

## Sharp solvability for singular SDEs\*

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### Abstract

The attracting inverse-square drift provides a prototypical counterexample to solvability of singular SDEs: if the coefficient of the drift is larger than a certain critical value, then no weak solution exists. We prove a positive result on solvability of singular SDEs where this critical value is attained from below (up to strict inequality) for the entire class of form-bounded drifts. This class contains e.g. the inverse-square drift, the critical Ladyzhenskaya-Prodi-Serrin class. The proof is based on a  $L^p$  variant of De Giorgi’s method.

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## 1 Introduction and main result

The paper addresses the question: what are the minimal assumptions on a locally unbounded vector field  $b : [0, \infty[ \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ ,  $d \geq 3$ , also called a drift, such that the stochastic differential equation (SDE)

$$dX_t = -b(t, X_t)dt + \sqrt{2}dB_t, \quad X_0 = x \in \mathbb{R}^d \tag{1.1}$$

admits a martingale (or weak) solution? Here  $B_t$  is a standard Brownian motion in  $\mathbb{R}^d$ . There is an extensive literature devoted to the search for such minimal assumptions, as well as to the question of what additional hypothesis on  $b$  is required in order to ensure the uniqueness of the solution. The interest is motivated, in particular, by physical applications and applications to the theory of stochastic optimal control.

It is known that if  $b$  is in the Ladyzhenskaya-Prodi-Serrin class

$$|b| \in L^q_{\text{loc}}([0, \infty[, L^r + L^\infty), \quad \frac{d}{r} + \frac{2}{q} < 1, \quad 2 < q \leq \infty \tag{LPS}$$

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then equation (1.1) has a unique in law martingale solution, see Portenko [15]. The strong existence and uniqueness is due to Krylov-Röckner [14]. In 2014, Beck-Flandoli-Gubinelli-Maurelli [1], extending an argument in [3], gave the following counterexample to weak solvability of (1.1) (among many results on the strong existence and uniqueness for (1.1) with  $b$  in the critical Ladyzhenskaya-Prodi-Serrin class

$$|b| \in L^q_{\text{loc}}([0, \infty[, L^r + L^\infty), \quad \frac{d}{r} + \frac{2}{q} \leq 1, \quad 2 < q \leq \infty). \quad (\text{LPS}_c)$$

**Example 1.1.** Consider the inverse-square drift  $b(x) = \sqrt{\delta} \frac{d-2}{2} |x|^{-2} x$  ( $d \geq 3$ ).

The following are true:

(a) If  $\delta > 4(\frac{d}{d-2})^2$ , then equation (1.1) with the initial point  $x = 0$  has no weak solution.

(b) If  $\delta > 4$ , then for every  $x \neq 0$  the solution to (1.1) arrives at the origin in finite time with positive probability.

The vector field in Example 1.1 has a stronger singularity than any vector field in  $(\text{LPS}_c)$ . Intuitively, when  $\delta > 4$ , the attraction to the origin is so strong that the process, even starting at  $x \neq 0$ , does not look like a Brownian motion. See also [19] regarding (b).

The fact that the value  $\delta = 4$  is critical was known, although in a different context: the theory of operator  $-\Delta + b \cdot \nabla$ . More precisely, let  $b$  be a form-bounded vector field, i.e.

$$|b|^2 \leq \delta(-\Delta) + g_\delta \quad \text{in the sense of quadratic forms (see below)}$$

(e.g.  $b$  in Example 1.1 is form-bounded). Consider Cauchy problem

$$(\partial_t - \Delta + b \cdot \nabla)u = 0, \quad u(0) = f \in L^p. \quad (1.2)$$

(Regarding the relationship with SDE (1.1), one expects, of course, in the time-homogeneous case  $b = b(x)$ ,  $u(t, x) = \mathbf{E}[f(X_t)]$ .) Multiplying the equation in (1.2) by  $u|u|^{p-2}$ , integrating by parts and using quadratic inequality and form-boundedness, one obtains that the admissible  $p$  that give e.g. an energy inequality turn out to be  $p > 2/(2 - \sqrt{\delta})$ . In fact, it was proved in [11] that if  $b = b(x)$  has form-bound  $\delta < 4$ , then there exists a quasi contraction strongly continuous Markov semigroup in  $L^p$ ,  $p > 2/(2 - \sqrt{\delta})$  that provides a strong solution to Cauchy problem (1.2). The interval of contraction solvability can be extended to  $[2/(2 - \sqrt{\delta}), \infty[$  and is sharp, see [7]. Now, as  $\delta \uparrow 4$ , this interval disappears, and with it the theory of the operator  $-\Delta + b \cdot \nabla$ . (Regarding time-inhomogeneous  $b$ , see [17].)

**Definition 1.2.** A Borel measurable vector field  $b : [0, \infty[ \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  is said to be form-bounded if  $|b| \in L^2_{\text{loc}}([0, \infty[ \times \mathbb{R}^d)$  and there exist a constant  $\delta > 0$  and a function  $0 \leq g_\delta \in L^1_{\text{loc}}([0, \infty[)$  such that

$$\int_0^\infty \|b(t, \cdot)\xi(t, \cdot)\|_2^2 dt \leq \delta \int_0^\infty \|\nabla \xi(t, \cdot)\|_2^2 dt + \int_0^\infty g_\delta(t) \|\xi(t, \cdot)\|_2^2 dt$$

for all  $\xi \in C_c^\infty([0, \infty[ \times \mathbb{R}^d)$  (written as  $b \in \mathbf{F}_\delta$ ).

Here  $\|\cdot\|_p := \|\cdot\|_{L^p(\mathbb{R}^d)}$ .

Examples of form-bounded vector fields include:  $b \in (\text{LPS}_c)$ , the vector field in Example 1.1 (by Hardy's inequality) or, more generally, vector fields  $b = b(x)$  with  $|b|$  in the weak  $L^d$  class or the scale-invariant Campanato-Morrey class with integrability constant  $> 2$ . One can construct, for every  $\varepsilon > 0$ , a form-bounded  $b = b(x)$  with  $|b| \notin L^{2+\varepsilon}$ . See [6, 7] for details and other examples.

The present paper deals with SDE (1.1) having form-bounded vector field  $b \in \mathbf{F}_\delta$ . In [6] it was proved that if the form-bound  $\delta$  satisfies  $\delta < d^{-2}$ , then, for every  $x \in \mathbb{R}^d$ , (1.1) has a weak solution, given by a Feller evolution family, and is unique in a large class. In

the time-homogeneous case  $b = b(x)$  the result is stronger, that is,  $b$  is required to be form-bounded with  $\delta < (2(d-2)^{-1} \wedge 1)^2$  [8], or even only weakly form-bounded [9], which allows to treat vector fields  $b$  that are a priori only in  $L^1_{loc}$ . In both cases the solutions are determined by a Feller semigroup, and are unique among weak solutions that are constructed using an approximation of  $b$  by smooth vector fields that do not increase the form-bound  $\delta$  of  $b$ .

See also Krylov [13] regarding Markov weak solvability of (1.1) for  $|b| \in L^q_{loc}([0, \infty[, L^r + L^\infty)$ ,  $\frac{d}{r} + \frac{1}{q} \leq 1$ ,  $r \geq d$ ,  $q \geq 1$  (in [13] the SDE can in fact have just measurable diffusion coefficients).

