mathfrak{p}(y)$,

$$\begin{cases} n(xy) = n(yx) - 1, & \text{if } \{x, y\} \in \mathfrak{p}(x_0, i); \\ n(xy) = n(yx), & \text{if } \{x, y\} \notin \mathfrak{p}(x_0, i). \end{cases}$$

Given $\lambda \in \mathfrak{R}$ and $i \in V\setminus\{x_0\}$, we call $\mathcal{N}$ an α-random network with sources (x_0, i) associated to λ if $\mathcal{N}$ is a network with sources (x_0, i) and $\{\mathcal{N}(xy) : x = \mathfrak{p}(y)\}$ are independent with $\mathcal{N}(xy)$ following the Bessel $(\alpha$, $2C^*_{xy}\sqrt{\lambda(x)\lambda(y)})$ distribution if $\{x, y\} \in \mathfrak{p}(x_0, i)$, and the Bessel $(\alpha - 1$, $2C^*_{xy}\sqrt{\lambda(x)\lambda(y)})$ distribution otherwise.

Remark. We will sometimes use the convention that a network with sources (x_0, x_0) is a sourceless network.

Every loop configuration $\mathscr{L}$ (i.e. a collection of unrooted, oriented loops) induces a network $\theta(\mathscr{L})$: for $x \sim y$,

$$\theta(\mathscr{L})(xy) := \#\ \text{crossings from } x \text{ to } y \text{ by the loops in } \mathscr{L}.$$

Due to the tree structure, it holds that $\theta(\mathscr{L})(xy) = \theta(\mathscr{L})(yx)$ for any $xy \in \vec{E}$, i.e. $\theta(\mathscr{L})$ is sourceless.

Let $\left(\phi^{(0)}, X_{[0,\tau_u]}, \phi^{(u)}\right)$ be the Ray-Knight triple associated to X, where $\phi^{(0)}$ is the local time field of a loop soup $\mathcal{L}^{(0)}$ independent of X. The path $X_{[0,\tau_u]}$ is also viewed as a loop configuration consisting of a single loop. Let $\mathcal{N}^{(u)} := \theta(\mathcal{L}^{(0)}) + \theta(X_{[0,\tau_u]})$. We have the following result, the proof of which is contained in §3.1.1. We mention that [12, Theorem 3.1] provides another proof for the case $\alpha = 0$.

Proposition 3.2. *For $\lambda \in \mathfrak{R}$ with $\lambda(x_0) = u$, conditionally on $\phi^{(u)} = \lambda$, $\mathcal{N}^{(u)}$ is a sourceless α-random network associated to λ.*

With this proposition, for the recovery of $X_{[0,\tau_u]}$, it suffices to derive the law of $X_{[0,\tau_u]}$ given $\phi^{(u)}$ and $\mathcal{N}^{(u)}$. For any $\lambda \in \mathfrak{R}$, set

$$\mathfrak{N} = \mathfrak{N}^\lambda := \begin{cases} \left\{n \in \mathbb{N}^{\vec{E}} : \partial n = \emptyset\right\}, & \text{if } \alpha > 0; \\ \left\{n \in \mathbb{N}^{\vec{E}} : \partial n = \emptyset, \forall xy \in \vec{E}, n(xy) \geq 1 \text{ iff } x, y \in \operatorname{supp}(\lambda)\right\}, & \text{if } \alpha = 0. \end{cases}$$

[2] For $\nu \geq -1$ and $z > 0$, the Bessel (ν, z) distribution is a distribution on $\mathbb{N}$ given by:

$$b_{\nu,z}(n) = I_\nu(z)^{-1}\frac{(z/2)^{2n+\nu}}{n!\,\Gamma(n+\nu+1)},\ n \in \mathbb{N}. \tag{3.1}$$

Bessel $(\nu, 0)$ distribution is defined to be the Dirac measure at 0.

The superscript λ in $\mathfrak{N}^\lambda$ is omitted whenever it is clear from the context. Given $n \in \mathbb{N}^{\vec{E}}$, for $\omega \in \Omega$ and $x, y \in V$ with $x \sim y$, set

$$\begin{aligned}\Theta_t(xy) = \Theta_t(n,\omega)(xy) &= \Theta_t(n,\omega_{[0,t]})(xy) \\ &:= n(xy) - \#\left\{0 < s \le t : \omega_{s-} = x, \omega_s = y\right\},\end{aligned} \tag{3.2}$$

which represents the remaining crossings while running the process ω until time t with the initial crossings n. Recall the notation introduced at the beginning of §2.4.1. Given $\lambda \in \mathfrak{R}$ and $n \in \mathfrak{N}$, the vertex-edge repelling jump process $X^{\lambda,n}$ is defined to be a process starting from x_0, behaves such that

(i) conditionally on $t < T^{X^{\lambda,n}}$ and $\left(X^{\lambda,n}_s : 0 \le s \le t\right)$ with $X^{\lambda,n}_t = x$, it jumps to a neighbour y of x with rate $r^{\lambda,n}_t\left(x, y, X^{\lambda,n}_{[0,t]}\right)$;

(ii) every time $\lim_{s\to t-} \Lambda_s(X^{\lambda,n}_s) = 0$ and $X^{\lambda,n}_{t-} \neq x_0$, it jumps to $\mathfrak{p}(x)$ at time t,

and stops at time $T^{X^{\lambda,n}}$ when the process exhausts the local time at x_0 or explodes. Here for $\omega \in \Omega$ with $T^\omega > t$,

$$r^{\lambda,n}_t(x, y, \omega) := \frac{\breve{\Theta}_t(xy)}{\Lambda_t(x)}. \tag{3.3}$$

In the above expression, Θ_t is viewed as a network, and $\breve{\Theta}_t$ is defined as before.

Intuitively, for this process, both the local time and crossings available are given at the beginning. The process eats the local time during its stay at vertices and consumes crossings at jumps over edges.

Theorem 3.3. *For any $\lambda \in \mathfrak{R}$ and $n \in \mathfrak{N}$, $X^{\lambda,n}$ has the law of $X_{[0,\tau_u]}$ conditioned on $\phi^{(u)} = \lambda$ and $\mathcal{N}^{(u)} = n$.*

3.1.1 Proof of Proposition 3.2

Recall that $\mathcal{L}^{(u)}$ consists of the partition of the path of $X_{[0,\tau_u]}$ and an independent loop soup $\mathcal{L}^{(0)}$. By the excursion theory (Cf. [23, §8 in Chapter VI]) (resp. basic properties of Poisson point processes), the prints of $X_{[0,\tau_u]}$ (resp. $\mathcal{L}^{(0)}$) on the different branches[3] at x_0 are independent. Here the print of a loop configuration is the collection of the prints of loops in the configuration. In particular, for any $x \sim x_0$, $\mathcal{N}^{(u)}(x_0x)$ is independent of the prints of $\mathcal{L}^{(u)}$ on the branches that do not contain x. By considering other vertices as the root of $\mathbb{T}$, we can readily obtain that given $\phi^{(u)} = \lambda$, $\left(\mathcal{N}^{(u)}(yz) : y = \mathfrak{p}(z)\right)$ are independent and the conditional law of $\mathcal{N}^{(u)}(yz)$ depends only on $\lambda(y)$ and $\lambda(z)$[4].

Now it reduces to considering the law of $\mathcal{N}^{(u)}(yz)$ conditioned on $\phi^{(u)}(y) = \lambda(y)$ and $\phi^{(u)}(z) = \lambda(z)$. We focus on the case $yz = x_0x$ only, since it is the same for other edges. Without conditioning, it holds that

(i) $\mathcal{N}^{(u)}(x_0x)$ has a Poisson distribution with parameter $uC_{x_0,x}$;

(ii) $L(\tau_u, x)$ equals a sum of $\mathcal{N}^{(u)}(x_0x)$ i.i.d. exponential random variables with parameter C_{x,x_0};

[3] A branch at x_0 is defined as a connected component of the tree when removing the vertex x_0, to which we add x_0.

[4] To see the root x_0 plays no special role, we add some killing rate at x_0. Then for any $\lambda \in \mathfrak{R}$ with $\lambda(x_0) = u$, $\mathcal{L}^{(u)}$ conditioned on $L_\cdot(\mathcal{L}^{(u)}) = \lambda$ has the same law as the loop soup $\mathcal{L}$ (associated to the process with killing rate) conditioned on $L_\cdot(\mathcal{L}) = \lambda$.

(iii) $L_x(\mathcal{L}^{(0)})$ follows a Gamma(α, C_{x,x_0}) distribution (a Gamma(0, β) r.v. is interpreted as a r.v. identically equal to 0). In fact, $L_x(\mathcal{L}^{(0)})$ follows a Gamma($\alpha, G^{\bar{x}_0}(x,x)^{-1}$) distribution, where $G^{\bar{x}_0}$ is the Green function of the process X killed at x_0. The recurrence of X implies that $G^{\bar{x}_0}(x,x) = C_{x,x_0}^{-1}$.

(iv) The above exponential random variables, $\mathcal{N}^{(u)}(x_0x)$ and $L_x(\mathcal{L}^{(0)})$ are mutually independent.

It follows that the conditional law of $\mathcal{N}^{(u)}(x_0x)$ is the same as the conditional law of U given

$$R_* + R_1 + \cdots + R_U = \lambda(x),$$

when U has Poisson($uC_{x_0,x}$) distribution, R_* has Gamma(α, C_{x,x_0}) distribution, $R_1, R_2, \cdots$ have Exponential(C_{x,x_0}) distribution and $U, R_*, R_1, R_2, \cdots$ are mutually independent. By directly writing the density of $R_* + R_1 + \cdots + R_U$ and then conditioning on the sum being $\lambda(x)$, we can readily obtain that the conditional distribution is Bessel ($\alpha - 1, 2C^*_{x_0x}\sqrt{u\lambda(x)}$) (see also [9, §2.7]). We have thus proved the proposition.

3.1.2 Proof of Theorem 3.3

The recovery of $X_{[0,\tau_u]}$ given $\phi^{(u)}$ and $\mathcal{N}^{(u)}$ is carried out by the following two steps:

(1) reconstruct the jump chain of $X_{[0,\tau_u]}$ conditionally on $\phi^{(u)}$ and $\mathcal{N}^{(u)}$;

(2) assign the holding times before every jump conditionally on $\phi^{(u)}$, $\mathcal{N}^{(u)}$ and the jump chain.

We shall prove the process recovered by the above two steps is exactly the vertex-edge repelling jump process.

For step (1), it is easy to see that the conditional law of the jump chain actually depends only on $\mathcal{N}^{(u)}$. Moreover, let $\bar{X}$ be distributed as the jump chain of X. For $k \geq 0$, let $\bar{\tau}_k := \inf\{l \geq 0 : \#\{1 \leq j \leq l : \bar{X}_j = x_0\} = k\}$. Fixing a sourceless network n, set $m = n(x_0)$. We consider $\bar{X}$ up to $\bar{\tau}_m$. Let $\bar{\mathcal{L}}^{(0)}$ have the law of the discrete loop soup induced by $\mathcal{L}^{(0)}$, and be independent of $\bar{X}$. Set $\bar{\mathcal{N}} := \theta(\bar{X}_{[0,\bar{\tau}_m]}) + \theta(\bar{\mathcal{L}}^{(0)})$, where $\theta(\bar{X}_{[0,\bar{\tau}_m]})$ and $\theta(\bar{\mathcal{L}}^{(0)})$ have the obvious meaning. Denote by $\bar{X}^n$ the process $\bar{X}$ conditioned on $\bar{\mathcal{N}} = n$. Then $\bar{X}^n$ has the same law as the jump chain of $X_{[0,\tau_u]}$ conditioned on $\mathcal{N}^{(u)} = n$. The following proposition plays a central role in the inversion. We present a combinatorial proof later.

Proposition 3.4. *The law of $\bar{X}^n$ can be described as follows. It starts from x_0. Conditionally on $\left(\bar{X}^n_k : 0 \leq k \leq l\right)$ with $l < T^{\bar{X}^n}$ and $\bar{X}^n_l = x$,*

- *if $\breve{\Theta}_l(x) > 0$, it jumps to a neighbour y of x with probability*

$$\frac{\breve{\Theta}_l(xy)}{\breve{\Theta}_l(x)},$$

- *if $\breve{\Theta}_l(x) = 0$ and $x \neq x_0$, it jumps to $\mathfrak{p}(x)$; (This can only happen when $\alpha = 0$.)*

And finally $\bar{X}^n$ stops at time $T^{\bar{X}^n}$. Here

$$\Theta_l(xy) = \Theta_l(n, \bar{X}^n_{[0,l]})(xy) := n(xy) - \#\{0 \leq k \leq l-1 : \bar{X}^n_k = x,\, \bar{X}^n_{k+1} = y\},$$
$$T^{\bar{X}^n} := \inf\{k \geq 0 : \Theta_k(x_0) = 0\}.$$

Now turn to step (2), i.e. recovering the jump times.

Proposition 3.5. *Given $\phi^{(u)} = \lambda$, $\mathcal{N}^{(u)} = n$ and the jump chain of $X_{[0,\tau_u]}$, denote by $r(x)$ the number of visits to x by the jump chain and h_i^x the i-th holding time at vertex x. Note that $r(x) = n(x)$ for each x in the case $\alpha = 0$. The following hold:*

(i) the holding times at different vertices are independent;

(ii) $\left(\sum_{j=1}^{i} h_j^{x_0} : 1 \le i \le n(x_0)\right)$ have the same law as $n(x_0)$ i.i.d. uniform random variables on $[0,u]$ ranked in ascending order and $h_{n(x_0)+1}^{x_0} = u - \sum_{j=1}^{n(x_0)} h_j^{x_0}$;

(iii) for any $x(\neq x_0)$ visited by $X_{[0,\tau_u]}$, $\left(\frac{h_1^x}{\lambda(x)}, \cdots, \frac{h_{r(x)}^x}{\lambda(x)}, 1 - \frac{\sum_{j=1}^{r(x)} h_j^x}{\lambda(x)}\right)$ is a Dirichlet distribution with parameter $(1, \cdots, 1, n(x) - r(x) + \alpha)$.

Here m-variable Dirichlet distribution with parameter $(1, \cdots, 1, 0)$ is interpreted as $(m-1)$-variable Dirichlet distribution with parameter $(1, \cdots, 1)$.

Proof. First, we recall the following fact of loop soups.

Fact 3.6. *Suppose $\mathcal{L}$ is a loop soup associated to some transient CTMC on $\mathbb{T}$. Then conditionally on the discrete skeleton of $\mathcal{L}$, $\mathcal{L}$ is obtained by including independent Exponential(C_x) holding times at each visit to $x \in V$ and adding one-point loops on which the holding times is a Poisson point process with intensity density $e^{-C_x t}/t$.*[5] *Consequently, the conditional law of the local time field $L_\cdot(\mathcal{L})$ of $\mathcal{L}$ can be described as follows:*

- *$\{L_x(\mathcal{L}) : x \in V\}$ are independent;*
- *suppose there is $m(x)$ visits at x by the discrete skeleton for each $x \in V$. Then for any $x \in V$, $L_x(\mathcal{L})$ follows a Gamma $(m(x) + \alpha, C_x)$ distribution.*

Since $\phi^{(0)}$ is the loop soup associated to X killed at x_0, the independence in (i) comes from the above facts and basic properties of CTMC.

$\left\{\sum_{j=1}^{i} h_j^{x_0} : 1 \le i \le n(x_0)\right\}$ is distributed as the jump times of a Poisson process on $[0,u]$ conditioned on that there are exactly $n(x_0)$ jumps during this time interval. That deduces (ii).

For (iii), again using the above fact, we have for any $x \neq x_0$, $\left\{h_j^x : 1 \le j \le r(x)\right\}$ has the law of $r(x)$ i.i.d exponential variables with parameter $C_x = k_x + \sum_{y: y \sim x} C_{xy}$ conditioned on the sum of them and an independent Gamma($n(x) - r(x) + \alpha$, C_x) random variable equal to $\lambda(x)$ (recall that a Gamma$(0, \beta)$ r.v is identically equal to 0). $\square$

Proposition 3.4 and 3.5 give a representation of the inversion of the Ray-Knight identity in terms of its jump chain and holding times. Using this, we can readily calculate the jump rates.

Proof of Theorem 3.3. It suffices to show that the jump chain and holding times of $X^{\lambda,n}$ are given by Proposition 3.4 and 3.5 respectively. As shown in the proof of Theorem A.6, $X^{\lambda,n}$ can be realized by a sequence of i.i.d exponential random variables. It is direct from this realization that the jump chain of $X^{\lambda,n}$ coincides with $\bar{X}^n$ in Proposition 3.4. To see this, one only needs to note that for any fixed $t \ge 0$ and $\omega \in \Omega$ with $T^\omega > t$ and $\omega_t = x$, the jump rate $r_t^{\lambda,n}(x, y, \omega)$ is proportional to $\breve{\Theta}_t(n, \omega)(xy)$ for $y \sim x$. So it remains to check the holding times. We consider $X_{[0,\tau_u]}$ given $\phi^{(u)} = \lambda$, $\mathcal{N}^{(u)} = n$, and its jump

[5] To get this, we consider an unrooted loop γ under the loop measure μ. It is not hard to deduce from [4, §3.1] (for example, one can use the definition of the pointed loop measure μ^{p*} and the fact that μ^{p*} induces μ on unrooted loops) that under μ, conditionally on the discrete skeleton of γ, if γ visits more than one vertices, then γ is obtained by adding independent exponential holding times at each visit to vertices; if γ is a one-point loop, then its holding time has density $e^{-C_x t}/t$. The above fact of $\mathcal{L}$ is then derived using the properties of Poisson point processes.

chain. Use the same notation $r(x)$ and h_i^x as before. For $1 \le i \le n(x)$, set $l_i^x := \sum_{j=1}^{i} h_j^x$. By Proposition 3.5, conditionally on all the holding times before the i-th visit at x, it holds that $h_{i+1}^x/(\lambda(x) - l_i^x)$ follows a Beta $(1, \check{n}(x) - i + 1)$ distribution. We can readily check that

$$(\lambda(x) - l_i^x) \cdot \mathrm{Beta}(1, \check{n}(x) - i + 1) \stackrel{d}{=} \left(u_x^{(i)}\right)^{-1}(\gamma),$$

where $u_x^{(i)}(t) = \int_0^t \frac{\check{n}(x)-i+1}{\lambda(x)-l_i^x-s}\,\mathrm{d}s$ ($0 \le t < \lambda(x) - l_i^x$) and γ is an exponential random variable with parameter 1. This is exactly the same holding times as $X^{\lambda,n}$. We have thus proved the theorem. $\square$

The remaining part is devoted to the proof of Proposition 3.4.

Proof of Proposition 3.4. **Case** $\alpha = 0$**.** Condition on $\left(\bar{X}_k^n : 0 \le k \le l\right)$ with $l < T^{\bar{X}^n}$ and $\bar{X}_l^n = x$. It holds that the remaining path of $\bar{X}$ is completed by uniformly choosing a path with edge crossings Θ_l since the conditional probability of any such path is the same, which equals

$$\prod_{yz \in \vec{E}} (C_{yz}/C_y)^{\Theta_l(yz)}.$$

So for the probability of the next jump, it suffices to count for any $y \sim x$, the number of all possible paths with edge crossings Θ_l and the first jump being to y. Here we use the same idea as the proof of [11, Proposition 2.1]. This number equals the number of relative orders of exiting each vertex satisfying (1) the first exit from x is to y; (2) the last exit from any $z \ne x_0$ is to $\mathfrak{p}(z)$. In particular, it is proportional to the number of relative orders of exiting x satisfying the above conditions at vertex x, which equals

$$\begin{cases} \dfrac{\left(\Theta_l(x)-1-1_{\{x \ne x_0\}}\right)!}{\prod_{z:z\sim x}\left(\Theta_l(xz)-1_{\{z=\mathfrak{p}(x)\}}-1_{\{z=y\}}\right)!}, & \text{if } \Theta_l(x) \ge 2; \\ 1_{\{y=\mathfrak{p}(x)\}}, & \text{if } \Theta_l(x) = 1, \end{cases}$$

where $(-1)! := \infty$. Thus the conditional probability that $\bar{X}_{l+1}^n = y$ is

$$\begin{cases} \dfrac{\check{\Theta}_l(xy)}{\check{\Theta}_l(x)}, & \text{if } \Theta_l(x) \ge 2; \\ 1_{\{y=\mathfrak{p}(x)\}}, & \text{if } \Theta_l(x) = 1. \end{cases} \tag{3.4}$$

Case $\alpha > 0$**. (1) The conditional transition probability at** x_0**.** We will first deal with the conditional transition probability given $\left(\bar{X}_k^n : 0 \le k \le l\right)$ with $l < T^{\bar{X}^n}$ and $\bar{X}_l^n = x_0$. We further condition on the remaining crossings of $\bar{X}^n$, i.e. $\theta' := \theta(\bar{X}^n_{[l,T^{\bar{X}^n}]})$. Note that $\Theta_l = \theta' + \theta(\bar{\mathcal{L}}^{(0)})$. In particular, it holds that $\theta'(x_0 y) = \Theta_l(x_0 y)$ for any $y \sim x_0$. By (3.4), the conditional probability that $\bar{X}_{l+1}^n = y$ is

$$\frac{\theta'(x_0 y)}{\theta'(x_0)} = \frac{\Theta_l(x_0 y)}{\Theta_l(x_0)},$$

which is independent of the further condition and hence gives the conclusion.

