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Multi-dimensional BSDEs with mean reflection*

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Abstract

In this paper, we consider multi-dimensional mean reflected backward stochastic differential equations (BSDEs) with possibly non-convex reflection domains along inward normal direction, which were introduced by Briand, Elie and Hu [6] in the scalar case. We first apply a fixed-point argument to establish the uniqueness and existence result under an additional bounded condition on the driver. Then, with the help of a priori estimates, we develop a successive approximation procedure to remove the additional bounded condition for the general case.

Keywords: mean reflected BSDE; non-convex domain; inward normal direction; a priori estimates.
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1 Introduction

Let $(\Omega, \mathscr{F}, \mathbf{P})$ be a complete probability space under which B is a d-dimensional standard Brownian motion. Suppose $(\mathscr{F}_t)_{0\leq t\leq T}$ is the natural filtration generated by B augmented by the $\mathbf{P}$-null sets and $\mathcal{P}$ is the sigma algebra of all progressive sets of $\Omega\times[0,T]$. The present paper is devoted to the study of the following multi-dimensional BSDE with mean reflection over the time interval $[0,T]$,

$$\begin{cases} Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s + \eta_T - \eta_t, & \forall t\in[0,T], \\ \mathbf{E}[Y_t] \in \overline{D}, & \forall t\in[0,T], \end{cases} \tag{1.1}$$

in which the terminal condition ξ is an $\mathbb{R}^m$-valued $\mathscr{F}_T$-measurable random vector, the driver $f:\Omega\times[0,T]\times\mathbb{R}^m\times\mathbb{R}^{m\times d}\to\mathbb{R}^m$ is a measurable map with respect to $\mathcal{P}\times$

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$\mathcal{B}(\mathbb{R}^m) \times \mathcal{B}(\mathbb{R}^{m\times d})$ and the mean of the solution Y_t is forced to stay within a possibly non-convex domain $D \subset \mathbb{R}^m$. Our aim is to investigate the solvability of the mean reflected BSDE (1.1) under a natural Skorokhod type condition (see Definition 3.1), where the solution η is a deterministic function with bounded variation.

When the constraint is not on the mean of the solution but on the paths of the solution, i.e., $Y_t \in \overline{D}$, the reflected BSDE (1.1) was first introduced by El Karoui et al. [21] in the scalar case. Since then, great progress has been made in this field, as it has rich connections with partial differential equations (PDEs) and mathematical finance. For instance, the authors applied the reflected BSDEs to provide a probabilistic interpretation for an obstacle problem of parabolic PDEs in [21] and El Karoui, Pardoux and Quenez [22] found that the price of an American option could be represented as the unique solution to the reflected BSDE under a Skorokhod type condition. Moreover, various types of scalar-valued constrained BSDEs have been formulated due to their financial motivations. For examples, Buckdahn and Hu [9] used the constrained BSDEs driven by both a Wiener process and a Poisson random measure to analyze the option pricing with constrained portfolios in an incomplete market; Cvitanić and Karatzas [17] considered a type of Dynkin games via the BSDEs with two reflecting barriers. For more research on this field, we refer the reader to [18, 25, 26, 27, 28, 29, 37, 39] and the references therein.

In the reflected BSDEs theory, the research of the multidimensional case is significantly more difficult than that of the scalar case. In [24], Gegout-Petit and Pardoux first obtained the existence and uniqueness results for the case of normal reflection in convex domains, which was generalized by Klimsiak, Rozkosz and Słomiński to the case of time-dependent random convex regions in [34]. On the other hand, motivated by the optimal switching problems, certain types of multi-dimensional reflected BSDEs with oblique direction of reflection were also investigated, see e.g., [1, 12, 13, 14, 30, 33]. Recently, Chassagneux and Richou [16] established the solvability of general obliquely reflected BSDEs in convex reflection domains.

The theory of multidimensional reflected stochastic differential equations (SDEs) with non-convex reflection domains has been studied systematically (cf. [20, 36, 40]). However, there are only few papers dealing with the multidimensional reflected BSDEs with non-convex reflection domains due to the complicated structure. We would like to mention that [8, 23] considered certain types of multidimensional reflected BSDEs in non-convex domains. Recently, Chassagneux, Nadtochiy and Richou [15] first established well-posedness results for general multidimensional reflected BSDEs with non-convex domains satisfying a weak star-shape property.

In contrast with the aforementioned paths constraints, the BSDEs with mean constraints were formulated recently to analyze partial hedging of financial derivatives in mathematical finance. In order to deal with quantile hedging problems, Bouchard, Elie and Réveillac [2] considered a new type of BSDEs, where the terminal value satisfies a type of mean constraint. Inspired by this, Briand, Elie and Hu [6] introduced the scalar-valued BSDEs with mean reflection to investigate the super-hedging problem under running risk management constraint. In this framework, the solution Y is required to satisfy the following type of mean constraint:

$$\mathbf{E}\big[\ell(t, Y_t)\big] \geq 0, \quad \forall t \in [0, T],$$

where $\ell(t, \cdot)$ is a collection of (possibly random) non-decreasing real-valued map.

Subsequently, Hibon et al. [31] considered quadratic BSDEs with mean reflection and Hu, Moreau and Wang [32] dealt with generalized mean reflected BSDEs, whose drivers also depend on the law of the solution Y. With the help of interacting particles systems, [5, 7, 8] studied the approximation of mean reflected SDEs and BSDEs. We also refer to Djehiche, Elie and Hamadène [19] which formulated a mean-filed type of reflected BSDEs motivated by applications in pricing life insurance contracts with surrender options.

Recently, Briand et al. [4] introduced multi-dimensional mean reflected BSDEs with normal reflection, in which the marginal probability distribution $\mathbf{P}_{Y_t}$ of the solution Y is required to stay within a subset of $\mathcal{P}_2(\mathbb{R}^m)$[1], i.e.,

$$\mathbf{P}_{Y_t} \in \big\{\mu \in \mathcal{P}_2\big(\mathbb{R}^m\big),\ H(\mu) \geq 0\big\},\ \ \forall t \in [0,T].$$

Here the function $H : \mathcal{P}_2(\mathbb{R}^m) \to \mathbb{R}$ is concave. Based on this, the authors also studied the associated propagation of chaos and established a probabilistic interpretation for an obstacle problem of PDEs stated on the Wasserstein space. Note that the framework of [4] coincides with the one of [6] when the constraint acts only on the mean of the solution, i.e.,

$$\ell(t,x) = x \text{ and } H(\mu) = \int x\mu(dx).$$

We refer to [4, Remark 4] for more details on their connections. Motivated by the results of [4, 6, 36], we want to investigate the multidimensional BSDE (1.1) with mean reflection in a possibly non-convex domain D.

In view of the arguments of [36, 40], we assume that the constraint domain D satisfies uniform exterior sphere and uniform interior cone conditions. Note that the well-posedness of deterministic multi-dimensional Skorokhod Problem is crucial for our main result as in [6]. Indeed, when the driver f does not depend on the unknowns y and z, it follows from (1.1) and the non-randomness of the solution η that

$$\mathbf{E}[Y_t] = \mathbf{E}\left[\xi + \int_t^T f_s ds\right] + \eta_T - \eta_t,\ \mathbf{E}[Y_t] \in \overline{D},\ \forall t \in [0,T],$$

which can be regarded as a deterministic Skorokhod problem. In this case, we can first define the component η and then solve a standard BSDE to find the components Y and Z. Based on this observation, we can construct an iteration map for the general case. The key point is to prove that the map is a contraction.

To this end, we firstly establish a priori estimates for solutions to the mean reflected BSDE (1.1) through Itô's formula and the associated Skorokhod condition. Compared with the one dimensional case, we cannot obtain the explicit form of the component η as in [6]. Fortunately, the solution η is deterministic and then we adapt the discretization technique introduced by [40] to obtain

$$d|\eta|_s^0 \leq C(r_0,\delta,\alpha)\mathbf{E}\big[|f(s,Y_s,Z_s)|\big]ds,\ ds\text{-a.e.}$$

which gives a priori estimate for the total variation $|\eta|_T^0$.

Note that the uniform estimate for $d|\eta^n|_t^0$ is crucial to construct a contraction map in the case of non-convex reflection domains, see Remark 4.2 for details. Then, we apply a fixed-point argument to establish the existence result under the following additional condition on the driver,

$$\mathbf{E}[|f(t,Y_t^n,Z_t^n)|] \text{ is uniformly bounded by some integrable function } g_t \in L^1([0,T]).$$

Finally, with the help of a priori estimates, we develop an approximation procedure for the general case through an appropriate truncation argument. Consequently, we are able to prove the existence and uniqueness of solution to the BSDE (1.1) with mean reflection in a non-convex domain, which extends the relevant results in [4]. In addition,

[1]The collection of all probability measures over $(\mathbb{R}^m, \mathcal{B}(\mathbb{R}^m))$ with finite second moment.

when the constraint domain is convex, we show that the solution can be constructed by a penalization approach as in [24].

Since the constraint is only on the mean of the solution, the structure of the multi-dimensional mean reflected BSDE (1.1) is simpler than that of [4]. Therefore, we can employ the arguments of [36, 40] to tackle the non-convex reflection domains case. On the other hand, compared with the case of multidimensional reflected BSDEs with non-convex domains in [15], the non-randomness of the solution η plays an important role in our arguments, which makes it easier to establish a priori estimates and construct a contraction map.

The paper is organized as follows. In Section 2, we recall some basic results of deterministic multi-dimensional Skorokhod Problem. In Section 3, we state the main result involving the existence and uniqueness for solution of the mean reflected BSDE (1.1), and some a priori estimates. Section 4 is devoted to the proof of the main result.

Let us finish this introduction by giving some notations which will be used frequently in this paper.

Notation

For each Euclidian space $\mathbb{E}$, we denote by $\langle\cdot,\cdot\rangle$ and $|\cdot|$ its scalar product and the associated norm, respectively. For each $p\geq 1$, consider the following collections:

- $L^p(\mathbb{E})$ is the space of $\mathbb{E}$-valued $\mathscr{F}_T$-measurable random vectors ξ satisfying $\mathbf{E}[|\xi|^p]<\infty$;
- $\mathcal{H}^p(\mathbb{E})$ is the space of $\mathbb{E}$-valued $\mathscr{F}$-progressively measurable processes $(z_t)_{0\leq t\leq T}$ satisfying

$$\mathbf{E}\Big[\Big(\int_0^T |z_t|^2 dt\Big)^{\frac{p}{2}}\Big]<\infty;$$

- $\mathcal{H}^{1,p}(\mathbb{E})$ is the space of $\mathbb{E}$-valued $\mathscr{F}$-progressively measurable processes $(z_t)_{0\leq t\leq T}$ satisfying

$$\mathbf{E}\Big[\Big(\int_0^T |z_t| dt\Big)^{p}\Big]<\infty;$$

- $\mathcal{S}^p(\mathbb{E})$ is the space of $\mathbb{E}$-valued $\mathscr{F}$-adapted continuous processes $(y_t)_{0\leq t\leq T}$ satisfying

$$\mathbf{E}\Big[\sup_{0\leq t\leq T}|y_t|^p\Big]<\infty;$$

- $\mathcal{C}(\mathbb{E})$ is the space of $\mathbb{E}$-valued continuous functions on $[0,T]$;
- $\mathcal{V}(\mathbb{E})$ is the space of $\mathbb{E}$-valued continuous functions $(\eta_t)_{0\leq t\leq T}$ satisfying $\eta_0=0$ and $|\eta|_T^0<\infty$, where $|\eta|_t^s$ is the total variation on $[s,t]$ for each $0\leq s\leq t\leq T$.

In what follows, for a given set of parameters α, $C(\alpha)$ will denote a positive constant only depending on these parameters and may change from line to line.

2 Multi-dimensional Skorokhod problem

In this section, we will review some basic notions and results about multi-dimensional Skorokhod problem, which will be used in subsequent discussions.

Let D be a non-empty open connected subset of $\mathbb{R}^m$ with the boundary ∂D. Denote by $\overline{D}$ the closure of D. For each $x \in \partial D$, we define K_x as the set of all inward normal unit vectors at x, i.e.,

$$K_x := \bigcup_{r>0} K(x,r) \text{ and } K(x,r) := \big\{\mathbf{n} \in \mathbb{R}^m : |\mathbf{n}| = 1, B(x - r\mathbf{n}, r) \cap D = \emptyset\big\}, \tag{2.1}$$

where $B(x,r) := \{y \in \mathbb{R}^m : |y - x| < r\}$.

Definition 2.1 (Skorokhod Problem). *Given a function $\psi \in \mathcal{C}(\mathbb{R}^m)$ satisfying $\psi_T \in \overline{D}$. Then $(\phi, \eta) \in \mathcal{C}(\overline{D}) \times \mathcal{V}(\mathbb{R}^m)$ is said to be a solution of Skorokhod problem for (D, ψ) (for short, SP(D,ψ)) if for each $t \in [0,T]$*

- **(i)** $\phi_t = \psi_t + \eta_t$;
- **(ii)** $|\eta|_t^0 = \int_0^t \mathbb{1}_{\{\phi(s)\in\partial D\}} d|\eta|_s^0$ *and* $\eta_t = \int_0^t \gamma_s d|\eta|_s^0$ *for some Borel measurable function* γ *satisfying* $\gamma_s \in K_{\phi_s}$ $d|\eta|_s^0$*-a.e.*

Next, we introduce the following two assumptions on the domain D, which have been used in Lions and Sznitman [36] and Saisho [40] to guarantee the well-posedness of SP(D,ψ).

(H1) (*uniform exterior sphere condition*) There exists a constant $r_0 > 0$ such that

$$K_x = K(x, r_0) \neq \emptyset \text{ for all } x \in \partial D.$$

(H2) There exist two constants $\delta > 0$ and $\alpha \in (0,1]$ satisfying the following property: for any $x \in \partial D$ there exists a unit vector $\mathbf{l}_x$ such that

$$\langle \mathbf{l}_x, \mathbf{n}\rangle \geq \alpha \text{ for any } \mathbf{n} \in \bigcup_{y \in B(x,\delta)\cap\partial D} K_y.$$

Remark 2.2. (i) It is easy to check that $\mathbf{n} \in K(x,r)$ if and only if $\langle x - y, \mathbf{n}\rangle \leq \frac{1}{2r}|x-y|^2$ for all $y \in \overline{D}$ (see Remark 1.2 in [36] or Remark 1.1 in [40]).

