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Oriented percolation in a random environment*

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Abstract

On the lattice $\widetilde{\mathbb{Z}}_+^2 := \{(x,y) \in \mathbb{Z} \times \mathbb{Z}_+ \colon x+y \text{ is even}\}$ we consider the following oriented (northwest-northeast) site percolation: the lines $H_i := \{(x,y) \in \widetilde{\mathbb{Z}}_+^2 \colon y = i\}$ are first declared to be *bad* or *good* with probabilities δ and $1-\delta$ respectively, independently of each other. Given the configuration of lines, sites on good lines are open with probability $p_G > p_c$, the critical probability for the standard oriented site percolation on $\mathbb{Z}_+ \times \mathbb{Z}_+$, and sites on bad lines are open with probability p_B, some small positive number, independently of each other. We show that given any pair $p_G > p_c$ and $p_B > 0$, there exists a $\delta(p_G, p_B) > 0$ small enough, so that for $\delta \leq \delta(p_G, p_B)$ there is a strictly positive probability of oriented percolation to infinity from the origin.

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1 Introduction

On the lattice $\widetilde{\mathbb{Z}}_+^2 := \{(x,y) \in \mathbb{Z} \times \mathbb{Z}_+ \colon x+y \text{ is even}\}$ with graph structure obtained by placing edges between any two sites of $\widetilde{\mathbb{Z}}_+^2$ at Euclidean distance $\sqrt{2}$ from each other, we consider the following oriented (northwest-northeast) site percolation model: the lines $H_i := \{(x,y) \in \widetilde{\mathbb{Z}}_+^2 \colon y = i\}$ are first declared to be *bad* or *good* with probabilities δ and $1-\delta$ respectively, independently of each other. Given the configuration of lines, sites on good lines are open with probability p_G, and sites on bad lines are open with probability p_B, independently of each other. More formally, on a suitable probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathbb{P} = \mathbb{P}_{p_G, p_B, \delta}$, we consider a Bernoulli sequence $\xi = (\xi_i \colon i \in \mathbb{Z}_+)$ with $\mathbb{P}(\xi_i = 1) = \delta = 1 - \mathbb{P}(\xi_i = 0)$, which determines H_i to be bad or good, and

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a family of occupation variables $(\eta_z\colon z \in \widetilde{\mathbb{Z}}^2_+)$ which are conditionally independent given ξ, with $\mathbb{P}(\eta_z = 1 \mid \xi) = p_B = 1 - \mathbb{P}(\eta_z = 0 \mid \xi)$ if $z \in H_i$ with $\xi_i = 1$, and $\mathbb{P}(\eta_z = 1 \mid \xi) = p_G = 1 - \mathbb{P}(\eta_z = 0 \mid \xi)$ if $z \in H_i$ with $\xi_i = 0$. If $\eta_z = 1$ the site z is *open*, and otherwise it is *closed*. An *open oriented path* on the graph $\widetilde{\mathbb{Z}}^2_+$ is a path along which the second coordinate is strictly increasing and all of whose vertices are open. The *open cluster* of a vertex $z \in \widetilde{\mathbb{Z}}^2_+$ is the collection of sites which can be reached from z by an oriented path for which all vertices are open except possibly the initial z. It is denoted by C_z and the open cluster of the origin is denoted by C_0. Thus, we always include z itself in C_z, whether z is open or not. We say that *percolation occurs* if C_0 is infinite with positive probability. This description of percolation is of course obtained by rotating the standard picture on $\mathbb{Z}_+ \times \mathbb{Z}_+$ by $\pi/4$ counterclockwise.

The interesting situation is when $p_G > p_c$, the critical probability for the standard oriented site percolation on $\mathbb{Z}_+ \times \mathbb{Z}_+$, and p_B is some small positive number. Given p_G and p_B we ask if $\delta > 0$ may be taken small enough so that there is a positive probability of oriented percolation to infinity from the origin. We prove the answer to be positive, provided $p_G > p_c$, as stated in the next theorem, which is the main result of this article.

Theorem 1.1. *In the setup described above, let*

$$\Theta(p_G, p_B, \delta) \;=\; \mathbb{P}(C_0 \textit{ is infinite}).$$

Then, if $p_G > p_c$ and $p_B > 0$, we can find $\delta_0 = \delta_0(p_G, p_B) > 0$ so that $\Theta(p_G, p_B, \delta) > 0$ for all $\delta \le \delta_0$. In fact, for $\delta \le \delta_0$,

$$\mathbb{P}(C_0 \textit{ is infinite}|\xi) > 0 \textit{ for almost all } \xi. \tag{1.1}$$

This work stems from attempts to understand and answer various questions which were naturally raised in probability, theoretical computer science and statistical physics. These questions lie on crossroads of various fields and have several quite distinct roots.

• Spatial growth processes such as percolation or contact process in random environment is a very well established topic. The situation is reasonably well understood when the environment has good space-time mixing properties. Much less is known for environments with long range dependencies. One source of inspiration is [5], where the contact process with spatial disorder persisting in time is considered. Shifting their setup to oriented percolation, the difference is that the (*good/bad*) layers in [5] are parallel to the growth direction. Our environment varies in an "orthogonal" fashion and it somehow generates more global effects. (See Figure 7.) It is worth comparing the situation treated here with that in [5], where survival (or percolation) is achieved by pushing the good lines to be *good enough*, given p_B and the frequency δ of bad lines. It is simple to see that this result cannot hold in the current situation, with the layers being transversal to the growth. Indeed, using a Peierls-type argument, it may be shown that there exist $p_B > 0$ and $\delta > 0$ such that for any p_G, $\Theta(p_G, p_B, \delta) = 0$.

• In late sixties, McCoy and Wu ([21, 22, 19, 20]) started the study of a specific class of disordered ferromagnets with random couplings that are constant along each horizontal line, for instance with randomly located layers of strongly and weakly coupled spin systems.

• A third set of questions comes from theoretical computer science. Among them, the clairvoyant scheduling problem or coordinate percolation, introduced by P. Winkler in early nineties: is it possible, in a complete graph with n vertices, to schedule two independently sampled random walks (by suitably delaying jumps), so that they never collide? This has a representation in terms of planar oriented percolation (due to Noga Alon). For results in this direction see [25, 1, 10]. The answer is negative for $n = 2$ or $n = 3$. Numerical simulations suggest a positive answer for $n \ge 4$. Recent progress in [3] gives a positive answer for n large enough.

In parallel, several questions of similar nature, such as compatibility of binary sequences, Lipshitz embedding and rough isometries of random one dimensional objects have been considered and answered in [4], and recently extended to the case of random fields [2]. We refer to the introduction of [2] for a more complete review of similar questions. See also [11, 13, 23, 15].

The approach undertaken in [4] and [2] is, as ours, based on multi-scale analysis. While the general concept is similar, both methods are quite different regarding technical implementation. The scheme developed in [4] relies more on the fine probabilistic block estimates. The approach taken in our work gives a precise geometric description of the random environment, describing the global picture in terms of increasing hierarchies and inter-relation between them. It is inspired by the much simpler situation of deterministic (hierarchical type) environments, as considered in [18] for two-dimensional bond percolation, and discussed in the lecture notes [9, 24] for oriented site percolation. In fact, [18] also describes a strategy to treat planar bond percolation in a class of random environments using the results of this paper.

Another related and more recent result is [7], that deals with an interesting problem for two-dimensional bond percolation in random environment. The proof therein indeed uses the main result of the current paper (available as preprint on arXiv since 2012).

A big portion of the paper deals with the construction of suitable renormalized lattices depending on the configuration of layers. For this, we let $\Gamma = \{x \in \mathbb{Z}_+ \colon \xi_x = 1\}$ be the set of indices that correspond to the bad layers, also called the *environment*. The starting point is a convenient grouping procedure of bad layers into what we call *blocks*, depending on a scale parameter L suitably related to the model parameter δ. This construction might have independent interest (it was used in [15] too).

The paper has two basically distinct parts: Sections 2-4 are focused on the environment and the construction of renormalized lattices. Sections 5-8 deal with the percolation issue.

Section 2 provides the details for the grouping procedure mentioned above, yielding an infinite sequence $(\mathbf{C}_k)_{k\geq 0}$ of increasingly coarse (in k) partitions of Γ into finite subsets. Lemma 2.3 gives a condition on δ for the convergence of this procedure, yielding a final partition $\mathbf{C}_\infty$. This allows to single out an event of positive probability of convenient environment configurations, that we call L-spaced (Definition 2.9), described by $\chi(\Gamma) = 0$, where χ is an a.s. finite random variable under the conditions of Lemma 2.3. In all the following sections we indeed work with a fixed L-spaced configuration. The basic strategy consists in proving that given $p_G > p_c$ and $p_B > 0$ we can take L suitably large so that the conditional probability $\mathbb{P}(C_0 \text{ is infinite } |\Gamma = \gamma)$ is a.s. positive on the event $\{\chi(\Gamma) = 0\}$. In combination with Lemma 2.8 this easily yields the proof of Theorem 1.1. (See the comment at the end of Section 2.)

In Section 3, and based on the partitions $(\mathbf{C}_k)_{k\geq 0}$, $\mathbb{Z}_+$ is split into a sequence of partitions $(\mathbf{H}_k)_{k\geq 0}$, again increasingly coarse, and inducing partitions of $\widetilde{\mathbb{Z}}^2_+$ into horizontal layers at various scales. Notions of good and bad layers will be introduced. The construction of the renormalized lattices is done in Section 4, and simply obtained by suitable vertical split of the horizontal layers corresponding to the partition $\mathbf{H}_k$ into cells $\{S^k_{u,v}\}_{(u,v)}$.

Sections 5-8 are dedicated to the study of the renormalized lattices. Depending on the percolation configuration, the sites of the renormalized lattice are declared passable or not. We state and prove Theorem 5.15, which describes the structure of passable sites at all scales and implies that we may take $p^* < 1$ so that given $p_B > 0$ and $p_G > p^*$, then for all L suitably large $\mathbb{P}(C_0 \text{ is infinite } |\Gamma = \gamma)$ is a.s. positive on the event $\{\chi(\Gamma) = 0\}$, as stated in Corollary 5.16. As already indicated, this implies the conclusion of Theorem 1.1 provided $p_B > 0, p_G > p^*$. The proof of Theorem 5.15 is given in Sections 6-7. The extension of our argument to all $p_G > p_c, p_B > 0$ requires a modification in the first scale of the renormalization procedure. We deal with this in Section 8, through Theorem 8.1.

The Appendix collects the proofs of some basic estimates that are used in the paper.

2 Construction of renormalized lattices: grouping

Recall that $\Gamma \equiv \Gamma(\omega) = \{x \in \mathbb{Z}_+ : \xi_x = 1\}$ denotes the set of indices that correspond to the bad layers. We label its elements in increasing order $\Gamma = \{x_j\}_{j\geq 1}$.

2.1 Definition of the grouping procedure

We will build an infinite sequence $\{\mathbf{C}_k\}_{k\geq 0}$ of partitions of Γ into finite subsets. These partitions will be increasingly coarse in k (the grouping step) and will play a crucial role in our renormalization procedure. The elements of each $\mathbf{C}_k$ are called *blocks*. The construction will depend on a parameter parameter L, a positive integer which will be fixed later so that Lemma 2.3 below holds. This Lemma guarantees the convergence of the grouping procedure yielding the final partition $\mathbf{C}_\infty$. We first set some general notation.

Notation.

(a) For any finite set $C \subset \mathbb{Z}_+$, span(C) denotes the smallest interval (in $\mathbb{Z}_+$) that contains C; $\min(C)$ ($\max(C)$) denotes the minimum (maximum, resp.) element of C; diam$(C) = \max\{|x-y| : x, y \in C\}$ denotes the diameter of C, and $|C|$ denotes its cardinality.

(b) We use $d(D_1, D_2)$ to denote the usual Euclidean distance between two sets D_1 and D_2.

The following will be the basic properties to hold for all blocks $\mathcal{C}$ at each grouping step k and for the limiting partition $\mathbf{C}_\infty$:

(i)
$$\mathcal{C} = \text{span}(\mathcal{C}) \cap \Gamma. \tag{2.1}$$

(ii) To each $\mathcal{C} \in \mathbf{C}_k$, we will attribute a *mass* $m(\mathcal{C}) = m$ in such a way that

$$d(\mathcal{C}, \mathcal{C}') \geq L^{\min\{m(\mathcal{C}), m(\mathcal{C}'), k\}}, \text{ for all } 1 \leq k \leq \infty \text{ and } \mathcal{C}, \mathcal{C}' \in \mathbf{C}_k. \tag{2.2}$$

(iii) If to each $\mathcal{C} \in \mathbf{C}_k$ we define its *level* $\ell(\mathcal{C})$, as the smallest j so that $\mathcal{C} \in \mathbf{C}_j$, then

$$\ell(\mathcal{C}) < m(\mathcal{C}) \text{ for all } \mathcal{C} \in \mathbf{C}_k, \text{ for all } k. \tag{2.3}$$

The construction.

Step 0. The elements of $\mathbf{C}_0$ are simply the subsets $\{x_j\}$ of Γ of cardinality one. To each of them we attribute mass one.

Step 1. For $n \geq 2$, we say that $x_i, x_{i+1}, \dots x_{i+n-1}$ form a 1-run of length n if

$$x_{j+1} - x_j < L, \; j = i, \dots, i+n-2,$$

and

$$x_{j+1} - x_j \geq L \; \begin{cases} \text{for } j = i-1, j = i+n-1, & \text{if } i > 1 \\ \text{for } j = i+n-1, & \text{if } i = 1. \end{cases}$$

The elements of $\mathbf{C}_0$, $\{x_i\}, \{x_{i+1}\}, \dots \{x_{i+n-1}\}$ will be called *constituents* of the 1-run. Note that there are no points in Γ between two consecutive points of a 1-run. Also note that x_j does not appear in any 1-run if and only if $d(\{x_j\}, \Gamma \setminus \{x_j\}) \geq L$.

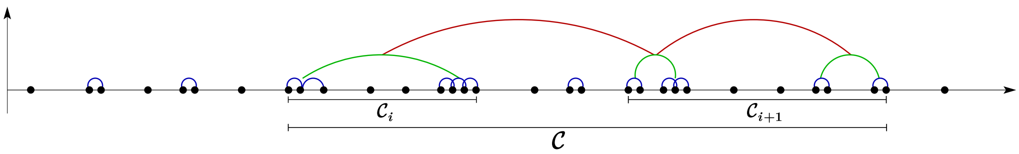

Figure 1: Illustration of the grouping procedure.

The blocks of level 1 are the sets of the form $\mathcal{C} = \{x_i, x_{i+1}, \ldots, x_{i+n-1}\}$ for some 1-run. To each such block we attribute the mass given by its cardinality.

It is obvious that $\mathbb{P}$-a.s. all 1-runs are finite, and that infinitely many such runs exist.

The elements of $\mathbf{C}_1$ are the blocks of level 1 and those $\{x_j\} \in \mathbf{C}_0$ such that x_j does not appear in any 1-run.

We automatically have the restriction of conditions (2.1) and (2.3) to $\mathbf{C}_1$. Condition (2.2) restricted to $k = 1$ is also trivially verified.

Step k+1. Let $k \geq 1$ and assume that the partitions $\mathbf{C}_{k'}$ have been defined for $k' \leq k$ and that properties (2.1), (2.2) and (2.3) hold when restricted to $k' \leq k$.

We now consider $(k+1)$-runs of large blocks in $\mathbf{C}_k$: for this let us write $\mathcal{C}_1, \mathcal{C}_2, \ldots$ for the sequence of all the blocks in $\mathbf{C}_k$ with mass at least $k+1$, labeled in increasing order. For $n \geq 2$, we say that $r = \{\mathcal{C}_i, \mathcal{C}_{i+1}, \ldots, \mathcal{C}_{i+n-1}\}$ forms a $(k+1)$-run of length $|r| = n$ if

$$d(\mathcal{C}_j, \mathcal{C}_{j+1}) \, < \, L^{k+1}, \; j = i, \ldots, i+n-2,$$

and in addition

$$d(\mathcal{C}_j, \mathcal{C}_{j+1}) \, \geq \, L^{k+1} \begin{cases} \text{for } \; j = i-1, j = i+n-1, & \text{if } i > 1 \\ \text{for } \; j = i+n-1, & \text{if } i = 1. \end{cases}$$

In this case we define a block of level $k+1$ as any set of the form

$$\mathcal{C} = \text{span}\Big(\bigcup_{C \in r} C\Big) \cap \Gamma, \tag{2.4}$$

where r is any $(k+1)$-run as above. To $\mathcal{C}$ we attribute the mass

$$m(\mathcal{C}) = \sum_{C \in r} m(C) - k(|r| - 1). \tag{2.5}$$

The blocks of $\mathbf{C}_k$ that form the $(k+1)$-run in (2.4) are called *constituents* of $\mathcal{C}$. This grouping procedure is illustrated in Figure 1.

The partition $\mathbf{C}_{k+1}$ is formed of the blocks of level $k+1$ and of all the blocks in $\mathbf{C}_k$ that are not contained in any block of level $k+1$.

Remark 2.1. (a) Again it is immediate that $\mathbb{P}$-a.s. all $(k+1)$-runs are finite and that infinitely many such runs exist.

(b) Notice that only the constituents of a block $\mathcal{C}$ contribute to its mass. For this reason, all blocks contained in $\mathcal{C}$ that are not constituents of $\mathcal{C}$ will be called *porous medium*.

At this point we need to check that (2.1), (2.2) and (2.3) hold up to $k+1$. The first is trivial, and so is (2.3). About (2.2), when $\min\{m(\mathcal{C}), m(\mathcal{C}')\} \geq k+1$, it follows at once from the definition of $(k+1)$-runs. On the other hand, if $\min\{m(\mathcal{C}), m(\mathcal{C}')\} \leq k$ we have that at least one of these blocks, say $\mathcal{C}$, belongs to $\mathbf{C}_k$ and it is not incorporated into a block of $\mathbf{C}_{k+1}$, so that independently of $\mathcal{C}'$ being obtained from a $(k+1)$-run of blocks in $\mathbf{C}_k$, which will imply $m(\mathcal{C}') \geq k+1 > m(\mathcal{C})$, or when $\mathcal{C}' \in \mathbf{C}_k$, we get $d(\mathcal{C},\mathcal{C}') \geq L^{\min\{m(\mathcal{C}),m(\mathcal{C}')\}}$.

Before proceeding to the next subsection, where we show the convergence of the grouping procedure and study its main properties, we state and prove a simple proposition. It is not essential here, but it will play a role to establish upper bounds on the length of the renormalized layers in the next section.

Proposition 2.2. *If $\mathcal{C} \in \cup_{\ell\geq 1}\boldsymbol{C}_\ell$, then*

$$\textit{diam}\big(\mathcal{C}\big) \ \leq 3L^{m(\mathcal{C})-1}. \tag{2.6}$$

Proof. The statement is trivially correct for $m=1$, since a block of mass one must be a singleton. We will use induction on m. Assume (2.6) holds for all blocks with mass at most $m-1$, where $m \geq 2$. Let $\mathcal{C}$ be a block with $m(\mathcal{C}) = m$ and $\ell(\mathcal{C}) = \ell$. Thus $1 \leq \ell \leq m-1$, by virtue of (2.3). If $\ell = 1$ then $\operatorname{diam}(\mathcal{C}) \leq (m-1)L < 3L^{m-1}$, for $m \geq 2$, provided we take $L \geq 2$. If $\ell \geq 2$, then there exist $n \geq 2$, and an ℓ-run $\mathcal{C}_{i_1},\dots\mathcal{C}_{i_n} \in \mathbf{C}_{\ell-1}$ which will be the constituents of $\mathcal{C}$. In particular, if $m_j = m(\mathcal{C}_{i_j})$, then $m_j \geq \ell$, and from (2.5) we see that $m_j \leq m-n+1$ for each j. From this and the induction hypothesis we get

$$\operatorname{diam}(\mathcal{C}) \leq \sum_{j=1}^{n}\operatorname{diam}(\mathcal{C}_{i_j}) + (n-1)L^{\ell} < 3nL^{m-n} + (n-1)L^{m-n+1} \leq 3L^{m-1} \tag{2.7}$$

for all $L \geq 3$ and $n \geq 2$. ◻

2.2 Properties of the grouping

From the construction of the $(\mathbf{C}_k)_k$ as successively coarser partitions of the set Γ, it is obvious that no $x \in \Gamma$ can belong to two distinct blocks of the same level, though x might belong to blocks of different levels, which occurs when a block of level ℓ containing x is incorporated into part of a block of higher level ℓ'. For the grouping to be useful, we would like that this process stops, yielding a limiting partition $\mathbf{C}_\infty$. Since the origin has a special role, we indeed would like to have a bit more, controlling how close to the origin can a block be in terms of its mass. For all this we need δ to be suitably small, depending on L, as stated below.

