

The rate of escape of the most visited site of Brownian motion

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Abstract

Let $\{L_t^z\}$ be the jointly continuous local times of a one-dimensional Brownian motion and let $L_t^* = \sup_{z \in \mathbb{R}} L_t^z$. Let V_t be any point z such that $L_t^z = L_t^*$, a most visited site of Brownian motion. We prove that if $\gamma > 1$, then

$$\liminf_{t \rightarrow \infty} \frac{|V_t|}{\sqrt{t}/(\log t)^\gamma} = \infty, \quad \text{a.s.},$$

with an analogous result for simple random walk. This proves a conjecture of Lifshits and Shi.

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1 Introduction

Let S_n be a simple random walk, let $N_n^k = \sum_{j=0}^n 1_{(S_j=k)}$ be the number of visits by the random walk to the point k by time n , and let $N_n^* = \sup_{k \in \mathbb{Z}} N_n^k$. Let $\mathcal{U}_n = \{k \in \mathbb{Z} : N_n^k = N_n^*\}$, the set of values k where N_n^k takes its maximum, and let U_n be any element of \mathcal{U}_n . We call \mathcal{U}_n the set of most visited sites of the random walk at time n . This concept was introduced in [4], and was simultaneously and independently defined by [13], who called U_n a favorite point of the random walk. In [4] it was proved that U_n is transient, and in fact

$$\liminf_{n \rightarrow \infty} \frac{|U_n|}{\sqrt{n}/(\log n)^\gamma} = \infty \tag{1.1}$$

if $\gamma > 11$ and

$$\liminf_{n \rightarrow \infty} \frac{|U_n|}{\sqrt{n}/(\log n)^\gamma} = 0 \tag{1.2}$$

if $\gamma < 1$. It has been of considerable interest since that time to prove that there exists γ_0 such that (1.1) holds if $\gamma > \gamma_0$ and (1.2) holds if $\gamma < \gamma_0$ and to find the value of γ_0 .

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One can state the analogous problem for Brownian motion, and [4] used Brownian motion techniques and an invariance principle for local times to derive the results for random walk from those of Brownian motion. Let $\{L_t^z\}$ be the jointly continuous local times of a Brownian motion and let $\mathcal{V}_t(\omega)$ be the set of values of z where the function $z \rightarrow L_t^z(\omega)$ takes its maximum. We call \mathcal{V}_t the set of most visited points or the set of favorite points of Brownian motion at time t . In [4] it was proved that if V_t is any element of \mathcal{V}_t , then

$$\liminf_{t \rightarrow \infty} \frac{|V_t|}{\sqrt{t}/(\log t)^\gamma} = \infty \tag{1.3}$$

if $\gamma > 11$ and

$$\liminf_{t \rightarrow \infty} \frac{|V_t|}{\sqrt{t}/(\log t)^\gamma} = 0 \tag{1.4}$$

if $\gamma < 1$.

The bounds in (1.2) and (1.4) have been improved somewhat. Lifshits and Shi [20] proved that the \liminf is 0 when $\gamma = 1$ as well as when $\gamma < 1$.

In [3] the most visited sites of symmetric stable processes of order α for $\alpha > 1$ were studied. As a by-product of the results there, the value of γ in (1.3) was improved from 11 to 9.

In Lifshits and Shi [20] it was asserted that the value of γ in (1.1) and (1.3) could be any value larger than 1, or equivalently, that γ_0 exists and is equal to 1. However, as Prof. Shi kindly informed us, there is a subtle but serious error in the proof; see Remark 2.5 for details.

Marcus and Rosen [22] subsequently showed that γ in (1.3) could be any value larger than 3.

In this paper we prove that the assertion of Lifshits and Shi is correct, that (1.1) and (1.3) hold whenever $\gamma > 1$. See Theorems 2.1 and 2.2. Our method relies mainly on the Ray-Knight theorems and a moving boundary estimate due to Novikov [23].

A few words about when \mathcal{U}_n and \mathcal{V}_t consist of more than one point are in order. Eisenbaum [10] and Leuridan [18] have shown that at any time t there are at most two values where L_t^z takes its maximum. Toth [27] has shown that for n sufficiently large, depending on ω , there are at most 3 values of k which are most visited sites for S_n , and more recently Ding and Shen [9] have shown that almost surely \mathcal{U}_n consists of 3 distinct points infinitely often. It turns out that the values of the \liminf in (1.1)–(1.4) do not depend on which value of the most visited site is chosen.

There are many results on the most visited sites of Brownian motion and of various other processes. See [5], [8], [11], [12], [14], [16], [19], [21], [24], and [26] for some of these.