In [6, 8, 9], the construction of the Feller evolution family (semigroup) and the weak solution to (1.1) is based on gradient estimates on solution  $u$  to the parabolic equation (1.2) (solution  $v$  to the elliptic equation  $(\lambda - \Delta + b \cdot \nabla)v = f$ ) with  $b \in \mathbf{F}_\delta$ . In [6] and [8] these estimates are, respectively,

$$\|\nabla u\|_{L^\infty([0,T],L^q)}, \|\nabla|\nabla u|^{\frac{q}{2}}\|_{L^2([0,T],L^2)}^{\frac{2}{q}} \leq C\|\nabla f\|_q \quad \text{for } q \in ]d, k(\delta)[ \quad \text{if } \delta < d^{-2}$$

and

$$\|\nabla v\|_{L^q}, \|\nabla|\nabla v|^{\frac{q}{2}}\|_{L^2}^{\frac{2}{q}} \leq C'\|f\|_q \quad \text{for } q \in ]2 \vee (d-2), k'(\delta)[ \quad \text{if } \delta < (2(d-2)^{-1} \wedge 1)^2.$$

It is not clear how to extend these estimates to  $\delta$  greater than  $(2(d-2)^{-1} \wedge 1)^2$ , not to mention  $\delta$  going up to 4. Thus, there is a gap between the hypothesis on  $\delta$  in [6, 8, 9] and in Example 1.1. The purpose of this paper is to fill this gap.

**Theorem 1.3.** *Let  $d \geq 3$ ,  $b \in \mathbf{F}_\delta$ . If  $\delta < 4$ , then for every  $x \in \mathbb{R}^d$  there exists a martingale solution of SDE (1.1).*

Theorem 1.3 shows that Example 1.1 is essentially sharp, at least as  $d \rightarrow \infty$ . A crucial feature of Theorem 1.3 is that it attains  $\delta = 4$ , up to the strict inequality, for the entire class of form-bounded vector fields.

We leave aside the important issues of the Feller property and the uniqueness of the constructed martingale solution when  $\delta < 4$ . Let us only mention that if one is willing to impose additional assumptions on  $\text{div } b$  (namely, the Kato class condition), then Nash's method allows to obtain two-sided Gaussian bounds on the heat kernel of (1.2), see [10], from which the Feller property of course follows.

We prove the main analytic result (Proposition 2.7 below) using De Giorgi's iterations. They are carried out in  $L^p$ ,  $p > \frac{2}{2-\sqrt{\delta}}$ ,  $p \geq 2$  rather than in the standard for the De Giorgi method  $L^2$  space, as is needed to handle  $1 \leq \delta < 4$ . In this regard, let us make a trivial observation that passing to  $L^p$  right away, using the fact that  $u^{\frac{p}{2}}$  is a subsolution and then applying to  $u^{\frac{p}{2}}$  the standard De Giorgi iteration procedure in  $L^2$ , does not allow to treat  $1 \leq \delta < 4$ . We will have to follow the iteration procedure from the very beginning and adjust it accordingly.

Earlier, De Giorgi's method in  $L^2$  was used by Zhang-Zhao [21], Zhao [23], Röckner-Zhao [16]. They prove, in particular, results on weak well-posedness of (1.1) with  $b$  having not too singular divergence and satisfying

$$|b| \in L^q_{loc}([0, \infty[, L^r + L^\infty), \quad \frac{d}{r} + \frac{2}{q} < 2.$$

Similarly to these papers, we apply a tightness argument to construct a martingale solution once Proposition 2.7 is established, see Section 2.2. We also refer to Hara [5] and Kinzebulatov-Vafadar [12] for the proofs of a posteriori Harnack inequality and Hölder continuity of solutions to divergence-form elliptic equations, using De Giorgi's and Moser's methods, with, respectively,  $b = b(x)$  in  $\mathbf{F}_\delta$ ,  $\delta < 1$ , and with  $b = b(x)$  in a

substantially larger class of vector fields satisfying the multiplicative form-boundedness condition

$$|b| \in L^1_{\text{loc}} \quad \text{and} \quad \| |b|^{\frac{1}{2}} \varphi \|_2^2 \leq \delta \| \nabla \varphi \|_2 \| \varphi \|_2 + c_\delta \| \varphi \|_2^2, \quad \forall \varphi \in C_c^\infty(\mathbb{R}^d)$$

with  $\delta < \infty$ ,  $c_\delta < \infty$ , but additionally having form-bounded divergence.

Finally, we note that passing to the  $L^p$  variant of De Giorgi’s method does not exclude other singular drift perturbations known to be amenable in  $L^2$  (cf. [5, 20]). For instance, the assertion of Theorem 1.3 is also valid for  $b = b_1 + b_2$ , where  $b_1 \in \mathbf{F}_{\delta_1}$ ,  $\delta_1 < 4$  and  $b_2$  satisfies:

1) there exists  $0 < a \leq 1$  such that  $|b_2| \in L^{1+a}_{\text{loc}}([0, \infty[\times \mathbb{R}^d)$  and

$$\int_0^\infty \langle |b_2(t, \cdot)|^{1+a} \xi^2(t, \cdot) \rangle dt \leq \delta_2 \int_0^\infty \| \nabla \xi(t, \cdot) \|_2^2 dt + \int_0^\infty g_\delta(t) \| \xi(t, \cdot) \|_2^2 dt \quad (1.3)$$

for all  $\xi \in C_c^\infty([0, \infty[\times \mathbb{R}^d)$ , for some  $0 < \delta_2 < \infty$  and  $0 \leq g_\delta \in L^1_{\text{loc}}([0, \infty[)$ . Here and everywhere below,

$$\langle f, g \rangle = \langle fg \rangle := \int_{\mathbb{R}^d} fg dx.$$

2) the divergence  $(\text{div } b_2)_+ \in L^1_{\text{loc}}([0, \infty[\times \mathbb{R}^d)$  and

$$\int_0^\infty \langle (\text{div } b_2)_+(t, \cdot) \xi^2(t, \cdot) \rangle dt \leq \nu \int_0^\infty \| \nabla \xi(t, \cdot) \|_2^2 dt + \int_0^\infty g_\nu(t) \| \xi(t, \cdot) \|_2^2 dt \quad (1.4)$$

with  $\nu < 4 - 2\sqrt{\delta_1}$  for some  $0 \leq g_\nu \in L^1_{\text{loc}}([0, \infty[)$ . We say that  $(\text{div } b_2)_+$  is a form-bounded potential (for instance, it can be a function in the weak  $L^{\frac{d}{2}}$  space). For details, see Remark 2.8 below. There we explain that  $\nu < 4 - 2\sqrt{\delta_1}$  suffices, provided that we prove the energy inequality in  $L^p$  for  $p$  such that  $\nu < 4\frac{p-1}{p} - 2\sqrt{\delta_1}$ . If we stay in  $L^2$ , then we have to impose more restrictive condition  $\nu < 2 - 2\sqrt{\delta_1}$ .

The class (1.3) is essentially twice more singular than  $\mathbf{F}_\delta$ . It first appeared in Q. S. Zhang [20], where the author used Moser’s method in  $L^2$  to prove, assuming that the vector field has zero divergence and satisfies (1.3), the local boundedness of weak solutions to the corresponding parabolic equation, and applied this result to study Navier-Stokes equations in  $\mathbb{R}^3$ .

Let us mention that the results of this paper in principle can be extended to SDEs

$$dX_t = -b(t, X_t)dt + \sqrt{2}\sigma(t, X_t)dB_t, \quad X_0 = x \in \mathbb{R}^d$$

having form-bounded  $b$  and “form-bounded diffusion coefficients”, i.e. such that matrix  $a = \sigma^\top \sigma$  is bounded, uniformly non-degenerate and such that the derivatives  $\nabla_i \sigma_{jk}$  are in  $\mathbf{F}_{\delta_1}$  with  $\delta_1$  sufficiently small. (Such diffusion coefficients in the time-homogeneous case  $\sigma = \sigma(x)$ , considered in [8], include e.g. the class  $\sigma_{jk} \in W^{1,d}$ , as well as some matrices having critical discontinuities.) Indeed, since  $\nabla_i \sigma_{jk}$  are form-bounded, we can rewrite the corresponding non-divergence form parabolic equation in divergence form, now having form-bounded drift  $\nabla \sigma + b$ , and then apply De Giorgi’s method which, of course, can handle  $-\nabla \cdot a \cdot \nabla$  instead of  $-\Delta$ .