(ii) If D is a convex set, it is easy to see that D satisfies **(H1)** for any $r_0 > 0$.

Remark 2.3. (i) The following uniform interior cone condition is very useful, which is slightly stronger than **(H2)**:

($\mathbf{H2'}$) (*uniform interior cone condition*) There exist two constants $\delta > 0$ and $\beta \in (0,1)$ satisfying the following property: for any $x \in \partial D$ there exists a unit vector $\mathbf{l}_x$ such that

$$C(y, \mathbf{l}_x, \beta) \cap B(x,\delta) \subset D \ \text{ for any } y \in B(x,\delta)\cap\partial D,$$

where $C(y, \mathbf{l}_x, \beta)$ is the convex cone with vertex y, i.e.,

$$C(y, \mathbf{l}_x, \beta) := \big\{z \in \mathbb{R}^m : \langle z - y, \mathbf{l}_x\rangle > \beta|z - y|\big\}.$$

(ii) It was shown in Bramson, Burdzy and Kendall (see [3, Section 2]) that the uniform interior cone condition is equivalent to the following Lipschitz boundary condition.

Lipschitz domain: A domain $D \subset \mathbb{R}^m$ is said to be Lipschitz if there exists $\delta > 0$ such that for all $x \in \partial D$, there exists an orthonormal basis $e_1, e_2, \ldots, e_m$ and a Lipschitz function $f : \mathbb{R}^{m-1} \to \mathbb{R}$ such that

$$B(x,\delta) \cap D = \big\{y \in B(x,\delta) : f(y_1, \ldots, y_{m-1}) < y_m\big\},$$

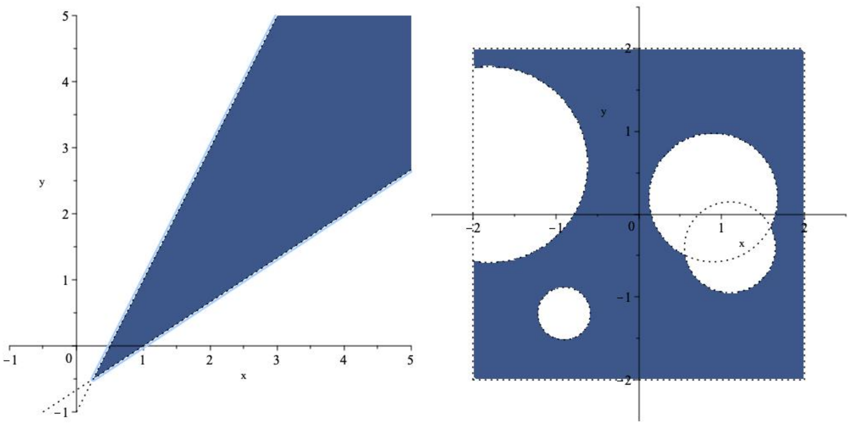
where $y_i := \langle y, e_i\rangle,\ i = 1, \ldots, m$.

According to Remark 2.3, we give some interesting examples of D which satisfy **(H1)** and **(H2)**.

Example 2.4. (1) Any bounded convex domain with piecewise smooth boundary satisfies **(H1)** and **(H2)**. Such as balls, polyhedrons and cylinders.

(2) (Unbounded) half planes, polyhedral cones and circular cones satisfy **(H1)** and **(H2)**, e.g., see Figure 1 (a).

(3) If the domain D satisfies **(H1)** and **(H2)**, and $\{B_i\}_{i=1}^n$ is a family of balls such that $\{\partial B_i\}_{i=1}^n, \partial D$ are non-tangential, then $D \setminus (\bigcup_{i=1}^n B_i)$ satisfies **(H1)** and **(H2)**, e.g., see Figure 1 (b).



(a) Cone. (b) Non-convex domain.

Figure 1: Domains satisfying **(H1)** and **(H2)**

In this paper, we need to consider the following multi-dimensional Backward Skorokhod Problem.

Definition 2.5 (Backward Skorokhod Problem)**.** *Given a function $\psi \in \mathcal{C}(\mathbb{R}^m)$ with $\psi_T \in \overline{D}$. Then $(\phi, \eta) \in \mathcal{C}(\overline{D}) \times \mathcal{V}(\mathbb{R}^m)$ is said to be a solution of Backward Skorokhod problem for (D, ψ) (for short, BSP(D,ψ)) if for each $t \in [0,T]$*

- **(i)** $\phi_t = \psi_t + \eta_T - \eta_t$;
- **(ii)** $|\eta|_t^0 = \int_0^t \mathbb{1}_{\{\phi(s)\in\partial D\}} d|\eta|_s^0$ *and* $\eta_t = \int_0^t \gamma_s d|\eta|_s^0$ *for some Borel measurable function* γ *satisfying* $\gamma_s \in K_{\phi_s}$ $d|\eta|_s^0$*-a.e.*

Remark 2.6. Note that (ϕ, η) is a solution to BSP(D, ψ) if and only if $(\widetilde{\phi}, \widetilde{\eta})$ is a solution to SP$(D, \widetilde{\psi})$, where $(\widetilde{\psi}_t, \widetilde{\phi}_t, \widetilde{\eta}_t) = (\psi_{T-t}, \phi_{T-t}, \eta_T - \eta_{T-t})$ for any $t \in [0,T]$.

Theorem 2.7. *Suppose that assumptions **(H1)** and **(H2)** are satisfied. Then there is a unique solution to BSP(D,ψ). Moreover, $\phi(t,\psi)$ and $\eta(t,\psi)$ are continuous in (t,ψ).*

Proof. The proof is immediate from Saisho [40, Theorem 4.1] and Remark 2.6. □

The following lemmas are crucial for our main results.

Lemma 2.8. *Assume that assumptions **(H1)** and **(H2)** are fulfilled. Suppose that (ϕ^n, η^n) is the unique solution of BSP(D, ψ_n) for each $n \geq 1$ and there exist three functions ϕ, ψ, η such that*

$$\sup_{n\geq 1} |\eta^n|_T^0 < \infty \quad \textit{and} \quad \lim_{n\to\infty} \sup_{0\leq t\leq T} \left(|\phi_t^n - \phi_t| + |\psi_t^n - \psi_t| + |\eta_t^n - \eta_t|\right) = 0.$$

Then (ϕ, η) is the unique solution of BSP(D,ψ).

Proof. The proof is immediate from the proof of [36, Theorem 1.1] or that of [40, Theorem 4.1]. □

Lemma 2.9. *Let assumptions **(H1)** and **(H2)** be satisfied and (ϕ,η) be the unique solution of BSP(D,ψ). Assume that ψ has finite total variation. Then, there exists a constant $C(r_0,\delta,\alpha)$ such that for any $0\le s\le t\le T$,*

$$|\eta|_t^s \le C(r_0,\delta,\alpha)|\psi|_t^s. \tag{2.2}$$

Proof. According to Theorem 2.7 and Remark 2.6, $(\widetilde{\phi},\widetilde{\eta})$ solves SP$(D,\widetilde{\psi})$ with

$$\widetilde{\phi}_t=\phi_{T-t},\ \widetilde{\psi}_t=\psi_{T-t},\ \widetilde{\eta}_t=\eta_T-\eta_{T-t},\ t\in[0,T].$$

Note that $|\eta|_t^s=|\widetilde{\eta}|_{T-s}^{T-t}$ and $|\psi|_t^s=|\widetilde{\psi}|_{T-s}^{T-t}$, it suffices to show that (2.2) holds for $\widetilde{\eta}$ and $\widetilde{\psi}$. The proof is from [40, Theorem 4.1] by some appropriate modifications.

For each $n\ge 1$, we denote ψ^n by

$$\widetilde{\psi}_t^n=\widetilde{\psi}_{k2^{-n}},\ k2^{-n}\le t<(k+1)2^{-n},\ k\ge 0.$$

In view of Remark 1.4 in [40], there exists a unique solution $(\widetilde{\phi}^n,\widetilde{\eta}^n)$ to SP$(D,\widetilde{\psi}^n)$, i.e.,

$$\widetilde{\phi}_t^n=\widetilde{\psi}_t^n+\widetilde{\eta}_t^n,\ t\in[0,T].$$

It follows from the proof of [40, p. 467] that $\widetilde{\eta}_n$ converges to $\widetilde{\eta}$ uniformly on $[0,T]$. Define

$$\begin{aligned}
T^{n,0}&=\inf\big\{t\ge 0:\widetilde{\psi}_t^n\in\partial D\big\}\wedge T;\\
t^{n,l}&=\inf\big\{t>T_{n,l-1}:|\widetilde{\psi}_t^n-\widetilde{\psi}_{T_{n,l-1}}^n|\ge\delta/2\big\}\wedge T;\\
T^{n,l}&=\inf\big\{t\ge t_{n,l}:\widetilde{\psi}_t^n\in\partial D\big\}\wedge T.
\end{aligned}$$

Recalling the proof of [40, pp. 465–466] and using the fact that $\Delta_{s,t}(\widetilde{\psi})\le\Delta_{0,T;|t-s|}(\widetilde{\psi})$, we can find an integer $n_0\ge 1$ and a constant $h>0$ such that for all $n\ge n_0$,

$$T^{n,l}-T^{n,l-1}\ge h,\ \text{if}\ T_{n,l}<T, \tag{2.3}$$

and for each $l\ge 1$, we have for all $s,t\in[T^{n,l-1},T^{n,l}]$

$$|\widetilde{\eta}^n|_t^s\le C(r_0,\delta,\alpha)\big(1+\exp\big\{C(r_0,\delta,\alpha)\big(1+\Delta_{0,T;|t-s|}\big(\widetilde{\psi}^n\big)\big)\big\}\big)\Delta_{s,t}\big(\widetilde{\psi}^n\big), \tag{2.4}$$

where for any $0\le s\le t\le T$ and $\theta>0$

$$\begin{aligned}
\Delta_{s,t}(\widetilde{\psi})&:=\sup\{|\widetilde{\psi}_{t_1}-\widetilde{\psi}_{t_2}|:s\le t_1,t_2\le t\},\\
\Delta_{0,T;\theta}(\widetilde{\psi})&:=\sup\big\{|\widetilde{\psi}(t_1)-\widetilde{\psi}(t_2)|:\ 0\le t_1,t_2\le T,\ |t_1-t_2|\le\theta\big\}.
\end{aligned}$$

It follows from (2.3) that for any $s,t\in[0,T]$, there exist integers $1\le i_n\le j_n$ such that $s\in[T^{n,i_n-1},T^{n,i_n}]$ and $t\in[T^{n,j_n-1},T^{n,j_n}]$. Then for any $\theta>0$, there is a partition $s=t_0\le t_1\le\cdots\le t_{n_\theta}=t$ such that

$$[t_{k-1},t_k]\subset\big[T^{n,l-1},T^{n,l}\big]\ \text{for some}\ i_n\le l\le j_n,\ \text{and}\ |t_k-t_{k-1}|\le\theta,\ \text{for all}\ k=1,2,\dots,n_\theta.$$

Thus applying (2.4) yields that for any $\theta>0$,

$$\begin{aligned}
|\widetilde{\eta}^n|_t^s&=\sum_{k=1}^{n_\theta}|\widetilde{\eta}^n|_{t_k}^{t_{k-1}}\\
&\le C(r_0,\delta,\alpha)\big(1+\exp\big\{C(r_0,\delta,\alpha)\big(1+\Delta_{0,T;\theta}\big(\widetilde{\psi}^n\big)\big)\big\}\big)\sum_{k=1}^{n_\theta}\Delta_{t_{k-1},t_k}\big(\widetilde{\psi}^n\big)\\
&\le C(r_0,\delta,\alpha)\big(1+\exp\big\{C(r_0,\delta,\alpha)\big(1+\Delta_{0,T;\theta+2^{-(n-1)}}(\widetilde{\psi})\big)\big\}\big)|\widetilde{\psi}^n|_t^s.
\end{aligned} \tag{2.5}$$

Sending $\theta \to 0$ in (2.5), we have

$$|\widetilde{\eta}^n|_t^s \le C(r_0,\delta,\alpha)\big(1+\exp\big\{C(r_0,\delta,\alpha)\big(1+\Delta_{0,T;2^{-(n-1)}}(\widetilde{\psi})\big)\big\}\big)|\widetilde{\psi}^n|_t^s.$$

Note that $\widetilde{\eta}_n$ converges to $\widetilde{\eta}$ uniformly on $[0,T]$ and

$$\lim_{n\to\infty}\Delta_{0,T;2^{-(n-1)}}(\widetilde{\psi})=0,\quad |\widetilde{\psi}^n|_t^s \le |\widetilde{\psi}|_{(t+2^{-n})\wedge T}^{(s-2^{-n})\vee 0}.$$

Consequently, we obtain

$$|\widetilde{\eta}|_t^s \le \liminf_{n\to\infty}|\widetilde{\eta}^n|_t^s \le C(r_0,\delta,\alpha)|\widetilde{\psi}|_t^s,$$

which ends the proof. □

3 Mean reflected BSDEs

The main purpose of this section is to study the solvability of the multidimensional mean reflected BSDE (1.1). In what follows, we make use of the following conditions on the terminal value ξ and the driver f.

(H3) The terminal condition $\xi \in L^2(\mathbb{R}^m)$ satisfies that $\mathbf{E}[\xi] \in \overline{D}$ and the driver $f(t,0,0)$ is in the space of $\mathcal{H}^{1,2}(\mathbb{R}^m)$.