(2) The conditional transition probability at $x \ne x_0$**.** When $\mathbb{T}$ is infinite, $\mathcal{L}$ contains infinitely many loops and the probability that $\mathcal{L}$ equals any loop configuration is zero. So to avoid talking about something with zero probability, we first assume that $\mathbb{T}$ is finite. Recall that the law of the Ray-Knight triple is independent of k_{x_0}. In this case, it is easier to consider the process X killed at x_0 with rate 1. With an abuse of notation, we still use X and $\mathcal{L}$ to denote this process and its associated loop soup respectively[6].

[6] We mention that such notation is only used in this part.

The main idea is to transform the recovery of the Markovian path to the recovery of the discrete loop soup. Roughly speaking, the latter recovery is just to glue the crossings together to reconstruct the discrete loop soup. The properties of discrete loop soups make the recovery quite intuitive as we will see later. Let us make some preparations first.

(a) Concatenation process L. First, we give a representation of the path $\bar{X}^n$ as a concatenation of loops in a loop soup as follows. Let $\bar{\mathcal{L}}$ be the discrete loop soup associated to $\mathcal{L}$. We focus on the loops in $\bar{\mathcal{L}}$ that visit x_0. The number of such loops is denoted by $\mathfrak{K}$. For each of them, we uniformly and independently root it at one of its visits at x_0. Then we choose uniformly at random (among all $\mathfrak{K}!$ choices) an order for the rooted loops labeled in order by $\{\bar{\gamma}_i : 1 \le i \le \mathfrak{K}\}$ and concatenate them:

$$\mathbf{L} := \bar{\gamma}_1 \circ \bar{\gamma}_2 \circ \cdots \circ \bar{\gamma}_{\mathfrak{K}}.$$

We call $\mathbf{L}$ the concatenation process of $\bar{\mathcal{L}}$. It can be easily deduced from the properties of discrete loop soup that the path between consecutive visits of x_0 in any $\bar{\gamma}_i$ has the same law as an excursion of $\bar{X}$ at x_0. Thus, conditionally on $\theta(\mathbf{L})(x_0) = m$, $\mathbf{L}$ has the same law as $\bar{X}_{[0,\bar{\tau}_m]}$. Consequently, we have the following corollary.

Corollary 3.7. *Given a network $n \in \mathfrak{N}$, denote by $\mathbf{L}^n$ the process $\mathbf{L}$ conditioned on $\theta(\bar{\mathcal{L}}) = n$. Then $\mathbf{L}^n$ has the same law as $\bar{X}^n$.*

(b) Pairing on the extended graph. To further explore the law of $\mathbf{L}^n$, we need to introduce the extended graphs of $\mathbb{T}$ and the definition of 'pairing'. We choose to work on the extended graph $\mathbb{T}^K$ (defined in the following) to make sure that the probability of a loop configuration on $\mathbb{T}^K$ is proportional to $\alpha^{\#\text{loops in the configuration}}$ as $K \to \infty$. Let $K \ge 1$. Replace each edge of $\mathbb{T}$ by K copies. The graph thus obtained, denoted by $\mathbb{T}^K = (V, E^K)$, is an extended graph (of $\mathbb{T}$). The collection of all directed edges in $\mathbb{T}^K$ is denoted by $\vec{E}^K$. The graph $\mathbb{T}^K$ is equipped with the killing measure δ_{x_0}, the Dirac measure at x_0, and for any $x \sim y$, the conductance on each one of the K directed edges from x to y is $C^K_{xy} := C_{xy}/K$. Any element in $\mathbb{N}^{\vec{E}^K}$ is called a network on $\mathbb{T}^K$. We will use N to denote a deterministic network on $\mathbb{T}^K$. For a network N on $\mathbb{T}^K$, the projection of N on $\mathbb{T}$ is a network on $\mathbb{T}$ defined by:

$$N^{\mathbb{T}}(xy) := \sum_{\vec{e} \in \vec{E}^K} N(\vec{e}) 1_{\{\vec{e} \text{ is from } x \text{ to } y\}}.$$

In the following, we only focus on the network $N \in \{0,1\}^{\vec{E}^K}$. For such a network N, denote $[N] := \{\vec{e} \in \vec{E}^K : N(\vec{e}) = 1\}$. For simplicity, we omit the superscript '$\to$' for directed edges throughout this part.

Definition 3.8. *Given a sourceless network $N \in \{0,1\}^{\vec{E}^K}$, a pairing of N is defined to be a bijection from $[N]$ to $[N]$, such that for any $e \in [N]$, the image of e is a directed edge whose head is the tail of e.*

Given $N \in \{0,1\}^{\vec{E}^K}$, a pairing b of N and a subset $[N_0]$ of $[N]$, $b|_{[N_0]} = \{b(e) : e \in [N_0]\}$ determines a set of loops and bridges. Precisely, for each $e \in [N_0]$, following the pairing on e, we arrive at a new (directed) edge $b(e)$. Continuing to keep track of the pairing on the new edge $b(e)$, we arrive at another edge $b(b(e))$. This procedure stops when arriving at either the initial edge e again, or an edge $\in [N] \setminus [N_0]$ because we lose the information about the pairing on $[N] \setminus [N_0]$. In the former case, a loop is obtained. In the latter case, we get a path whose first edge is e and last edge $\in [N] \setminus [N_0]$. Any two such paths are either disjoint, or one is a part of the other. This naturally determines a partial order. All the maximal elements with respect to this partial order and the loops obtained in the former case form the set of bridges and loops determined by $b|_{[N_0]}$.

Now we start the proof of the conditional transition probability at $x \neq x_0$. Let X^K be the CTMC on $\mathbb{T}^K$ starting from x_0 induced by the conductances $\left(C^K_{xy}\right)$ and killing measure δ_{x_0}. Consider the discrete loop soup $\bar{\mathcal{L}}^K$ associated to X^K, the projection of which on $\mathbb{T}$ has the same law as $\bar{\mathcal{L}}$. Let $\mathbf{L}^K$ be its concatenation process. By Corollary 3.7, it suffices to consider the law of (the projection of) $\mathbf{L}^K$ given $\left(\theta(\bar{\mathcal{L}}^K)\right)^{\mathbb{T}} = n$.

Observe that conditionally on $l < T^{\mathbf{L}^K}$ and $\left(\mathbf{L}^K_k : 0 \le k \le l\right)$ with $\mathbf{L}^K_l = x\,(\neq x_0)$, for the next step, it will definitely jump along its present loop in $\bar{\mathcal{L}}^K$, which is a loop visiting both x and x_0. Note that the probability that $\bar{\mathcal{L}}^K$ uses every edge at most once tends to 1 as $K \to \infty$. So by the standard arguments (see [26, 27]), it suffices to show that for any $N \in \{0,1\}^{\vec{E}^K}$ with $N^{\mathbb{T}}(yz) = n(yz)$ for any $y \sim z$, the transition probability of $\mathbf{L}^K$ at x is given by the statements in the proposition, i.e.

$$\mathbb{P}\left(\mathbf{L}^K_{l+1} = y \;\middle|\; \theta(\bar{\mathcal{L}}^K) = N, \mathbf{L}^K_{[0,l]}, \mathbf{L}^K_l = x, l < T^{\mathbf{L}^K}\right) = \frac{\breve{\Theta}_l(n, \mathbf{L}^K_{[0,l]})(xy)}{\breve{\Theta}_l(n, \mathbf{L}^K_{[0,l]})(x)}, \tag{3.5}$$

where on the right-hand side, the path $\mathbf{L}^K_{[0,l]}$ is considered as its projection on $\mathbb{T}$.

Next, we shall translate conditioning in (3.5) into the language of pairings. Let $\mathfrak{b}$ be the pairing of $[N]$ induced by $\bar{\mathcal{L}}^K$ and

$$[N_0] := \left\{e \in [N] : e \text{ is crosses by } \mathbf{L}^K_{[0,l-1]} \text{ or the tail of } e \text{ is not } x\right\}.$$

Then the information of $\mathbf{L}^K_{[0,l]}$ is contained in $\mathfrak{b}|_{[N_0]}$ and the (relative) concatenation order of the loops crossed by $\mathbf{L}^K_{[0,l]}$. Observe that since $x \neq x_0$, conditionally on $\theta(\bar{\mathcal{L}}^K) = N$ and $\mathfrak{b}|_{[N_0]}$, we have $\mathbf{L}^K_l\mathbf{L}^K_{l+1} = \mathfrak{b}(\mathbf{L}^K_{l-1}\mathbf{L}^K_l)$ is independent of the concatenation order of the loops crossed by $\mathbf{L}^K_{[0,l]}$. Thus it is enough to show that

$$\mathbb{P}\left(\mathfrak{b}(\mathbf{L}^K_{l-1}\mathbf{L}^K_l) \text{ is from } x \text{ to } y \;\middle|\; \mathfrak{b}|_{[N_0]}\right) = \frac{\breve{\Theta}_l(n, \mathbf{L}^K_{[0,l]})(xy)}{\breve{\Theta}_l(n, \mathbf{L}^K_{[0,l]})(x)}. \tag{3.6}$$

The condition $\mathfrak{b}|_{[N_0]}$ determines a set of loops and bridges from and to x. Denote $N_l(e) = 1_{\left\{e \in [N] \text{ and is not crossed by } \mathbf{L}^K_{[0,l]}\right\}}$ for $e \in \vec{E}^K$. Then there are exactly $N^{\mathbb{T}}_l(x)$ bridges, including exactly $N^{\mathbb{T}}_l(xy)$ bridges whose first edge enters y, for any $y \sim x$. Moreover, there is exactly one bridge partly crossed by the path $\mathbf{L}^K_{[0,l]}$. This bridge is a part of the loop that $\mathbf{L}^K$ is walking along at time l. So it visits x_0 by the construction of $\mathbf{L}^K$, which implies that the first edge of the bridge enters $\mathfrak{p}(x)$ due to the tree structure.

Now we focus on the conditional law of the pairing of these bridges, i.e. the conditional law of $\mathfrak{b}^* := \mathfrak{b}|_{[N^*]}$, where $[N^*] = [N] \setminus [N_0]$ is the collection of the last edges in the above $N^{\mathbb{T}}_l(x)$ bridges. Note that $\mathfrak{b}^*([N^*])$ consists of the first edges in these bridges. Denote $[N^*] = \{e_1, \cdots, e_r\}$ and $\mathfrak{b}^*([N^*]) = \{f_1, \cdots, f_r\}$, where $r = N^{\mathbb{T}}_l(x)$ and we assign the subscripts such that

- e_i and f_i are in the same bridge ($i = 1, \cdots, r$);
- $e_1 = \mathbf{L}^K_{l-1}\mathbf{L}^K_l$. (So the bridge containing e_1 and f_1 is exactly the unique bridge partly crossed by $\mathbf{L}^K_{[0,l]}$.)

The totality of bijections from $\{e_1, \cdots, e_r\}$ to $\{f_1, \cdots, f_r\}$ is denoted by $\mathcal{B}$. Every $b \in \mathcal{B}$ pairs the bridges into a loop configuration. We simply call it the configuration completed by b. It is easy to see that this defines a one-to-one correspondence between $\mathcal{B}$ and all the possible configurations obtained by pairing these bridges.

A key observation is that conditionally on $\theta(\bar{\mathcal{L}}^K) = N$, the probability of a loop configuration of $\bar{\mathcal{L}}^K$ is proportional to $\alpha^{\#\text{loops in the configuration}}$. In fact, $\bar{\mathcal{L}}^K$ is a Poisson point process on the space of loops on $\mathbb{T}$. The intensity of a loop is α times the product of the transition probabilities of the edges divided by the multiplicity of the loop. Note that for any configuration $\mathscr{L}$ with $\theta(\mathscr{L}) = N$, the multiplicities of the loops in $\mathscr{L}$ are all 1. Hence the probability that $\bar{\mathcal{L}}^K = \mathscr{L}$ is

$$P_\emptyset \cdot \alpha^{\#\text{loops in } \mathscr{L}} \prod_{yz \in \vec{E}} (C_{yz}/KC_y)^{N^{\mathbb{T}}(yz)},$$

where $P_\emptyset$ is the probability that $\bar{\mathcal{L}}^{K,x}$ is empty. So we have

$$\mathbb{P}\big(\mathfrak{b}^* = b \mid \mathfrak{b}|_{[N_0]}\big) \propto \alpha^{\#(b)},$$

where $\#(b) := \#$loops in the configuration completed by b. Set $\mathcal{B}_i := \{b \in \mathcal{B} : b(e_1) = f_i\}$. For $1 \le i, j \le r$ with $i \ne j$, a bijection Υ_{ij} from B_i to B_j can be defined as follows. For any $b \in B_i$, $\Upsilon_{ij}(b)$ is defined by exchanging the image of e_1 and $b^{-1}(f_j)$. Precisely,

$$(\Upsilon_{ij}(b))\,(e) := \begin{cases} f_j, & \text{if } e = e_1; \\ f_i, & \text{if } e = b^{-1}(f_j); \\ b(e), & \text{otherwise.} \end{cases}$$

We can readily check that for $b \in B_i$,

$$\begin{cases} \#(b) = \#(\Upsilon_{ij}(b)), & \text{if } i, j \ne 1; \\ \#(b) = \#(\Upsilon_{ij}(b)) + 1, & \text{if } i = 1 \text{ and } j \ne 1. \end{cases}$$

Therefore if we denote by p_i the conditional probability that $\mathfrak{b}^* \in B_i$, then $p_i = p_j$ for $i, j \ne 1$, and $p_1 = \alpha p_j$ for $j \ne 1$. Namely,

$$\begin{cases} p_1 = \alpha/(r + \alpha - 1); \\ p_i = 1/(r + \alpha - 1), \text{ for } i \ne 1. \end{cases}$$

It follows that the conditional probability of $\mathbf{L}^K_{l+1} = y$ is

$$\sum_{i=1}^{N_l(x)} 1_{\{\text{the tail of } f_i \text{ is } y\}} \cdot p_i = \frac{\breve{N}^{\mathbb{T}}_l(xy)}{\breve{N}^{\mathbb{T}}_l(x)} = \frac{\breve{\Theta}_l(n, \mathbf{L}^K_{[0,l]})(xy)}{\breve{\Theta}_l(n, \mathbf{L}^K_{[0,l]})(x)},$$

where the first equality is due to the fact that f_1 enters $\mathfrak{p}(x)$. That completes the proof in the case of the finite $\mathbb{T}$.

Finally, let us tackle the case where $\mathbb{T}$ is infinite. The result follows from the local property: for any finite subtree $\mathbb{T}_0 = (V_0, E_0)$ of $\mathbb{T}$ containing x_0, the print of $\mathbf{L}$ on V_0 is independent of $\theta(\mathbf{L})|_{\vec{E}\setminus\vec{E}_0}$. Indeed, denote by $\mathbf{L}^{\mathbb{T}_0}$ the print of $\mathbf{L}$ on V_0 and by $\mathbf{L}^{\mathbb{T}_0,n}$ the process $\mathbf{L}^{\mathbb{T}_0}$ conditioned on $\theta(\mathbf{L}^{\mathbb{T}_0}) = n|_{\vec{E}_0}$. Since the print of $\mathcal{L}$ on V_0 is a loop soup associated to the print of X on V_0 (Cf. [14, §3.2]), the previous proof in the finite case tells us that the law of $\mathbf{L}^{\mathbb{T}_0,n}$ is given by that in the statement of Proposition 3.4 with X replaced by its print on V_0. Then the local property implies the result we want.

To get the local property, we consider two finite subtrees $\mathbb{T}_i = (V_i, E_i)$ $(i = 0, 1)$ containing x_0 with $V_0 \subset V_1$. It is directly seen from the laws of $\mathbf{L}^{\mathbb{T}_0,n}$ and $\mathbf{L}^{\mathbb{T}_1,n}$ that $\mathbf{L}^{\mathbb{T}_0,n}$ is the print of $\mathbf{L}^{\mathbb{T}_1,n}$ on V_0. In other words, conditionally on $n|_{\vec{E}_1}$, $\mathbf{L}^{\mathbb{T}_0}$ has the same law as $\mathbf{L}^{\mathbb{T}_0,n}$. This implies that $\mathbf{L}^{\mathbb{T}_0}$ is independent of $n|_{\vec{E}_1\setminus\vec{E}_0}$. The local property then follows immediately. $\square$

3.2 The representation of the inversion as a vertex repelling jump process

Let $\mathcal{N}^\lambda$ be a sourceless α-random network associated to λ as in Definition 3.1 and X^λ be a process distributed, conditionally on $\mathcal{N}^\lambda = n$, as $X^{\lambda,n}$. By Proposition 3.2 and Theorem 3.3, X^λ has the law of $X_{[0,\tau_u]}$ conditioned on $\phi^{(u)} = \lambda$. The goal of this subsection is to show the following proposition, so as to obtain Theorem 2.9. Recall the distribution $\mathbb{P}^\lambda_{x_0}$ introduced in §2.4.1.

Proposition 3.9. *For any $\lambda \in \mathfrak{R}$, X^λ has the law $\mathbb{P}^\lambda_{x_0}$.*

In other words, X^λ is a jump process on V with jump rates given by (2.7). Recall the definition of Λ_t and Θ_t in (2.5) and (3.2) respectively. The key to Proposition 3.9 is the following lemma, which reveals a renewal property of the remaining crossings of X^λ, i.e. $\Theta_t(\mathcal{N}^\lambda, X^\lambda_{[0,t]}) = \Theta_t(n, X^\lambda_{[0,t]})\big|_{n=\mathcal{N}^\lambda}$. The proof is given in §3.2.1.

Lemma 3.10. *For any $\lambda \in \mathfrak{R}$, conditionally on $t < T^{X^\lambda}$ and $\left(X^\lambda_s : 0 \le s \le t\right)$, the network $\Theta_t(\mathcal{N}^\lambda, X^\lambda_{[0,t]})$ is an α-random network with sources (x_0, X^λ_t) associated to Λ_t.*

Remark. In the statement of the lemma and below, a network with sources (x_0, x_0) has to be understood as a sourceless network.

Proof of Proposition 3.9. By Lemma 3.10, for $\lambda \in \mathfrak{R}$, conditionally on $t < T^{X^\lambda}$ and $\left(X^\lambda_s : 0 \le s \le t\right)$ with $X^\lambda_t = x$, for any $y \sim x$,

- if $x = \mathfrak{p}(y)$, $\Theta_t(\mathcal{N}^\lambda)(xy)$ follows a Bessel ($\alpha - 1$, $\varphi_t(xy)$) distribution;
- if $y = \mathfrak{p}(x)$, $\Theta_t(\mathcal{N}^\lambda)(xy) - 1$ follows a Bessel (α, $\varphi_t(xy)$) distribution.

Note that if further conditioned on $\Theta_t(\mathcal{N}^\lambda)$, the process jumps to y at time t with rate

$$\frac{\check{\Theta}_t(\mathcal{N}^\lambda)(xy)}{\Lambda_t(x)}.$$

So using Corollary A.10 and averaging over $\Theta_t(\mathcal{N}^\lambda)(xy)$, we get the probability of a jump of X^λ from x to y during $[t, t + \Delta t]$ is

$$\begin{cases} \Big(I_\alpha\left(\varphi_t(xy)\right)^{-1} \sum_{k\ge 0} \dfrac{k+\alpha}{\Lambda_t(x)} \dfrac{\left(\varphi_t(xy)/2\right)^{2k+\alpha}}{k! \cdot \Gamma(k+\alpha+1)}\Big)\Delta t + o(\Delta t), & \text{if } y = \mathfrak{p}(x); \\ \Big(I_{\alpha-1}\left(\varphi_t(xy)\right)^{-1} \sum_{k\ge 0} \dfrac{k}{\Lambda_t(x)} \dfrac{\left(\varphi_t(xy)/2\right)^{2k+\alpha-1}}{k! \cdot \Gamma(k+\alpha)}\Big)\Delta t + o(\Delta t), & \text{if } x = \mathfrak{p}(y), \end{cases}$$

which gives the jump rates (2.7). □

3.2.1 Proof of Lemma 3.10

For any $x \in V$, set

$$\begin{aligned} \mathfrak{R}^x &:= \{\lambda \in \mathfrak{R} : \lambda(x) > 0\}, \\ \mathfrak{N}^x = \mathfrak{N}^{\lambda,x} &:= \big\{n \in \mathfrak{N} : \partial n = (x_0, x), \forall xy \in \vec{E} \text{ with } y = \mathfrak{p}(x), \\ &\qquad n(xy) \ge 1 \text{ if and only if } x, y \in \text{supp}\lambda\big\}. \end{aligned} \tag{3.7}$$

In particular, $\mathfrak{R} = \mathfrak{R}^{x_0}$ and $\mathfrak{N} = \mathfrak{N}^{x_0}$. First, let us generalize the notation X^λ, $X^{\lambda,n}$ and $\mathcal{N}^\lambda$. Forgetting the original definition before, we construct $\Big\{X^{\lambda,n}_{[0,T^{X^{\lambda,n}}]} : \lambda \in \mathfrak{R}, n \in \bigcup_{x\in V} \mathfrak{N}^x\Big\}$, $\left\{\mathcal{N}^\lambda : \lambda \in \mathfrak{R}\right\}$ and $\left\{\mathbb{P}_x : x \in V\right\}$ a family of stochastic processes, random networks, and probability measures respectively on the same measurable space, such that for any $x \in V$, $\lambda \in \mathfrak{R}^x$, and $n \in \mathfrak{N}^x$, under $\mathbb{P}_x$,

- $X^{\lambda,n}$ is a process that starts at x, has the jump rates $r_t^{\lambda,n}$ (as defined in (3.3)) and the same resurrect mechanism as the vertex-edge repelling jump process, and stops at $T^{X^{\lambda,n}}$ the time when the process exhausts the local time at x_0 or explodes;
- $\mathcal{N}^\lambda$ is an α-random network with sources (x_0, x) associated to λ;
- X^λ is a process distributed, conditionally on $\mathcal{N}^\lambda = n$, as $X^{\lambda,n}$.