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## 2 Proof of Theorem 1.3

### 2.1 De Giorgi's iterations in $L^p$

In the next two propositions,  $u$  is the solution to Cauchy's problem for inhomogeneous Kolmogorov equation

$$(\partial_t - \Delta + b \cdot \nabla)u = |h|f, \quad u(0) = 0, \tag{2.1}$$

where

$$b \in \mathbf{F}_\delta \cap C_c^\infty([0, \infty[\times \mathbb{R}^d), \quad \delta < 4,$$

$$h \in \mathbf{F}_\nu \cap C_c^\infty([0, \infty[\times \mathbb{R}^d), \quad \nu < \infty \quad \text{and } f \in C_c.$$

Since the coefficients of (2.1) are smooth with compact support, the solution  $u$  exists and is sufficiently regular to justify the manipulations with the equation below.

Set

$$p_\delta := \frac{2}{2 - \sqrt{\delta}}.$$

**Definition 2.1.** We will call a constant generic if it only depends on  $d, p, \delta, \nu, g_\delta, g_\nu$  (and  $T > 0$ , in case we work over a fixed finite time interval  $[0, T]$ ).

**Proposition 2.2** (Caccioppoli's inequality). Let  $u$  be the solution to Cauchy problem (2.1). Let  $p > p_\delta, p \geq 2$ . Set  $u_c := (u - c)_+, c \in \mathbb{R}$ . Fix  $T > 0$  and  $\eta \in C_c^\infty(\mathbb{R}^d)$ . Then,  $0 < t - s \leq T$ ,

$$\sup_{\vartheta \in [s, t]} \langle u_c^p(\vartheta)\eta^2 \rangle + \int_s^t \langle |\nabla(\eta u_c^{\frac{p}{2}})|^2 \rangle \tag{2.2}$$

$$\leq C_1 \langle u_c^p(s)\eta^2 \rangle + C_2 \int_s^t \langle u_c^p |\nabla \eta|^2 \rangle + C_3 \int_s^t \langle (\mathbf{1}_{\{|h| \geq 1\}} + \mathbf{1}_{\{|h| < 1\}} |h|^p) \mathbf{1}_{\{u > c\}} |f|^p \eta^2 \rangle$$

for generic constants  $C_1 - C_3 > 0$ .

The last term in the RHS has this form because this is what will be needed in the next section. There we will fix a general discontinuous  $b \in \mathbf{F}_\delta$  and will apply Proposition 2.2 (or rather its consequence, Proposition 2.9) to bounded smooth vector field  $\{b_n\} \subset \mathbf{F}_\delta \cap C_c^\infty([0, \infty[\times \mathbb{R}^d)$  that approximate  $b$  in  $L_{loc}^2([0, \infty[\times \mathbb{R}^d, \mathbb{R}^d)$  and preserve its form-bound  $\delta$  (and have  $g$  independent of  $n$ ). The right-hand side in (2.1) will be chosen as follows:

- 1)  $h = b_n$ , in order to apply a tightness argument to construct a candidate for the martingale solution to (1.1)
- 2)  $h = b_{m_1} - b_{m_2}$ , in order to pass to the limit in the martingale problem; we will take  $f = |\nabla \varphi|$ , where  $\varphi \in C_c^2$  is a test function in the martingale problem.

The construction of such  $b_n$  is discussed in Appendix A.

**Remark 2.3.** Let us emphasize that the constants  $C_1 - C_3$  in Proposition 2.2 are generic, so, by definition, they are allowed to depend on form-bounds  $\delta$  and  $\nu$  of  $b$  and  $h$ , respectively. However, since  $b_n$ 's preserve form-bound  $\delta$  of  $b$ , once we apply Proposition 2.2 with choices 1), 2), we will be able to take  $n \rightarrow \infty$  while still controlling constants  $C_1 - C_3$ . This remark also applies to Propositions 2.7 and 2.9.

**Remark 2.4.** If we were to multiply equation (2.1) by a constant, this would scale the LHS in (2.2) and the first two terms in the RHS, but this would also change the form-bound  $\nu$  of  $h$  (and function  $g_\nu$ ) and hence would change constant  $C_3$  in front of the third term in the RHS.)

*Proof of Proposition 2.2.* Put for brevity  $v := u_c$ . It suffices to prove

$$\begin{aligned} \sup_{\vartheta \in [s,t]} \langle v^p(\vartheta)\eta^2 \rangle + \int_s^t \langle |\nabla v^{\frac{p}{2}}|^2 \eta^2 \rangle & \tag{2.3} \\ & \leq C_1 \langle v^p(s)\eta^2 \rangle + C_2 \int_s^t \langle v^p |\nabla \eta|^2 \rangle + C_3 \int_s^t \langle (\mathbf{1}_{\{|h| \geq 1\}} + \mathbf{1}_{\{|h| < 1\}} |h|^p) \mathbf{1}_{\{v > 0\}} |f|^p \eta^2 \rangle. \end{aligned}$$

We multiply equation (2.1) by  $v^{p-1}\eta^2$  and integrate to obtain

$$\begin{aligned} \langle v^p(t)\eta^2 \rangle - \langle v^p(s)\eta^2 \rangle + \frac{4(p-1)}{p} \int_s^t \langle |\nabla v^{\frac{p}{2}}|^2 \eta^2 \rangle \\ \leq 4 \left| \int_s^t \langle \nabla v^{\frac{p}{2}}, v^{\frac{p}{2}} \eta \nabla \eta \rangle \right| + 2 \left| \int_s^t \langle b \cdot \nabla v^{\frac{p}{2}}, v^{\frac{p}{2}} \eta^2 \rangle \right| + \left| \int_s^t \langle |h| f v^{p-1} \eta^2 \rangle \right| \end{aligned}$$

where we estimate in the RHS:

1.

$$4 \left| \int_s^t \langle \nabla v^{\frac{p}{2}}, v^{\frac{p}{2}} \eta \nabla \eta \rangle \right| \leq 2\varepsilon_1 \int_s^t \langle |\nabla v^{\frac{p}{2}}|^2 \eta^2 \rangle + \frac{2}{\varepsilon_1} \int_s^t \langle v^p |\nabla \eta|^2 \rangle \quad (\varepsilon_1 > 0).$$

2.

$$\begin{aligned} 2 \left| \int_s^t \langle b \cdot \nabla v^{\frac{p}{2}}, v^{\frac{p}{2}} \eta^2 \rangle \right| & \leq \frac{1}{\sqrt{\delta}} \int_s^t \langle |b|^2 v^p \eta^2 \rangle + \sqrt{\delta} \int_s^t \langle |\nabla v^{\frac{p}{2}}|^2 \eta^2 \rangle \\ & \text{(we are using } b \in \mathbf{F}_\delta) \\ & \leq \frac{1}{\sqrt{\delta}} \left( \delta \int_s^t \langle |\nabla(\eta v^{\frac{p}{2}})|^2 \rangle + \int_s^t g_\delta \langle v^p \eta^2 \rangle \right) + \sqrt{\delta} \int_s^t \langle |\nabla v^{\frac{p}{2}}|^2 \eta^2 \rangle. \end{aligned}$$

We bound  $|\nabla(\eta v^{\frac{p}{2}})|^2 \leq (1 + \varepsilon_2) \langle |\nabla v^{\frac{p}{2}}|^2 \eta^2 \rangle + (1 + \varepsilon_2^{-1}) \langle v^p |\nabla \eta|^2 \rangle$ ,  $\varepsilon_2 > 0$ . Then

$$\begin{aligned} \sup_{\vartheta \in [s,t]} \langle v^p(\vartheta)\eta^2 \rangle + \left[ \frac{4(p-1)}{p} - (2 + \varepsilon_2)\sqrt{\delta} - 2\varepsilon_1 \right] \int_s^t \langle |\nabla v^{\frac{p}{2}}|^2 \eta^2 \rangle \\ \leq \langle v^p(s)\eta^2 \rangle + \frac{1}{\sqrt{\delta}} \int_s^t g_\delta \langle v^p \eta^2 \rangle + C'_2 \int_s^t \langle v^p |\nabla \eta|^2 \rangle + \left| \int_s^t \langle |h| f v^{p-1} \eta^2 \rangle \right|. \end{aligned}$$

Now, assuming first that  $T > 0$  is sufficiently small so that  $\frac{1}{\sqrt{\delta}} \left( \int_s^t g_\delta \right) < \frac{1}{3}$ , we obtain

$$\begin{aligned} \frac{2}{3} \sup_{\vartheta \in [s,t]} \langle v^p(\vartheta)\eta^2 \rangle + \left[ \frac{4(p-1)}{p} - (2 + \varepsilon_2)\sqrt{\delta} - 2\varepsilon_1 \right] \int_s^t \langle |\nabla v^{\frac{p}{2}}|^2 \eta^2 \rangle \\ \leq C'_1 \langle v^p(s)\eta^2 \rangle + C'_2 \int_s^t \langle v^p |\nabla \eta|^2 \rangle + C'_3 \left| \int_s^t \langle |h| f v^{p-1} \eta^2 \rangle \right|. \tag{2.4} \end{aligned}$$

Next, using the reproduction property, we extend the last inequality to arbitrary  $T > 0$  (at expense of increasing  $C'_i = C'_i(T)$ ,  $i = 1, 2, 3$ ).