Lemma 2.3. *Let us assume that $\delta > 0$ and $3 \leq L < (64\delta)^{-1/2}$. Under such conditions there exist constants $c_1 > 0$ and $c_2 > \log L$ such that*

$$\mathbb{P}\left(\exists \mathcal{C} \in \bigcup_{\ell\geq 1}\boldsymbol{C}_\ell\colon \textit{min}(\mathcal{C}) = z, m(\mathcal{C}) = m\right) \leq c_1 e^{-c_2 m}, \tag{2.8}$$

for each m and each z. In particular we may assume

$$c_1(L^m+1)e^{-c_2 m} \leq 2c_1 e^{-c_3 m} \tag{2.9}$$

for some constant $c_3 > 0$.

Before proving the above lemma, we state and prove the following important consequence.

Lemma 2.4. *Let*

$$\chi(\gamma) = \inf\left\{k \geq 0\colon\ \textit{min}(\mathcal{C}) \geq L^{m(\mathcal{C})} \ \ \textit{for all}\ \mathcal{C} \in \bigcup_{\ell\geq 1}\mathbf{C}_\ell \ \textit{with}\ m(\mathcal{C}) > k\right\}, \tag{2.10}$$

and set $\chi(\gamma) = \infty$ if the above set is empty. Under the conditions of Lemma 2.3 we have

$$\mathbb{P}(\chi < \infty) = 1. \tag{2.11}$$

Proof. In fact, (2.8) and (2.9) show that

$$\mathbb{P}(\chi(\gamma) \geq k) \leq \sum_{m>k} c_1 L^m e^{-c_2 m} \to 0 \text{ as } k \to \infty. \quad \square \tag{2.12}$$

For each $x \in \Gamma$, define the random index

$$\kappa(x) = \sup\left\{\ell(\mathcal{C})\colon x \in \mathcal{C} \in \bigcup_{0\leq k<\infty} \mathbf{C}_k\right\}. \tag{2.13}$$

Since $m(\mathcal{C}) > \ell(\mathcal{C})$ for each $\mathcal{C}$ as in (2.13), the corollary below is an immediate consequence of Lemma 2.4.

Corollary 2.5. *Let δ and L be as in Lemma 2.3. For each $x \in \Gamma$ we have*

$$\mathbb{P}(\kappa(x) < \infty) = 1. \tag{2.14}$$

Remark 2.6. (Convergence of the grouping) The property described in Corollary 2.5 is not the main point, as we indeed need something stronger. But it is useful to observe that:

(i) Corollary 2.5 says that the grouping procedure stops a.s., yielding a natural definition of $\mathbf{C}_\infty$. Indeed, it guarantees that on a set of probability one, for each $x \in \Gamma$, there exists a unique block of level $\kappa(x) \in \mathbb{Z}_+$ which contains x. We may call it the *maximal block of* x and denote it by $\mathcal{C}(x)$. Moreover, for $x, x' \in \Gamma$, if $x' \in \mathcal{C}(x)$, then $\kappa(x) = \kappa(x')$ and $\mathcal{C}(x) = \mathcal{C}(x')$. This immediately allows us to set $\mathbf{C}_\infty$ as the partition of Γ into such maximal blocks. In other words, a block $\mathcal{C} \in \mathbf{C}_k$, for some $0 \leq k < \infty$, belongs to $\mathbf{C}_\infty$ if and only if it is not contained in any block of strictly larger level. Thus, a.s. (in ξ), the blocks $\mathcal{C} \in \mathbf{C}_\infty$ form a partition of Γ into finite sets, and conditions (2.1), (2.2), (2.3) are verified also for $k = \infty$.

(ii) For the proof of (2.14), less than Lemma 2.3 is needed.[1] It would suffice to have (2.8) for some $c_2 > 0$, as it follows from an application of the Borel-Cantelli lemma.

We now turn back to the proof of Lemma 2.3.

Proof of Lemma 2.3. Let $m \geq 2$ and $z \in \mathbb{Z}_+$. (Of course it suffices to take $m \geq 2$.)

To any given $\mathcal{C} \in \cup_{\ell\geq 1}\mathbf{C}_\ell$ we associate a "genealogical weighted tree". It describes the successive merging processes which lead to the creation of $\mathcal{C}$, i.e., it tells the levels at which some blocks formed runs, merging into larger blocks and how many constituents entered each run, down to level 1, and finally the masses of such level 1 blocks. We represent it as a tree with the root corresponding to $\mathcal{C}$; the leaves correspond to blocks of level 1, which are the basic constituents at level 1. This weighted tree gives the basic information on the block, neglecting what was incorporated as "porous medium", on the way.

More formally, we construct the tree iteratively. The root of the tree corresponds to $\mathcal{C}$. If this block is of level 1, the procedure is stopped. For notational consistency such a tree will be called a 1-leaf tree. To the root we attribute the index 1, as well as another index which equals the mass of the block.

If the resulting block $\mathcal{C}$ is of level $\ell > 1$, we attribute to the root the index ℓ and add to the graph n_1 edges (children) going out from the root, where $n_1 \geq 2$ is the number of

[1] MEV thanks the referee for this remark.

constituents which form the ℓ-run leading to $\mathcal{C}$. Each endvertex of a newly added edge will correspond to a constituent of the run, i.e., if $\mathcal{C}$ has constituents $\mathcal{C}_{i_1},\dots,\mathcal{C}_{i_{n_1}}\in\mathbf{C}_{\ell-1}$, for suitable $i_1,\dots,i_{n_1}$, then there is a vertex at the end of an edge going out from the root corresponding to $\mathcal{C}_{i_j}$ for each $j=1,\dots,n_1$. If the constituent corresponding to a given endvertex is a level 1 block, the procedure at this endvertex is stopped (producing a leaf on the tree), and to this leaf we attribute an index, which equals the mass of the corresponding constituent.

If a given endvertex corresponds to a block $\mathcal{C}'$ of level ℓ' with $1<\ell'<\ell$, then to this endvertex we attribute the index ℓ', and add to the graph n_2 new edges going out of this endvertex, where n_2 is the number of constituents of $\mathcal{C}'$ in $\mathbf{C}_{\ell'-1}$ which make up $\mathcal{C}'$ at step ℓ.

The procedure continues until we reach the state that all constituents corresponding to newly added edges are of level 1. In this way we obtain a tree with the following properties:

i) each vertex of the tree has either 0 or at least two offspring; in case of 0 offspring we say that the vertex is a *leaf* of the tree. Otherwise we call it a *branch node*.

ii) to each branch node we attribute an index ℓ; these indices are strictly decreasing to 1 along any self-avoiding path from the root to a leaf of the tree.

iii) to each leaf is associated a mass. This defines a map

$$g\colon \mathcal{C}\in\cup_{\ell\geq1}\mathbf{C}_\ell\mapsto g(\mathcal{C})\equiv(\Upsilon(\mathcal{C}),\bar{l}(\mathcal{C}),\overline{m}(\mathcal{C})),$$

where $\Upsilon(\mathcal{C})$ is a finite tree with $\mathcal{L}(\Upsilon(\mathcal{C}))$ leaves and $\mathcal{N}(\Upsilon(\mathcal{C}))$ branching nodes. We use the following notation:

$\bar{l}(\mathcal{C})=\{\ell_1(\mathcal{C}),\dots,\ell_{\mathcal{N}(\Upsilon(\mathcal{C}))}(\mathcal{C})\}$ is a multi-index with one component for each branching node of $\Upsilon(\mathcal{C})$, which indicates the level at which branches "merge" into the block corresponding to the node;

$\overline{m}(\mathcal{C})=\{m_1(\mathcal{C}),\dots,m_{\mathcal{L}(\Upsilon(\mathcal{C}))}(\mathcal{C})\}$ a multi-index with one component for each leaf of $\Upsilon(\mathcal{C})$, which gives the mass of the block corresponding to the leaf;

$\bar{n}(\mathcal{C})=\{n_1(\mathcal{C}),\dots,n_{\mathcal{N}(\Upsilon(\mathcal{C}))}(\mathcal{C})\}$ is a multi-index with one component for each vertex of $\Upsilon(\mathcal{C})$, which gives the degree of the vertex minus 1. Note that $\bar{n}(\mathcal{C})$ is determined by $\Upsilon(\mathcal{C})$.

To lighten the notation, we will omit the argument $\mathcal{C}$ in situations where confusion is unlikely. Thus we occasionally write $g(\mathcal{C})\equiv(\Upsilon,\bar{l},\overline{m})$ instead of $(\Upsilon(\mathcal{C}),\bar{l}(\mathcal{C}),\overline{m}(\mathcal{C}))$.

In order to prove (2.8) we decompose the event

$$\big[\exists\,\mathcal{C}\in\cup_{\ell\geq1}\mathbf{C}_\ell\colon\ \min(\mathcal{C})=z,m(\mathcal{C})=m\big] \tag{2.15}$$

according to the possible values for $g(\mathcal{C})$; we shall abbreviate the number of leaves of $\Upsilon(\mathcal{C})$ by $\mathcal{L}$. Since the resulting block $\mathcal{C}$, obtained after all merging process "along the tree", has mass m, it imposes the following relation between the multi-indices $\overline{m}$ and $\bar{l}$:

$$\sum_{i=1}^{\mathcal{L}}m_i-\sum_{j=1}^{\mathcal{N}}(n_j-1)(\ell_j-1)=m. \tag{2.16}$$

Here the first sum runs over all leaves, while the second sum runs over all branching nodes. This relation follows from (2.5) by induction on the number of vertices, by writing the tree as the "union" of the root and the subtrees which remain after removing the root. We note that Υ also has to satisfy

$$\sum_{j=1}^{\mathcal{N}}(n_j-1)=\mathcal{L}-1, \tag{2.17}$$

because it is a tree, as one easily sees by induction on the number of leaves. This implies the further restriction

$$\sum_{i=1}^{\mathcal{L}} m_i \geq m + \mathcal{L} - 1,$$

because $\ell_j \geq 2$ in each term of the second sum in (2.16) (recall that we stop our tree construction at each node corresponding to a block of level 1). Thus the probability of the event in (2.15) equals to

$$\sum_{r\geq 1} \sum_{\substack{\Upsilon: \\ \mathcal{L}(\Upsilon)=r}} \sum_{\bar{l},\overline{m}}^{\Upsilon} \mathbb{P}\big(\exists\, \mathcal{C} \in \cup_{\ell\geq 1}\mathbf{C}_\ell \colon\ \min(\mathcal{C}) = z, m(\mathcal{C}) = m, g(\mathcal{C}) = (\Upsilon, \bar{l}, \overline{m})\big), \tag{2.18}$$

where the third sum $\sum_{\bar{l},\overline{m}}^{\Upsilon}$ is taken over all possible values of $\bar{l}, \overline{m}$, satisfying (2.16).

A decomposition according to the value of the sum $\sum_i m_i$, shows that the expression (2.18) equals

$$\sum_{r\geq 1} \sum_{\substack{\Upsilon: \\ \mathcal{L}(\Upsilon)=r}} \sum_{s\geq r-1} \sum_{\substack{\overline{m}: \\ \sum_i m_i \\ =m+s}} \sum_{\bar{l}} \mathbb{P}\left(\exists\, \mathcal{C} \in \cup_{\ell\geq 1}\mathbf{C}_\ell \colon\ \min(\mathcal{C}) = z,\ m(\mathcal{C}) = m, \gamma(\mathcal{C}) = (\Upsilon, \bar{l}, \overline{m})\right), \tag{2.19}$$

the sum $\sum_{\bar{l}}$ being taken over possible choices of $\bar{l}$ such that $\sum_j (n_j - 1)(\ell_j - 1) = s$. The multiple sum in (2.19) can be bounded from above by

$$\sum_{r\geq 1} \sum_{\substack{\Upsilon: \\ \mathcal{L}(\Upsilon)=r}} \sum_{s\geq r-1} \sum_{\substack{\overline{m}: \\ \sum_i m_i = m+s}} \sum_{\bar{l}} \delta^{m+s} L^{m+2s}. \tag{2.20}$$

Indeed, for fixed z, m and $(\Upsilon, \bar{l}, \overline{m})$, the probability

$$\mathbb{P}\left(\exists\, \mathcal{C} \colon\ \min(\mathcal{C}) = z,\ m(\mathcal{C}) = m, g(\mathcal{C}) = (\Upsilon, \bar{l}, \overline{m})\right)$$

is easily estimated by the following argument: the probability to find a level 1 block of mass m_i which corresponds to some leaf of the tree, and which starts at a given point x, is bounded from above by $\delta^{m_i} L^{m_i - 1}$. Indeed, such a block has to come from a 1-run $x_u, x_{u+1}, \ldots, x_{u+m_i-1}$ of elements of Γ, with $x_u = x$ and $x_{j+1} - x_j \leq L$ for $j = u, \ldots, u + m_i - 2$. The number of choices for such a run is at most L^{m_i-1}, and given the x_j, the probability that they all lie in Γ is δ^{m_i}. Similarly, the probability to find two level 1 blocks of mass m_{i_1} and m_{i_2} which merge at level ℓ_j can be bounded above by $\delta^{m_{i_1}} L^{m_{i_1}-1} \delta^{m_{i_2}} L^{m_{i_2}-1} L^{\ell_j}$. The factor L^{ℓ_j} here is an upper bound for the number of choices for the distance between the two blocks; if they are to merge at level ℓ_j, their distance can be at most L^{ℓ_j}. Iterating this argument we get that

$$\mathbb{P}\left(\exists \mathcal{C} \in \cup_{\ell\geq 1}\mathbf{C}_\ell \colon\ \min(\mathcal{C}) = z,\ m(\mathcal{C}) = m, g(\mathcal{C}) = (\Upsilon, \bar{l}, \overline{m})\right) \leq \delta^{\sum_i m_i} L^{\sum_i (m_i - 1)} L^{\sum_j (n_j - 1)\ell_j},$$

and taking into account that

$$\sum_i (m_i - 1) + \sum_j (n_j - 1)\ell_j = m + s + s - r + \sum_j (n_j - 1),$$

as well as (2.17), we get the bound (2.20).

The number of terms in the sums over $\overline{m}$ and $\bar{l}$ in (2.20) are respectively bounded by 2^{m+s} and 2^s (since $\sum_j (\ell_j - 1) \leq \sum_j (n_j - 1)(\ell_j - 1) = s$ and $\ell_j \geq 2$). Thus we can bound (2.20) from above by

$$\begin{aligned} &\textstyle\sum_{r\geq 1} \sum_{\Upsilon\colon \mathcal{L}(\Upsilon)=r} \sum_{s\geq r-1} 2^{m+s} 2^s \delta^{m+s} L^{m+2s} \\ &\quad \leq (2\delta L)^m \textstyle\sum_{r\geq 1} \sum_{\Upsilon\colon\ \mathcal{L}(\Upsilon)=r} \sum_{s\geq r-1} (4\delta L^2)^s \\ &\qquad \leq (2\delta L)^m \textstyle\sum_{r\geq 1} \sum_{\Upsilon\colon \mathcal{L}(\Upsilon)=r} \frac{(4\delta L^2)^{r-1}}{1-4\delta L^2}, \end{aligned} \tag{2.21}$$

provided we take $4\delta L^2 < 1$. Now the number of planted plane trees of u vertices is at most 4^u (see [12]). Our trees have r leaves, but all vertices which are not leaves have degree at least 3 (except, possibly, the root). Thus, by virtue of (2.17), these trees have at most $2r$ vertices. The number of possibilities for Υ in the last sum is therefore at most $\sum_{u=r+1}^{2r} 4^u \leq \frac{4}{3}4^{2r} \leq 2 \cdot 4^{2r}$. It follows that (2.21) is further bounded by

$$2\frac{(2\delta L)^m}{1-4\delta L^2}\sum_{r\geq 1} 4^{2r}(4\delta L^2)^{r-1} = \frac{32(2\delta L)^m}{1-4\delta L^2}\sum_{r\geq 1}(64\delta L^2)^{r-1}.$$

If we take $64\delta L^2 < 1$, this can be bounded by

$$\frac{32(2\delta L)^m}{(1-4\delta L^2)(1-64\delta L^2)},$$

which proves (2.8) and (2.9) with $c_2 = -\log(2\delta) - \log L > \log L$ for our choice of δ, L. $\square$

Remark 2.7. Note that $\mathbf{C}_\infty$ depends on the collection Γ only. We shall occasionally write $\mathbf{C}_\infty(\gamma)$ for the partition $\mathbf{C}_\infty$ at a sample point with $\Gamma = \gamma$.

Lemma 2.8. *Under the conditions of Lemma 2.3, and for χ as in Lemma 2.4 we have*

$$\mathbb{P}(\chi = 0) > 0. \tag{2.22}$$

Proof. If $\chi(\gamma)$ is finite and non-zero, then there exists a unique block $\mathcal{C}^* \in \mathbf{C}_\infty(\gamma)$ such that $m(\mathcal{C}^*) = \chi(\gamma)$ and $\min(\mathcal{C}^*) < L^{\chi(\gamma)}$. The existence of $\mathcal{C}^*$ follows at once from the definition of χ. For the uniqueness we observe that if two such blocks, say $\mathcal{C}'$ and $\mathcal{C}''$, would exist, then they would have to satisfy $d(\mathcal{C}', \mathcal{C}'') < L^{\chi(\gamma)} = L^{\min\{m(\mathcal{C}'), m(\mathcal{C}'')\}}$, which contradicts (2.2) by virtue of the assumption $\mathcal{C}', \mathcal{C}'' \in \mathbf{C}_\infty$.

We use $\mathcal{C}^*$ to construct a new environment $\widetilde{\gamma}$ corresponding to the following sequence $\{\widetilde{\xi}\}_{i\geq 0}$ of zeroes and ones:

$$\text{if } \chi(\gamma) = 0, \text{ then } \widetilde{\xi}_i = \xi_i \text{ for all } i \geq 0;$$

$$\text{if } 0 < \chi(\gamma) < \infty, \text{ then } \widetilde{\xi}_i = \begin{cases} 0 & \text{if } i \leq \max(\mathcal{C}^*) \\ \xi_i & \text{if } i > \max(\mathcal{C}^*). \end{cases} \tag{2.23}$$

We shall now show that

$$\chi(\widetilde{\gamma}) = 0. \tag{2.24}$$

Of course we only have to check this in the case $0 < \chi(\gamma) < \infty$. We claim that in this case all blocks in $\mathbf{C}_k(\widetilde{\gamma})$ (which are of course located in $[\max(\mathcal{C}^*)+1, \infty)$) belong also to $\mathbf{C}_k(\gamma)$ for all k, and the masses of such blocks in the two environments γ and $\widetilde{\gamma}$ are the same.

To check the claim we simply run through the construction of $\cup_{\ell\geq 1}\mathbf{C}_\ell(\widetilde{\gamma})$, until we would see an element that wouldn't be a block the environment γ. We apply induction with respect to the level of the blocks. Clearly any block of level 0 in $\widetilde{\gamma}$ is simply a single point of Γ which lies in $[\max(\mathcal{C}^*)+1, \infty)$, and has mass 1. This is also a block of level 0 and mass 1 in γ. Since $\xi_i = 0$ for $i \leq \max(\mathcal{C}^*)$ in the environment $\widetilde{\gamma}$, the span of any k-run in $\widetilde{\gamma}$ has to be contained in $[\max(\mathcal{C}^*)+1, \infty)$, for any $k \geq 1$. Therefore the span of any block of level k in environment $\widetilde{\gamma}$ also has to be contained in $[\max(\mathcal{C}^*)+1, \infty)$. In addition, since the two environments γ and $\widetilde{\gamma}$ agree in this interval, a difference in the constructions or masses of some block of level k can arise only because in γ there is a k-run which contains blocks of level $k-1$ which lie in $[\max(\mathcal{C}^*)+1, \infty)$ as well as blocks which intersect $[0, \max(\mathcal{C}^*)]$. But then these blocks will be constituents of a single $\mathcal{C} \in \mathbf{C}_k(\gamma)$ say, and span($\mathcal{C}$) has to contain points in both intervals $[0, \max(\mathcal{C}^*)]$ and

$[\max(\mathcal{C}^*)+1,\infty)$ in γ, which is in clear contradiction with the fact that $\mathcal{C}^* \in \mathbf{C}_\infty$ and establishes our last claim.