(H4) There exists a constant $\lambda > 0$ such that for any $t \in [0,T]$, $y_1, y_2 \in \mathbb{R}^m$ and $z_1, z_2 \in \mathbb{R}^{m\times d}$

$$|f(t,y_1,z_1)-f(t,y_2,z_2)| \le \lambda(|y_1-y_2|+|z_1-z_2|).$$

Definition 3.1. *A triplet $(Y,Z,\eta) \in \mathcal{S}^2(\mathbb{R}^m)\times\mathcal{H}^2(\mathbb{R}^{m\times d})\times\mathcal{V}(\mathbb{R}^m)$ is said to be a solution to the BSDE* (1.1) *with mean reflection if it satisfies equation* (1.1) *and the component η_t changes only when $\mathbf{E}[Y_t]$ is on the boundary of D such that*

(i) $|\eta|_t^0 = \int_0^t \mathbb{1}_{\{\mathbf{E}[Y_s]\in\partial D\}} d|\eta|_s^0$*;*

(ii) *there exists a measurable function $\gamma : [0,T] \to \mathbb{R}^m$ such that $\gamma_s \in K_{\mathbf{E}[Y_s]}$ $d|\eta|_s^0$-a.e. and*

$$\eta_t = \int_0^t \gamma_s d|\eta|_s^0.$$

Remark 3.2. In [4], Briand et al. introduced the following multi-dimensional mean reflected BSDEs with normal reflection:

$$\begin{cases} Y_t = \xi + \int_t^T f(s,Y_s,Z_s)ds - \int_t^T Z_s dB_s + \int_t^T D_\mu H(\mathbf{P}_{Y_s})(Y_s)dK_s, \quad \forall t\in[0,T], \\ \mathbf{P}_{Y_t} \in \big\{\mu \in \mathcal{P}_2\big(\mathbb{R}^m\big),\ H(\mu)\ge 0\big\}, \quad \forall t\in[0,T],\ \ \int_0^T H(\mathbf{P}_{Y_t})dK_t = 0, \end{cases} \tag{3.1}$$

in which $D_\mu H$ denotes the Lions' derivative (see [11, 35]). When the constraint acts only on the mean of the solution, i.e., $H(\mu) = \int x\mu(dx)$, (3.1) reduces to

$$\begin{cases} Y_t = \xi + \int_t^T f(s,Y_s,Z_s)ds - \int_t^T Z_s dB_s + (K_T - K_t), \forall t\in[0,T], \\ \mathbf{E}[Y_t] \ge 0, \forall t\in[0,T], \qquad\qquad \int_0^T \mathbf{E}[Y_t]dK_t = 0, \end{cases}$$

which is the same as the equation (1.1) with the convex domain $D = \{x\in\mathbb{R}^m,\ x>0\}$. We refer to [4, Remark 4] for more details on this topic.

Now we are ready to state the main result of this paper.

Theorem 3.3. *Suppose that assumptions **(H1)–(H4)** hold. Then the BSDE* (1.1) *with mean reflection admits a unique square integrable solution $(Y,Z,\eta) \in \mathcal{S}^2(\mathbb{R}^m)\times\mathcal{H}^2(\mathbb{R}^{m\times d})\times\mathcal{V}(\mathbb{R}^m)$.*

Remark 3.4. In [15], Chassagneux, Nadtochiy and Richou studied the multidimensional reflected BSDEs with the following possibly non-convex domain $\mathcal{D}$:

$$\mathcal{D} = \{y \in \mathbb{R}^m : \ \phi(y) < 0\},$$

where $\phi \in C^2(\mathbb{R}^m)$ satisfies compactness, smoothness and a weak star-shape property (see [15, Assumption 1.1]). It is easy to check that the domain $\mathcal{D}$ satisfies conditions **(H1)** and **(H2)**. Due to the adaptedness issues, the a priori estimate for the total variation $d|\eta|_t^0$ is much more complicated in this case, which has quadratic terms in z (see [15, Lemma 2.1]).

In what follows, we are going to prove Theorem 3.3. Firstly, we state some useful a priori estimates for solutions to the BSDE (1.1) with mean reflection, which is much more delicate and involved compared with the scalar-valued case.

Lemma 3.5. *Assume that assumptions **(H1)–(H4)** are satisfied. Let (Y, Z, η) be a square integrable solution to the mean reflected BSDE* (1.1). *Then*

$$\mathbf{E}\left[\sup_{0\le s\le T}|Y_s|^2 + \int_0^T |Z_s|^2 ds\right] + \left(|\eta|_T^0\right)^2 \le C(r_0,\delta,\alpha,\lambda,T)\mathbf{E}\left[|\xi|^2 + \left(\int_0^T |f(s,0,0)|ds\right)^2\right]. \tag{3.2}$$

Proof. Note that the Lipschitz continuity of f implies

$$2\big\langle y, f(t,y,z)\big\rangle \le 2|y||f(t,0,0)| + \big(2\lambda+3\lambda^2\big)|y|^2 + \frac{1}{3}|z|^2, \ \forall (y,z)\in\mathbb{R}^m\times\mathbb{R}^{m\times d}. \tag{3.3}$$

Setting $a = (2\lambda+3\lambda^2)+b$ for some constant $b>0$ which is to be determined later. Using the inequality (3.3) and applying Itô's formula to $e^{at}|Y_t|^2$ yields that

$$\begin{aligned}
&e^{at}|Y_t|^2 + b\int_t^T e^{as}|Y_s|^2 ds + \frac{2}{3}\int_t^T e^{as}|Z_s|^2 ds \\
&\le e^{aT}|\xi|^2 + 2\int_t^T e^{as}|Y_s||f(s,0,0)|ds + 2\int_t^T e^{as}\langle Y_s, d\eta_s\rangle - 2\int_t^T e^{as}\langle Y_s, Z_s dB_s\rangle \\
&\le e^{aT}|\xi|^2 + 2\sup_{0\le s\le T}|Y_s|\int_t^T e^{as}|f(s,0,0)|ds + 2\int_t^T e^{as}\langle Y_s, d\eta_s\rangle - 2\int_t^T e^{as}\langle Y_s, Z_s dB_s\rangle.
\end{aligned} \tag{3.4}$$

Now set $\phi_t = \mathbf{E}[Y_t]$ and $\psi_t = \mathbf{E}[\xi + \int_t^T f(s,Y_s,Z_s)ds]$. It is easy to check that (ϕ,η) is the unique solution to BSP(D,ψ). Since ψ has finite total variation and

$$|\psi|_t^s \le \int_s^t \big|\mathbf{E}\big[f(r,Y_r,Z_r)\big]\big|dr \text{ for all } 0\le s\le t\le T,$$

it follows from (2.2) in Lemma 2.9 that

$$d|\eta|_t^0 \le C(r_0,\delta,\alpha)\big|\mathbf{E}\big[f(t,Y_t,Z_t)\big]\big|dt. \tag{3.5}$$

Note that $\eta_t = \int_0^t \gamma_s d|\eta|_s^0$ for some Borel measurable function γ satisfying $\gamma_s \in K_{\mathbf{E}[Y_s]}$ $d|\eta|_s^0$-a.e. Thus, we obtain

$$\begin{aligned}
&2\mathbf{E}\left[\int_t^T e^{as}\langle Y_s, d\eta_s\rangle\right] \\
&\le 2C(r_0,\delta,\alpha)\int_t^T e^{as}\mathbf{E}[|Y_s|]\big|\mathbf{E}\big[f(s,Y_s,Z_s)\big]\big|ds \\
&\le C(r_0,\delta,\alpha)\big(2\lambda+3C(r_0,\delta,\alpha)\lambda^2\big)\int_t^T e^{as}\mathbf{E}\big[|Y_s|^2\big]ds + \frac{1}{3}\int_t^T e^{as}\mathbf{E}\big[|Z_s|^2\big]ds \\
&\quad + 2C(r_0,\delta,\alpha)\mathbf{E}\Big[\sup_{0\le s\le T}|Y_s|\Big]\int_t^T e^{as}\mathbf{E}\big[|f(s,0,0)|\big]ds.
\end{aligned} \tag{3.6}$$

Setting $b = C(r_0,\delta,\alpha)(2\lambda + 3C(r_0,\delta,\alpha)\lambda^2)$ and taking expectations on both sides of (3.4), we have

$$\begin{aligned}
&\sup_{0\le s\le T} e^{as}\mathbf{E}\big[|Y_s|^2\big] + \mathbf{E}\Big[\int_0^T e^{as}|Z_s|^2 ds\Big] \\
&\le 3\mathbf{E}\big[e^{aT}|\xi|^2\big] + 6\mathbf{E}\Big[\sup_{0\le s\le T}|Y_s|\int_t^T e^{as}|f(s,0,0)|ds\Big] \\
&\quad + 6C(r_0,\delta,\alpha)\mathbf{E}\Big[\sup_{0\le s\le T}|Y_s|\Big]\mathbf{E}\Big[\int_0^T e^{as}|f(s,0,0)|ds\Big].
\end{aligned}$$

Then we deduce that for any $\varepsilon\in(0,1)$

$$\begin{aligned}
&\sup_{0\le s\le T}\mathbf{E}\big[|Y_s|^2\big] + \mathbf{E}\Big[\int_0^T |Z_s|^2 ds\Big] \\
&\le \frac{1}{\varepsilon}C(r_0,\delta,\alpha,\lambda,T)\mathbf{E}\Big[|\xi|^2 + \Big(\int_0^T |f(s,0,0)|ds\Big)^2\Big] + \varepsilon\mathbf{E}\Big[\sup_{0\le s\le T}|Y_s|^2\Big].
\end{aligned} \tag{3.7}$$

According to (3.5) and (3.7), we derive that

$$|\eta|_T^0 \le C(r_0,\delta,\alpha,\lambda,T)\Big\{\frac{1}{\sqrt{\varepsilon}}\Big(\mathbf{E}\Big[|\xi|^2 + \Big(\int_0^T |f(s,0,0)|ds\Big)^2\Big]\Big)^{\frac12} + \sqrt{\varepsilon}\Big(\mathbf{E}\Big[\sup_{0\le s\le T}|Y_s|^2\Big]\Big)^{\frac12}\Big\}. \tag{3.8}$$

On the other hand, it follows from the definition of (1.1) that for any $s\in[0,T]$,

$$|Y_s| \le |\xi| + |\eta|_T^0 + \int_0^T |f(r,0,0)|dr + \lambda\int_0^T |Z_r|dr + \sup_{0\le r\le T}\Big|\int_r^T Z_u dB_u\Big| + \lambda\int_s^T |Y_r|dr.$$

Applying Gronwall's inequality, we have

$$|Y_s| \le e^{\lambda(T-s)}\Big(|\xi| + |\eta|_T^0 + \int_0^T |f(r,0,0)|dr + \lambda\int_0^T |Z_r|dr + \sup_{0\le r\le T}\Big|\int_r^T Z_u dB_u\Big|\Big),$$

which together with Burkholder-Davis-Gundy's (BDG's) inequality implies

$$\mathbf{E}\Big[\sup_{0\le s\le T}|Y_s|^2\Big] \le C(\lambda,T)\mathbf{E}\Big[|\xi|^2 + \big(|\eta|_T^0\big)^2 + \Big(\int_0^T |f(s,0,0)|ds\Big)^2 + \int_0^T |Z_s|^2 ds\Big]. \tag{3.9}$$

Putting (3.7), (3.8) and (3.9) together, we conclude that

$$\mathbf{E}\Big[\sup_{0\le s\le T}|Y_s|^2\Big] \le C(r_0,\delta,\alpha,\lambda,T)\Big(\frac{1}{\varepsilon}\mathbf{E}\Big[|\xi|^2 + \Big(\int_0^T |f(s,0,0)|ds\Big)^2\Big] + \varepsilon\mathbf{E}\Big[\sup_{0\le s\le T}|Y_s|^2\Big]\Big).$$

Taking ε small enough so that $C(r_0,\delta,\alpha,\lambda,T)\varepsilon = \frac12$, we can get the desired result, which completes the proof. □

Corollary 3.6. *Assume that the same conditions hold as in Lemma 3.5. Then for any* $p\ge 2$*,*

$$\begin{aligned}
&\mathbf{E}\Big[\sup_{0\le s\le T}|Y_s|^p + \Big(\int_0^T |Z_s|^2 ds\Big)^{\frac{p}{2}}\Big] + \big(|\eta|_T^0\big)^p \\
&\le C(p,r_0,\delta,\alpha,\lambda,T)\mathbf{E}\Big[|\xi|^p + \Big(\int_0^T |f(s,0,0)|ds\Big)^p\Big].
\end{aligned}$$

Proof. Without loss of generality, assume that the right-side hand of the above inequality is finite. Since η is a deterministic function, it follows from Lemma 3.5 that

$$\big(|\eta|_T^0\big)^p \leq C(p,r_0,\delta,\alpha,\lambda,T)\mathbf{E}\bigg[|\xi|^p + \bigg(\int_0^T |f(s,0,0)|ds\bigg)^p\bigg]. \tag{3.10}$$

Note that $(Y_t+\eta_t-\eta_T, Z_t)_{0\leq t\leq T}$ solves the following standard BSDE:

$$\widetilde{Y}_t = \xi + \int_t^T \widetilde{f}(s,\widetilde{Y}_s,\widetilde{Z}_s)ds - \int_t^T \widetilde{Z}_s dB_s,\ \forall t\in[0,T],$$

whose generator is given by $\widetilde{f}(s,y,z) = f(s,y+\eta_T-\eta_s,z)$. In view of [41, Theorem 4.4.4], we deduce that

$$\begin{aligned}
&\mathbf{E}\bigg[\sup_{0\leq s\leq T}|Y_s|^p + \bigg(\int_0^T |Z_s|^2 ds\bigg)^{\frac{p}{2}}\bigg]\\
&\leq 2^{p-1}\mathbf{E}\bigg[\sup_{0\leq s\leq T}|\widetilde{Y}_s|^p + \big(|\eta|_T^0\big)^p + \bigg(\int_0^T |\widetilde{Z}_s|^2 ds\bigg)^{\frac{p}{2}}\bigg]\\
&\leq C(p,\lambda,T)\mathbf{E}\bigg[|\xi|^p + \bigg(\int_0^T |f(s,\eta_T-\eta_s,0)|ds\bigg)^p + \big(|\eta|_T^0\big)^p\bigg]\\
&\leq C(p,\lambda,T)\mathbf{E}\bigg[|\xi|^p + \bigg(\int_0^T |f(s,0,0)|ds\bigg)^p + \big(|\eta|_T^0\big)^p\bigg],
\end{aligned}$$

which together with (3.10) indicates the desired result. The proof is complete. □

Lemma 3.7. *Assume that assumptions **(H1)**, **(H3)** and **(H4)** hold. Let (Y^i,Z^i,η^i) be a square integrable solution to* (1.1) *corresponding to the data (ξ^i,f^i), $i=1,2$. Then, there exists a constant $C(\lambda,T)$ such that*

$$\begin{aligned}
&\mathbf{E}\bigg[\sup_{t\in[0,T]}|\widehat{Y}_t|^2 + \int_0^T |\widehat{Z}_s|^2 ds\bigg]\\
&\leq C(\lambda,T)e^{\frac{2}{r_0}(|\eta^1|_T^0+|\eta^2|_T^0)}\mathbf{E}\bigg[\bigg(|\xi^1-\xi^2| + \int_0^T |f^1\big(s,Y_s^2,Z_s^2\big) - f^2\big(s,Y_s^2,Z_s^2\big)|ds\bigg)^2\bigg],
\end{aligned}$$

where $\widehat{\ell}_t = \ell_t^1 - \ell_t^2$ for $\ell_t = Y_t, Z_t, \eta_t$.