It remains to estimate the last term in the RHS of (2.4):

$$\begin{aligned} \left| \int_s^t \langle |h| f v^{p-1} \eta^2 \rangle \right| \\ \leq \int_s^t \langle \mathbf{1}_{|h| \geq 1} |h|^{\frac{2}{p'}} |f| |v|^{p-1} \eta^2 \rangle + \int_s^t \langle \mathbf{1}_{|h| < 1} |h| |f| |v|^{p-1} \eta^2 \rangle =: I_1 + I_2, \tag{2.5} \end{aligned}$$

where, by Young's inequality,

$$\begin{aligned}
 I_1 &\leq \frac{\varepsilon_3^{p'}}{p'} \int_s^t \langle \mathbf{1}_{|h| \geq 1} |h|^2 v^p \eta^2 \rangle + \frac{\varepsilon_3^{-p}}{p} \int_s^t \langle \mathbf{1}_{|h| \geq 1} \mathbf{1}_{\{v > 0\}} |f|^p \eta^2 \rangle \quad (\varepsilon_3 > 0) \\
 &\quad (\text{we are using } h \in \mathbf{F}_\nu) \\
 &\leq \frac{\varepsilon_3^{p'}}{p'} \left( \nu \int_s^t |\nabla(\eta v^{\frac{p}{2}})|^2 + \int_s^t g_\nu \langle v^p \eta^2 \rangle \right) + \frac{\varepsilon_3^{-p}}{p} \int_s^t \langle \mathbf{1}_{|h| \geq 1} \mathbf{1}_{\{v > 0\}} |f|^p \eta^2 \rangle,
 \end{aligned}$$

and

$$I_2 \leq \frac{\varepsilon_4^{p'}}{p'} \int_s^t \langle \mathbf{1}_{|h| < 1} v^p \eta^2 \rangle + \frac{\varepsilon_4^{-p}}{p} \int_s^t \langle \mathbf{1}_{|h| < 1} |h|^p \mathbf{1}_{\{v > 0\}} |f|^p \eta^2 \rangle \quad (\varepsilon_4 > 0).$$

Inserting these estimates in (2.4) and taking care of the term  $\int_s^t g_\nu \langle v^p \eta^2 \rangle$  in the same way as we did above, we arrive at

$$\begin{aligned}
 &C \sup_{r \in [s, t]} \langle v^p(r) \eta^2 \rangle + \left[ \frac{4(p-1)}{p} - (2 + \varepsilon_2) \sqrt{\delta} - 2\varepsilon_1 - \frac{\varepsilon_3^{p'}}{p'} \nu - \frac{\varepsilon_4^{p'}}{p'} \right] \int_s^t \langle |\nabla v^{\frac{p}{2}}|^2 \eta^2 \rangle \\
 &\leq C_1'' \langle v^p(s) \eta^2 \rangle + C_2'' \int_s^t \langle v^p |\nabla \eta|^2 \rangle + C_3'' \int_s^t \langle (\mathbf{1}_{\{|h| \geq 1\}} + \mathbf{1}_{\{|h| < 1\}} |h|^p) \mathbf{1}_{\{v > 0\}} |f|^p \eta^2 \rangle,
 \end{aligned}$$

where the appropriate constant  $C$  is positive provided that  $\varepsilon_3, \varepsilon_4$  are sufficiently small. Note that  $\frac{4(p-1)}{p} - 2\sqrt{\delta} > 0$  if and only if  $p > p_\delta$ . Since the latter is a strict inequality, we can and will select  $\varepsilon_i$  ( $i = 1, 2, 3, 4$ ) sufficiently small so that the coefficient of  $\int_s^t \langle |\nabla v^{\frac{p}{2}}|^2 \eta^2 \rangle$  is positive. We arrive at (2.3), as needed.  $\square$

**Remark 2.5.** Apart from the weight  $\eta$  with compact support, we will also consider the weight

$$\rho(x) = (1 + \kappa|x|^{-2})^{-\beta}, \quad \beta > \frac{d}{4}, \quad \kappa > 0$$

(clearly, with our choice of  $\beta$ , we have  $\rho \in L^2$ ). Then, in the assumptions of Proposition 2.2, assuming that  $\kappa$  is chosen sufficiently small, we have for every  $p > p_c, p \geq 2$ , for all  $0 \leq s \leq t$ ,

$$\begin{aligned}
 &\sup_{\vartheta \in [s, t]} \langle u_c^p(\vartheta) \rho^2 \rangle + \int_s^t \langle |\nabla(\rho u_c^{\frac{p}{2}})|^2 \rangle \\
 &\leq C_1 \langle u_c^p(s) \rho^2 \rangle + C_2 \int_s^t \langle (\mathbf{1}_{\{|h| \geq 1\}} + \mathbf{1}_{\{|h| < 1\}} |h|^p) \mathbf{1}_{\{u > c\}} |f|^p \rho^2 \rangle. \quad (2.6)
 \end{aligned}$$

The proof essentially repeats the proof of (2.2) (we use  $|\nabla \rho| \leq \beta \sqrt{\kappa} \rho$  at the last step to get rid of the  $C_2$  term in (2.2)).

**Lemma 2.6** ([4, Sect.7.2]). *If  $\{y_m\}_{m=0}^\infty \subset \mathbb{R}_+$  is a nondecreasing sequence such that*

$$y_{m+1} \leq N C_0^m y_m^{1+\alpha}$$

for some  $C_0 > 1, \alpha > 0$ , and

$$y_0 \leq N^{-\frac{1}{\alpha}} C_0^{-\frac{1}{\alpha^2}}.$$

Then

$$\lim_m y_m = 0.$$

**Proposition 2.7.** *Let  $u$  be the solution to Cauchy problem (2.1). Fix  $T > 0$  and  $1 < \theta < \frac{d}{d-1}$ . For all  $p > p_\delta$ ,  $p \geq 2$ , there exists a generic constant  $K$  such that*

$$\begin{aligned} \sup_{[0,T] \times B(0, \frac{1}{2})} u_+ &\leq 2 \left( \int_0^T \langle (\mathbf{1}_{\{|h| \geq 1\}} + \mathbf{1}_{\{|h| < 1\}} |h|^p)^{\theta'} |f|^{p\theta'} \mathbf{1}_{B(0,1)} \rangle \right)^{\frac{1}{p\theta'}} \\ &+ K \left( \int_0^T \langle u_+^p \mathbf{1}_{B(0,1)} \rangle + \left( \int_0^T \langle u_+^{p\theta} \mathbf{1}_{B(0,1)} \rangle \right)^{\frac{1}{\theta}} \right)^{\frac{1}{p}}, \quad \theta' = \frac{\theta}{\theta - 1}. \end{aligned} \tag{2.7}$$