Now, by definition of χ, (2.24) is equivalent to

$$\min(\mathcal{C}) \geq L^{m(\mathcal{C})} \tag{2.25}$$

for all $\mathcal{C}$ in $\mathbf{C}_\infty(\widetilde{\gamma})$. In view of our claim this will be implied by (2.25) for all $\mathcal{C}$ in $\mathbf{C}_\infty(\gamma)$ located in $[\max(\mathcal{C}^*),\infty)$. Now, if $\mathcal{C}$ is such a block with $m(\mathcal{C}) \leq m(\mathcal{C}^*)$, then (2.25) holds, because, by virtue of (2.3),

$$\min(\mathcal{C}) \geq \min(\mathcal{C}) - \max(\mathcal{C}^*) = d(\mathcal{C},\mathcal{C}^*) \geq L^{m(\mathcal{C})}.$$

On the other hand, if $m(\mathcal{C}) > m(\mathcal{C}^*) = \chi(\gamma)$, then the definition of χ shows that we have $\min(\mathcal{C}) \geq L^{m(\mathcal{C})}$. This proves (2.25) in all cases, and therefore also proves (2.24).

We now have

$$1 = \mathbb{P}(\chi(\Gamma)<\infty) \leq \mathbb{P}(\chi(\Gamma)=0) + \sum_{n=0}^{\infty} \mathbb{P}(\max(\mathcal{C}^*)=n, \chi(\Gamma^{(n)})=0),$$

where $\mathcal{C}^*$ is as above with γ denoting the value of Γ, and if Γ corresponds to the sequence $\{\xi_i\}$, then $\Gamma^{(n)}$ corresponds to the sequence $\xi_i^{(n)}$ given by

$$\xi_i^{(n)} = \begin{cases} 0 & \text{if } i \leq n \\ \xi_i & \text{if } i > n. \end{cases}$$

Thus, either $\mathbb{P}(\chi(\Gamma)=0)>0$ or there is some non-random $n \in \mathbb{Z}_+$ for which $P(\chi(\Gamma^{(n)}) = 0) > 0$. However,

$$\begin{aligned} \mathbb{P}(\chi(\Gamma)=0) &\geq \mathbb{P}(\chi(\Gamma)=0,\ \xi_i = 0 \text{ for } 0 \leq i \leq n) \\ &= \mathbb{P}(\chi(\Gamma^{(n)})=0,\ \xi_i = 0 \text{ for } 0 \leq i \leq n) \\ &= \mathbb{P}(\xi_i = 0 \text{ for } 0 \leq i \leq n)\mathbb{P}(\chi(\Gamma^{(n)})=0) \end{aligned}$$

(since $\Gamma^{(n)}$ is determined by $(\xi_i; i>n)$). This proves the validity of (2.22) and concludes the argument. □

Definition 2.9. *Given $L \geq 3$ an integer, an environment configuration γ is said to be* L-spaced *if $\chi(\gamma)=0$ for the given choice of scale parameter L.*

A comment on the proof strategy. Since the environment is given by an i.i.d. sequence and the event $\{\xi\colon \mathbb{P}(C_0 \text{ is infinite}|\xi)>0\}$ is a tail event in the ξ sequence, Lemma 2.8 implies that, in order to prove (1.1) in Theorem 1.1, it suffices to show that given p_G, p_B as in the statement, $L = L(p_G,p_B)$ may be taken so that $\mathbb{P}(C_0 \text{ is infinite } |\gamma) > 0$ for each L-spaced environment γ. All the construction that follows is done for a fixed L-spaced environment, and we consider the percolation problem in terms of the conditional probability $\mathbb{P}(\cdot|\gamma)$, which we shall denote by $P^\gamma(\cdot)$ simply.

3 Construction of renormalized lattices: Layers

Assumptions. From now on we restrict ourselves to δ and L as in Lemma 2.3, with environments γ such that $\chi(\gamma)=0$. For the construction below, we also assume $L \geq 36$ and (for convenience) divisible by 3.

3.1 Partitions

Having fixed an L-spaced environment, we have not only that the blocks are suitably separated, but also that there is enough space between the origin and the first block of any given mass. Later on we may need to increase L (and therefore reduce δ). Our first goal is now to reach a definition of renormalized sites on $\widetilde{\mathbb{Z}}^2_+$ suitably adapted to our environment γ. This is done through the construction below.

We start by recursively defining a sequence $(\mathbf{H}_k)_{k\geq 0}$ of partitions of $\mathbb{Z}_+$ determined by the environment through the blocks $(\mathbf{C}_k)_{k\geq 0}$. To achieve a convenient regularity, we might split the space in between consecutive blocks. These partitions of $\mathbb{Z}_+$ will correspond to the horizontal layers at all scales. During the construction, each $\mathbf{H}_k$ will be shown to have the following properties:

1. the elements of $\mathbf{H}_k$ are finite intervals of $\mathbb{Z}_+$;
2. the partitions $\mathbf{H}_k$ are increasingly coarse in k;
3. each block of $\mathbf{C}_k$ is contained in an element of $\mathbf{H}_k$;
4. an element of $\mathbf{H}_k$ may contain zero, one, or several blocks of $\mathbf{C}_k$; it will contain at most one block of $\mathbf{C}_k$ of mass larger than or equal to k;
5. if $\mathcal{C} \in \mathbf{C}_k$ and $m(\mathcal{C}) > k$, then span$(\mathcal{C}) \in \mathbf{H}_k$. Such $\mathcal{C}$ are called *large* (at scale k);
6. the cardinality of blocks of $\mathbf{H}_k$ not in point 5 is at most L^k.

Definition 3.1. *We again refer to the elements of* $\mathbf{H}_k$ *as* blocks*, also called* k*-blocks. Those blocks in point 5 above are called* bad*. The others, except the block that contains* $\{0\}$*, are said to be* good*; a good block in* $\mathbf{H}_k$ *is called* good of type 1 *if it contains a block of* $\mathbf{C}_k$ *of mass* k*, and* good of type 2 *otherwise.*

Remark: As it will follow from the construction, a good block of type 1 of $\mathbf{H}_k$ will contain a unique $\mathcal{C} \in \mathbf{C}_k$ with $m(\mathcal{C}) = k$ and possibly several blocks of $\mathbf{C}_k$ of smaller masses.

Since $\mathbf{H}_k$ will be a partition of $\mathbb{Z}_+$ into finite intervals, we define at each step the set of all right endpoints of the intervals $\mathcal{H}_i \in \mathbf{H}_k$. Properties 1, ..., 6 will be checked by induction on k.

Though it is not truly relevant we may think of $\mathbf{H}_0$ as the partition of $\mathbb{Z}_+$ into singletons $\{i\}$, which are good or bad according to $\xi_i = 0$ or $\xi_i = 1$, respectively.

Step 1. We define the set $\mathcal{X} \subset \mathbb{Z}_+$ containing the right endpoints of the intervals in the partition $\mathbf{H}_1$. It contains three types of points, reflecting the three types of blocks:

• *start and endpoints of large blocks of* $\mathbf{C}_1$; all points of the form $\min \mathcal{C} - 1$ and $\max \mathcal{C}$, for $\mathcal{C} \in \mathbf{C}_1$ with $m(\mathcal{C}) > 1$;

• *approximate endpoints of blocks with mass one in* $\mathbf{C}_1$: all points of the form $\max \mathcal{C} + 3$, for $\mathcal{C} \in \mathbf{C}_1$ with $m(\mathcal{C}) = 1$;

• *points between the blocks of* $\mathbf{C}_1$: enumerating the elements of $\mathbf{C}_1$ from left to right as $\mathcal{C}_1, \mathcal{C}_2, \dots$, and setting $\mathcal{C}_0 = \{0\}$ the set $\mathcal{X}$ contains all points of the form

$$\max \mathcal{C}_i + jL/3, \text{ for } i \geq 0 \text{ and } 1 \leq j < \lfloor \frac{\min \mathcal{C}_{i+1} - \max \mathcal{C}_i}{L/3} \rfloor. \tag{3.1}$$

For convenience we take the first (exceptional) block as $\mathcal{H}_1 = \{0, 1\}$, so that we also add $x_1 = 1$ to the set $\mathcal{X}$ which we now write $\mathcal{X} = \{x_1 < x_2 < \dots\}$, and setting $x_0 = -1$ the partition $\mathbf{H}_1$ is formed by the intervals (see Figure 2).

$$\mathcal{H}_i = (x_{i-1}, x_i] \cap \mathbb{Z}, \text{ for } i \geq 1. \tag{3.2}$$

We see at once that $\mathbf{H}_1$ satisfies properties 1, 3, 4, 5 for $k = 1$. Property 2 is also trivial with the previous definition of $\mathbf{H}_0$. About property 6, the construction shows that the cardinality of a good block of $\mathbf{H}_1$ is at most $2L/3 + 4 \leq L$ for any $L \geq 12$.

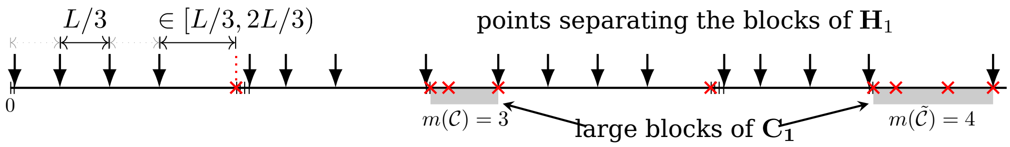

Figure 2: Illustration of the blocks in $\mathbf{H}_1$.

Definition 3.2. *If $\mathcal{H}$ is a good 1-block we set $\mathcal{D}(\mathcal{H}) = \{\max \mathcal{H} - 1, \max \mathcal{H}\}$. When it is good of type 1, i.e. $\mathcal{H} = [\min \mathcal{H}, z+3] \cap \mathbb{Z}$ with $\{z\} = \mathcal{C} \in \mathbf{C}_1$ with $m(\mathcal{C}) = 1$, we write $\mathcal{K}(\mathcal{H}) = [\min H, z-1] \cap \mathbb{Z}$ and distinguish $\mathcal{D}_{\mathcal{K}}(\mathcal{H}) = \{z-1\}$.*

Step k. Let $k \geq 2$. To recursively define the partition $\mathbf{H}_k$ we again identify the right endpoints of its elements, calling once more $\mathcal{X} \subset \mathbb{Z}_+$ the set of such points. The partition $\mathbf{H}_k$ will be coarser than $\mathbf{H}_{k-1}$. So we start with $\mathcal{Y}$ the set of right endpoints of the blocks in $\mathbf{H}_{k-1}$, here listed as $\mathcal{Y} = \{y_1 < y_2 < \dots\}$, and need to determine $\mathcal{X} \subset \mathcal{Y}$. For the construction we bring in all the elements of $\mathbf{C}_k$ with mass at least k. Let these be listed in increasing order as $\mathcal{C}_1, \mathcal{C}_2, \dots$. Let us also assume the validity of properties 1-6 up to scale $k-1$.

We begin by pointing out that $\{\min \mathcal{C}_i - 1, \max \mathcal{C}_i\} \subset \mathcal{Y}$ for all i, a property needed in the construction. Indeed, when $\ell(\mathcal{C}_i) < k$ we have that $\mathcal{C}_i \in \mathbf{C}_{k-1}$ and therefore span$(\mathcal{C}_i)$ is a bad block in $\mathbf{H}_{k-1}$ which implies that $\min \mathcal{C}_i - 1$ and $\max \mathcal{C}_i$ belong to $\mathcal{Y}$. On the other hand, if $\ell(\mathcal{C}_i) = k$, it means that $\mathcal{C}_i$ was formed from a $(k-1)$-run of blocks in $\mathbf{C}_{k-1}$ with mass at least k, implying that $\max \mathcal{C}_i$ is the maximum of a bad block in $\mathbf{H}_{k-1}$ and $\min \mathcal{C}_i$ is the minimum of a bad block in $\mathbf{H}_{k-1}$ and the conclusion follows.

To define the set $\mathcal{X}$ we proceed as follows:

(a) The leftmost element of $\mathcal{X}$ is $x_1 = y_3$. Also add to $\mathcal{X}$ all the points of the form

$$y_{u_s}, \text{ where } y_{u_s - 1} < sL^k/3 \leq y_{u_s}, 1 \leq s < \lfloor \frac{\min \mathcal{C}_1}{L^k/3} \rfloor, \tag{3.3}$$

i.e. y_{u_s} is the right endpoint of the $(k-1)$-block that contains $sL^k/3$.

(b) If $m(\mathcal{C}_i) = k$ and $\max \mathcal{C}_i = y_j$, we add y_{j+3} to $\mathcal{X}$, and also all the points of the form:

$$y_{u_s}, \text{ where } y_{u_s - 1} < y_j + sL^k/3 \leq y_{u_s}, \quad 1 \leq s < \lfloor \frac{\min \mathcal{C}_{i+1} - \max \mathcal{C}_i}{L^k/3} \rfloor =: b, \tag{3.4}$$

i.e. y_{u_s} is the right endpoint of the $(k-1)$-block that contains $y_j + sL^k/3$.

(c) If $m(\mathcal{C}_i) > k$ and $\max \mathcal{C}_i = y_j$, we add y_j and $\min \mathcal{C}_i - 1$ to $\mathcal{X}$, as well the points y_{u_s} defined as in (3.4).

Remarks.

(i) We could unify (a) and (b) by thinking of an artificial $\mathcal{C}_0 = \{0\}$ with mass k.

(ii) For the above construction to make sense, we need $y_3 < y_{u_1}$ in (a), and $y_{j+3} < y_{u_1}$ in (b) above. This is fine under our assumption, as we see from the upper bound of L^{k-1} for the cardinality of any good block of $\mathbf{H}_{k-1}$, as stated in point 6.

(iii) Since $d(\mathcal{C}_{i+1}, \mathcal{C}_i) \geq L^k$ one sees that $b \geq 3$. This implies the existence of at least two good k-blocks of type 2 contained in the interval $[\max \mathcal{C}_i + 1, \min \mathcal{C}_{i+1} - 1]$. Indeed all the selected points $y_{u_s} \in \mathcal{Y}$ are distinct i.e. $j + 3 < u_1 < u_2 < \cdots < u_{b-1}$. This follows from the same upper bound used in (ii) above. In particular, $y_j + s\frac{L^k}{3} \leq y_{u_s} \leq y_j + s\frac{L^k}{3} + L^{k-1} + 1 < y_j + (s+1)\frac{L^k}{3}$ for $1 \leq s \leq b-1$.

(iv) As defined above, when $m(\mathcal{C}_{i+1}) > k$, there is also a good type 2 k-block that ends at $\min \mathcal{C}_{i+1} - 1$. Otherwise the interval starting at $y_{u_{b-1}} + 1$ is incorporated as

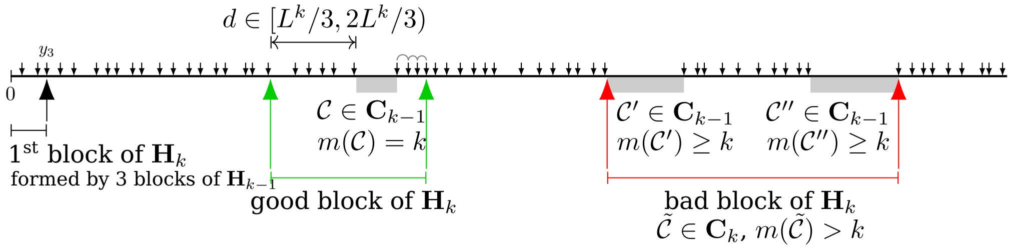

Figure 3: Illustration of the construction of $\mathbf{H}_k$. The small arrows indicate the right endpoints of the blocks in $\mathbf{H}_{k-1}$. The environment within the good blocks in $\mathbf{H}_{k-1}$ is not marked in this picture.

part of a good k block of type 1 with right endpoint y_{l+3}, where $y_l = \max \mathcal{C}_{i+1}$. Also, all $(k-1)$-blocks in between $y_j = \max \mathcal{C}_i$ and $\min \mathcal{C}_{i+1} - 1$ are good, and there are least $L/3$ of them, as we see at once from $d(\mathcal{C}_{i+1}, \mathcal{C}_i) \geq L^k$ and the upper bound for the cardinality of a good block in $\mathbf{H}_{k-1}$, as in (ii).

Now, writing $\mathcal{X} = \{x_1 < x_2 < \dots\}$ and setting $x_0 = -1$, the partition $\mathbf{H}_k$ is defined through the blocks (see Figure 3).

$$\mathcal{H}_i = (x_{i-1}, x_i] \cap \mathbb{Z}, i \geq 1. \tag{3.5}$$

It remains to check that $\mathbf{H}_k$ satisfies properties 1-6 listed before. For this we proceed by induction, assuming they hold up to $\mathbf{H}_{k-1}$

Properties 1 and 2 are immediate from the above construction. We now verify property 5. If $\mathcal{C} \in \mathbf{C}_k$ with $m(\mathcal{C}) > k$, we consider the two possibilities: when $\ell(\mathcal{C}) < k$, then by induction we see that $\mathrm{span}(\mathcal{C}) \in \mathbf{H}_{k-1}$ and the above construction shows that it is kept in $\mathbf{H}_k$. When $\ell(\mathcal{C}) = k$ the construction shows that $\mathrm{span}(\mathcal{C}) \in \mathbf{H}_k$. Property 3 is again an almost immediate consequence of the construction and property 2. It follows from property 5 in the case of a block $\mathcal{C} \in \mathbf{C}_k$ with $\ell(\mathcal{C}) = k$. On the other hand, if $\mathcal{C} \in \mathbf{C}_k$ and $\ell(\mathcal{C}) < k$, we have $\mathcal{C} \in \mathbf{C}_{k-1}$, so that by induction $\mathcal{C} \subset \mathcal{H}$ for some $\mathcal{H} \in \mathbf{H}_{k-1}$ and the conclusion follows due to property 2. We now verify property 4: when a block $\mathcal{H}$ in $\mathbf{H}_k$ contains $\mathcal{C} \in \mathbf{C}_k$ of $\ell(\mathcal{C}) = k$ then $\mathcal{H} = \mathrm{span}(\mathcal{C})$. The remaining case follows at once from the above construction, when examining the two consecutive blocks $\mathcal{C}_i$ ad $\mathcal{C}_{i+1}$ with $\min\{m(\mathcal{C}_i), m(\mathcal{C}_{i+1})\} \geq k$. It remains to check property 6. From the construction, and writing

$$A_k := \max\{|\mathcal{H}| \colon \mathcal{H} \in \mathbf{H}_k \text{ is a good } k\text{-block}\}, \tag{3.6}$$

we have

$$A_k \leq 2L^k/3 + \max\{\mathrm{diam}(\mathcal{C}) \colon \mathcal{C} \in \mathbf{C}_k, m(\mathcal{C}) = k\} + 1 + 3A_{k-1}, \tag{3.7}$$

from which we easily obtain $A_k \leq L^k$ for all $L \geq 36$. This concludes the proof of the properties 1,..., 6 at all scales.

We now extend Definition 3.2 to scales $k \geq 2$.

Definition 3.3. *Let $\mathcal{H}$ be a good k-block with $k \geq 2$ as defined above and let $\mathcal{Y} = \{y_1 < y_2 < \dots\}$ be the set of right endpoints of the blocks in $\mathbf{H}_{k-1}$. We define four subintervals $\mathcal{K}(\mathcal{H})$, $\mathcal{D}_{\mathcal{K}}(\mathcal{H})$, $\mathcal{D}(\mathcal{H})$ and $\mathcal{F}(\mathcal{H})$ of $\mathcal{H}$ as follows:*

(a) If $\mathcal{H}$ is good of type 1, which means that we may write $\mathcal{H} = (y_s, y_{t+3}] \cap \mathbb{Z}$, for suitable s and t, where $(y_{t-1}, y_t] \cap \mathbb{Z} = \mathrm{span}(\mathcal{C})$, $\mathcal{C}$ being the unique block in $\mathbf{C}_k$ contained in $\mathcal{H}$ with $m(\mathcal{C}) = k$, we set: $\mathcal{K}(\mathcal{H}) = (y_s, y_{t-1}] \cap \mathbb{Z}$, $\mathcal{D}_{\mathcal{K}}(\mathcal{H}) = (y_{t-2}, y_{t-1}] \cap \mathbb{Z}$, $\mathcal{D}(\mathcal{H}) = (y_{t+1}, y_{t+3}] \cap \mathbb{Z}$, and $\mathcal{F}(\mathcal{H}) = (y_s, y_{s+1}] \cap \mathbb{Z}$, i.e. $\mathcal{K}(\mathcal{H})$ denotes the part of the block that stays to the left of

$\mathcal{C}$, $\mathcal{D}_{\mathcal{K}}(\mathcal{H})$ the last $(k-1)$-block of $\mathcal{K}(\mathcal{H})$, $\mathcal{D}(\mathcal{H})$ the two rightmost $(k-1)$-blocks of $\mathcal{H}$, and $\mathcal{F}(\mathcal{H})$ the leftmost $(k-1)$-block in $\mathcal{H}$.