It is easy to see that for $\lambda \in \mathfrak{R}$ and $n \in \mathfrak{N}$, under $\mathbb{P}_{x_0}$, X^λ, $X^{\lambda,n}$ and $\mathcal{N}^\lambda$ are consistent with the original definition.

We start the proof with an observation. To emphasize the tree where X^λ is defined, let us write $X^\lambda = X^{\lambda,\mathbb{T}}$. Any subtree $\mathbb{T}_0 = (V_0, E_0)$ containing x_0 is automatically equipped with conductances $(C_{xy})_{xy\in\vec{E}_0}$ and no killing. The induced CTMC is exactly the print of X on V_0 (recall §2.1). The following restriction principle is obtained using a similar argument to the last part of the proof of Proposition 3.4 (the proof of the local property).

Proposition 3.11 (Restriction principle)**.** *For $\lambda \in \mathfrak{R}$ and any subtree $\mathbb{T}_0 = (V_0, E_0)$ containing x_0, the print of $X^{\lambda,\mathbb{T}}$ on V_0 has the same law as $X^{\lambda|_{\mathbb{T}_0},\mathbb{T}_0}$.*

Observe that by the restriction principle, it suffices to tackle the case where $\mathbb{T}$ is finite, which we will assume henceforth.

For $\lambda \in \mathfrak{R}^x$, consider X^λ under $\mathbb{P}_x$. In the following, we simply write $\Theta_t(\mathcal{N}^\lambda)$ for $\Theta_t(\mathcal{N}^\lambda, X^\lambda_{[0,t]})$ and denote by $J_1 = J_1^{X^\lambda}$ the first jump time of X^λ. For $0 < t \le \lambda(x)$, let $\lambda^t(y) := \lambda(y) - t \cdot 1_{\{y=x\}}$ for $y \in V$. We will show that under $\mathbb{P}_x$,

(i) for any $0 < t \le \lambda(x)$, conditionally on $X^\lambda_s = x$ on $[0,t]$, $\Theta_t(\mathcal{N}^\lambda)$ is an α-random network with sources (x_0, x) associated to λ^t;

(ii) for any $y \sim x$, conditionally on $J_1 \le \lambda(x)$ and $X^\lambda_{J_1} = y$, $\Theta_{J_1}(\mathcal{N}^\lambda)$ is an α-random network with sources (x_0, y) associated to λ^{J_1}.

Note that once (i) and (ii) are proved, we have conditionally on a stay or jump at the beginning of X^λ,

(a) the remaining crossings $\Theta_t(\mathcal{N}^\lambda)$ is distributed as an α-random network associated to the remaining local time field;

(b) the process in the future is distributed as $X^{\lambda'}$ under $\mathbb{P}_y$, where λ' is the remaining local time and y equals x or the vertex it jumps to accordingly. In fact, it is simple to deduce from the strong renewal property of $X^{\lambda,n}$ (see Corollary A.9) an analogous property for X^λ that reads as follows: for any stopping time S, conditionally on $S < T^{X^\lambda}$ and $\left(X^\lambda_t : 0 \le t \le S\right)$ with $X^\lambda_S = y$ and $\Lambda_S(\cdot, X^\lambda_{[0,S]}) = \lambda'$, the process after S i.e. $\left(X^\lambda_{S+t} : 0 \le t \le T^{X^\lambda} - S\right)$ has the same law as $X^{\lambda',\mathcal{N}'}$ under $\mathbb{P}_y$, where $\mathcal{N}'$ is a random network following the conditional distribution of $\Theta_S(\mathcal{N}^\lambda)$. Then the statement follows from (a).

Under $\mathbb{P}_{x_0}$, iteratively using (a) and (b), we have after a chain of stays or jumps, $\Theta_t(\mathcal{N}^\lambda)$ keeps being distributed as an α-random network associated to the remaining local time, which leads to the conclusion.

We present the proof of (ii), and the proof of (i) is similar. First, consider the case $J_1 = \lambda(x)$. Notice that this event has a positive probability under $\mathbb{P}_x$ only when $\alpha = 0$, $x \neq x_0$, $y = \mathfrak{p}(x)$ and $\mathcal{N}^\lambda(x) = \mathcal{N}^\lambda(xy) = 1$. In this case, $\mathcal{N}^\lambda$ is a 0-random network with sources (x_0, x) associated to λ conditioned on $\mathcal{N}^\lambda(x) = 1$. Since $\Theta_{J_1}(\mathcal{N}^\lambda)(ij) = \mathcal{N}^\lambda(ij) - 1_{\{ij=xy\}}$ for $ij \in \vec{E}$, it is easily seen that $\Theta_{J_1}(\mathcal{N}^\lambda)$ is a 0-random network with sources (x_0, x) associated to λ^{J_1}.

Now we focus on the case where $J_1 < \lambda(x)$. Recall the α-random network defined in Definition 3.1. A simple calculation shows that the law of an α-random network with sources (x_0, i) associated to λ is given by: for any $n \in \mathbb{N}^{\vec{E}}$,

$$K^{\lambda,i}(n) = 1_{\{\partial n=(x_0,i)\}}\sigma_\alpha^i(\lambda)^{-1}\sqrt{\frac{\lambda(x_0)}{\lambda(i)}}\prod_{x\in\mathrm{supp}(\lambda)}\lambda(x)^{n(x)+\frac{\alpha-1}{2}\deg(x)}\prod_{x,y\in\mathrm{supp}(\lambda)}\frac{\big(C^*_{xy}\big)^{\check{n}(xy)}}{\Gamma(\check{n}(xy)+1)},$$

where $\deg(x) := \#\{y \in V : y \sim x\}$ and

$$\sigma_\alpha^i := \prod_{\{x,y\}\in\mathfrak{p}(x_0,i)} I_\alpha\big(2C^*_{xy}\sqrt{\lambda(x)\lambda(y)}\big)\prod_{\{x,y\}\in\mathrm{supp}(\lambda)\setminus\mathfrak{p}(x_0,i)} I_{\alpha-1}\big(2C^*_{xy}\sqrt{\lambda(x)\lambda(y)}\big).$$

For $r \in \mathfrak{N}^y$, set $r'(ij) = r(ij) + 1_{\{ij=xy\}}$ for $ij \in \vec{E}$. Then for any Borel subset $D \subset (0, u)$,

$$\begin{aligned}&\mathbb{P}_x\left(J_1\in D,\, X^\lambda_{J_1}=y,\, \Theta_{J_1}(\mathcal{N}^\lambda)=r\right)\\ &= K^{\lambda,x}(r')\cdot\mathbb{P}_x\Big(J_1(X^{\lambda,r'})\in D,\, X^{\lambda,r'}_{J_1(X^{\lambda,r'})}=y\Big)\\ &= K^{\lambda,x}(r')\int_D\left(\frac{\lambda^s(x)}{\lambda(x)}\right)^{\check{r}'(x)}\frac{\check{r}'(xy)}{\lambda^s(x)}\,\mathrm{d}s = \int_D C(\lambda,s)K^{\lambda^s,y}(r)\,\mathrm{d}s,\end{aligned} \tag{3.8}$$

where the second equality is due to (A.1) and in the last expression,

$$C(\lambda,s) := \frac{K^{\lambda,x}(r')}{K^{\lambda^s,y}(r)}\left(\frac{\lambda^s(x)}{\lambda(x)}\right)^{\check{r}'(x)}\frac{\check{r}'(xy)}{\lambda^s(x)} = \frac{\sigma^y_\alpha(\lambda^s)}{\sigma^x_\alpha(\lambda)}\sqrt{\frac{\lambda(x_0)\lambda(y)}{\lambda^s(x_0)\lambda(x)}}\left(\frac{\lambda(x)}{\lambda^s(x)}\right)^{(1-\alpha)1_{\{x\neq x_0\}}},$$

which is independent of r. Summing over $r \in \mathfrak{N}^x$ in (3.8), we get

$$\mathbb{P}_x\left(J_1\in D,\, X^\lambda_{J_1}=y\right) = \int_D C(\lambda,s)\,\mathrm{d}s. \tag{3.9}$$

By the monotone class methods, we can replace '1_D' in (3.9) by any non-negative measurable function on $\mathbb{R}^+$ vanishing on $[u,\infty)$. In particular,

$$\mathbb{E}_x\Big(K^{\lambda^{J_1},y}(r);J_1\in D,\, X^\lambda_{J_1}=y\Big) = \int_D C(\lambda,s)K^{\lambda^s,y}(r)\,\mathrm{d}s. \tag{3.10}$$

Comparing (3.8) and (3.10), we have

$$\mathbb{P}_x\left(\Theta_{J_1}(\mathcal{N}^\lambda)=r,\, J_1\in D,\, X^\lambda_{J_1}=y\right) = \mathbb{E}_x\Big(K^{\lambda^{J_1},y}(r);\, J_1\in D,\, X^\lambda_{J_1}=y\Big).$$

We have thus reached (ii).

Remark 3.12 (Connection to the process of Sabot and Tarrès [24])**.** When V is finite, $\alpha = 1/2$, and the conductances are symmetric, the jump rates in (2.7) coincide with that in [24, Theorem 5]. Precisely, in [24], the authors express the jump rates as $C_{xy}\sqrt{\frac{\Lambda_t(y)}{\Lambda_t(x)}}\frac{\langle\sigma_y\rangle_t}{\langle\sigma_x\rangle_t}$. Here $(\sigma_z)_{z\in V}$ is the Ising model with boundary condition $\sigma_{x_0} = +1$ on the graph G with interaction $\varphi_t(zw)$ at two neighboured vertices z and w. And $\langle\sigma_x\rangle_t$ is the expectation of σ_x under the law of the Ising model. By the expansion of Ising model in [7, (9)] and the tree structure in our case, we have

$$\langle\sigma_x\rangle_t = 2^{|V|}\sum_{n\in\mathbb{N}^{\vec{E}}:\partial n=(x_0,x)}\prod_{\{i,j\}\in E}\frac{(\varphi_t(ij))^{n(ij)+n(ji)}}{\big(n(ij)+n(ji)\big)!}$$

$$
\begin{aligned}
&= 2^{|V|} \sum_{n\in\mathbb{N}^E} \prod_{\{i,j\}\in\mathfrak{p}(x_0,x)} \frac{(\varphi_t(ij))^{2n_{\{i,j\}}+1}}{(2n_{\{i,j\}}+1)!} \prod_{\{i,j\}\in E\setminus\mathfrak{p}(x_0,x)} \frac{(\varphi_t(ij))^{2n_{\{i,j\}}}}{(2n_{\{i,j\}})!} \\
&= 2^{|V|}\pi^{\frac{|E|}{2}} \sum_{n\in\mathbb{N}^E} \prod_{\{i,j\}\in\mathfrak{p}(x_0,x)} \frac{(\varphi_t(ij)/2)^{2n_{\{i,j\}}+1}}{n_{\{i,j\}}!\Gamma(n_{\{i,j\}}+\frac{3}{2})} \prod_{\{i,j\}\in E\setminus\mathfrak{p}(x_0,x)} \frac{(\varphi_t(ij)/2)^{2n_{\{i,j\}}}}{n_{\{i,j\}}!\Gamma(n_{\{i,j\}}+\frac{1}{2})} \\
&= 2^{|V|}\left(\frac{\pi}{2}\varphi_t(ij)\right)^{\frac{|E|}{2}} \prod_{\{i,j\}\in\mathfrak{p}(x_0,x)} I_{1/2}\left(\varphi_t(ij)\right) \prod_{\{i,j\}\in E\setminus\mathfrak{p}(x_0,x)} I_{-1/2}\left(\varphi_t(ij)\right).
\end{aligned}
$$

It follows that $C_{xy}\sqrt{\frac{\Lambda_t(y)}{\Lambda_t(x)}\frac{\langle\sigma_y\rangle_t}{\langle\sigma_x\rangle_t}}$ equals the rates in (2.7).

3.3 Continuity with respect to initial local time fields

Recall the notation T_0^λ, T_∞^λ defined in §2.4.1. Under $\mathbb{P}_{x_0}^\lambda$, there are two possible cases that the process stops at a finite time: either the process exhausts the local time at x_0, i.e. $T_0^\lambda<\infty$, or there is an explosion, i.e. $T_\infty^\lambda<\infty$. In this part, we will show that the measure $\mathbb{P}_{x_0}^\lambda 1_{\{T_0^\lambda<\infty\}}$ is continuous with respect to λ. The result will be used in §5.

Consider an increasing sequence of finite subtrees $\mathbb{T}_i=(V_i,E_i)$ of $\mathbb{T}$, such that $x_0\in V_i$ for all i and $V_i\uparrow V$. (When $\mathbb{T}$ is finite, we can directly take $\mathbb{T}_i=\mathbb{T}$ for all i.) For any $\lambda\in(\mathbb{R}^+)^V$, the local uniform norm on $\mathbb{T}$ is defined as:

$$
\|\lambda\|_{\mathbb{T},\mathrm{loc}}=\sum_{i\geq 1}2^{-i}\big(1\wedge\max\{\lambda(x):x\in V_i\}\big), \tag{3.11}
$$

where for different choices of $(\mathbb{T}_i)_{i\geq 1}$, the norms are equivalent.

Theorem 3.13. *Let $\{\lambda^{(i)}\}_{i\geq1}$ and λ be in $\mathfrak{R}$, such that $\lambda^{(i)}\xrightarrow{\|\cdot\|_{\mathbb{T},loc}}\lambda$. If $\alpha=0$, further require that supp$(\lambda^{(i)})=$ supp(λ) for all i. Then $\mathbb{P}_{x_0}^{\lambda^{(i)}}1_{\{T_0^\lambda<\infty\}}$ converges vaguely to $\mathbb{P}_{x_0}^{\lambda}1_{\{T_0^\lambda<\infty\}}$.*

Remark. Here for the topology in Ω, we consider every path in Ω with a finite lifetime to stay at a cemetery point Δ after the death. Then the topology on $D_{V_\Delta}[0,\infty)$ ($V_\Delta:=V\cup\{\Delta\}$) induces the topology on Ω.

Proof. We only give the proof for the case $\lambda(x_0)=\lambda^{(i)}(x_0)$, from which the general case easily follows. Fix $\{\lambda^{(i)}\}_{i\geq1}$, λ in $\mathfrak{R}$ satisfying the above condition and the conditions in the statement. It suffices to prove that for any $A:=A\big(x_0\overset{[0,\infty)}{\to}x_1\overset{[0,\infty)}{\to}\cdots\overset{[0,\infty)}{\to}x_l\overset{\infty}{\to}\big)$ with $x_{l-1}=x_0$ and $x_l=\Delta$ (see the beginning of Appendix A and (A.2) for the notation σ and A),

$$
\mathbb{P}_{x_0}^{\lambda^{(i)}}(\,\cdot\,;A)\to\mathbb{P}_{x_0}^{\lambda}(\,\cdot\,;A)\ \text{ in total variation.} \tag{3.12}
$$

Proof of (3.12) **in the case $\alpha>0$.** We denote $u=\lambda(x_0)=\lambda^{(i)}(x_0)$. Note that conditionally on $T_0^\lambda<\infty$, it holds that $T_0^\lambda(\omega)=\tau_{u-}^\omega$ a.s. By the path probability (A.7) shown in Appendix A (also recall Remark A.7), we have for any bounded measurable function Φ on Ω,

$$
\mathbb{E}_{x_0}^{\lambda}(\Phi\,;A)=\int\Phi(\sigma)d_\sigma^\lambda\prod_{j=1}^{l-1}\mathrm{d}t_j \tag{3.13}
$$

where $\sigma=\sigma(x_0\xrightarrow{t_1}x_1\xrightarrow{t_2}\cdots\xrightarrow{t_l}x_l\xrightarrow{\infty})$ and

$$
d_\sigma^\lambda:=1_{E_\lambda}(\sigma)\exp\left(-\int_0^{\tau_{u-}^\sigma}\sum_{y:y\sim\sigma_v}r_v^\lambda(\sigma_v,y,\sigma)\,\mathrm{d}v\right)\prod_{j=1}^{l-1}r_{t_j}^\lambda(x_{j-1},x_j,\sigma), \tag{3.14}
$$

with $E_\lambda = \{\sigma \in \Omega : t_1 < \cdots < t_l = \tau^\sigma_{u-} \text{ and } \Lambda_{\tau^\sigma_{u-}}(x,\sigma) > 0 \;\forall x \neq x_0\}$. Note that $\sigma \in A$ is determined by its jump times $t_1, \cdots, t_{l-1}$. By Scheffé's lemma, it suffices to prove

(i) $d^{\lambda^{(i)}}_\sigma \to d^\lambda_\sigma$ for Lebesgue-a.s. $t_1, \cdots, t_{l-1}$;

(ii) $\mathbb{P}^{\lambda^{(i)}}_{x_0}(A) \to \mathbb{P}^\lambda_{x_0}(A)$.

Since $\{\lambda^{(i)}\}$ and λ with $\lambda^{(i)} \to \lambda$, any $\sigma \in E_\lambda$ is also in $E_{\lambda^{(i)}}$ for i sufficiently large. For such i, by (2.8), we have $\lim\limits_{v\uparrow\tau^\sigma_{u-}} r^\lambda_v(\sigma_v, y, \sigma)$ and $\lim\limits_{v\uparrow\tau^\sigma_{u-}} r^{\lambda^{(i)}}_v(\sigma_v, y, \sigma)$ exists for any $y \sim x_0$, and $\lim\limits_{v\uparrow\tau^\sigma_{u-}} r^{\lambda^{(i)}}_v(\sigma_v, y, \sigma) \to \lim\limits_{v\uparrow\tau^\sigma_{u-}} r^\lambda_v(\sigma_v, y, \sigma)$ as $i \uparrow \infty$. So it follows easily from the dominated convergence theorem that $d^{\lambda^{(i)}}_\sigma \to d^\lambda_\sigma$ for any $\sigma \in E_\lambda$. This concludes (i).

For (ii), recall that X^λ (defined in the beginning of §3.2) has the law $\mathbb{P}^\lambda_{x_0}$. We take any finite subtree $\mathbb{T}_0 = (V_0, E_0)$ of $\mathbb{T}$ such that $x_i \in V_0 \setminus \partial V_0$[7] for all $i = 0, 1, \cdots, l$. The conclusion follows immediately from the fact that

- the law of $\mathcal{N}^\lambda|_{\vec{E}_0}$ is continuous with respect to λ;
- $\mathbb{P}(X^{\lambda,n} \in A)$, as a function of (λ, n), depends only on $n|_{\vec{E}_0}$. (The space of networks on $\mathbb{T}_0$ is equipped with the discrete topology. So any function of these networks is automatically continuous.)

Proof of (3.12) **in the case $\alpha = 0$.** In this case, we can further assume $\{x_i\}_{0\le i\le l} = \text{supp}(\lambda)$. The proof is quite similar to the case $\alpha > 0$, except that one needs to note that after the last visit at any vertex x, it will definitely exhaust the remaining local time at x and then jump to $\mathfrak{p}(x)$ if $x \neq x_0$. So (3.13) should be modified as follows: For $0 \le j < l$, we keep those j's such that there exists $j < j' < l$ such that $x_{j'} = x_j$, i.e. the j-th jump is not the last visit to x_j. Denote the collection of all these j's as $\{i_1, \cdots, i_d\}$, where $d = l - \#\text{supp}(\lambda)$. The integral on the right-hand side of (3.13) should be replaced by the integral with respect to $\prod_{j=1}^d \mathrm{d}t_{i_j+1}$, that is the Lebesgue measure on $\mathbb{R}^d$, and the other t_j's in the expression are determined by: ($t_0 := 0$)

$$\sum_{i=1}^{l} 1_{\{x_{i-1}=x\}}(t_i - t_{i-1}) = \lambda(x) \text{ for any } x \in \text{supp}(\lambda).$$

The remaining part of the proof is exactly the same as the case $\alpha > 0$. ☐

4 Inverting percolation Ray-Knight identity

In this section, we only consider the case where the conductances are symmetric and $\alpha \in (0, 1)$. In §4.1, we will introduce the metric-graph Brownian motion. It turns out that the metric-graph Brownian motion together with the associated percolation process gives a realization of $\mathcal{X}$ defined in §2.3, which leads to the percolation Ray-Knight identity (Theorem 2.5). §4.2 is devoted to the proof of the inversion of percolation Ray-Knight identity (Theorem 2.10).

