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# Intertwining and duality for consistent Markov processes*

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**Abstract**

In this paper we derive intertwining relations for a broad class of conservative particle systems both in discrete and continuous setting. Using the language of point process theory, we are able to derive a new framework in which duality and intertwining can be formulated for particle systems evolving in general spaces. These new intertwining relations are formulated with respect to factorial and orthogonal polynomials.

Our novel approach unites all the previously found self-dualities in the context of discrete consistent particle systems and provides new duality results for several interacting systems in the continuum, such as interacting Brownian motions. We also introduce a process that we call generalized inclusion process, consisting of interacting random walks in the continuum, for which our method applies and yields generalized Meixner polynomials as orthogonal self-intertwiners.

**Keywords:** intertwiner; self-duality; orthogonal polynomials; consistency; point processes.
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## 1 Introduction

### 1.1 Duality for interacting particle systems: a short overview

Duality and self-duality are important technical tools in the study of interacting particle systems and models of population dynamics. E.g. in [42] self-duality is the key tool to analyze the ergodic properties of the symmetric exclusion process (SEP), in [32] duality is the key tool to infer properties of a non-equilibrium steady state in the so-called KMP model of heat conduction and in [14], [18, Chapter 10] duality is used to study the long-time behaviour in stochastic models of population genetics.

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Self-duality can be expressed as the property that the time evolution of well-chosen polynomials of degree $n$ boils down to the time evolution of $n$ (dual) particles. This is a significant simplification, because properties of a system of possibly infinitely many particles are reduced to properties of a finite number of particles. In its simplest setting (one particle duality in discrete systems), it is the property that the expected number of particles at a given location $x$ at time $t > 0$ can be expressed in terms of the initial configuration and the location at time $t > 0$ of *a single particle starting from* $x$. The three main classical discrete interacting particle systems for which self-duality has been proved are SEP, simple symmetric inclusion process (SIP) and independent random walks (IRW).

Mostly dualities and self-dualities in the context of conservative interacting particle systems are studied on lattices, i.e., in a discrete setting where the particle configuration specifies at each site of the lattice the number of particles. The duality functions are then usually products over lattice sites of polynomials in the number of particles, depending on the number of dual particles (the number of dual particles corresponds to the degree of the polynomial). The self-duality functions are usually categorized in "classical" self-duality functions, corresponding to (modified) factorial moments, and "orthogonal" self-duality functions which are products of orthogonal polynomials (where orthogonality is with respect to an underlying reversible product measure).

Self-duality relations with respect to orthogonal polynomials are particularly useful in the study of fluctuation fields, the Boltzmann Gibbs principle, and cumulants in non-equilibrium steady states ([4], [5], [15], [23]). So far, for the classical discrete systems (SEP, SIP, IRW) orthogonal polynomial self-duality functions were obtained via various methods: the three term recurrence relations [24], Lie algebra representation theory [27], unitary symmetries [10], and a direct relation between "classical" (factorial moment) self-duality functions and orthogonal duality functions [23].

### 1.2 Self-duality beyond the discrete setting

The language and formulation of duality in terms of occupation variables at discrete lattice sites clearly breaks down in many natural settings of e.g. particles moving in the continuum, such as interacting Brownian motions or more general interacting Markov processes. Even for one of the simplest examples such as independent Brownian motions, it is not immediately clear how to formulate and obtain self-duality. The naive approach of using the scaling limit of self-dualities of independent random walks does not lead to useful results. However, it is very natural to expect that all the classical discrete systems with self-duality properties have counterparts in the continuum. It is therefore important to develop a more general approach to self-duality that can lead to results also in the continuum, on very general state spaces. First, one has to find a language in which the basic duality properties of discrete systems, including the orthogonal duality functions, can be restated in such a way that they make sense in the continuum. Second, one has to understand under which assumptions these generalized relations are valid, hoping to include many more systems in the class of self-dual Markov processes. These two steps are part of the contributions of this work.

### 1.3 Consistent particle systems

In [12], the notion of consistency (see also [32]) was connected to self-duality in the context of discrete interacting particle systems. In particular, for the three basic particle systems having self-duality (SEP, SIP and IRW), the "classical" duality relations can all be derived from the same intertwining relation, which in turn is derived from consistency. Consistency roughly means that the time evolution commutes with the operation of randomly selecting a given number of particles out of the system. Equivalently, up to permutations, it implies that in a system of $n$ particles, the $k$ particle evolution coincides with the evolution of $k$ particles out of these $n$ particles, i.e., the effect of the interactions with the other $n-k$ particles is “wiped out”. This is a remarkable property, trivially valid for independent particles, but also for interacting systems with special symmetries, such as the SEP and SIP.

The consistency property appeared (under a slightly different form) in the literature on stochastic flows [40], [51] including e.g. interacting Brownian motions, the Brownian web, and the Howitt-Warren flow. It also played a crucial role in the analysis of the KMP model [32]. Therefore, the consistency property seems the natural starting point for establishing self-duality relations for conservative particle systems in a general state space. Because we want to consider evolution of configurations of particles, we are naturally led to the context of point processes [38].

### 1.4 Summary of main results

We summarize below our main contributions.

(i) We introduce a new framework in which self-duality type relations, more precisely self-intertwining relations, with respect to polynomials can be formulated for particle systems evolving on a general Borel space, thus also on $\mathbb{R}^d$. This framework also provides a new approach to self-duality.

(ii) We provide a necessary and sufficient condition to have self-intertwining relations with generalized falling factorial polynomials as intertwiners. In particular, we provide new self-intertwining results for systems such as independent and interacting Brownian motions. Moreover, from this new approach, the known self-duality functions for classical conservative interacting particle systems (i.e., SEP, IRW, SIP and the inhomogeneous version of these processes) are recovered. Our approach is thus unifying and avoids the need of ad hoc computations for each system when proving self-duality.

(iii) We prove that, assuming reversibility for the particle system, the Gram-Schmidt orthogonalization procedure, viewed as the linear map acting as a projection on a proper subspace and responsible for the orthogonalization procedure, is a symmetry for the particle dynamics of a consistent process, i.e., this linear map commutes with the semigroup of the process. This property is new not only in our general setting but even in the discrete setting (i.e., for SEP, IRW, SIP and the inhomogeneous version of these processes) where orthogonal polynomial duality is proved via direct (rather tedious) computations. As a consequence, orthogonalizing the previously introduced falling factorial polynomial self-intertwinings, we show orthogonal self-intertwinings in the same context of consistent particle systems on general state spaces. In doing so, we also show some properties of generalized orthogonal polynomials which are of independent interest. Again, our new machinery allows to recover all the known orthogonal self-duality functions for classical consistent interacting particle systems.

(iv) We introduce and study a new process in the continuum, called *generalized symmetric inclusion process*, for which all our self-intertwining results apply. It turns out that the distributions of the so-called Pascal point processes are reversible measures of the generalized inclusion process. We prove that generalized Meixner polynomials are self-intertwiners for the generalized symmetric inclusion process and some properties of these orthogonal polynomials are derived in a novel and simple way.

These self-intertwining results open doors to many potential future applications to the study of properties of particles systems in general state spaces, including characterization of the stationary measures and their attractors (see, e.g., [42, Chapter 8]), hydrodynamic limits (see, e.g., [15, Chapter 2]) and fluctuations (see, e.g., [5]), and boundary driven non-equilibrium systems (see, e.g., [21], [32]).

### 1.5 Organization of the paper

Our paper is organized as follows. In Section 2 we introduce the general setting and the class of Markov processes under consideration. We then state the two main theorems, the two self-intertwining results where the intertwiners are respectively, generalized falling factorial and orthogonal polynomials. We also provide the proof of some properties of the generalized orthogonal polynomials. In Section 3 we list some examples of known processes which satisfy the assumptions of our main theorems. In particular we show how the known self-duality relations for exclusion and inclusion process follow from our general results. In Section 4 we introduce and study a continuum version of the inclusion process. In particular we identify its reversible distribution, we show that it satisfies the assumptions of the two intertwiner results, and finally we exhibit the relation between the generalized orthogonal polynomials and the Meixner polynomials.

Finally, Appendix A is mainly intended for the reader familiar with the language of interacting particle systems and "interpolates" between the usual notations of that context and the point process notations. We revisit self-duality for independent random walks and link it to factorial moment measures of point processes. This allows us to rewrite the self-duality relation in such a way that it makes sense for independent Markov processes on general state spaces, provided a symmetry condition is fulfilled.

## 2 Self-intertwining relations

In this section, we start by introducing the setting and the class of processes that we consider, namely the consistent and conservative Markov processes. Then, in Section 2.2, we introduce the generalized falling factorial polynomials and we state and prove our first main result, a self-intertwining relation. In Section 2.3, after providing the construction of generalized orthogonal polynomials, we state and prove a second self-intertwining relation. In Section 2.4 we provide the proof of some properties of the generalized orthogonal polynomials.

### 2.1 Setting and consistent Markov processes

Throughout this article we investigate Markov processes whose state space consists of configurations of non-labeled particles in some general measurable space $(E,\mathcal{E})$ and are thus usually denoted by *configuration processes*. To avoid the technical difficulties associated with infinitely many particles (for example, a rigorous construction of interacting dynamics), we consider configurations of finitely many particles only.

We follow modern point process notation in modelling such configurations as finite counting measures on $(E,\mathcal{E})$. Thus, let $\mathbf{N}_{<\infty}$ be the space of finite counting measures, i.e., the space of finite measures that assign values in $\mathbb{N}_0 := \{0,1,2,\dots\}$ to every set $B \in \mathcal{E}$. The space is equipped with the $\sigma$-algebra $\mathcal{N}_{<\infty}$ generated by the counting variables $\mathbf{N}_{<\infty} \ni \eta \mapsto \eta(B)$, $B \in \mathcal{E}$. Assumptions on $(E,\mathcal{E})$ are needed to ensure that every counting measure is a sum of Dirac measures, therefore we assume throughout the article that $(E,\mathcal{E})$ is a Borel space (see [38, Definition 6.1]). The reader may think of a Polish space or $\mathbb{R}^d$ endowed with the Borel $\sigma$-algebra. It is well-known (see, e.g., [29, Section 1.1] or [38, Chapter 6]) that for a Borel space, every finite counting measure $\eta \in \mathbf{N}_{<\infty}$ is either zero or of the form $\eta = \delta_{x_1} + \cdots + \delta_{x_n}$ for some $n \in \mathbb{N} = \{1,2,3,\dots\}$ and

$x_1, \dots, x_n \in E$ not necessarily distinct. In particular, the total mass $\eta(E)$ corresponds to the total number of particles.

For our purpose, a Markov process with state space $\mathbf{N}_{<\infty}$ is a collection $(\Omega, \mathfrak{F}, (\eta_t)_{t\geq 0}, (\mathbb{P}_\eta)_{\eta\in\mathbf{N}_{<\infty}})$, where $(\Omega, \mathfrak{F})$ is a measurable space, $\eta_t : (\Omega, \mathfrak{F}) \to (\mathbf{N}_{<\infty}, \mathcal{N}_{<\infty})$ is a measurable map for all $t \geq 0$ and for $\eta \in \mathbf{N}_{<\infty}$, $\mathbb{P}_\eta$ are probability measures on $(\Omega, \mathfrak{F})$ such that $\mathbb{P}_\eta(\eta_0 = \eta) = 1$ and the map $\eta \to \mathbb{P}_\eta(\eta_t \in B)$ is measurable for each $B \in \mathcal{E}$ and $t \geq 0$. The Markov property is implicitly assumed to be satisfied with respect to the natural filtration $\mathfrak{F}_t := \sigma(\eta_s,\ 0 \leq s \leq t)$. We denote by $\mathbb{E}_\eta$ the expectation with respect to $\mathbb{P}_\eta$.

We focus on a special class of Markov processes, which has been considered in [12], [32], [40], [51], namely consistent Markov processes. In order to precisely define the concept of consistent Markov process we introduce the lowering operator

$$\mathcal{A}f(\eta) := \int f(\eta - \delta_x)\eta(\mathrm{d}x), \quad \eta \in \mathbf{N}_{<\infty}$$

acting on non-negative measurable functions $f : \mathbf{N}_{<\infty} \to \mathbb{R}_+$ and on functions $f \in \mathcal{G}$, where $\mathcal{G}$ denotes the set of measurable functions $f : \mathbf{N}_{<\infty} \to \mathbb{R}$ such that the restriction of $f$ to every $n$-particle sector $\mathbf{N}_n := \{\eta \in \mathbf{N}_{<\infty} :\ \eta(E) = n\}$ is bounded. Note $\mathcal{A}$ is well-defined and that $\mathcal{A}f \in \mathcal{G}$ for $f \in \mathcal{G}$.

**Definition 2.1** (Consistent Markov process)**.** *Let $(\eta_t)_{t\geq 0}$ be a Markov process on $\mathbf{N}_{<\infty}$ with Markov semigroup $(P_t)_{t\geq 0}$. The process $(\eta_t)_{t\geq 0}$ is said to be consistent if for all $t \geq 0$ and measurable function $f : \mathbf{N}_{<\infty} \to \mathbb{R}_+$*

$$P_t\mathcal{A}f(\eta) = \mathcal{A}P_tf(\eta), \quad \eta \in \mathbf{N}_{<\infty}. \tag{2.1}$$

Notice that (2.1) can be written as

$$\mathbb{E}_\eta\left(\int f(\eta_t - \delta_x)\eta_t(\mathrm{d}x)\right) = \int \mathbb{E}_{\eta-\delta_x}(f(\eta_t))\eta(\mathrm{d}x).$$

Thus, in the context of conservative dynamics (see Assumption 2.2 (ii) below), by dividing the left-hand side by $\eta_t(E)$ and the right-hand side by $\eta(E)$, (2.1) reads as follows: on the left-hand side we first evolve the system and afterwards we remove uniformly at random a particle, while on the right-hand side we first remove uniformly at random a particle from the initial configuration and then we let the process evolve from the new initial state.

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where on the left-hand side we first evolve the system and afterwards we remove uniformly at random a particle, while on the right-hand side we first remove uniformly at random a particle from the initial configuration and then we let the process evolve from the new initial state. We refer to [12, Theorem 2.7 and Theorem 3.2] for further characterizations of consistency in terms of the infinitesimal generator $L$, namely $L\mathcal{A} = \mathcal{A}L$, and to Section 3 and 4 for some examples of consistent Markov processes.

For our results we need the following set of assumptions.

**Assumption 2.2.** *We assume that $(\eta_t)_{t\geq 0}$ is a Markov process on $\mathbf{N}_{<\infty}$ with Markov semigroup $(P_t)_{t\geq 0}$, such that*

(i) *it is consistent;*

(ii) *it is conservative, i.e. for all $\eta \in \mathbf{N}_{<\infty}$ it holds $\mathbb{P}_\eta(\eta_t(E) = \eta_0(E)) = 1$ for all $t > 0$.*

Notice that Assumption 2.2 (ii) yields $P_t f \in \mathcal{G}$ for all $f \in \mathcal{G}$ and thus, by Assumption 2.2 (i), we obtain $P_t \mathcal{A} f(\eta) = \mathcal{A} P_t f(\eta)$ for $f \in \mathcal{G}$ and $\eta \in \mathbf{N}_{<\infty}$.

Let us briefly explain how consistency as defined in Definition 2.1 relates to a stronger form of consistency reminiscent of Kolmogorov's consistency theorem. Often the process $(\eta_t)_{t\geq 0}$ comes from a process for *labeled* particles, as is the case for the independent random walks in Appendix A below. *Strong consistency*, called *compatibility* by Le Jan and Raimond [40, Definition 1.1], roughly means that time evolution and removal of any *deterministic* particle commute—there is no need to choose the particle to be removed uniformly at random.

Precisely, suppose that for each $n \in \mathbb{N}$, we are given a transition function $(p_t^{[n]})_{t\geq 0}$ on $(E^n, \mathcal{E}^{\otimes n})$, where $\mathcal{E}^{\otimes n}$ denotes the $n$ fold product $\sigma$ algebra of $\mathcal{E}$, that preserves permutation invariance. Then one can define a transition function $(P_t)_{t\geq 0}$ on $(\mathbf{N}_{<\infty}, \mathcal{N}_{<\infty})$ by $P_t(0, B) = \mathbb{1}_B(0)$, where $0$ denotes the null counting measure, and

$$P_t(\delta_{x_1} + \cdots + \delta_{x_n}, B) = p_t^{[n]}\big(x_1, \dots, x_n; \iota_n^{-1}(B)\big), \quad (x_1, x_2, \dots, x_n) \in E^n, B \in \mathcal{N}_{<\infty} \quad (2.2)$$

where $\iota_n : E^n \to \mathbf{N}_{<\infty}$ is the map given by $\iota_n(x_1, \dots, x_n) = \delta_{x_1} + \cdots + \delta_{x_n}$.