(b) If $\mathcal{H}$ is good of type 2, we set $\mathcal{K}(\mathcal{H}) = \mathcal{H}$, keep the same definition for $\mathcal{F}(\mathcal{H})$, we still call $\mathcal{D}(\mathcal{H})$ the interval that corresponds to the two rightmost $(k-1)$-blocks contained in $\mathcal{H}$, and $\mathcal{D}_{\mathcal{K}}(\mathcal{H})$ is the rightmost block of $\mathbf{H}_{k-1}$ contained in $\mathcal{H}$.

The lemma below is stated for convenience.

Lemma 3.4. *Let $L \geq 36$ and let γ be an environment with $\chi(\gamma) = 0$. The following properties hold for all $k \geq 1$: Let A_k be defined as in (3.6) and*

$$a_k := \min\{|\mathcal{H}| \colon \mathcal{H} \in \mathbf{H}_k \text{ is a good } k\text{-block}\}. \tag{3.8}$$

Then,

$$L^k/4 \leq a_k \leq A_k \leq L^k. \tag{3.9}$$

Proof. The upper bound for A_k has already been proved as property 6 above. We now prove the lower estimate in (3.9). Consider first the block immediately to the right of the one that contains the origin. In this case we get the lower bound $L^k/3 - 3L^{k-1} \geq L^k/4$ for all $L \geq 36$. The same estimate works for all other good k-blocks as easily seen from the construction and the upper bound for A_{k-1}. □

Remark 3.5. Recall that the block of $\mathbf{H}_k$ which contains the origin (call it $\mathcal{H}_1^k$) is exceptional and it has not been classified as good or bad. There would be no true harm in calling it good (of type 2). Its length is shorter, as for $k = 1$ we took it as $\{0, 1\}$ simply, and at each step the first k-layer is formed by the first three $k-1$-layers. In particular, under the same assumptions of Lemma 3.4 one has:

$$|\mathcal{H}_1^k| \leq 2\sum_{j=0}^{k-1} L^j \leq 3L^{k-1}. \tag{3.10}$$

Remark 3.6. Recalling the construction of $\mathbf{H}_k$ and considering two consecutive blocks $\mathcal{C}_i$ and $\mathcal{C}_{i+1}$ of $\mathbf{C}_k$ with mass at least k, it follows that the number of $(k-1)$-blocks contained in the interval $(\max \mathcal{C}_i, \min \mathcal{C}_{i+1})$ is at least $L/4$ under our assumptions.

The proposition below was indeed proven during the construction of the partitions $\mathbf{H}_k$. We state it here in order to facilitate its later usage.

Proposition 3.7. *(a) Let $\mathcal{H} \in \mathbf{H}_k$, with $\mathcal{K}(\mathcal{H})$ as in Definition 3.3. All the $(k-1)$-blocks contained in $\mathcal{K}(\mathcal{H})$ are good.*

(b) Let $\mathcal{H} \in \mathbf{H}_k$ be good of type 1 and let $\mathcal{C}$ be the unique block of $\mathbf{C}_k$ of mass k contained in $\mathcal{H}$. Then $\min \mathcal{C} - 1$ is the right endpoint of a good $(k-1)$-block of type 2 contained in $\mathcal{H}$.

3.2 Reverse partitions

The good k-blocks were defined with a clear orientation in mind. It will be relevant to consider an appropriate modification of layers that reverses the orientation. This will be denoted by $\widehat{\mathbf{H}}_k$ and will be used later in the paper when estimating crossing probabilities over bad layers. A sufficiently long good layer allows the percolation process to grow before facing a bad layer. We use this in both directions benefiting from the planarity of our graph. The reverse blocks are taken as close as possible to the $\mathbf{H}_k$ and also satisfy the same properties 1-6. The main difference comes from the k-blocks around a $\mathcal{C} \in \mathbf{C}_k$ with $m(\mathcal{C}) = k$, for all $k \geq 1$.

As before, and though not relevant we may set $\widehat{\mathbf{H}}_0 = \mathbf{H}_0$, the partition of $\mathbb{Z}_+$ into singletons.

Step 1. Let us recall the construction of $\mathbf{H}_1$. At this scale the only blocks that change when passing to $\widehat{\mathbf{H}}_1$ are those around each $\mathcal{C} \in \mathbf{C}_1$ with $m(\mathcal{C}) = 1$. For $\mathbf{H}_1$, they have the form $\mathcal{H}_i = (y, z+3] \cap \mathbb{Z}$ and $\mathcal{H}_{i+1} = (z+3, \bar{y}] \cap \mathbb{Z}$, for some i, where $\mathcal{C} = \{z\}$, and where $L/3 < z - y < 2L/3$ and $\bar{y} = z + L/3$. In the partition $\widehat{\mathbf{H}}_1$, we replace these two blocks by $\widehat{\mathcal{H}}_i = (y, z-4] \cap \mathbb{Z}$ and $\widehat{\mathcal{H}}_{i+1} = (z-4, \bar{y}] \cap \mathbb{Z}$. Each of the other blocks of $\widehat{\mathbf{H}}_1$ coincides with the corresponding one of $\mathbf{H}_1$; those that coincide with span$(\mathcal{C})$ for some $\mathcal{C} \in \mathbf{C}_1$ with $m(\mathcal{C}) > 1$ and all the other intermediate blocks.

Step k. Assume $\widehat{\mathbf{H}}_{k-1}$ already settled. First of all, the bad k-blocks coincide in both partitions. Following what was done for $\widehat{\mathbf{H}}_1$, we imitate the definition of $\mathbf{H}_k$ except that whenever we have a $\mathcal{C}$ in $\mathbf{C}_k$ with $m(\mathcal{C}) = k$ the "approximate endpoint" stay now to the left of $\mathcal{C}$, and to the right of $\mathcal{C}$ the block extends for at least $L^k/3$. For each such $\mathcal{C}$, we focus on the two blocks around $\mathcal{C}$. Continuing what was done before and for the partition $\mathbf{H}_k$, let $\tilde{\mathcal{C}}$ be the rightmost element of $\mathbf{C}_k$ to the left of $\mathcal{C}$, with $m(\tilde{\mathcal{C}}) \geq k$ (i.e. $\tilde{\mathcal{C}}$ and $\mathcal{C}$ are two consecutive blocks in $\mathbf{C}_k$ with mass at least k). Let $\widehat{\mathcal{Y}} = \{\widehat{y}_1 < \widehat{y}_2 < \dots\}$ be the right endpoints of the blocks in $\widehat{\mathbf{H}}_{k-1}$. Thus, $\max \mathcal{C} = \widehat{y}_v \in \widehat{\mathcal{Y}}$ and $\min \mathcal{C} - 1 = \widehat{y}_{v-1} \in \widehat{\mathcal{Y}}$ for some v. Similarly $\max \tilde{\mathcal{C}} \in \widehat{\mathcal{Y}}$. The rule for $\widehat{\mathbf{H}}_k$ follows the method indicated above: we set $\widehat{\mathcal{H}}_{i+1} = (\widehat{y}_{v-4}, \widehat{y}_u] \cap \mathbb{Z}$ where $\widehat{y}_u = \min\{x \in \widehat{\mathcal{Y}} \colon x \geq \max \mathcal{C} + L^k/3\}$, and $\widehat{\mathcal{H}}_i = (y, \widehat{y}_{v-4}] \cap \mathbb{Z}$, where $y = \min\{x \in \widehat{\mathcal{Y}}, \colon x \geq \max \tilde{\mathcal{C}} + (\bar{b}-1)L^k/3\}$, with $\bar{b} = \lfloor \frac{\min \mathcal{C} - \max \tilde{\mathcal{C}}}{L/3} \rfloor$. The label i is irrelevant, but we include it just to indicate the correspondence in the two partitions, i.e. in the direct partition $\mathcal{H}_i$ will be the layer of $\mathbf{H}_k$ that contains $\mathcal{C}$, while $\mathcal{H}_{i+1}$ is the one immediately to the right of $\mathcal{C}$.

The next lemma shows the relation between the partitions $\mathbf{H}_k$ and $\widehat{\mathbf{H}}_k$. Each interval in one of these partitions differs not too much from one (and only one) of the intervals in the other partition.

Lemma 3.8. *Write $\mathbf{H}_k = \{\mathcal{H}_i^k, i \geq 1\}$ and $\widehat{\mathbf{H}}_k = \{\widehat{\mathcal{H}}_i^k, i \geq 1\}$ with the intervals labelled in increasing order. There exists a universal constant C so that for all $k \geq 1$,*

$$|\mathcal{H}_i^k \triangle \widehat{\mathcal{H}}_i^k| \leq C L^{k-1}, \tag{3.11}$$

where $\triangle$ denotes symmetric difference and $|\cdot|$ the cardinality.

Proof. It follows from the construction that the symmetric difference is contained in the union of eight good blocks of $\mathbf{H}_{k-1} \cup \widehat{\mathbf{H}}_{k-1}$ and a bad $(k-1)$-block which coincides with span$(\mathcal{C})$, where $\mathcal{C} \in \mathbf{C}_k$ has mass k. The estimate then follows from (2.6) and the rightmost inequality in (3.9). □

Remark. Proposition 3.7 has a clear analogue (with the same proof) for reverse partitions.

4 Construction of renormalized lattices: sites

Recall that we work under the assumptions stated at the beginning of Section 3, with fixed L and γ (L-spaced environment), and the partitions $\mathbf{C}_k$ and $\mathbf{H}_k$ defined for such fixed L, γ. We now complete the construction of the renormalized lattices. For this, we also fix a small positive constant c to be made explicit later (depending on the parameter p_G). For the moment we simply assume:

$$1/c \in \mathbb{N} \text{ and } cL/2 \in \mathbb{N}. \tag{4.1}$$

Let $\mathbf{H}_k = \{\mathcal{H}_1^k, \mathcal{H}_2^k, \dots\}$, with the blocks $\mathcal{H}_i^k$ listed in increasing order. For later use, we write

$$H_i^k = \{(x, y) \in \widetilde{\mathbb{Z}}_+^2 \colon y \in \mathcal{H}_i^k\}, \tag{4.2}$$

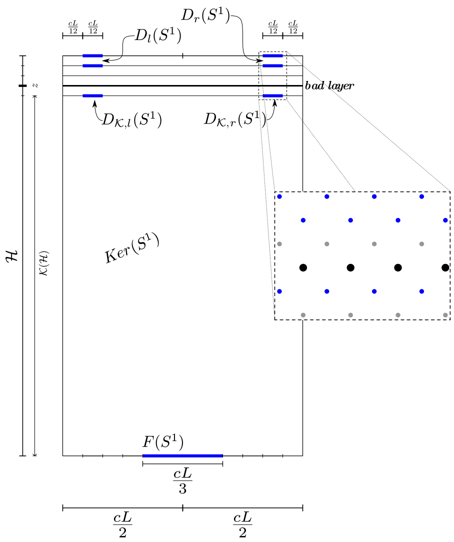

Figure 4: Illustration of a good 1-site of type 1.

for the corresponding horizontal layer defined by the k-block $\mathcal{H}_i^k$, also called k-layer. When the scale k is clearly understood, we may omit it from the notation.

We then define the renormalized k-sites $S_{u,v}^k$ as follows:

Step 0. $S_{u,v}^0 = (u,v)$, for $(u,v) \in \widetilde{\mathbb{Z}}_+^2$;

Step k. For $k \geq 1, (u,v) \in \widetilde{\mathbb{Z}}_+^2$, with $v \geq 2$

$$S_{u,v}^k = \left(\left(\frac{u-1}{2}(cL)^k, \frac{u+1}{2}(cL)^k \right] \times \mathcal{H}_v^k \right) \bigcap \widetilde{\mathbb{Z}}_+^2, \tag{4.3}$$

also called a k-site. (See Figure 4 for the case $k=1$.)

Remark. Since $\mathbf{H}_k$ is a partition of $\mathbb{Z}_+$, it is obvious that for any fixed $k \geq 1$, the sets $S_{u,v}^k$ just defined form a partition of $\widetilde{\mathbb{Z}}_+^2 \setminus H_1^k$, in the notation (4.2). We also see that for any $k \geq 2$, a k-site $S_{u,v}^k$ differs from the union of the $(k-1)$-sites that it contains. Defining

$$\mathring{S}_{u,v}^k := \bigcup S_{x,y}^{k-1}, \tag{4.4}$$

where the union runs over all (x,y) such that $S_{x,y}^{k-1} \subset S_{u,v}^k$, we have that $S_{u,v}^k \setminus \mathring{S}_{u,v}^k$ is contained in a strip of width $(cL)^{k-1}$ along the vertical boundary of the k-site. There is no true nuisance from the fact that the partitions in k-sites are not exactly coarser as k increases.

Definition 4.1. *A renormalized site $S_{u,v}^k$ with $v \geq 2$ is called good of type 1 (type 2) when the corresponding k-block $\mathcal{H}_v^k$ is good of type 1 (type 2 respectively).*

The next properties follow at once from the definition and the properties of $\mathbf{H}_k$:

i) the number of horizontal $(k-1)$-layers intersecting a good k-site $S_{u,v}^k$ obviously does not depend on u, and it is between $L/4$ and $8L$. A good k-site intersects at most one bad $(k-1)$-layer.

ii) for $k \geq 2$, the intersection of $S_{u,v}^k$ with a $(k-1)$-layer, if not empty, contains exactly cL $(k-1)$-sites. Each horizontal line of a 1-site $S_{u,v}^1$ contains $cL/2$ sites of $\widetilde{\mathbb{Z}}_+^2$.

Recall that

$$\mathcal{H}_1^k = \mathcal{H}_1^{k-1} \cup \mathcal{H}_2^{k-1} \cup \mathcal{H}_3^{k-1}.$$

Thus, by iterating this relation we see that for any $k \geq 1$, the layer H_1^k is the union of the layers $H_2^t \cup H_3^t$ for $1 \leq t \leq k-1$ and of $H_1^1 = \{(x,y) \in \widetilde{\mathbb{Z}}_+^2 \colon y \in \{0,1\}\}$.

Structure of good sites.

If $S_{u,v}^k$ is a good k-site we let

$$D_l(S_{u,v}^k) = \left[\left(\frac{u-1}{2} + \frac{1}{12}\right)(cL)^k, \left(\frac{u}{2} - \frac{1}{3}\right)(cL)^k\right] \times \mathcal{D}_v^k,$$
$$D_r(S_{u,v}^k) = \left[\left(\frac{u}{2} + \frac{1}{3}\right)(cL)^k, \left(\frac{u+1}{2} - \frac{1}{12}\right)(cL)^k\right] \times \mathcal{D}_v^k,$$

as well as

$$D_{\mathcal{K},l}(S_{u,v}^k) = \left[\left(\frac{u-1}{2} + \frac{1}{12}\right)(cL)^k, \left(\frac{u}{2} - \frac{1}{3}\right)(cL)^k\right] \times \mathcal{D}_{\mathcal{K},v}^k,$$
$$D_{\mathcal{K},r}(S_{u,v}^k) = \left[\left(\frac{u}{2} + \frac{1}{3}\right)(cL)^k, \left(\frac{u+1}{2} - \frac{1}{12}\right)(cL)^k\right] \times \mathcal{D}_{\mathcal{K},v}^k, \tag{4.5}$$

where $\mathcal{D}_v^k = \mathcal{D}(\mathcal{H}_v^k)$ and $\mathcal{D}_{\mathcal{K},v}^k = \mathcal{D}_{\mathcal{K}}(\mathcal{H}_v^k)$ (see Definition 3.3).

We next define

$$Ker(S_{u,v}^k) = S_{u,v}^k \cap (\mathbb{Z} \times \mathcal{K}(\mathcal{H}_v^k)). \tag{4.6}$$

This set is called the *kernel of* $S_{u,v}^k$. Note that $Ker(S_{u,v}^k)$ equals $S_{u,v}^k$ if this site is good of type 2, but is a strict subset if $S_{u,v}^k$ is good of type 1. If $S_{u,v}^k$ is a good k-site of type 1, then its projection on the vertical axis, $\mathcal{H}_v^k$, contains exactly one block $\mathcal{C}$ of $\mathbf{C}_k$ of mass k (and none of mass greater than k). The kernel is then the portion of $S_{u,v}^k$ that stays just below $\mathbb{Z} \times \mathcal{C}$. Thus, if H_v^k is a good layer and $k \geq 1$, then

$$\text{top line of } Ker(S_{u,v}^k) = \text{top line of } \mathcal{D}_{\mathcal{K},v}^k = \text{ top line of } D_{\mathcal{K},\vartheta}(S_{u,v}^k), \tag{4.7}$$

for $\vartheta = l$ or r, coinciding with the intersection of each of these sets with the horizontal line $\{(x,y)\colon x \in \mathbb{Z}\}$, where y is the right endpoint of $\mathcal{H}_v^k$.

Finally,

$$F(S_{u,v}^k) = [(u/2 - 1/6)(cL)^k, (u/2 + 1/6)(cL)^k] \times \mathcal{F}(\mathcal{H}_v^k), \tag{4.8}$$

which is, roughly speaking, the middle third of the lowest $(k-1)$-layer in $S_{u,v}^k$. The $(k-1)$-sites contained in $F(S_{u,v}^k)$ are said to be *centrally located* in $S_{u,v}^k$. (See Figure 5 for a schematic illustration of a good k-site $S^k = S_{u,v}^k$ where the regions $F(S^k)$, $D_{\mathcal{K},l}(S^k)$, $D_{\mathcal{K},r}(S^k)$, $D_l(S^k)$, $D_r(S^k)$ and $Ker(S^k)$ are all marked. Further details in the picture will play a role in the next section.)

The reversed sites $\widehat{S}_{u,v}^k$ are defined as in (4.3), with $\mathcal{H}_v^k$ replaced by $\widehat{\mathcal{H}}_v^k$.

Remark 4.2. In view of Lemma 3.8, it becomes natural to write $\widehat{S}(S_{u,v}^k) := \widehat{S}_{u,v}^k$ as well as $S(\widehat{S}_{u,v}^k) := S_{u,v}^k$ for all (u,v).

Before proceeding to the next section, where we start examining the events that describe percolation, we briefly discuss the general motivation behind what has been done so far.

The goal of our renormalization procedure is to have a successive way to define the notion that corresponds to being *open* at all scales, and which we shall call *passable*. The crucial thing is that the existence of a *passable path* at any scale should imply the existence of open paths on the original lattice, at scale 0. Two general comments:

- For a fixed L-spaced environment γ, the definition of the renormalized layers $\mathbf{H}_k$ had the purpose of dividing the space in a way that for an oriented path moving upwards there is enough space before it meets the bad layer of the previous scale.