In Section 2 we state our main theorems precisely and give some preliminaries. Section 3 contains some estimates on local times and squared Bessel processes of dimension 0. These are used in Section 4 to establish a lower bound on the supremum of local time at certain random times, and in Section 5 we move from random times to fixed times to obtain our result for Brownian motion. Finally in Section 6 we prove the result for random walks.

2 Preliminaries

Let W_t be a one-dimensional Brownian motion and let $\{L_t^z\}$ be a jointly continuous version of its local times. Let

$$L_t^* = \sup_{z \in \mathbb{R}} L_t^z.$$

We define the collection of most visited sites of W by

$$\mathcal{V}_t = \{x \in \mathbb{R} : L_t^x = L_t^*\}.$$

Let $V_t^s = \inf\{|x| : x \in \mathcal{V}_t\}$ and $V_t^\ell = \sup\{|x| : x \in \mathcal{V}_t\}$.

Our main theorem can be stated as follows.

Theorem 2.1. (1) If $\gamma > 1$, then

$$\liminf_{t \rightarrow \infty} \frac{V_t^s}{\sqrt{t}/(\log t)^\gamma} = \infty, \quad \text{a.s.}$$

(2) If $\gamma \leq 1$,

$$\liminf_{t \rightarrow \infty} \frac{V_t^\ell}{\sqrt{t}/(\log t)^\gamma} = 0, \quad \text{a.s.}$$

We have the corresponding theorem for a simple random walk S_n . Let

$$N_n^k = \sum_{j=0}^n 1_{(S_j=k)},$$

the number of times S_j is equal to k up to time n . Let $N_n^* = \max_{k \in \mathbb{Z}} N_n^k$ and let

$$\mathcal{U}_t = \{k \in \mathbb{Z} : N_n^k = N_n^*\}.$$

Let $U_t^s = \inf\{|x| : x \in \mathcal{N}_t\}$ and $U_t^\ell = \sup\{|x| : x \in \mathcal{N}_t\}$.

Our second theorem is the following.

Theorem 2.2. (1) If $\gamma > 1$, then

$$\liminf_{n \rightarrow \infty} \frac{U_n^s}{\sqrt{n}/(\log n)^\gamma} = \infty, \quad \text{a.s.}$$

(2) If $\gamma \leq 1$,

$$\liminf_{n \rightarrow \infty} \frac{U_n^\ell}{\sqrt{n}/(\log n)^\gamma} = 0, \quad \text{a.s.}$$

A process X_t is called the square of a Bessel process of dimension 0 started at $x \geq 0$, denoted $BES(0)^2$, if it is the unique solution to the stochastic differential equation

$$X_t = x + 2\sqrt{X_t} dW_t,$$

where $X_t \geq 0$ a.s. for each t and W is a one-dimensional Brownian motion with filtration $\{\mathcal{F}_t\}$. When X_t hits 0, which it does almost surely, it then stays there forever. X has a scaling property: for $r > 0$ and X is started at x , the process $\frac{1}{r}X_t$ has the same law as the process $X_{t/r}$ started at x/r . If Y_t is the nonnegative square root of X_t and $x > 0$, then Y is the unique solution to the stochastic differential equation

$$Y_t = \sqrt{x} + W_t - \frac{1}{2Y_t} dt.$$

See [25] for details.

For any process ξ_t let

$$\tau_a = \tau_a^\xi = \inf\{t > 0 : \xi_t = a\}, \quad (2.1)$$

the hitting time of a by the process ξ_t .

Let

$$T_r = T(r) = \inf\{t > 0 : L_t^0 \geq r\}, \quad (2.2)$$

the inverse local time at 0.

The main preliminary result we need is the following version of a special case of the Ray-Knight theorems. See [17], [22], and [25].

Theorem 2.3. Suppose $r > 0$. The processes $\{L_{T_r}^z, z \geq 0\}$ and $\{L_{T_r}^{-z}, z \geq 0\}$ are each BES(0)² processes with time parameter z started at r and are independent of each other.

We also need

Proposition 2.4. Let $0 < r < s$. The processes $\{L_{T_s}^z - L_{T_r}^z, z \geq 0\}$ and $\{L_{T_s}^{-z} - L_{T_r}^{-z}, z \geq 0\}$ are each BES(0)² processes started at $s - r$, are independent of each other, and are independent of the processes $\{L_{T_r}^z, z \geq 0\}$ and $\{L_{T_r}^{-z}, z \geq 0\}$.

Proof. Since the local time at 0 of a Brownian motion increases only when the Brownian motion is at 0, then $W_{T_r} = 0$ for all $r > 0$. Proposition 2.4 follows easily from this, the strong Markov property applied at time T_r , and Theorem 2.3. \square

We use the letter c with or without subscripts to denote finite positive constants whose exact value is unimportant and whose value may change from line to line.