**Definition 2.3.** *The family $(p_t^{[n]})_{t\geq 0}$ is* strongly consistent *if for all $n \in \mathbb{N}$, $i \in \{1, \dots, n\}$, and $(x_1, \dots, x_n) \in E^n$, the image of the measure $\mathcal{E}^{\otimes n} \ni B \mapsto p_t^{[n]}(x_1, \dots, x_n; B)$ under the map $E^n \ni (y_1, \dots, y_n) \to (y_1, \dots, y_{i-1}, y_{i+1} \dots, y_n) \in E^{n-1}$ is equal to the measure $\mathcal{E}^{n-1} \ni B \mapsto p_t^{[n-1]}(x_1, \dots, x_{i-1}, x_{i+1} \dots, x_n; B)$.*

An elementary but important observation is that strong consistency of the family $(p_t^{[n]})_{t\geq 0}$ implies consistency of $(P_t)_{t\geq 0}$ in the sense of Definition 2.1. The observation yields a whole class of consistent processes, see Section 3.3.

Theorem 2.6 uses both $(P_t)_{t\geq 0}$ and a semigroup $(p_t^{[n]})_{t\geq 0}$ for labeled particles. As we wish to use the semigroup $(P_t)_{t\geq 0}$ as our starting point, let us mention that (2.2) implies

$$(P_t f)(\delta_{x_1} + \cdots + \delta_{x_n}) = \big(p_t^{[n]} f_n\big)(x_1, \dots, x_n) \quad (2.3)$$

whenever $f_n = f \circ \iota_n$ and $f : \mathbf{N}_{<\infty} \to \mathbb{R}$ is measurable and non-negative or bounded. This determines the action of $(p_t^{[n]})_{t\geq 0}$ on the space $\mathcal{F}_n$ of bounded, measurable, permutation invariant functions $f_n : E^n \to \mathbb{R}$ uniquely. Therefore, given a conservative semigroup $(P_t)_{t\geq 0}$ on $\mathbf{N}_{<\infty}$ we may take (2.3) as the *definition* of an associated semigroup of $n$ unlabeled distinct particles (which, depending on the specific dynamics under consideration, may occupy the same location) on the space of bounded permutation invariant functions $\mathcal{F}_n$. For $n = 0$, we set $\mathcal{F}_0 := \mathbb{R}$ and let $p_t^{[0]}$ be the identity operator on $\mathbb{R}$, for all $t \geq 0$. Notice that the semigroup property of $(p_t^{[n]})_{t\geq 0}$ on $\mathcal{F}_n$ is thus a direct consequence of the assumed semigroup property of $(P_t)_{t\geq 0}$, (2.3) and Assumption 2.2 (ii). We also remark that, since we start from the semigroup $(P_t)_{t\geq 0}$, we only deal with unlabeled particles and thus $(p_t^{[n]})_{t\geq 0}$ is defined only on permutation invariant functions. Defining $(p_t^{[n]})_{t\geq 0}$ on non-permutation invariant functions would require the specification of a labeling of the particles that does not follow from $(P_t)_{t\geq 0}$ and may depend on the specific example under consideration. For our purposes, we are not interested in a labeling and $(p_t^{[n]})_{t\geq 0}$ has to be thought as the semigroup of the configuration process associated to the system with $n$ particles.

## 2.2 Generalized falling factorial polynomials

For $\eta = \sum_{i=1}^m \delta_{x_i} \in \mathbf{N}$ and $n \in \mathbb{N}$, we recall (see, e.g., [38, Eq. (4.5)]) that the $n$-th factorial measure of $\eta$ on $(E^n, \mathcal{E}^{\otimes n})$ is given by

$$\eta^{(n)} := \sum_{1\leq i_1, \dots, i_n \leq m}^{\neq} \delta_{(x_{i_1}, \dots, x_{i_n})}, \quad (2.4)$$

where $\eta = 0$ yields $\eta^{(r)} = 0$. Using the notation adopted in [38], the superscript $\neq$ indicates summation over $n$-tuples $(i_1, \ldots, i_n)$ with pairwise different entries in $\{1, \ldots, m\}$.

**Definition 2.4.** *For $n \in \mathbb{N}$ and measurable $f_n : E^n \to \mathbb{R}$ we define the associated generalized falling factorial polynomial as follows*

$$J_n(f_n, \eta) := \int f_n(x_1, \ldots, x_n) \eta^{(n)}(\mathrm{d}(x_1, \ldots, x_n)), \quad \eta \in \mathbf{N}_{<\infty}.$$

*For $n = 0$ and $f_0 \in \mathbb{R}$ we set $J_0(f_0, \eta) := \int f_0 \, \mathrm{d}\eta^{(0)} := f_0$.*

In particular, we have $J_n(f_n, \cdot\,) \in \mathcal{G}$ for $f_n : E^n \to \mathbb{R}$ bounded and measurable.

**Remark 2.5.** The fact that $J_n$ generalizes falling factorial polynomials becomes evident when considering $f_n = \mathbb{1}_{B_1^{d_1} \times \cdots \times B_N^{d_N}}$ for pairwise disjoint sets $B_1, \ldots, B_N \in \mathcal{E}$, $N \in \mathbb{N}$ and $d_1, \ldots, d_N \in \mathbb{N}_0$, $d_1 + \ldots + d_N =: n$. Indeed, it follows from the definition of the factorial measure that

$$J_n(\mathbb{1}_{B_1^{d_1} \times \cdots \times B_N^{d_N}}, \eta) = (\eta(B_1))_{d_1} \cdots (\eta(B_N))_{d_N}, \quad \eta \in \mathbf{N}_{<\infty}, \tag{2.5}$$

where $(a)_k := a(a-1)\cdots(a-k+1)$, $a \in \mathbb{R}$, $k \in \mathbb{N}$, $(a)_0 := 1$, denotes the falling factorial. Equation (2.5) will be used in Section 3 below to recover known self-duality functions for particle systems on a finite set from Theorem 2.6 below. We refer to [20] for further properties of the generalized falling factorial polynomials.

Our first main result is an intertwining relation between the Markov semigroups $(P_t)_{t\geq 0}$ and $(p_t^{[n]})_{t\geq 0}$, with the generalized falling factorial polynomials $J_n$ defined above as intertwiner. Thus, we view the result as a generalization of the self-duality relations for interacting particle systems on a finite set where the self-duality functions consist in (weighted) falling factorial moments of the occupation variables (see Section 3.1 below and (A.1) below). Notice that in the literature one refers to *self*-intertwining (or *self*-duality) when the two processes involved in the relation are exactly the same. In our context the Markov semigroups $(P_t)_{t\geq 0}$ and $(p_t^{[n]})_{t\geq 0}$ act on two different function spaces and indeed refers to different Markov processes. However, in view of the equality (2.3) which builds on Assumption 2.2 (ii) and indicates that the dynamics of the particles in terms of permutations invariant observable is the same, we still refer to the relation in Theorem 2.6 below as *self-intertwining*.

**Theorem 2.6** (Self-intertwining relation)**.** *Let $(\eta_t)_{t\geq 0}$ be a Markov process satisfying Assumption 2.2. We then have*

$$P_t J_n(f_n, \cdot\,)(\eta) = J_n\left(p_t^{[n]} f_n, \eta\right), \quad \eta \in \mathbf{N}_{<\infty} \tag{2.6}$$

*for each $f_n \in \mathcal{F}_n$, $n \in \mathbb{N}_0$ and $t \geq 0$.*

*Proof.* Let us define the lowering operator $\mathcal{A}_{r-1,r}$ acting on functions $f_{r-1} \in \mathcal{F}_{r-1}$ as

$$(\mathcal{A}_{r-1,r} f_{r-1})(x_1, \ldots, x_r) := \sum_{k=1}^{r} f_{r-1}(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_r)$$

for $x_1, \ldots, x_r \in E$ and $r \geq 2$ and $\mathcal{A}_{0,1} f_0 := f_0 \mathbb{1}$, $f_0 \in \mathbb{R}$ for $r = 1$. We then have, as a direct consequence of consistency of $(\eta_t)_{t\geq 0}$, that $p_t^{[r]} \mathcal{A}_{r-1,r} f_{r-1} = \mathcal{A}_{r-1,r} p_t^{[r-1]} f_{r-1}$, $r \in \mathbb{N}$. Denoting for all $r \geq n \geq 0$,

$$\mathcal{A}_{n,r} f_n := \begin{cases} \mathcal{A}_{r-1,r} \cdots \mathcal{A}_{n,n+1} f_n & \text{if } r > n \\ f_n & \text{if } n = r \end{cases},$$

for all $f_n \in \mathcal{F}_n$, one obtains, by induction, that

$$p_t^{[r]}\mathcal{A}_{n,r}f_n = \mathcal{A}_{n,r}p_t^{[n]}f_n.$$

The proof is concluded by noticing that for all $n \le r$, $x_1, \ldots, x_r \in E$,

$$J_n(f_n, \delta_{x_1} + \ldots + \delta_{x_r}) = \frac{n!}{(r-n)!}(\mathcal{A}_{n,r}f_n)(x_1, \ldots, x_r). \qquad \square$$

**Remark 2.7.** A close look at the proof reveals that the relation in Theorem 2.6 is in fact an equivalence: a conservative process is consistent if and only if the self-intertwining relation (2.6) holds true for all $n, f_n, t$. The equivalence is closely related to Theorem 4.3 in [12] in the discrete setting.

Theorem 2.6 can be rephrased in a number of ways. The first rephrasing is in terms of kernels and justifies the denomination *intertwining*. Let $\Lambda_n : \mathbf{N}_{<\infty} \times \mathcal{E}^{\otimes n} \to \mathbb{R}_+$ be the kernel given by $\Lambda_n(\eta, B) := \eta^{(n)}(B) = J_n(\mathbb{1}_B, \eta)$. Then, $P_t\Lambda_n = \Lambda_n p_t^{[n]}$ means that

$$\int P_t(\eta, \mathrm{d}\xi)\Lambda_n(\xi, B) = \int \Lambda_n(\eta, \mathrm{d}x)p_t^{[n]}(x, B)$$

for all $\eta \in \mathbf{N}_{<\infty}$ and all permutation invariant sets $B \in \mathcal{E}^{\otimes n}$, where $p_t^{[n]}(x, B) = p_t^{[n]}\mathbb{1}_B(x)$. Hence, the kernel $\Lambda_n(\eta, B) = J_n(\mathbb{1}_B, \eta)$ intertwines the semigroups $(P_t)_{t\ge 0}$ and $(p_t^{[n]})$. The second rephrasing uses the semigroup $(P_t)_{t\ge 0}$ only, which makes the “self” in self-intertwining spring to the eye. Set

$$\mathcal{K}(f, \eta) := f(0) + \sum_{n=1}^{\infty} \frac{1}{n!} \int f(\delta_{x_1} + \ldots + \delta_{x_n})\eta^{(n)}(\mathrm{d}(x_1, \ldots, x_n))$$

for measurable bounded $f : \mathbf{N}_{<\infty} \to \mathbb{R}$ and $\eta \in \mathbf{N}_{<\infty}$. Note that the integral vanishes for $n > \eta(E)$ and $\mathcal{K}(f, \cdot\,) \in \mathcal{G}$ for $f \in \mathcal{G}$. The function $\mathcal{K}(f, \cdot\,)$ is also known as $K$-transform of $f$ (see, e.g., [35, Section 3.2] and last equation in p.243 of [41]) and by linearity, it follows from (2.3) that (2.6) is equivalent to the fact that $\mathcal{K}$ intertwines $(P_t)_{t\ge 0}$ with itself, i.e.,

$$P_t\mathcal{K}(f, \cdot\,)(\eta) = \mathcal{K}(P_t f, \eta). \tag{2.7}$$

for $f \in \mathcal{G}, \eta \in \mathbf{N}_{<\infty}$. For free Kawasaki dynamics, which is a special case of independent particles, this result is in fact known (see [35, Section 3.2]).

In terms of expectations, the self-intertwining relation becomes

$$\mathbb{E}_\eta\left[\int f(\delta_{x_1} + \ldots + \delta_{x_n})\eta_t^{(n)}(\mathrm{d}(x_1, \ldots, x_n))\right] = \int \mathbb{E}_{\delta_{x_1}+\ldots+\delta_{x_n}}\left[f(\eta_t)\right]\eta^{(n)}(\mathrm{d}(x_1, \ldots, x_n))$$

for measurable, bounded $f : \mathbf{N}_n \to \mathbb{R}$, $n \in \mathbb{N}_0$ and $t \ge 0$.

To conclude we note a corollary on the time-evolution of correlation functions.

**Corollary 2.8.** *Under the assumptions of Theorem 2.6, the following holds true for every initial condition $\eta \in \mathbf{N}_{<\infty}$. Let $\alpha_n^t(B) := \mathbb{E}_\eta[\eta_t^{(n)}(B)]$ be the $n$-th factorial moment measure of the process $(\eta_t)_{t\ge 0}$ started in $\eta$. Then*

$$\alpha_n^t(B) = \int \alpha_n^0(\mathrm{d}x)p_t^{[n]}(x, B)$$

*for all $n \in \mathbb{N}$, $t \ge 0$, and permutation invariant sets $B \in \mathcal{E}^{\otimes n}$.*

Notice that given the initial condition $\eta$ we have $\alpha_n^0 = \eta^{(n)}$.

*Proof.* We have

$$\alpha_n^t(B) = \mathbb{E}_\eta\Big[J_n(\mathbb{1}_B, \eta_t)\Big] = J_n\left(p_t^{[n]}\mathbb{1}_B, \eta\right) = \int \eta^{(n)}(\mathrm{d}x)(p_t^{[n]}\mathbb{1}_B)(x) = \int \alpha_n^0(\mathrm{d}x)p_t^{[n]}(x,B)$$

where in the second equality we used Theorem 2.6. □

A reformulation of (2.6) which emphasizes the connection with classical self-duality relations is recovered under the additional condition that for some $\sigma$-finite measure $\lambda$ on $E$ and each $n \in \mathbb{N}$, there exists a measurable function $u_t^{[n]} : E^n \times E^n \to \mathbb{R}_+$ with $u_t^{[n]}(x,y) = u_t^{[n]}(y,x)$ on $E^n \times E^n$ and

$$p_t^{[n]}(x,B) = \int_B u_t^{[n]}(x,y)\lambda^{\otimes n}(\mathrm{d}y) \tag{2.8}$$

for all $t > 0$, $x \in E^n$, and permutation invariant set $B \in \mathcal{E}^{\otimes n}$. Notice that from (2.8) and the symmetry of $u_t^{[n]}$ it follows that $\lambda^{\otimes n}$ is a reversible measure. The additional condition is satisfied for example by independent reversible diffusions on $E = \mathbb{R}^d$ where $\lambda$ is the Lebesgue measure and $u_t^{[n]}$ is the product from $i = 1$ to $n$ of the densities of the transition kernels of the one particle dynamics. Corollary 2.8, (2.8), and the symmetry of $u_t^{[n]}$ yield

$$\mathbb{E}_\eta\left[\eta_t^{(n)}(B)\right] = \int_B \left(\int_{E^n} u_t^{[n]}(y,x)\eta^{(n)}(\mathrm{d}x)\right)\lambda^{\otimes n}(\mathrm{d}y). \tag{2.9}$$

This relation generalizes Proposition A.2 below, in which the classical self-duality relation for independent random walks is reformulated. Finally we remark that (2.8) shares similarities with the notion of duality (with respect to a measure) from probabilistic potential theory, see Blumenthal and Getoor [8, Chapter VI].

### 2.3 Generalized orthogonal polynomials

In this section we generalize the orthogonal self-duality relation (see, e.g., [24, Theorem 1], [50, Section 4.1] and Appendix A.3 below) to the class of Markov processes on $\mathbf{N}_{<\infty}$ satisfying Assumption 2.2. More precisely, assuming that there exists a reversible measure $\rho$, we show another self-intertwining relation where the intertwiner satisfies an orthogonality relation with respect to this measure. The intertwiner is a so-called *generalized orthogonal polynomial*, a well studied object in the infinite dimensional analysis literature (see, e.g., [44, Section 5], [52, Chapter 5] and [55, p.678]). We thus start by constructing the generalized orthogonal polynomials, following closely [44, Section 5].

Let $\rho$ be a probability measure on $(\mathbf{N}_{<\infty}, \mathcal{N}_{<\infty})$. We use the shorthand $L^2(\rho) := L^2(\mathbf{N}_{<\infty}, \mathcal{N}_{<\infty}, \rho)$. Through the rest of the section we assume that all moments of the total number of particles are finite.

**Assumption 2.9.** *Assume* $\int \eta(E)^n \rho(\mathrm{d}\eta) < \infty$ *for all* $n \in \mathbb{N}$.

Assumption 2.9 implies that every map $\eta \mapsto \eta^{\otimes n}(f_n) = \int f_n \mathrm{d}\eta^{\otimes n}$, with $f_n : E^n \to \mathbb{R}$ a bounded measurable function, is in $L^2(\rho)$.

Orthogonal polynomials in a single real variable can be constructed by an orthogonalization procedure. This definition extends to the infinite-dimensional setting: generalized orthogonal polynomials are defined by taking an orthogonal projection onto a proper subspace of generalized polynomials, see [44, Section 5] and references therein. We thus define the space $\mathcal{P}_n$ of generalized polynomials (with bounded coefficients) of degree less or equal than $n \in \mathbb{N}_0$ as the set of linear combinations of maps $\eta \mapsto \int f_k \mathrm{d}\eta^{\otimes k}$, $k \leq n$, with bounded measurable $f_k : E^k \to \mathbb{R}$, with the convention $\eta^{\otimes 0}(f_0) := f_0 \in \mathbb{R}$. Thus the set $\mathcal{P}_0$ consists of the constant functions. We refer to the functions $f_k$ as coefficients.