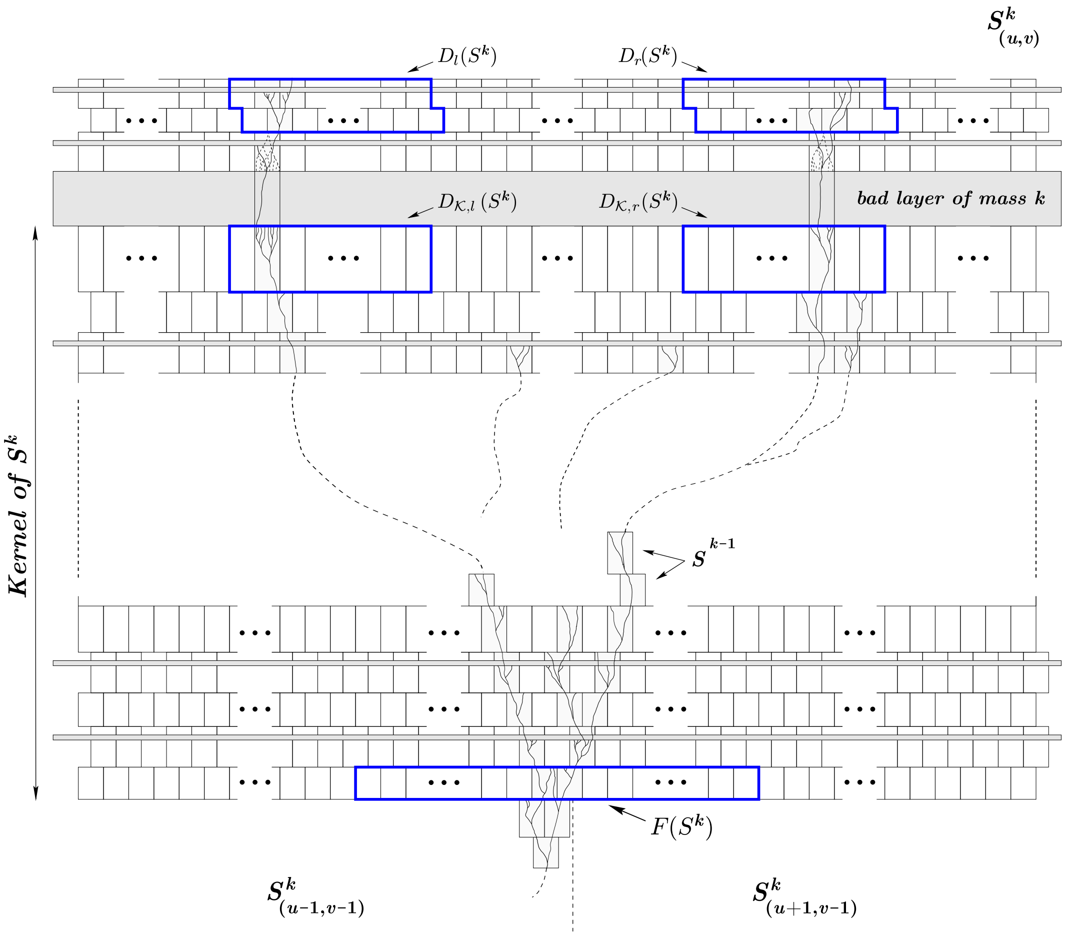

Figure 5: A good k-site with marked regions and illustration of the procedure. On the top part of the picture we see a very schematic representation of matching pairs and of the event related to property (b_k) in the statement of Theorem 5.15. It is properly constructed in Section 6 and its probability under P^γ (γ being L-spaced) is estimated in Section 7. In order to mark the special regions $D_{\mathcal{K},l}, D_{\mathcal{K},r}$, etc., the picture was drawn much wider than the true k-site really is; the ratio between length and width of a good k-site quickly increases with k since $c < 1$ is kept fixed in our argument.

- When meeting the bad layer, many of the available paths should fail to percolate. On the other hand, we need not only to find a successful path, but we need that such path should connect back to a percolative path at scale k. It is at this point that we look at the other side of the bad layer and consider the reverse site that follows it. Being on a planar graph will be crucial for this strategy. (See detail on the upper left part of Figure 5.)

For this approach to work, besides assuming that L is sufficiently large (depending on the two parameters p_G, p_B), we need to have two things:

(a) The constant c in (4.1) should be related to the percolation model, so as to guarantee a good density of open paths within the renormalized site, just below the bad layer. Starting with $k = 1$ this is just the original percolation model within a spatial region where $p = p_G$, in this case the kernel of a good 1-site. Therefore we should relate c to the asymptotic shape of the oriented site percolation on $\widetilde{\mathbb{Z}}^2_+$ (assuming that L has been sufficiently large) so as to guarantee a good density of paths. As we move to larger scales the notion of a site being open is replaced by that of what we call *passable*; the corresponding probability should suitably increases with the scale and we may hope for the same behavior to be preserved at larger scales k with the same large L and c. We recall again that the relation between δ and L is given by Lemma 2.3 so that the event of L-spaced environments has positive probability (Lemma 2.8) and, as already mentioned, we work for such a fixed environment configuration.

(b) Considering the just mentioned strategy to deal with the crossing of a bad layer of previous scale (with paths from both directions), it will be convenient to make sure that there is a positive density of matching (on opposite sides of the bad layer) passable sites. This demands the asymptotic density of the percolation cluster in a totally good environment, $\rho(p_G)$, to be larger than $1/2$ as our main extra condition on p_G. As mentioned at the end of the Introduction, the extra conditions will be released in Section 8.

5 Passable sites. Probability estimates for large p_G

Having fixed the environment γ, which we assume to be L-spaced (for L suitably large) and the parameters p_G, p_B of our main theorem, i.e. $p_G > p_c > p_B > 0$, the first goal of this section is to set various notions that involve the percolation configuration, and which should be suitably defined at all scales. We then state probability estimates which, once verified at all scales, will imply:

$$\mathbb{P}(C_0 \text{ is infinite}|\gamma) > 0. \tag{5.1}$$

Notation: Recall that P^γ will be used for the conditional probability given $\{\Gamma = \gamma\}$, i.e. $\mathbb{P}(\cdot|\gamma)$.

The probability estimates will involve a multiscale argument which is simpler if $p_G > p^*$ for a suitable $p^* < 1$ and $p_B > 0$. Under these conditions, (5.1) will be proven provided $L \geq L_1 = L_1(p_G, p_B)$. This will be the content of Theorem 5.15. Its proof will take most of our efforts during this and the next two sections. The extension to all $p_G > p_c, p_B > 0$ (and $L \geq L_0(p_G, p_B)$) will be treated in Section 8.

Before setting all these notions mentioned above, we make the following assumptions: Let P_p denote the homogeneous Bernoulli oriented site percolation model on $\widetilde{Z}^2_+$, and let $\theta(p)$ denote its percolation probability,

$$\theta(p) := P_p(C_0 \text{ is infinite}), \tag{5.2}$$

so that $p_c = \inf\{p \in (0,1) \colon \theta(p) > 0\}$ and $\theta(p) \uparrow 1$ as $p \uparrow 1$. For $p > p_c$ we also write $s(p) \in (0,1]$ for the asymptotic edge speed for P_p, as defined in Sec. 3 in [8], where it is denoted by $\alpha(p)$. [2]

Assumption 5.1. *i) We first take p_G large enough so that $\theta(p_G) > 1/2$, and ρ will be some fixed number in the interval $(1/2, \theta(p_G))$.*

ii) Regarding the constant c in (4.1)*, we add:*

$$c < \frac{3}{14} s(p_G). \tag{5.3}$$

5.1 Definitions: s-passable and c-passable sites. Rooted seed

We remind the reader that $\widetilde{\mathbb{Z}}^2_+$ is oriented upwards in the second coordinate. We shall therefore say that A is connected to B by an open path π only if π is an open path which

[2] In [8], the author considers oriented bond percolation, but the same definition and basic properties apply to the case of supercritical oriented site percolation.

respects the orientation and with initial and endpoint in A and B, respectively. We call such a path simply an *open path from A to B*. As in (4.7), for $A \subset \widetilde{\mathbb{Z}}_+^2$, we call *top line of* A the subset $\{(x,y) \in A : y = y_0\}$, where y_0 has the maximal value for which this subset is nonempty. If y_0 takes the smallest value for which $\{(x,y) \in A : y = y_0\}$ is non empty, then we call this subset the *bottom line* of A. Note that these are not complete lines, not even intervals, in general.

Step 0. A 0-site is called *s-passable* if and only it is open.

Rooted 0-seed. The rooted 0-seed $Q_{u,v}^{(0)}$, with root at (u,v), is the set of three open 0-sites in $\widetilde{\mathbb{Z}}_+^2$:

$$Q^{(0)} = Q_{u,v}^{(0)} = \{(u,v), (u+1, v+1), (u-1, v+1)\}.$$

The site (u,v) is called the *root* of $Q_{u,v}^{(0)}$, and we write $R(Q^{(0)}) = \{(u,v)\}$; the sites $(u-1,v+1)$ and $(u+1,v+1)$ are called the *active sites* of $Q^{(0)}$, and we set $A(Q^{(0)}) = \{(u-1,v+1),\ (u+1,v+1)\}$. (When the location of the seed is not important we will suppress the subscript.)

Open cluster of a rooted 0-seed. (a) The open cluster of a rooted 0-seed $Q^{(0)} = Q_{u,v}^{(0)}$ is the collection of 0-sites w for which there exists an open path of 0-sites from $A(Q^{(0)})$ to w. It is denoted by $U(Q^{(0)})$.

(b) We also need the definition of open cluster of a 0-seed $Q^{(0)}$ restricted to the kernel of a 1-site S^1 located in such a way that all 0-sites of $A(Q^{(0)})$ are below and adjacent to $F(S^1)$. (See Figure 4.) This is simply the collection of good 0-sites w for which there exists an open path of (good) 0-sites entirely contained in S^1, from a 0-site adjacent to $A(Q^{(0)})$ to w.

We next define open clusters, passability and rooted seeds for a general $k \geq 1$. These definitions have to be used in sequence. Having defined a rooted 0-seed and open cluster of a 0-site, we can define passability of a good 1-site, then a rooted 1-seed and the open cluster of a 1-site; next passability of a good 2-site and a rooted 2-seed, and so on.

s-Passable k-site. A good k-site S^k is said to be *s-passable from a rooted $(k-1)$-seed* $Q^{(k-1)}$ if the following conditions (s1), (s2) and (s3) are satisfied:

- (s1) All 0-sites of $A(Q^{(k-1)})$ are below and adjacent to the middle third of the bottom layer of S^k, i.e., adjacent to $F(S^k)$ (see (4.8) for the definition of $F(S^k)$).
- (s2) There exist two rooted $(k-1)$-seeds, $\widetilde{Q}_l^{(k-1)}$ and $\widetilde{Q}_r^{(k-1)}$ say, such that their top lines are contained in the top line of $D_l(S^k)$ and the top line of $D_r(S^k)$, respectively, and such that there exist open [3] paths of 0-sites, entirely contained in S^k, from 0-sites adjacent to $A(Q^{(k-1)})$ to $R(\widetilde{Q}_l^{(k-1)})$ and to $R(\widetilde{Q}_r^{(k-1)})$.
- (s3) If $k \geq 2$, the number of $(k-1)$-sites in the open cluster of $Q^{(k-1)}$ restricted to $Ker(S^k)$ that lie on each of $D_{\mathcal{K},l}(S^k)$ and $D_{\mathcal{K},r}(S^k)$ is not smaller than $\rho cL/12$ (see (4.5) for the definition). When $k = 1$ we have a similar condition, but $\rho cL/12$ is replaced by $\rho cL/24$.

Definition 5.2. *In the previous setup, we say that S^k has s-dense kernel from $Q^{(k-1)}$ if the above condition (s3) holds.*

Notation. We shall denote the leftmost rooted $(k-1)$-seed which fulfills the requirements for $\widetilde{Q}_l^{(k-1)}$ in (s2) as $Q_l(S^k)$. Similarly $Q_r(S^k)$ denotes the rightmost rooted $(k-1)$-seed which fulfills the requirements for $\widetilde{Q}_r^{(k-1)}$. We further define $A(S^k) = A(Q_l(S^k)) \cup A(Q_r(S^k))$ and call the sites in this set the *active sites of* S^k. Note that in these definitions $Q_l(S^k), Q_r(S^k)$ and $A(S^k)$ also depend on $Q^{(k-1)}$, even though the notation

[3] always oriented

does not indicate this. However, in the definition of the open cluster of a rooted k seed we shall use the more explicit notation $Q_\vartheta(S^k, Q^{(k-1)})$ with $\vartheta = l$ or r to indicate this dependence.

Rooted k-seed. A rooted k-seed is formed by a rooted $(k-1)$-seed $Q^{(k-1)}$ and three good k-sites

$$S^k_{u,v}, S^k_{u-1,v+1} \text{ and } S^k_{u+1,v+1},$$

such that

- (i) $S^k_{u,v}$ is s-passable from $Q^{(k-1)}$,
- (ii) $S^k_{u-1,v+1}$ and $S^k_{u+1,v+1}$ are passable from $Q_l(S^k_{u,v})$ and $Q_r(S^k_{u,v})$, respectively.

The corresponding k-seed is denoted by

$$Q^{(k)} = S^k_{u,v} \cup S^k_{u-1,v+1} \cup S^k_{u+1,v+1} \cup Q^{(k-1)}. \tag{5.4}$$

We set

$$\begin{aligned} R(Q^{(k)}) = & \quad R(Q^{(k-1)}), \\ A(Q^{(k)}) = & \quad A(Q_l(S^k_{u-1,v+1})) \cup A(Q_r(S^k_{u-1,v+1})) \\ & \cup A(Q_l(S^k_{u+1,v+1})) \cup A(Q_r(S^k_{u+1,v+1})). \end{aligned}$$

The 0-site $R(Q^{(k)})$ is called the root of $Q^{(k)}$; the sites in $A(Q^{(k)})$ are called the active sites of $Q^{(k)}$.

Remark. We point out that the locations of $D_l(S^k_{u,v})$ and $D_r(S^k_{u,v})$ are such that the definition of a rooted k-seed makes sense. Specifically, the top line of $D_l(S^k_{u,v})$ is adjacent to and just below $F(S^k_{u-1,v+1})$ and so, if $S^k_{u,v}$ is s-passable, then also $Q_l(S^k_{u,v})$ is adjacent to and just below $F(S^k_{u-1,v+1})$. Thus, it makes sense to speak of s-passability of $S^k_{u-1,v+1}$ from $Q_l(S^k_{u,v})$. Similar statements hold for $S^k_{u+1,v+1}$ and $Q_r(S^k_{u,v})$.

Remark. Note that in the definition of a rooted 0-seed we required the three 0-sites which make up the seed to be open. Starting from this fact we deduce the following lemma.

Lemma 5.3. *In a rooted k-seed $Q^{(k)}$ there exists, for each $x \in A(Q^{k)})$, an open oriented path of 0-sites lying in $Q^{(k)}$, and going from $R(Q^{(k)})$ to x.*

Proof. We use a proof by induction on k. For $k = 0$ the conclusion of the lemma is obvious. For the induction step, let $k \geq 1$ and assume that the conclusion of the lemma with k replaced by $k-1$ has already been proven. Let further $Q^{(k)} = S^k_{u,v} \cup S^k_{u-1,v+1} \cup S^k_{u+1,v+1} \cup Q^{(k-1)}$ be a rooted k-seed and let $x \in A(Q_l(S^k_{u-1,v+1}))$. The other possible locations for x in $A(Q^{(k)})$ can be handled in the same way. Then there exist open paths of 0-sites π_i as follows:

π_1 from $y := R(Q_l(S^k_{u-1,v+1}))$ to x (by the induction hypothesis);

π_2 from some point z in $A(Q_l(S^k_{u,v}))$ to y (because $S^k_{u-1,v+1}$ is s-passable from $Q_l(S^k_{u,v})$);

π_3 from $w := R(Q_l(S^k_{u,v}))$ to z (by the induction hypothesis again);

π_4 from some point a in $A(Q^{(k-1)})$ to w (because $S^k_{u,v}$ is s-passable from $Q^{(k-1)}$);

π_5 from $R(Q^{(k-1)})$ to a (by the induction hypothesis once more).

Now concatenation of the paths $\pi_5, \pi_4, \ldots, \pi_1$ gives an open path of 0-sites from $R(Q^{(k-1)})$ to x, as desired, noticing that all these paths can be taken entirely lying in $Q^{(k)}$. □

Remark. If the origin is connected to $R(Q^{(k-1)})$ by an open path, and S^k is s-passable from $Q^{(k-1)}$, it follows that the origin is connected by an open path of 0-sites to all sites in $A(S^k)$.

Open cluster of a rooted k-seed with $k \geq 1$. (a) The open cluster of a rooted k-seed $Q^{(k)} = S^k_{u,v} \cup S^k_{u-1,v+1} \cup S^k_{u+1,v+1} \cup Q^{(k-1)}$ as in (5.4) is defined as the collection of k-sites consisting of $S^k_{u,v}, S^k_{u-1,v+1}, S^k_{u+1,v+1}$ and the k-sites S for which there exists a sequence $S(1), \dots, S(n)$ of k-sites with the following properties:

$$\text{each } S(j) \text{ is good,} \tag{5.5}$$

$$S(n) = S, \tag{5.6}$$

$$\text{for } 0 \leq j \leq n, S(j) \text{ is s-passable from a rooted } (k-1)\text{-seed } \widetilde{Q}(j-1), \tag{5.7}$$

$$\text{where, in the notation of the remark following condition (s3),} \tag{5.8}$$

$$\widetilde{Q}(j-1) = Q^{(k-1)}_{\vartheta(j-1)}\big(S(j-1), Q^{(k-1)}_{\vartheta(j-2)}(S(j-2)\big). \tag{5.9}$$

Here $\vartheta(i)$ can be l or r, independently of each other, and $S(-1) = S^k_{u,v}$, and $S(0) = S^k_{u+\phi,v+1}$ with $\phi = -1$ if $\vartheta(0) = l$ and $\phi = +1$ if $\vartheta(0) = r$. Also, $\widetilde{Q}(-1) = Q^{(k-1)}$.

(b) We define the *open cluster of the rooted k-seed $Q^{(k)}$ restricted to $Ker(S^{k+1})$* as in (a), but now with the added restriction that all $S(j), 1 \leq j \leq n$, are contained in $Ker(S^{k+1}) \cap \mathring{S}^{k+1}$ (see (4.4) and (4.6)), and we remove $Q^{(k)}$ from this restricted cluster.[4]

c-Passable k-site. A good 0-site is said to be *c-passable* if it is open. For $k \geq 1$, a good k-site S^k is said to be *c-passable* if:

(c1) There exist two rooted $(k-1)$-seeds, $\widetilde{Q}_l^{(k-1)}$ and $\widetilde{Q}_r^{(k-1)}$ say, such that their top lines are contained in the top line of $D_l(S^k)$ and the top line of $D_r(S^k)$, respectively, and such that there exist open oriented paths of 0-sites, entirely contained in S^k, from the lowest 0-level layer of $F(S^k)$ to $R(\widetilde{Q}_l^{(k-1)})$ and to $R(\widetilde{Q}_r^{(k-1)})$.

(c2) If $k \geq 2$, the number of $(k-1)$-sites in the open cluster of $F(S^k)$ restricted to $Ker(S^k)$ that lie on each of $D_{\mathcal{K},l}(S^k)$ and $D_{\mathcal{K},r}(S^k)$ is not smaller than $\rho cL/12$ (see (4.5)). When $k = 1$ we have a similar condition, but $\rho cL/12$ is replaced by $\rho cL/24$.

Definition 5.4. *In the above setup, we say that S^k has c-dense kernel if condition (c2) holds.*

Remark 5.5. Taking into account the reversed partition, we analogously define the notions of $\hat{c}$- and $\hat{s}$-passable sites.

Lemma 5.6. *Let $k \geq 1$. If a good k-site S^k has an s-dense kernel from a rooted seed $Q^{(k-1)}$, then $A(Q^{(k-1)})$ is connected by open paths of 0-sites to at least $n(k) := \lceil \rho cL/24 \rceil \lceil \rho cL/12 \rceil^{k-1}$ 0-sites in the top line of $D_{\mathcal{K},\vartheta}(S^k)$, for $\vartheta = l$ and $\vartheta = r$, i.e. to the left and to the right of the middle third of the top line of $Ker(S^k)$. Except for each initial site (in $A(Q^{(k-1)})$), these open paths are contained in $Ker(S^k)$.*

Proof. The proof goes by induction on k. Start with $k = 1$. If S^1 has an s-dense kernel from the 0-rooted seed $Q^{(0)}$, then there are at least $n(1)$ 0-sites in the open cluster of $Q^{(0)}$ restricted to $Ker(S^1)$ which belong to $D_{\mathcal{K},\vartheta}(S^1)$, for $\vartheta = l, r$. Each such 0-site is just a vertex $w \in D_{\mathcal{K},\vartheta}(S^1)$ for which there is a path in $Ker(S^1)$ of open good 0-sites starting at a site adjacent to $A(Q^{(0)})$ and ending at w. Moreover, such w automatically stays in the top line of $Ker(S^1_{u,v})$, because (since $k = 1$) the cardinality of $\mathcal{D}_{\mathcal{K}}(\mathcal{H}^1_v)$ equals 1 in this

[4] This does not affect the construction. It is just for consistency with the case $k = 0$.

case. Thus, for $k=1$, the conclusion of the lemma is immediate from the definitions of an s-dense kernel and of the open cluster of $Q^{(0)}$.