*Proof of Proposition 2.7.* Set

$$\begin{aligned} R_m &:= \frac{1}{2}(1 + 2^{-m}), \quad B_m := B(0, R_m), \\ M_m &:= M(2 - 2^{-m}) \end{aligned}$$

for a constant  $M > 0$  to be determined later. Put  $\eta_m := \eta_{R_m, R_{m-1}}$  where  $\eta_{r,R}$  is a fixed family of smooth cutoff functions

$$\eta_{r,R} = 1 \text{ in } B(0, r), \quad \eta_{r,R} = 0 \text{ in } \mathbb{R}^d - B(0, R), \quad |\nabla \eta_{r,R}| \leq \frac{C_0}{4}(R - r)^{-1} \text{ for } 0 < r < R. \tag{2.8}$$

Then  $|\nabla \eta_m| \leq c_0 2^m$ . Define

$$u_m := (u - M_m)_+$$

and

$$\begin{aligned} E_m &:= \sup_{\vartheta \in [0, T]} \langle u_m^p(\vartheta) \eta_m^2 \rangle + \int_0^T \langle |\nabla(\eta_m u_m^{\frac{p}{2}})|^2 \rangle, \\ U_m &:= \int_0^T \langle u_m^p \mathbf{1}_{B_m} \rangle + \left( \int_0^T \langle u_m^{p\theta} \mathbf{1}_{B_m} \rangle \right)^{\frac{1}{\theta}}. \end{aligned}$$

By Proposition 2.2, using Hölder’s inequality, we have for all  $0 \leq t \leq T$

$$\begin{aligned} \sup_{\vartheta \in [0, t]} \langle u_{m+1}^p(\vartheta) \eta_{m+1}^2 \rangle + \int_0^t \langle |\nabla(\eta_{m+1} u_{m+1}^{\frac{p}{2}})|^2 \rangle \\ \leq C_2 c_0^2 4^m \int_0^t \langle u_{m+1}^p \mathbf{1}_{B_m} \rangle + C_3 H^{\frac{1}{\theta'}} |\{u_{m+1} > 0\} \cap [0, t] \times B_m|^{\frac{1}{\theta}}, \end{aligned} \tag{2.9}$$

where

$$H := \int_0^T \langle (\mathbf{1}_{\{|h| \geq 1\}} + \mathbf{1}_{\{|h| < 1\}} |h|^p)^{\theta'} |f|^{p\theta'} \mathbf{1}_{B(0,1)} \rangle.$$

We estimate the last term in (2.9):

$$\begin{aligned} |\{u_{m+1} > 0\} \cap [0, t] \times B_m|^{\frac{1}{\theta}} &= \left( \int_0^t \langle \mathbf{1}_{\{u_m > M2^{-m-1}\}} \mathbf{1}_{B_m} \rangle \right)^{\frac{1}{\theta}} \\ &\leq (M2^{-m-1})^{-p} \left( \int_0^t \langle u_m^{p\theta} \mathbf{1}_{\{u_m > M2^{-m-1}\}} \mathbf{1}_{B_m} \rangle \right)^{\frac{1}{\theta}}. \end{aligned} \tag{2.10}$$

We assume from now on that  $M$  satisfies  $M^p \geq H^{\frac{1}{\theta'}}$ . Then (2.9) and (2.10) yield

$$E_{m+1} \leq C_4^m U_m \quad \text{for appropriate constant } C_4. \tag{2.11}$$

Next, using the Sobolev Embedding Theorem, we have

$$\sup_{[0, T]} \langle u_{m+1}^p \mathbf{1}_{B_{m+1}} \rangle + c_S \int_0^T \|\mathbf{1}_{B_{m+1}} u_{m+1}\|_{\frac{pd}{d-2}}^p \leq E_{m+1}$$



Applying Hölder’s inequality and Young’s inequality, we have

$$c\|\mathbf{1}_{B_{m+1}}u_{m+1}\|_{L^{2p}([0,T],L^{\frac{pd}{d-1}})}^p \leq \sup_{[0,T]} \langle u_{m+1}^p \mathbf{1}_{B_{m+1}} \rangle + c_S \int_0^T \|\mathbf{1}_{B_{m+1}}u_{m+1}\|_{L^{\frac{pd}{d-2}}}^p$$

for a  $c > 0$ . Next, applying Hölder’s inequality to both terms in the definition of  $U_{m+1}$ , we obtain, for appropriate  $\alpha > 0$ ,

$$U_{m+1} \leq c_2\|\mathbf{1}_{B_{m+1}}u_{m+1}\|_{L^{2p}([0,T],L^{\frac{pd}{d-1}})}^p |\{u_{m+1} > 0\} \cap [0, T] \times B_m|^{\frac{\alpha}{\theta}}.$$

Combining this with the previous estimate, we have

$$U_{m+1} \leq c_2c^{-1}E_{m+1}|\{u_{m+1} > 0\} \cap [0, T] \times B_m|^{\frac{\alpha}{\theta}} \tag{2.12}$$

Now, (2.11) and (2.12) yield

$$U_{m+1} \leq c_2c^{-1}C_4^mU_m|\{u_{m+1} > 0\} \cap [0, T] \times B_m|^{\frac{\alpha}{\theta}}.$$

Applying (2.10) to the last multiple, we obtain

$$U_{m+1} \leq M^{-p\alpha}C_5^mU_m^{1+\alpha}$$

for constant  $C_5 = C_5(C_4, c, c_2, \alpha)$ .

To end the proof, we fix  $M$  by  $M = H^{\frac{1}{p\theta'}} + C_5^{\frac{1}{p\alpha^2}}U_0^{\frac{1}{p}}$ . Then  $U_0 \leq C_5^{-\frac{1}{\alpha^2}}M^p$ . We now apply Lemma 2.6 (with  $N = M^{-p\alpha}$ ) to obtain

$$\lim_m U_m = 0.$$

On the other hand,

$$\int_0^T (u - 2M)_+^p \mathbf{1}_{B(0, \frac{1}{2})} \leq \lim_m U_m.$$

It follows that

$$\begin{aligned} \sup_{[0,T] \times B(0, \frac{1}{2})} u_+ \leq 2M \leq 2 \left( \int_0^T \langle (\mathbf{1}_{\{|h| \geq 1\}} + \mathbf{1}_{\{|h| < 1\}}|h|^p)^{\theta'} |f|^{p\theta'} \mathbf{1}_{B(0,1)} \rangle \right)^{\frac{1}{p\theta'}} \\ + K \left( \int_0^T \langle u_+^p \mathbf{1}_{B(0,1)} \rangle + \left( \int_0^T \langle u_+^{p\theta} \mathbf{1}_{B(0,1)} \rangle \right)^{\frac{1}{p}} \right)^{\frac{1}{p}} \end{aligned}$$

for a generic constant  $K$ , as claimed. □

**Remark 2.8.** Let  $\eta \in C_c^\infty$  be a refined cutoff function satisfying  $|\nabla\eta| \leq c\eta^{1-\gamma}$  for some  $0 < \gamma < 1, c > 0$ . In fact, the weights  $\eta = \eta_{r,R}$  in (2.8) can be chosen to satisfy this bound with generic  $\gamma, c_0$ :

$$|\nabla\eta_{r,R}| \leq c_0(R - r)^{-1}\eta_{r,R}^{1-\gamma}, \quad 0 < r < R$$