Assumption 2 guarantees that every polynomial is square-integrable, i.e., $\mathcal{P}_n$ is a subspace of $L^2(\rho)$. In general it is not closed, we write $\overline{\mathcal{P}_n}$ for its closure in $L^2(\rho)$. The linear space $\mathcal{P}_n$ and its closure have the same orthogonal complement $\mathcal{P}_n^\perp = \overline{\mathcal{P}_n}^\perp$ in $L^2(\rho)$.

The next definition is equivalent to a definition from [44, Section 5].

**Definition 2.10** (Generalized orthogonal polynomials)**.** *For $n \in \mathbb{N}$ and $f_n : E^n \to \mathbb{R}$ a bounded measurable function we define the associated generalized orthogonal polynomial as follows*

$$I_n(f_n, \cdot\,) := \textit{orthogonal projection of } (\eta \mapsto \eta^{\otimes n}(f_n)) \textit{ onto } \overline{\mathcal{P}_{n-1}}^\perp.$$

Equivalently,

$$I_n(f_n, \eta) = \eta^{\otimes n}(f_n) - Q(\eta)$$

with $Q \in \overline{\mathcal{P}_{n-1}}$ the orthogonal projection of $\eta \mapsto \eta^{\otimes n}(f_n)$ onto $\overline{\mathcal{P}_{n-1}}$. Notice that $I_n(f_n, \eta)$ is only defined up to $\rho$-null sets.

**Remark 2.11** (Wick dots and multiple stochastic integrals)**.** In the literature (see, e.g., [44, Section 5]) the generalized orthogonal polynomial $I_n(f_n, \eta)$ is often denoted by $:\eta^{\otimes n}(f_n):$ ("Wick dots"). When $\rho$ is the distribution of a Poisson point process with intensity measure $\lambda$, the generalized orthogonal polynomial is given by a multiple stochastic integral with respect to the compensated Poisson measure $\eta - \lambda$, hence we use the notation $I_n(f_n, \eta)$ similarly to [37, Eq. (25)]. The notation has the advantage of being analogous to the one used for the self-intertwiner $J_n$ in Section 2.2, which is why we keep it.

**Remark 2.12** (Orthogonality relation)**.** It follows from Definition 2.10 that for each $k \in \mathbb{N}$ and bounded measurable $f_k : E^k \to \mathbb{R}$, $I_k(f_k, \cdot)$ is in the orthogonal difference between $\overline{\mathcal{P}_k}$ and $\overline{\mathcal{P}_{k-1}}$ and thus

$$\int I_n(f_n, \cdot\,) I_m(g_m, \cdot\,)\, \mathrm{d}\rho = 0$$

for $n \neq m$.

**Remark 2.13** (Chaos decompositions and Lévy white noise)**.** Generalized orthogonal polynomials appear naturally in the study of non-Gaussian white noise [6], [7], they are used to prove chaos decompositions. The relation between polynomial chaos and chaos decompositions in terms of multiple stochastic integrals with respect to *power jump martingales* [48] is investigated in detail [44]. Chaos decompositions play a role in the study of Lévy white noise and stochastic differential equations driven by Lévy white noise [16], [43], [47]. Moreover $f_n \mapsto I_n(f_n, \cdot\,)$ extends to a unitary operator on the space of permutation invariant functions that are square integrable with respect to some measure $\lambda_n$ (see, e.g., [44, Corollary 5.2] for further details). When $\rho$ is the distribution of a Poisson process with intensity measure $\lambda$, the measure $\lambda_n$ is the product $\lambda_n = \lambda^{\otimes n}$, but in general the measure $\lambda_n$ is more complicated.

We complement the definition of the generalized orthogonal polynomials by two propositions on their properties. The first proposition says that the orthogonal polynomials can also be obtained by an orthogonal projection of the generalized falling factorial polynomials $\eta \mapsto J_n(f_n, \eta)$ instead of $\eta \mapsto \eta^{\otimes}(f_n)$. This observation plays an important role in the proof of Theorem 2.17.

**Proposition 2.14.** *The following identities hold*

$$\mathcal{P}_n = \left\{ \eta \mapsto \sum_{k=0}^{n} J_k(f_k, \eta) : f_k \in \mathcal{F}_k, k \in \{0, \dots, n\} \right\}, \quad n \in \mathbb{N}_0, \tag{2.10}$$

$$I_n(f_n,\,\cdot\,) = \textit{orthogonal projection of } J_n(f_n,\,\cdot\,) \textit{ onto } \overline{\mathcal{P}_{n-1}}^{\perp}, \quad f_n \in \mathcal{F}_n,\ n \in \mathbb{N}. \tag{2.11}$$

We note that (2.10) is a direct consequence of the fact that $J_k(f_k,\,\cdot\,)$ can be written as linear combination of integrals with respect to the product measure of degree $\leq k$ and vice versa, see [20, Eq. (3.1)-(3.3)]. We provide a complete proof of the above proposition in Section 2.4 below.

The second proposition applies under an additional assumption of complete independence. A finite point process $\zeta$ is *completely independent* (or *completely orthogonal*) [38, Section 6.4] if the counting variables $\zeta(A_1),\ldots,\zeta(A_m)$ associated with pairwise disjoint regions $A_1,\ldots,A_m \in \mathcal{E}$, $m \in \mathbb{N}$, are independent. Complete independence implies a factorization property of generalized orthogonal polynomials with disjointly supported coefficients. Recall that given $f_i : E^{d_i} \to \mathbb{R}$, $i = 1,\ldots,N$, $f_1 \otimes \ldots \otimes f_N$ is the function that maps the vector $(z_1,\ldots,z_N)$ with $z_i \in E^{d_i}$ for each $i \in \{1,\ldots,N\}$ to $\prod_{i=1}^N f_i(z_i)$.

**Proposition 2.15.** *Suppose that $\rho$ is the distribution of some finite completely independent point process. Let $N \geq 2$, $A_1,\ldots,A_N \in \mathcal{E}$ be pairwise disjoint, and $d_1,\ldots,d_N \in \mathbb{N}_0$. Further let $f_i : E^{d_i} \to \mathbb{R}$, $i = 1,\ldots,N$ be bounded measurable functions that vanish on $E^{d_i} \setminus A_i^{d_i}$. Set $n := d_1 + \cdots + d_N$. Then*

$$I_n(f_1 \otimes \ldots \otimes f_N, \eta) = I_{d_1}(f_1,\eta) \cdots I_{d_N}(f_N,\eta) \tag{2.12}$$

*for $\rho$-almost all $\eta \in \mathbf{N}_{<\infty}$.*

The proposition is proven in Section 2.4. For special cases of measures $\rho$ that give rise to orthogonal polynomials of Meixner's type, a similar factorization property is found, for example, in [45, Lemma 3.1]. Our proposition instead holds true for all distributions of completely independent point processes.

**Remark 2.16.** A particularly relevant case is when $f_i$ is the indicator of $A_i^{d_i}$. Then Proposition 2.15 says that the orthogonalized version of $\eta \mapsto \prod_{i=1}^n \eta(A_i)^{d_i}$ is equal to the product of the orthogonalized versions of $\eta \mapsto \eta(A_i)^{d_i}$. When $\rho$ is the distribution of a Poisson or Pascal point process (see Sections 3 and 4 below), the orthogonalized version of $\eta(A_i)^{d_i}$ is in fact a univariate orthogonal polynomial in the variable $\eta(A_i) \in \mathbb{N}_0$ and we obtain a product of univariate orthogonal polynomials, see (3.6) and (4.3). In general, however, the orthogonalized version of $\eta(A_i)^{d_i}$ need not be a univariate polynomial.

We now state the second theorem of this section, which is the analogue of Theorem 2.6 but where the self-intertwiner is the generalized orthogonal polynomial introduced above.

**Theorem 2.17** (Self-intertwining relation)**.** *Let $(\eta_t)_{t\geq 0}$ be a Markov process on $\mathbf{N}_{<\infty}$ that satisfies Assumption 2.2, i.e. it is consistent and conservative. Let $\rho$ be a reversible probability measure for $(\eta_t)_{t\geq 0}$ that satisfies Assumption 2.9. Then,*

$$P_t I_n(f_n,\cdot)(\eta) = I_n\left(p_t^{[n]} f_n, \eta\right) \tag{2.13}$$

*for $\rho$-almost all $\eta \in \mathbf{N}_{<\infty}$, all $t \geq 0$, and all $f_n \in \mathcal{F}_n$.*

*Proof.* To lighten notation, we drop the second variable of $I_n(f_n,\,\cdot\,)$ and write $I_n(f_n)$ when we refer to the function in $L^2(\rho)$, similarly for $J_n(f_n)$. Let $\Pi_{n-1}$ be the orthogonal projection from $L^2(\rho)$ onto $\overline{\mathcal{P}_{n-1}}$, and $\mathrm{id}$ the identity operator in $L^2(\rho)$. By Proposition 2.14,

$$I_n(f_n) = (\mathrm{id} - \Pi_{n-1}) J_n(f_n).$$

The theorem follows once we know that the semigroup $P_t$ commutes with the projection $\Pi_{n-1}$ i.e.

$$P_t \Pi_{n-1} = \Pi_{n-1} P_t \tag{2.14}$$

since then, (2.13) is obtained as follows

$$\begin{aligned} P_t I_n(f_n) &= P_t(\mathrm{id} - \Pi_{n-1}) J_n(f_n) \\ &= (\mathrm{id} - \Pi_{n-1}) P_t J_n(f_n) = (\mathrm{id} - \Pi_{n-1}) J_n\left(p_t^{[n]} f_n\right) = I_n\left(p_t^{[n]} f_n\right) \end{aligned}$$

where we used Proposition 2.14 in the first and the fourth equality and Theorem 2.6 in the third equality.

Let $k \leq n-1$ and let us recall the characterization of $\mathcal{P}_n$ given in Proposition 2.14. Using Theorem 2.6 combined with the fact that $p_t^{[k]} f_k \in \mathcal{F}_k$ for all $f_k \in \mathcal{F}_k$, we have that $P_t J_k(f_k, \cdot) = J_k(p_t^{[k]} f_k, \cdot) \in \mathcal{P}_{n-1}$. Thus, for all $t \geq 0$ and $n \in \mathbb{N}_0$, $P_t \mathcal{P}_{n-1} \subset \mathcal{P}_{n-1}$ and by the boundedness of $P_t$ on $L^2(\rho)$ we obtain

$$P_t \overline{\mathcal{P}_{n-1}} \subset \overline{\mathcal{P}_{n-1}}. \tag{2.15}$$

The operator $P_t$ is self-adjoint in $L^2(\rho)$ because of the reversibility of $\rho$. It is a general fact that a bounded self-adjoint operator that leaves a closed vector space invariant commutes with the orthogonal projection onto that space. Let us check this fact for our concrete operators and spaces. For $f \in \overline{\mathcal{P}_{n-1}}^{\perp}$, by the self-adjointness of $P_t$ on $L^2(\rho)$ and (2.15), we have, for all $g \in \overline{\mathcal{P}_{n-1}}$, that $\int (P_t f) g \mathrm{d}\rho = \int f (P_t g) \mathrm{d}\rho = 0$ and thus

$$P_t \overline{\mathcal{P}_{n-1}}^{\perp} \subset \overline{\mathcal{P}_{n-1}}^{\perp}. \tag{2.16}$$

We then have, using (2.15), (2.16) and $f - \Pi_{n-1} f \in \overline{\mathcal{P}_{n-1}}^{\perp}$ that, for all $f \in L^2(\rho)$,

$$\Pi_{n-1} P_t f = \Pi_{n-1} P_t \Pi_{n-1} f + \Pi_{n-1} P_t (f - \Pi_{n-1} f) = P_t \Pi_{n-1} f.$$

This completes the proof of (2.14) and the proof of the theorem. □

### 2.4 Properties of generalized orthogonal polynomials. Proof of Propositions 2.14 and 2.15

This section is devoted to the proof of Propositions 2.14 and 2.15.

**Orthogonalization of generalized falling factorial polynomials** Proposition 2.14 follows from explicit formulas that link factorial measures $\eta^{(n)}$ and product measure $\eta^{\otimes n}$. These relations are similar to relations between moments and factorial moments of integer-valued random variables with Stirling numbers, see [13, Chapter 5]. A systematic treatment in terms of Stirling operators is found in [20].

*Proof of* (2.10). In order to show that $\mathcal{P}_n$ is the linear hull of generalized falling factorials $J_k(f_k, \eta)$, $k \leq n$, it is enough to check that every monomial $\eta \mapsto \eta^{\otimes n}(f_n)$ is a linear combination of falling factorials of degree $k \leq n$ and vice-versa.

Let $\eta = \delta_{x_1} + \cdots + \delta_{x_\kappa} \in \mathbf{N}_{<\infty}$ and $f_n : E^n \to \mathbb{R}$ be a bounded measurable function. Then

$$\eta^{\otimes n}(f_n) = \sum_{1 \leq i_1, \ldots, i_n \leq \kappa} f_n(x_{i_1}, \ldots, x_{i_n}).$$

Every multi-index $(i_1, \ldots, i_n)$ on the right-hand side gives rise to a set partition $\sigma$ of $\{1, \ldots, n\}$ in which $k$ and $\ell$ belong to the same block if and only if $i_k = i_\ell$. Denote by $\Sigma_n$ the set of partitions of $\{1, \ldots, n\}$. For $\sigma \in \Sigma_n$, let $|\sigma|$ be the number of blocks of the set partition. Further let $(f_n)_\sigma : E^{|\sigma|} \to \mathbb{R}$ be the function obtained from $f_n$ by identifying, in order of occurrence, those arguments which belong to the same block of $\sigma$. As an example,

$$(f_4)_{\{\{1,3\},\{2\},\{4\}\}}(x_1, x_2, x_3) = f_4(x_1, x_2, x_1, x_3).$$

Grouping multi-indices $(i_1,\dots,i_n)$ that give rise to the same partition $\sigma$, we find

$$\int f_n \mathrm{d}\eta^{\otimes n} = \sum_{\sigma\in\Sigma_n} \int (f_n)_\sigma \,\mathrm{d}\eta^{(|\sigma|)}$$

(compare [13, Exercise 5.4.5]) and conclude that $\eta^{\otimes n}(f_n)$ is a linear combination of generalized falling factorials of degrees $|\sigma|\le n$.

Conversely,

$$\int f_n \mathrm{d}\eta^{(n)} = \sum_{\sigma\in\Sigma_n} (-1)^{n-|\sigma|} \int (f_n)_\sigma \,\mathrm{d}\eta^{\otimes|\sigma|} \tag{2.17}$$

hence the falling factorial of degree $n$ on the left-hand side is a linear combination of monomials $\eta^{\otimes k}(g_k)$ of degree $k\le n$. $\square$

*Proof of* (2.11). For $n\in\mathbb{N}$, we notice that (2.17) implies

$$\int f_n \mathrm{d}\eta^{(n)} = \int f_n \mathrm{d}\eta^{\otimes n} + Q(\eta)$$

for some $Q\in\mathcal{P}_{n-1}$, given by a sum over set partitions with a number of blocks $|\sigma|\le n-1$. It follows that $\eta\mapsto J_n(f_n,\eta)$ and $\eta\mapsto\eta^{\otimes n}(f_n)$ have the same orthogonal projections onto $(\mathcal{P}_{n-1})^\perp$. $\square$

**Factorization property of generalized orthogonal polynomials** In order to exploit the complete independence, it is helpful to check that if $f: E^n\to\mathbb{R}$ is supported in $A^n$, then $I_n(f,\eta)$ depends only on what happens inside $A$. We show a bit more. Let $\mathcal{P}_n(A)\subset\mathcal{P}_n$ be the space of linear combinations of maps $\eta\mapsto\eta^{\otimes k}(f_k)$, $k\le n$, with bounded measurable $f_k: E^k\to\mathbb{R}$ vanishing on $E^k\setminus A^k$. Notice that every function $F\in\mathcal{P}_n(A)$ depends only on the restriction $\eta_A$, defined by $\eta_A(B):=\eta(A\cap B)$.

**Lemma 2.18.** *Let $d\in\mathbb{N}$, $A\in\mathcal{E}$, and $f: E^d\to\mathbb{R}$ be a bounded measurable function that vanishes on $E^d\setminus A^d$. Then there exists a map $Q\in\overline{\mathcal{P}_{d-1}(A)}$ such that $I_d(f,\eta)=\eta^{\otimes d}(f)-Q(\eta)$ for $\rho$-almost all $\eta\in\mathbf{N}_{<\infty}$.*

*Proof.* Let $Q$ be the orthogonal projection of $\eta\mapsto\eta^{\otimes d}(f)$ onto $\overline{\mathcal{P}_{d-1}(A)}$. Then $Q\in\overline{\mathcal{P}_{d-1}(A)}$ and the difference $F(\eta):=\eta^{\otimes d}(f)-Q(\eta)$ is orthogonal to $\overline{\mathcal{P}_{d-1}(A)}$. We exploit the complete independence to show that $F$ is actually orthogonal to the bigger space $\overline{\mathcal{P}_{d-1}}$.