Remark 2.5. The error in [20] is that inequality (2.12) of that paper need not hold. Let $a > 0$. Note that $\sup_{y > a\sqrt{t}} L_t^y$ can be decreasing in t at some times because the supremum is over decreasing sets. This can happen even when $W_t > a\sqrt{t}$. Similarly, $\sup_{x < a\sqrt{t}} L_t^x$ can be increasing in t at some times even when $W_t > a\sqrt{t}$ because the supremum is over increasing sets.

3 Some estimates

Define

$$I^+(t, h) = \sup_{0 \leq z \leq h} L_t^z.$$

Proposition 3.1. Let $\theta > 0$. There exists a positive real number M depending on θ such that

$$\limsup_{t \rightarrow \infty} \frac{\sup_{s \leq t} [I^+(s, \sqrt{t}/(\log t)^\theta) - L_s^0]}{\sqrt{t} \log \log t / (\log t)^{\theta/2}} \leq M, \quad \text{a.s.}$$

Proof. Let A_n be the event

$$A_n = \left\{ \sup_{s \leq 2^{n+1}} [I^+(s, 2^{(n+1)/2}/(\log 2^n)^\theta) - L_s^0] \geq M \frac{2^{n/2} \log \log 2^n}{(\log 2^{n+1})^{\theta/2}} \right\},$$

where M is a positive real to be chosen in a moment. By scaling, the probability of A_n is the same as the probability of

$$B_n = \left\{ \sup_{s \leq 1} [I^+(s, 1/(\log 2^n)^\theta) - L_s^0] \geq M \frac{2^{-1/2} \log \log 2^n}{(\log 2^{n+1})^{\theta/2}} \right\}.$$

Lemma 5.2 of [4] says that if $\delta \leq 1$ and $t \geq 1$, then

$$\mathbb{P}(\sup_{s \leq t} \sup_{0 \leq x, y \leq 1, |x-y| \leq \delta} |L_s^y - L_s^x| \geq \lambda) \leq \frac{c_1}{\delta} e^{-\lambda/c_2 \delta^{1/2} t^{1/4}}.$$

Applying this with $t = 1$, $\delta = 1/(\log 2^n)^\theta$, $x = 0$, and

$$\lambda = 2^{-1/2} M \log \log 2^n / (\log 2^{n+1})^{\theta/2},$$

and recalling $\mathbb{P}(A_n) = \mathbb{P}(B_n)$, we see that $\mathbb{P}(A_n)$ is summable provided we choose M large enough. By the Borel-Cantelli lemma, $\mathbb{P}(A_n \text{ i.o.}) = 0$. If $2^n \leq t \leq 2^{n+1}$ and t is large

enough (depending on ω), then

$$\begin{aligned} \sup_{s \leq t} [I^+(s, \sqrt{t}/(\log t)^\theta) - L_s^0] &\leq \sup_{s \leq 2^{n+1}} [I^+(s, 2^{(n+1)/2}/(\log 2^n)^\theta) - L_s^0] \\ &\leq M \frac{2^{n/2} \log \log 2^n}{(\log 2^{n+1})^{\theta/2}} \\ &\leq M \sqrt{t} \log \log t / (\log t)^{\theta/2}. \end{aligned}$$

The proposition follows. □

Proposition 3.2. *Let X_t be a BES(0)² and let \mathbb{P}^x denote the law of X started at x . Then*

$$\mathbb{P}^1(\tau_0 < \tau_{1+a}) = \frac{a}{1+a}.$$

Proof. We know $\tau_0 < \infty$ a.s. Now X is a continuous martingale, hence a time change of a Brownian motion, and thus the hitting probabilities are the same as those for a Brownian motion. □

The next two propositions show that in many respects a BES(0)² is similar to a Brownian motion as long as it is not too close to 0.

Proposition 3.3. *For X a BES(0)² and $x > 0$,*

$$\mathbb{P}^x(\inf_{s \leq t} X_s < x - \lambda) \leq c_1 e^{-c_2 \lambda^2 / xt}.$$

Proof. Since $X \geq 0$, there is nothing to prove unless $\lambda \leq x$. By a scaling argument, it suffices to suppose $x = 1$.

We start by writing

$$\mathbb{P}^1(\tau_{1-\lambda}^X \leq t) \leq \mathbb{P}^1(\tau_2^X \leq t) + \mathbb{P}^1(\tau_{1-\lambda}^X \leq t, \tau_2^X > t). \tag{3.1}$$

To estimate the terms on the right hand side of (3.1) we use Doob's inequality. Recalling that $dX_t = 2\sqrt{X_t} dW_t$, we have $d\langle X \rangle_t = 4X_t dt$.