4.1 Percolation Ray-Knight identity

Replacing every edge $e = \{x, y\}$ of $\mathbb{T}$ by an interval I_e with length $1/2C_{xy}$, it defines the metric graph associated to $\mathbb{T}$, denoted by $\widetilde{\mathbb{T}}$. V is considered to be a subset of $\widetilde{\mathbb{T}}$. One can naturally construct a metric-graph Brownian motion B on $\widetilde{\mathbb{T}}$, i.e. a diffusion that behaves like a Brownian motion inside each edge, performs Brownian excursion inside each adjacent edge when hitting a vertex in V and is killed at each vertex $x \in V$

[7] $\partial V_0 := \{x \in V_0 : \text{there exists } y \in V \setminus V_0 \text{ such that } y \sim x\}$.

with rate k_x. Let $\widetilde{\mathcal{L}}$ be the loop soup associated to B. Then X and B, resp. $\mathcal{L}$ and $\widetilde{\mathcal{L}}$, can be naturally coupled through restriction (i.e. X, resp. $\mathcal{L}$, is the print of B, resp. $\widetilde{\mathcal{L}}$, on V), which we will assume from now on. See [15] for more details of metric graphs, metric-graph Brownian motions, and the above couplings. Notation such as $L_\cdot(\widetilde{\mathcal{L}})$ and $\widetilde{\mathcal{L}}^{(u)}$ $(u \ge 0)$ are similarly defined for $\widetilde{\mathcal{L}}$ as for $\mathcal{L}$. Assume that B starts from x_0. We also have the Ray-Knight identity for B, that is to replace $\mathcal{L}^{(u)}, X, \tau_u$ in Theorem 2.2 by $\widetilde{\mathcal{L}}^{(u)}, B, \tau_u^B$ respectively.

Let B and $\widetilde{\mathcal{L}}^{(0)}$ be independent. On $\{\tau_u < T^X\} = \{\tau_u^B < T^B\}$, we set

$$\widetilde{\phi}_t(x) := L_x(\widetilde{\mathcal{L}}^{(0)}) + L^B(Q^{-1}(t), x), \text{ for } x \in \widetilde{\mathbb{T}} \text{ and } 0 \le t \le \tau_u.$$

where $Q(t) = \sum_{x\in V} L^B(t,x)$ and Q^{-1} is the right-continuous inverse. By the coupling of B and X, it holds that $\widetilde{\phi}_t|_V = \phi_t$ (defined in (2.2)). Next, for $t \ge 0$, $\widetilde{\mathcal{O}}_t$ will denote a percolation on E defined as: for $e \in E$,

$$\widetilde{\mathcal{O}}_t(e) = 1_{\left\{\widetilde{\phi}_t \text{ has no zero on } I_e\right\}}.$$

Recall the notation defined in §2.3. We will show the following proposition, which immediately implies the percolation Ray-Knight identity in Theorem 2.5.

Proposition 4.1. *Conditionally on $\tau_u < T^X$,*

(i) $(\widetilde{\phi}_{\tau_u}|_V, \widetilde{\mathcal{O}}_{\tau_u})$ *has the same law as* $(L_\cdot(\mathcal{L}^{(u)}), \mathcal{O}^{(u)})$*;*

(ii) $\widetilde{\mathcal{X}} := (X_t, \widetilde{\mathcal{O}}_t)_{0\le t\le\tau_u}$ *has the same law as* $\mathcal{X} = (X_t, \mathcal{O}_t)_{0\le t\le\tau_u}$.

Before the proof of Proposition 4.1, we present two lemmas in terms of the loop soups $\mathcal{L}$ and $\widetilde{\mathcal{L}}$, which will later be translated into the analogous versions associated to $\widetilde{\mathcal{X}}$ using the Ray-Knight identity.

Define $O(\mathcal{L}), O(\widetilde{\mathcal{L}})$ on E as follows: for $e \in E$,

$$O(\widetilde{\mathcal{L}})(e) := 1_{\{L_\cdot(\widetilde{\mathcal{L}}) \text{ has no zero on } I_e\}},\ O(\mathcal{L})(e) := 1_{\{e \text{ is crossed by } \mathcal{L}\}}.$$

Remember that $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ are naturally coupled. So it holds that $O(\widetilde{\mathcal{L}}) \ge O(\mathcal{L})$. Moreover, they have the following relations.

Lemma 4.2. *Conditionally on $\mathcal{L}$,* $\Big(O(\widetilde{\mathcal{L}})(e) : e \in E,\, O(\mathcal{L})(e) = 0\Big)$ *is a family of independent random variables, and*

$$\begin{aligned}&\mathbb{P}\left(O(\widetilde{\mathcal{L}})(\{x,y\}) = 0 \,\middle|\, L_\cdot(\mathcal{L}) = \ell, O(\mathcal{L})(\{x,y\}) = 0\right)\\ &= \begin{cases} 2\Gamma(1-\alpha)^{-1}\left(C_{xy}\sqrt{\ell_x\ell_y}\right)^{1-\alpha} K_{1-\alpha}\left(2C_{xy}\sqrt{\ell_x\ell_y}\right), & \text{if } \ell_x,\ell_y > 0;\\ 1, & \text{if } \ell_x \wedge \ell_y = 0.\end{cases}\end{aligned} \tag{4.1}$$

Remark. This lemma is still true on general graphs.

Lemma 4.3. *Conditionally on $L_\cdot(\mathcal{L})$,* $\Big(O(\widetilde{\mathcal{L}})(e) : e \in E\Big)$ *is a family of independent random variables, and*

$$\mathbb{P}\left(O(\widetilde{\mathcal{L}})(\{x,y\}) = 1 \,\middle|\, L_\cdot(\mathcal{L}) = \ell\right) = \begin{cases} \dfrac{I_{1-\alpha}\left(2C_{xy}\sqrt{\ell_x\ell_y}\right)}{I_{\alpha-1}\left(2C_{xy}\sqrt{\ell_x\ell_y}\right)}, & \text{if } \ell_x,\ell_y > 0,\\ 0, & \text{if } \ell_x\wedge\ell_y = 0.\end{cases} \tag{4.2}$$

Proof of Lemma 4.2. The independence follows from an argument using the Ray-Knight identity of B and the excursion theory similar to that in the proof of Proposition 3.2.

With the same idea as [15, §3], conditionally on $O(\mathcal{L})(\{x,y\}) = 0$, the print of $\widetilde{\mathcal{L}}$ in $I_{\{x,y\}}$ consists of the loops entirely contained in $I_{\{x,y\}}$, excursions from and to x (resp. y) inside $I_{\{x,y\}}$ of loops in $\widetilde{\mathcal{L}}$ visiting x (resp. y). By considering the contribution to the local time field by each part, we have that the left-hand side of (4.1) is the same as the probability that the sum of three independent processes

$$\left(a_t^{(h)} + b_t^{(h,\ell_x)} + b_{h-t}^{(h,\ell_y)}\right)_{0\le t\le h} \tag{4.3}$$

has a zero on $(0,h)$. Here $h = 1/2C_{xy}$, $(a_t^{(h)})_{0\le t\le h}$ is a $\mathrm{BESQ}^{2\alpha,h}_{0\to 0}$ (i.e. a 2α-dimensional BESQ bridge from 0 to 0 over $[0,h]$) and $(b_t^{(h,l)})_{0\le t\le h}$ is a $\mathrm{BESQ}^{0,h}_{l\to 0}$.

For (4.3) to have a zero, the process $b_t^{(h,\ell_x)}$ has to hit 0 before the last zero of $(a_t^{(h)} + b_{h-t}^{(h,\ell_y)})_{0\le t\le h}$. The density of the first zero of $b_t^{(h,\ell_x)}$ is

$$1_{\{0<t<h\}}\frac{\ell_x}{2t^2}\exp\left(-\frac{(h-t)\ell_x}{2ht}\right)\mathrm{d}t. \tag{4.4}$$

To get this, one can start with the well-known fact that the first zero of a 2δ-dimensional BESQ process starting from x is distributed as $x^2/2\mathrm{Gamma}(1-\delta,1)$ (Cf. for example [13, Proposition 2.9]). Then use the fact that for a 2δ-dimensional BESQ process ρ_t starting from x, the process $h(1-u/h)^2\rho_{u/(h-u)}$ $(0\le u\le h)$ is a $\mathrm{BESQ}^{2\delta,h}_{x\to 0}$.

Since $(a_t^{(h)} + b_{h-t}^{(h,\ell_y)})_{0\le t\le h}$ has the same law as a $\mathrm{BESQ}^{2\alpha,h}_{0\to\ell_y}$, its last zero has the same law as h minus the first zero of a $\mathrm{BESQ}^{2\alpha,h}_{\ell_y\to 0}$, which has the density

$$1_{\{0<t<h\}}\frac{2^{\alpha-1}}{\Gamma(1-\alpha)}h^{\alpha}\ell_y^{1-\alpha}t^{-\alpha}(h-t)^{\alpha-2}\exp\left(-\frac{\ell_y t}{2h(h-t)}\right)\mathrm{d}t. \tag{4.5}$$

Gathering (4.4), (4.5) and taking $h = 1/2C_{xy}$, we get the probability of (4.3) having a zero is

$$\begin{aligned}
&\int_0^h\int_0^t \frac{\ell_x}{2s^2}\exp\left(-\frac{(h-s)\ell_x}{2hs}\right)\frac{2^{\alpha-1}}{\Gamma(1-\alpha)}h^{\alpha}\ell_y^{1-\alpha}t^{-\alpha}(h-t)^{\alpha-2}\exp\left(-\frac{\ell_y t}{2h(h-t)}\right)\mathrm{d}s\,\mathrm{d}t\\
&= \frac{2^{\alpha-1}}{\Gamma(1-\alpha)}\ell_y^{1-\alpha}h^{\alpha}\exp\left(\frac{\ell_y+\ell_x}{2h}\right)\int_0^h \exp\left(-\frac{\ell_x}{2t}-\frac{\ell_y}{2(h-t)}\right)t^{-\alpha}(h-t)^{\alpha-2}\,\mathrm{d}t\\
&= \frac{1}{\Gamma(1-\alpha)}\int_0^\infty \exp\left(-\frac{\ell_x\ell_y}{(2h)^2 s}-s\right)\frac{\mathrm{d}s}{s^{\alpha}}\\
&= \begin{cases} 2\Gamma(1-\alpha)^{-1}\left(C_{xy}\sqrt{\ell_x\ell_y}\right)^{1-\alpha}K_{1-\alpha}\left(2C_{xy}\sqrt{\ell_x\ell_y}\right), & \text{if } \ell_x,\ell_y>0,\\ 1, & \text{if } \ell_x\wedge\ell_y = 0,\end{cases}
\end{aligned}$$

where in the second equality, we use the change of variable $s = \dfrac{\ell_y t}{2h(h-t)}$ and the last equality follows from [22, (136)]. □

Proof of Lemma 4.3. By Proposition 3.2,

$$\mathbb{P}\left(O(\mathcal{L})(\{x,y\}) = 0 | L_{\cdot}(\mathcal{L}) = \ell\right) = \begin{cases} \dfrac{\left(C_{xy}\sqrt{\ell_x\ell_y}\right)^{\alpha-1}}{\Gamma(\alpha)I_{\alpha-1}(2C_{xy}\sqrt{\ell_x\ell_y})}, & \text{if } \ell_x,\ell_y>0,\\ 1, & \text{if } \ell_x\wedge\ell_y=0.\end{cases} \tag{4.6}$$

The probability we are interested in equals 1 minus the multiplication of (4.1) and (4.6). □

Proof of Proposition 4.1. By the Ray-Knight identity of B, we have $\widetilde{\phi}_{\tau_u}$ has the law of $L_\cdot(\widetilde{\mathcal{L}}^{(u)})$. So it follows from Lemma 4.3 that $(\widetilde{\phi}_{\tau_u}|_V, \widetilde{\mathcal{O}}_{\tau_u})$ has the same law as $(L_\cdot(\mathcal{L}^{(u)}), \mathcal{O}^{(u)})$. That concludes (i).

For (ii), if X jumps through the edge $\{x,y\}$, then B crosses the interval $I_{\{x,y\}}$, which makes $\widetilde{\phi}_t$ positive on this interval. So $\widetilde{\mathcal{O}}_t(\{x,y\})$ turns to 1.

It remains to calculate the rate of opening an edge without X jumping. The key observation is that conditionally on $\widetilde{\mathcal{X}}_{[0,t]}$, if $\widetilde{\mathcal{O}}_t(\{x,y\}) = 0$, then $\widetilde{\phi}_t|_{I_{\{x,y\}}}$ has the same law as the occupation field of $\widetilde{\mathcal{L}}$ on $I_{\{x,y\}}$ conditioned on $L_x(\widetilde{\mathcal{L}}) = \phi_t(x)$ and $L_y(\widetilde{\mathcal{L}}) = \phi_t(y)$, which is obtained by considering the print of B on the branch at x containing y and using the Ray-Knight identity of B. It follows easily that $(\widetilde{\mathcal{X}}, \phi)$ is a Markov process, which allows us to condition only on $(\widetilde{\mathcal{X}}_t, \phi_t)$ when calculating the rate. Note that the case of opening $\{x,y\}$ without X jumping can happen only when X stays at x and $x = \mathfrak{p}(y)$. We use $\{X_{[t,t+\Delta t]} = x\}$ to stand for the event that X stays at x during $[t, t+\Delta t]$. Denote by A_1 the event that at some time in $[t, t+\Delta t]$, $\{x,y\}$ is opened without X jumping. And let $A_2 := A_1 \cap \{X_{[t,t+\Delta t]} = x\}$. For $(\widetilde{\mathcal{X}}_t, \phi_t)$ with $X_t = x$ and $\widetilde{\mathcal{O}}_t(\{x,y\}) = 0$, if we write $\mathbb{Q} = \mathbb{P}(\cdot \mid X_t = x, \phi_t)$, then

$$
\begin{aligned}
\mathbb{P}(A_2 \mid \widetilde{\mathcal{X}}_t, \phi_t) &= \mathbb{Q}(\widetilde{\mathcal{O}}_{t+\Delta t}(\{x,y\}) = 1, X_{[t,t+\Delta t]} = x \mid \widetilde{\mathcal{O}}_t(\{x,y\}) = 0) \\
&= \mathbb{Q}(\widetilde{\mathcal{O}}_t(\{x,y\}) = 0)^{-1} \cdot \Big[\mathbb{Q}(\widetilde{\mathcal{O}}_t(\{x,y\}) = 0, X_{[t,t+\Delta t]} = x) \\
&\quad - \mathbb{Q}(\widetilde{\mathcal{O}}_{t+\Delta t}(\{x,y\}) = 0, X_{[t,t+\Delta t]} = x)\Big] \\
&= \mathbb{Q}(X_{[t,t+\Delta t]} = x) \cdot \left[1 - \frac{\mathbb{Q}(\widetilde{\mathcal{O}}_{t+\Delta t}(\{x,y\}) = 0 \mid X_{[t,t+\Delta t]} = x)}{\mathbb{Q}(\widetilde{\mathcal{O}}_t(\{x,y\}) = 0)}\right].
\end{aligned}
$$

Again using the observation above, we have the fraction in the above square brackets equals

$$
\frac{\mathbb{P}\big(O(\widetilde{\mathcal{L}})(\{x,y\}) = 0 \mid O(\mathcal{L})(\{x,y\}) = 0, L_x(\mathcal{L}) = \phi_t(x) + \Delta t, L_y(\mathcal{L}) = \phi_t(y)\big)}{\mathbb{P}\big(O(\widetilde{\mathcal{L}})(\{x,y\}) = 0 \mid O(\mathcal{L})(\{x,y\}) = 0, L_x(\mathcal{L}) = \phi_t(x), L_y(\mathcal{L}) = \phi_t(y)\big)},
$$

which further equals $f(\phi_t(x) + \Delta t)/f(\phi_t(x))$ with

$$
f(s) := 2\Gamma(1-\alpha)^{-1} \left(C_{xy}\sqrt{s\phi_t(y)}\right)^{1-\alpha} K_{1-\alpha}\left(2C_{xy}\sqrt{s\phi_t(y)}\right)
$$

by Lemma 4.2. Therefore,

$$
\mathbb{P}(A_2 \mid \widetilde{\mathcal{X}}_t, \phi_t) = -\frac{f'(\phi_t(x))}{f(\phi_t(x))}\Delta t + o(\Delta t). \tag{4.7}
$$

Observe that

$$
\begin{aligned}
A_1 \setminus A_2 \subset &\{X \text{ has at least 2 jumps in } [t, t+\Delta t]\} \bigcup \\
&\{\exists\, 0 < \delta_1 < \delta_2 \le \Delta t, \text{ s.t. } \{x,y\} \text{ is opened at } t+\delta_1, \\
&X_{[t,t+\delta_2)} = x, \text{ and } X \text{ jumps at } t+\delta_2\}.
\end{aligned}
$$

The (conditional) probability of the first event on the right-hand side is obviously $o(\Delta t)$; while we can use the same argument as the proof of Lemma A.11 to show the probability of the second event is also $o(\Delta t)$. So

$$
\mathbb{P}(A_1 \mid \widetilde{\mathcal{X}}_t, \phi_t) = \mathbb{P}(A_2 \mid \widetilde{\mathcal{X}}_t, \phi_t) + o(\Delta t) = -\frac{f'(\phi_t(x))}{f(\phi_t(x))}\Delta t + o(\Delta t),
$$

which leads to the rate in (2.3) by using $[z^\nu K_\nu(z)]' = -z^\nu K_{\nu-1}(z)$ and $K_{-\nu} = K_\nu$. $\square$

Remark 4.4. In this remark, we will average over $\mathcal{O}_{(0,\tau_u)}$ in the process $\overleftarrow{X}_{[0,\tau_u]}$ and describe the annealed law of $(X_{[0,\tau_u]}, \mathcal{O}_0, \mathcal{O}_{\tau_u})$. In particular, we will see that when $\alpha = 1/2$, the annealed law coincides with that in Theorem 8 in the paper of Lupu, Sabot, and Tarrès [18].

In the process $\overleftarrow{X}_t = (X_t, \mathcal{O}_t)$, an edge $\{x,y\}$, with $x = \mathfrak{p}(y)$, is open in $\mathcal{O}_{\tau_u}$ if either one of the following cases happens: (1) it is open in $\mathcal{O}_0$; (2) it is crossed by $X_{[0,\tau_u]}$; (3) it is opened during X staying at x.

Observe that conditionally on $\tau_u < T^X$ and $(L_\cdot(\mathcal{L}^{(0)}), \mathcal{O}_0, X_{[0,\tau_u]})$, if neither case (1) or (2) happens, then the probability that case (3) happens equals

$$\begin{cases} 1 - \exp\Big(- \int_{\phi_0(x)}^{\phi_{\tau_u}(x)} C_{xy} \sqrt{\frac{\phi_0(y)}{s}} \cdot \frac{K_\alpha(2C_{xy}\sqrt{s\phi_0(y)})}{K_{1-\alpha}(2C_{xy}\sqrt{s\phi_0(y)})} \,\mathrm{d}s \Big)^8, & \text{if } \phi_0(y) > 0; \\ 0, & \text{if } \phi_0(y) = 0. \end{cases} \tag{4.8}$$

It can be seen from the proof of Proposition 4.1 that the integrand on the first line in (4.8) equals $-\frac{f'(s)}{f(s)}$ with f defined as (4.7) (recall that $\phi_t(y) \equiv \phi_0(y)$ under the condition). It follows that the first probability in (4.8) further equals

$$1 - \sqrt{\frac{\phi_{\tau_u}(x)}{\phi_0(x)}} \cdot \frac{K_{1-\alpha}(2C_{xy}\sqrt{\phi_{\tau_u}(x)\phi_0(y)})}{K_{1-\alpha}(2C_{xy}\sqrt{\phi_0(x)\phi_0(y)})}.$$

The above results yield that if we average over $\mathcal{O}_{(0,\tau_u)}$, then $(\mathcal{O}_0, X_{[0,\tau_u]})$ has the same law as before and conditionally on them, the open edges in $\mathcal{O}_{\tau_u}$ consist of the open edges in $\mathcal{O}_0$, the edges crossed by $X_{[0,\tau_u]}$, and additional edges opened conditionally independently with probability

$$\begin{cases} 1 - \sqrt{\frac{\phi_{\tau_u}(x)}{\phi_0(x)}} \cdot \frac{K_{1-\alpha}(2C_{xy}\sqrt{\phi_{\tau_u}(x)\phi_0(y)})}{K_{1-\alpha}(2C_{xy}\sqrt{\phi_0(x)\phi_0(y)})}, & \text{if } \phi_0(y) > 0; \\ 0, & \text{if } \phi_0(y) = 0. \end{cases} \tag{4.9}$$

When $\alpha = 1/2$, using $K_{1/2}(z) = \left(\frac{\pi}{2z}\right)^{1/2} e^{-z}$, we have the first probability in (4.9) equals

$$1 - \exp\Big\{ - 2C_{xy}\sqrt{\phi_0(y)}(\sqrt{\phi_{\tau_u}(x)} - \sqrt{\phi_0(x)}) \Big\}. \tag{4.10}$$

The relation between the above coupling of $\mathcal{O}_0$ and $\mathcal{O}_{\tau_u}$ and Theorem 8 in [18] is as follows. In [18] the authors consider the coupling of two Gaussian free fields with different boundary conditions. The key part is the coupling between the sign clusters of the Gaussian free fields on the metric graph with boundary conditions 0 and $\sqrt{2u}$. By Lupu's isomorphism theorem (see [18, Theorem 3] or [15, §2]), it is equivalent to the coupling of $\mathcal{O}_0$ and $\mathcal{O}_{\tau_u}$. With this points of view, if we assume the Gaussian free fields in [18, Theorem 8] are coupled with metric-graph loop soups as in Lupu's isomorphism theorem, then the probability in (4.10) coincides with that in Theorem 8 in [18]. Hence the two couplings are essentially the same.