Proof. Denote by

$$I = \mathbf{E}\bigg[\bigg(|\xi^1-\xi^2| + \int_0^T |f^1\big(s,Y_s^2,Z_s^2\big) - f^2\big(s,Y_s^2,Z_s^2\big)|ds\bigg)^2\bigg],$$

and

$$a_t = \big(2\lambda+2\lambda^2\big)t + r_0^{-1}\big(|\eta^1|_t^0 + |\eta^2|_t^0\big).$$

Applying Itô's formula to $e^{a_t}|\widehat{Y}_t|^2$ on $[t,T]$, we have

$$\begin{aligned}
e^{a_T}|\widehat{Y}_T|^2 - e^{a_t}|\widehat{Y}_t|^2 &= \int_t^T e^{a_s}|\widehat{Y}_s|^2 da_s - 2\int_t^T e^{a_s}\big\langle\widehat{Y}_s, f^1\big(s,Y_s^1,Z_s^1\big) - f^1\big(s,Y_s^2,Z_s^2\big)\big\rangle ds\\
&\quad + \int_t^T e^{a_s}|\widehat{Z}_s|^2 ds - 2\int_t^T e^{a_s}\big\langle\widehat{Y}_s, f^1\big(s,Y_s^2,Z_s^2\big) - f^2\big(s,Y_s^2,Z_s^2\big)\big\rangle ds\\
&\quad - 2\int_t^T e^{a_s}\langle\widehat{Y}_s, d\widehat{\eta}_s\rangle + 2\int_t^T e^{a_s}\langle\widehat{Y}_s, \widehat{Z}_s dB_s\rangle,
\end{aligned}$$

which together with

$$2\left\langle Y_s^1 - Y_s^2, f^1\left(s, Y_s^1, Z_s^1\right) - f^1\left(s, Y_s^2, Z_s^2\right)\right\rangle \le \left(2\lambda + 2\lambda^2\right)|Y_s^1 - Y_s^2|^2 + \frac{1}{2}|Z_s^1 - Z_s^2|^2$$

indicates

$$\begin{aligned} & e^{a_t}|\widehat{Y}_t|^2 + \frac{1}{2}\int_t^T e^{a_s}|\widehat{Z}_s|^2 ds \\ & \le e^{a_T}|\xi^1 - \xi^2|^2 + 2\int_t^T e^{a_s}|\widehat{Y}_s||f^1\left(s, Y_s^2, Z_s^2\right) - f^2\left(s, Y_s^2, Z_s^2\right)|ds \\ & \quad - \frac{1}{r_0}\int_t^T e^{a_s}|\widehat{Y}_s|^2 d\left(|\eta^1|_s^0 + |\eta^2|_s^0\right) + 2\int_t^T e^{a_s}\langle \widehat{Y}_s, d\widehat{\eta}_s\rangle - 2\int_t^T e^{a_s}\langle \widehat{Y}_s, \widehat{Z}_s dB_s\rangle. \end{aligned} \tag{3.11}$$

Note that $|\eta^i|_t^0 = \int_0^t \mathbb{1}_{\{\mathbf{E}[Y_s^i]\in\partial D\}} d|\eta|_s^0$ and $\eta_t^i = \int_0^t \gamma_s^i d|\eta^i|_s^0$ for some Borel measurable function γ^i satisfying $\gamma_s^i \in K_{\mathbf{E}[Y_s^i]}$ $d|\eta^i|_s^0$-a.e. Then recalling assumption (H1) and assertion (i) of Remark 2.2, we obtain

$$\begin{aligned} \mathbf{E}\left[\int_t^T e^{a_s}\langle \widehat{Y}_s, d\hat{\eta}_s\rangle\right] &= \int_t^T e^{a_s}\left\langle \mathbf{E}\left[Y_s^1\right] - \mathbf{E}\left[Y_s^2\right], \gamma_s^1\right\rangle d|\eta^1|_s^0 \\ &\quad + \int_t^T e^{a_s}\left\langle \mathbf{E}\left[Y_s^2\right] - \mathbf{E}\left[Y_s^1\right], \gamma_s^2\right\rangle d|\eta^2|_s^0 \\ &= \int_t^T e^{a_s}\left\langle \mathbf{E}\left[Y_s^1\right] - \mathbf{E}\left[Y_s^2\right], \gamma_s^1\right\rangle \mathbb{1}_{\{\mathbf{E}[Y_s^1]\in\partial D\}} d|\eta^1|_s^0 \\ &\quad + \int_t^T e^{a_s}\left\langle \mathbf{E}\left[Y_s^2\right] - \mathbf{E}\left[Y_s^1\right], \gamma_s^2\right\rangle \mathbb{1}_{\{\mathbf{E}[Y_s^2]\in\partial D\}} d|\eta^2|_s^0 \\ &\le \frac{1}{2r_0}\int_t^T e^{a_s}\mathbf{E}\left[|\widehat{Y}_s|^2\right] d\left(|\eta^1|_s^0 + |\eta^2|_s^0\right). \end{aligned} \tag{3.12}$$

In view of (3.11), we derive that for any $t \in [0, T]$

$$\begin{aligned} & \mathbf{E}\left[e^{a_t}|\widehat{Y}_t|^2 + \int_t^T e^{a_s}|\widehat{Z}_s|^2 ds\right] \\ & \le 4e^{a_T}\mathbf{E}\left[|\xi^1 - \xi^2|^2 + \int_t^T |\widehat{Y}_s||f^1\left(s, Y_s^2, Z_s^2\right) - f^2\left(s, Y_s^2, Z_s^2\right)|ds\right]. \end{aligned} \tag{3.13}$$

On the other hand, it follows from

$$\widehat{\eta}_t = \widehat{Y}_0 - \mathbf{E}[\widehat{Y}_t] - \mathbf{E}\left[\int_0^t \left(f^1\left(s, Y_s^1, Z_s^1\right) - f^2\left(s, Y_s^2, Z_s^2\right)\right)ds\right]$$

and (3.13) that

$$\begin{aligned} & \sup_{t\in[0,T]} |\widehat{\eta}_t|^2 \\ & \le C(\lambda, T)\left(\sup_{t\in[0,T]} \mathbf{E}\left[|\widehat{Y}_t|^2 + \int_0^T |\widehat{Z}_s|^2 ds + \left(\int_0^T |f^1\left(s, Y_s^2, Z_s^2\right) - f^2\left(s, Y_s^2, Z_s^2\right)|ds\right)^2\right]\right) \\ & \le C(\lambda, T)e^{a_T}\left(I + \mathbf{E}\left[\int_0^T |\widehat{Y}_s||f^1\left(s, Y_s^2, Z_s^2\right) - f^2\left(s, Y_s^2, Z_s^2\right)|ds\right]\right). \end{aligned}$$

Recalling the definition of (1.1) and using BDG's inequality and (3.13), we have

$$
\begin{aligned}
&\mathbf{E}\Big[\sup_{0\le s\le T}|\widehat{Y}_s|^2\Big]\\
&\le C(\lambda,T)e^{a_T}\bigg(I+\mathbf{E}\bigg[\int_0^T|\widehat{Y}_s||f^1\big(s,Y_s^2,Z_s^2\big)-f^2\big(s,Y_s^2,Z_s^2\big)|ds\bigg]\bigg)\\
&\le C(\lambda,T)e^{a_T}\bigg(I+\frac{1}{2}C(\lambda,T)e^{a_T}\mathbf{E}\bigg[\bigg(\int_0^T|f^1\big(s,Y_s^2,Z_s^2\big)-f^2\big(s,Y_s^2,Z_s^2\big)|ds\bigg)^2\bigg]\bigg)\\
&\quad+\frac{1}{2}\mathbf{E}\Big[\sup_{0\le s\le T}|\widehat{Y}_s|^2\Big].
\end{aligned}
$$

It follows that

$$\mathbf{E}\Big[\sup_{0\le s\le T}|\widehat{Y}_s|^2\Big]\le C(\lambda,T)e^{2a_T}I,$$

which is the desired result. □

The following lemma is a direct consequence of Lemma 3.7.

Lemma 3.8. *Suppose assumptions **(H1)**, **(H3)** and **(H4)** hold. Then the BSDE* (1.1) *with mean reflection has at most one square integrable solution.*

Proof. The proof is immediate from Lemma 3.7 by taking $(\xi^i,f^i)=(\xi,f), i=1,2$. □

Compared with the uniqueness, the existence of solution to the mean reflected BSDE (1.1) is much more complicated, which will be stated in the next section. In what follows, we deal with the case of convex reflection domains to illustrate our main idea.

Lemma 3.9. *Suppose that assumptions **(H1)–(H4)** are fulfilled. Assume also that the generator f is independent of the first unknown y. Then the BSDE* (1.1) *with mean reflection admits a unique square integrable solution.*

Proof. Let $(\widetilde{Y},\widetilde{Z})$ be the $\mathcal{S}^2(\mathbb{R}^m)\times\mathcal{H}^2(\mathbb{R}^{m\times d})$-solution to the following standard BSDE:

$$\widetilde{Y}_t=\xi+\int_t^T f(s,\widetilde{Z}_s)ds-\int_t^T\widetilde{Z}_sdB_s,\ \forall t\in[0,T]. \tag{3.14}$$

Set $\psi_t=\mathbf{E}[\xi+\int_t^T f(s,\widetilde{Z}_s)ds]$. It is easy to check that $\psi\in\mathcal{C}(\mathbb{R}^m)$ with $\psi_T=\mathbf{E}[\xi]\in\overline{D}$. By Theorem 2.7, there exists a unique solution (ϕ,η) to BSP(D,ψ). Set $Y_t=\widetilde{Y}_t-\eta_t+\eta_T$ and $Z_t=\widetilde{Z}_t$. It follows that $(Y,Z)\in\mathcal{S}^2(\mathbb{R}^m)\times\mathcal{H}^2(\mathbb{R}^{m\times d})$ solves the following BSDE

$$Y_t=\xi+\int_t^T f(s,Z_s)ds-\int_t^T Z_sdB_s+\eta_T-\eta_t,\ t\in[0,T].$$

Note also that

$$\mathbf{E}[Y_t]=\mathbf{E}\bigg[\xi+\int_t^T f(s,Z_s)ds\bigg]+\eta_T-\eta_t=\psi_t+\eta_T-\eta_t=\phi_t\in\overline{D}.$$

Therefore, (Y,Z,η) is a square integrable solution of the BSDE (1.1). The uniqueness follows from Lemma 3.8 and the proof is complete. □

Note that a non-empty convex set satisfies assumption (H1) with any $r_0>0$. Then it follows from assertion (i) of Remark 2.2 that for any $x\in\partial D$ and $n\in K_x$

$$\langle x-y,\mathbf{n}\rangle\le 0,\ \forall y\in\overline{D}. \tag{3.15}$$

Theorem 3.10. *Suppose that assumptions **(H2)–(H4)** are satisfied. If the domain D is convex, then the mean reflected BSDE* (1.1) *has a unique square integrable solution.*

Proof. For any given $(U,V)\in\mathcal{H}^2(\mathbb{R}^m)\times\mathcal{H}^2(\mathbb{R}^{m\times d})$, it follows from Lemma 3.9 that the following BSDE with mean reflection

$$\begin{cases} Y_t=\xi+\int_t^T f(s,U_s,V_s)ds-\int_t^T Z_s dB_s+\eta_T-\eta_t, & \forall t\in[0,T],\\ \mathbf{E}[Y_t]\in\overline{D}, & \forall t\in[0,T]\end{cases}\tag{3.16}$$

has a unique solution $(Y,Z,\eta)\in\mathcal{S}^2(\mathbb{R}^m)\times\mathcal{H}^2(\mathbb{R}^{m\times d})\times\mathcal{V}(\mathbb{R}^m)$. Now we define a map

$$\begin{array}{cccc}\Phi: & \mathcal{H}^2\big(\mathbb{R}^m\big)\times\mathcal{H}^2\big(\mathbb{R}^{m\times d}\big) & \to & \mathcal{H}^2\big(\mathbb{R}^m\big)\times\mathcal{H}^2\big(\mathbb{R}^{m\times d}\big),\\ & (U,V) & \mapsto & (Y,Z).\end{array}\tag{3.17}$$

Then it suffices to show that the map Φ is a contraction.