(such weights were used in [2, 20]). With such choice of  $\eta$ , Proposition 2.2 and thus Proposition 2.7 are also valid for  $b = b_1 + b_2$  where  $b_1 \in \mathbf{F}_{\delta_1}, \delta_1 < 4$ , and  $b_2$  satisfies (1.3), (1.4) with  $\nu < \frac{4(p-1)}{p} - 2\sqrt{\delta_1}, p > \frac{2}{2-\sqrt{\delta_1}}$ , and  $h$  satisfies (1.3) with some form-bound  $\nu < \infty$ . Indeed, we only need to complement the proof of Proposition 2.2 by evaluating, using the integration by parts,

$$-2 \int_s^t \langle b_2 \cdot \nabla v^{\frac{p}{2}}, v^{\frac{p}{2}} \eta^2 \rangle = \int_s^t \langle \operatorname{div} b_2, v^p \eta^2 \rangle + 2 \int_s^t \langle b_2 v^p \eta \nabla \eta \rangle$$

and then estimating the RHS from above as follows. We apply the form-boundedness condition on  $(\operatorname{div} b_2)_+$ , i.e. (1.4). As for the last term, we have for every  $\varepsilon_5 > 0$ , by Young's inequality,

$$\begin{aligned} 2 \int_s^t \langle b_2 v^p, \eta \nabla \eta \rangle &\leq \frac{\varepsilon_5^{1+a}}{1+a} \int_s^t \langle |b_2|^{1+a} v^p \eta^2 \rangle + \frac{a}{a+1} \varepsilon_5^{-\frac{a+1}{a}} \int_s^t \langle v^p |\nabla \eta|^{\frac{a+1}{a}} \rangle \\ &\leq \frac{\varepsilon_5^{1+a}}{1+a} \int_s^t \langle |b_2|^{1+a} v^p \eta^2 \rangle + C_{\varepsilon_5} (R-r)^{-\frac{a+1}{a}} \int_s^t \langle v^p \eta^{\frac{a+1}{a}(1-\gamma)} \rangle. \end{aligned}$$

We apply (1.3) in the first term and  $\eta^{\frac{a+1}{a}(1-\gamma)} \leq \mathbf{1}_{\{\eta>0\}}$  in the second term. Finally, assuming that  $p$  is chosen so that  $\frac{1+a}{p'} \geq 1$ , we modify (2.5) as

$$\left| \int_s^t \langle |h| f v^{p-1} \eta^2 \rangle \right| \leq \int_s^t \langle \mathbf{1}_{|h|\geq 1} |h|^{\frac{1+a}{p'}} |f| |v|^{p-1} \eta^2 \rangle + \int_s^t \langle \mathbf{1}_{|h|< 1} |h| |f| |v|^{p-1} \eta^2 \rangle,$$

so, after applying Young's inequality as in the proof, we can use condition (1.3) for  $h$ . Now we can repeat the rest of the proof of Proposition 2.2. (We arrive at (2.2) with  $(R-r)^{-\frac{a+1}{a}} \mathbf{1}_{\{\eta>0\}}$  instead of  $|\nabla \eta|^2$ , but this is what we need in Proposition 2.7 anyway.) Of course, the form-bound  $\delta_2$  of  $b_2$  can be arbitrarily large since we can choose  $\varepsilon_5$  as small as needed.

Recall:  $\rho(x) = (1 + \kappa|x|^{-2})^{-\beta}$ ,  $\beta > \frac{d}{4}$ ,  $\kappa > 0$  is sufficiently small.

**Proposition 2.9.** *Let  $u$  be the solution to Cauchy problem (2.1). Fix  $T > 0$  and  $\theta' \in ]d, \infty[$ . Then, for all  $p > p_\delta$ ,  $p \geq 2$ , there exists a generic constant  $C$  such that*

$$\|u\|_{L^\infty([0,T] \times \mathbb{R}^d)} \leq C \sup_{z \in \mathbb{Z}^d} \left( \int_0^T \left\langle (\mathbf{1}_{\{|h|\geq 1\}} + \mathbf{1}_{\{|h|< 1\}} |h|^p)^{\theta'} |f|^{p\theta'} \rho_z^2 \right\rangle \right)^{\frac{1}{p\theta'}}$$

where  $\rho_z(x) := \rho(x - z)$ .

*Proof of Proposition 2.9.* Applying  $\rho \geq c_0 \mathbf{1}_{B(0,1)}$  and (2.6) to the last term in (2.7) of Proposition 2.7, we arrive at

$$\begin{aligned} \sup_{[0,T] \times B(0, \frac{1}{2})} u_+ &\leq C' \left( \int_0^T \left\langle (\mathbf{1}_{\{|h|\geq 1\}} + \mathbf{1}_{\{|h|< 1\}} |h|^p)^{\theta'} |f|^{p\theta'} \rho^2 \right\rangle \right)^{\frac{1}{p\theta'}} \\ &\quad + C'' \left( \int_0^T \left\langle (\mathbf{1}_{\{|h|\geq 1\}} + \mathbf{1}_{\{|h|< 1\}} |h|^p) |f|^p \rho^2 \right\rangle \right)^{\frac{1}{p}} \\ &\quad + C''' \left( \int_0^T \left\langle (\mathbf{1}_{\{|h|\geq 1\}} + \mathbf{1}_{\{|h|< 1\}} |h|^p)^\theta |f|^{p\theta} \rho^2 \right\rangle \right)^{\frac{1}{p\theta}} \equiv I_1 + I_2 + I_3. \end{aligned}$$

Applying Hölder's inequality to  $I_2$  and  $I_3$  (using that  $\theta' > \theta > 1$ ), we arrive at

$$\|u\|_{L^\infty([0,T] \times B(0, \frac{1}{2}))} \leq C \left( \int_0^T \left\langle (\mathbf{1}_{\{|h|\geq 1\}} + \mathbf{1}_{\{|h|< 1\}} |h|^p)^{\theta'} |f|^{p\theta'} \rho^2 \right\rangle \right)^{\frac{1}{p\theta'}}$$

Since the choice of the centre of the ball  $B(0, \frac{1}{2})$  was arbitrary, this ends the proof.  $\square$

## 2.2 Proof of Theorem 1.3

Once Proposition 2.9 is established, one can construct a martingale solution to (1.1) via a standard tightness argument. The proof below, included for reader's convenience, follows [16, 21, 23].

**Definition 2.10.** A probability measure  $\mathbb{P}_x$  on the canonical space  $(C([0, 1], \mathbb{R}^d), \mathcal{B}_t = \sigma\{\omega_s \mid 0 \leq s \leq t\})$  is called a martingale solution to the SDE (1.1) if

- 1)  $\mathbb{P}_x[\omega_0 = x] = 1$ .
- 2)

$$\mathbb{E}_x \int_0^t |b(s, \omega_s)| < \infty, \quad 0 < t \leq 1 \quad (\mathbb{E}_x := \mathbb{E}_{\mathbb{P}_x}).$$

- 3) For every  $\varphi \in C_c^2$  the process

$$M_t^\varphi := \varphi(\omega_t) - \varphi(\omega_0) + \int_0^t (-\Delta\varphi + b \cdot \nabla\varphi)(s, \omega_s) ds$$

is a martingale:

$$\mathbb{E}_x[M_{t_1}^\varphi \mid \mathcal{B}_{t_0}] = M_{t_0}^\varphi$$

for all  $0 \leq t_0 < t_1 \leq 1$   $\mathbb{P}_x$ -a.s.

Let  $b$  be a vector field in  $\mathbf{F}_\delta$ ,  $\delta < 4$ , so in general it is locally unbounded. Let us fix bounded smooth vector fields  $b_n \in C_c^\infty([0, \infty[ \times \mathbb{R}^d, \mathbb{R}^d) \cap \mathbf{F}_\delta$  (with  $g = g_\delta$  independent of  $n$ ) such that

$$b_n \rightarrow b \quad \text{in } L_{loc}^2([0, \infty[ \times \mathbb{R}^d, \mathbb{R}^d)$$

and also  $b_n \rightarrow b$  a.e. Such vector fields can be constructed by multiplying  $b$  by  $\mathbf{1}_{\{0 \leq |t| \leq n, |x| \leq n, |b(x)| \leq n\}}$ , which preserves form-bound  $\delta$  and function  $g$ , and then applying a K. Friedrichs mollifier in  $(t, x)$ , see Appendix A for details. (In fact, we don't even need to include the indicator function, which allows to control the form-bound of  $\text{div } b$  [10, Sect. 3, 4], cf. Remark 2.8.)

Fix  $x \in \mathbb{R}^d$ . By a classical result, there exist strong solutions  $X^n$  to the SDEs

$$X_t^n = x - \int_0^t b_n(s, X_s^n) ds + \sqrt{2}B_t, \quad n = 1, 2, \dots,$$

where  $B_t$  is a Brownian motion in  $\mathbb{R}^d$  on a fixed complete probability space  $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})$ .