Suppose $a > 0$. Then

$$\begin{aligned} \mathbb{P}^1(\tau_2^X \leq t) &= \mathbb{P}^1(\sup_{s \leq t \wedge \tau_2^X} X_s \geq 2) = \mathbb{P}^1(\sup_{s \leq t \wedge \tau_2^X} a(X_s - 1) \geq a) \\ &\leq e^{-a} \mathbb{E}^1 \exp(a(X_{t \wedge \tau_2^X} - 1)). \end{aligned}$$

To bound the expectation,

$$\begin{aligned} &\mathbb{E}^1 \exp(a(X_{t \wedge \tau_2^X} - 1)) \\ &= \mathbb{E}^1 \left[\exp(a(X_{t \wedge \tau_2^X} - 1) - \frac{1}{2} a^2 \langle X \rangle_{t \wedge \tau_2^X}) \exp(\frac{1}{2} a^2 \langle X \rangle_{t \wedge \tau_2^X}) \right] \\ &\leq \mathbb{E}^1 \exp(a(X_{t \wedge \tau_2^X} - 1) - \frac{1}{2} a^2 \langle X \rangle_{t \wedge \tau_2^X}) e^{4a^2 t}. \end{aligned}$$

Setting $a = 1/8t$ yields

$$\mathbb{P}^1(\tau_2^X \leq t) \leq e^{-1/16t}.$$

The second term of (3.1) is slightly more complicated, but quite similar. Let \tilde{X}_t be X_t stopped at time τ_2^X and use (2.1) to define $\tau_{1-\lambda}^{\tilde{X}}$. Suppose $a > 0$ and write

$$\begin{aligned} \mathbb{P}^1(\tau_{1-\lambda}^X \leq t, \tau_2^X > t) &\leq \mathbb{P}^1(\inf_{s \leq t \wedge \tau_{1-\lambda}^{\tilde{X}}} (\tilde{X}_s - 1) \leq -\lambda) \\ &= \mathbb{P}^1(\sup_{s \leq t \wedge \tau_{1-\lambda}^{\tilde{X}}} (-a(\tilde{X}_s - 1)) \geq a\lambda) \\ &\leq e^{-a\lambda} \mathbb{E}^1 \exp(a(-(\tilde{X}_{t \wedge \tau_{1-\lambda}^{\tilde{X}}} - 1))) \end{aligned}$$

and the expectation on the last line is equal to

$$\mathbb{E}^1 \left[\exp(-a(\tilde{X}_{t \wedge \tau_{1-\lambda}^{\tilde{X}}} - 1) - \frac{1}{2}a^2 \langle \tilde{X} \rangle_{t \wedge \tau_{1-\lambda}^{\tilde{X}}}) \exp(\frac{1}{2}a^2 \langle \tilde{X} \rangle_{t \wedge \tau_{1-\lambda}^{\tilde{X}}}) \right],$$

which is bounded by e^{4a^2t} . Setting $a = \lambda/8t$ we see the second term on the right of (3.1) is bounded by $e^{-\lambda^2/16t}$.

Combining the two estimates for the terms on the right hand side of (3.1) and recalling that we are supposing $\lambda \leq 1$ yields the proposition. \square

Another approach to the preceding proposition is to use the results of [6].

Proposition 3.4. *Let $R > 0$, let X_t be a BES(0)², and let g be a non-negative absolutely continuous function on $[0, R]$ with $g(0) > 0$. Let $p > 1$. Then*

$$\begin{aligned} \mathbb{P}^1(X_t \leq 1+g(t), 0 \leq t \leq R) & \tag{3.2} \\ & \leq c_1 e^{c_2(p)R} \left(\frac{g(0)}{\sqrt{R}}\right)^{1/p^2} \exp\left(\frac{1}{2(p-1)p} \int_0^R g'(s)^2 ds\right) + c_3 e^{-c_4/R}. \end{aligned}$$

Proof. By Novikov [23], Theorem 6,

$$\begin{aligned} \mathbb{P}^0(W_t \leq g(t), 0 \leq t \leq R) & \tag{3.3} \\ & \leq c_1 \left(\Phi_0\left(\frac{g(0)}{\sqrt{R}}\right)\right)^{1/p} \exp\left(\frac{1}{2(p-1)} \int_0^R g'(s)^2 ds\right), \end{aligned}$$

where W is a Brownian motion, $\Phi_0(x) = 2\Phi(x) - 1$, and $\Phi(x)$ is the distribution function of a standard normal random variable. Note $\Phi_0(x) \leq cx$ for $x \geq 0$.