4.2 Proof of Theorem 2.10

Let $\overleftarrow{\mathcal{X}} = (\overleftarrow{X}_t, \overleftarrow{\mathcal{O}}_t)_{0 \le t \le \tau_u}$ be the time reversal of $\mathcal{X}$, i.e. $\overleftarrow{\mathcal{X}} = (\mathcal{X}_{\tau_u - t})_{0 \le t \le \tau_u}$. In this part, we will verify that for any $\lambda \in \mathfrak{R}$ and configuration O on E, the jump rate of $\overleftarrow{\mathcal{X}}$ conditionally on $\tau_u < T^X$ and $\left(\phi^{(u)}, \mathcal{O}^{(u)}\right) = (\lambda, O)$ is given by (2.9), which leads to Theorem 2.10.

[8] Note that under the condition, y is not visited by $X_{[0,\tau_u]}$. Thus $\phi_t(y) \equiv \phi_0(y)$, $t \in [0,\tau_u]$.

Recall the notation in (2.5). In the following, we will use Λ_t and φ_t to represent $\Lambda_t(\lambda, \overleftarrow{X}_{[0,t]})$ and $\varphi_t(\lambda, \overleftarrow{X}_{[0,t]})$ respectively. Note that given $\phi^{(u)} = \lambda$, the aggregated local time $\phi_{\tau_u - t}$ equals Λ_t. Let us condition on $\tau_u < T^X$, $\left(\phi^{(u)}, \mathcal{O}^{(u)}\right) = (\lambda, O)$ and the path $\overleftarrow{\mathcal{X}}_{[0,t]}$ with $\overleftarrow{X}_t = x$, and calculate the jump rates related to the edge $\{x,y\}$, i.e. the rate of the jump from x to y by $\overleftarrow{X}_t$ without modifying $\overleftarrow{\mathcal{O}}_t$, or the closure of $\{x,y\}$ in $\overleftarrow{\mathcal{O}}_t$ without $\overleftarrow{X}_t$ jumping, or the jump and closure simultaneously. Due to the Markov property of $\mathcal{X}$, it suffices to condition only on $(\overleftarrow{\mathcal{X}}_t, \Lambda_t)$. For simplicity, denote by $\mathbb{Q} = \mathbb{P}(\cdot \mid \overleftarrow{X}_t = x, \Lambda_t) = \mathbb{P}(\cdot \mid X_{\tau_u - t} = x, \phi_{\tau_u - t})$ henceforth. Be careful that in the definition of $\mathbb{Q}$, we do not condition on $\overleftarrow{\mathcal{O}}_t$. The jump rates are analyzed in the following two cases.

(1) if $y = \mathfrak{p}(x)$, then there are two possible jumps: $\overleftarrow{X}_t$ jumps from x to y without modifying $\overleftarrow{\mathcal{O}}_t$, or it jumps with the closre of $\{x,y\}$ in $\overleftarrow{\mathcal{O}}_t$. Since $\overleftarrow{X}_t = x$, it holds that $\overleftarrow{\mathcal{O}}_t(\{x,y\}) = 1$. This allows us to ignore the condition on $\overleftarrow{\mathcal{O}}_t(\{x,y\})$. Note that the law of $\overleftarrow{X}$ given $\phi^{(u)} = \lambda$ and $\tau_u < T^X$ is also $\mathbb{P}^{\lambda}_{x_0}$ (defined in §2.4). So the rate of the jump from x to y by $\overleftarrow{X}$ at time t is $C_{xy}\sqrt{\dfrac{\Lambda_t(y)}{\Lambda_t(x)}} \cdot \dfrac{I_{\alpha-1}(\varphi_t(xy))}{I_{\alpha}(\varphi_t(xy))}$ as shown in (2.7).

We further consider the probability that the jump is accompanied by the closure of $\{x,y\}$ in $\overleftarrow{\mathcal{O}}_t$. Observe that this happens if and only if $\mathcal{O}_{(\tau_u - t)-}(\{x,y\}) = 0$. Conditionally on Λ_t and $\overleftarrow{X}$ jumping from x to y at time t, $\mathcal{O}_{(\tau_u - t)-}(\{x,y\})$ has the same law as $O(\widetilde{\mathcal{L}})(\{x,y\})$ given $L_x(\widetilde{\mathcal{L}}) = \Lambda_t(x)$ and $L_y(\widetilde{\mathcal{L}}) = \Lambda_t(y)$. By Lemma 4.3, the conditional probability that $\mathcal{O}_{(\tau_u - t)-}(\{x,y\}) = 0$ is

$$1 - \frac{I_{1-\alpha}(\varphi_t(xy))}{I_{\alpha-1}(\varphi_t(xy))}. \tag{4.11}$$

Therefore, $\overleftarrow{X}$ jumps from x to y and $\overleftarrow{\mathcal{O}}(\{x,y\})$ turns to 0 at time t with rate

$$\begin{aligned} &C_{xy}\sqrt{\frac{\Lambda_t(y)}{\Lambda_t(x)}} \cdot \frac{I_{\alpha-1}(\varphi_t(xy))}{I_{\alpha}(\varphi_t(xy))} \cdot \Big(1 - \frac{I_{1-\alpha}(\varphi_t(xy))}{I_{\alpha-1}(\varphi_t(xy))}\Big) \\ &= C_{xy}\sqrt{\frac{\Lambda_t(y)}{\Lambda_t(x)}} \cdot \frac{I_{\alpha-1}(\varphi_t(xy)) - I_{1-\alpha}(\varphi_t(xy))}{I_{\alpha}(\varphi_t(xy))}; \end{aligned}$$

while $\overleftarrow{X}$ jumps from x to y at time t without modifying $\overleftarrow{\mathcal{O}}_t$ with rate

$$C_{xy}\sqrt{\frac{\Lambda_t(y)}{\Lambda_t(x)}} \cdot \frac{I_{1-\alpha}(\varphi_t(xy))}{I_{\alpha}(\varphi_t(xy))}.$$

(2) if $x = \mathfrak{p}(y)$, the possible jumps are: $\overleftarrow{X}_t$ jumps from x to y without modifying $\overleftarrow{\mathcal{O}}_t$, or $\{x,y\}$ is closed without $\overleftarrow{X}_t$ jumping. Denote by A_1 (resp. A_2) the event that there is a jump of the first (resp. second) kind during $[t, t+\Delta t]$. Note that A_1 can happen only when $\overleftarrow{\mathcal{O}}_{t-}(\{x,y\}) = \overleftarrow{\mathcal{O}}_t(\{x,y\}) = 1$, since there exists $s > t$ such that $\overleftarrow{X}$ jumps from y to x at time s. We have

$$\begin{aligned} &\mathbb{P}(A_1 \mid \overleftarrow{X}_t = x, \overleftarrow{\mathcal{O}}_t(\{x,y\} = 1, \Lambda_t) = \mathbb{Q}(A_1)/\mathbb{Q}(\overleftarrow{\mathcal{O}}_t(\{x,y\}) = 1) \\ &= C_{xy}\sqrt{\frac{\Lambda_t(y)}{\Lambda_t(x)}} \cdot \frac{I_{\alpha}\left(\varphi_t(xy)\right)}{I_{\alpha-1}\left(\varphi_t(xy)\right)}\Delta t \cdot \left(\frac{I_{1-\alpha}(\varphi_t(xy))}{I_{\alpha-1}(\varphi_t(xy))}\right)^{-1} + o(\Delta t) \\ &= C_{xy}\sqrt{\frac{\Lambda_t(y)}{\Lambda_t(x)}} \cdot \frac{I_{\alpha}\left(\varphi_t(xy)\right)}{I_{1-\alpha}\left(\varphi_t(xy)\right)}\Delta t + o(\Delta t). \end{aligned} \tag{4.12}$$

Now we turn to A_2. Observe that A_2 is also the event that at some time in $[\tau_u - t - \Delta t, \tau_u - t]$, $\{x, y\}$ is opened in $\mathcal{O}$ without X jumping. Let $A_3 = A_2 \cap \{X_{[\tau_u - t - \Delta t, \tau_u - t]} = x\}$. Then

$$\begin{aligned}&\mathbb{P}(A_3 \,|\, \overleftarrow{X}_t = x, \overleftarrow{\mathcal{O}}_t(\{x,y\}) = 1, \Lambda_t)\\ &= \mathbb{Q}(\mathcal{O}_{\tau_u - t - \Delta t}(\{x,y\}) = 0, X_{[\tau_u - t - \Delta t, \tau_u - t]} = x \,|\, \mathcal{O}_{\tau_u - t}(\{x,y\}) = 1)\\ &= \mathbb{Q}(X_{[\tau_u - t - \Delta t, \tau_u - t]} = x \,|\, \mathcal{O}_{\tau_u - t}(\{x,y\}) = 1)\\ &- \mathbb{Q}(\mathcal{O}_{\tau_u - t - \Delta t}(\{x,y\}) = 1, X_{[\tau_u - t - \Delta t, \tau_u - t]} = x \,|\, \mathcal{O}_{\tau_u - t}(\{x,y\}) = 1) =: q_1 - q_2.\end{aligned}$$

We observe that

$$q_1 = 1 - C_{xy}\sqrt{\frac{\Lambda_t(y)}{\Lambda_t(x)}} \cdot \frac{I_\alpha\left(\varphi_t(xy)\right)}{I_{1-\alpha}\left(\varphi_t(xy)\right)}\Delta t + o(\Delta t)$$

by (4.12); while if we denote $h(s) = \frac{I_{1-\alpha}(2C_{xy}\sqrt{s\Lambda_t(y)})}{I_{\alpha-1}(2C_{xy}\sqrt{s\Lambda_t(y)})}$, then by Lemma 4.3,

$$\begin{aligned} q_2 &= q_1 \mathbb{P}(\mathcal{O}_{\tau_u - t - \Delta t} = 1 \,|\, X_{\tau_u - t - \Delta t} = x, \Lambda_{\tau_u - t - \Delta t} = \ell - \Delta t \cdot \delta_x)|_{\ell = \Lambda_t}\\ &= q_1 h(\Lambda_t - \Delta t) = q_1[h(\Lambda_t) - h'(\Lambda_t)\Delta t]\\ &= 1 - C_{xy}\sqrt{\frac{\Lambda_t(y)}{\Lambda_t(x)}} \cdot \frac{I_{-\alpha}\left(\varphi_t(xy)\right)}{I_{1-\alpha}\left(\varphi_t(xy)\right)}\Delta t + o(\Delta t).\end{aligned}$$

So $\mathbb{P}(A_3 \,|\, \overleftarrow{X}_t = x, \overleftarrow{\mathcal{O}}_t(\{x,y\}) = 1, \Lambda_t)$ equals

$$C_{xy}\sqrt{\frac{\Lambda_t(y)}{\Lambda_t(x)}} \cdot \frac{I_{-\alpha}(\varphi_t(xy)) - I_\alpha(\varphi_t(xy))}{I_{1-\alpha}(\varphi_t(xy))}\Delta t + o(\Delta t). \tag{4.13}$$

Similar to the last part of the proof of Proposition 4.1, we can show that $\mathbb{P}(A_2 \setminus A_3 \,|\, \overleftarrow{X}_t = x, \overleftarrow{\mathcal{O}}_t(\{x,y\}) = 1, \Lambda_t) = o(\Delta t)$. Hence $\mathbb{P}(A_2 \,|\, \overleftarrow{X}_t = x, \overleftarrow{\mathcal{O}}_t(\{x,y\}) = 1, \Lambda_t)$ also equals (4.13). We have thus shown all the rates in (2.9). The rates already determine the process (Cf. Appendix A). So we are done.

Remark 4.5 (Connection to the process of Lupu, Sabot, and Tarrès [18, §3])**.** When $\alpha = 1/2$, using $I_{1/2}(z) = \left(\frac{2}{\pi z}\right)^{1/2}\sinh(z)$ and $I_{-1/2}(z) = \left(\frac{2}{\pi z}\right)^{1/2}\cosh(z)$, we get the jump rates in (2.9) can be simplified as

$$\begin{cases} 1_{\{O_1(\{x,y\})=1\}} C_{xy}\sqrt{\dfrac{\Lambda_t(y)}{\Lambda_t(x)}}, & \text{if } x \neq y, O_1 = O_2,\\ 2C_{xy}\sqrt{\dfrac{\Lambda_t(y)}{\Lambda_t(x)}} \cdot \left(e^{2\varphi_t(xy)} - 1\right)^{-1}, & \text{if } O_2 = O_1 \setminus \{x, z\} \text{ for some } z \sim x.\end{cases}$$

As in Remark 4.4, if we assume the Gaussian free fields in [18] are coupled with the metric-graph loop soups as in Lupu's isomorphism theorem, then the above jump rates coincide with that of $(\check{X}_t, \check{\mathcal{C}}_t)$ defined in Proposition 3.4 in [18].

5 Mesh limits of repelling jump processes

Recall the setting in §2.5. In this section, we first introduce the self-repelling diffusion B^λ which inverts the Ray-Knight identity of reflected Brownian motion, and then show that B^λ is the mesh limit of vertex repelling jump processes as stated in Theorem 2.11.

Throughout the section, B is a reflected Brownian motion on $\mathbb{R}^+$; $\widetilde{\mathcal{L}}^{(0)}$ and $\widetilde{\mathcal{L}}^{(u)}$ are the loop soups associated to B conditioned on their local time at 0 are 0 and u respectively; $\left(\widetilde{\phi}^{(0)}, B_{[0,\tau_u^B]}, \widetilde{\phi}^{(u)}\right)$ is a Ray-Knight triple associated to B with starting point 0.

5.1 Self-repelling diffusions

First, we define a process $M = M^\alpha$ closely related to the Ray-Knight triple. When $\alpha = 0$, $M = (M_t : t \geq 0)$ is a reflected Brownian motion on $\mathbb{R}^+$; when $\alpha > 0$, $M = (M_t : t \in \mathbb{R})$ is the print of a two-sided perturbed reflected Brownian motion with parameter $1/\alpha$ on $\mathbb{R}^+$ which can be constructed as

$$M_t = \begin{cases} |\beta'_{-t}| + \alpha^{-1} L^{\beta'}(-t, 0), & t < 0, \\ |\beta_t|, & t \geq 0, \end{cases}$$

where $(\beta_t, t \geq 0)$ and $(\beta'_t, t \geq 0)$ are two independent Brownian motions.

When $\alpha > 0$, $M_{(-\infty,0]}$ and $\widetilde{\mathcal{L}}^{(0)}$ is linked as follows (Cf. [16, §5]): by reading off the excursions of M above the minimum process $t \mapsto \inf_{s \in (-\infty,t]} M_s$, $t \in (-\infty, 0]$, and forgetting the starting points of the excursions, we get a collection of loops which have the same law as $\widetilde{\mathcal{L}}^{(0)}$; conversely, by rooting the loops in $\widetilde{\mathcal{L}}^{(0)}$ at their minimum and 'concatenating' them together (see [16, §5] for the details of this concatenation), we get a process distributed as $M_{(-\infty,0]}$. In particular, it can be deduced from the above link that the local time field of $M_{(-\infty,0]}$ has the same law as $\widetilde{\mathcal{L}}^{(0)}$. (Both are distributed as a BESQ$^{2\alpha}$ process starting from 0. See [16, Proposition 4.6].)

We denote by $L^M(\tau_u^M, \cdot)$ the local time field of $M_{[0,\tau_u^M]}$ (when $\alpha = 0$) or $M_{(-\infty,\tau_u^M]}$ (when $\alpha > 0$). It is seen that

- $M_{[0,\tau_u^M]}$ has the law of $B_{[0,\tau_u^B]}$;
- when $\alpha > 0$, the local time field of $M_{(-\infty,0]}$ is independent of $M_{[0,\tau_u^M]}$ and distributed as $\widetilde{\mathcal{L}}^{(0)}$.

This implies that $M_{[0,\tau_u^M]}$ conditioned on $L^M(\tau_u^M, \cdot)$ has the same law as $B_{[0,\tau_u^B]}$ conditioned on $\widetilde{\phi}^{(u)}$.

In the following, we shall present the conditional law of $M_{[0,\tau_u^M]}$ shown in [25] ($\alpha = 0$) and [2] ($\alpha > 0$). It is shown that by performing a time-space transformation of $M_{[0,\tau_u^M]}$ which depends only on the local time $L^M(\tau_u^M, \cdot)$ (and some extra randomness in the case $\alpha = 0$), we get a process independent of $L^M(\tau_u^M, \cdot)$, which we call burglar. So to get the conditional law, one can start with the burglar and make the inverse transformation using the given time field. The idea also comes from the Perkins' disintegration theorem (Cf. [1, §3]). The details are as follows.

For any non-negative continuous function f on $\mathbb{R}^+$ with $f(0) > 0$, denote by $\mathfrak{d}_f$ the hitting time of 0 by f, i.e. $\mathfrak{d}_f := \inf\{x > 0 : f(x) = 0\} \in (0, \infty]$. We call f admissible if

$$\int_0^{\mathfrak{d}_f} f(x)^{-1} \, \mathrm{d}x = \infty.$$

Let $\widetilde{\mathfrak{R}}$ be the set of non-negative, continuous, admissible functions λ on $\mathbb{R}^+$ with $\lambda(0) > 0$ when $\alpha > 0$ and further with $\mathfrak{d}_\lambda < \infty$ and $\lambda(x) = 0$ for all $x \geq \mathfrak{d}_\lambda$ when $\alpha = 0$.

For $f \in \widetilde{\mathfrak{R}}$, define the following change of scale η_f and change of time C_f associated to a deterministic continuous process R on $\mathbb{R}^+$ starting from 0:

$$\eta_f(y) = \int_0^y f(x)^{-1} \, \mathrm{d}x, y \in [0, \mathfrak{d}_f), \; C_f(t) = C_f^R(t) = \int_0^t f(R_s)^{-2} ds, \; t \in [0, H_{\mathfrak{d}_f}^R).$$

For $0 < a \leq H_{\mathfrak{d}_f}^R$, we define $\Phi(R_{[0,a)}, f)$ to be the process $\Big(\eta_f\big(R_{C_f^{-1}(t)}\big) : t \in [0, C_f(a))\Big)$. And $\Phi(R_{[0,a]}, f)$ can be defined for $0 < a < H_{\mathfrak{d}_f}^R$.

Denote $h(x) = L^M\big(\tau_u^M, x\big)$, which has the law of a BESQ$^{2\alpha}$ process starting from u. It follows from the properties of BESQ processes that $h \in \widetilde{\mathfrak{R}}$ a.s. When $\alpha > 0$, define $\mathcal{Z} := \Phi\big(M_{[0,\tau_u^M]}, h\big)$.

Proposition 5.1 ([2, Theorem 5.15]). *$\mathcal{Z}$ is independent of h.*

When $\alpha = 0$, we divide $M_{[0,\tau_u^M]}$ into two parts: the processes before and after $T_{\max}$, where $T_{\max}$ is the time when $M_{[0,\tau_u^M]}$ attains its maximum. Set

$$h^{(1)}(x) = L^M(T_{\max}, x), \quad h^{(2)}(x) = L^M\big(\tau_u^M, x\big) - L^M(T_{\max}, x),$$
$$\mathcal{Z}^{(1)} = \Phi(M_{[0,T_{\max})}, h^{(1)}), \quad \mathcal{Z}^{(2)} = \Phi\big((M_{\tau_u^M - t} : 0 \le t < \tau_u^M - T_{\max}), h^{(2)}\big).$$

Proposition 5.2 ([25, Theorem 9]). *The three processes $\mathcal{Z}^{(1)}$, $\mathcal{Z}^{(2)}$ and h are independent. $\mathcal{Z}^{(1)}$ and $\mathcal{Z}^{(2)}$ have the same law. Moreover, there exists an independent Jacobi diffusion $\mathcal{J}^{2,2}$ with dimensions 2 and 2 starting from a uniform point in $[0,1]$, i.e. a diffusion on $[0,1]$ with infinitesimal generator $2x(1-x)\,\mathrm{d}^2x + 2(1-2x)\,\mathrm{d}x$, such that*

$$h^{(1)}(x)/h(x) = \mathcal{J}^{2,2}_{\eta_h(x)},\ 0 \le x < \mathfrak{d}_h = M_{T_{\max}}.$$

The above $\mathcal{Z}^{(i)}$, resp. $\mathcal{Z}$, is called the burglar with parameter 0, resp. with parameter $\alpha(>0)$. Now given h, we can start with $\mathcal{Z}^{(1)}, \mathcal{Z}^{(2)}, \mathcal{J}^{2,2}$ (when $\alpha = 0$) or $\mathcal{Z}$ (when $\alpha > 0$), and perform the inverse transformation to derive the conditional law of $M_{[0,\tau_u^M]}$ given h. The detailed construction is as follows:

For $f \in \widetilde{\mathfrak{R}}$, define the following change of time K_f associated to a deterministic continuous process R on $\mathbb{R}^+$ starting from 0:

$$K_f(t) = K_f^R(t) = \int_0^t \big(f \circ \eta_f^{-1}(R_s)\big)^2\,\mathrm{d}s,\ t \in [0, H^R_{\mathfrak{d}_f}).$$

Set $\Phi^{-1}(R_{[0,a)}, f) := \Big(\eta_f^{-1}\big(R_{K_f^{-1}(t)}\big) : t \in [0, K_f(a))\Big)$ for $0 < a \le H^R_{\mathfrak{d}_f}$. We can check that for any a and f with $0 < a \le H^R_{\mathfrak{d}_f}$,

$$\Phi^{-1}\big(\Phi(R_{[0,a)}, f), f\big) = R_{[0,a)}.$$

In fact, denote $S_t = \Phi(R_{[0,a)]}, f) = \eta_f\big(R_{(C_f^R)^{-1}(t)}\big)$. It reduces to showing $R_{(K_f^S \circ C_f^R)^{-1}(t)} = R_t$. This follows from the fact that $(K_f^S \circ C_f^R)'(t) = \big(f \circ \eta_f^{-1}(S_{C_f^R(t)})\big)^2 \cdot f(R_t)^{-2} = 1$.