Now assume that the lemma has already been proven for k replaced by $k-1$. Assume further that S^k has an s-dense kernel from the rooted seed $Q^{(k-1)} = S^{k-1}_{u,v} \cup S^{k-1}_{u-1,v+1} \cup S^{k-1}_{u+1,v+1} \cup Q^{(k-2)}$. Let $\widetilde{S}^{k-1}$ be a $(k-1)$-site which belongs to the open cluster of $Q^{(k-1)}$. Further, for the sake of argument, let $\widetilde{S}^{k-1}$ lie in $D_{\mathcal{K},l}(S^k)$. Then there exists some n and sequences $S^{k-1}(0),\dots,S^{k-1}(n)$ and $\vartheta(0),\dots,\vartheta(n)$ such that (5.5)-(5.9) with k replaced by $k-1$ and S by $\widetilde{S}^{k-1}$ hold. In particular, $S^{k-1}(j)$ is passable from the rooted $(k-2)$-seed $\widetilde{Q}(j-1) = Q^{(k-2)}_{\vartheta(j-1)}(S^{k-1}(j-1), Q^{(k-2)}_{\vartheta(j-2)}(S^{k-1}(j-2))$. Also, part of the definition of s-passability gives that the top line of $\widetilde{Q}(j-1)$ will be equal to the top line of $S^{k-1}(j-1)$. A simple induction argument (with respect to j), similar to the proof of Lemma 5.3, then shows that there exists a path of open 0-sites in $Ker(S^k)$ from a site adjacent to $A(Q^{(k-1)})$ to $R(\widetilde{Q}(j))$. For $j=n$, an application of Lemma 5.3 then shows that for each vertex x in $A(\widetilde{Q}(n))$ there exists an open path of 0-sites from $A(Q^{(k-1)})$ to x. Since S^k has a dense kernel there are at least $\lceil \rho c L/12 \rceil$ choices for $\widetilde{S}^{k-1}$ which are contained in $D_{\mathcal{K},l}(S^k)$ (respectively in $D_{\mathcal{K},r}(S^k)$). Since different $(k-1)$-sites are disjoint, there are at least $\lceil \rho c L/12 \rceil$ disjoint choices for $\widetilde{S}^{k-1}$ in each of $D_{\mathcal{K},l}(S^k)$ and $D_{\mathcal{K},r}(S^k)$. Moreover, if $\widetilde{S}^{k-1} = S^{k-1}(n)$ is any fixed one of the possible choices, then $\widetilde{S}^{k-1}$ is s-passable from a rooted seed $\widetilde{Q}(n-1)$, as we just showed. By the induction hypothesis, there exist at least $n(k-1)$ 0-sites y in the top line of $Ker(\widetilde{S}^{k-1})$, with the property that there exists an open path in $Ker(\widetilde{S}^{k-1})$ from some x adjacent to $A(\widetilde{Q}(n-1))$ to y. Such a connection can be concatenated with the connection from $A(Q^{(k-1)})$ to x, to obtain an open path in $Ker(S^k)$ from a vertex adjacent to $A(Q^{(k-1)})$ to y. But then there at least $n(k-1)$ choices for y in each possible $\widetilde{S}^{k-1}$ and $\rho c L/12$ choices for $\widetilde{S}^{k-1}$. In total this gives at least $n(k)$ 0-sites with the required open connection from $R(A^{(k-1)})$.

The 0-sites y constructed in the preceding paragraph lie in the top line of $Ker(\widetilde{S}^{k-1})$ for some $\widetilde{S}^{k-1}$, which itself lies in $D_{\mathcal{K},l}(S^k) \cup D_{\mathcal{K},r}(S^k)$. It remains to show that these y lie in the top line of $Ker(S^k)$ itself. However, as recalled in Proposition 3.7,

$$\text{each of the possible } \widetilde{S}^{k-1} \text{ is a good } (k-1)\text{-site of type 2.} \tag{5.10}$$

As observed right after the definition (4.6), the validity of (5.10) implies $Ker(\widetilde{S}^{k-1}) = \widetilde{S}^{k-1}$. Thus (5.10) implies that the possible y lie in the top lines of the possible $\widetilde{S}^{k-1}$ and these latter top lines are contained in the top line of $Ker(S^k)$. This is so because, by construction, the projections of the sets $\widetilde{S}^{k-1}$ and $D_{\mathcal{K},\vartheta}(S^k)$, for $\vartheta = l$ or r, on the vertical axis coincide with a same interval $\mathcal{H} \in \mathbf{H}_{k-1}$. The induction step easily follows. □

Notation. For $\mathcal{C} \subset \mathbb{Z}_+$ we write

$$B(\mathcal{C}) = \{(x,y) \in \widetilde{\mathbb{Z}}^2_+ : y \in \mathrm{span}(\mathcal{C})\}. \tag{5.11}$$

Definition 5.7. *(Matching pair) Let $\mathcal{C} \in \mathbf{C}_\ell$ be such that $\ell(\mathcal{C}) = \ell$ and $m(\mathcal{C}) = m$. Since $\mathcal{C}$ is formed at level ℓ there are two blocks $\mathcal{C}_1$ and $\mathcal{C}_2$ in $\mathbf{C}_{\ell-1}$ with $m(\mathcal{C}_i) \ge \ell$, $i=1,2$, which are respectively the leftmost and rightmost constituents of $\mathcal{C}$. Thus $\mathrm{span}(\mathcal{C}) = [\min \mathcal{C}_1, \max \mathcal{C}_2] \cap \mathbb{Z}$. In particular, for each $k \in \{\ell-1,\dots,m-1\}$, there is a k-block $\mathcal{H}^k_{v-1}$ with $\max \mathcal{H}^k_{v-1} = \min \mathcal{C}_1 - 1$ and a k-block $\widehat{\mathcal{H}}^k_{v'+1} \in \widehat{\mathbf{H}}_k$ so that $\min \widehat{\mathcal{H}}^k_{v'+1} = \max \mathcal{C}_2 + 1$. Note that $\mathcal{H}^k_{v-1}$ and $\widehat{\mathcal{H}}^k_{v'+1}$ are good k-blocks (of type 2). We then say that two k-sites $S^k_{(u,v-1)}$, $\widehat{S}^k_{(u',v'+1)}$ form a matching pair with respect to $B(\mathcal{C})$ if either $u' = u$ or $u' = u \pm 1$, according as $v' - v$ is even or odd.*

Remark. For $k = \ell - 1$, $\mathcal{H}^k_v = \mathrm{span}(\mathcal{C}_1)$ and $\widehat{\mathcal{H}}^k_{v'} = \mathrm{span}(\mathcal{C}_2)$. On the other hand, for $k \in \{\ell,...,m-1\}$ we simply have $v = v'$, corresponding to the bad k-block that coincides with $\mathrm{span}(\mathcal{C})$. (Figure 5 contains an illustration of this last situation.)

5.2 Some classical estimates

Before formulating our basic set of estimates we state a number of properties of supercritical oriented site percolation on $\widetilde{\mathbb{Z}}^2_+$. We start with a simple observation which holds for any Bernoulli percolation as an immediate consequence of coupling.

Lemma 5.8. *Consider site percolation on a graph $\mathcal{G}$ (possibly partially oriented). Denote the probability measure under which all sites are independently open with probability p by P_p, and let $\mathcal{E}$ be some increasing event. If $p_0, p_0' \in [0,1]$ and $\tilde{p} = 1-(1-p_0)(1-p_0')$, then*

$$P_p\{\mathcal{E}\} \geq 1-(1-P_{p_0}\{\mathcal{E}\})(1-P_{p_0'}\{\mathcal{E}\}) \quad \textit{for all } p \geq \tilde{p}. \tag{5.12}$$

Now let us go back to oriented site percolation on $\widetilde{\mathbb{Z}}^2_+$ and let P_p be as in the preceding lemma. For $\mathcal{A} \subset \widetilde{\mathbb{Z}}^2_+$ define

$$\Theta(\mathcal{A}) = \{\text{all vertices in } \mathcal{A} \text{ are open}\} \tag{5.13}$$

and let $|\mathcal{A}|$ denotes the cardinality of $\mathcal{A}$.

Lemma 5.9. *There exists $\tilde{p} \in (0,1)$ and a universal constant $c_5 < \infty$ such that for all $p \geq \tilde{p}$ and all subsets $\mathcal{A}$ of $2\mathbb{Z} \times \{0\}$ it holds*

$$P_p\{\textit{there is an open path from } \mathcal{A} \textit{ to } \infty \big| \Theta(\mathcal{A})\} \geq 1 - c_5[9(1-p)]^{|\mathcal{A}|+1}. \tag{5.14}$$

This is a well-known argument in percolation. For sake of completeness we give the proof in the Appendix.

Again consider oriented site percolation on $\widetilde{\mathbb{Z}}^2_+$. Write $\mathbf{0}$ for the origin and define

$$\begin{aligned} r_n &:= \quad \sup\{x\colon (x,n) \in \widetilde{\mathbb{Z}}^2_+ \text{ and there exists an open path from } \mathbf{0} \text{ to } (x,n)\}, \\ \ell_n &:= \quad -\inf\{x\colon (x,n) \in \widetilde{\mathbb{Z}}^2_+ \text{ and there exists an open path from } \mathbf{0} \text{ to } (x,n)\}, \end{aligned} \tag{5.15}$$

$$r_n = \ell_n = 0 \text{ if there is no open path from } \mathbf{0} \text{ to } \mathbb{Z} \times n.$$

We further remind the reader that the percolation probability $\theta(p)$ was defined in (5.2). It is known (see [8]) that for all $p \geq \tilde{p} > p_c$ we have $\theta(p) \geq \theta(\tilde{p}) > 0$ and there exists $s(p) \in (0, +\infty)$ for which

$$\begin{gathered} \lim_{n\to\infty} \tfrac{1}{n} r_n = \lim_{n\to\infty} -\tfrac{1}{n}\ell_n = s(p) \text{ a.s. } [P_p] \text{ on the event} \\ \Omega_0 := \{\text{there exists an open path from } \mathbf{0} \text{ to } \infty\}; \end{gathered} \tag{5.16}$$

$s(p)$ is called the edge-speed (see [8] or [17]).

Finally we need the existence of a positive density in $[-ns(p), ns(p)] \times \{n\}$ of sites which have an open connection from a fixed finite nonempty set. The next lemma gives the precise meaning of this statement. We need the following definition: Let $\alpha \leq \beta$ and $\eta > 0$. Also let $\mathcal{A} = \{0, 2, 4, \dots 2a-2\}$ be some nonempty interval of a even integers. Then

$$\begin{aligned} \nu_n(\alpha,\beta) = \quad & \nu_n(\alpha,\beta,\mathcal{A},\eta) := \text{ number of points } (x,n) \text{ with } \alpha n \leq x \leq \beta n, \\ & x+n \text{ even, for which there exists an open path from} \\ & \mathcal{A}\times\{0\} \text{ to } (x,n) \text{ which stays inside } [-\eta n, \eta n] \times [0,n]. \end{aligned} \tag{5.17}$$

Lemma 5.10. *Let $0 < \varepsilon, \eta \leq 1$. There exists some $\bar{p} = \bar{p}(\varepsilon,\eta) < 1$ such that for $p \geq \bar{p}$ there exists an $n_0 = n_0(\varepsilon,\eta,p)$, such that for $n \geq n_0$ and $-s(p) \leq \alpha \leq \beta \leq s(p)$,*

$$P_p\left(\frac{1}{n}\nu_n(\alpha,\beta,\mathcal{A},\eta) \geq [\theta(p)(\beta-\alpha)-\varepsilon]\frac{\eta}{17} \textit{ for all } -s(p) \leq \alpha \leq \beta \leq s(p)\right) \geq 1-\varepsilon. \tag{5.18}$$

The proof of this lemma is shifted to the Appendix.

Corollary 5.11. *Let* $\Theta(\mathcal{A})$ *be as in* (5.13) *and*

$$\Omega(\mathcal{A}) := \{\textit{there is an open path from } \mathcal{A} \textit{ to infinity}\}.$$

Then under the conditions of Lemma 5.10, if n$\geq n_0$*, it holds for any finite set* $\mathcal{A} \subset \widetilde{\mathbb{Z}}^2_+$

$$P_p\left(\frac{1}{n}\nu_n(\alpha,\beta,\mathcal{A},\eta) \geq [\theta(p)(\beta-\alpha)-\varepsilon]\frac{\eta}{17} \textit{ for all } -s(p)\leq\alpha\leq\beta\leq s(p)\Big|\Theta(\mathcal{A})\right) \geq 1-\varepsilon. \quad (5.19)$$

and

$$P_p\left(\frac{1}{n}\nu_n(\alpha,\beta,\mathcal{A},\eta) \geq [\theta(p)(\beta-\alpha)-\varepsilon]\frac{\eta}{17} \textit{ for all } -s(p)\leq\alpha\leq\beta\leq s(p)\Big|\Omega(\mathcal{A})\right) \geq 1-\varepsilon. \quad (5.20)$$

Proof. Since $\Theta(\mathcal{A})$ and $\Omega(\mathcal{A})$ are increasing events of the environment, these inequalities are immediate from (5.18) and the Harris-FKG inequality. □

Comment: The previous lemma is used in order to achieve the dense kernel property when growing a cluster constrained to stay in a site. To understand it, it suffices to consider the first scale. Since the sites S^1 have dimension $cL/2\times L'$ where L' is determined by the length of the corresponding $\mathcal{H}^1$ block, and can vary between $L/4$ and $2L/3+4$, we need a restriction to stay within S^1 reaching the required density at the top line of $Ker(S^1)$. Thus, we apply the lemma for an initial growth (for a suitable fraction of the total height) and then release the restriction to reach the desired density ρ, using standard results in oriented percolation, for all suitably large L. Recall that we are using L as a parameter and then δ is taken suitably small, according to Lemma 2.3.

We now state and give a sketchy proof of another estimate that will be used in Section 7.

For $n\geq 1$ a large integer, consider the "rectangle" $R_n = ([0,n]\times[0,n^2])\cap\widetilde{\mathbb{Z}}^2_+$, let $\mathbf{x}=(x,k),\mathbf{y}=(y,n^2)\in R_n$ be such that $|x-n/2|,|y-n/2|\leq n/10$, and define the following event of vertical crossing:

$$V(\mathbf{x},\mathbf{y},R_n) = \{\text{there exists an open oriented path from } \mathbf{x} \text{ to } \mathbf{y} \text{ lying entirely in } R_n\}.$$

Lemma 5.12. *There exist* $n_0\geq 1, 0<\tilde{p}<1$ *and* $\varkappa'>0$*, such that for any* $n\geq n_0$ *and* $p\geq\tilde{p}$ *we have*

$$P_p(V(\mathbf{x},\mathbf{y},R_n)) \geq p^{\varkappa'}.$$

Proof. The proof of the above inequality is rather standard. We sketch it briefly. Let $a_n = \lfloor\frac{4n}{10}\rfloor$ and let A be the event that there is an open oriented path from (x,k) to the horizontal segment $\{(z,k+a_n)\in\widetilde{\mathbb{Z}}^2_+ : z\in[0,n]\}$ and that on this segment the open cluster from (x,k) is close to its asymptotic shape and asymptotic density. We can chose n_0, $\varkappa''$ and $\tilde{p}$ so that $P_p(A)\geq p^{\varkappa''}$ for all $p\geq\tilde{p}$ and all $n\geq n_0$. Similarly define B for a downward oriented cluster starting from (y,n^2) and going down to the segment $\{(z,n^2-a_n)\in\widetilde{\mathbb{Z}}^2_+ : z\in[0,n]\}$, so that $P_p(B)\geq p^{\varkappa''}$ for all $p\geq\tilde{p}$. If $k+a_n\geq n^2$ it follows at once from the planarity of the graph (concatenation of paths) that the occurrence of $A\cap B$ implies the existence of a path from (x,k) to (y,n^2) contained in R_n. Otherwise, if A occurs, we can restart, but now from order $\tilde{c}n$ points centrally located in the segment at height $k+a_n$ and repeat the argument for successive rectangles of height a_n, until we reach height at least n^2. The probability of succeeding in each of them will be bounded from below by $1-\exp(-c_1 n)$, with constant $c_1>0$, and uniformly bounded away from 0, for such p. Putting them together and using the concatenation of paths we construct an open path lying in R_n that goes from (x,k) to (y,n^2). The lemma follows for $\varkappa' > 2\varkappa''$. $\square$

Remark 5.13. It is obvious that by increasing $\varkappa'$ the above statement extends uniformly on $p \geq p'$, for any fixed $p' > p_c$.

5.3 Basic step

At this point we are ready to give a more detailed description of the inductive step. Recall that the environment will be a fixed γ with $\chi(\gamma)=0$.

As already explained, and seen from the construction made in Section 3, our option has been to use the same renormalization procedure for all scales. The authors believed this simplifies a bit the whole structure of the proof. On the other hand, it demands p_G to be larger than what is truly needed. We shall collect such needed assumptions in the main statement in this section (Theorem 5.15). In Section 8 we will discuss how to relax it to $p_G > p_c$. Thus, complementing Assumptions 5.1, we add

Assumption 5.14. *Assume $p_G > 2/3$ and that N is a fixed integer for which*

$$[3(1-p_G)]^{N/5-2} \leq 1/72.$$

From Assumption 5.14 we then have

$$8(1-p^3)^{N/5} \leq (1-p)^2, \text{ for all } p \geq p_G. \tag{5.21}$$

If $k \geq 1$ and S^k is a good k-site of type 1 we will be looking at a very particular way to obtain its s-passability from a given $(k-1)$-seed $Q_0^{(k-1)}$. It will turn out to be enough for Theorem 1.1 to consider the situation when S^k is good of type 1 and the bad $(k-1)$-sites contained in S^k lie in a layer $B(\mathcal{C})$, where $\mathcal{C}$ is a bad block with $\ell(\mathcal{C})=\ell<k$ and $m(\mathcal{C})=k$. Thus, the projection of $B(\mathcal{C})$ on the vertical axis equals $\mathcal{H}_v^{k-1}$ for some v. The top line of the kernel of S^k is contained in the line $y=\min(\mathcal{H}_v^{k-1})-1$. The $(k-1)$-sites with the same top line are those $(k-1)$-sites with projection onto the vertical axis equal to $\mathcal{H}_{v-1}^{k-1}$, i.e. of the form $S_{u,v-1}^{k-1}$ for some suitable u.

In each case, passability of S^k will be built from the occurrence of three events W_1^k, W_2^k and W_3^k which we define now. Further properties of these W_i^k will be given in Theorem 5.15 at the end of this section.

For $k \geq 1$

$$W_1^k(s) = W_1^k(s, S^k, Q_0^{(k-1)}) = \{S^k \text{ has } s\text{-dense kernel from a seed } Q_0^{(k-1)}\},$$

where $Q_0^{(k-1)}$ is a given rooted $(k-1)$-seed which fulfills condition (s1) for s-passability of S^k. If $W_1^k(s)$ occurs, then there exists for $\vartheta=l$ (left) and for $\vartheta=r$ (right) in $D_{\mathcal{K},\vartheta}(S^k)$ a collection $\mathcal{R}_\vartheta^{k-1}$ with at least[5] $\lceil \rho c L/12 \rceil$ $(k-1)$-sites S^{k-1} in the open cluster of $Q_0^{(k-1)}$ restricted to $Ker(S^k)$. Each of these is s-passable from some rooted $(k-2)$-seed. We remind the reader that this implies that each of these S^{k-1} in $\mathcal{R}_\vartheta^{k-1}$ contains for $\lambda=l$ and for $\lambda=r$ a rooted $(k-2)$-seed $Q_\lambda^{(k-2)} = Q_\lambda^{k-2}(S^{k-1})$ with top line contained in $D_{\mathcal{K},\lambda}(S^{k-1})$ for which there exists an open path of 0-sites in $Ker(S^k)$ from a site adjacent to $A(Q_0^{(k-1)})$ to $R(Q_\lambda^{(k-2)})$ (see the proof of Lemma 4.3). The union of the active sites of $Q_l^{(k-2)}(S^{k-1})$ and $Q_r^{(k-2)}(S^{k-1})$ is denoted by $A(S^{k-1})$.

For $k \geq 2$ the event $W_2^k(s)$ occurs if and only if $W_1^k(s)$ occurs and for $\vartheta=l$ and for $\vartheta=r$ there exist a collection $\mathcal{L}_\vartheta^{k-1}$ of $(k-1)$-sites with the following properties:

[5] For $k=1$ one needs to replace 12 by 24 here.

- (i) $\mathcal{L}_\vartheta^{k-1} \subset \mathcal{R}_\vartheta^{k-1}$ and the cardinality of $\mathcal{L}_\vartheta$ is at least N;
- (ii) for each $S^{k-1} = S_{u,v-1}^{k-1} \in \mathcal{L}_\theta$ there exists an index $\tilde{u}$ with $|\tilde{u} - u| \leq 2$ and a rooted $(k-2)$-seed $Q^{(k-2)}(\tilde{u})$ say, in $S_{\tilde{u},v+1}^{k-1}$ and with top line contained in the $D_l(S_{\tilde{u},v+1}^{k-1}) \cup D_r(S_{\tilde{u},v+1}^{k-1})$ and such that there is an open path inside S^k from $A(S^{k-1}))$ to $R(Q^{(k-2)}(\tilde{u}))$.