Let Z be the unique solution to

$$dZ_t = dW_t - a(Z_t) dt,$$

where $a(x) = 1/2x$ for $x \geq 1/2$ and $a(x) = 1$ for $x < 1/2$. Let $Y_t = X_t^{1/2}$.

We start by writing

$$\begin{aligned} \mathbb{P}^1(X_t \leq 1 + g(t), 0 \leq t \leq R) & \tag{3.4} \\ & \leq \mathbb{P}^1(X_t \leq 1 + g(t), 0 \leq t \leq R, \tau_{1/4}^X > R) + \mathbb{P}^1(\tau_{1/4}^X \leq R). \end{aligned}$$

The second term on the right is bounded by $c_1 e^{-c_2/R}$ by Proposition 3.3. The first term on the right is equal to

$$\begin{aligned} \mathbb{P}^1(Y_t \leq (1 + g(t))^{1/2}, 0 \leq t \leq R, \tau_{1/2}^Y > R) & \\ & \leq \mathbb{P}^1(Y_t \leq 1 + \frac{1}{2}g(t), 0 \leq t \leq R, \tau_{1/2}^Y > R) \\ & = \mathbb{P}^1(Z_t \leq 1 + \frac{1}{2}g(t), 0 \leq t \leq R, \tau_{1/2}^Z > R) \\ & \leq \mathbb{P}^1(B), \end{aligned}$$

where

$$B = \{Z_t \leq 1 + \frac{1}{2}g(t), 0 \leq t \leq R\}$$

and $\tau_{1/2}^Z$ is defined by (2.1); we use the fact that $Z_t = Y_t$ for $t < \tau_{1/2}^Y$.

Let

$$M_t = \exp\left(\int_0^t a(Z_s) dW_s - \frac{1}{2} \int_0^t a(Z_s)^2 ds\right).$$

Let \mathbb{Q} be defined by $d\mathbb{Q}/d\mathbb{P}^1 = M_t$ on \mathcal{F}_t . By the Girsanov theorem, $Z_t = W_t - \int_0^t a(Z_s) ds$ is a Brownian motion under \mathbb{Q} .

By Hölder's inequality,

$$\mathbb{P}^1(B) = \mathbb{E}_{\mathbb{Q}}[M_R^{-1}; B] \leq (\mathbb{E}_{\mathbb{Q}}M_R^{-r})^{1/r}(\mathbb{Q}(B))^{1/p},$$

where $r = p/(p - 1)$. We bound the second factor by (3.3).

It remains to bound

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}[M_R^{-r}] &= \mathbb{E}_{\mathbb{P}}^1[M_R^{1-r}] \\ &= \mathbb{E}_{\mathbb{P}}^1\left[\exp\left((1-r)\int_0^R a(Z_s) dW_s - \frac{1-r}{2}\int_0^R a(Z_s)^2 ds\right)\right] \\ &= \mathbb{E}_{\mathbb{P}}^1\left[\exp\left((1-r)\int_0^R a(Z_s) dW_s - \frac{(1-r)^2}{2}\int_0^R a(Z_s)^2 ds\right)\right. \\ &\quad \left.\times \exp\left(\frac{(1-r)^2 - (1-r)}{2}\int_0^R a(Z_s)^2 ds\right)\right] \\ &\leq \exp\left(\frac{r^2 - r}{2}R\right). \end{aligned}$$

Combining our estimates yields the proposition. □

4 Growth of local times

Suppose $\varepsilon \in (0, \frac{1}{2})$ and $0 < \delta \leq \frac{1}{2}$. Choose $p > 1$ close to 1 so that $1/p^2 \geq 1 - \varepsilon$. Choose $\beta \in (0, \frac{1}{2})$ small so that $\beta^2/4p(p - 1) < \varepsilon/2$. Let

$$U_t = L_{T_1}^t - 1. \tag{4.1}$$

Recall that here t is actually the space variable for local time. Set

$$g(t) = \begin{cases} 4\delta, & t \leq 16\delta^2/\beta^2; \\ \beta\sqrt{t}, & t > 16\delta^2/\beta^2. \end{cases}$$

Let

$$A = \{\exists t \in [0, \delta^\varepsilon] : U_t \geq g(t)\}. \tag{4.2}$$

Proposition 4.1.

$$\mathbb{P}(A^c) \leq c_1\delta^{1-2\varepsilon}.$$

Proof. We estimate the right hand side of (3.2) with $R = \delta^\varepsilon$ and $g(0) = 4\delta$. Observe that $g'(t)$ is zero unless $t > 16\delta^2/\beta^2$, in which case $g'(t) = \beta/2\sqrt{t}$. Hence

$$\begin{aligned} \frac{1}{2p(p-1)} \int_0^{\delta^\varepsilon} g'(t)^2 dt &\leq \frac{\beta^2}{8p(p-1)} \int_{16\delta^2/\beta^2}^1 \frac{1}{t} dt \\ &= \frac{\beta^2}{4p(p-1)} \log(1/\delta) + c(p, \beta), \end{aligned}$$

where $c(p, \beta)$ depends on p and β , but not δ .