For each $i\in\{1,2\}$, denote by (Y^i,Z^i,η^i) the solution to (3.16) corresponding to the data $(U^i,V^i)\in\mathcal{H}^2(\mathbb{R}^m)\times\mathcal{H}^2(\mathbb{R}^{m\times d})$. Denote by $\widehat{\ell}=\ell^1-\ell^2$ for $\ell=Y,Z,\eta,U,V$ and $a=4\lambda^2+1$. Then applying Itô's formula to $e^{a_t}|\widehat{Y}_t|^2$ and using a similar analysis as in Lemma 3.7, we obtain

$$\begin{aligned}-|\widehat{Y}_0|^2&=a\int_0^T e^{as}|\widehat{Y}_s|^2ds-2\int_0^T e^{as}\big\langle\widehat{Y}_s,f\big(s,U_s^1,V_s^1\big)-f\big(s,U_s^2,V_s^2\big)\big\rangle ds\\&\quad+\int_0^T e^{as}|\widehat{Z}_s|^2ds-2\int_0^T e^{as}\langle\widehat{Y}_s,d\widehat{\eta}_s\rangle+2\int_0^T e^{as}\widehat{Y}_s\widehat{Z}_s dB_s\\&\geq\int_0^T e^{as}\big(|\widehat{Y}_s|^2+|\widehat{Z}_s|^2\big)ds-\frac{1}{2}\int_0^T e^{as}\big(|\widehat{U}_s|^2+|\widehat{V}_s|^2\big)ds\\&\quad-2\int_0^T e^{as}\langle\widehat{Y}_s,d\widehat{\eta}_s\rangle+2\int_0^T e^{as}\widehat{Y}_s\widehat{Z}_s dB_s,\end{aligned}\tag{3.18}$$

where we have used the fact that

$$2\big\langle\widehat{Y}_s,f\big(s,U_s^1,V_s^1\big)-f\big(s,U_s^2,V_s^2\big)\big\rangle\leq 4\lambda^2|\widehat{Y}_s|^2+\frac{1}{2}\big(|\widehat{U}_s|^2+|\widehat{V}_s|^2\big)$$

in the last inequality. In view of (3.15), we get

$$\begin{aligned}\mathbf{E}\Big[\int_0^T e^{as}\langle\widehat{Y}_s,d\widehat{\eta}_s\rangle\Big]&=\int_0^T e^{as}\big\langle\mathbf{E}\big[Y_s^1\big]-\mathbf{E}\big[Y_s^2\big],\gamma_s^1\big\rangle\mathbb{1}_{\{\mathbf{E}[Y_s^1]\in\partial D\}}d|\eta^1|_s^0\\&\quad+\int_0^T e^{as}\big\langle\mathbf{E}\big[Y_s^2\big]-\mathbf{E}\big[Y_s^1\big],\gamma_s^2\big\rangle\mathbb{1}_{\{\mathbf{E}[Y_s^2]\in\partial D\}}d|\eta^2|_s^0\\&\leq 0.\end{aligned}\tag{3.19}$$

Putting (3.18) and (3.19) together yields that

$$\mathbf{E}\Big[\int_0^T e^{as}\big(|\widehat{Y}_s|^2+|\widehat{Z}_s|^2\big)ds\Big]\leq\frac{1}{2}\mathbf{E}\Big[\int_0^T e^{as}\big(|\widehat{U}_s|^2+|\widehat{V}_s|^2\big)ds\Big].$$

Therefore, the map Φ has a unique fixed point $(Y,Z)\in\mathcal{H}^2(\mathbb{R}^m)\times\mathcal{H}^2(\mathbb{R}^{m\times d})$. We denote η by

$$\eta_t=Y_0-Y_t-\int_0^t f(s,Y_s,Z_s)ds+\int_0^t Z_s dB_s,\ t\in[0,T].$$

It follows from the definition of the map Φ that $\eta\in\mathcal{V}(\mathbb{R}^m)$ satisfies (i) and (ii) in Definition 3.1. Finally, with the help of BDG's inequality, we have $Y\in\mathcal{S}^2(\mathbb{R}^m)$, which ends the proof. □

Remark 3.11. Note that (3.15) is crucial to construct a contraction map in the proof of Theorem 3.10. In the non-convex reflection domains case, (3.19) should be replaced by (3.12), and then we need to estimate $d|\eta|_s^0$, which results in the main difficulty.

Remark 3.12. In the case of convex reflection domains, the solution to the mean reflected BSDE (1.1) can be also constructed through a penalization approach as in [24], see Theorem A.5 in the appendix.

4 The existence

This section is devoted to the study of the existence results of multi-dimensional mean reflected BSDEs with non-convex reflection domains. In what follows, we shall combine a fixed-point argument and a truncation technique to deal with it through a priori estimates established in Section 3.

Firstly, we employ a fixed-point argument to show the existence of solutions under an additional condition on the generator.

Lemma 4.1. *Suppose that assumptions **(H1)–(H4)** are satisfied. Assume in addition that there exists a nonnegative process $g \in \mathcal{H}^{1,1}(\mathbb{R})$ such that*

$$|f(t,y,z)| \leq g_t, \ \forall (t,y,z) \in [0,T] \times \mathbb{R}^m \times \mathbb{R}^{m\times d}.$$

Then the BSDE (1.1) *with mean reflection admits a unique square integrable solution.*

Proof. Using the same notations as in the proof of Theorem 3.10. In particular, we can also define the map Φ as in (3.17). Setting

$$a_t = \big(4\lambda^2+1\big)t + b\int_0^t \mathbf{E}[g_s]ds$$

for some constant $b>0$ which is to be determined later. According to the derivation of (3.18), we have

$$\begin{aligned}
-|\widehat{Y}_0|^2 &= \int_0^T e^{a_s}|\widehat{Y}_s|^2 da_s - 2\int_0^T e^{a_s}\big\langle \widehat{Y}_s, f\big(s,U_s^1,V_s^1\big) - f\big(s,U_s^2,V_s^2\big)\big\rangle ds \\
&\quad + \int_0^T e^{a_s}|\widehat{Z}_s|^2 ds - 2\int_0^T e^{a_s}\langle \widehat{Y}_s, d\widehat{\eta}_s\rangle + 2\int_0^T e^{a_s}\widehat{Y}_s\widehat{Z}_s dB_s \\
&\geq \int_0^T e^{a_s}\big(|\widehat{Y}_s|^2 + |\widehat{Z}_s|^2\big)ds - \frac{1}{2}\int_0^T e^{a_s}\big(|\widehat{U}_s|^2 + |\widehat{V}_s|^2\big)ds \\
&\quad + b\int_0^T e^{a_s}|\widehat{Y}_s|^2\mathbf{E}[g_s]ds - 2\int_0^T e^{a_s}\langle \widehat{Y}_s, d\widehat{\eta}_s\rangle + 2\int_0^T e^{a_s}\widehat{Y}_s\widehat{Z}_s dB_s.
\end{aligned} \tag{4.1}$$

Taking expectations on both sides of (4.1) yields that

$$\begin{aligned}
&\mathbf{E}\left[\int_0^T e^{a_s}\big(|\widehat{Y}_s|^2 + |\widehat{Z}_s|^2\big)ds\right] \\
&\leq \frac{1}{2}\mathbf{E}\left[\int_0^T e^{a_s}\big(|\widehat{U}_s|^2 + |\widehat{V}_s|^2\big)ds\right] + 2\mathbf{E}\left[\int_0^T e^{a_s}\big\langle \widehat{Y}_s, d\widehat{\eta}(s)\big\rangle\right] \\
&\quad - b\int_0^T e^{a_s}\mathbf{E}\big[|\widehat{Y}_s|^2\big]\mathbf{E}[g_s]ds.
\end{aligned} \tag{4.2}$$

In view of (3.12), we derive

$$\mathbf{E}\left[\int_0^T e^{a_s}\big\langle \widehat{Y}_s, d\widehat{\eta}(s)\big\rangle\right] \leq \frac{1}{2r_0}\int_0^T e^{a_s}\mathbf{E}\big[|\widehat{Y}_s|^2\big]d\big(|\eta^1|_s^0 + |\eta^2|_s^0\big). \tag{4.3}$$

Next we need to estimate the right-hand side of (4.3), where Lemma 2.9 plays an important role. Indeed, denote by $\phi^i_t = \mathbf{E}[Y^i_t]$ and $\psi^i_t = \mathbf{E}[\xi + \int_t^T f(s, U^i_s, V^i_s)ds]$ for each $t \in [0,T]$, $i = 1,2$. Then we see that (ϕ^i, η^i) is the unique solution of BSP(D, ψ^i). It is easy to check that ψ^i has finite total variation with

$$|\psi^i|^s_t \leq \mathbf{E}\left[\int_s^t |f\big(r, U^i_r, V^i_r\big)|ds\right] \leq \int_s^t \mathbf{E}[g_r]dr,\ 0 \leq s \leq t \leq T,\ i = 1,2. \tag{4.4}$$

Thus it follows from (4.4) and Lemma 2.9 that there exists a constant $C(r_0, \delta, \alpha)$ such that

$$d|\eta^i|^0_s \leq C(r_0,\delta,\alpha)\mathbf{E}[g_s]ds,\ ds\text{-a.e.},\ i = 1,2.$$

Consequently, setting $b = \frac{2C(r_0,\delta,\alpha)}{r_0}$ and recalling (4.2), (4.3), we deduce that

$$\mathbf{E}\left[\int_0^T e^{a_s}\big(|\widehat{Y}_s|^2 + |\widehat{Z}_s|^2\big)ds\right] \leq \frac{1}{2}\mathbf{E}\left[\int_0^T e^{a_s}\big(|\widehat{U}_s|^2 + |\widehat{V}_s|^2\big)ds\right],$$

which implies that the map Φ defined in (3.17) has a unique fixed point (Y, Z). Finally, by a similar analysis as in the proof of Theorem 3.10, we can get the desired result. □

Remark 4.2. In Lemma 4.1, we assume an additional bounded condition on the driver, which implies that

$$d|\eta^n|^0_s \leq C(r_0,\delta,\alpha)\mathbf{E}\big[|f\big(s, Y^n_s, Z^n_s\big)|\big]ds \leq C(r_0,\delta,\alpha)\mathbf{E}[g_s]ds,\ ds\text{-a.e.} \tag{4.5}$$

Here, (Y^n, Z^n, η^n) is Picard iteration sequence with respect to the mean reflected BSDE (1.1). Then, we are able to prove the convergence of (Y^n, Z^n) under the following norm:

$$\mathbf{E}\left[\int_0^T \exp\left(\big(4\lambda^2+1\big)t + \frac{2C(r_0,\delta,\alpha)}{r_0}\int_0^t \mathbf{E}[g_s]ds\right)\big(|Y^n_t|^2 + |Z^n_t|^2\big)dt\right]^{\frac{1}{2}}.$$

In general, it is difficult to establish a priori estimate: $\mathbf{E}[|f(s, Y^n_s, Z^n_s)|] \leq \mathbf{E}[g_s]$ for some process $g \in \mathcal{H}^{1,1}(\mathbb{R})$.

Next, we utilize an approximation approach to remove the above additional condition when the terminal value has a finite moment of order $p > 2$. For this purpose, we introduce the following approximating sequence $(f^n)_{n\geq 1}$:

$$f^n(t,y,z) = f\big(t, \Pi_{B_n}(y), \Pi_{B_n}(z)\big),\ \forall (t,y,z) \in [0,T]\times\mathbb{R}^m\times\mathbb{R}^{m\times d}, \tag{4.6}$$

where Π_{B_n} is the projection on $B_n := \{x \in \mathbb{E} : |x| \leq n\}$ for each Euclidian space $\mathbb{E}$.

Remark 4.3. Note that $|\Pi_{B_n}(x) - \Pi_{B_n}(y)| \leq |x - y|$ for any $x, y \in \mathbb{E}$. Hence under assumption **(H4)**, f^n also satisfies **(H4)** with the same Lipschitz constant λ.

Lemma 4.4. *Suppose that assumptions **(H1)**, **(H2)** and **(H4)** are satisfied. Assume in addition that $\xi \in L^p(\mathbb{R}^m)$ and $f(\cdot,0,0) \in \mathcal{H}^{1,p}(\mathbb{R}^m)$ for some $p > 2$. Then the mean reflected BSDE* (1.1) *has a unique square integrable solution.*

Proof. Without loss of generality, assume that $p < 4$. It is easy to check that for any $(t,y,z) \in [0,T]\times\mathbb{R}^m\times\mathbb{R}^{m\times d}$,

$$|f^n(t,y,z)| \leq |f(t,0,0)| + 2\lambda n \in \mathcal{H}^{1,p}\big(\mathbb{R}^m\big) \subset \mathcal{H}^{1,1}\big(\mathbb{R}^m\big),$$

where f^n is given by (4.6). Thus, it follows from Lemma 4.1 that the following BSDE with mean reflection

$$\begin{cases} Y^n_t = \xi + \int_t^T f^n\big(s, Y^n_s, Z^n_s\big)ds - \int_t^T Z^n_s dB_s + \eta^n_T - \eta^n_t, & \forall t \in [0,T], \\ \mathbf{E}\big[Y^n_t\big] \in \overline{D}, & \forall t \in [0,T] \end{cases} \tag{4.7}$$

admits a unique square integrable solution (Y^n, Z^n, η^n) for each $n \geq 1$. In view of Corollary 3.6 and noting that $f^n(t,0,0) = f(t,0,0)$, we have

$$\sup_{n\geq 1}\mathbf{E}\left[\sup_{0\leq t\leq T}|Y_t^n|^p + \left(\int_0^T |Z_s^n|^2 ds\right)^{\frac{p}{2}}\right] + \sup_{n\geq 1}\left(|\eta^n|_T^0\right)^p \leq M \tag{4.8}$$

with

$$M := C(p, r_0, \delta, \alpha, \lambda, T)\mathbf{E}\left[|\xi|^p + \left(\int_0^T |f(s,0,0)|ds\right)^p\right].$$

In the rest of the proof, we will prove that the limit of (Y^n, Z^n, η^n) is the desired solution, which will be divided into two steps.