Let $n\in\{1,\dots,d-1\}$. If $C\in\mathcal{E}^{\otimes n}$ is of the form $C_1\times C_2$ with $C_i\in\mathcal{E}^{\otimes s_i}$ where $s_1,s_2\in\mathbb{N}_0$ and $C_1\subset A^{s_1}$, $C_2\subset(A^c)^{s_2}$, then $\eta^{\otimes n}(C)=\eta^{\otimes s_1}(C_1)\eta^{\otimes s_2}(C_2)$ and by the complete independence (notice $F(\eta)=F(\eta_A)$)

$$\int F(\eta)\eta^{\otimes n}(C)\rho(\mathrm{d}\eta) = \Big(\int F(\eta)\eta^{\otimes s_1}(C_1)\rho(\mathrm{d}\eta)\Big)\Big(\int \eta^{\otimes s_2}(C_2)\rho(\mathrm{d}\eta)\Big).$$

The first integral on the right-hand side vanishes because of $C_1\subset A^{s_1}$, $s_1\le d-1$, and $F\perp\overline{\mathcal{P}_{d-1}(A)}$. Therefore $F$ is orthogonal to $\eta\mapsto\eta^{\otimes n}(C)$.

More generally, every set $C\in\mathcal{E}^{\otimes n}$ is the disjoint union of Cartesian products $C_1\times\cdots\times C_n$ in which every $C_i$ is either contained in $A$ or in $A^c$. Taking linear combinations and exploiting that $\eta^{\otimes n}(g)$ does not change if we permute variables in $g$, we find that $F$ is orthogonal to $\eta^{\otimes n}(C)$ for all $C\in\mathcal{E}^{\otimes n}$ and then, by the usual measure-theoretic arguments, to all maps $\eta\mapsto\eta^{\otimes n}(g)$, $g: E^n\to\mathbb{R}$ bounded and measurable. The map $F$ is also orthogonal to all constant functions because every constant function is in $\overline{\mathcal{P}_{d-1}(A)}$.

Hence, taking linear combinations of maps $\eta^{\otimes n}(f_n)$, $n\in\{0,\dots,d-1\}$, we see that $F$ is orthogonal to the space $\overline{\mathcal{P}_{d-1}}$. As $F(\eta)=\eta^{\otimes d}(f)-Q(\eta)$ with $Q\in\overline{\mathcal{P}_{d-1}}$, it follows that $I_d(f,\eta)=F(\eta)$ for $\rho$-almost all $\eta$. $\square$

When evaluating the product of two generalized orthogonal polynomials $I_n(f,\eta)$ using Lemma 2.18, it is important to know that the product of two polynomials is again a polynomial.

**Lemma 2.19.** *Let $A$ and $B$ be two disjoint measurable subsets of $E$ and $m,n \in \mathbb{N}_0$. Pick $F \in \overline{\mathcal{P}_m(A)}$ and $G \in \overline{\mathcal{P}_n(B)}$. Then $FG$ is in $\overline{\mathcal{P}_{m+n}(A \cup B)}$.*

*Proof.* Write $||\cdot||$ for the $L^2(\rho)$-norm. Let $(F_k)_{k\in\mathbb{N}}$ and $(G_k)_{k\in\mathbb{N}}$ be sequences in $\mathcal{P}_m(A)$ and $\mathcal{P}_n(B)$, respectively, with $||F - F_k|| \to 0$ and $||G - G_k|| \to 0$. We have $F_k(\eta) = F_k(\eta_A)$ for all $k$ and $\eta$ hence $F(\eta) = F(\eta_A)$ for $\rho$-almost all $\eta$. Similarly $G_k$ and $G$ depend on $\eta_B$ only. The triangle inequality and the complete independence yield

$$\begin{aligned} ||FG - F_kG_k|| &\le ||(F - F_k)G|| + ||F_k(G - G_k)|| \\ &= ||F - F_k||\,||G|| + ||F_k||\,||G - G_k|| \to 0. \end{aligned}$$

As each product $F_kG_k$ is in $\mathcal{P}_{m+n}(A \cup B)$, the limit $FG$ is in the closure $\overline{\mathcal{P}_{m+n}(A \cup B)}$. □

*Proof of Proposition 2.15.* It is enough to treat the case $N = 2$; the general case follows by an induction over $N$. Let $A_1$ and $A_2$ be two disjoint measurable subsets in $\mathcal{E}$. Let $d_1, d_2$ be two integers and $f_i : E^{d_i} \to \mathbb{R}$, $i = 1,2$ be two bounded measurable functions that vanish outside $A_1^{d_1}$ and $A_2^{d_2}$ respectively. By Lemma 2.18, there exist maps $Q_i \in \overline{\mathcal{P}_{d_i-1}(A_i)}$, $i = 1,2$, such that

$$I_{d_1}(f_1,\eta) = \eta^{\otimes d_1}(f_1) - Q_1(\eta), \quad I_{d_2}(f_2,\eta) = \eta^{\otimes d_2}(f_2) - Q_2(\eta)$$

for $\rho$-almost all $\eta$. Therefore by Lemma 2.19, we have

$$I_{d_1}(f_1,\eta)I_{d_2}(f_2,\eta) = \eta^{\otimes d_1}(f_1)\eta^{\otimes d_2}(f_2) - Q(\eta) \tag{2.18}$$

with $Q \in \overline{\mathcal{P}_{d_1+d_2-1}}$. Let $s_1, s_2 \in \mathbb{N}_0$ and $C_1 \in \mathcal{E}^{s_1}$, $C_2 \in \mathcal{E}^{s_2}$ with $s_1 + s_2 \le d_1 + d_2 - 1$ and $C_1 \subset A_1^{s_1}$, $C_2 \subset (A_1^c)^{s_2}$. Then, by the complete independence,

$$\int I_{d_1}(f_1,\eta)I_{d_2}(f_2,\eta)\,\eta^{\otimes(s_1+s_2)}(C_1 \times C_2)\rho(\mathrm{d}\eta) = \prod_{i=1}^{2}\int I_{d_i}(f_i,\eta)\,\eta^{\otimes s_i}(C_i)\rho(\mathrm{d}\eta).$$

We must have $s_1 \le d_1 - 1$ or $s_2 \le d_2 - 1$, therefore at least one of the integrals on the right-hand side vanishes and the product $I_{d_1}(f_1,\eta)I_{d_2}(f_2,\eta)$ is orthogonal to $\eta^{\otimes(d_1+d_2-1)}(C)$. We conclude with an argument similar to the proof of Lemma 2.18 that $I_{d_1}(f_1,\eta)I_{d_2}(f_2,\eta)$ is in fact orthogonal to $\overline{\mathcal{P}_{d_1+d_2-1}}$. It follows that the product is equal to $I_{d_1+d_2}(f_1 \otimes f_2,\eta)$ for $\rho$-almost all $\eta$. □

## 3 Examples

In this section we provide some examples of known consistent and conservative Markov processes, i.e. of processes satisfying Assumption 2.2. Moreover, we also provide the reversible distribution of those processes, when known, and we specify when the assumptions of Theorem 2.17 are also satisfied. In particular, we recover known self-duality functions of systems of particles hopping on a finite set. In the next section, we introduce a new process, which generalizes the inclusion process (see, e.g., [26, Section 3.3]) for which both main theorems apply.

Before doing that, we recall the definition of the Charlier and Meixner polynomials, see e.g. [33, Section 9.14 and Section 9.10], which are polynomials orthogonal with respect to the Poisson and negative binomial distribution, respectively. Differently from the usual definition in the literature, we normalize orthogonal polynomials to be monic where a polynomial $p(x) = a_0 + a_1 x + \ldots + a_n x^n$ is called monic if $a_n = 1$. These sequences of orthogonal polynomials can be expressed by using the generalized hypergeometric function given by

$$ {}_pF_q\left(\begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix}\,\middle|\, z\right) := \sum_{k=0}^{\infty} \frac{(a_1)^{(k)}\cdots(a_p)^{(k)}}{(b_1)^{(k)}\cdots(b_q)^{(k)}} \frac{z^k}{k!} $$

for $a_1,\ldots,a_p,b_1,\ldots,b_q,z \in \mathbb{R}$, $p,q \in \mathbb{N}$, where we remind the reader that $(a)^{(0)} := 1$ and $(a)^{(k)} := a(a+1)\cdots(a+k-1)$ denotes the rising factorial (also called Pochhammer symbol). Similarly, we recall the falling factorial defined by $(a)_k := a(a-1)\cdots(a-k+1)$, $(a)_0 := 1$.

(i) The monic Charlier polynomials are given by

$$ \mathscr{C}_n(x;\alpha) := (-\alpha)^n\, {}_2F_0\left(\begin{matrix} -n,-x \\ - \end{matrix}\,\middle|\, -\frac{1}{\alpha}\right) = \sum_{k=0}^{n} \binom{n}{k} (-\alpha)^{n-k} (x)_k, \quad x \in \mathbb{N}_0 $$

for $n \in \mathbb{N}_0$ and $\alpha > 0$ and they satisfy the orthogonality relation

$$ \sum_{\ell=0}^{\infty} \mathscr{C}_n(\ell;\alpha)\mathscr{C}_m(\ell;\alpha)\mathrm{Poi}(\alpha)(\{\ell\}) = \mathbb{1}_{\{n=m\}} \alpha^n n! $$

for $n,m \in \mathbb{N}_0$, i.e., $\mathscr{C}_n(\,\cdot\,;\alpha)$ are orthogonal polynomials with respect to the Poisson distribution $\mathrm{Poi}(\alpha)(\{\ell\}) = e^{-\alpha}\frac{\alpha^\ell}{\ell!}$. $\ell \in \mathbb{N}_0$.

(ii) The monic Meixner polynomials are given by

$$ \begin{aligned} \mathscr{M}_n(x;a;p) &:= (a)^{(n)} \left(1-\frac{1}{p}\right)^{-n} {}_2F_1\left(\begin{matrix} -x,-n \\ a \end{matrix}\,\middle|\, 1-\frac{1}{p}\right) \\ &= \sum_{k=0}^{n} \binom{n}{k} \left(1-\frac{1}{p}\right)^{k-n} (a+k)^{(n-k)} (x)_k, \quad x \in \mathbb{N}_0 \end{aligned} $$

for $n \in \mathbb{N}_0$, $a > 0$, $p \in (0,1)$ and they satisfy the orthogonality relation

$$ \sum_{\ell=0}^{\infty} \mathscr{M}_n(\ell;a;p)\mathscr{M}_m(\ell;a;p)\mathrm{NB}(a,p)(\{\ell\}) = \mathbb{1}_{\{n=m\}} \frac{p^n n! (a)^{(n)}}{(1-p)^{2n}} \tag{3.1} $$

for $n,m \in \mathbb{N}_0$, i.e., $(\mathscr{M}_n(\,\cdot\,;a;p))_{n\in\mathbb{N}_0}$ are orthogonal polynomials with respect to the generalized negative binomial distribution

$$ \mathrm{NB}(a,p)(\{\ell\}) = (a)^{(\ell)} \frac{p^\ell}{\ell!} (1-p)^a, \quad \ell \in \mathbb{N}_0. $$

### 3.1 Reversible interacting particle systems on a finite set

Let $E$ be a non-empty finite set and identify $\xi \in \mathbf{N}_{<\infty}$ with $(\xi_k)_{k\in E} := (\xi(\{x\}))_{x\in E} \in \mathbb{N}_0^E$. Let $(\eta_t)_{t\geq 0}$ be a Markov process on $\mathbf{N}_{<\infty}$ satisfying Assumption 2.2 and $\rho$ be a reversible probability measure satisfying Assumption 2.9. We then have that $D^{\mathrm{cheap}}(\xi,\eta) := \frac{\mathbb{1}_{\{\eta=\xi\}}}{\rho(\{\xi\})}$, for $\eta,\xi \in \mathbf{N}_{<\infty}$ is the so-called cheap or trivial self-duality function ([12, Eq. (4.2)]). In this section we recover well-known self-duality functions of systems of particles hopping on a finite set by applying the intertwiners $J_n$ and $I_n$ to the cheap duality function. Note that

$$ D_n^{\mathrm{cheap}}(\xi,x) := D^{\mathrm{cheap}}\left(\xi, \sum_{k=1}^{n} \delta_{x_k}\right), \quad \xi \in \mathbf{N}_{<\infty}, x = (x_1,\ldots,x_n) \in E^n, n \in \mathbb{N}, $$

is a duality functions for $(P_t)_{t\geq 0}$ and the $n$-particle semigroup $(p_t^{[n]})_{t\geq 0}$, i.e.,

$$P_t D_n^{\mathrm{cheap}}(\,\cdot\,,x)(\xi) = p_t^{[n]} D_n^{\mathrm{cheap}}(\xi,\,\cdot\,)(x)$$

for each $\xi \in \mathbb{N}_0^E$, $x \in E^n$, $n \in \mathbb{N}$. Putting $D_0^{\mathrm{cheap}}(\xi,) := D^{\mathrm{cheap}}\,(\xi,0)$ yields $P_t D_0^{\mathrm{cheap}}(\,\cdot\,,)(\xi) = p_t^{[0]} D_0^{\mathrm{cheap}}(\xi,)$.

**Proposition 3.1.** *Let $\rho = \bigotimes_{k\in E}\rho_k$ where $\rho_k$ are probability measures on $\mathbb{N}_0$ satisfying $\rho_k(\{\ell\}) > 0$ for each $\ell \in \mathbb{N}_0$. Consider for each $\rho_k$ the sequence of monic orthogonal polynomials denoted by $(\mathscr{P}_n(\,\cdot\,,\rho_k))_{n\in\mathbb{N}_0}$. Then,*

1. *applying $J_n$ to $D_n^{\mathrm{cheap}}(\xi,\cdot)$ yields*

$$\mathfrak{D}_n^{\mathrm{cl}}(\xi,\eta) := \frac{1}{n!} J_n\left(D_n^{\mathrm{cheap}}(\xi,\,\cdot\,),\eta\right) = \mathbb{1}_{\{\xi(E)=n\}} \prod_{x\in E} \frac{1}{\rho_x(\{\xi_x\})\xi_x!}(\eta_x)_{\xi_x}$$

*for all $n \in \mathbb{N}_0$ and $\xi,\eta \in \mathbf{N}_{<\infty}$;*

2. *applying $I_n$ to $D_n^{\mathrm{cheap}}(\xi,\cdot)$ yields*

$$\mathfrak{D}_n^{\mathrm{ort}}(\xi,\eta) := \frac{1}{n!} I_n\left(D_n^{\mathrm{cheap}}(\xi,\,\cdot\,),\eta\right) = \mathbb{1}_{\{\xi(E)=n\}} \prod_{x\in E} \frac{1}{\rho_x(\{\xi_x\})\xi_x!}\mathscr{P}_{\xi_x}(\eta_x,\rho_x)$$

*for all $n \in \mathbb{N}_0$ and $\xi,\eta \in \mathbf{N}_{<\infty}$.*

It is well-known that applying an intertwiner to a duality function, for instance $D_n^{\mathrm{cheap}}(\xi,x)$, yields again a duality function, see e.g. [10, Theorem 2.5] or [26, Remark 2.7]. As a consequence of the above proposition, Theorem 2.6 and Theorem 2.17 yield that $\mathfrak{D}_n^{\mathrm{cl}}$ and $\mathfrak{D}_n^{\mathrm{ort}}$ are duality functions for $(P_t)_{t\geq 0}$ and $(p_t^{[n]})_{t\geq 0}$ for each $n \geq \mathbb{N}_0$. Moreover, summing over $n$ in $\mathfrak{D}_n^{\mathrm{cl}}$ and $\mathfrak{D}_n^{\mathrm{ort}}$, we obtain the self-duality functions

$$\mathfrak{D}^{\mathrm{cl}}(\xi,\eta) := \prod_{x\in E} \frac{1}{\rho_x(\{\xi_x\})\xi_x!}(\eta_x)_{\xi_x}, \quad \xi,\eta \in \mathbf{N}_{<\infty}, \tag{3.2}$$

$$\mathfrak{D}^{\mathrm{ort}}(\xi,\eta) := \prod_{x\in E} \frac{1}{\rho_x(\{\xi_x\})\xi_x!}\mathscr{P}_{\xi_x}(\eta_x,\rho_x), \quad \xi,\eta \in \mathbf{N}_{<\infty}, \tag{3.3}$$

i.e. they satisfy the relation

$$P_t\mathfrak{D}(\xi,\cdot)(\eta) = P_t\mathfrak{D}(\cdot,\eta)(\xi)$$

for all $\eta,\xi \in \mathbb{N}_0^E$, $t \geq 0$ and $\mathfrak{D} \in \{\mathfrak{D}^{\mathrm{cl}},\mathfrak{D}^{\mathrm{ort}}\}$. In the following, given a function $f_n : E^n \to \mathbb{R}$, we define its symmetrization by

$$\tilde{f}_n(x_1,\ldots,x_n) = \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n} f_n(x_{\sigma(1)},\ldots,x_{\sigma(n)}), \tag{3.4}$$

for $x_1,\ldots,x_n \in \mathbb{E}$, where $\mathfrak{S}_n$ denotes the set of permutations of $\{1,\ldots,n\}$.

*Proof.* Without loss of generality, let $E = \{1,\ldots,N\}$ and fix $\xi \in \mathbb{N}_0^N$, $n \in \mathbb{N}$.