Therefore

$$\mathbb{P}(A^c) \leq c_1(\delta^{1-\varepsilon/2})^{1/p^2} (1/\delta)^{\beta^2/4p(p-1)} + c_2e^{-c_3\delta^{-\varepsilon}} \leq c_4\delta^{1-2\varepsilon}. \tag{4.3}$$

For $s \in [0, 1]$ let

$$X_t^s = L_{T(1+s)}^t - L_{T(1)}^t - s. \tag{4.3}$$

Let

$$B_s = \{\exists t \in [0, \delta^\varepsilon] : X_t^s \leq -\frac{1}{4}g(t)\}. \tag{4.4}$$

For U , an estimate involving a power of δ close to 1 is the best we can expect. However the exponential estimate we obtain in the next proposition allows us to take the supremum over a large number of values of s .

Proposition 4.2. For $s \in [0, \delta^\varepsilon]$

$$\mathbb{P}(B_s) \leq c_1 \log(1/\delta) e^{-c_2/\delta^\varepsilon}.$$

Proof. Let $I_0 = [0, 16\delta^2/\beta^2]$. Let M be the smallest positive integer such that $2^M(16\delta^2/\beta^2)$ is larger than δ^ε . For $1 \leq m \leq M$ let

$$I_m = [2^{m-1}(16\delta^2/\beta^2), 2^m(16\delta^2/\beta^2)].$$

For $0 \leq m \leq M$ let

$$C_m = \{\exists t \in I_m : X_t^s \leq -\frac{1}{4}g(t)\}.$$

By Proposition 3.3, for $1 \leq m \leq M$,

$$\mathbb{P}(C_m) \leq c_1 \exp\left(-c_2 \frac{2^{m-1}\delta^2}{s2^m\delta^2}\right).$$

Because $s \leq \delta^\varepsilon$, this is bounded by $c_1 e^{-c_2\delta^{-\varepsilon}}$. Similarly

$$\mathbb{P}(C_0) \leq c_1 \exp\left(-c_2 \frac{\delta^2}{s\delta^2}\right) \leq c_3 e^{-c_4\delta^{-\varepsilon}}.$$

Since $M \leq c \log(\delta^{\varepsilon-2})$,

$$\mathbb{P}(\cup_{m=0}^M C_m) \leq c_1 \log(1/\delta) e^{-c_2\delta^{-\varepsilon}}.$$

Observing that $B_s \subset \cup_{m=0}^M C_m$ completes the proof. □

Proposition 4.3. There exists c such that

$$\mathbb{P}(\exists u \in [1, 1 + \delta^\varepsilon] : (L_{T_u}^* - u) \leq \delta) \leq c\delta^{2-4\varepsilon}.$$

c depends on ε but not δ .

Proof. Let $J = \lceil \delta^{\varepsilon-1} \rceil + 1$ and let $0 = s_0 < s_1 < \dots < s_J = \delta^\varepsilon$ be points of the interval $[0, \delta^\varepsilon]$ such that $s_{j+1} - s_j \leq \delta$ for all j . Let

$$D_j = \{\sup_{t \geq 0} (U_t + X_t^{s_j}) \leq 2\delta\}.$$

We know $\mathbb{P}(D_0) \leq 2\delta$ by Proposition 3.2.

Suppose $1 \leq j \leq J$. If $\omega \in A \cap B_{s_j}^c$, then there exists $t \in [0, \delta^\varepsilon]$ such that $U_t(\omega) \geq g(t)$ but $X_t^{s_j}(\omega) \geq -\frac{1}{4}g(t)$. But then

$$U_t(\omega) + X_t^{s_j}(\omega) \geq g(t) - \frac{1}{4}g(t) \geq 3\delta,$$

which implies $\omega \notin D_j$. Therefore $D_j \subset A^c \cup B_{s_j}$. It follows that

$$\cup_{j=1}^J D_j \subset A^c \cup (\cup_{j=1}^J B_{s_j}).$$

Using Propositions 4.1 and 4.2 and the fact that $J \leq c\delta^{\varepsilon-1}$, we then have

$$\begin{aligned} \mathbb{P}(\exists j \leq J : \sup_{t \geq 0} (U_t + X_t^{s_j}) \leq 2\delta) &\leq 2\delta + c_1\delta^{1-2\varepsilon} + c_2\delta^{\varepsilon-1} \log(1/\delta) e^{-c_3\delta^{-\varepsilon}} \\ &\leq c_4\delta^{1-2\varepsilon}. \end{aligned}$$