Let  $0 \leq t_0 < t_1 \leq 2$ . Consider the terminal-value problem for  $t \leq t_1$

$$\partial_t u_n + \Delta u_n + b_n \cdot \nabla u_n + F = 0, \quad u_n(t_1) = 0,$$

where  $F \in C_c(\mathbb{R}^{d+1})$ . Then the Itô formula yields

$$\mathbf{E} \int_{t_0}^{t_1} F(r, X_r^n) dr = u_n(t_0, X_{t_0}^n).$$

Hence, selecting  $F = |h|f$ , where  $h \in \mathbf{F}_\nu \cap C_c^\infty(\mathbb{R}^d, \mathbb{R}^d)$  and  $f \in C_c$  are as in the previous section, we have by Proposition 2.9

$$\left| \mathbf{E} \int_{t_0}^{t_1} |h(s, X_s^n)| f(s, X_s^n) ds \right| \leq C \sup_{z \in \mathbb{Z}^d} \left( \int_{t_0}^{t_1} \left\langle (\mathbf{1}_{\{|h| \geq 1\}} + \mathbf{1}_{\{|h| < 1\}} |h|^p)^{\theta'} |f|^{p\theta'} \rho_z^2 \right\rangle \right)^{\frac{1}{p\theta'}}. \tag{2.13}$$

The following two instances of estimate (2.13) will yield the sought martingale solution:

1. (2.13) with  $h = b_n$  and  $f \equiv 1$  (here  $f \in C_c \Rightarrow f \equiv 1$  using Fatou's Lemma):

$$\begin{aligned} \mathbf{E} \int_{t_0}^{t_1} |b_n(s, X_s^n)| ds &\leq C \sup_{z \in \mathbb{Z}^d} \left( \int_{t_0}^{t_1} \left\langle (\mathbf{1}_{\{|b_n| \geq 1\}} + \mathbf{1}_{\{|b_n| < 1\}} |b_n|^p)^{\theta'} \rho_z^2 \right\rangle \right)^{\frac{1}{p\theta'}} \\ &\leq C_0(t_1 - t_0)^\mu \quad \text{for generic } \mu > 0 \text{ and } C_0 \text{ (indep. of } x). \end{aligned} \tag{2.14}$$

(Note that since  $C_0$  is a generic constant, Remark 2.3 also applies to inequality (2.14), and yields that  $C_0$  does not depend on  $n$ .) Inequality (2.14) allows to verify tightness of probability measures

$$\mathbb{P}_x^n := (\mathbf{P} \circ X^n)^{-1}$$

on  $(C([0, 1], \mathbb{R}^d), \mathcal{B}_t)$ . Here we argue as in [16, proof of Theorem 1.1]. Namely, first note that we have, for a stopping time  $0 \leq \tau \leq 1$ ,

$$X_{\tau+\sigma, x}^n - X_{\tau, x}^n = \int_{\tau}^{\tau+\sigma} b_n(s, X_{s, x}^n) ds + \sqrt{2}(B_{\tau+\sigma} - B_{\tau}), \quad 0 < \sigma < 1. \quad (2.15)$$

Next, note that (2.14) yields

$$\mathbf{E} \int_{\tau}^{\tau+\sigma} |b_n(s, X_{s, x}^n)| ds \leq C_0 \sigma^\mu, \quad (2.16)$$

see Remark 1.2 in [21] (to show that (2.14)  $\Rightarrow$  (2.16), the authors of [21] use a decreasing sequence of stopping times  $\tau_m$  converging to  $\tau$  and taking values in  $S = \{k2^{-m} \mid k \in \{0, 1, 2, \dots\}\}$ , and note that the proof of estimate (2.16) with  $\tau_m$  in place of  $\tau$  can be reduced to applying (2.14) on intervals  $[t_0, t_1] := [c, c + \sigma]$ ,  $c \in S$ .) Thus, applying (2.16) in (2.15), one obtains

$$\mathbf{E} \sup_{\sigma' \in [0, \sigma]} |X_{\tau+\sigma', x}^n - X_{\tau, x}^n| \leq C_0 \sigma^\mu + C_1 \sigma^{\frac{1}{2}} =: H(\sigma).$$

Now, applying [22, Lemma 2.7] (a Chebyshev-type argument), we obtain: there exists a constant  $\hat{C}$  independent of  $n$  such that

$$\sup_n \sup_{x \in \mathbb{R}^d} \mathbf{E} \left[ \sup_{t \in [0, 1], \sigma' \in [0, \sigma]} |X_{t+\sigma', x}^n - X_{t, x}^n|^{\frac{1}{2}} \right] \leq \hat{C} H(\sigma). \quad (2.17)$$

Now, applying Chebyshev's inequality in (2.17), since  $H(\sigma) \downarrow 0$  as  $\sigma \downarrow 0$ , we have for every  $\epsilon > 0$

$$\lim_{\sigma \downarrow 0} \sup_n \sup_{x \in \mathbb{R}^d} \mathbf{P} \left[ \sup_{t \in [0, 1], \sigma' \in [0, \sigma]} |X_{t+\sigma', x}^n - X_{t, x}^n| > \epsilon \right] = 0.$$

It follows that  $\{\mathbb{P}_x^n\}_{n=1}^\infty$  is a tight family of probability measures (see [18, Theorem 1.3.2]). Therefore, for every  $x \in \mathbb{R}^d$  there exists a subsequence  $\{\mathbb{P}_x^{n_k}\}$  and a probability measure  $\mathbb{P}_x$  such that

$$\mathbb{P}_x^{n_k} \rightarrow \mathbb{P}_x \text{ weakly.} \quad (2.18)$$

Let us rewrite (2.13) as

$$\left| \mathbb{E}_x^n \int_{t_0}^{t_1} |h(s, \omega_s)| f(s, \omega_s) ds \right| \leq C \sup_{z \in \mathbb{Z}^d} \left( \int_{t_0}^{t_1} \left\langle (\mathbf{1}_{\{|h| \geq 1\}} + \mathbf{1}_{\{|h| < 1\}} |h|^p)^{\theta'} |f|^{p\theta'} \rho_z^2 \right\rangle \right)^{\frac{1}{p\theta'}}, \quad (2.19)$$

where  $\mathbb{E}_x^n := \mathbb{E}_{\mathbb{P}_x^n}$ .

Using (2.19) with  $h := b_m$  and then applying (2.18), we obtain

$$\mathbb{E}_x \int_{t_0}^{t_1} |b_m(s, \omega_s)| ds \leq C_0 (t_1 - t_0)^\mu,$$

and hence, using e.g. Fatou's Lemma,

$$\mathbb{E}_x \int_{t_0}^{t_1} |b(s, \omega_s)| ds \leq C_0 (t_1 - t_0)^\mu$$

(in view of (2.19) and (2.18), changing  $b$  on a measure zero set does not affect the previous estimate). Our goal now is to show that the limit measure  $\mathbb{P}_x$  solves the martingale problem for (1.1). It suffices to show that  $\mathbb{E}_x[M_{t_1}^\varphi G] = \mathbb{E}_x[M_{t_0}^\varphi G]$  for every  $\mathcal{B}_{t_0}$ -measurable  $G \in C_b(C([0, T], \mathbb{R}^d))$ . The task reduces to passing to the limit in  $n$  in  $\mathbb{E}_x^n[M_{t_1}^{\varphi, n} G] = \mathbb{E}_x^n[M_{t_0}^{\varphi, n} G]$ , where

$$M_t^{\varphi, n} = \varphi(\omega_t) - \varphi(\omega_0) + \int_0^t (-\Delta\varphi + b_n \cdot \nabla\varphi)(s, \omega_s) ds.$$

That is, we need to prove

$$\lim_{n_k} \mathbb{E}_x^{n_k} \int_0^t (b_{n_k} \cdot \nabla\varphi)(s, \omega_s) G(\omega) ds = \mathbb{E}_x \int_0^t (b \cdot \nabla\varphi)(s, \omega_s) G(\omega) ds. \tag{2.20}$$

This is done using the weak convergence (2.18) and the next estimate.