1. Note that

$$\mathbb{1}_{\{\xi=\delta_{x_1}+\ldots+\delta_{x_n}\}} = \mathbb{1}_{\{\xi(E)=n\}}\frac{n!}{\xi_1!\cdots\xi_N!}\widetilde{\mathbb{1}_{\{1\}}^{\otimes\xi_1}\otimes\cdots\otimes\mathbb{1}_{\{N\}}^{\otimes\xi_N}}(x_1,\ldots,x_n) \tag{3.5}$$

for $x_1, \ldots, x_n \in E$, where $\mathbb{1}_{\{1\}}^{\otimes \xi_1} \widetilde{\otimes \cdots \otimes} \mathbb{1}_{\{N\}}^{\otimes \xi_N}$ denotes the symmetrization of the function $\mathbb{1}_{\{1\}}^{\otimes \xi_1} \otimes \cdots \otimes \mathbb{1}_{\{N\}}^{\otimes \xi_N}$. Hence, using (2.5), we obtain

$$\begin{aligned}\frac{1}{n!} J_n \left( D_n^{\mathrm{cheap}}(\xi, \cdot), \eta \right) &= \frac{\mathbb{1}_{\{\xi(E)=n\}}}{\rho(\{\xi\}) \xi_1! \cdots \xi_N!} \int \mathbb{1}_{\{1\}}^{\otimes \xi_1} \otimes \cdots \otimes \mathbb{1}_{\{N\}}^{\otimes \xi_N} \, \mathrm{d}\eta^{(n)} \\ &= \mathbb{1}_{\{\xi(E)=n\}} \prod_{x=1}^{N} \frac{1}{\rho_x(\{\xi_x\}) \xi_x!} (\eta_x)_{\xi_x}.\end{aligned}$$

for each $\xi \in \mathbb{N}_0^N$.

2. Let $\mathbf{P}_n := \overline{\mathcal{P}_n} \cap \overline{\mathcal{P}_{n-1}}^{\perp}$. By the orthogonal decomposition

$$\mathbf{P}_n = \bigoplus_{d_1 + \ldots + d_N = n} \mathrm{span}\{ \mathscr{P}_{d_1}(\cdot, \rho_1) \otimes \cdots \otimes \mathscr{P}_{d_N}(\cdot, \rho_N) \}$$

we obtain that the projection of $\mathbb{N}_0^N \ni \eta \mapsto \int \mathbb{1}_{\{1\}}^{\otimes \xi_1} \cdots \mathbb{1}_{\{N\}}^{\otimes \xi_N} \, \mathrm{d}\eta^{\otimes n} = \eta_1^{\xi_1} \cdots \eta_N^{\xi_N}$ onto $\mathbf{P}_n$ is equal to $\eta \mapsto \mathscr{P}_{\xi_1}(\eta_1, \rho_1) \cdots \mathscr{P}_{\xi_N}(\eta_1, \rho_N)$. Therefore, using (3.5)

$$\begin{aligned}\frac{1}{n!} I_n \left( D_n^{\mathrm{cheap}}(\xi, \cdot), \eta \right) &= \frac{\mathbb{1}_{\{\xi(E)=n\}}}{\rho(\{\xi\}) \xi_1! \cdots \xi_N!} I_n(\mathbb{1}_{\{1\}}^{\otimes \xi_1} \otimes \cdots \otimes \mathbb{1}_{\{N\}}^{\otimes \xi_N}, \eta) \\ &= \mathbb{1}_{\{\xi(E)=n\}} \prod_{x=1}^{N} \frac{1}{\rho_x(\{\xi_x\}) \xi_x!} \mathscr{P}_{\xi_x}(\eta_x, \rho_x)\end{aligned}$$

for each $\eta \in \mathbb{N}_0^N$.

□

We consider three prominent examples of consistent and conservative Markov processes on $\mathbb{N}_0^E$. For a characterization of consistent particle system on countable $E$ we refer to [12, Theorem 3.3]. Let $|E| \geq 2$, $c = \{c_{\{x,y\}}, x, y \in E\}$ be a set of symmetric and non-negative conductances, such that $(E, c)$ is connected and let $(\alpha_y)_{y \in E} \in \mathbb{N}^E$. Then, for $\sigma \in \{-1, 0, 1\}$, the Markov process with infinitesimal generator acting on functions $f : \mathbf{N}_{<\infty} \to \mathbb{R}$ as

$$Lf(\eta) = \sum_{x,y \in E} c_{\{x,y\}} \left( f(\eta - \delta_x + \delta_y) - f(\eta) \right) (\alpha_y + \sigma \eta(\{y\})) \eta(\{x\}), \quad \eta \in \mathbf{N}_{<\infty}$$

is a consistent and conservative process. In particular, for $\sigma = -1$, we obtain the inhomogeneous partial exclusion process (SEP) (see, e.g., [22, Eq. (1.3)]), for $\sigma = 0$ a system of independent random walks (IRW) and for $\sigma = 1$ the inhomogeneous inclusion process SIP (see, e.g., [23, Eq. (2.2)]).

By a simple detailed balance computation one can show that, for those processes, there exists a one parameter family $\{\rho_\theta, \ \theta \in \Theta\}$ with $\Theta = (0, 1]$ for $\sigma = -1$ and $\Theta = (0, \infty)$ for $\sigma \in \{0, 1\}$ of reversible measures, namely (cf. [23, Eq. (3.1)]) $\rho_\theta := \bigotimes_{x \in E} \rho_{x,\theta}$ with

$$\rho_{x,\theta} = \begin{cases} \mathrm{Bin}(\alpha_x, \theta) & \text{if } \sigma = -1 \\ \mathrm{Poi}(\alpha_x \theta) & \text{if } \sigma = 0 \\ \mathrm{NB}\left( \alpha_x, \frac{\theta}{1+\theta} \right) & \text{if } \sigma = 1 \ . \end{cases}$$

Using that $\rho_{x,\theta}(\{n\}) = \frac{w_x(n)}{z_{x,\theta}} \left( \frac{\theta}{1+\sigma\theta} \right)^n \frac{1}{n!}$ where

$$w_x(n) := \begin{cases} (\alpha_x)_n & \text{if } \sigma = -1 \\ \alpha_x^n & \text{if } \sigma = 0 \\ (\alpha_x)^{(n)} & \text{if } \sigma = 1 \end{cases} \quad \text{and } z_{x,\theta} := \begin{cases} (1-\theta)^{-\alpha_x} & \text{if } \sigma = -1 \\ e^{\alpha_x \theta} & \text{if } \sigma = 0 \\ (1+\theta)^{\alpha_x} & \text{if } \sigma = 1 \end{cases}$$

in (3.2) we obtain

$$\mathfrak{D}^{\mathrm{cl}}(\xi,\eta) = \left(\frac{\theta}{1+\sigma\theta}\right)^{-\xi(E)} \left(\prod_{x\in E} z_{x,\theta}\right) \prod_{x\in E} \frac{(\eta_x)_{\xi_x}}{w_x(\xi_x)},$$

which are the classical duality functions for $(\eta_t)_{t\geq 0}$ (see, e.g., [23, Eq. (2.16)]). Notice that, due to Assumption 2.2 (ii), the term $\left(\frac{\theta}{1+\sigma\theta}\right)^{-\xi(E)} \left(\prod_{k\in E} z_{k,\theta}\right)$ is constant in time and, thus, it does not play any role in the duality relation.

For these systems, the self-duality functions provided by (3.3) coincide (up to a multiplicative constant depending on the total number of particles which is a conserved quantity) with the orthogonal self-duality functions studied in [23, Theorem 4.1], [24, Theorem 1] and [50, Section 4.1] which are given by product of Charlier polynomials for $\sigma = 0$, products of Meixner polynomials for $\sigma = 1$ and products of Krawtchouk polynomials (see, e.g., [33, Eq. (9.11.1)]) for $\sigma = -1$. Indeed, considering, for instance, the system of independent random walks, the self-duality function of (3.3) turns into

$$\begin{aligned}
\mathfrak{D}^{\mathrm{ort}}(\xi,\eta) &= \prod_{k\in E} \frac{1}{\rho_k(\{\xi_k\})\xi_k!} \mathscr{C}_{\xi_k}(\eta_k,\alpha_k) \\
&= \prod_{k\in E} \frac{1}{e^{-\alpha_k}\alpha_k^{\xi_k}} (-\alpha_k)^{\xi_k} {}_2F_0 \left( \begin{matrix} -\xi_k, -\eta_k \\ - \end{matrix} \middle| -\frac{1}{\alpha_k} \right) \\
&= e^{\alpha(E)} (-1)^{\xi(E)} \prod_{k\in E} {}_2F_0 \left( \begin{matrix} -\xi_k, -\eta_k \\ - \end{matrix} \middle| -\frac{1}{\alpha_k} \right),
\end{aligned}$$

coinciding with the orthogonal self-duality functions given in literature mentioned above. The same holds also for the exclusion and the inclusion process.

### 3.2 Independent Markov processes

Every system of independent Markov processes (e.g. the free Kawasaki dynamics [35], independent Brownian motions) is consistent and conservative. For independent particles, our results allow us to recover known results on intertwining with Lenard's $K$-transform and multiple stochastic integrals, see [35, Sections 3.2 and 4], [54] and the references therein. Our contribution is the proof that these intertwining relations correspond exactly to classical and orthogonal duality relations for independent random walks on lattices from [15, Proposition 2.9.4] and [24, Theorem 4].

Let $(p_t)_{t\geq 0}$ be a Markov transition function on $(E,\mathcal{E})$. The transition function for $n$ independent labeled particles with one-particle evolution governed by $(p_t)_{t\geq 0}$ has transition function $p_t^{\otimes n}$ uniquely determined by

$$p_t^{\otimes n}(x_1,\dots,x_n; A_1\times\cdots\times A_n) = \prod_{i=1}^n p_t(x_i;A_i) \quad x_1,\dots,x_n\in E,\ A_1,\dots,A_n\in\mathcal{E}.$$

The family of transition functions $(p_t^{\otimes n})_{t\geq 0}$, $n\in\mathbb{N}$ is strongly consistent and therefore the associated conservative transition function $(P_t)_{t\geq 0}$ (see (2.2)) is consistent.

Hence, Theorem 2.6 applied to the process $(\eta_t)_{t\geq 0}$ with transition function $(P_t)_{t\geq 0}$ yields the self-intertwining relation $P_t J_n(f_n,\cdot)(\eta) = J_n(p_t^{[n]} f_n, \eta)$ or more concretely,

$$\mathbb{E}_\eta\Big[\int f_n \mathrm{d}\eta_t^{(n)}\Big] = \int \big(p_t^{\otimes n} f_n\big)\,\mathrm{d}\eta.$$

The relation holds true for all $t\geq 0$, all initial values $\eta\in\mathbf{N}_{<\infty}$, and all $f_n\in\mathcal{F}_n$. As noted in (2.7), it implies that Lenard's $K$-transform and the semigroup $(P_t)_{t\geq 0}$ commute. Hence, for free Kawasaki dynamics, we recover a relation from [35, Section 3.2].

If we find a $\sigma$-finite reversible measure $\lambda$ for the one-particle dynamics $(p_t)_{t\geq 0}$, then the distribution of a Poisson process with intensity measure $\lambda$, denoted by $\pi_\lambda$, is reversible for $(\eta_t)_{t\geq 0}$. This property is a version of Doob's Theorem (cf. [15, Theorem 2.9.5]) and of the displacement theorem (cf. [31, p.61]). Moreover, $\lambda^{\otimes n}$ is reversible for $(p_t^{\otimes n})_{t\geq 0}$. For finite $\lambda$, the assumptions of Theorem 2.17 are satisfied and the self-intertwining relation $P_t I_n(f_n,\,\cdot\,)(\eta) = I_n(p_t^{\otimes n} f_n, \eta)$ holds for $\pi_\lambda$-almost all $\eta \in \mathbf{N}_{<\infty}$, all $f_n \in \mathcal{F}_n$ and all $t \geq 0$.

The construction of the generalized orthogonal polynomial with respect to the Poisson point process is standard and it is well-known that the orthogonality relation

$$\int I_n(f_n,\,\cdot\,) I_m(g_m,\,\cdot\,)\,\mathrm{d}\pi_\lambda = \mathbb{1}_{\{n=m\}} n! \int f_n g_m \mathrm{d}\lambda^{\otimes n}$$

holds for bounded $f_n \in \mathcal{F}_n, g_m \in \mathcal{F}_m$, $n, m \in \mathbb{N}_0$, with $\int f_0 g_0 \mathrm{d}\lambda^{\otimes 0} := f_0 g_0$, and they generalize the Charlier polynomial in the following sense (see, e.g., [39, Eq. (3.3)],

$$I_n\left(\mathbb{1}_{B_1}^{\otimes d_1} \otimes \cdots \otimes \mathbb{1}_{B_N}^{\otimes d_N}, \eta\right) = \prod_{k=1}^{N} \mathscr{C}_{d_k}(\eta(B_k); \lambda(B_k)) \tag{3.6}$$

for $\pi_\lambda$-almost all $\eta \in \mathbf{N}_{<\infty}$, $d_1 + \ldots + d_N = n$ and all pairwise disjoint $B_1, \ldots, B_N \in \mathcal{E}$. Yet another viewpoint is that $I_n(f_n,\,\cdot\,)$ are multiple stochastic integrals with respect to the compensated Poisson measure $\eta - \lambda$. The reader interested in the relation between the generalized orthogonal polynomials $I_n(f_n,\,\cdot\,)$ and multiple Wiener-Itô integrals, chaos decompositions, and Fock spaces is referred to [37], [44] and [46].

In the language of multiple stochastic integrals, the intertwining relation from Theorem 2.17 says that applying the semigroup to the $n$-fold integral of $f_n$ is the same as the $n$-fold integral of $p_t^{\otimes n} f_n$.

### 3.3 The Howitt–Warren flow and a consistent family of sticky Brownian motions

As noted in Section 2.1, every strongly consistent family $(p_t^{[n]})_{t\geq 0}$, $n \in \mathbb{N}$, of transition functions induces a consistent semigroup $(P_t)_{t\geq 0}$. Strongly consistent families have been studied in the context of stochastic flows: Le Jan and Raimond [40] investigate a one-to-one correspondence between strong consistency families and stochastic flows of kernels.

A particular and well studied case is the Howitt–Warren flow. It is a stochastic flow of kernels whose $n$ point motions is given by a family of $n$ interacting Brownian motions that interact, roughly, by sticking together for a while when they meet. The interacting diffusions can be constructed, for example, as solutions to a martingale problem [28, Section 2]. Theorem 2.6 applies to the semigroup $(P_t)_{t\geq 0}$ induced by the strongly consistent family of transitions functions $(p_t^{[n]})_{t\geq 0}$, $n \in \mathbb{N}$, for $n$ sticky Brownian motions.

The dynamics of sticky Brownian motions, whose precise definition we do not recall in detail because it is rather technical and would not add any further insights for our purposes (see [51, Definition 2.2] for further details), depends on a choice of parameters and includes diffusions with a drift. For zero drift and a special choice of parameters, Brockington and Warren [9] prove, using a Bethe ansatz, an explicit formula for transition probabilities (see [9, Theorem 1.2]) and the reversibility of the $n$-point motions with respect to some explicit measure $m_\theta^{(n)}$ (see [9, Definition 1.1]). They work on the Weyl chambers $\overline{W^n} := \{x \in \mathbb{R}^n : x_1 \geq \ldots \geq x_n\}$ and show that the transition function is of the form $p_t^{[n]}(x, \mathrm{d}y) = u_t^{(n)}(x,y) m_\theta^{(n)}(\mathrm{d}y)$ for some symmetric function $u_t^{(n)}(x,y) = u_t^{(n)}(y,x)$. With this the self-intertwining relation from Theorem 2.6 can be rewritten as

$$\mathbb{E}_\eta\left[\int_{\bar{W}^r} f_r(y)\eta_t^{(r)}(\mathrm{d}y)\right] = \int_{\bar{W}^r} f_r(y)\left[\int_{\bar{W}^r} u_t^{(r)}(y,x)\eta^{(r)}(\mathrm{d}x)\right] m_\theta^{(r)}(\mathrm{d}y). \tag{3.7}$$

Thus we obtain an identity analogous to (2.9) and (A.8).

As the reversible measures $m_\theta^{(n)}$ from [9, Definition 1.1] have infinite total mass, it is not possible to construct from them a reversible measure supported on configurations of finitely many particles and Theorem 2.17 on orthogonal intertwining relations is not applicable. The detailed study of the orthogonal self-intertwining relation for the system of sticky Brownian motions has been carried out by the fourth author of the present manuscript building on the results developed above (see [56, Section 3]).

For other examples of strongly consistent families, beyond sticky Brownian motions, we refer to [40] and [51].

# 4 Generalized symmetric inclusion process

As an example of interacting system of particles jumping on a general Borel space $(E, \mathcal{E})$, we introduce here a new process which is a natural extension of the SIP. Coherently with the setting of this paper, we consider the finite particle case only. Extension to the infinite particle case is not part of the scope of the present work and it is left for future research.