Step 1. The convergence. For any fixed $k \geq n$, set $\widehat{\ell}_t^{k,n} = \ell_t^k - \ell_t^n$ for $\ell_t = Y_t, Z_t, \eta_t$. Note that

$$|f^k\big(t, Y_t^n, Z_t^n\big) - f^n\big(t, Y_t^n, Z_t^n\big)| \leq \lambda\big(|Y_t^n|\mathbb{1}_{\{|Y_t^n|>n\}} + |Z_t^n|\mathbb{1}_{\{|Z_t^n|>n\}}\big).$$

Then with the help of Lemma 3.7, we derive that

$$\begin{aligned}&\mathbf{E}\left[\sup_{s\in[0,T]}|\widehat{Y}_s^{k,n}|^2 + \int_0^T |\widehat{Z}_s^{k,n}|^2 ds\right]\\&\leq C(\lambda,T)e^{\frac{2}{r_0}(|\eta^k|_T^0 + |\eta^n|_T^0)}\mathbf{E}\left[\left(\int_0^T \big(|Y_s^n|\mathbb{1}_{\{|Y_s^n|>n\}} + |Z_s^n|\mathbb{1}_{\{|Z_s^n|>n\}}\big)ds\right)^2\right].\end{aligned} \tag{4.9}$$

Applying Hölder's inequality yields that

$$\begin{aligned}&\mathbf{E}\left[\left(\int_0^T \big(|Y_s^n|\mathbb{1}_{\{|Y_s^n|>n\}} + |Z_s^n|\mathbb{1}_{\{|Z_s^n|>n\}}\big)ds\right)^2\right]\\&\leq C(p,T)\mathbf{E}\left[\left(\int_0^T \big(|Y_s^n|^{\frac{4}{p}}\mathbb{1}_{\{|Y_s^n|>n\}} + |Z_s^n|^{\frac{4}{p}}\mathbb{1}_{\{|Z_s^n|>n\}}\big)ds\right)^{\frac{p}{2}}\right]\\&\leq \frac{C(p,T)}{n^{p-2}}\mathbf{E}\left[\sup_{0\leq t\leq T}|Y_t^n|^p + \left(\int_0^T |Z_s^n|^2 ds\right)^{\frac{p}{2}}\right]\\&\leq \frac{C(p,T)M}{n^{p-2}},\end{aligned} \tag{4.10}$$

where we have used (4.8) in the last inequality. Since $|\eta^n|_T^0$ is uniformly bounded, it follows from (4.9) that

$$\lim_{k,n\to\infty}\mathbf{E}\left[\sup_{0\leq s\leq T}|Y_s^n - Y_s^k|^2 + \int_0^T |Z_s^n - Z_s^k|^2 ds\right] = 0. \tag{4.11}$$

Note that

$$\widehat{\eta}_t^{k,n} = \widehat{Y}_0^{k,n} - \mathbf{E}\big[\widehat{Y}_t^{k,n}\big] - \mathbf{E}\left[\int_0^t \big(f^k\big(s, Y_s^k, Z_s^k\big) - f^n\big(s, Y_s^n, Z_s^n\big)\big)ds\right].$$

Recalling (4.10) and (4.11), we obtain

$$\begin{aligned}&\lim_{k,n\to\infty}\sup_{0\leq t\leq T}|\widehat{\eta}_t^{k,n}|^2\\&\leq C(\lambda,T)\lim_{k,n\to\infty}\mathbf{E}\left[\sup_{0\leq s\leq T}|\widehat{Y}_s^{k,n}|^2 + \int_0^T |\widehat{Z}_s^{k,n}|^2 ds + \frac{C(p,T)M}{n^{p-2}}\right]\\&= 0.\end{aligned} \tag{4.12}$$

Consequently, putting (4.11) and (4.12) together, we can find a triple of processes $(Y,Z,\eta)\in\mathcal{S}^2(\mathbb{R}^m)\times\mathcal{H}^2(\mathbb{R}^{m\times d})\times\mathcal{V}(\mathbb{R}^m)$ so that

$$\lim_{n\to\infty}\left(\mathbf{E}\left[\sup_{0\le s\le T}|Y^n_s-Y_s|^2+\int_0^T|Z^n_s-Z_s|^2ds\right]+\sup_{0\le t\le T}|\eta^n_t-\eta_t|\right)=0. \tag{4.13}$$

Step 2. The solution. Recalling the definition of f^n, we have

$$|f^n\big(t,Y^n_t,Z^n_t\big)-f(t,Y_t,Z_t)|\le\lambda\big(|Y^n_t-Y_t|+|Z^n_t-Z_t|\big)+\lambda(|Y_t|\mathbb{1}_{\{|Y_t|>n\}}+|Z_t|\mathbb{1}_{\{|Z_t|>n\}}),$$

which together with (4.13) indicates that

$$\begin{aligned}&\lim_{n\to\infty}\mathbf{E}\left[\int_0^T|f^n\big(s,Y^n_s,Z^n_s\big)-f(s,Y_s,Z_s)|ds\right]\\&\le\lim_{n\to\infty}\left\{C(\lambda,T)\mathbf{E}\left[\sup_{0\le s\le T}|Y^n_s-Y_s|^2+\int_0^T|Z^n_s-Z_s|^2ds\right]^{\frac12}\right.\\&\qquad\qquad\left.+\frac{C(\lambda,T)}{n}\mathbf{E}\left[\sup_{0\le s\le T}|Y_s|^2+\int_0^T|Z_s|^2ds\right]\right\}\\&=0.\end{aligned}\tag{4.14}$$

Thus, letting $n\to\infty$ in (4.7), we conclude that (Y,Z,η) satisfies equation (1.1). It remains to show that the component η satisfies (i) and (ii) in Definition 3.1, which is equivalent to prove that (ϕ,η) is the solution of $\mathrm{BSP}(D,\psi)$ with $\phi_t:=\mathbf{E}[Y_t]$ and $\psi_t:=\mathbf{E}[\xi+\int_t^T f(s,Y_s,Z_s)ds]$.

Set

$$\phi^n_t:=\mathbf{E}\big[Y^n_t\big]\text{ and }\psi^n_t:=\mathbf{E}\left[\xi+\int_t^T f^n\big(s,Y^n_s,Z^n_s\big)ds\right]\text{ for all }n\ge1.$$

It is clear that (ϕ^n,η^n) is the solution of $\mathrm{BSP}(D,\psi^n)$ for each $n\ge1$. Recalling (4.8), (4.13) and (4.14), we have

$$\sup_{n\ge1}|\eta^n|^0_T<\infty\ \text{ and }\ \lim_{n\to\infty}\sup_{0\le t\le T}\big(|\phi^n_t-\phi_t|+|\psi^n_t-\psi_t|+|\eta^n_t-\eta_t|\big)=0,$$

which together with Lemma 2.8 indicates the desired result. The proof is complete. □

Finally, we are ready to complete the proof of Theorem 3.3.

The proof of Theorem 3.3. It suffices to prove the existence. Set

$$\xi^{(n)}:=\xi\mathbb{1}_{\{|\xi|\le n\}}+\mathbf{E}[\xi\mathbb{1}_{\{|\xi|>n\}}],\ f^{(n)}(t,y,z):=f(t,y,z)-f(t,0,0)\mathbb{1}_{\{|f(t,0,0)|>n\}},\ \forall n\ge1. \tag{4.15}$$

It is easy to check that the data $(\xi^{(n)},f^{(n)})$ satisfies the following condition:

- $\xi^{(n)}$ is bounded and $\mathbf{E}[\xi^{(n)}]=\mathbf{E}[\xi]\in\overline{D}$.
- $f^{(n)}(t,0,0)$ is bounded and f^n satisfies **(H4)**.

By Lemma 4.4, there exists a unique square integrable solution $(Y^{(n)},Z^{(n)},\eta^{(n)})$ to the following BSDE with mean reflection:

$$\begin{cases}Y^{(n)}_t=\xi^{(n)}+\int_t^T f^{(n)}\big(s,Y^{(n)}_s,Z^{(n)}_s\big)ds-\int_t^T Z^{(n)}_s dB_s+\eta^{(n)}_T-\eta^{(n)}_t, & \forall t\in[0,T],\\ \mathbf{E}\big[Y^{(n)}_t\big]\in\overline{D}, & \forall t\in[0,T].\end{cases}\tag{4.16}$$

It is obvious that

$$|f^{(n)}(t,0,0)|\le|f(t,0,0)|,\ \mathbf{E}\big[|\xi^{(n)}|^2\big]\le4\mathbf{E}\big[|\xi|^2\big],\ n\ge1,$$

which together with Lemma 3.5 implies

$$\begin{aligned}&\sup_{n\geq 1}\mathbf{E}\Big[\sup_{0\leq t\leq T}|Y_t^{(n)}|^2+\int_0^T|Z_s^{(n)}|^2ds\Big]+\sup_{n\geq 1}\big(|\eta^{(n)}|_T^0\big)^2\\&\leq C(r_0,\delta,\alpha,\lambda,T)\mathbf{E}\bigg[|\xi|^2+\bigg(\int_0^T|f(s,0,0)|ds\bigg)^2\bigg].\end{aligned}\tag{4.17}$$

The rest of the proof is similar to that of Lemma 4.4. For readers' convenience, we give the sketch of the proof.

Step 1. The convergence. For any fixed $k\geq n$, set $\widehat{\ell}_t^{(k,n)}=\ell_t^{(k)}-\ell_t^{(n)}$ for $\ell_t=Y_t,Z_t,\eta_t$. Note that

$$|f^{(k)}\big(t,Y_t^{(n)},Z_t^{(n)}\big)-f^{(n)}\big(t,Y_t^{(n)},Z_t^{(n)}\big)|\leq|f(t,0,0)|\mathbb{1}_{\{|f(t,0,0)|>n\}}.$$

Applying Lemma 3.7 yields that

$$\begin{aligned}&\mathbf{E}\Big[\sup_{0\leq s\leq T}|\widehat{Y}_s^{(k,n)}|^2+\int_0^T|\widehat{Z}_s^{(k,n)}|^2ds\Big]\\&\leq C(\lambda,T)e^{\frac{2}{r_0}(|\eta^{(k)}|_T^0+|\eta^{(n)}|_T^0)}\mathbf{E}\bigg[|\xi^{(k)}-\xi^{(n)}|^2+\bigg(\int_0^T|f(s,0,0)|\mathbb{1}_{\{|f(s,0,0)|>n\}}ds\bigg)^2\bigg].\end{aligned}\tag{4.18}$$

Since $|\xi^{(n)}|\leq|\xi|+\mathbf{E}[|\xi|]$, we can use dominated convergence theorem to obtain

$$\lim_{n\to\infty}\mathbf{E}\bigg[|\xi^{(n)}-\xi|^2+\bigg(\int_0^T|f(s,0,0)|\mathbb{1}_{\{|f(s,0,0)|>n\}}ds\bigg)^2\bigg]=0.\tag{4.19}$$

Then it follows from (4.17)–(4.19) that

$$\lim_{k,n\to\infty}\bigg(\mathbf{E}\Big[\sup_{0\leq s\leq T}|\widehat{Y}_s^{(k,n)}|^2\Big]+\mathbf{E}\Big[\int_0^T|\widehat{Z}_s^{(k,n)}|^2ds\Big]\bigg)=0.$$

Consequently, in view of the derivation of (4.12), we also have

$$\lim_{k,n\to\infty}\sup_{0\leq t\leq T}|\widehat{\eta}_t^{(k,n)}|=0.$$

Step 2. The solution. It is clear that there exists a triple of processes $(Y,Z,\eta)\in\mathcal{S}^2(\mathbb{R}^m)\times\mathcal{H}^2(\mathbb{R}^{m\times d})\times\mathcal{V}(\mathbb{R}^m)$ such that

$$\lim_{n\to\infty}\bigg(\mathbf{E}\Big[\sup_{0\leq s\leq T}|Y_s^n-Y_s|^2\Big]+\mathbf{E}\Big[\int_0^T|Z_s^n-Z_s|^2ds\Big]+\sup_{0\leq t\leq T}|\eta_t^n-\eta_t|\bigg)=0.\tag{4.20}$$

By a similar analysis as in the proof of Lemma 4.4 (**Step 2**), we obtain

$$\lim_{n\to\infty}\mathbf{E}\bigg[\bigg|\int_0^T|f^{(n)}\big(s,Y_s^{(n)},Z_s^{(n)}\big)-f(s,Y_s,Z_s)|ds\bigg|\bigg]=0,\tag{4.21}$$

which indicates that that (Y,Z,η) satisfies (1.1) by letting $n\to\infty$ in (4.16). Moreover, η satisfies (i) and (ii) in Definition 3.1. The proof is complete. □

A Approximation by penalization method

In this appendix, we shall use a penalization method to construct the unique solution to the mean reflected BSDE (1.1) when the reflection domain is convex inspired by the results of [24]. Indeed, the unique solution can be represented as the limit of a sequence of penalized mean-filed BSDEs. Moreover, we can remove assumption **(H2)** in the convex case.

Assume that the domain D is convex. Then, for any $x \in \mathbb{R}^m$, there exists a unique point $\Pi(x) \in \overline{D}$ such that

$$|x - \Pi(x)| = d(x, D) := \inf\{|x-y| : y \in D\}.$$

The following properties of the inward normal reflection is important for our subsequent discussions.

Lemma A.1 ([24, 38]). **(i)** *For any $x \in \mathbb{R}^m$, $x' \in \overline{D}$,*

$$\big\langle x' - x, x - \Pi(x) \big\rangle \leq 0. \tag{A.1}$$

(ii) *For any $x, x' \in \mathbb{R}^m$,*

$$\big\langle x' - x, x - \Pi(x) \big\rangle \leq \big\langle x' - \Pi\big(x'\big), x - \Pi(x) \big\rangle. \tag{A.2}$$

(iii) *There exist a point $x_0 \in D$ and a constant $\kappa > 0$ such that for any $x \in \mathbb{R}^m$,*

$$\big\langle x - x_0, x - \Pi(x) \big\rangle \geq \kappa |x - \Pi(x)|. \tag{A.3}$$

Since ∂D is not regular, let us recall the approximation of D in [24]: For any $\epsilon > 0$, there exists a convex regular domain D_ϵ (with smooth boundary) such that

$$\sup_{x \in D} d(x, D_\epsilon) < \epsilon \ \text{ and } \ \sup_{x \in D_\epsilon} d(x, D) < \epsilon.$$

It is easy to check that $|d(x, D) - d(x, D_\epsilon)| \leq \epsilon$. Denote by Π_ϵ the projection from $\mathbb{R}^m$ to D_ϵ.

Lemma A.2 (Lemma 2.2 and Corollary 2.3 in [24]). *There exists a constant $\gamma > 0$ such that for any $\epsilon < 1$ and $x \in \mathbb{R}^m$,*

$$|\Pi(x) - \Pi_\epsilon(x)| \leq \gamma \sqrt{\epsilon^2 + \epsilon d(x, D_\epsilon)},$$

and

$$|\Pi(x) - \Pi_\epsilon(x)| \mathbb{1}_{\{d(x, D_\epsilon) > \epsilon\}} \leq \gamma \sqrt{\epsilon} \sqrt{d(x, D_\epsilon)} \mathbb{1}_{\{d(x, D_\epsilon) > \epsilon\}}.$$

Now, we introduce the following penalized mean-filed BSDEs:

$$Y_t^n = \xi + \int_t^T f\big(s, Y_s^n, Z_s^n\big) ds - n \int_t^T \big(\mathbf{E}\big[Y_s^n\big] - \Pi\big(\mathbf{E}\big[Y_s^n\big]\big)\big) ds - \int_t^T Z_s^n dB_s, \ 0 \leq t \leq T. \tag{A.4}$$

According to the results of [10] up to a slight modification, the BSDE (A.4) has a unique solution $(Y^n, Z^n) \in \mathcal{S}^2(\mathbb{R}^m) \times \mathcal{H}^2(\mathbb{R}^{m \times d})$ under assumptions **(H3)** and **(H4)**. Then, we define

$$\eta_t^n = -n \int_0^t \big(\mathbf{E}\big[Y_s^n\big] - \Pi\big(\mathbf{E}\big[Y_s^n\big]\big)\big) ds, \ \forall t \in [0, T]. \tag{A.5}$$

The penalized term η^n forces the mean of the solution Y^n to stay within the domain $\overline{D}$. We will show that (Y^n, Z^n, η^n) converges as $n \to \infty$ and the limit is the unique solution to the BSDE (1.1) with mean reflection.