Definition 5.3. *For any $\lambda \in \widetilde{\mathfrak{R}}$, define the self-repelling diffusion B^λ as follows:*

- *when $\alpha > 0$, set*
$$B^\lambda := \Phi^{-1}(\mathcal{Z}, \lambda);$$
- *when $\alpha = 0$, let $\mathcal{Z}^{(1)}$, $\mathcal{Z}^{(2)}$ and $\mathcal{J}^{2,2}$ be three independent processes such that $\mathcal{Z}^{(1)}$ and $\mathcal{Z}^{(2)}$ are identically distributed as the burglar with parameter 0 and $\mathcal{J}^{2,2}$ is a Jacobi diffusion with dimensions 2 and 2 starting from a uniform point in $[0,1]$. Set $\lambda^{(1)} = \lambda(x)\mathcal{J}^{2,2}_{\eta_\lambda(x)}$, $\lambda^{(2)}(x) = \lambda(x) - \lambda^{(1)}(x)$ and $t^{(i)} = K^{Z^{(i)}}_{\lambda^{(i)}}(\infty)$[9] $(i = 1, 2)$. Define*

$$B_t^\lambda := \begin{cases} \Phi^{-1}(\mathcal{Z}^{(1)}, \lambda^{(1)}, \infty)(t), & 0 \le t \le t^{(1)}, \\ \Phi^{-1}(\mathcal{Z}^{(2)}, \lambda^{(2)}, \infty)(t^{(1)} + t^{(2)} - t), & t^{(1)} < t \le t^{(1)} + t^{(2)}. \end{cases} \tag{5.1}$$

Proposition 5.4 ([2, 25]). *For any $\lambda \in \widetilde{\mathfrak{R}}$, B^λ has the law of $M_{[0,\tau_u^M]}$ conditioned on $L^M(\tau_u^M, \cdot)$. Hence it also has the law of $B_{[0,\tau_u^B]}$ conditioned on $\widetilde{\phi}^{(u)} = \lambda$.*

It is known that B^λ admits bicontinuous local times $L^{B^\lambda}(t,x)$. We naturally generalize the notation Λ_t in (2.5) to the setting of continuous state space. In particular, $\Lambda_t(\lambda, B^\lambda)(x) = \lambda(x) - L^{B^\lambda}(t,x)$. The next proposition shows the continuity of the self-repelling diffusion with respect to the initial local time field, which is crucial in the proof of the convergence.

[9] It holds that $T^{Z^{(i)}} = \infty$ (Cf. [25]).

Proposition 5.5. *Let $\{a_k\}_{k\geq 1}$ be a sequence of random functions in $\widetilde{\mathfrak{R}}$ that converges weakly to a deterministic function λ in $\widetilde{\mathfrak{R}}$ for the local uniform topology on $[0,\infty)$. If $\alpha=0$, further require that $\mathfrak{d}_{a_k}$ converges weakly to $\mathfrak{d}_\lambda$. Let B^{a_k} and B^λ be the corresponding self-repelling diffusions. Then*

$$\left(B_t^{a_k}, T^{B^{a_k}}, \Lambda_t(a_k, B^{a_k})(x)\right)_{t\geq 0, x\in\mathbb{R}^+}$$

converge weakly to $\left(B_t^\lambda, T^{B^\lambda}, \Lambda_t(\lambda, B^\lambda)(x)\right)_{t\geq 0, x\in\mathbb{R}^+}$ for the uniform topology in t and the local uniform topology on $[0,\infty)$ in x.

Remark. In the statement of the proposition and below, we assume that all the vertex repelling jump processes and self-repelling diffusions stay at 0 after the lifetimes, and their local time fields stay still after that time.

Proof. By Skorokhod's representation theorem, we can assume that all a_k are defined on the same probability space and the convergences in the proposition are all almost sure. Besides, we assume that B^{a_k} and B^λ are constructed from the same processes $\mathcal{Z}^{(1)}$, $\mathcal{Z}^{(2)}$, $\mathcal{J}^{2,2}$. We will show that

$$\left(B_t^{a_k}, T^{B^{a_k}}, \Lambda_t(a_k, B^{a_k})(x)\right)_{t\geq 0, x\in\mathbb{R}^+} \tag{5.2}$$

converges almost surely to $\left(B_t^\lambda, T^{B^\lambda}, \Lambda_t(\lambda, B^\lambda)(x)\right)_{t\geq 0, x\in\mathbb{R}^+}$.

(1) Case $\boldsymbol{\alpha=0}$. Note that in this case, we have $T^{B^\lambda}=\int_0^{\mathfrak{d}}\lambda(x)\,\mathrm{d}x$ by Proposition 5.4. So the above assumptions imply that

$$T^{B^{a_k}}=\int_0^{\mathfrak{d}_{a_k}} a_k(x)\,\mathrm{d}x \to T^{B^\lambda}=\int_0^{\mathfrak{d}_\lambda}\lambda(x)\,\mathrm{d}x, \text{ a.s..} \tag{5.3}$$

From the definition (5.1), B^{a_k} and B^λ can be constructed from each other by a time-space transformation. Precisely, if we write $\Lambda_t^{a_k}$ and Λ_t^λ for $\Lambda_t(a_k, B^{a_k})$ and $\Lambda_t(\lambda, B^\lambda)$ respectively, then for $0\leq t\leq C_k^{(1)}(T_{\max}^{B^\lambda})$ and $x\in[0,\mathfrak{d}_{a_k})$,

$$\begin{cases} B_t^{a_k}=\left(\eta_k^{(1)}\right)^{-1}\left(B^\lambda_{(C_k^{(1)})^{-1}(t)}\right); \\ \Lambda_t^{a_k}(x)=a_k^{(1)}(x)\left(\lambda^{(1)}\circ\eta_k^{(1)}(x)\right)^{-1}\left[\Lambda^\lambda_{(C_k^{(1)})^{-1}(t)}\left(\eta_k^{(1)}(x)\right)-\lambda^{(2)}(x)\right]+a_k^{(2)}(x), \end{cases}$$

where

- $T_{\max}^{B^\lambda}$ is the time when B^λ attains its maximum;
- for $i=1,2$, $\eta_k^{(i)}=(\eta_{\lambda^{(i)}})^{-1}\circ\eta_{a_k^{(i)}}$ with $\lambda^{(1)}(x)=\lambda(x)\mathcal{J}^{2,2}_{\eta_\lambda(x)}$, $\lambda^{(2)}(x)=\lambda(x)-\lambda^{(1)}(x)$ and $a_k^{(1)}(x)=a_k(x)\mathcal{J}^{2,2}_{\eta_{a_k}(x)}$, $a_k^{(2)}(x)=a_k(x)-a_k^{(1)}(x)$;
- $C_k^{(1)}(t)=\int_0^t\left(a_k^{(1)}\circ\eta_k^{(1)}(B_s^\lambda)\right)^2\lambda^{(1)}(B_s^\lambda)^{-2}\,\mathrm{d}s$ for $t\in[0,T_{\max}^{B^\lambda}]$.

Observe that for any $m<\mathfrak{d}_\lambda$, we have $\inf_{0\leq x\leq m}\lambda^{(1)}(x)>0$ a.s.. Using this, we can readily check that for any $m<\mathfrak{d}_\lambda$ and $T<T_{\max}^{B^\lambda}$,

$$\sup_{x\in[0,m]}\left|\eta_k^{(1)}(x)-x\right|\vee\left|\left(\eta_k^{(1)}\right)^{-1}(x)-x\right|\to 0,\ \sup_{t\in[0,T]}\left|C_k^{(1)}(t)-t\right|\to 0. \tag{5.4}$$

In particular, fixing $\varepsilon>0$, we let $\zeta_\varepsilon^{(1)}$ be the last visit time of $\mathfrak{d}_\lambda-\varepsilon$ by B^λ before time $T_{\max}^{B^\lambda}$, and take $T=\zeta_\varepsilon^{(1)}$ and $m=\max\left\{B_t^\lambda: t\in[0,\zeta_\varepsilon^{(1)}]\right\}$ in (5.4). Further note the facts: (1) $\zeta_\varepsilon^{k,(1)}:=C_k^{(1)}(\zeta_\varepsilon^{(1)})$ is the last visit time of $\left(\eta_k^{(1)}\right)^{-1}(\mathfrak{d}_\lambda-\varepsilon)$ by B^{a_k} before $T_{\max}^{B^{a_k}}$ (the time when B^{a_k} attains its maximum), (2) $L^{B^\lambda}(t,x)$ is bicontinuous and hence so is $\Lambda_t(x)$. Using these, it can be deduced from (5.4) that a.s.

$$\begin{cases} \sup\left\{\left|B_t^{a_k}-B_t^\lambda\right| : t\in[0,\zeta_\varepsilon^{k,(1)}]\right\}\to 0, \\ \sup\left\{\left|\Lambda_t^{a_k}(x)-\Lambda_t^\lambda(x)\right| : t\in[0,\zeta_\varepsilon^{k,(1)}], x\in[0,\mathfrak{d}_\lambda-\varepsilon]\right\}\to 0. \end{cases} \tag{5.5}$$

We can also consider processes after time $T_{\max}^{B^\lambda}$. A similar argument together with (5.3) yields that

$$\begin{cases} \sup\left\{\left|B_t^{a_k}-B_t^\lambda\right| : t\in[\zeta_\varepsilon^{k,(2)},\infty]\right\}\to 0, \\ \sup\left\{\left|\Lambda_t^{a_k}(x)-\Lambda_t^\lambda(x)\right| : t\in[\zeta_\varepsilon^{k,(2)},\infty], x\in[0,\mathfrak{d}_\lambda-\varepsilon]\right\}\to 0, \end{cases} \tag{5.6}$$

where $\zeta_\varepsilon^{k,(2)}$ is the first hitting time of $\left(\eta_k^{(2)}\right)^{-1}(\mathfrak{d}_\lambda-\varepsilon)$ by B^{a_k} after $T_{\max}^{B^{a_k}}$.

Furthermore, observe that for k large enough (depending on ω), it holds that

(i) $\left(\eta_k^{(1)}\right)^{-1}(\mathfrak{d}_\lambda-\varepsilon)\wedge\left(\eta_k^{(2)}\right)^{-1}(\mathfrak{d}_\lambda-\varepsilon)>\mathfrak{d}_\lambda-2\varepsilon$ and $\mathfrak{d}_{a_k}<\mathfrak{d}_\lambda+\varepsilon$;

(ii) $\zeta_\varepsilon^{k,(1)}=C_k^{(1)}(\zeta_\varepsilon^{(1)})>\zeta_{2\varepsilon}^{(1)}:=$ the last visit time of $\mathfrak{d}_\lambda-2\varepsilon$ before $T_{\max}^{B^\lambda}$ by B^λ; similarly, $\zeta_\varepsilon^{k,(2)}<\zeta_{2\varepsilon}^{(2)}:=$ the first hitting time of $\mathfrak{d}_\lambda-2\varepsilon$ after $T_{\max}^{B^\lambda}$ by B^λ;

(iii) $a_k(x)\le\lambda(x)+\varepsilon$ for all x.

(i) and (ii) tell us $B_t^{a_k},B_t^\lambda\in(\mathfrak{d}_\lambda-2\varepsilon,\mathfrak{d}_\lambda+\varepsilon)$ for any $t\in(\zeta_\varepsilon^{k,(1)},\zeta_\varepsilon^{k,(2)})$; while (iii) implies that $\Lambda_t^{a_k}(x),\Lambda_t^\lambda(x)\le\sup_{x\in[\mathfrak{d}_\lambda-2\varepsilon,\mathfrak{d}_\lambda)}\lambda(x)+\varepsilon$ for all $t\ge 0$ and $x\ge\mathfrak{d}_\lambda-2\varepsilon$. These together with (5.5) and (5.6) lead to the conclusion.

(2) Case $\boldsymbol{\alpha>0}$. In this case, the time-space transformation is: for $t\ge 0$ and $x\in[0,\mathfrak{d}_{a_k})$,

$$\begin{cases} B_t^{a_k}=\eta_k\big(B_{C_k(t)}^\lambda\big); \\ \Lambda_t^{a_k}(x)=a_k(x)(\lambda\circ\eta_k(x))^{-1}\Lambda_{C_k(t)}^\lambda(\eta_k(x)); \\ T^{B^{a_k}}=C_k\big(T^{B^\lambda}\big), \end{cases}$$

where $\eta_k:=(\eta_\lambda)^{-1}\circ\eta_{a_k}$ and $C_k:=\int_0^t\big(a_k\circ\eta_k(B_s^\lambda)\big)^2\lambda(B_s^\lambda)^{-1}\,\mathrm{d}s$ for $t\in[0,T^{B^\lambda}]$. We can deduce from the last equality that $T^{B^{a_k}}\to T^{B^\lambda}$ a.s.. The other parts of the proof follow a similar way to the case $\alpha=0$. So we omit the details. □

5.2 Convergence of vertex repelling jump processes

Recall the notation in §2.5. Henceforth we fix $\lambda_k\in\mathfrak{R}_k$ and $\lambda\in\widetilde{\mathfrak{R}}$ such that λ_k (linearly interpolated outside $\mathbb{N}_k$) convergence to λ for the local uniform topology. Let Ω_k be the collection of the right-continuous, minimal paths on $\mathbb{T}_k$. For $k\ge 1$, $x,y\in\mathbb{N}_k$ with $x\sim y$ and $\omega\in\Omega_k$ with $T^\omega>t$, set

$$\begin{aligned} \Lambda_t^k(x,\omega)&=\lambda_k(x)-2^k\int_0^t 1_{\{\omega_s=x\}}\,\mathrm{d}s, \\ \varphi_t^k(xy,\omega)&=2C_{xy}^k\sqrt{\Lambda_t^k(x,\omega)\Lambda_t^k(y,\omega)}. \end{aligned} \tag{5.7}$$

Using Theorem 2.9, we can readily check that $X^{\lambda_k,(k)}$ is a jump process on $\mathbb{N}_k$ with jump rates $r^{\lambda,(k)}(x,y)$ and the similar resurrect mechanism and lifetime as before, where

$$r^{\lambda,(k)}(x,y):=\begin{cases} 2^{2k-1}\sqrt{\dfrac{\Lambda_t^k(y)}{\Lambda_t^k(x)}}\cdot\dfrac{I_{\alpha-1}(\varphi_t^k(xy))}{I_\alpha(\varphi_t^k(xy))}, & \text{if } x=\mathfrak{p}(y); \\ 2^{2k-1}\sqrt{\dfrac{\Lambda_t^k(y)}{\Lambda_t^k(x)}}\cdot\dfrac{I_\alpha(\varphi_t^k(xy))}{I_{\alpha-1}(\varphi_t^k(xy))}, & \text{if } y=\mathfrak{p}(x). \end{cases} \tag{5.8}$$

Our proof of Theorem 2.11 follows a similar way to [18]. The key step is Proposition 5.7 which enables us to realize vertex repelling jump processes as the prints of self-repelling diffusions on dyadic grids.

For any $k \geq 1$, let ϕ^{λ_k} be distributed as $\widetilde{\phi}^{(u)}$ conditioned on $\widetilde{\phi}^{(u)}(x) = \lambda_k(x)$ for all $x \in \mathbb{N}_k$. Precisely, ϕ^{λ_k} can be obtained by interpolating between values $\phi^{\lambda_k}(x) = \lambda_k(x)$ for consecutive points $x \in \mathbb{N}_k$ with independent $\mathrm{BESQ}^{2\alpha}$ bridges.

Remark 5.6. Note that if $\lambda_k(x) = 0$ for some $x \in \mathbb{N}_k \setminus \{0\}$, then the properties of BESQ bridge yield that ϕ^{λ_k} is admissible a.s..

Proposition 5.7. *Suppose the field λ_k satisfies ϕ^{λ_k} is admissible a.s.. Let $B^{\phi^{\lambda_k}}$ be the self-repelling diffusion with random initial local time field ϕ^{λ_k}. Denote $Q^k(t) := \sum_{x\in\mathbb{N}_k} L^{B^{\phi^{\lambda_k}}}(t,x)$. Then*

$$\left(B^{\phi^{\lambda_k}}_{(Q^k)^{-1}(2^k t)}\right)_{t\geq 0} \stackrel{d}{=} X^{\lambda_k,(k)}. \tag{5.9}$$

Proof. Recall $X^{(k)}$ defined in §2.5. It holds that

$$\left(X^{(k)}_{2^k t}, L^{X^{(k)}}(2^k t, x)\right)_{t\geq 0, x\in\mathbb{N}_k} \stackrel{d}{=} \left(B_{(\widetilde{Q}^k)^{-1}(2^k t)}, L^B\big((\widetilde{Q}^k)^{-1}(2^k t), x\big)\right)_{t\geq 0, x\in\mathbb{N}_k},$$

where $\widetilde{Q}^k(t) = \sum_{x\in\mathbb{N}_k} L^B(t,x)$. So for λ_k not fixed but randomly distributed as $\widetilde{\phi}^{(u)}|_{\mathbb{N}_k}$, (5.9) is a direct consequence of Theorem 2.9 and Theorem 5.4, which implies that (5.9) holds for a.s. λ_k with respect to the law of $\widetilde{\phi}^{(u)}|_{\mathbb{N}_k}$. Further note that by Theorem 2.9, we have for a.s. λ_k, $X^{\lambda_k,(k)}$ ends up exhausting the local time at x_0. Then the identity for all λ_k with ϕ^{λ_k} admissible a.s. follows from the continuity with respect to the initial time field of both sides (see Theorem 3.13 and Proposition 5.5). □

Proof of Theorem 2.11. First, we additionally assume that for any $k \geq 1$, λ_k satisfies that ϕ^{λ_k} is admissible a.s.. (The assumption automatically holds in the case $\alpha = 0$ by Remark 5.6) Then by Proposition 5.7, it suffices to prove $B^{\phi^{\lambda_k}}_{(Q^k)^{-1}(2^k\cdot)}$ converges weakly to B^λ. For this convergence, by Proposition 5.5, it is enough to prove:

(i) ϕ^{λ_k} converges weakly to λ for the local uniform topology on $[0,\infty)$. Besides, in the case $\alpha = 0$, $\mathfrak{d}_{\phi^{\lambda_k}}$ converges weakly to $\mathfrak{d}_\lambda$;

(ii) $t \mapsto \left(Q^k\right)^{-1}\left[2^k t \wedge Q^k\left(T^{B^{\phi^{\lambda_k}}}\right)\right]$ converges weakly to $t \mapsto t \wedge T^{B^\lambda}$ for the uniform topology.

The second convergence in (i) is immediate from the convergence of $\mathfrak{d}_{\lambda_k}$. For the first convergence, recall that ϕ^{λ_k} is obtained by interpolating with independent $\mathrm{BESQ}^{2\alpha}$ bridges. Denote by $\mathrm{Q}^{\delta,t}_{x,y}$ a BESQ^δ bridge from x to y over $[0,t]$. Fix $r \in \cup_k \mathbb{N}_k$. Let

$$M := \sup_{0\leq x\leq r} \lambda(x) \vee \sup\{\lambda_k(x) : k \geq 1,\ 0 \leq x \leq r\}.$$

For the convergence of ϕ^{λ_k}, it suffices to prove for any $\varepsilon > 0$, there exists $0 < \sigma < \varepsilon$, such that for all $x, y \in [0, M]$ with $|x - y| < \sigma$, the probability that a $\mathrm{Q}^{\delta,t}_{x,y}$ bridge deviates more than ε from x is $o(t)$, where $o(t)$ is uniform for all x, y. In fact, once it is proved, we consider k large enough such that $\sup_{x\in\mathbb{N}_k\cap[0,r]} |\lambda_k(x) - \lambda(x)| < \sigma/3$ and $|\lambda(x) - \lambda(y)| < \sigma/3$ for any $x, y \in [0, r]$ and $|x - y| < 2^{-k}$. Then the probability of the difference between λ and ϕ^{λ_k} on a subinterval $[i/2^k, (i+1)/2^k]$ of $[0, r]$ greater than 2ε is $o\left(2^{-k}\right)$, where $o\left(2^{-k}\right)$ depends on ε and is independent of i. The probability beats the 2^k factor, which implies the convergence.

For the proof of the above statement, we note that the BESQ bridge has the following independent decomposition (Cf. [21, (1.f)])

$$\mathrm{Q}^{\delta,t}_{x\to y} = \mathrm{Q}^{0,t}_{x\to 0} \oplus \mathrm{Q}^{0,t}_{0\to y} \oplus \mathrm{Q}^{\delta,t}_{0\to 0} \oplus \sum_{n=1}^{\infty} b_{\delta/2-1}(\sqrt{xy}, n)\mathrm{Q}^{4n,t}_{0\to 0},$$

where b is defined in (3.1). We shall first show that for any $\delta \geq 0$ and $\varepsilon > 0$,

$$\mathbb{P}\left(\mathrm{Q}^{\delta,t}_{0\to 0} \oplus \sum_{n=1}^{\infty} b_{\delta/2-1}(\sqrt{xy}, n)\mathrm{Q}^{4n,t}_{0\to 0} > \varepsilon\right) = o(t), \tag{5.10}$$

which implies that if the statement holds for a certain $\delta \geq 0$, then it automatically holds for all $\delta \geq 0$. We will then prove the statement in the special case $\delta = 1$.