The SIP on countable sets was introduced in [25, Eq. 3.2] as a dual process of a model of heat conduction, which shares some features with the well-studied KMP model (see [32]). The process also appears, with a different interpretation, in mathematical population genetics. Indeed, in [11, Section 5], it is proved that the generator of the SIP coincides with the generator of an instance of the Moran model, which is dual to the Wright-Fisher diffusion process. Moreover, the scaling limit of the Moran model is the celebrated Fleming-Viot superprocess (see [17, p.25] and references therein) which has been studied using duality as well.

## 4.1 Introducing the gSIP

Let $\alpha$ be a finite, non-zero measure on $E$ and $c : E \times E \to \mathbb{R}_+$ be a bounded, measurable and symmetric function with $c(x, x) = 0$ for all $x \in E$. The generalized symmetric inclusion process (gSIP) is a continuous-time jump process on $\mathbf{N}_{<\infty}$ with jump kernel

$$Q(\eta, B) = \iint \mathbb{1}_B(\eta - \delta_x + \delta_y) c(x, y)(\alpha + \eta)(\mathrm{d}y)\eta(\mathrm{d}x), \tag{4.1}$$

for $\eta \in \mathbf{N}_{<\infty}$ and $B \in \mathcal{N}_{<\infty}$. It can be viewed, when $E = \mathbb{R}^d$, as a particular case of a Kawasaki dynamics (see, e.g., [34]). Bypassing the precise description of the domain, the generator of the process is formally given by

$$Lf(\eta) = \iint \big(f(\eta - \delta_x + \delta_y) - f(\eta)\big) c(x, y)(\alpha + \eta)(\mathrm{d}y)\eta(\mathrm{d}x). \tag{4.2}$$

Notice that $Q(\eta, E) < \infty$ for finite measures $\alpha$ and finite configurations $\eta \in \mathbf{N}_{<\infty}$. The dynamics can be described informally as follows. Starting from an initial configuration $\eta_0 = \eta$ with $n = \eta(E)$ points $x_1, \dots, x_n$, set

$$q_{i0} := \int c(x_i, y)\alpha(\mathrm{d}y), \quad q_{ij} := c(x_i, x_j), \quad z_i := \sum_{j=0}^n q_{ij}, \quad z := \sum_{i=1}^n z_i$$

and do the following:

(i) Wait for an exponential time with parameter $Q(\eta, E) = z$.

(ii) When time is up, choose one out of the $n$ points $x_1, \dots, x_n$ randomly, where the point $x_i$ is chosen with probability $z_i/z$. Move the chosen point $x = x_i$ to a new location $y$:

- With probability $q_{ij}/z_i$, the new location $y$ is equal to $y = x_j$.
- With probability $q_{i0}/z_i$, the new location $y$ is chosen according to the probability measure $\alpha(E)^{-1}\alpha(\mathrm{d}y)$.

Then, repeat. Accordingly the process $(\eta_t)_{t\geq 0}$ can be constructed with the usual jump-hold construction and the semigroup $(P_t)_{t\geq 0}$ is the minimal solution of the backward Kolmogorov equation, see, e.g., [19, Chapter X Section 10].

**Remark 4.1.** 1. As we will see later, the gSIP $(\eta_t)_{t\geq 0}$ has the following connection to the well-known SIP of particles hopping on a finite set. Let $A_1, \ldots, A_m \in \mathcal{E}$, $m \in \mathbb{N}$ be a partition of $E$ and let $c$ be constant on $A_i \times A_j$ with $c(x, y) = d_{ij}$ for all $x \in A_i$ and $y \in A_j$, for each $i, j$. Then, the process $(\eta_t(A_1), \ldots, \eta_t(A_m))$ starting at $\eta_0 \in \mathbf{N}_{<\infty}$ behaves like a SIP on the finite set $\{1, \ldots, m\}$ with initial configuration $(\eta_0(A_1), \ldots, \eta_0(A_m))$ and transition rates $d_{ij}$.

2. Notice that a direct generalization of the Exclusion process analogous to the gSIP, would not be meaningful in general, because the probability to jump on already occupied points is zero whenever the jumping kernel of the single particle is not atomic. Thus an exclusion rule miming the one in the discrete setting cannot be modelled in the continuum.

### 4.2 Reversibility and intertwiners for the gSIP

Fix $p \in (0, 1)$. A *Pascal point process* with parameters $p$ and $\alpha$ is a point process $\zeta$ with the following properties:

(i) If $B_1, \ldots, B_m \in \mathcal{E}$ are disjoint then $\zeta(B_1), \ldots, \zeta(B_m)$ are independent.

(ii) For every $B \in \mathcal{E}$, the distribution of $\zeta(B)$ is given by a negative binomial law:

$$\mathbb{P}\big(\zeta(B) = k\big) = \big(1 - p\big)^{\alpha(B)}\alpha(B)\big(\alpha(B) + 1\big) \cdots \big(\alpha(B) + k - 1\big)\, \frac{p^k}{k!}, \quad k \in \mathbb{N}_0.$$

For $k = 0$, the equation is to be read as $\mathbb{P}(\zeta(B) = 0) = (1 - p)^{\alpha(B)}$.

The *Pascal distribution* is the distribution of a Pascal point process and it is a direct generalization of the product measure of negative binomial distributions that is reversible for SIP. Indeed, if $E$ is countable and $\alpha_x := \alpha(\{x\}) > 0$ for all $x \in E$, the measure $\otimes_{x\in E}\mathrm{NB}(\alpha_x, p)$, can be seen as a Pascal distribution. Property (i) follows immediately whereas (ii) follows from the fact that if $n_x \sim \mathrm{NB}(\alpha_x, p)$ and $n_y \sim \mathrm{NB}(\alpha_y, p)$, with $n_x$ and $n_y$ independent for $x \neq y \in E$, then $n_x + n_y \sim \mathrm{NB}(\alpha_x + \alpha_y, p)$.

**Theorem 4.2.** *Let $\alpha$ be a finite measure on $E$. Then*

(i) *the generalized symmetric inclusion process with formal generator* (4.2) *is a consistent Markov process and thus the intertwining relation* (2.6) *with generalized falling factorials holds;*

(ii) *for every $p \in (0, 1)$, the Pascal distribution $\rho$ with parameters $\alpha$ and $p$ is reversible and thus, the intertwining relation* (2.13) *with generalized orthogonal polynomials holds.*

Notice that we have a family of reversible measures, indexed by $p \in (0, 1)$, moreover the reversible Pascal distributions do not depend on the function $c(x, y)$ in the dynamics.

Theorem 4.2(ii) is complemented by a concrete relation of the abstract generalized orthogonal polynomials $I_n(f_n, \cdot)$ with the univariate Meixner polynomials defined in Section 3.1. Generalized orthogonal polynomials of Meixner's type have been studied intensely in the context of non-Gaussian white noise [6], [7]. Connections with quantum probability and representations of $^*$-Lie algebras and current algebras are investigated in [1], [2].

The following proposition is a variant of Lemma 3.1 in [45]. We give a self-contained proof in Section 4.4 that does not use the machinery of Jacobi fields or distribution theory.

**Proposition 4.3.** *The intertwiner $I_n$ is related to the Meixner polynomials via*

$$I_n\left(\mathbb{1}_{B_1}^{\otimes d_1}\otimes\cdots\otimes\mathbb{1}_{B_N}^{\otimes d_N},\eta\right)=\prod_{k=1}^{N}\mathscr{M}_{d_k}\left(\eta(B_k);\alpha(B_k);p\right). \tag{4.3}$$

*for $\rho$-almost all $\eta\in\mathbf{N}_{<\infty}$ and all pairwise disjoint $B_1,\dots,B_N\in\mathcal{E}$, $n\in\mathbb{N}_0$, $d_1,\dots,d_N$, $N\in\mathbb{N}$ with $d_1+\ldots+d_N=n$.*

We define a measure $\lambda_n$ on $E^n$ that replaces the product measure $\lambda^{\otimes n}$ in the Poisson-Charlier case. Let $\Sigma_n$ be the collection of set partitions of $\{1,\dots,n\}$. For $\sigma\in\Sigma_n$ and $g:E^n\to\mathbb{R}$, let $|\sigma|$ be the number of blocks of the partition $\sigma$ and $g_\sigma:E^{|\sigma|}\to\mathbb{R}$ the function obtained by identifying, in order of occurrence, those arguments that belong to the same block of $\sigma$. Define

$$\lambda_n(B)=\sum_{\sigma\in\Sigma_n}\left(\prod_{A\in\sigma}(|A|-1)!\right)\int(\mathbb{1}_B)_\sigma\,\mathrm{d}\alpha^{\otimes|\sigma|},\quad B\in\mathcal{E}^{\otimes n}. \tag{4.4}$$

For example $\lambda_1=\alpha$ and

$$\lambda_2(B)=\iint\mathbb{1}_B(x,y)\alpha(\mathrm{d}x)\alpha(\mathrm{d}y)+\int\mathbb{1}_B(x,x)\alpha(\mathrm{d}x)$$

for all $B\in\mathcal{E}^{\otimes 2}$. Further set $\int f_0g_0\,\mathrm{d}\lambda_0:=f_0g_0$ for $f_0,g_0\in\mathcal{F}_0=\mathbb{R}$.

The following proposition generalizes the univariate orthogonality relation (3.1). It is similar to Corollary 5.2 in [45], we provide a self-contained proof in Section 4.4.

**Proposition 4.4.** *The following orthogonality relations holds*

$$\int I_n(f_n,\,\cdot\,)I_m(g_m,\,\cdot\,)\,\mathrm{d}\rho=\mathbb{1}_{\{n=m\}}\frac{p^n n!}{(1-p)^{2n}}\int f_ng_m\,\mathrm{d}\lambda_n \tag{4.5}$$

*for $f_n\in\mathcal{F}_n$, $g_m\in\mathcal{F}_m$, $n,m\in\mathbb{N}_0$.*

**Remark 4.5** (Sequential construction of $\lambda_n$)**.** For $n\in\mathbb{N}$, define a kernel $k_{n,n+1}:E^n\times\mathcal{E}^{\otimes(n+1)}\to\mathbb{R}_+$ by

$$k_{n,n+1}(x_1,\dots,x_n;B)=\int\mathbb{1}_B(x_1,\dots,x_n,y)\alpha(\mathrm{d}y)+\sum_{i=1}^{n}\mathbb{1}_B(x_1,\dots,x_n,x_i).$$

Then $\lambda_{n+1}=\lambda_nk_{n,n+1}$ meaning that $\lambda_{n+1}(B)=\int_{E^n}\lambda_n(\mathrm{d}x)k_{n,n+1}(x,B)$ for all $B\in\mathcal{E}^{\otimes(n+1)}$. Thus $\lambda_n$ is formed by adding points one by one; at each step, a new point either joins a pile of existing points or is placed at a new location $y$. This relation on the one hand connects to the very definition of the dynamics of the gSIP and on the other hand is reminiscent of the Chinese restaurant process used in sequential constructions for random partitions [49, Chapter 3]. Notice that, upon normalization by the total mass of $\lambda_n$, (4.4) gives rise to a probability measure on the set $\Sigma_n$ of partitions, related to the Ewens sampling formula.

### 4.3 Proof of Theorem 4.2

Here we prove Theorem 4.2. In addition, we remind the reader of an explicit description of the Pascal process as a compound Poisson process.

**Consistency (Proof of Theorem 4.2(i))** We start by proving that the gSIP is consistent (see Definition 2.1). Since we consider the finite particle case only, it is enough to check the commutation property in Definition 2.1 for the generator instead of the semigroup, i.e., $\mathcal{A}Lf(\eta) = L\mathcal{A}f(\eta)$ for all $f \in \mathcal{G}$ and $\eta \in \mathbf{N}_{<\infty}$. Indeed, decompose the generator in (4.2) as $L = L_1 + L_2$ with

$$L_1 f(\eta) := \iint \left(f(\eta - \delta_x + \delta_y) - f(\eta)\right) c(x,y)\alpha(\mathrm{d}y)\eta(\mathrm{d}x)$$

and

$$L_2 f(\eta) := \iint \left(f(\eta - \delta_x + \delta_y) - f(\eta)\right) c(x,y)\eta(\mathrm{d}y)\eta(\mathrm{d}x).$$

Notice that $L_1$ is the generator of a system of independent Markov processes, namely, independent random walks with transition kernel given by $c(x,y)\alpha(\mathrm{d}y)$. Thus, it is straightforward to check that $\mathcal{A}L_1 f(\eta) = L_1\mathcal{A}f(\eta)$. It remains to show that

$$\mathcal{A}L_2 f(\eta) = L_2\mathcal{A}f(\eta). \tag{4.6}$$

First, we compute

$$\begin{aligned}
&L_2\mathcal{A}f(\eta)\\
&= \iiint f(\eta - \delta_x + \delta_y - \delta_z)\eta(\mathrm{d}z)c(x,y)\eta(\mathrm{d}y)\eta(\mathrm{d}x)\\
&- \iint f(\eta - 2\delta_x + \delta_y)c(x,y)\eta(\mathrm{d}y)\eta(\mathrm{d}x) + \iint f(\eta - \delta_x)c(x,y)\eta(\mathrm{d}y)\eta(\mathrm{d}x)\\
&- \iiint f(\eta - \delta_z)\eta(\mathrm{d}z)c(x,y)\eta(\mathrm{d}y)\eta(\mathrm{d}x),
\end{aligned}$$

second,

$$\begin{aligned}
\mathcal{A}L_2 f(\eta) &= \iiint \left(f(\eta - \delta_z - \delta_x + \delta_y) - f(\eta - \delta_z)\right) c(x,y)(\eta - \delta_z)(\mathrm{d}y)(\eta - \delta_z)(\mathrm{d}x)\eta(\mathrm{d}z)\\
&= L_2\mathcal{A}f(\eta)\\
&- \iint \left(f(\eta - \delta_x) - f(\eta - \delta_z)\right) c(x,z)\eta(\mathrm{d}x)\eta(\mathrm{d}z)\\
&+ \int \left(f(\eta - \delta_z) - f(\eta - \delta_z)\right) c(z,z)\eta(\mathrm{d}z).
\end{aligned}$$

Because the last two integrals above are both $0$, we obtain (4.6) and the proof is concluded. □

**Explicit representation of the pascal process** The Pascal process, also called negative binomial process, is a well-known point process (cf. [36, Proposition 1.1], [53, Section 2.7]). For the reader's convenience we recall the construction of that process.

Fix $p \in (0,1)$ and a finite measure $\alpha$. Note that the Pascal point process has the structure of a measure-valued Lévy process, since $\zeta(A_1), \dots, \zeta(A_n)$ are independent for pairwise disjoint $A_1, \dots, A_n \in \mathcal{E}$ and the distribution of $\zeta(A)$ only depends on $\alpha(A)$, $A \in \mathcal{E}$. For more details, see [29, Chapter 3 Section 3], [30], [31, Chapter 8].

More precisely, the Pascal process can be constructed as a compound Poisson process (see [38, Chapter 15]) with Lévy measure $\nu := \sum_{n=1}^{\infty} \frac{p^n}{n}\delta_n$, i.e.,

$$\zeta(A) := \int_{A\times\mathbb{N}} y\,\xi(\mathrm{d}(x,y)), A \in \mathcal{E}$$

where $\xi$ is a Poisson point process on $E \times \mathbb{N}$ with intensity measure $\lambda := \alpha \otimes \nu$. The Laplace functional of $\zeta$ is then given by

$$L_\zeta(f) := \mathbb{E}\left[e^{-\int f\,\mathrm{d}\zeta}\right] = \exp\left(\int \left(e^{-yf(x)} - 1\right) \lambda(\mathrm{d}(x,y))\right) = \exp\left(-\int \Phi(f(x))\alpha(\mathrm{d}x)\right) \tag{4.7}$$

for all measurable $f : E \to [0,\infty)$, where $\Phi(y) := \log\left(\frac{1-pe^{-y}}{1-p}\right)$, $y \geq 0$. Above, we used [38, (15.3)] in the first equality. Equation (4.7) implies for $A \in \mathcal{E}$

$$\mathbb{E}\left(e^{-\zeta(A)s}\right) = \exp\left(-\int \Phi(s\mathbb{1}_A(x))\alpha(\mathrm{d}x)\right) = \exp\left(-\alpha(A)\Phi(s)\right) = \left(\frac{1-p}{1-pe^{-s}}\right)^{\alpha(A)}$$

for $s > 0$ which is the Laplace transform of a negative binomial distributed random variable with parameters $\alpha(A)$ and $p$. Moreover, (4.7) implies the independence of $\zeta(A_1), \dots, \zeta(A_n)$ immediately.

**Reversible measure (Proof of Theorem 4.2(ii))** Let $Q_c = Q$ be the jump kernel from (4.1). It is enough to check the detailed balance relation

$$\rho \otimes Q_c(\mathscr{A} \times \mathscr{B}) = \rho \otimes Q_c(\mathscr{B} \times \mathscr{A}) \quad \mathscr{A}, \mathscr{B} \in \mathcal{N}_{<\infty}, \tag{4.8}$$

where

$$\rho \otimes Q_c(\mathscr{A} \times \mathscr{B}) = \int_{\mathscr{A}} \rho(\mathrm{d}\eta) \iint \mathbb{1}_{\mathscr{B}}(\eta - \delta_x + \delta_y) c(x,y)(\alpha + \eta)(\mathrm{d}y)\eta(\mathrm{d}x).$$

The idea of the proof is that for particularly simple choices of $c(x,y)$ and $\mathscr{A}, \mathscr{B}$, the relation (4.8) boils down to a detailed balance relation for a discrete inclusion process.