When $k=1$ we modify (ii) somewhat. Recall that a 0-site is just a vertex of $\widetilde{\mathbb{Z}}_+^2$. For $k=1$, $\mathcal{R}_\vartheta^0$ will just be taken as the collection of 0-sites $(u, v-1)$ in $D_{\mathcal{K},\theta}$ for which there exists an open path in $Ker(S^k)$ from a site adjacent to $A(Q_0^{k-1})$ to $(u, v-1)$. We then replace (ii) by

(ii, k=1) for each $S_{u,v-1}^0 = (u, v-1) \in \mathcal{L}_\theta$, there exists a $\tilde{u}$ with $|\tilde{u} - u| \leq 2$ such that there is an open path in S^1 from $(u, v-1)$ to $S_{\tilde{u},v+1}^0 = (\tilde{u}, v+1)$.

Finally, if $W_2^k(s)$ occurs, and $k \geq 2$, then $W_3^k(s)$ occurs if and only if there exist (at least) two rooted $(k-1)$-seeds in S^k, $Q_{l,1}^{k-1}$ with top line in $D_l(S^k)$ and $Q_{r,1}^{k-1}$ with top line in $D_r(S^k)$, and open connections of 0-sites in S^k from the collection of the rooted $(k-2)$-seeds $Q^{(k-2)}(\tilde{u})$ mentioned in (ii) above to $R(Q_{l,1}^{k-1})$ as well as to $R(Q_{r,1}^{k-1})$. When $k=1$ we merely replace the collection of rooted $(k-2)$-seeds $Q^{(k-2)}(\tilde{u})$ here by the collection of 0-sites $S_{\tilde{u},v+1}^0$ mentioned in (ii,k=1).

The definitions of the W_i^k are unfortunately quite involved due to the multi-scale argument. The reader should think of W_1^k as providing open connections from the middle third of the bottom of S^k to the top of its kernel; then W_2^k will provide open connections from the top of the kernel to the top of the bad layer, and finally W_3^k from the top of the bad layer to the top of S^k. The connections required for W_2^k from the bottom of the bad layer to its top are the most difficult to come by. They will be constructed in the next section.

With the definitions just described we now state

Theorem 5.15. *There exists $p^* < 1$ so that for all $p_G \geq p^*$, $p_B > 0$, one can find $L_1 = L_1(p_G, p_B)$ such that for every $L \geq L_1$, every $k \geq 1$, and good k-site of type 1 which intersects a bad layer $B(\mathcal{C})$ with $\mathcal{C} \in \mathbf{C}_\ell$ with $\ell(\mathcal{C}) = \ell$ and $m(\mathcal{C}) = k$ for some $\ell \leq k-1$, the following bounds hold for any environment γ with $\chi(\gamma) = 0$:*

(a) If $Q^{(k-1)}$ is a rooted $(k-1)$-seed and S^k a good k-site of type 1 which satisfy condition (s1) for an s-passable k-site, then

$$P^\gamma(W_1^k(s) \mid Q^{(k-1)} \text{ is a rooted } (k-1)\text{-seed }) \geq 1 - \frac{(1-p_G)^{k+1}}{4}. \tag{5.22}$$

(b)

$$P^\gamma(W_2^k | W_1^k) \geq 1 - \frac{(1-p_G)^{k+1}}{4}. \tag{5.23}$$

(c)

$$P^\gamma(W_3^k | W_2^k) \geq 1 - \frac{(1-p_G)^{k+1}}{4}. \tag{5.24}$$

(d)

$$\begin{aligned} &P^\gamma(S^k \text{ is } s\text{-passable from } Q^{(k-1)} \big| Q^{(k-1)} \text{is a rooted } (k-1) - \text{seed}) \\ &\quad \geq 1 - (1-p_G)^{k+1}, \\ &P^\gamma(S^k \text{ is } c\text{-passable}) \geq 1 - (1-p_G)^{k+1}. \end{aligned} \tag{5.25}$$

The following is an immediate consequence of Theorem 5.15.

Corollary 5.16. *Let p_B, p_G and $L \geq L_1(p_G, p_B)$ as in the statement of Theorem 5.15. Then*

$$P^\gamma(C_0 \textit{ is infinite}) > 0, \tag{5.26}$$

provided the environment γ is L-spaced.

Proof. One simply recursively uses the lower bounds for conditional probabilities in (5.25) and notice that their product over all $k \geq 1$ is positive. The conclusion follows immediately. □

Outline of the proof of Theorem 5.15. The proof goes by induction. For simplicity we write $(a_k), (b_k), (c_k)$ and (d_k) for the corresponding statement at level k. We shall prove $(a_1), (b_1)$ and (c_1) and the following implications:

$$(a_k), (b_k) \text{ and } (c_k) \text{ together } \Longrightarrow (d_k); \tag{5.27}$$

$$(d_k) \Longrightarrow (a_{k+1}) \text{ and } (c_{k+1}). \tag{5.28}$$

$$(a_j)_{j \leq k+1}, (b_j)_{j \leq k} \text{ and } (c_j)_{j \leq k} \Longrightarrow (b_{k+1}) \tag{5.29}$$

We first observe that (5.27) is immediate from the definitions. The most difficult and involved step is (5.29). This is indeed the core of the proof and will be concluded in Section 7. Before proving (5.28) we focus on the proof of $(a_1), (b_1)$ and (c_1).

For (a_1), assume for the sake of argument that $S^1 = S^1_{0,v} = \big((-cL/2, cL/2] \times \mathcal{H}^1_v\big) \cap \widetilde{\mathbb{Z}}^2_+$ (see (4.3)) with $Ker(S^1) \cap (\mathbb{Z} \times \mathcal{K}^1_v)$ (see (4.6)).

Since S^1 is assumed to be good of type 1, it intersects a unique bad layer, which indeed corresponds to a block in $\mathbf{C}_1$ of mass 1, i.e. a singleton. Let this be $\{z\}$ so that $\mathcal{H}^1_v = [\min \mathcal{H}^1, z+3] =: [y_0, z+3]$ and $\mathcal{K}^1_v = [y_0, z-1]$ and $|\mathcal{K}^1_v| \geq L/4$ for all L under consideration. Moreover all the lines that lie in $Ker(S^1)$ are good lines.

The condition that $Q^{(0)}$ is a rooted 0-seed gives us two adjacent open vertices $(x, y_0 - 1)$ and $(x+2, y_0-1)$ for some $x \in [-cL/6 - 1, cL/6 + 1]$. For $W^1(s, S^1, Q^{(0)})$ to hold it suffices to have open paths in $Ker(S^1)$ from sites adjacent to a site in $A(Q^{(0)}) = \{(x, y_0 - 1), (x+2, y_0 - 1)\}$ to at least $\lceil \rho cL/24 \rceil$ 0-sites in $D^{\mathcal{K}}_\vartheta$ for $\vartheta = l$ and $\vartheta = r$. In the simple case of $k = 1$ these are just open paths in S^1 to $([-\frac{5}{12}cL, -\frac{1}{3}cL] \times \{z-1\}) \cap \widetilde{\mathbb{Z}}^2_+$ (if $\vartheta = l$) and to $([\frac{1}{3}cL, \frac{5}{12}cL] \times \{z-1\}) \cap \widetilde{\mathbb{Z}}^2_+$. Thus (5.22) for $k=1$ can now be satisfied for large L by an application of Lemma 5.10 that guarantees positive density of the open oriented cluster restricted to S^1 at a suitable height proportional to L, and then using unrestricted growth to achieve density ρ in $D^{\mathcal{K}}_\vartheta$ for $\vartheta = l$ and $\vartheta = r$.

Next, (5.23) for $k = 1$ is easy. If $W^1_1(s)$ occurs, then for $\vartheta = l, r$ there exist sets $\mathcal{R}^0_\vartheta$ containing at least $\lceil \rho cL/24 \rceil$ open 0-sites which have an open connection from the origin. These sets are contained in the top line of $Ker(S^1)$, that is, in the horizontal line $\{y = z-1\}$. It is important that these sets $\mathcal{R}^0_\vartheta$ are determined by the occupation variables $\eta_{(a,b)}$ with $b \leq z-1$. For $W^1_2(s)$ to occur, there should be at least N (see (5.21)) sites $(a, z-1)$ in each of $\mathcal{R}^0_\vartheta, \vartheta = l, r$ which have an open connection to $(x, z+1)$ (which is on the line just above the bad line $\{y = z\}$) for some $x \in [a-2, a+2]$. But the cardinality of $\mathcal{R}^0_\vartheta$ is at least $\lceil \rho cL/24 \rceil$ and hence goes to infinity with L. Thus if we keep $0 < p_B, p_G$ and N fixed, then the conditional probability in the left hand side of (5.23) tends to 1 as $L \to \infty$. Indeed, given $W^0_1(s)$, the event that $(a, z-1)$ has an open connection to $[a-2, a+2] \times \{z+1\}$ has a strictly positive conditional probability, and these events for $a = a'$ and $a = a''$ are conditionally independent when $|a' - a''| \geq 5$. Thus, by raising L_1 if necessary, (5.23) for $k = 1$ follows.

We turn to (5.24) for $k = 1$. If $W^1_2(s)$ occurs, then let $\mathcal{L}^0_\vartheta$ be the subset of 0-sites $(a, z-1)$ in $\mathcal{R}^0_\vartheta$ which have an open connection to $(x, z+1)$ for some $x \in [a-2, a+2]$.

On the event $W_2^1(s)$ the cardinality of $\mathcal{L}_\vartheta^0$ is at least N. Denote by $\widetilde{\mathcal{L}}_\vartheta^0$ the collection of 0-sites $(x, z+1)$ with an open connection from some $(a, z-1) \in \mathcal{L}_\vartheta^0$, with separated values a as just mentioned. On the event $W_2^1(s)$, the cardinality of $\widetilde{\mathcal{L}}_\vartheta^0$ is at least $N/5$. Then $W_3^1(s)$ occurs if for $\vartheta = l$ as well as $\vartheta = r$, there is a $(x, z+1) \in \widetilde{\mathcal{L}}_\vartheta^0$ which has an open connection to a rooted 0-seed with top line in the top line of S^1. Note that the top line of S^1 is contained in the line $\{y = z+3\}$. Therefore, if $(x, z+1) \in \widetilde{\mathcal{L}}_\vartheta^0$, then the conditional probability that it has such an open connection is bounded below by p_G^3. Since the cardinality of $\widetilde{\mathcal{L}}_\vartheta^0$ is at least $N/5$, we see at once that (5.21) suffices to guarantee (5.24) when $k = 1$. Thus, having $p_G > 2/3$ and such that Lemma 5.10 applies, N as in Assumption 5.14 (so that (5.21) holds), we then have $L_1 = L_1(p_G, p_B)$ so that both (5.23) and (5.24) are valid for $k = 1$.

The implication (5.28) is also simple. If $p_G > 2/3$ and N has been chosen as above, we see at once from (5.21) that (d_k) implies (c_{k+1}). The coupling argument given in Lemma 5.8 easily shows that (a_{k+1}) follows from (d_k).

As already mentioned, we postpone the proof of (5.29). It requires a more detailed study of blocks introduced in Section 2, and which is the object of the next section. Figure 6 illustrates the drilling process, for which the fact that we deal with a planar graph plays a crucial role.

6 Towards drilling. Structure of bad layers

Lemma 6.1. *If $\mathcal{C} \in \mathbf{C}_\ell$ is a block of mass m and level ℓ, then it has at most $m - \ell + 1$ constituents.*

Proof. $\mathcal{C}$ is formed from an ℓ-run of r constituents: $\mathcal{C}_1, \ldots, \mathcal{C}_r$, $r \geq 2$, each $\mathcal{C}_i$ being of level $\ell_i < \ell$, and mass $m_i \geq \ell$, $i = 1, \ldots, r$. On the other hand, using the definition of the mass of a block, see (2.5) we see that $m \geq m_1 + (r-1)$. The statement follows at once. $\square$

Notation. Given $\Gamma(\omega) = \{x \in \mathbb{Z}_+ \colon \xi_x = 1\}$ and an interval [6] $[a, b]$ we set $\Gamma_{[a,b]} \equiv \Gamma_{[a,b]}(\omega) = \Gamma(\omega) \cap [a, b]$. Equivalently, $\xi_{[a,b]}(x) = \xi_x$ if $x \in [a, b]$, and equal zero otherwise.

Definition 6.2. (Porous medium) *We say that a segment $[a, b] \subset \mathbb{Z}_+$ is porous medium of level k (with respect to Γ) if:*

1) $\boldsymbol{C}_\infty(\Gamma_{[a,b]})$ contains no blocks of mass strictly larger than k;

2) for any $\mathcal{C} \in \boldsymbol{C}_\infty(\Gamma_{[a,b]})$ we have:

$$d(\mathcal{C}, a) \geq L^{m(\mathcal{C})} - 1 \quad \textit{and} \quad d(\mathcal{C}, b) \geq L^{m(\mathcal{C})} - 1.$$

In particular, $a, b \notin \Gamma$. When $k = 0$ the definition reduces to $\Gamma \cap [a, b] = \emptyset$.

Lemma 6.3. *a) If $\ell \geq 1$ and $\mathcal{C}$, $\tilde{\mathcal{C}} \in \boldsymbol{C}_{\ell-1}(\Gamma)$ are two consecutive constituents of an ℓ-run, then the interval $[\max(\mathcal{C}) + 1, \min(\tilde{\mathcal{C}}) - 1]$ is porous medium of level $\ell - 1$ with respect to Γ.*

b) If $k \geq 1$ and $\mathcal{C}$, $\tilde{\mathcal{C}} \in \boldsymbol{C}_\infty(\Gamma)$ are two consecutive blocks of mass at least k, then the interval $[\max(\mathcal{C}) + 1, \min(\tilde{\mathcal{C}}) - 1]$ is porous medium of level $k - 1$ with respect to Γ.

Proof. It follows at once from the construction of $\mathbf{C}_{\ell-1}$ and $\mathbf{C}_\infty$. $\square$

Lemma 6.4. (Descending decomposition) *Each block $\mathcal{C} \in \bigcup_\ell \boldsymbol{C}_\ell(\Gamma)$ of mass $m \geq 2$ has the following representation: there exists an increasing sequence of integers*

$$\min(\mathcal{C}) = f_1 < g_1 < f_2 < g_2 < \cdots < f_v < g_v \leq \max(\mathcal{C}) - 1,$$

[6] For simplicity, we shall from now on denote $[a, b] \cap \mathbb{Z}_+$ by $[a, b]$ simply.

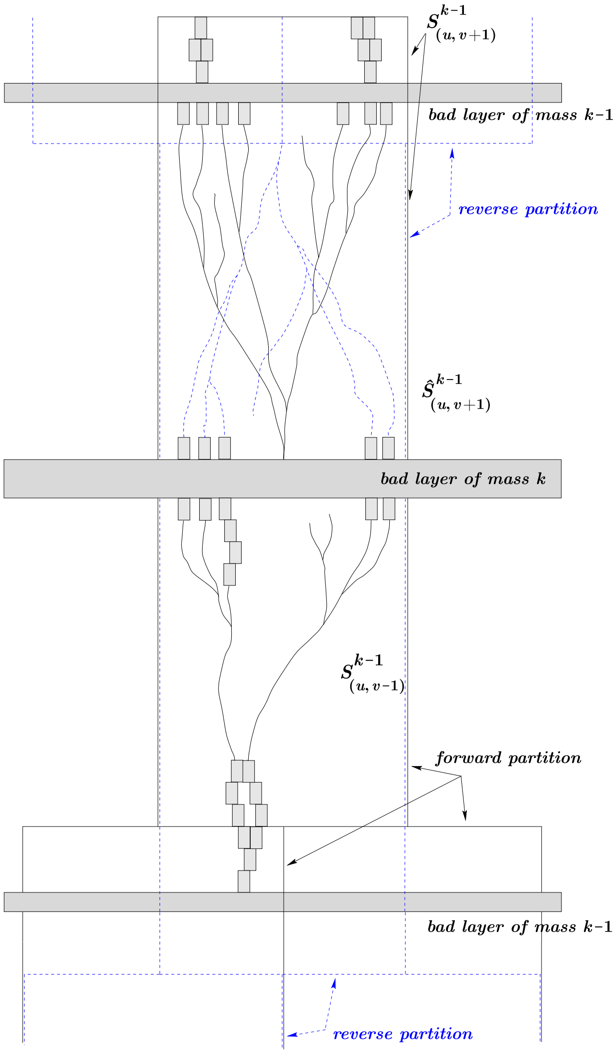

Figure 6: Schematic illustration of matching pairs and the drilling process. In the picture we represent a bad layer of mass k. Call it $B(\mathcal{C})$, where $m(\mathcal{C}) = k$. It corresponds to a bad block of $\mathbf{H}_{k-1}$ that we label as $\mathcal{H}_v^{k-1} = \hat{\mathcal{H}}_v^{k-1}$. The $(k-1)$-sites $S^{k-1}_{u,v-1}$ and $\hat{S}^{k-1}_{u,v+1}$ are then good of type 2 and form a matching pair with respect to $B(\mathcal{C})$. In order to investigate the percolation through this bad layer, we first examine matching pairs of smaller scales, depending on the level $\ell(\mathcal{C})$; these are represented by the smaller colored sites around $B(\mathcal{C})$. Then one moves to smaller scales which enter the bad layer through a selection of well positioned sets, have their connections checked as explained in Section 6, and their probabilities (under P^γ) estimated in Section 7. The use of forward and reverse paths, represented respectively in black and blue, is crucial to keep good crossing properties. As one may guess from the picture, this depends heavily on the fact that we have a planar graph. The choice of the positions of the regions $F(S^j)$, $D_{\mathcal{K},l}(S^j)$, $D_{\mathcal{K},r}(S^j)$ for all j and all good j-sites S^j enforces the crossing of the corresponding paths at scale 0. A few neighboring $(k-1)$-sites are also drawn (in black for the forward partition and in blue for the reverse partition).

so that for each $1 \le s \le v$*, the partition* $\mathbf{C}_\infty(\Gamma_{[f_s,g_s]})$ *consists of the unique block* $[f_s, g_s] \cap \Gamma$ *denoted by* $\widetilde{\mathcal{C}}_s$*, and the following holds:*

1) $m(\widetilde{\mathcal{C}}_1) = m - 1$*,* $m(\widetilde{\mathcal{C}}_s) = \widetilde{m}_s$ *for* $2 \le s \le v$*, where* $m - 1 \equiv \widetilde{m}_1 > \widetilde{m}_2 > \cdots > \widetilde{m}_v$*;*

2) *the intervals* $[g_{s-1}+1, f_s - 1]$ *are porous media of level* $\widetilde{m}_s$ *with respect to* Γ*,* $2 \le s \le v$*, and*

$$L^{\widetilde{m}_s} \le f_s - g_{s-1} \le L^{\widetilde{m}_s+1}; \tag{6.1}$$

$$\max(\mathcal{C}) - L < g_v, \quad [g_v + 1, \max(\mathcal{C}) - 1] \cap \Gamma = \emptyset. \tag{6.2}$$

Proof. Observe that the statement is obvious for blocks of level 1 and mass $m \ge 2$, in which case $v = 1$. We therefore consider blocks of level at least 2. The proof uses induction on the mass. Assuming the statement to be true for every block of mass at most m we prove it for mass $m+1$. Fix $\mathcal{C} \in \bigcup_\ell \mathbf{C}_\ell(\Gamma)$, such that $m(\mathcal{C}) = m + 1$. We split the proof in two sub-cases.