If $\sup_{x \geq 0} L_{T(1+s_j)}^x - (1 + s_j) \leq 2\delta$, then $\sup_{t \geq 0} (U_t + X_t^{s_j}) \leq 2\delta$, and so

$$\mathbb{P}(\exists j \leq J : \sup_{x \geq 0} L_{T(1+s_j)}^x - (1 + s_j) \leq 2\delta) \leq c_4\delta^{1-2\varepsilon}. \tag{4.5}$$

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Let $L_t^+ = \sup_{x>0} L_t^x$ and $L_t^- = \sup_{x<0} L_t^x$. If $L_{T(1+s_j)}^* - (1 + s_j) \leq 2\delta$, then

$$L_{T(1+s_j)}^+ - (1 + s_j) \leq 2\delta \quad \text{and} \quad L_{T(1+s_j)}^- - (1 + s_j) \leq 2\delta.$$

By independence, symmetry, and (4.5),

$$\mathbb{P}(E) \leq (c_1 \delta^{1-2\varepsilon})^2 = c_2 \delta^{2-4\varepsilon},$$

where

$$E = \{\exists j \leq J : L_{T(1+s_j)}^* - (1 + s_j) \leq 2\delta\}.$$

If $u \leq \delta^\varepsilon$ and $u \in [s_j, s_{j+1}]$, then

$$\begin{aligned} L_{T(1+u)}^* - (1 + u) &\geq L_{T(1+s_j)}^* - (1 + s_j) + (s_j - u) \\ &\geq L_{T(1+s_j)}^* - (1 + s_j) - \delta. \end{aligned}$$

We conclude that on the event E^c

$$L_{T(1+u)}^* - (1 + u) > 2\delta - \delta = \delta.$$

Therefore

$$\mathbb{P}(\exists u \in [0, \delta^\varepsilon] : L_{T(1+u)}^* - (1 + u) \leq \delta) \leq c\delta^{2-4\varepsilon}. \quad \square$$

Theorem 4.4. *If $\gamma > 1/2$, then*

$$\liminf_{t \rightarrow \infty} \frac{L_{T_t}^* - t}{t/(\log t)^\gamma} = \infty, \quad \text{a.s.}$$

Proof. Let $r_K = 2^K$, $a > 0$, and

$$\delta_K = \frac{a}{(\log r_K)^\gamma}.$$

Divide $[r_K, r_{K+1}]$ into $\lceil \delta_K^{-\varepsilon} \rceil + 1$ equal subintervals. Each subinterval will have length less than or equal to $\delta_K^\varepsilon r_K$. Let

$$F_K = \{\exists t \in [r_K, r_{K+1}] : (L_{T_t}^* - t) \leq \delta_K r_K\}.$$

Then by scaling, Proposition 4.3, and our bound on the number of subintervals,

$$\mathbb{P}(F_K) \leq c_1 \delta_K^{-\varepsilon} \delta_K^{2-4\varepsilon} = c_1 \delta_K^{2-5\varepsilon}.$$

If $\gamma > \frac{1}{2}$, choose ε small enough so that $(2 - 5\varepsilon)\gamma > 1$. By the Borel-Cantelli lemma, $\mathbb{P}(F_K \text{ i.o.}) = 0$. This implies

$$\mathbb{P}\left(L_{T_t}^* - t \leq \frac{at}{(\log t)^\gamma} \text{ i.o.}\right) = 0.$$

Since a is arbitrary, the theorem follows. □

5 From random times to fixed times

Now we derive our results for fixed times from Theorem 4.4. For values r where T_r is approximately r^2 , the argument is straightforward, but for other values of r a different argument is necessary to avoid an extraneous power of logarithm.

Let

$$I(t, h) = \sup_{|z| \leq h} L_t^z.$$

Theorem 5.1. *Let $\gamma > 1$. There exists $\rho > 0$ such that with probability one,*

$$L_t^* > I(t, \sqrt{t}/(\log t)^\gamma) + \frac{c\sqrt{t}}{(\log t)^\rho}$$

for all t sufficiently large.

Proof. Without loss of generality assume $\gamma \leq 2$. Choose $1/2 < b < \gamma/2$ and then choose $a < \gamma$ such that $\gamma/2 - a/2 > b$. Suppose

$$T_{r-} \leq t \leq T_r,$$

where $T_{r-} = \lim_{s \rightarrow r-} T_s$. Then $L_t^0 = r$.