We start with some preliminary observations. First, it is enough to prove (4.8) for functions $c$ of the form

$$c(x,y) = \sum_{i,j=1}^{r} d_{ij} \mathbb{1}_{C_i}(x)\mathbb{1}_{C_j}(y) \tag{4.9}$$

with $r \in \mathbb{N}$, symmetric non-negative weights $d_{ij} = d_{ji} \geq 0$, and sets $C_1, \dots, C_r \in \mathcal{E}$. Indeed, the set $\mathcal{M}$ of non-negative measurable functions $f : E \times E \to \mathbb{R}_+$ for which the symmetrized function $c(x,y) := \frac{1}{2}(f(x,y) + f(y,x))$ satisfies (4.8) is closed under pointwise monotone limits. If (4.8) holds true for all functions $c$ of the form (4.9), then $\mathcal{M}$ contains all indicators $\mathbb{1}_{A \times B}$, $A, B \in \mathcal{E}$. The monotone class theorem then implies that $\mathcal{M}$ contains all bounded non-negative measurable functions.

Second, by the $\pi$-$\lambda$ theorem, it is enough to check (4.8) for sets of the form

$$\mathscr{A} = \bigcap_{j=1}^{k} \left\{\eta \in \mathbf{N}_{<\infty} : \eta(A_j) = m_j\right\}, \quad \mathscr{B} = \bigcap_{j=1}^{\ell} \left\{\eta \in \mathbf{N}_{<\infty} : \eta(B_j) = n_j\right\} \tag{4.10}$$

with $k, \ell \in \mathbb{N}$, $A_i, B_j \in \mathcal{E}$, and $m_i, n_j \in \mathbb{N}_0$.

Third, for the relation (4.8) to hold true for all $c(x,y)$ of the form (4.9) and all sets $\mathscr{A}$, $\mathscr{B}$ of the form (4.10), it is enough to consider the situation where $r = k = \ell$, $A_i = B_i = C_i$, and the sets $A_1, \dots, A_r$ are pairwise disjoint, as the general case follows by taking linear combinations.

In the situation of the last paragraph and assuming all diagonal elements $d_{ii}$ vanish, we compute

$$Q_c(\eta, \mathscr{B}) = \sum_{i,j=1}^{r} d_{ij} \int_{A_i} \left(\int_{A_j} \mathbb{1}_{\mathscr{B}}(\eta - \delta_x + \delta_y)(\alpha + \eta)(\mathrm{d}y)\right) \eta(\mathrm{d}x)$$

$$= \sum_{i,j=1}^{r} d_{ij}\eta(A_i)\big(\alpha(A_j)+\eta(A_j)\big)\mathbb{1}_{\{\eta(A_i)-1=n_i\}}\mathbb{1}_{\{\eta(A_j)+1=n_j\}}\prod_{s\notin\{i,j\}}\mathbb{1}_{\{\eta(A_s)=n_s\}}$$

$$= \sum_{\substack{1\le i,j\le r:\\ i\neq j}} d_{ij}m_i(\alpha(A_j)+m_j)\delta_{m_i-1,n_i}\delta_{m_j+1,n_j}\prod_{s\notin\{i,j\}}\delta_{m_s,n_s}, \tag{4.11}$$

for $\eta\in\mathscr{A}$. In the last equation above, we recognize the transition rates of the SIP with state space $\mathbb{N}_0^r$. For non-zero diagonal elements, we need to add

$$\sum_{i=1}^{r} d_{ii}m_i(\alpha(A_i)+m_i)\prod_{s=1}^{r}\delta_{m_i,n_i}. \tag{4.12}$$

We denote the sum of (4.11) and (4.12) $q(m,n)$. Notice that for non-zero $d_{ii}$ we may have $q(m,m)>0$.

Abbreviate $\alpha(A_j)=:\alpha_j$. For $j\in\{1,\dots,r\}$ and $m_j\in\mathbb{N}_0$, set

$$\pi_j(m_j) := \rho\big(\{\eta:\ \eta(A_j)=m_j\}\big) = (1-p)^{\alpha_j}\alpha_j(\alpha_j+1)\cdots(\alpha_j+m_j-1)\frac{p^{m_j}}{m_j!}.$$

Further set $\pi(m)=\pi_1(m_1)\cdots\pi_r(m_r)$. Then

$$\rho\otimes Q_c(\mathscr{A},\mathscr{B}) = \pi(m)q(m,n).$$

A similar computation shows $\rho\otimes Q_c(\mathscr{B},\mathscr{A})=\pi(n)q(n,m)$. The symmetry relation (4.8) now reads $\pi(m)q(m,n)=\pi(n)q(n,m)$ which is the detailed balance relation for the SIP. □

### 4.4 Properties of generalized Meixner polynomials. Proof of Propositions 4.3 and 4.4

Let $p\in(0,1)$. Note that the generating function of monic Meixner polynomials, given by (see, e.g., [33, Eq. 9.10.11])

$$e_t(x,a) := \sum_{n=0}^{\infty}\frac{t^n}{n!}\mathscr{M}_n(x;a;p) = \left(\frac{1-p+t}{1-p+tp}\right)^{x}\left(\frac{1-p}{1-p+tp}\right)^{a}, \quad t,a>0, x\in\mathbb{N}_0,$$

satisfies $e_t(x+y,a+b)=e_t(x,a)e_t(y,b)$ for each $t>0$, $x,y\in\mathbb{N}_0$, $a,b>0$. As a consequence, we get the convolution property (see, e.g., [3])

$$\mathscr{M}_n(x+y;a+b;p) = \sum_{k=0}^{n}\binom{n}{k}\mathscr{M}_k(x;a;p)\mathscr{M}_{n-k}(y;b;p). \tag{4.13}$$

*Proof of Proposition 4.3.* By the factorization property from Proposition 2.15 it is enough to show

$$I_d\left(\mathbb{1}_{A^d},\eta\right) = \mathscr{M}_d(\eta(A);\alpha(A);p) \tag{4.14}$$

for all $d\in\mathbb{N}$ and $A\in\mathcal{E}$. As we have chosen our univariate Meixner polynomials $\mathscr{M}_d$ to have leading coefficient one, we know that $\mathscr{M}_d(\eta(A);\alpha(A);p)$ is equal to $\eta(A)^d$ plus some polynomial in $\eta(A)$ of degree $\le d-1$. Therefore (4.14) follows once we know that the map $\eta\mapsto\mathscr{M}_d(\eta(A);\alpha(A);p)$ is orthogonal to the space $\mathcal{P}_{d-1}$. We shall see that this identity follows from the convolution property (4.13) and the complete independence.

We check first that $\eta\mapsto\mathscr{M}_d(\eta(A);\alpha(A);p)$ is orthogonal in $L^2(\rho)$ to all maps $\eta\mapsto\eta^{\otimes m}(C)$, for every $m\le d-1$ and $C\in\mathcal{E}^{\otimes m}$ with $C\subset A^m$. When $C=A^m$, we are looking at two univariate polynomials in the variable $x=\eta(A)$ and the orthogonality relation follows from the orthogonality of the univariate Meixner polynomials $x\mapsto\mathscr{M}_d(\eta(A);\alpha(A);p)$ to the monomial $x \mapsto x^m$. The orthogonality to constant functions ($m = 0$) follows from univariate orthogonality as well.

Next consider the case $C = C_1^{d_1} \times \cdots \times C_N^{d_N}$ with $N \in \mathbb{N}$, $d_1, \ldots, d_N \in \mathbb{N}$ with $d_1 + \cdots + d_N \le d - 1$ and pairwise disjoint measurable sets $C_i \subset A$. Suppose first that $C_1 \cup \cdots \cup C_N = A$. We use the convolution property (4.13) and the complete independence of the Pascal point process to find

$$\int \mathscr{M}_d(\eta(A); \alpha(A); p)\eta^{\otimes m}(C)\rho(\mathrm{d}\eta) \\ = \sum_{k_1+\cdots+k_N=m} \binom{m}{k_1, \ldots, k_N} \prod_{i=1}^N \int \mathscr{M}_{k_i}\big(\eta(C_i); \alpha(C_i); p\big)\eta^{\otimes d_i}(C_i)\rho(\mathrm{d}\eta). \quad (4.15)$$

In each summand, we must have $d_i < k_i$ for at least one $i \in \{1, \ldots, N\}$ and therefore by the orthogonality of univariate Meixner polynomials, at least one of the integrals on the right-hand side above vanishes. As a consequence,

$$\int \mathscr{M}_d(\eta(A); \alpha(A); p)\eta^{\otimes m}(C)\rho(\mathrm{d}\eta) = 0. \quad (4.16)$$

This holds true as well when each $C_i$ is contained in $A$ and $C_{N+1} := A \setminus (C_1 \cup \cdots \cup C_N)$ is non-empty. In that case we use a similar decomposition but now the sum on the right-hand side of (4.15) is over $(k_1, \ldots, k_{N+1})$ and the product has an additional factor $\int \mathscr{M}_{k_{N+1}}(\eta(C_{N+1}); \alpha(C_{N+1}); p)\rho(\mathrm{d}\eta)$.

Every Cartesian product $C = D_1 \times \cdots \times D_m$ contained in $A^m$ is a disjoint union of finitely many Cartesian products in which any two factors are either disjoint or equal. Therefore, by linearity, the orthogonality relation (4.16) extends to all such sets. The functional monotone class theorem yields the orthogonality of the generalized Meixner polynomial to all maps of the form $\eta \mapsto \eta^{\otimes m}(f_m)$ with bounded measurable $f_m : E^m \to \mathbb{R}$ supported in $A^m$ and then, by linearity, the orthogonality to all linear combinations of such maps.

In the notation of Lemma 2.18 below, we have checked the orthogonality of $\mathscr{M}_d(\eta(A); \alpha(A); p)$ to $\mathcal{P}_{d-1}(A)$. Using complete independence and arguments similar to the proof of Lemma 2.18, we conclude that the Meixner polynomial is in fact orthogonal to $\mathcal{P}_{d-1}$. This completes the proof of the proposition. □

*Proof of Proposition 4.4.* The orthogonality of $I_n(f_n, \cdot\,)$ and $I_m(g_m, \cdot\,)$ for $m \ne n$ is an immediate consequence of the definition of generalized orthogonal polynomials, it does not use any properties of the Pascal distribution $\rho$. Thus we need only treat the case $m = n$.

Using linearity and the monotone class theorem as in the proof of Proposition 4.3, one finds that it suffices to show the orthogonality relation for functions $\tilde{f}_n$, $\tilde{g}_n$ that are symmetrized versions (see (3.4)) of indicator functions $f_n, g_n : E^n \to \mathbb{R}$ of the form

$$f_n = \mathbb{1}_{B_1}^{\otimes d_1} \otimes \cdots \otimes \mathbb{1}_{B_N}^{\otimes d_N}, \quad g_n = \mathbb{1}_{B_1}^{\otimes d_1'} \otimes \cdots \otimes \mathbb{1}_{B_N}^{\otimes d_N'}$$

with $B_1, \ldots, B_N \in \mathcal{E}$ disjoint, and $\sum_{i=1}^N d_i = \sum_{i=1}^N d_i' = n$. Notice that $I_n(\tilde{f}_n, \eta) = I_n(f_n, \eta)$ and $I_n(\tilde{g}_n, \eta) = I_n(g_n, \eta)$, but in general $\int \tilde{f}_n \tilde{g}_n \mathrm{d}\lambda_n \ne \int f_n g_n \mathrm{d}\lambda_n$.

Proposition 4.3, the complete independence, and the orthogonality relation (3.1) for univariate Meixner polynomials yield

$$\int I_n(\tilde{f}_n, \eta)\, I_n(\tilde{g}_n, \eta)\, \rho(\mathrm{d}\eta) = \prod_{i=1}^N \mathbb{1}_{\{d_i = d_i'\}} \frac{d_i!\, p^{d_i}}{(1-p)^{2d_i}} \left(\alpha(B_i)\right)^{(d_i)}. \quad (4.17)$$

If $d_i \neq d_i'$ for at least one $i$, then the right-hand side is zero, moreover $\tilde{f}_n\, \tilde{g}_n$ vanishes identically. Hence in that case

$$\int I_n(\tilde{f}_n, \eta)\, I_n(\tilde{g}_n, \eta)\, \rho(\mathrm{d}\eta) = 0 = \int \tilde{f}_n \tilde{g}_n \mathrm{d}\lambda_n.$$

and the required equality holds true.

If $d_i = d_i'$ for all $i$, then $f_n = g_n$ on $E^n$. By the definition of $\lambda_n$, we have

$$\int f_n^2\, \mathrm{d}\lambda_n = \lambda_n(B_1^{d_1} \times \cdots \times B_n^{d_n}) = \prod_{i=1}^{N} (\alpha(B_i))^{(d_i)}$$

hence (4.17) gives

$$\int \big(I_n(\tilde{f}_n, \eta)\big)^2\, \rho(\mathrm{d}\eta) = \frac{p^n}{(1-p)^{2n}} \Big(\prod_{i=1}^{N} d_i!\Big) \int f_n^2 \mathrm{d}\lambda_n. \tag{4.18}$$

Next we check that the product of factorials on the right-hand side disappears when $f_n$ is replaced by the symmetrized function $\tilde{f}_n$. For $\sigma \in \mathfrak{S}_n$ and $x = (x_1, \ldots, x_n) \in E^n$, let $x_\sigma := (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. Then, by the permutation invariance of the measure $\lambda_n$, we have

$$\int \tilde{f}_n^2\, \mathrm{d}\lambda_n = \frac{1}{n!^2} \sum_{\sigma, \tau \in \mathfrak{S}_n} \int f_n(x_\sigma) f_n(x_\tau) \lambda_n(\mathrm{d}x) = \frac{1}{n!} \sum_{\pi \in \mathfrak{S}_n} \int f_n(x_\pi) f_n(x) \lambda_n(\mathrm{d}x).$$

Because of the disjointness of the sets $B_i$, the product $f_n(x_\pi) f_n(x)$ vanishes unless $\pi$ leaves the sets $\{1, \ldots, d_1\}$, $\{d_1 + 1, \ldots, d_1 + d_2 - 1\}$ etc. invariant, and in the latter case $f_n(x_\pi) f_n(x) = f_n(x)^2$. The number of relevant permutations is equal to $d_1! \cdots d_N!$. As a consequence,

$$\int \tilde{f}_n^2\, \mathrm{d}\lambda_n = \frac{1}{n!} \Big(\prod_{i=1}^{N} d_i!\Big) \int f_n^2 \mathrm{d}\lambda_n.$$

By (4.18), we get

$$\int \big(I_n(\tilde{f}_n, \eta)\big)^2 \rho(\mathrm{d}\eta) = \frac{n! p^n}{(1-p)^{2n}} \int \tilde{f}_n^2\, \mathrm{d}\lambda_n$$

which is the required equality (remember $\tilde{f}_n = \tilde{g}_n$). □

## A Self-duality for independent random walks with symmetric jump rates on a finite set

In this appendix we consider the simplest case of a system of independent random walks with symmetric jump rates on a finite set, and show how the self-duality properties of this process (see, e.g., [15, Section 2.9.3], [26, Section 3.4]) correspond exactly to the intertwining relations of Theorem 2.6 and Theorem 2.17. This is mostly intended for the reader familiar with duality in the context of interacting particle systems, such as in [15, Section 2.9.3], and intends to make a smooth transition between this notational context and the point-process notation adopted in our paper.

Let $E$ be a finite set and $(\eta_t)_{t \geq 0}$, $\eta_t = (\eta_t(x))_{x \in E}$, be the Markov process on $\mathbb{N}_0^E$ generated by

$$Lf(\eta) = \sum_{x, y \in E} \eta(x) c(x, y) (f(\eta - \delta_x + \delta_y) - f(\eta))$$

for $f : \mathbb{N}_0^E \to \mathbb{R}$, $c : E \times E \to \mathbb{R}_+$ a symmetric function ($c(x, x) = 0$ for any $x \in E$ without loss of generality) and where $\delta_x$ denotes the configuration with a single particle at $x$ and no particles at other locations. That is, $\delta_x$ is the configuration $\eta \in \mathbb{N}_0^E$ given by $\eta(y) = \delta_{x,y}$. Then $(\eta_t)_{t\geq 0}$ is called the configuration process with $\eta_t(x)$ denoting the numbers of particles at time $t \geq 0$ in $x \in E$. We denote by $p_t(x,y)$ the transition probability of a single random walk, which is a symmetric function due to the symmetry of the rates $c : E \times E \to \mathbb{R}_+$.