Lemma A.3. *Suppose that assumptions **(H3)** and **(H4)** hold. Then,*

$$\mathbf{E}\left[\sup_{0 \leq s \leq T} |Y_s^n|^2 + \int_0^T |Z_s^n|^2 ds\right] \leq C(\lambda, T) \mathbf{E}\left[|x_0|^2 + |\xi|^2 + \left(\int_0^T |f(s, 0, 0)| ds\right)^2\right],$$
$$|\eta^n|_T^0 \leq C(\lambda, T, \kappa) \mathbf{E}\left[|x_0|^2 + |\xi|^2 + \left(\int_0^T |f(s, 0, 0)| ds\right)^2\right].$$

Proof. The main idea is from [24]. For reader's convenience, we give the sketch of the proof. By a standard calculus, applying Itô's formula to $|Y_t^n - x_0|^2$ yields that for any $t \in [0,T]$,

$$\begin{aligned}\mathbf{E}\Big[|Y_t^n - x_0|^2 + \int_t^T |Z_s^n|^2 ds\Big] &= \mathbf{E}\big[|\xi - x_0|^2\big] + 2\mathbf{E}\Big[\int_t^T \big\langle Y_s^n - x_0, f\big(s, Y_s^n, Z_s^n\big)\big\rangle ds\Big] \\ &\quad - 2n\int_t^T \big\langle \mathbf{E}\big[Y_s^n\big] - x_0, \mathbf{E}\big[Y_s^n\big] - \Pi\big(\mathbf{E}\big[Y_s^n\big]\big)\big\rangle ds.\end{aligned} \tag{A.6}$$

Since the third term in the right-hand side of (A.6) is non-positive according to (A.3), we have

$$\begin{aligned}\mathbf{E}\Big[|Y_t^n - x_0|^2 + \int_t^T |Z_s^n|^2 ds\Big] &\le \mathbf{E}\big[|\xi - x_0|^2\big] + 2\mathbf{E}\Big[\sup_{0\le t\le T} |Y_t^n - x_0| \int_0^T |f(s, x_0, 0)| ds\Big] \\ &\quad + \big(2\lambda + 2\lambda^2\big)\mathbf{E}\Big[\int_t^T |Y_t^n - x_0|^2 ds\Big] + \frac{1}{2}\mathbf{E}\Big[\int_t^T |Z_s^n|^2 ds\Big].\end{aligned}$$

In view of Gronwall's inequality, we obtain

$$\sup_{0\le t\le T}\mathbf{E}\big[|Y_t^n - x_0|^2\big] \le C(\lambda, T)\Big(\mathbf{E}\big[|\xi - x_0|^2\big] + \mathbf{E}\Big[\sup_{0\le t\le T} |Y_t^n - x_0| \int_0^T |f(s, x_0, 0)| ds\Big]\Big), \tag{A.7}$$

and then

$$\mathbf{E}\Big[\int_0^T |Z_s^n|^2 ds\Big] \le C(\lambda, T)\Big(\mathbf{E}\big[|\xi - x_0|^2\big] + \mathbf{E}\Big[\sup_{0\le t\le T} |Y_t^n - x_0| \int_0^T |f(s, x_0, 0)| ds\Big]\Big). \tag{A.8}$$

On the other hand, recalling the definition of (A.4) and noting that η^n is deterministic, we get

$$\begin{aligned}&|\eta_T^n - \eta_t^n|^2 \\ &= \Big|\mathbf{E}\Big[\xi - Y_t^n + \int_t^T f\big(s, Y_s^n, Z_s^n\big) ds\Big]\Big|^2 \\ &\le C(\lambda, T)\Big(|x_0|^2 + \mathbf{E}\Big[|\xi|^2 + \Big(\int_0^T |f(s,0,0)| ds\Big)^2 + \sup_{0\le t\le T} |Y_t^n - x_0| \int_0^T |f(s, x_0, 0)| ds\Big]\Big),\end{aligned} \tag{A.9}$$

where we have used (A.7) and (A.8) in the last inequality. Note that

$$\begin{aligned}&\sup_{0\le t\le T} |Y_t^n - x_0|^2 \\ &\le 4\Big(|\xi - x_0|^2 + \Big(\int_0^T |f\big(s, Y_s^n, Z_s^n\big)| ds\Big)^2 + \sup_{0\le t\le T} |\eta_T^n - \eta_t^n|^2 + \sup_{0\le t\le T}\Big|\int_t^T Z_s^n dB_s\Big|^2\Big).\end{aligned}$$

It follows from (A.7)–(A.9) and BDG's inequality that

$$\begin{aligned}&\mathbf{E}\Big[\sup_{0\le t\le T} |Y_t^n - x_0|^2\Big] \\ &\le C(\lambda, T)\mathbf{E}\Big[|x_0|^2 + |\xi|^2 + \Big(\int_0^T |f(s,0,0)| ds\Big)^2 + \sup_{0\le t\le T} |Y_t^n - x_0| \int_0^T |f(s, x_0, 0)| ds\Big] \\ &\le C(\lambda, T)\mathbf{E}\Big[|x_0|^2 + |\xi|^2 + \Big(\int_0^T |f(s,0,0)| ds\Big)^2\Big] + \frac{1}{2}\mathbf{E}\Big[\sup_{0\le t\le T} |Y_t^n - x_0|^2\Big].\end{aligned} \tag{A.10}$$

Putting (A.8) and (A.10) together, we deduce that the first inequality holds.

Finally, recalling (A.3) and (A.6)–(A.8), we obtain

$$\begin{aligned}
&2n\kappa \int_0^T |\mathbf{E}\big[Y_s^n\big] - \Pi\big(\mathbf{E}\big[Y_s^n\big]\big)|ds \\
&\leq C(\lambda,T)\bigg(\mathbf{E}\big[|\xi - x_0|^2\big] + \mathbf{E}\bigg[\sup_{0\leq t\leq T} |Y_t^n - x_0| \int_0^T |f(s,x_0,0)|ds\bigg]\bigg).
\end{aligned}$$

It follows that

$$|\eta^n|_T^0 \leq \frac{C(\lambda,T)}{2\kappa}\mathbf{E}\bigg[|x_0|^2 + |\xi|^2 + \bigg(\int_0^T |f(s,0,0)|ds\bigg)^2\bigg],$$

which ends the proof. □

Lemma A.4. *Suppose that assumptions* ***(H3)*** *and* ***(H4)*** *are fulfilled. Let* $f(\cdot,0,0)$ *be in the space of* $\mathcal{H}^2(\mathbb{R}^m)$*. Then, there exists a constant* $C(\lambda,T)$ *such that*

$$\sup_{0\leq t\leq T} |\mathbf{E}\big[Y_t^n\big] - \Pi\big(\mathbf{E}\big[Y_t^n\big]\big)|^2 \leq \frac{C(\lambda,T)}{n}\mathbf{E}\bigg[|x_0|^2 + |\xi|^2 + \int_0^T |f(s,0,0)|^2 ds\bigg],$$

and

$$\int_0^T |\mathbf{E}\big[Y_t^n\big] - \Pi\big(\mathbf{E}\big[Y_t^n\big]\big)|^2 dt \leq \frac{C(\lambda,T)}{n^2}\mathbf{E}\bigg[|x_0|^2 + |\xi|^2 + \int_0^T |f(s,0,0)|^2 ds\bigg].$$

Proof. Consider the distance between $\mathbf{E}[Y^n]$ and the regular domain D_ϵ. Set $\varphi_\epsilon(x) = d^2(x, D_\epsilon)$. It is clear that $\varphi_\epsilon \in C^1(\mathbb{R}^m)$ with $\bigtriangledown\varphi_\epsilon(x) = 2(x - \Pi_\epsilon(x))$. Note that

$$\mathbf{E}\big[Y_t^n\big] = \mathbf{E}[\xi] + \int_t^T \mathbf{E}\big[f\big(s, Y_s^n, Z_s^n\big)\big]ds - n\int_t^T \big(\mathbf{E}\big[Y_s^n\big] - \Pi\big(\mathbf{E}\big[Y_s^n\big]\big)\big)ds$$

which indicates that

$$\begin{aligned}
\varphi_\epsilon\big(\mathbf{E}\big[Y_t^n\big]\big) = \varphi_\epsilon\big(\mathbf{E}[\xi]\big) &+ 2\int_t^T \big\langle \mathbf{E}\big[Y_s^n\big] - \Pi_\epsilon\big(\mathbf{E}\big[Y_s^n\big]\big), \mathbf{E}\big[f\big(s, Y_s^n, Z_s^n\big)\big]\big\rangle ds \\
&- 2n\int_t^T \big\langle \mathbf{E}\big[Y_s^n\big] - \Pi_\epsilon\big(\mathbf{E}\big[Y_s^n\big]\big), \mathbf{E}\big[Y_s^n\big] - \Pi\big(\mathbf{E}\big[Y_s^n\big]\big)\big\rangle ds.
\end{aligned}$$

Similar to the derivation of (32) in [24], with the help of Lemma A.2, we have

$$\begin{aligned}
&-2n\int_t^T \big\langle \mathbf{E}\big[Y_s^n\big] - \Pi_\epsilon\big(\mathbf{E}\big[Y_s^n\big]\big), \mathbf{E}\big[Y_s^n\big] - \Pi\big(\mathbf{E}\big[Y_s^n\big]\big)\big\rangle ds \\
&\leq C(\gamma)n\epsilon^2 - n\int_t^T \varphi_\epsilon\big(\mathbf{E}\big[Y_s^n\big]\big)\mathbb{1}_{\{d(\mathbf{E}[Y_s^n],D)>\epsilon\}}ds.
\end{aligned}$$

Note that $\varphi_\epsilon(\mathbf{E}[\xi]) \leq \epsilon^2$ since $\mathbf{E}[\xi] \in \overline{D}$. Therefore, we deduce that

$$\begin{aligned}
&\varphi_\epsilon\big(\mathbf{E}\big[Y_t^n\big]\big) + n\int_t^T \varphi_\epsilon\big(\mathbf{E}\big[Y_s^n\big]\big)\mathbb{1}_{\{d(\mathbf{E}[Y_s^n],D)>\epsilon\}}ds \\
&\leq C(\gamma)n\epsilon^2 + 2\int_t^T |\varphi_\epsilon\big(\mathbf{E}\big[Y_s^n\big]\big)|^{\frac{1}{2}}\mathbf{E}\big[|f\big(s, Y_s^n, Z_s^n\big)|\big]ds \\
&\leq C(\gamma)n\epsilon^2 + 4\epsilon\mathbf{E}\bigg[\int_t^T |f\big(s, Y_s^n, Z_s^n\big)|ds\bigg] \\
&\quad + 2\int_t^T |\varphi_\epsilon\big(\mathbf{E}\big[Y_s^n\big]\big)|^{\frac{1}{2}}\mathbf{E}\big[|f\big(s, Y_s^n, Z_s^n\big)|\big]\mathbb{1}_{\{d(\mathbf{E}[Y_s^n],D)>\epsilon\}}ds \\
&\leq C(\gamma)n\epsilon^2 + 4\epsilon\mathbf{E}\bigg[\int_t^T |f\big(s, Y_s^n, Z_s^n\big)|ds\bigg] + \frac{2}{n}\mathbf{E}\bigg[\int_t^T |f\big(s, Y_s^n, Z_s^n\big)|^2 ds\bigg] \\
&\quad + \frac{n}{2}\int_t^T \varphi_\epsilon\big(\mathbf{E}\big[Y_s^n\big]\big)\mathbb{1}_{\{d(\mathbf{E}[Y_s^n],D)>\epsilon\}}ds.
\end{aligned}$$

Letting $\epsilon \to 0$, we have for any $t \in [0,T]$

$$d^2\big(\mathbf{E}\big[Y_t^n\big], D\big) + \frac{n}{2}\int_t^T d^2\big(\mathbf{E}\big[Y_s^n\big], D\big)ds \le \frac{2}{n}\mathbf{E}\bigg[\int_t^T |f\big(s, Y_s^n, Z_s^n\big)|^2 ds\bigg],$$

which implies the desired result by Lemma A.3. □

Now we are in a position to state the main result.