Case 1. $t \leq r^2(\log r)^a$. By [15], for t sufficiently large (depending on ω),

$$r = L_t^0 \leq c\sqrt{t \log \log t},$$

so $\log r \leq c \log t$. By Proposition 3.1 and symmetry, for sufficiently large t (also depending on ω),

$$\begin{aligned} I(t, \sqrt{t}/(\log t)^\gamma) - L_t^0 &\leq c \frac{\sqrt{t} \log \log t}{(\log t)^{\gamma/2}} \\ &\leq c \frac{r(\log r)^{a/2} \log \log r}{(\log r)^{\gamma/2}} \\ &= c \frac{r \log \log r}{(\log r)^{\gamma/2 - a/2}}. \end{aligned}$$

For r sufficiently large, for all $s \in [r/2, r)$, by Theorem 4.4 we have

$$L_{T_s}^* - s \geq \frac{s}{2(\log s)^b}.$$

Letting s increase up to r ,

$$\begin{aligned} L_t^* - r &\geq L_{T_{r-}}^* - r \geq \frac{r}{2(\log r)^b} \\ &\geq I(t, \sqrt{t}/(\log t)^\gamma) - r + c \frac{r}{(\log r)^b} \\ &\geq I(t, \sqrt{t}/(\log t)^\gamma) - r + c \frac{\sqrt{t}}{(\log t)^{b+a/2}} \end{aligned}$$

for t sufficiently large.

Case 2. $t > r^2(\log r)^a$. Then

$$L_t^0 = r \leq c_1 \frac{\sqrt{t}}{(\log t)^{a/2}}.$$

By this, Proposition 3.1, and symmetry, there exists $K > c_1$ such that

$$I(t, \sqrt{t}/(\log t)^\gamma) \leq L_t^0 + K \frac{\sqrt{t} \log \log t}{(\log t)^{\gamma/2}} \leq 2K \frac{\sqrt{t}}{(\log t)^{a/2}}$$

for t large. By Kesten's law of the iterated logarithm (see [15] and also [7]), there exists $\kappa > 0$ such that for t sufficiently large,

$$\begin{aligned} L_t^* &\geq \kappa \sqrt{t}/(\log \log t)^{1/2} \\ &\geq 3K \frac{\sqrt{t}}{(\log t)^{a/2}} \geq I(t, \sqrt{t}/(\log t)^\gamma) + K \frac{\sqrt{t}}{(\log t)^{a/2}}. \end{aligned}$$

In either case,

$$L_t^* \geq I(t, \sqrt{t}/(\log t)^\gamma) + c \frac{\sqrt{t}}{(\log t)^{b+a/2}}, \quad (5.1)$$

and we may take $\rho = b + a/2$. □

Proof of Theorem 2.1. Theorem 2.1(2) is already known; see [20]. For (1), let $\gamma > 1$. For large enough t ,

$$L_t^* > I(t, \sqrt{t}/(\log t)^\gamma),$$

which means that L_t^z takes its maximum for z outside the interval

$$[-\sqrt{t}/(\log t)^\gamma, \sqrt{t}/(\log t)^\gamma].$$

Theorem 2.1(1) now follows. □

6 Random walks

Proof of Theorem 2.2. (2) follows from [20], so we only consider (1). By the invariance principle of [24] we can find a simple random walk S_n and a Brownian motion W_t such that for each $\varepsilon > 0$,

$$\sup_{k \in \mathbb{Z}} |L_n^k - N_n^k| = o(n^{1/4+\varepsilon}), \quad \text{a.s.} \quad (6.1)$$

If $\gamma > 1$ and $K_n = \max_{k \in \mathbb{Z}, |k| \leq \sqrt{n}/(\log n)^\gamma} N_n^k$, by (6.1), Lemma 5.3 of [4], and Theorem 5.1, there exists $\rho > 0$ such that

$$\begin{aligned} N_n^* &\geq L_n^* - cn^{1/4+\varepsilon} \\ &\geq I(n, \sqrt{n}/(\log n)^\gamma) + c_1 \frac{\sqrt{n}}{(\log n)^\rho} - c_2 n^{1/4+\varepsilon} \\ &\geq K_n + c_1 \frac{\sqrt{n}}{(\log n)^\rho} - 2c_2 n^{1/4+\varepsilon} \\ &> K_n \end{aligned}$$

for n sufficiently large. We conclude the most visited site of S_n must be larger in absolute value than $\sqrt{n}/(\log n)^\gamma$ for n large. □

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