Let $\xi \in \mathbb{N}_0^E$, we then define the polynomials

$$D(\xi,\eta) := \prod_{x\in E} \frac{\eta(x)!}{(\eta(x)-\xi(x))!}\, \mathbb{1}_{\{\xi(x)\leq\eta(x)\}}, \quad \eta \in \mathbb{N}_0^E. \tag{A.1}$$

We refer to (A.1) as the *classical self-duality functions* and to $\xi$ as the dual configuration. The self-duality relation for the system of independent walks then reads as follows

$$\mathbb{E}_\eta\left(D(\xi,\eta_t)\right) = \mathbb{E}_\xi\left(D(\xi_t,\eta)\right) \tag{A.2}$$

for all $\eta, \xi \in \mathbb{N}_0^E$ and $t \geq 0$, where $\mathbb{E}_\eta$ denotes the expectation in the configuration process started from $\eta$ (see, e.g., [15, Section 2.9.3], [26, Section 3.4]). Notice here that we have restricted to the case of finite $E$ for convenience but (A.2) can be extended to countable $E$, a suitable set of allowed starting configurations $\eta \in \mathbb{N}_0^E$, and finite dual configurations $\xi$ (see [15, Section 2.9.3]). In Section A.1, by a change of notation, we reformulate the relation (A.2) with one dual particle (i.e., $\xi = \delta_x$) in such a way that it is meaningful in contexts more general than random walks on a finite set, namely also in the continuum. Thus we get rid of the configuration process notation. In Section A.2 we proceed by reformulating (A.2) in the general case with $n$ dual particles and finally, in Section A.3, we recall and reformulate the orthogonal self-duality functions for independent random walks.

### A.1 The labeled configuration notation and the associated point configuration: self-duality with one dual particle

Let $\mathcal{X} := (\mathcal{X}_0(1), \ldots, \mathcal{X}_0(N))$ be an arbitrary labelling of the initial positions of $N < \infty$ independent random walks with jump rates $c(x,y)$. We then denote $\mathcal{X}_t$ the positions of these particles at time $t \geq 0$, with $\mathcal{X}_t(i)$ the position of the $i$-th particle at time $t \geq 0$. The correspondence between the labeled system $(\mathcal{X}_t)_{t\geq 0}$ and the previously introduced configuration process $(\eta_t)_{t\geq 0}$ is given by $\eta_t(x) = \sum_{i=1}^N \mathbb{1}_{\{\mathcal{X}_t(i)=x\}}$.

We describe the system also via the counting measure $\sum_{i=1}^N \delta_{\mathcal{X}_t(i)}$. Notice that in this discrete setting, this is simply a change of notation for the configuration: indeed, for $x \in E$, we have $\left(\sum_{i=1}^N \delta_{\mathcal{X}_t(i)}\right)(\{x\}) = \eta_t(x)$. In view of the generalization of self-duality in the next sections, from now on, we identify $\eta_t$ with the counting measure

$$\eta_t = \sum_{i=1}^N \delta_{\mathcal{X}_t(i)}.$$

This is the same as identifying a measure $\eta$ on the finite $E$ with the vector $\eta(\{x\}), x \in E$. The advantage of this change of notation is that it generalizes to arbitrary measurable state spaces $E$, and it also allows to produce a simple but insightful proof of the self-duality (A.2).

Let us start with self-duality with a single dual particle, i.e. (A.1) with dual configuration $\xi = \delta_x$, which reads as

$$\mathbb{E}_\eta(\eta_t(\{x\})) = \mathbb{E}_x^{\mathrm{IRW}}\left(\eta_0(\{Y_t\})\right) = \sum_{y\in E} p_t(x,y)\eta(\{y\}),$$

where $\mathbb{E}_x^{\mathrm{IRW}}$ denotes the expectation with respect to the random walk $(Y_t)_{t\geq 0}$ with transition rates $c(x,y)$ starting at $x \in E$.

Let us denote by $\mathbb{E}_{\mathcal{X}}(\eta_t)$ the measure defined as $\mathbb{E}_{\mathcal{X}}\left(\eta_t\right)(A) := \mathbb{E}_{\mathcal{X}}\left(\eta_t(A)\right)$ for $A \subset E$, where $\mathbb{E}_{\mathcal{X}}$ denotes the expectation when starting $(\mathcal{X}_t(1), \ldots, \mathcal{X}_t(N))_{t\geq 0}$ at $\mathcal{X}$. We then have

$$\mathbb{E}_{\mathcal{X}}\left(\eta_t\right) = \mathbb{E}_{\mathcal{X}}\left(\sum_{i=1}^N \delta_{\mathcal{X}_t(i)}\right) = \sum_{i=1}^N \mathbb{E}_{\mathcal{X}}(\delta_{\mathcal{X}_t(i)}) = \sum_{i=1}^N \mathbb{E}^{\mathrm{IRW}}_{\mathcal{X}_0(i)}(\delta_{\mathcal{X}_t(i)}), \tag{A.3}$$

where in the third equality in (A.3) we used that the particles are independent, i.e., the distribution of the position of the $i$-th particle is only depending on its initial position $\mathcal{X}_0(i)$ and not on the other particles. Using that $\mathbb{E}^{\mathrm{IRW}}_{\mathcal{X}_0(i)}(\delta_{\mathcal{X}_t(i)}) = \sum_{y\in E} p_t(\mathcal{X}_0(i), y)\delta_y$, we can rewrite (A.3) as

$$\mathbb{E}_{\mathcal{X}}\left(\eta_t\right) = \sum_{i=1}^N \sum_{y\in E} p_t(\mathcal{X}_0(i), y)\delta_y = \sum_{y\in E}\delta_y \sum_{i=1}^N p_t(y, \mathcal{X}_0(i)) = \sum_{y\in E}\left(\int p_t(y,z)\eta_0(\mathrm{d}z)\right)\delta_y,$$

where in the second equality we used the symmetry of the transition probabilities $p_t(x,y)$. If we denote by $\lambda(\mathrm{d}y)$ the counting measure on $E$ we obtain

$$\left(\mathbb{E}_{\mathcal{X}}\left(\eta_t\right)\right)(\mathrm{d}y) = \left(\int p_t(y,z)\eta_0(\mathrm{d}z)\right)\lambda(\mathrm{d}y). \tag{A.4}$$

The above reformulation of the self-duality relation (A.2) with one dual particle now makes sense on general measurable state spaces $E$.

### A.2 Reformulation of self-duality with $n$ dual particles

As a next step we want to generalize (A.4) to the case of $n$ dual particles. Let $\eta = \sum_{i=1}^N \delta_{x_i}$ and recall (see (2.4) above) that $\eta^{(n)}$ denotes the $n$-th factorial measure of $\eta$, i.e.

$$\eta^{(n)} = \sum_{1\leq i_1,\ldots,i_n\leq N}^{\neq} \delta_{(x_{i_1},\ldots,x_{i_n})}. \tag{A.5}$$

The reason why the above measure is called falling factorial is clearly explained by the elementary combinatorial lemma below, where the relation with the classical self-duality functions defined in (A.1) (consisting of products of falling factorial polynomials) is given. We leave the simple proof to the reader.

**Lemma A.1.** *Let $\eta = \sum_{i=1}^N \delta_{x_i}$. Then, for all $(y_1,\ldots,y_n) \in E^n$, we have*

$$\eta^{(n)}(\{(y_1,\ldots,y_n)\}) = D\left(\sum_{k=1}^n \delta_{y_k}, \eta\right), \tag{A.6}$$

*where $D(\,\cdot\,,\,\cdot\,)$ is the self-duality function given in* (A.1)*. As a consequence, the $n$-th factorial measure can be rewritten as follows*

$$\eta^{(n)} = \sum_{y_1,\ldots,y_n\in E} \delta_{(y_1,\ldots,y_n)} D\left(\sum_{k=1}^n \delta_{y_k}, \eta\right). \tag{A.7}$$

We can then generalize (A.4) to the expectation of the $n$-th factorial measure $\eta_t^{(n)}$ of the counting measure valued process $\eta_t = \sum_i \delta_{\mathcal{X}_t(i)}$ introduced above.

**Proposition A.2.** *Let $\lambda$ be the counting measure on $E$. Then, for all $t > 0$ and $n \in \mathbb{N}$,*

$$\mathbb{E}_{\mathcal{X}}(\eta_t^{(n)})(\mathrm{d}(y_1\ldots y_n)) = \left(\int_{E^n}\prod_{i=1}^n p_t(y_i,z_i)\eta_0^{(n)}(\mathrm{d}(z_1,\ldots z_n))\right)\lambda^{\otimes n}(\mathrm{d}(y_1,\ldots,y_n)). \tag{A.8}$$

*Proof.* Let $f : E^n \to \mathbb{R}$. We then have

$$\mathbb{E}_{\mathcal{X}}\left(\int f(y_1,\ldots,y_n)\eta_t^{(n)}(\mathrm{d}(y_1\ldots y_n))\right) = \sum_{1\le i_1,\ldots,i_n\le N}^{\neq} \mathbb{E}_{\mathcal{X}} f(\mathcal{X}_t(i_1),\ldots,\mathcal{X}_t(i_n)) \tag{A.9}$$

$$= \sum_{1\le i_1,\ldots,i_n\le N}^{\neq} \int f(y_1,\ldots,y_n)\prod_{k=1}^n p_t(\mathcal{X}_0(i_k),y_k)\prod_{k=1}^n \lambda(\mathrm{d}y_k)$$

$$= \sum_{1\le i_1,\ldots,i_n\le N}^{\neq} \int f(y_1,\ldots,y_n)\prod_{k=1}^n p_t(y_k,\mathcal{X}_0(i_k))\prod_{k=1}^n \lambda(\mathrm{d}y_k)$$

$$= \int f(y_1,\ldots,y_n)\left(\int \prod_{k=1}^n p_t(y_k,z_k)\eta_0^{(n)}(\mathrm{d}(z_1\ldots z_n))\right)\prod_{k=1}^n \lambda(\mathrm{d}y_k),$$

where we used the definition of the $n$-th factorial measure in the first and the last equality, the independence of the particles in the second equality and the symmetry of the transition probabilities in the third equality. Because $f$ is arbitrary, this proves (A.8). □

**Remark A.3.** (i) Equation (A.8) holds for each system of independent reversible random walks where the reversible measure $\lambda_{\mathrm{rev}}$ for the single random walk is used in place of the counting measure $\lambda$ (see (2.9) above).

(ii) Without assuming the symmetry of the rates $c : E \times E \to \mathbb{R}$, from (A.9) and the independence of the particles, we still have the relation

$$\mathbb{E}_{\mathcal{X}}\left(\int_{E^n} f\mathrm{d}\eta_t^{(n)}\right) = \int_{E^n} \mathbb{E}^{\mathrm{IRW}}_{y_1,\ldots,y_n}\left(f(Y_t(1),\ldots,Y_t(n))\right)\eta_0^{(n)}(\mathrm{d}(y_1\ldots,y_n)), \tag{A.10}$$

where $f : E^n \to \mathbb{R}$ is a permutation invariant function and $\mathbb{E}^{\mathrm{IRW}}_{y_1,\ldots,y_n}$ denotes expectation with respect to $n$ independent random walks initially starting from $(y_1,\ldots,y_n)$. Equation (A.10) has to be read as a self-intertwining relation and it is generalized in Section 2.2.

iii) For any $(y_1,\ldots,y_n) \in E^n$, (A.8) implies

$$\mathbb{E}_{\mathcal{X}}(\eta_t^{(n)}(\{(y_1,\ldots,y_n)\}) = \mathbb{E}^{\mathrm{IRW}}_{y_1,\ldots,y_n}\left(\eta_0^{(n)}(\{(Y_t(1),\ldots,Y_t(n))\})\right)$$

which, in view of (A.6), reads as

$$\mathbb{E}_{\mathcal{X}}\left(D\left(\sum_{k=1}^n \delta_{y_k},\eta_t\right)\right) = \mathbb{E}^{\mathrm{IRW}}_{y_1,\ldots,y_n}\left(D\left(\sum_{k=1}^n \delta_{Y_t(k)},\eta\right)\right),$$

which is precisely the *classical self-duality* relation given in (A.2).

### A.3 Orthogonal self-duality

In this section we turn to orthogonal self-duality functions for random walks in a finite set. In [23], [24] and [50] it has been shown (using, respectively, generating functions method, three term recurrence relations and algebraic methods) that, for all $\theta > 0$, the following self-duality relation holds

$$\mathbb{E}_\eta(D_\theta(\xi,\eta_t)) = \mathbb{E}_\xi(D_\theta(\xi_t,\eta)) \tag{A.11}$$

with respect to the self-duality functions

$$D_\theta(\xi,\eta) = \prod_{x\in E} d^{\mathrm{or}}_{\xi(\{x\})}(\eta(\{x\});\theta). \tag{A.12}$$

$\{d_n^{\mathrm{or}}(\,\cdot\,;\theta)\}_{n\in\mathbb{N}}$ are the Charlier polynomials. These polynomials satisfy the following orthogonality relation

$$\int d_n^{\mathrm{or}}(\eta(\{x\});\theta)d_m^{\mathrm{or}}(\eta(\{x\});\theta)\rho_\theta(\mathrm{d}\eta)=\mathbb{1}_{\{n=m\}}\frac{n!}{\theta^n}$$

with $\rho_\theta=\otimes_{x\in E}\rho_{x,\theta}$ and $\rho_{x,\theta}=\mathrm{Poi}(\theta)$ for each $x\in E$. We refer to the functions in (A.12) as *orthogonal self-duality functions*. Let $[n]:=\{1,\dots,n\}$. In this setting, the relation between orthogonal and classical duality functions is simple and given by (see [23, Remark 4.2])

$$D_\theta(\xi,\eta)=\sum_{\xi'\leq\xi}(-\theta)^{|\xi|-|\xi'|}\binom{\xi}{\xi'}D(\xi',\eta)=\sum_{I\subset[n]}(-\theta)^{n-|I|}D\left(\sum_{i\in I}\delta_{y_i},\eta\right), \tag{A.13}$$

from which it follows that (A.11) is a direct consequence of (A.2) and the independence of the particles. We can now reformulate the self-duality relation (A.11) in terms of a counting measure notation. First, we introduce the orthogonalized version of the falling factorial measure associated to a counting measure $\eta=\sum_{i=1}^N\delta_{x_i}$, namely

$$\eta^{(n),\theta}(\mathrm{d}(x_1,\dots,x_n)):=\sum_{r=0}^n(-\theta)^{n-r}\sum_{I\subset[n]:|I|=r}\eta^{(r)}(\mathrm{d}(x_1,\dots,x_n)_I)\otimes\lambda^{\otimes(n-r)}(\mathrm{d}(x_1,\dots,x_n)_{[n]\setminus I}),$$

where $\lambda$ denotes the counting measure, $\int f_0\,\mathrm{d}\eta^{(0)}:=f_0$ for all $f_0\in\mathbb{R}$ and $(x_1,\dots,x_n)_I$ denotes the subvector of $(x_1,\dots,x_n)$ with components in $I\subset[n]$. The relation between $\eta^{(n),\theta}$ and the orthogonal self-duality functions is expressed in the following result.

**Lemma A.4.** *Let $\eta=\sum_{i=1}^N\delta_{x_i}$. Then, for all $(y_1,\dots,y_n)\in E^n$, we have*

$$\eta^{(n),\theta}(\{(y_1,\dots,y_n)\})=D_\theta\left(\sum_{i=1}^n\delta_{y_i},\eta\right), \tag{A.14}$$

*where $D_\theta(\,\cdot\,,\,\cdot\,)$ is the orthogonal self-duality function given in* (A.13). *As a consequence*

$$\eta^{(n),\theta}=\sum_{y_1,\dots,y_n\in E}D_\theta\left(\sum_{i=1}^n\delta_{y_i},\eta\right)\delta_{(y_1,\dots,y_n)}.$$

*Proof.* For $I\subset[n]$ with $|I|=r$, we have, using (A.6),

$$\begin{aligned}D\left(\sum_{i\in I}\delta_{y_i},\eta\right)&=\eta^{(r)}((y_1,\dots,y_n)_I)=\int\mathbb{1}_{(y_1,\dots,y_n)_I}(x_1,\dots,x_r)\eta^{(r)}(\mathrm{d}(x_1,\dots,x_r))\\&=\int\mathbb{1}_{(y_1,\dots,y_n)}(x_1,\dots,x_n)\eta^{(r)}(\mathrm{d}(x_1,\dots,x_n)_I)\otimes\lambda^{\otimes(n-r)}(\mathrm{d}(x_1,\dots,x_n)_{[n]\setminus I}).\end{aligned}$$

Therefore, (A.14) follows from (A.13). □

We then state the analogue of Proposition A.2 for $\eta^{(n),\theta}$ in a notation which makes sense in the context of general measurable state space $E$. The result follows from (A.8) combined with the definition of $\eta^{(n),\theta}$ and the reversibility of $\lambda$ for the single random walk: we omit here the simple proof and we refer to Section 2.3 above for the proof of the self-intertwining formulation of this result in a much more general setting.

**Proposition A.5.** *For all $t > 0$ and $n \in \mathbb{N}$*

$$\mathbb{E}_{\mathcal{X}}\left[\eta_t^{(n),\theta}\right](\mathrm{d}(y_1,\ldots,y_n)) = \left(\int_{E^n}\prod_{i=1}^n p_t(y_i,x_i)\eta_0^{(n),\theta}(\mathrm{d}(x_1,\ldots,x_n))\right)\lambda^{\otimes n}(\mathrm{d}(y_1,\ldots,y_n)). \tag{A.15}$$

It was observed in [24] (just above equation (8) in [24]), that the orthogonal self-duality functions given in (A.12) coincide with the polynomials obtained by the Gram-Schmidt orthogonalization procedure initialized with the classical duality functions given in (A.1). In the present context, the Gram-Schmidt orthogonalization applied to (A.1) is (A.13).

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