Theorem A.5. *Suppose that assumptions **(H3)** and **(H4)** hold and $f(\cdot,0,0)$ is in the space of $\mathcal{H}^2(\mathbb{R}^m)$. Then the mean reflected BSDE* (1.1) *has a unique square integrable solution (Y,Z,η). Moreover, there exists a constant $C(\lambda,T)$ such that*

$$\begin{aligned}&\mathbf{E}\bigg[\sup_{0\le s\le T}|Y_s^n - Y_s|^2 + \int_0^T |Z_s^n - Z_s|^2 ds\bigg] + \sup_{0\le t\le T}|\eta_t^n - \eta_t|^2\\ &\le \frac{C(\lambda,T)}{n}\mathbf{E}\bigg[|x_0|^2 + |\xi|^2 + \int_0^T |f(s,0,0)|^2 ds\bigg].\end{aligned}$$

Proof. Since the convex domain D satisfies **(H1)**, the uniqueness follows from Lemma 3.8. It suffices to show that the limit of (Y^n, Z^n, η^n) is a solution to the BSDE (1.1) with mean reflection. Applying Itô's formula to $|Y_t^k - Y_t^n|^2$ yields that for any $t \in [0,T]$

$$\begin{aligned}&\mathbf{E}\big[|Y_t^k - Y_t^n|^2\big] + \mathbf{E}\bigg[\int_t^T |Z_s^k - Z_s^n|^2 ds\bigg]\\ &= 2\mathbf{E}\bigg[\int_t^T \big\langle Y_s^k - Y_s^n, f\big(s, Y_s^k, Z_s^k\big) - f\big(s, Y_s^n, Z_s^n\big)\big\rangle ds\bigg]\\ &\quad + 2k\int_t^T \big\langle \mathbf{E}\big[Y_s^n\big] - \mathbf{E}\big[Y_s^k\big], \mathbf{E}\big[Y_s^k\big] - \Pi\big(\mathbf{E}\big[Y_s^k\big]\big)\big\rangle ds\\ &\quad + 2n\int_t^T \big\langle \mathbf{E}\big[Y_s^k\big] - \mathbf{E}\big[Y_s^n\big], \mathbf{E}\big[Y_s^n\big] - \Pi\big(\mathbf{E}\big[Y_s^n\big]\big)\big\rangle ds\\ &\le 2\big(\lambda + \lambda^2\big)\mathbf{E}\bigg[\int_t^T |Y_s^k - Y_s^n|^2 ds\bigg] + \frac{1}{2}\mathbf{E}\bigg[\int_t^T |Z_s^k - Z_s^n|^2 ds\bigg]\\ &\quad + 2(k+n)\int_t^T \big\langle \mathbf{E}\big[Y_s^k\big] - \Pi\big(\mathbf{E}\big[Y_s^k\big]\big), \mathbf{E}\big[Y_s^n\big] - \Pi\big(\mathbf{E}\big[Y_s^n\big]\big)\big\rangle ds,\end{aligned}$$

where we have used (A.2) in the last inequality. By Lemma A.4, we have

$$\begin{aligned}&\int_t^T \big\langle \mathbf{E}\big[Y_s^k\big] - \Pi\big(\mathbf{E}\big[Y_s^k\big]\big), \mathbf{E}\big[Y_s^n\big] - \Pi\big(\mathbf{E}\big[Y_s^n\big]\big)\big\rangle ds\\ &\le \frac{C(\lambda,T)}{kn}\mathbf{E}\bigg[|x_0|^2 + |\xi|^2 + \int_0^T |f(s,0,0)|^2 ds\bigg].\end{aligned}$$

It follows that

$$\begin{aligned}&\sup_{0\le t\le T}\mathbf{E}\big[|Y_t^k - Y_t^n|^2\big] + \mathbf{E}\bigg[\int_0^T |Z_s^k - Z_s^n|^2 ds\bigg]\\ &\le \bigg(\frac{1}{k} + \frac{1}{n}\bigg)C(\lambda,T)\mathbf{E}\bigg[|x_0|^2 + |\xi|^2 + \int_0^T |f(s,0,0)|^2 ds\bigg].\end{aligned}$$

In view of the fact that

$$\eta_t^l = \mathbf{E}\big[\eta_t^l\big] = \mathbf{E}\bigg[Y_t^l - \xi - \int_t^T f\big(s, Y_s^l, Z_s^l\big)ds\bigg],\ l = k, n,$$

we deduce that

$$\sup_{0\le t\le T}|\eta_t^k-\eta_t^n|^2\le C(\lambda,T)\bigg(\sup_{0\le t\le T}\mathbf{E}\big[|Y_t^k-Y_t^n|^2\big]+\mathbf{E}\bigg[\int_0^T|Z_s^k-Z_s^n|^2ds\bigg]\bigg).$$

On the other hand, by the definition of (A.4), we get

$$\begin{aligned}&\sup_{0\le t\le T}|Y_t^k-Y_t^n|^2\\&\le C(\lambda,T)\bigg\{\int_0^T\big(|Y_s^k-Y_s^n|^2+|Z_s^k-Z_s^n|^2\big)ds\\&\qquad+\sup_{0\le t\le T}|\eta_t^k-\eta_t^n|^2+\sup_{0\le t\le T}\bigg|\int_t^T|Z_s^k-Z_s^n|dB_s\bigg|^2\bigg\}.\end{aligned}$$

Putting the above three inequalities together, we derive that

$$\begin{aligned}&\mathbf{E}\bigg[\sup_{0\le s\le T}|Y_s^k-Y_s^n|^2+\int_0^T|Z_s^k-Z_s^n|^2ds\bigg]+\sup_{0\le t\le T}|\eta_t^k-\eta_t^n|^2\\&\le\bigg(\frac{1}{k}+\frac{1}{n}\bigg)C(\lambda,T)\mathbf{E}\bigg[|x_0|^2+|\xi|^2+\int_0^T|f(s,0,0)|^2ds\bigg].\end{aligned}$$

Consequently, we can find a triple of processes $(Y,Z,\eta)\in\mathcal{S}^2(\mathbb{R}^m)\times\mathcal{H}^2(\mathbb{R}^{m\times d})\times\mathcal{V}(\mathbb{R}^m)$ so that (Y^n,Z^n,η^n) converges to (Y,Z,η).

It remains to show that (Y,Z,η) is a solution to the BSDE (1.1) with mean reflection. Passing $n\to\infty$ in (A.4), we see that (Y,Z,η) satisfies the first equation in (1.1). In view of Lemma A.4, $d(\mathbf{E}[Y_t],D)=\lim_{n\to\infty}d(\mathbf{E}[Y_t^n],D)=0$, and then $\mathbf{E}[Y_t]\in\overline{D}$.

For any $\zeta\in\mathcal{C}(\overline{D})$, in view of (A.1) we have

$$\int_0^T\big\langle\zeta_t-\mathbf{E}\big[Y_t^n\big],d\eta_t^n\big\rangle=-n\int_0^T\big\langle\zeta_t-\mathbf{E}\big[Y_t^n\big],\mathbf{E}\big[Y_t^n\big]-\Pi\big(\mathbf{E}\big[Y_t^n\big]\big)\big\rangle dt\ge0.$$

Note that $\mathbf{E}[Y_t^n]$ and η_t^n converge to $\mathbf{E}[Y_t]$ and η_t uniformly in t respectively, and

$$\sup_{n\ge1}|\eta^n|_T^0\le C(\lambda,T,\kappa)\mathbf{E}\bigg[|x_0|^2+|\xi|^2+\bigg(\int_0^T|f(s,0,0)|ds\bigg)^2\bigg].$$

In the spirit of Lemma 5.8 in [24], we have

$$\int_0^T\big\langle\zeta_t-\mathbf{E}[Y_t],d\eta_t\big\rangle=\lim_{n\to\infty}\int_0^T\big\langle\zeta_t-\mathbf{E}\big[Y_t^n\big],d\eta_t^n\big\rangle\ge0.$$

Finally, by a similar analysis as in the proof of [40, pp. 469–470] and Remark 2.2 (or see [24, Lemma 2.1]), we can prove that η satisfies (i) and (ii) in Definition 3.1. The proof is complete. □

References

[1] Bénézet, C., Chassagneux, J.-F. and Richou, A.: Switching problems with controlled randomisation and associated obliquely reflected BSDEs. *Stochastic Processes and their Applications* **144** (2022), 23–71. MR4343905

[2] Bouchard, B., Elie, R. and Réveillac, A.: BSDEs with weak terminal condition. *The Annals of Probability* **43** (2015), 572–604. MR3306000

[3] Bramson, M., Burdzy, K. and Kendall, W.: Shy couplings, $\mathrm{CAT}(0)$ spaces, and the Lion and Man. *The Annals of Probability* **41** (2013), 744–784. MR3077525

[4] Briand, P., Cardaliaguet, P., Chaudru de Raynal, P.-É. and Hu, Y.: Forward and backward stochastic differential equations with normal constraints in law. *Stochastic Processes and their Applications* **130** (2020), 7021–7097. MR4167200

[5] Briand, P., Chaudru de Raynal, P.-É., Guillin, A. and Labart, C.: Particles systems and numerical schemes for mean reflected stochastic differential equations. *The Annals of Applied Probability* **30** (2020), 1884–1909. MR4132640

[6] Briand, P., Elie, R. and Hu, Y.: BSDEs with mean reflection. *The Annals of Applied Probability* **28** (2018), 482–510. MR3770882

[7] Briand, P., Ghannoum, A. and Labart, C.: Mean reflected stochastic differential equations with jumps. *Advances in Applied Probability* **52** (2020), 523–562. MR4123645

[8] Briand, P. and Hibon, H.: Particles Systems for mean reflected BSDEs. *Stochastic Processes and their Applications* **131** (2021), 253–275. MR4157714

[9] Buckdahn, R. and Hu, Y.: Pricing of American contingent claims with jump stock price and constrained portfolios. *Mathematics of Operations Research* **23** (1998), 177–203. MR1606470

[10] Buckdahn, R., Li, J. and Peng, S.: Mean-field backward stochastic differential equations and related partial differential equations. *Stochastic Processes and their Applications* **119** (2009), 3133–3154. MR2568268

[11] Carmona, R. and Delarue, F.: Probabilistic theory of mean field games with applications. I. Mean field FBSDEs, control, and games. Probab. Theory Stoch. Model., 83, *Springer*, Cham, 2018. xxv+713 pp. MR3752669

[12] Carmona, R. and Ludkovski, M.: Pricing asset scheduling flexibility using optimal switching. *Applied Mathematical Finance* **15** (2008), 405–447. MR2489967

[13] Chassagneux, J.-F., Elie, R. and Kharroubi, I.: A note on existence and uniqueness for solutions of multidimensional reflected BSDEs. *Electronic Communications in Probability* **16** (2011), 120–128. MR2775350

[14] Chassagneux, J.-F., Elie, R. and Kharroubi, I.: Discrete-time approximation of multidimensional BSDEs with oblique reflections. *The Annals of Applied Probability* **22** (2012), 971–1007. MR2977983

[15] Chassagneux, J.-F., Nadtochiy, S. and Richou, A.: Reflected BSDEs in non-convex domains. *Probability Theory and Related Fields* **183** (2022), 1237–1284. MR4453325

[16] Chassagneux, J.-F. and Richou, A.: Obliquely reflected backward stochastic differential equations. *Annales de l'Institut Henri Poincaré. Probabilités et Statistiques* **56** (2020), 2868–2896. MR4164859

[17] Cvitanić, J. and Karatzas, I.: Backward stochastic differential equations with reflection and Dynkin games. *The Annals of Probability* **24** (1996), 2024–2056. MR1415239

[18] Cvitanić, J., Karatzas, I. and Soner, H. M.: Backward stochastic differential equations with constraints on the gains-process. *The Annals of Probability* **26** (1998), 1522–1551. MR1675035

[19] Djehiche, B., Elie, R. and Hamadène, S.: Mean-field reflected backward stochastic differential equations. *The Annals of Applied Probability*, in press. MR4612648

[20] Dupuis, P. and Ishii, H.: SDEs with oblique reflection on nonsmooth domains. *The Annals of Probability* **21** (1993), 554–580. MR1207237

[21] El Karoui, N., Kapoudjian, C., Pardoux, E., Peng, S. and Quenez, M. C.: Reflected solutions of backward SDE's, and related obstacle problems for PDE's. *The Annals of Probability* **25** (1997), 702–737. MR1434123

[22] El Karoui, N., Pardoux, E. and Quenez, M. C.: Reflected backward SDEs and American options. Numerical methods in finance, Publ. Newton Inst., 13, *Cambridge University Press*, Cambridge, 1997, 215–231. MR1470516

[23] Gayduk, R. and Nadtochiy, S.: Control-stopping games for market microstructure and beyond. *Mathematics of Operations Research* **45** (2020), 1289–1317. MR4166750

[24] Gegout-Petit, A. and Pardoux, E.: Equations différentielles stochastiques rétrogrades réfléchies dans un convexe. *Stochastics and Stochastics Reports* **57** (1996), 111–128. MR1407950

[25] Grigorova, M., Imkeller, P., Ouknine, Y. and Quenez, M.-C.: Doubly reflected BSDEs and $\mathcal{E}^f$-Dynkin games: beyond the right-continuous case. *Electronic Journal of Probability* **23** (2018), 1–38. MR3896859

[26] Hamadène, S.: Mixed zero-sum stochastic differential game and American game options. *SIAM Journal on Control and Optimization* **45** (2007), 496–518. MR2246087

[27] Hamadène, S. and Hassani, M.: BSDEs with two reflecting barriers: The general result. *Probability Theory and Related Fields* **132** (2005), 237–264. MR2199292

[28] Hamadène, S., Hassani, M. and Ouknine, Y.: Backward SDEs with two rcll reflecting barriers without Mokobodski's hypothesis. *Bulletin des Sciences Mathématiques* **134** (2010), 874–899. MR2737357

[29] Hamadène, S. and Jeanblanc, M.: On the starting and stopping problem: Application in reversible investments. *Mathematics of Operations Research* **32** (2007), 182–192. MR2292506

[30] Hamadène, S. and Zhang, J.: Switching problem and related system of reflected backward SDEs *Stochastic Processes and their Applications* **120** (2010), 403–426. MR2594364

[31] Hibon, H., Hu, Y, Lin, Y., Luo, P. and Wang, F.: Quadratic BSDEs with mean reflection. *Mathematical Control and Related Fields* **8** (2018), 721–738. MR3917461

[32] Hu, Y., Moreau, R. and Wang, F.: General mean reflected BSDEs, arXiv:2211.01187

[33] Hu, Y. and Tang, S. Multi-dimensional BSDE with oblique reflection and optimal switching. *Probability Theory and Related Fields* **147** (2010), 89–121. MR2594348

[34] Klimsiak, T., Rozkosz, A. and Słomiński, L.: Reflected BSDEs in time-dependent convex regions. *Stochastic Processes and their Applications* **125** (2015), 571–596. MR3293295

[35] Lions, P.-L.: In cours au collège de france, www.college-de-france.fr (2008).

[36] Lions, P.-L. and Sznitman, A.-S.: Stochastic differential equations with reflecting boundary conditions. *Communications on Pure and Applied Mathematics* **37** (1984), 511–537. MR0745330

[37] Luo, P.: Reflected BSDEs with time-delayed generators and nonlinear resistance. *Statistics & Probability Letters* **163** (2020), 108765, 8 pp. MR4083851

[38] Menaldi, J. L.: Stochastic variational inequality for reflected diffusion. *Indiana University Mathematics Journal* **32** (1983), 733–744. MR0711864

[39] Peng, S. and Xu, M.: Reflected BSDE with a constraint and its applications in an incomplete market. *Bernoulli* **16** (2010), 614–640. MR2730642

[40] Saisho, Y.: Stochastic differential equations for multidimensional domain with reflecting boundary. *Probability Theory and Related Fields* **74** (1987), 455–477. MR0873889

[41] Zhang, J.: Backward stochastic differential equations. From linear to fully nonlinear theory. Probability Theory and Stochastic Modelling, 86, *Springer*, New York, 2017. xv+386 pp. MR3699487

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