Te get (5.10), note that the tail probability of the maximum of a standard BESQ bridge is (Cf. for example [10, Remark 3.1])

$$\mathbb{P}\big(\sup \mathrm{Q}^{\delta,t}_{0\to 0} > m\big) \sim \frac{2^{\delta/2}\sqrt{2\pi}}{\Gamma(\delta/2)}(m/t)^{(\delta-3)/2}e^{-2m/t}, \text{ as } t \downarrow 0. \tag{5.11}$$

Besides, set $\xi = \sum_{n=0}^{\infty} b_{\delta/2-1}(\sqrt{xy}, n)\mathrm{Q}^{4n,t}_{0\to 0}$ and denote $c(x, y, \delta) = \sum_{n\geq 1} n^2 b_{\delta/2-1}(\sqrt{xy}, n)$, which is uniformly bounded for $x, y \in [0, M]$. Then using the scaling property and additivity property of BESQ bridge, we get

$$\begin{aligned}
\mathbb{P}(\sup \xi > m) &\leq \sum_{n=1}^{\infty} \mathbb{P}\big(\sup \mathrm{Q}^{4n,t}_{0\to 0} > c(x, y, \delta)^{-1}mn^2\big) \\
&\leq \sum_{n=1}^{\infty} n\mathbb{P}\big(\sup \mathrm{Q}^{4,t}_{0\to 0} > c(x, y, \delta)^{-1}mn\big) \\
&= \sum_{n=1}^{\infty} n\mathbb{P}\big(\sup \mathrm{Q}^{4,t/n}_{0\to 0} > c(x, y, \delta)^{-1}m\big) \\
&\leq C_1 e^{-C_2/t} \text{ for some constant } C_1, C_2 > 0.
\end{aligned} \tag{5.12}$$

Here the last line comes from (5.11). Combining (5.11) and (5.12), we get (5.10).

Now we consider the case $\delta = 1$. Observe that a $Q^{1,t}_{x,y}$ bridge can be constructed by first sampling $\eta \in \{\pm 1\}$ with

$$\mathbb{P}(\eta = \pm 1) = \frac{e^{-(\sqrt{x}\mp\sqrt{y})^2/2t}}{e^{-(\sqrt{x}-\sqrt{y})^2/2t} + e^{-(\sqrt{x}+\sqrt{y})^2/2t}},$$

and then sampling a Brownian bridge from $\sqrt{x}$ to $\eta\sqrt{y}$ over $[0, t]$ and taking the square. Then it is a simple exercise to deduce the conclusion from the basic properties of Brownian bridges. That concludes the convergence of ϕ^{λ_k}.

Since we have proved (i), the convergences in the conclusion of Proposition 5.5 hold. As in the proof of Proposition 5.5, we assume the convergences in (i) and the conclusion of Proposition 5.5 are all almost sure. For (ii), first, let us show that

$$t \mapsto 2^{-k}Q^k(t \wedge T^{B^{\phi^{\lambda_k}}}) \text{ converges uniformly to } t \mapsto t \wedge T^{B^\lambda}\, a.s.. \tag{5.13}$$

It follows from the definition of Q^k that

$$\begin{aligned}
&2^{-k}Q^k(t \wedge T^{B^{\phi^{\lambda_k}}}) = 2^{-k} \sum_{x\in\mathbb{N}_k} \big(\lambda(x) - \Lambda_t(\phi^{\lambda_k}, B^{\phi^{\lambda_k}})(x)\big) \\
&= 2^{-k} \sum_{x\in\mathbb{N}_k} \big(\Lambda_t(\lambda, B^\lambda)(x) - \Lambda_t(\phi^{\lambda_k}, B^{\phi^{\lambda_k}})(x)\big) + 2^{-k} \sum_{x\in\mathbb{N}_k} \big(\lambda(x) - \Lambda_t(\lambda, B^\lambda)(x)\big).
\end{aligned}$$

The convergence of the paths implies that a.s. $B^{\phi^{\lambda_k}}$ and B^{λ} are bounded by a common constant m (depending on ω). It follows that the terms in the first sum on the right-hand side of the last equality are all zero for $x > m$. Then the convergence of Λ_t implies that the first sum converges to 0 a.s.. Besides, it is direct from the occupation times formula that the second sum goes to $t \wedge T^{B^{\lambda}}$. This gives (5.13).

Using the substitution $t = 2^{-k}Q^k(s)$ for $0 \le t \le 2^{-k}Q^k(T^{B^{\phi^{\lambda_k}}})$, for s and t in the corresponding range, we have

$$\begin{aligned}&\sup\left|(Q^k)^{-1}\left[2^k t \wedge Q^k(T^{B^{\phi^{\lambda_k}}})\right] - t \wedge T^{B^{\lambda}}\right| \\ &= \sup\left|s \wedge T^{B^{\phi^{\lambda_k}}} - 2^{-k}Q^k(s \wedge T^{B^{\phi^{\lambda_k}}}) \wedge T^{B^{\lambda}}\right|,\end{aligned}$$

which goes to 0 as $k \uparrow \infty$ by (5.13) and the convergence $T^{B^{\phi^{\lambda_k}}} \to T^{B^{\lambda}}$. For $t > 2^{-k}Q^k(T^{B^{\phi^{\lambda_k}}})$, the above quantity is bounded by

$$\left|T^{B^{\phi^{\lambda_k}}} - T^{B^{\lambda}}\right| + \left|T^{B^{\phi^{\lambda_k}}} - 2^{-k}Q^k(T^{B^{\phi^{\lambda_k}}})\right|,$$

which also goes to 0. That completes the proof of (ii).

Finally, we remove the additional assumption. Without loss of generality, we assume $\alpha > 0$. For $k \ge 1$, let

$$o_k = o_k(\lambda) := \inf\left\{x \in \mathbb{N}_k : x \ge \mathfrak{d}_{\lambda}\right\}.$$

Given λ_k, we define another sequence of fields λ'_k on $\mathbb{N}_k$ as follows: for each $k \ge 1$,

$$\lambda'_k(x) = \begin{cases} 0, & \text{if } x = o_k; \\ \lambda_k(x), & \text{otherwise.} \end{cases}$$

Then λ'_k satisfies the additional assumption and converges to λ for the local uniform topology. So the previous proof implies that

$$X^{\lambda'_k,(k)} \xrightarrow{d} B^{\lambda}. \tag{5.14}$$

In particular, since a.s. $\sup_{t\ge 0} B^{\lambda}_t < \mathfrak{d}_{\lambda}$, we have for any $\varepsilon > 0$, there exists $o_* \in (0, \mathfrak{d}_{\lambda})$ such that for k sufficiently large,

$$\mathbb{P}\left(\sup_{t\ge 0} X^{\lambda'_k,(k)} \ge o_*\right) < \varepsilon.$$

On the other hand, we observe that $\lambda_k(x) = \lambda'_k(x)$ for $x \in [0, o_k) \cap \mathbb{N}_k$ and the event $\left\{\sup_{t\ge 0} X^{\lambda'_k,(k)} \ge o_*\right\}$ (resp. $\left\{\sup_{t\ge 0} X^{\lambda'_k,(k)} \ge o_*\right\}$) depends only on the print of $X^{\lambda_k,(k)}$ (resp. $X^{\lambda'_k,(k)}$) on $[0, o_k) \cap \mathbb{N}_k$. By the restriction principle (see Proposition 3.11), we can couple $X^{\lambda_k,(k)}$ and $X^{\lambda'_k,(k)}$ such that their prints on $x \in [0, o_k) \cap \mathbb{N}_k$ coincide. Then

$$\begin{aligned}\mathbb{P}\left(X^{\lambda_k,(k)} \ne X^{\lambda'_k,(k)}\right) &\le \mathbb{P}\left(\sup_{t\ge 0} X^{\lambda_k,(k)} \ge o_*\right) + \mathbb{P}\left(\sup_{t\ge 0} X^{\lambda'_k,(k)} \ge o_*\right) \\ &= 2\mathbb{P}\left(\sup_{t\ge 0} X^{\lambda'_k,(k)} \ge o_*\right) < 2\varepsilon.\end{aligned}$$

Combining with (5.14), we get $X^{\lambda_k,(k)} \xrightarrow{d} B^{\lambda}$. $\square$

Remark 5.8. In this remark, we explain how to generalize the convergence result on $\mathbb{R}^+$ in Theorem 2.11 to the case of general trees. We won't give the proof of this. For details supporting what we explain see [1, §3]. The self-repelling diffusion B^λ on $\mathbb{R}^+$ enjoys the local property. Namely, for any $[a,b]$ on B^λ, the law of the print of B^λ on $[a,b]$ depends only on $\lambda|_{[a,b]}$. So we can make sense of $B|_{[a,b]}$ conditioned on a time field $\lambda = (\lambda_x : x \in [a,b])$ on $[a,b]$, which we denote by $B^\lambda_{[a,b]}$. Moreover, it holds that $B^\lambda_{[a,b]} = B^{\lambda\circ\theta_b}_{[0,b-a]}$, where $\lambda \circ \theta_b = (\lambda_{b+x} : x \in [0, b-a])$ is a time field on $[0, b-a]$.

For a general metric graph $\widetilde{\mathbb{T}}$ associated to a tree, write it as the union of the intervals $\cup_n I_{\{a_n,b_n\}}$ (recall the notation I_e defined in §4), where we assume $a_n = \mathfrak{p}(b_n)$ on the tree. Each $I_{\{a_n,b_n\}}$ is also identified with the interval $\big[0, \big|I_{\{a_n,b_n\}}\big|\big]$ (a_n is identified with 0), where $|\cdot|$ is the length of the interval. For any time field λ on $\widetilde{\mathbb{T}}$, we associate each $I_{\{a_n,b_n\}}$ with the process $B^{\lambda|_{I_{\{a_n,b_n\}}}}_{I_{\{a_n,b_n\}}}$. The self-repelling diffusion on $\widetilde{T}$ is constructed by gluing the excursions of these processes at vertices a_n and b_n, $n \ge 1$, indexed by the local times[10], which inverts the Ray-Knight identity on $\widetilde{T}$ with initial time field λ.

A Processes with terminated jump rates

Let $G = (V, E)$ be a finite or countable graph where each vertex has finite degree. All the processes considered in this part are assumed to be right-continuous, minimal, nearest-neighbour jump processes on G with a finite or infinite lifetime. We consider the process to stay at a cemetery point Δ after the lifetime. The collection of all such sample paths is denoted by Ω, and $\Omega_\infty := \{\omega \in \Omega : \omega \text{ has infinite lifetime}\}$. Let $\{\mathscr{F}_t : t \ge 0\}$ be the natural filtration of the coordinate process on Ω_∞.

Let us introduce some notation that will be used throughout this part. For $0 \le a < b \le \infty$, $[a, b\rangle$ represents the interval $[a,b]$ or $[a,b)$ respectively as $b < \infty$ or $b = \infty$ respectively. Let $l \ge 0$, $t \in (0,\infty]$, $\{t_i\}_{1\le i\le l}$ be positive real numbers and $\{x_i\}_{0\le i\le l}$ be vertices in V, such that $0 =: t_0 < t_1 < \cdots < t_l \le t_{l+1} := t$ and $x_0 \sim x_1 \sim \cdots \sim x_l$. Denote by $\sigma(x_0 \xrightarrow{t_1} x_1 \xrightarrow{t_2} \cdots \xrightarrow{t_l} x_l \xrightarrow{t})$ the function: $[0,t\rangle \to V$, that equals x_i on $[t_i, t_{i+1})$ (resp. $[t_i, t_{i+1}\rangle$) for $i = 0, \cdots, l-1$ (resp. $i = l$).

Definition A.1 (LSC stopping time)**.** *Let T be a stopping time with respect to $(\mathscr{F}_t : t \ge 0)$, such that it is lower semi-continuous under the Skorokhod topology. We simply say T is a LSC stopping time. The notation $1_{\{t<T(\omega_{[0,t]})\}}$ has the natural meaning when T is a stopping time. We further say T is regular if for any $\omega \in \Omega_\infty$ with $t < T(\omega)$ and $\omega_t = x$, there exists $l = l(\omega_{[0,t]}) > 0$, such that for any $0 < s \le l$ and $y \sim x$, it holds that $T\big(\omega_{[0,t]} \circ \sigma(x \xrightarrow{s} y \xrightarrow{\infty})\big) \ge t + l$. Roughly speaking, for a regular T, if the process ω does not 'reach' the stopping time $T(\omega)$ at time t, then as long as it has exactly one jump during $[t, t+l]$, it will not 'reach' $T(\omega)$ until time $t + l$.*

Definition A.2 (Terminated jump rates)**.** *Given a LSC stopping time T. Denote*

$$\mathscr{D}_T := \big\{(t,x,y,\omega) \in \mathbb{R}^+ \times V \times V \times \Omega_\infty : 0 \le t < T(\omega),\, x \sim y,\, \omega_t = x\big\}.$$

A function $r = r_t(x,y) = r(t,x,y,\omega)$: $\mathscr{D}_T \to \mathbb{R}^+$ is called (a family of) T-terminated jump rates if (1) it is continuous with respect to the product topology, where Ω_∞ is equipped with the Skorokhod topology; (2) for any $x \sim y$, the process $\big(r_t(x,y)1_{\{t<T,\,\omega_t=x\}} : t \ge 0\big)$ is adapted to $(\mathscr{F}_t)$. For such r, we also write $r_t(x,y,\omega) = r_t(x,y,\omega_{[0,t]})$ on $\{t < T(\omega),\, \omega_t = x\}$.

From now on, we always assume that T is a LSC stopping time and r is T-terminated jump rates.

[10] for edges $e_1, \cdots, e_n$ with common vertices a, let $\ell_j(a)$ be the local time at a of the processes associated to I_{e_j} for $j = 1, \cdots, n$. In the gluing procedure, we glue their excursions at vertices a up to the local time $\min\{\ell_j(a) : j = 1, \cdots, n\}$. So some of the excursions are not used in this procedure.

Definition A.3 (Processes with terminated jump rates)**.** *A process $Z = (Z_s : 0 \le s < T^Z)$ is said to have jump rates r if for any $t \ge 0$, conditionally on $t < T^Z$ and $(Z_s : 0 \le s \le t)$ with $Z_t = x$,*

(R1) the probability that the first jump of Z after time t occurs in $[t, t + \Delta t]$ and the jump is to a neighbour y is $r_t(x, y, Z_{[0,t]})\Delta t + o(\Delta t)$, where $o(\Delta t)$ depends on the conditioned path $Z_{[0,t]}$,

and the process finally stops at time T^Z. Here $T^Z = T_0^Z \wedge T_\infty^Z$ with

$$T_0^Z := \sup\{t \ge 0 : t < T(Z_{[0,t]})\};$$
$$T_\infty^Z := \sup\{t \ge 0 : Z_{[0,t]} \text{ has finitely many jumps}\}.$$

Remark A.4. In the above definition, if it holds that T is regular, then the condition (R1) can be replaced by (R2) below without affecting the law of the defined process (see Remark A.13 for details).

(R2) the probability of a jump of Z from x to a neighbour y in $[t, t+\Delta t]$ is $r_t(x, y, Z_{[0,t]})\Delta t+ o(\Delta t)$, where $o(\Delta t)$ depends on the conditioned path $Z_{[0,t]}$.

The following renewal property is direct from the definition.

Proposition A.5 (Renewal property)**.** *Suppose Z is a process with jump rates r. Then for any $t \ge 0$, conditionally on $t < T^Z$ and $(Z_s : 0 \le s \le t)$, the process $(Z_{t+s} : 0 \le s < T^Z - t)$ is a process with jump rates $r'_u(x, y, \omega) = r_{t+u}(x, y, Z_{[0,t)} \circ \omega)$ starting from Z_t, where $Z_{[0,t)} \circ \omega$ is a function that equals $Z_{[0,t)}$ on $[0, t)$ and equals $\omega(\cdot - t)$ on $[t, \infty)$.*

Now we tackle the basic problem: the existence and uniqueness of the processes with jump rates r. Let $\mathscr{B}$ be the σ-field on Ω generated by the coordinate maps.

Theorem A.6. *For any $x_0 \in V$, there exists a process with jump rates r starting from x_0. Moreover, if T is regular, then all such processes have the same distribution on $(\Omega, \mathscr{B})$.*

Proof of the existence part of Theorem A.6. Similar to the case of CTMC, we can use a sequence of i.i.d. exponential random variables to construct the process. Precisely, let $(\gamma_x : x \sim x_0)$ be a family of i.i.d. exponential random variables with parameter 1. For $x \sim x_0$, we set

$$u_{x_0x}(t) = \int_0^t r_s(x_0, x, \sigma(x_0 \overset{\infty}{\to})) \, \mathrm{d}s \quad \Big(0 \le t < T(\sigma(x_0 \overset{\infty}{\to})\Big)$$

and $\Gamma_{x_0x} = (u_{x_0x})^{-1}(\gamma_x)$, where $(u_{x_0x})^{-1}$ is the right-continuous inverse of u_{x_0x} and if $\gamma_x \ge u_{x_0x}(T(\sigma(x_0 \overset{\infty}{\to})-)$, then $\Gamma_{x_0x} := \infty$. The process is constructed as follows. It starts from x_0. If $\Gamma_{x_0} := \min_{x\sim x_0} \Gamma_{x_0x} = \infty$, the process stays at x_0 until the lifetime $T(\sigma(x_0 \overset{\infty}{\to}))$. Otherwise, i.e., $\Gamma_{x_0} < \infty$, it jumps at time $J_1 := \Gamma_{x_0}$ to x_1, which is the unique $x \sim x_0$ such that $\Gamma_{x_0x} = \Gamma_{x_0}$. For the second jump, the protocol is the same except that x_0 and $\sigma(x_0 \overset{\infty}{\to})$ are replaced by x_1 and $\sigma(x_0 \overset{J_1}{\to} x_1 \overset{\infty}{\to})$ respectively.

In the same way that one verifies the jump rates of a CTMC constructed from a sequence of i.i.d. exponential random variables, it is simple to check that the process constructed above has jump rates r. ☐

The proof of the uniqueness part is delayed to §A.1 after the introduction of path probability.

Remark A.7. In this remark, we give the rigorous definitions of repelling processes presented in §2 ∼§4. We only construct the process with law $\mathbb{P}^\lambda_{x_0}$ (defined in §2.4.1) and it is similar for the others. For $\lambda \in \mathfrak{R}$, let $T^\lambda(\omega) := \sup\{t \ge 0 : \Lambda_t(\omega_{t-}, \omega) > 0\}$ and consider r^λ (defined as (2.7)) restricted to $\mathscr{D}_{T^\lambda}$. Intuitively, $T^\lambda(\omega)$ represents the time when the process ω uses up the local time at some vertex. We can readily check that T^λ is a regular LSC stopping time and r^λ is T^λ-terminated jump rates. We first run a process Z^1 with jump rates r^λ starting from x_0 up to T^{Z^1}. If the death is due to the exhaustion of local time at some vertex $x \neq x_0$, i.e. $T^{Z^1} = T_0^{Z^1}$ and $Z^1_{T^{Z^1}-} = x$, we record the remaining local time at each vertex, say ϱ. Then we resurrect the process by letting it jump to $\mathfrak{p}(x)$ and running an independent process Z^2 with jump rates r^ϱ starting from $\mathfrak{p}(x)$ up to T^{Z^2}. After the death, we resurrect the process again. This continues until it explodes or exhausts the local time at x_0. This procedure defines a process, the law of which is defined to be $\mathbb{P}^\lambda_{x_0}$. By Theorem A.6, such a process always exists and the law is already determined by the definition.

By the analysis in §2, when $\alpha > 0$, Z^1 cannot die at $x \neq x_0$. So in this case, we actually do not need the resurrect procedure, and Z^1 up to T^{Z^1} has the law $\mathbb{P}^\lambda_{x_0}$.

We assume the regularity of T henceforth. Based on Theorem A.6, when considering a process with terminated jump rates, we can always focus on its special construction presented in the previous proof of the existence part. The following three corollaries are immediately derived from this perspective.

Corollary A.8. *Suppose Z is a process with jump rates r. Let $J_1 = J_1^Z$ be the first jump time of Z. Then*

$$\begin{aligned}&\mathbb{P}\left(J_1 \in \mathrm{d}t,\, Z_{J_1} = x\right)\\&= 1_{\{t<T(\sigma(x_0 \overset{\infty}{\rightarrow}))\}} \exp\Big(-\int_0^t \sum_{y:y\sim x_0} r_s(x_0, y, \sigma(x_0 \overset{\infty}{\rightarrow}))\,\mathrm{d}s\Big) \cdot r_t(x_0, x, \sigma(x_0 \overset{\infty}{\rightarrow}))\,\mathrm{d}t.\end{aligned} \tag{A.1}$$

Corollary A.9 (Strong renewal property)**.** *Suppose Z is a process with jump rates r starting from x_0. Then for any stopping time S with respect to the natural filtration of Z, conditionally on $S < T^Z$ and $(Z_s : 0 \le s \le S)$, the process $(Z_{S+s} : 0 \le s \le T^Z)$ is a process with jump rates $r'_t(x, y, \omega) = r_{t+S}(x, y, Z_{[0,S)} \circ \omega)$ starting from Z_S.*

The proof of the above strong renewal property is almost a word-by-word copy of the proof of the strong Markov property of CTMC presented in [19, Theorem 6.5.4].