2. (2.19) with  $h := b_{m_1} - b_{m_2} \in \mathbf{F}_{\sqrt{2}\delta}$ ,  $f := |\nabla\varphi|$ :

$$\begin{aligned} & \mathbb{E}_x^n \int_{t_0}^{t_1} |b_{m_1}(s, \omega_s) - b_{m_2}(s, \omega_s)| |\nabla\varphi(s, \omega_s)| ds \\ & \leq \sup_{z \in \mathbb{Z}^d} \left( \int_{t_0}^{t_1} \left\langle (\mathbf{1}_{\{|b_{m_1} - b_{m_2}| \geq 1\}} + \mathbf{1}_{\{|b_{m_1} - b_{m_2}| < 1\}} |b_{m_1} - b_{m_2}|^p |\nabla\varphi|^{p\theta'})^{\theta'} \rho_z^2 \right\rangle \right)^{\frac{1}{p\theta'}}. \end{aligned}$$

Since  $\varphi$  has compact support, the RHS converges to 0 as  $m_1, m_2 \rightarrow \infty$ . It follows from the weak convergence (2.18) (and an argument similar to the one in 1) that

$$\begin{aligned} & \mathbb{E}_x \int_{t_0}^{t_1} |b(s, \omega_s) - b_m(s, \omega_s)| |\nabla\varphi(s, \omega_s)| ds \\ & \leq \sup_{z \in \mathbb{Z}^d} \left( \int_{t_0}^{t_1} \left\langle (\mathbf{1}_{\{|b - b_m| \geq 1\}} + \mathbf{1}_{\{|b - b_m| < 1\}} |b - b_m|^p |\nabla\varphi|^{p\theta'})^{\theta'} \rho_z^2 \right\rangle \right)^{\frac{1}{p\theta'}}, \end{aligned}$$

where the RHS converges to 0 as  $m \rightarrow \infty$ . Now, we prove (2.20):

$$\begin{aligned} & \left| \mathbb{E}_x^{n_k} \int_0^t (b_{n_k} \cdot \nabla\varphi)(s, \omega_s) G(\omega) ds - \mathbb{E}_x \int_0^t (b \cdot \nabla\varphi)(s, \omega_s) G(\omega) ds \right| \\ & \leq \left| \mathbb{E}_x^{n_k} \int_0^t |b_{n_k} - b_m| |\nabla\varphi|(s, \omega_s) |G(\omega)| ds \right| \\ & \quad + \left| \mathbb{E}_x^{n_k} \int_0^t (b_m \cdot \nabla\varphi)(s, \omega_s) G(\omega) ds - \mathbb{E}_x \int_0^t (b_m \cdot \nabla\varphi)(s, \omega_s) G(\omega) ds \right| \\ & \quad + \left| \mathbb{E}_x \int_0^t |b_m - b| |\nabla\varphi|(s, \omega_s) |G(\omega)| ds \right|, \end{aligned}$$

where the first and the third terms in the RHS can be made arbitrarily small using the estimates above and the boundedness of  $G$  by selecting  $m$ , and then  $n_k$ , sufficiently large. The second term can be made arbitrarily small in view of (2.18) by selecting  $n_k$  even larger. This ends the proof of Theorem 1.3.

**Remark 2.11.** Let  $b = b(x)$ ,  $b \in \mathbf{F}_\delta$ ,  $\delta < 4$ . Let  $u_n$  be defined by

$$(\partial_t - \Delta + b_n \cdot \nabla)u_n = 0, \quad u_n(0) = g \in C_b \cap L^1,$$

where  $b_n$  are as above.

1. For every  $p > p_\delta$ , the limit

$$u := s\text{-}L^p\text{-}\lim_n u_n \quad \text{loc. uniformly in } t \geq 0, \tag{2.21}$$

exists and determines a weak solution (in  $L^p$ ) to Cauchy problem  $(\partial_t - \Delta + b \cdot \nabla)u = 0$ ,  $u(0+) = g$ . For the proof see [7] and [11].

2. One can apply Moser's method in  $L^p$ ,  $p > p_\delta$ ,  $p \geq 2$  to show Hölder continuity of the weak solution  $u$ . Combined with (2.21), this allows to conclude that  $u_n \rightarrow u$  everywhere on  $\mathbb{R}^d$ . We plan to address these matters in detail elsewhere.

## A Appendix

Let  $b \in \mathbf{F}_\delta$  with function  $g = g_\delta^b \in L^1_{\text{loc}}([0, \infty[)$ . The following construction of bounded smooth vector fields  $\{b_m\}$  satisfying

$$b_m \rightarrow b \quad \text{in } L^2_{\text{loc}}([0, \infty[ \times \mathbb{R}^d, \mathbb{R}^d) \tag{A.1}$$

and

$$\int_0^\infty \|b_m(t, \cdot)\xi(t, \cdot)\|_2^2 dt \leq \delta \int_0^\infty \|\nabla \xi(t, \cdot)\|_2^2 dt + \int_0^\infty g(t)\|\xi(t, \cdot)\|_2^2 dt \tag{A.2}$$

for all  $\xi \in C_c^\infty([0, \infty[ \times \mathbb{R}^d)$ , for the same function  $g$ , is taken from [6, Sect.3.].

Define  $b_m$  by the following formula (extending  $b$  to  $t < 0$  by 0):

$$b_m := c_m \gamma_{\varepsilon_m} * (\mathbf{1}_m b), \quad c_m, \varepsilon_m > 0,$$

where  $\mathbf{1}_m$  is the indicator of  $\{(t, x) \mid |b(t, x)| \leq m, |x| \leq m, |t| \leq m\}$  and  $\gamma_\varepsilon$  is a Friedrichs mollifier on  $\mathbb{R} \times \mathbb{R}^d$ . Let us show that, for appropriate choice of  $\varepsilon_m \downarrow 0$  and  $c_m \uparrow 1$ , (A.1) and (A.2) are satisfied. Put  $\tilde{b}_m = \gamma_{\varepsilon_m} * (\mathbf{1}_m b)$  and write

$$\tilde{b}_m = \mathbf{1}_m b + (\tilde{b}_m - \mathbf{1}_m b).$$

Clearly, the first term  $\mathbf{1}_m b \in \mathbf{F}_\delta$  with the same  $g = g_\delta^b$ . In turn, since for all  $m = 1, 2, \dots$  we have

$$\mathbf{1}_m b \in L^\infty([0, \infty[, L^r) \quad \text{for every } d \leq r < \infty, \quad \text{sprt } \mathbf{1}_m b \subset [0, m] \times B(0, m),$$

the following is true: given any  $\alpha_m \downarrow 0$  we can select  $\varepsilon_m \downarrow 0$  so that e.g.

$$\|\tilde{b}_m - \mathbf{1}_m b\|_{L^r([0, \infty[, L^r)} \leq \alpha_m, \tag{A.3}$$

and so the second term  $\tilde{b}_m - \mathbf{1}_m b \in \mathbf{F}_{C_S \alpha_m^2}$  with  $g \equiv 0$ , where  $C_S$  is the constant in the Sobolev inequality (for details, if needed, see Appendix A in [6]). Hence,

$$\tilde{b}_m \in \mathbf{F}_{\delta_m} \quad \text{with } \delta_m = (\sqrt{\delta} + \sqrt{C_S \alpha_m^2})^2 \text{ and } g = g_\delta^b,$$

Now, selecting  $c_m = \frac{\delta}{\delta_m}$  (clearly,  $c_m \uparrow 1$ ) and recalling that  $b_m = c_m \tilde{b}_m$ , we have  $b_m \in \mathbf{F}_\delta$  with the same  $g = g_\delta^b$ , i.e. (A.2) holds. In turn, (A.1) is clear from  $\mathbf{1}_m \uparrow 1$  and the Dominated Convergence Theorem, (A.3) and  $c_m \uparrow 1$ .

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