The next corollary gives a more accurate bound of the probability in (R1) and (R2). We only state in terms of the event in (R2). For $t \ge 0$ and $\sigma \in \Omega_t := \{\omega_{[0,t]} : \omega \in \Omega_\infty\}$, we use $\sigma^\rightarrow$ to represent the function in Ω_∞ that equals σ on $[0,t]$ and stays at σ_t after time t.

Corollary A.10. *Suppose Z is a process with jump rates r starting from x_0. Then for any $\sigma \in \Omega_t$ with $\sigma_t = x$ and $t < T(\sigma)$, if we denote $R(z) = \exp\Big(-\int_t^{t+\Delta t} r_s(x, z, \sigma^\rightarrow)\,\mathrm{d}s\Big)$ for $z \sim x$, it holds that for $y \sim x$,*

$$\begin{aligned}&\mathbb{P}\Big(\text{there is a jump of } Z \text{ from } x \text{ to } y \text{ in } [t, t+\Delta t] \,\Big|\, Z_{[0,t]} = \sigma\Big)\\&\in \Big[\Big(\prod_{z:z\sim x,\, z\neq y} R(z)\Big) \cdot \big(1 - R(y)\big),\ 1 - R(y)\Big].\end{aligned}$$

A.1 Path probability

Assume T is regular. Let Z be a process with jump rates r starting from x_0. For any Borel subsets $\{D_i\}_{1\le i\le l}$ of $\mathbb{R}^+$, $t \in (0,\infty]$ and $\{x_i\}_{1\le i\le l}$ with $x_0 \sim x_1 \sim \cdots \sim x_l$, denote

$$\begin{aligned}A\big(x_0 \overset{D_1}{\rightarrow} x_1 \overset{D_2}{\rightarrow} \cdots \overset{D_l}{\rightarrow} x_l \overset{t}{\rightarrow}\big) := \Big\{&\sigma(x_0 \overset{s_1}{\rightarrow} x_1 \overset{s_2}{\rightarrow} \cdots \overset{s_l}{\rightarrow} x_l \overset{t}{\rightarrow}) : 0 < s_1 <\\&\cdots < s_l \le t \text{ and } s_i \in D_i\ \forall 1 \le i \le l\Big\}.\end{aligned} \tag{A.2}$$

The goal of this part is to calculate the path probability

$$\mathbb{P}\Big(Z_{[0,t]} \in A\big(x_0 \overset{D_1}{\to} x_1 \overset{D_2}{\to} \cdots \overset{D_l}{\to} x_l \overset{t}{\to}\ )\Big). \tag{A.3}$$

The key point of the calculation is the following lemma. At first glance, the lemma seems obvious. We mention that the main difficulty comes from the dependence of $o(\Delta t)$ in (R1) on the conditioned path $Z_{[0,t]}$.

Lemma A.11. *Conditionally on $t < T^Z$ and $(Z_s : 0 \le s \le t)$, the probability that Z has at least 2 jumps during $[t, t+\Delta t]$ is $O((\Delta t)^2)$, where $O((\Delta t)^2)$ depends on the conditioned path.*

Remark. Without the regularity of T, the conditional probability of two jumps in $[t, t+\Delta t]$ may be comparable to Δt. For example, we consider a path-dependent Poisson process as follows. Starting from 0, it jumps to 1 with rate 1. Its jump rate at 1 is given by:

$$r_s\big(1, 2, \sigma(0 \overset{t}{\to} 1 \overset{\infty}{\to})\big) := \frac{1}{2t-s} \ \text{ for } t > 0 \text{ and } t \le s \le 2t.$$

Let us consider the special realization presented in Theorem A.6 with the above rates. Then it holds that conditionally on the process jumping from 0 to 1 at time t, it is bound to jump from 1 to 2 before time $2t$. So the probability of two jumps in $[0, \Delta t]$ is no less than $1 - e^{-\Delta t/2}$.

Before the proof, we present a simple result. Recall the notation $\sigma^{\to}$. It is easy to see that given $s > t \ge 0$ and $\sigma \in \Omega_t$ with $\sigma_t = x$ and $s < T(\sigma^{\to})$,

$$\mathbb{P}\left(Z_v = x \text{ on } [t,s] \,\middle|\, Z_{[0,t]} = \sigma\right) = \exp\Big(-\int_t^s \sum_{y:y\sim x} r_u(x, y, \sigma^{\to})\,\mathrm{d}u\Big). \tag{A.4}$$

To this end, one can consider the left-hand side of (A.4) as a function of s and formulate a differential equation.

Proof of Lemma A.11. The renewal property (in Proposition A.5) enables us to consider only the case $t = 0$. It suffices to fix a neighbour y of x_0 and prove the probability that Z has at least 2 jumps during $[0, \Delta t]$ and the first jump is to y is $O((\Delta t)^2)$. Let J_1 and J_2 be the first and second jump time of Z respectively, and $B := \big\{J_1, J_2 \in [0, \Delta t],\, Z_{J_1} = y\big\}$. The event B can be divided according to the jump times as follows. For $n \ge 1$, define

$$B_{j,n} := \Big\{J_1 \in \Big(\frac{j-1}{2^n}\Delta t, \frac{j}{2^n}\Delta t\Big],\, J_2 \in \Big(\frac{j}{2^n}\Delta t, \Delta t\Big],\, Z_{J_1} = y\Big\}\ (1 \le j \le 2^n - 1).$$

Set $B_n := \bigcup_{j=1}^{2^n-1} B_{j,n}$. Then it holds that $B_n \uparrow B$. So it suffices to show that there exists a constant C independent of n, such that

$$\mathbb{P}(B_n) \le C(\Delta t)^2, \text{ for all } n \ge 1.$$

For any j and n with $1 \le j \le 2^n - 1$, let $B_{j,n}^{(1)}$ be the event that $J_1 \in \Big(\frac{j-1}{2^n}\Delta t, \frac{j}{2^n}\Delta t\Big]$ and $Z_u = y$ for $u \in \Big[J_1, \frac{j}{2^n}\Delta t\Big]$. Then

$$\mathbb{P}\left(B_{j,n}\right) \le \mathbb{P}\Big(J_2 \in \Big(\frac{j}{2^n}\Delta t, \Delta t\Big] \,\Big|\, B_{j,n}^{(1)},\, J_2 > \frac{j}{2^n}\Delta t\Big) \cdot \mathbb{P}\Big(B_{j,n}^{(1)}\Big).$$

For the first probability on the right-hand side, observe that if we further condition on J_1, then we have conditioned on the whole path $(Z_s : 0 \le s \le \frac{j}{2^n}\Delta t)$. By the regularity of T, for Δt sufficiently small, $T(\sigma(x_0 \overset{u}{\to} y \overset{\infty}{\to})) > \Delta t$ for any $0 < u \le \Delta t$. Then it follows from (A.4) that on $B_{j,n}^{(1)} \cup \{J_2 > \frac{j}{2^n}\Delta t\}$,

$$\begin{aligned}
&\mathbb{P}\Big(J_2 \in \Big(\frac{j}{2^n}\Delta t, \Delta t\Big] \,\Big|\, J_1\Big) \\
&= \mathbb{P}\Big(J_2 \in \Big(\frac{j}{2^n}\Delta t, \Delta t\Big] \,\Big|\, Z_{[0,\frac{j}{2^n}\Delta t]} = \sigma\big(x_0 \overset{t_1}{\to} y \overset{\frac{j}{2^n}\Delta t}{\to}\,)\Big)\Big|_{t_1=J_1} \\
&= 1 - \exp\Big(-\int_{(\frac{j}{2^n}\Delta t, \Delta t]} \sum_{z:z\sim y} r_v\big(y, z, \sigma\big(x_0 \overset{t_1}{\to} y \overset{\frac{j}{2^n}\Delta t}{\to}\,)\big)\,\mathrm{d}v\Big)\Big|_{t_1=J_1} \le C\Delta t,
\end{aligned}$$

where $C = \sup\Big\{\sum_{z:z\sim y} r_v(y, z, \sigma(x_0 \overset{u}{\to} x_1 \overset{\infty}{\to})) : 0 \le u \le v \le \Delta t\Big\}$, which is finite by the continuity of r. The same bound also holds for $\mathbb{P}\Big(J_2 \in \Big(\frac{j}{2^n}\Delta t, \Delta t\Big] \,\Big|\, B_{j,n}^{(1)},\, J_2 > \frac{j}{2^n}\Delta t\Big)$. So

$$\begin{aligned}
\mathbb{P}(B_n) &= \sum_{j=1}^{n-1} \mathbb{P}(B_{j,n}) \le C\Delta t \sum_{j=1}^{n-1} \mathbb{P}\Big(B_{j,n}^{(1)}\Big) \le C\Delta t \cdot \mathbb{P}(J_1 \in [0, \Delta t],\, Z_{J_1} = y) \\
&= C\Delta t \cdot \Big(r_0(x_0, y, \sigma(x_0 \overset{\infty}{\to}))\Delta t + o(\Delta t)\Big) \le C'(\Delta t)^2.
\end{aligned}$$

That completes the proof. □

Corollary A.12. *Conditionally on $t < T^Z$ and $(Z_s : 0 \le s \le t)$ with $Z_t = x$, the probability that Z has exactly one jump during $[t, t+\Delta t]$ and the jump is to y is $r_t(x, y, Z_{[0,t]})\Delta t + o(\Delta t)$, where $o(\Delta t)$ depends on the conditioned path.*

Let us start the calculation of path probability (A.3). The case $l = 0$ has been covered by (A.4). For $l \ge 1$, Corollary A.12 enables us to use the methods of formulating differential equations to calculate path probabilities. Fix $x_1 \sim x_0$ and $0 < t_1 < t_2 \le \infty$ with $t_2 < T(\sigma(x_0 \overset{t_1}{\to} x_1 \overset{\infty}{\to}))$. By the lower semi-continuity of T, there exists $0 < u < t_2 - t_1$, such that for any $0 \le s < u$, $t_2 < T\Big(\sigma(x_0 \overset{t_1+s}{\to} x_1 \overset{\infty}{\to})\Big)$. For such s, let us calculate

$$q(s) := \mathbb{P}\Big(Z_{[0,t_2]} \in A\big(x_0 \overset{[t_1,t_1+s]}{\to} x_1 \overset{t_2}{\to}\,)\Big).$$

For $0 < \Delta s < u - s$,

$$q(s+\Delta s) - q(s) = \mathbb{P}\Big(Z_{[0,t_2]} \in A\big(x_0 \overset{[t_1+s,t_1+s+\Delta s]}{\to} x_1 \overset{t_2}{\to}\,)\Big) =: q_1 q_2 q_3,$$

where $q_1 = \mathbb{P}(E_1)$, $q_2 = \mathbb{P}(E_2 \,|\, E_1)$, $q_3 = \mathbb{P}(E_3 \,|\, E_1 \cap E_2)$ with

$E_1 = \{Z_u = x_0$ for $u \in [0, t_1 + s]\}$,
$E_2 = \{Z$ has exactly one jump during $[t_1 + s, t_1 + s + \Delta s]$ and the jump is to $x_1\}$,
$E_3 = \{Z_u = x_1$ for $u \in [t_1 + s + \Delta s, t_2]\}$.

It holds that

$$\begin{cases}
q_1 = \exp\Big(-\int_0^{t_1+s} \sum_{y:y\sim x_0} r_v\big(x_0, y, \sigma(x_0 \overset{\infty}{\to})\big)\,\mathrm{d}v\Big); \\
q_2 = r_{t_1+s}\big(x_0, x_1, \sigma(x_0 \overset{\infty}{\to})\big) \cdot \Delta s + o(\Delta s); \\
q_3 = \exp\Big(-\int_{t_1+s}^{t_2} \sum_{y:y\sim x_1} r_v\big(x_1, y, \sigma(x_0 \overset{t_1+s}{\to} x_1 \overset{\infty}{\to})\big)\,\mathrm{d}v\Big) + o(1),
\end{cases} \tag{A.5}$$

where q_1 and q_2 follows from (A.4) and Corollary A.12 respectively. For q_3, it suffices to note that for any $0 \le h \le \Delta s$, conditionally on $Z_{[0,t_1+s+\Delta s]} = \sigma(x_0 \overset{t_1+s+h}{\to} x_1 \overset{t_1+s+\Delta s}{\to})$, the probability of E_3 is

$$\exp\Big(- \int_{t_1+s+\Delta s}^{t_2} \sum_{y:y\sim x_1} r_v\big(x_1, y, \sigma(x_0 \overset{t_1+s+h}{\to} x_1 \overset{\infty}{\to})\big)\Big) \,\mathrm{d}v,$$

which together with the continuity of r easily leads to the probability q_3.

By further considering the case $\Delta s < 0$, we can formulate a differential equation, the solution of which gives: for $0 \le s < u$,

$$q(s) = \int_{t_1}^{t_1+s} \exp\Big(- \int_0^{t_2} \sum_{y:y\sim\sigma_v} r_v(\sigma_v, y, \sigma)\,\mathrm{d}v\Big) \cdot r_{s'}(x_0, x_1, \sigma)\,\mathrm{d}s',$$

where $\sigma = \sigma(x_0 \overset{s'}{\to} x_1 \overset{\infty}{\to})$ which varies with s'. Observe that the lower semi-continuity of T implies that $\{s' \in [0, t_2] : t_2 < T(\sigma(x_0 \overset{s'}{\to} x_1 \overset{t_2}{\to}))\}$ is a relatively open subset of $[0, t_2]$. So we can readily deduce that for any Borel subset D in $\mathbb{R}^+$ and $x_1 \in V$,

$$\begin{aligned}
&\mathbb{P}\left(Z_{[0,t]} \in A\left(x_0 \overset{D}{\to} x_1 \overset{t}{\to}\right)\right) \\
&= \int_D 1_{\{s'<t<T(\sigma)\}} \exp\Big(- \int_0^t \sum_{y:y\sim\sigma_v} r_v(\sigma_v, y, \sigma)\,\mathrm{d}v\Big) \cdot r_{s'}(x_0, x_1, \sigma)\,\mathrm{d}s'.
\end{aligned}$$

Similarly, we can inductively check that for Borel subsets D_i in $\mathbb{R}^+$ and $x_i \in V$ $(1 \le i \le l)$,

$$\begin{aligned}
\mathbb{P}\left(Z_{[0,t]} \in A_t\right) = \int \cdots \int_{D_1\times\cdots\times D_l} 1_{\{s_1<\cdots<s_l<t<T(\sigma)\}} \cdot \prod_{j=1}^{l} r_{t_j}(x_{j-1}, x_j, \sigma) \\
\cdot \exp\Big(- \int_0^t \sum_{y:y\sim\sigma_v} r_v(\sigma_v, y, \sigma)\,\mathrm{d}v\Big) \cdot \prod_{j=1}^{l} \mathrm{d}s_j,
\end{aligned} \tag{A.6}$$

where $\sigma = \sigma(x_0 \overset{s_1}{\to} x_1 \overset{s_2}{\to} \cdots \overset{s_l}{\to} x_l \overset{t}{\to})$ and $A_t = A\big(x_0 \overset{D_1}{\to} x_1 \overset{D_2}{\to} \cdots \overset{D_l}{\to} x_l \overset{t}{\to}\big)$.

When (x_i), (D_i) and t vary, all the sets $A_t = A\big(x_0 \overset{D_1}{\to} x_1 \overset{D_2}{\to} \cdots \overset{D_l}{\to} x_l \overset{t}{\to}\big)$ constitute a π-system. The σ-field generated by these sets on $(\Omega, \mathscr{B})$ contains sets in the form $\{\omega : \omega_s = x\}$ ($x \in V$, $s \ge 0$) and hence equals $\mathscr{B}$. So (A.6) determines the law of Z. That completes the proof of the uniqueness part of Theorem A.6.

Remark A.13. Observe that if we replace (R1) by (R2) in the Definition A.3, then following the same routine, we obtain the same path probability (A.6). In fact, the only difference in the proof is that in (A.4) '$=$' should be replaced by '$\ge$', and the later proofs still work with minor modifications. This easily leads to the statement in Remark A.4.

In the following, we present several other forms of path probability and give an application. Note that (A.6) can also be written as an equality of measures:

$$\mathbb{P}\left(Z_{[0,t]} \in A_t\right) = d_\sigma^{(t)} \prod_{j=1}^{l} \mathrm{d}t_j,$$

where $A_t = A\big(x_0 \overset{\mathrm{d}t_1}{\to} x_1 \overset{\mathrm{d}t_2}{\to} \cdots \overset{\mathrm{d}t_l}{\to} x_l \overset{t}{\to}\big)$, $\sigma = \sigma(x_0 \overset{t_1}{\to} x_1 \overset{t_2}{\to} \cdots \overset{t_l}{\to} x_l \overset{t}{\to})$ and

$$d_\sigma^{(t)} = 1_{\{t_1<\cdots<t_l<t<T(\sigma)\}} \exp\Big(- \int_0^t \sum_{y:y\sim\sigma_v} r_v(\sigma_v, y, \sigma)\,\mathrm{d}v\Big) \cdot \prod_{j=1}^{l} r_{t_j}(x_{j-1}, x_j, \sigma).$$

An argument of the monotone class methods yields that for any bounded measurable function Φ on Ω_t,

$$\mathbb{E}\left(\Phi(Z_{[0,t]});\, Z_{[0,t]} \in A_t\right) = d_\sigma^{(t)}\Phi(\sigma)\prod_{j=1}^{l} \mathrm{d}t_j.$$

We further give another path probability including the information of the lifetime. Recall that we consider the process to stay at the cemetery point Δ after the lifetime. We naturally generalize to allow $x_l = \Delta$ in the definition of A and σ. Then we can obtain that for any bounded measurable function Φ on Ω,

$$\mathbb{E}\big(\Phi(Z);\, Z \in A\big) = d_\sigma^{\Delta}\Phi(\sigma)\delta_{T(\sigma)}(\,\mathrm{d}t_l)\prod_{j=1}^{l-1} \mathrm{d}t_j, \tag{A.7}$$

where $A = A\big(x_0 \overset{\mathrm{d}t_1}{\to} x_1 \overset{\mathrm{d}t_2}{\to} \cdots \overset{\mathrm{d}t_l}{\to} x_l = \Delta \overset{\infty}{\to}\big)$, $\sigma = \sigma(x_0 \overset{t_1}{\to} x_1 \overset{t_2}{\to} \cdots \overset{t_l}{\to} x_l = \Delta \overset{\infty}{\to})$ and

$$d_\sigma^{\Delta} = 1_{\{t_1 < \cdots < t_l = T(\sigma)\}} \exp\Big(-\int_0^{t_l} \sum_{y:y\sim\sigma_v} r_v(\sigma_v, y, \sigma)\,\mathrm{d}v\Big)\cdot \prod_{j=1}^{n} r_{t_j}(x_{j-1}, x_j, \sigma).$$

We mention that $\delta_{T(\sigma)}(\,\mathrm{d}t_l)$ makes sense since $T(\sigma)$ depends only on $t_1, \cdots, t_{l-1}$.

Remark A.14. More generally, we may allow the terminated jump rates to further contain the rates from any $x \in V$ to the cemetery point Δ. In this case, we can similarly define the processes with terminated jump rates. In particular, such processes also stop when jumping to Δ. The results of the path probability can be easily generalized to this case.

Now we give an application of the path probability. With the same notation as §2, let $\big(\phi^{(0)}, X_{[0,\tau_u]}, \phi^{(u)}\big)$ be a Ray-Knight triple associated to X. We will show that $X_{[0,\tau_u]}$ conditioned on $\tau_u < T^X$ is distributed as $Y_{[0,\tau_u^Y]}$ defined in the beginning of §3.

Proposition A.15.
- *Y is recurrent.*
- *For any $u > 0$, $X_{[0,\tau_u]}$ conditioned on $\tau_u < T^X$ has the same law as $Y_{[0,\tau_u^Y]}$.*

Proof. In this proof, we always consider the processes with terminated jump rates in the sense of Remark A.14. Fix $u > 0$. We note that X and Y are the processes with jump rates given by the conductances and killing measure. Let $T(\omega) = \tau_{u-}^{\omega}$ for $\omega \in \Omega_\infty$. Restrict the above two jump rates to $\mathscr{D}_T$. Then the processes with the terminated jump rates are $X_{[0,\tau_{u-}\wedge T^X)}$ and $Y_{[0,\tau_{u-}^Y\wedge T^Y)}$ respectively. For both processes, using the generalized version of (A.7), we can deduce that for any bounded measurable function Φ on Ω,

$$\mathbb{E}\left(\Phi\big(Y_{[0,\tau_{u-}^Y)}\big);\, \tau_u^Y < T^Y\right) = e^{u(C_{x_0} - C_{x_0}^h)}\mathbb{E}\Big(\Phi\left(X_{[0,\tau_{u-})}\right);\, \tau_u < T^X\Big), \tag{A.8}$$

where $X_{[0,\tau_{u-})}$ and $Y_{[0,\tau_{u-}^Y)}$ are considered to stay at Δ after time τ_{u-} and τ_{u-}^Y respectively. The standard results in the electric network theory tell us

$$\mathbb{P}^x\left(\tau_u < T^X\right) = \mathbb{P}^x\left(H_{x_0} < T^X\right)\mathbb{P}^{x_0}\left(\tau_u < T^X\right) = h(x)e^{-u(C_{x_0} - C_{x_0}^h)}.$$

This together with (A.8) leads to the conclusions. $\square$

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Acknowledgments. The authors are grateful to Prof. Elie Aïdékon for introducing this project and inspiring discussions. The authors would also like to thank Prof. Jiangang Ying and Shuo Qin for the helpful discussions and valuable suggestions.