Case $\ell \equiv \ell(\mathcal{C}) = m$*.* In this case it follows from Lemma 6.1 that $\mathcal{C}$ is formed as an m-run of only two constituents, $\mathcal{C}_1$ and $\mathcal{C}_2$, where we assume $\mathcal{C}_1$ to the left of $\mathcal{C}_2$, with $m(\mathcal{C}_1) = m(\mathcal{C}_2) = m$, and we take $f_1 = \min(\mathcal{C})$ and $g_1 = \max(\mathcal{C}_1)$. By Lemma 6.3 we have that $[\max(\mathcal{C}_1) + 1, \min(\mathcal{C}_2) - 1]$ is porous media of level $m - 1$, and from the definition of the run we have that $L^{m-1} \le \min(\mathcal{C}_2) - \max(\mathcal{C}_1) < L^m$. On the other hand, from the induction assumption we know that there exists a sequence of integers

$$\min(\mathcal{C}_2) = f'_1 < g'_1 < f'_2 < g'_2 < \cdots < f'_{v'} < g'_{v'} \le \max(\mathcal{C}_2) - 1,$$

such that for each $1 \le s \le v'$ the partition $\mathbf{C}_\infty(\Gamma_{[f'_s,g'_s]})$ consists of unique block, denoted by $\widetilde{\mathcal{C}}'_s$ with $\min(\widetilde{\mathcal{C}}'_s) = f_s$ and $\max(\widetilde{\mathcal{C}}'_s) = g_s$, with

$$m(\widetilde{\mathcal{C}}'_1) = m - 1, \quad \text{and} \quad m(\widetilde{\mathcal{C}}'_s) = \widetilde{m}'_s, \quad 2 \le s \le v',$$

and the intervals $[g_{s-1}+1, f_s - 1]$ are porous media with respect to Γ of level $\widetilde{m}'_s,\ 2 \le s \le v'$, and

$$L^{\widetilde{m}'_s} \le f_s - g_{s-1} \le L^{\widetilde{m}'_s+1}. \tag{6.3}$$

Taking $f_s = f'_{s-1}$ and $g_s = g'_{s-1},\ 2 \le s \le v'$, we get the desired representation of $\mathcal{C}$.

Case $2 \le \ell \equiv \ell(\mathcal{C}) < m$*.* In this case $\mathcal{C}$ is formed as an ℓ-run of r constituents $\mathcal{C}_1, \ldots, \mathcal{C}_r$, $2 \le r \le m - \ell + 2$, with $m(\mathcal{C}_i) \ge \ell,\ 1 \le i \le r$, and so $\mathbf{C}_\infty(\Gamma_{[\min(\mathcal{C}_1), \max(\mathcal{C}_{r-1})]})$ consists of a unique block which we denote by $\widehat{\mathcal{C}}$.

If $m(\widehat{\mathcal{C}}) = m$, from the construction in Section 2 we know that $m(\mathcal{C}_r) = \ell < m$. In this case we set $g_1 = \max(\mathcal{C}_{r-1})$, and using the inductive assumption for $\mathcal{C}_r$, we complete the representation as in the previous case.

If $m(\widehat{\mathcal{C}}) < m$, we have that $\ell + 1 \le m(\mathcal{C}_r) = m - m(\widehat{\mathcal{C}}) + \ell \le m$. By the inductive assumption applied to $\mathcal{C}_r$ as the unique element of $\mathbf{C}_\infty(\Gamma_{[\min(\mathcal{C}_r), \max(\mathcal{C}_r)]})$ there are integers

$$\min(\mathcal{C}_r) = \widetilde{f}_1 < \widetilde{g}_1 < \widetilde{f}_2 < \widetilde{g}_2 < \cdots < \widetilde{f}_{\tilde{v}} < \widetilde{g}_{\tilde{v}} \le \max(\mathcal{C}_r) - 1$$

for which properties $1) - 2)$ of the lemma hold. Moreover, the unique block $\widetilde{\mathcal{C}}_1$ of $\mathbf{C}_\infty(\Gamma_{[\widetilde{f}_1, \widetilde{g}_1]})$ has mass $m - m(\widehat{\mathcal{C}}) + \ell - 1 \ge \ell$. In the configuration $\Gamma_{[\alpha(\mathcal{C}_1), \widetilde{g}_1]}$ the blocks $\mathcal{C}_1, \ldots, \mathcal{C}_{r-1}$ and $\widetilde{\mathcal{C}}_1$ will form an ℓ-run, producing a block of mass m. Therefore, taking $f_s = \widetilde{f}_s,\ s \ge 2$ and $g_s = \widetilde{g}_s,\ 1 \le s \le \tilde{v}$, we get the desired representation of $\mathcal{C}$. □

Definition 6.5. (Itinerary of a bad layer) *In the notation of the previous lemma, the sequence* $\{\widetilde{m}_s\}_{s=1}^v$ *will be called the itinerary of the descending decomposition.*

It follows from the construction that if $\mathcal{C} \in \mathbf{C}_\ell$ is such that $\ell(\mathcal{C}) = \ell$ and $m(\mathcal{C}) = m$, then for any $k \le m-1$ one can find $i_k \le i'_k$ so that $B(\mathcal{C}) = \cup_{i_k \le s \le i'_k} H^k_s$, and if $\ell \le k$ we have $i_k = i'_k$. In particular it exists j so that $B(\mathcal{C}) = H^{m-1}_j$. (For $\ell = 0$ this is a single bad line, and $m = 1$.) We keep this in mind while setting the next definitions that will play an important role in the proof.

Definition 6.6. (Zones and tunnels) *If $S^k_{(i,i_k-1)}, \widehat{S}^k_{(i',i'_k+1)}$ form a matching pair with respect to $B(\mathcal{C})$, where $\mathcal{C}$ is a block of mass m and level ℓ, we set*

$$\mathcal{Z}(S^k_{(i,i_k-1)}, \widehat{S}^k_{(i',i'_k+1)}) = \left[(cL)^k \left(\frac{i - L^{1/2}}{2}\right), (cL)^k \left(\frac{i + L^{1/2}}{2}\right)\right] \times \mathcal{H}^{m-1}_j,$$

which will be called the zone associated to $S^k_{(i,i_k-1)}$ and $\widehat{S}^k_{(i',i'_k+1)}$.

For $k \ge 1$ and if $i = i'$, we set

$$T(S^k_{(i,i_k-1)}, \widehat{S}^k_{(i',i'_k+1)}) = \left[\frac{i-1}{2}(cL)^k, \frac{i+1}{2}(cL)^k\right] \times \mathcal{H}^{m-1}_j,$$

which we call the tunnel associated to $S^k_{(i,i_k-1)}$ and $\widehat{S}^k_{(i',i'_k+1)}$.

If $|i' - i| = 1$, the tunnel associated with $S^k_{(i,i_k-1)}$ and $\widehat{S}^k_{(i',i'_k+1)}$ is defined in the following way:

$$T(S^k_{(i,i_k-1)}, \widehat{S}^k_{(i',i'_k+1)}) = \left[\frac{i \wedge i'}{2}(cL)^k, \frac{i \vee i'}{2}(cL)^k\right] \times \mathcal{H}^{m-1}_j.$$

And finally, when $k = 0$ we set

$$T((i, i_0 - 1), (i', i'_0 + 1)) = \begin{cases} [i-1, i] \times \mathcal{H}^{m-1}_j \text{ if } i = i' \\ [i \wedge i', i \vee i'] \times \mathcal{H}^{m-1}_j \text{ if } |i - i'| = 1. \end{cases}$$

Remark. Notice $\ell - 1 \le k \le m-1$ in the previous definition. For $k \ge \ell$, it is always the case that $i' = i$; the case $i' = i \pm 1$ may occur only for $k = \ell - 1$. (See Definition 5.7 and the remark that follows it.)

Definition 6.7. (Vertical sequences) *A collection of k-sites $\{S^k_{(u_s,v_s)}\}_{s=1}^r$, with $v_{n+1} = v_n + 1,\ n = 1, \dots, r-1$, is called a vertical sequence if $|u_1 - u_s| \le 1$ for all $1 < s < r$.*

Definition 6.8. *We say that the k-site $S^k_{(u_2,v_2)}$ lies above $S^k_{(u_1,v_1)}$, or, equivalently, $S^k_{(u_1,v_1)}$ lies below $S^k_{(u_2,v_2)}$, if $v_1 < v_2$, and $|u_1 - u_2| \le 1$.*

We will use the above definition also in the case of sequences of reversed sites.

Definition 6.9. (Separated pairs) *Two matching pairs $S^k_{(i,i_k-1)}, \widehat{S}^k_{(i',i'_k+1)}$ and $S^k_{(j,i_k-1)}$, $\widehat{S}^k_{(j',i'_k+1)}$ are said to be separated if $|j - i| > 2L^{1/2}$.*

Notice that if two matching pairs are separated, their corresponding zones do not intersect.

Notation. For an horizontal segment $I = \{(x,y) \in \widetilde{\mathbb{Z}}^2_+ : a \le x \le b\}$ we denote

$$\begin{aligned} I_{\urcorner} &= \{(x,y) \in \widetilde{\mathbb{Z}}^2_+ : a + \lfloor (b-a)/12 \rfloor + 1 \le x \le a + 2\lfloor (b-a)/12 \rfloor - 1\}, \\ I_{\ulcorner} &= \{(x,y) \in \widetilde{\mathbb{Z}}^2_+ : a + 10\lfloor (b-a)/12 \rfloor + 1 \le x \le a + 11\lfloor (b-a)/12 \rfloor - 1\}. \end{aligned}$$

Definition 6.10. *An horizontal segment $I = \{(x,y) \in \widetilde{\mathbb{Z}}^2_+ : a \le x \le b\}$ with $b - a = (cL)^k$ is called k-segment either if it is contained in some good k-site S^k, or if there is $\mathcal{C}$ a block in $\mathbf{C}_\ell$ with level $\ell < k$ and two good k-sites S^k, $\widehat{S}^k$ forming a matching pair with respect to $B(\mathcal{C})$ such that $I \subset T_{(S^k, \widehat{S}^k)}$. We denote a k-segment I by I^k.*

Definition 6.11. (Hierarchical k-set) *Given a k-segment I^k, a collection $\overline{I^k}$ of ℓ-segments $\{I_j^\ell\}_j$, $\ell = 0, \ldots, k$, contained in I^k is a hierarchical k-set associated with I^k if:*

i) I^k is the unique k-segment in the collection;

ii) $I_j^\ell \cap I_{j'}^\ell = \emptyset$ if $j \neq j'$, for any $\ell \in \{0, \ldots, k-1\}$;

iii) for $k \geq 1$ and $\ell \in \{1, \ldots, k\}$, each interval $(I_j^\ell)_\urcorner$ and $(I_j^\ell)_\ulcorner$ contains at least $\frac{1}{12}\rho c L$ $(\ell-1)$-segments in $\overline{I^k}$.

When $k = 0$ we simply have $\overline{I^0} = \{I^0\} = \{S^0\}$ for a 0-site S^0 and we identify $\overline{I^0}$ with I^0. For $k \geq 1$ and having fixed I^k, it is convenient to label the elements of $\overline{I^k}$: going down, from $\ell = k-1$ to $\ell = 0$, we label all ℓ-segments contained in each $I^{\ell+1}$ from left to right, starting the numbering within each $I^{\ell+1}$ every time from 1. Proceeding in this way, we have a multi-index $\mu_{\langle k,\ell\rangle} = \langle \mu_{k-1}, \mu_{k-2}, \ldots, \mu_\ell \rangle$ which indicates the "genealogical tree" down to scale ℓ. We denote the corresponding ℓ-segment with this index by $I^\ell_{\mu_{\langle k,\ell\rangle}}$. We shall also use the following convention. If $\mu_\ell = j$, we will write:

$$\langle \mu_{k-1}, \ldots, \mu_{\ell+1}, j \rangle = \langle \mu_{\langle k,\ell+1\rangle}, j \rangle. \tag{6.4}$$

Definition 6.12. *(a) For any type 2 good k-site S^k, $k \geq 1$, $\Psi^k(S^k)$ denotes its top 0-layer. Analogously, if $\widehat{S}^k$ is a good reverse k-site of type 2, $\Upsilon^k(\widehat{S}^k)$ denotes its bottom 0-layer. When $k = 0$, we set $\Psi^0(S^0) = S^0$ and $\Upsilon^0(\widehat{S}^0) = \widehat{S}^0$.*

(b) If two k-sites S^k and $\widehat{S}^k$ form a matching pair with respect to $B(\mathcal{C})$ in the sense of Definition 5.7, where $\mathcal{C} \in \mathbf{C}_\ell$ has mass m and level ℓ, and $\ell - 1 \leq k \leq m-1$, we say that $\Psi^k(S^k)$ and $\Upsilon^k(\widehat{S}^k)$ also form a matching pair with respect to $B(\mathcal{C})$.

(c) Two hierarchical k-sets $\overline{\Psi^k}$ and $\overline{\Upsilon^k}$ whose k-segments Ψ^k and Υ^k form a matching pair with respect to $B(\mathcal{C})$ as in (b) are also called a matching pair with respect to $B(\mathcal{C})$.

For the proof in Section 7 we shall use the following hierarchical k-sets: let S^k be a good k-site with dense kernel. In this case, there will be at least $\lceil \frac{1}{12}\rho c L \rceil$ $(k-1)$-sites[7] in $D_{\mathcal{K},l}(S^k)$ and $D_{\mathcal{K},r}(S^k)$ respectively, and each of them will have dense kernel. The same happens at all smaller scales. The top 0-layers of the kernel of these dense kernel sites at all scales form a hierarchical k-set, which we denote as $\overline{\Psi^k}(S^k)$. The analogous hierarchical k-set for a reverse k-site $\widehat{S}^k$ we will denote by $\overline{\Upsilon^k}(\widehat{S}^k)$. We shall use this in the case when S^k is of type 2, lying immediately below a bad layer of mass larger than k (so that S^k coincides with its kernel) or when it contains a bad layer of mass k (type 1).

Notation. It will be convenient to single out the class of bad 1-layers $B(\mathcal{C})$ that consist of m consecutive bad lines. We call such bad layers *monolithic*, and refer to them as bad 1_M-layers, writing $B(m)$ for $B(\mathcal{C})$ in this particular case.

The following concept of *chaining* plays an important role in the proof in Section 7. We split it into two definitions, for *large* and *small* hierarchical sets, where the distinction has to do with the level of the bad layer, as made precise below.

Definition 6.13. (Large chained hierarchical k-sets) *Let $\mathcal{C} \in \mathbf{C}_\ell$ with $\ell(\mathcal{C}) = \ell$ and $m(\mathcal{C}) = m > \ell$, and let $k \in \{\ell-1, \ldots, m-1\}$. Two hierarchical k-sets $\overline{\Psi^k}$ and $\overline{\Upsilon^k}$ forming a matching pair with respect to $B(\mathcal{C})$ are said to be chained through $B(\mathcal{C})$ if the following holds:*

***The case** $k = 0$. In this situation $\ell = 1$, and we distinguish the 1_M-layers.*

• Monolithic layer. We say that $\overline{\Psi^0}$ and $\overline{\Upsilon^0}$ are chained if there exists an open vertical path of 0-sites from a nearest neighbor of Ψ^0 to nearest neighbor of Υ^0.

• Non-monolithic layer. In this case $B(\mathcal{C})$ is formed by m bad lines grouped into $r > 1$ 1_M-layers[8], separated among themselves by at most $L-1$ good lines. We denote these

[7] For $k = 1$ this number is replaced by $\lceil \frac{1}{24}\rho c L \rceil$.

[8] this is slight abuse of our previous notation

parts by $B_v(m_v),\ 1 \le v \le r$, where m_v is the number of bad lines it contains. We say that $\overline{\Psi^0}$ and $\overline{\Upsilon^0}$ are chained through $B(\mathcal{C})$ if there exist a vertical sequence of hierarchical 0-sets

$$\widehat{S}^0(1), S^0(2), \widehat{S}^0(2), \dots, S^0(r),$$

such that

a) all sites $S^0(v),\ v = 2, \dots, r$, and $\widehat{S}^0(s)$, $v = 1, \dots, r-1$, are passable;

b) the sites $S^0(v)$ and $\widehat{S}^0(v),\ v = 2, \dots, r-1$, form a matching pair with respect to $B_v(m_v)$;

c) Ψ^0 and $S^0(1)$ are chained through $B_1(m_1)$; $S^0(v)$ and $\widehat{S}^0(v)$ are chained through $B_v(m_v)$, for each $v = 2, \dots, r-1$; $S^0(r)$ and Υ^0 are chained through $B_r(m_r)$.

d) each pair of sites $\widehat{S}^0(s)$ and $S^0(s+1)$, $1 \le s \le r-1$, is connected by an open path of 0-sites lying within $\mathcal{Z}(S^0_{(i,i_k-1)}, \widehat{S}^0_{(i',i'_k+1)})$.

***The case** $k \ge 1$**.** Again we distinguish two cases:*

• $k \ge \ell$. Since $\rho > 1/2$, letting $\widehat{\rho} = \rho - 1/2$, the definition of hierarchical set implies the existence of at least $\widehat{\rho}\frac{c}{6}L$ matching pairs $\overline{\Psi^{k-1}}$, $\overline{\Upsilon^{k-1}}$ with respect to $B(\mathcal{C})$, with $\overline{\Psi^{k-1}} \subset \overline{\Psi^k}$ and $\overline{\Upsilon^{k-1}} \subset \overline{\Upsilon^k}$. Let $\mathcal{M}$ be the set formed by the first (from left to right) $\lfloor \widehat{\rho}\frac{c}{48}L^{1/2} \rfloor$ such pairs which are separated. We say that $\overline{\Psi^k}$ and $\overline{\Upsilon^k}$ are chained if at least one matching pair in $\mathcal{M}$ is chained through $B(\mathcal{C})$.

• $k = \ell - 1$. Assume that $\mathcal{C}$ has $r > 1$ constituents of masses m_v and levels $\ell_v < \ell$, hereby denoted as $\mathcal{C}_v$, $v = 1, \dots, r$. We say that $\overline{\Psi^k}$ and $\overline{\Upsilon^k}$ are chained if there exist a vertical sequence of good k-sites

$$\widehat{S}^k(1), S^k(2), \widehat{S}^k(2), \dots, S^k(r),$$

such that

a) For each $1 \le v \le r-1$, $S(\widehat{S}^k(v))$ and $S^k(v+1)$ are connected by a passable k-path, lying entirely in $\mathcal{Z}(S^k_{(i,i_k-1)}, \widehat{S}^k_{(i',i'_k+1)})$. (In particular, $S^k(v)$ has s-dense kernel, $v = 2, \dots, r$.)

b) $\widehat{S}^k(v)$ has c-dense kernel (reversed), $v = 1, \dots, r-1$.

c) $S^k(v)$ and $\widehat{S}^k(v)$ form a matching pair which respect to $B(\mathcal{C}_v)$, $v = 2, \dots, r-1$.

d) $\overline{\Psi^k}$ and $\overline{\Upsilon^k}(\widehat{S}^k(1))$ are chained through $B(\mathcal{C}_1)$; $\overline{\Psi^k}(S^k(v))$ and $\overline{\Upsilon^k}(\widehat{S}^k(v))$ are chained through $B(\mathcal{C}_v)$, $v = 2, \dots, r-1$; and finally $\overline{\Psi^k}(S^k(r))$ and $\overline{\Upsilon^k}$ are chained through $B(\mathcal{C}_r)$.

The connections by open oriented paths that are examined using the iterative procedure just defined will be called restricted.

Notation. For easiness of notation we shall write $J = \lfloor \widehat{\rho}\frac{c}{48}L^{1/2} \rfloor$.

Remarks.

a) Let $\mathcal{C} \in \mathbf{C}_\ell$ with level ℓ and mass m, and $\ell - 1 \le k \le m-1$. If a matching pair of hierarchical k-sets as above $\overline{\Psi^k}$ and $\overline{\Upsilon^k}$ is chained through $B(\mathcal{C})$ and $k \ge \ell$, there must exist an open oriented (restricted) path of 0-sites crossing $B(\mathcal{C})$ and lying in $T(S^k_{(i,i_k-1)}, \widehat{S}^k_{(i',i'_k+1)})$; if $k = \ell - 1$ such a path exists in $\mathcal{Z}(S^k_{(i,i_k-1)}, \widehat{S}^k_{(i',i'_k+1)})$.

b) Notice that in Definition 6.13, for each $j \in \{0, \dots, k-1\}$, we examine at each step (according to the set $\mathcal{M}$ in Definition 6.13) exactly J j-segments within each checked $j+1$-segment in $\overline{\Upsilon^k}$, and similarly for $\overline{\Psi^k}$; each checked to be connected to different j-segments within $B(\mathcal{C})$. The algorithm for selecting $\mathcal{M}$ at each smaller scale depends (except in the trivial case of monolithic layers) on what happens within $B(\mathcal{C})$ as explained therein. With some abuse of notation we call $\mathcal{M}(\overline{\Upsilon^k})$ and similarly $\mathcal{M}(\overline{\Psi^k})$ the collection of these checked segments at all scales (J at each scale).

c) The estimates in the next section become easier to formulate once the number of j-segments to be examined within a $j+1$-segment is fixed at all times. The exact algorithm to define the set $\mathcal{M}$ is not so relevant. For the construction in Section 7 this will be slightly different then the one used in the above definition, though we shall use the same notation.

Definition 6.14. (Small chained hierarchical sets) *Let $\mathcal{C} \in \mathbf{C}_\ell$ be a block of mass m and level $\ell \geq 2$. Let $k \in \{\ell-1, \dots, m-1\}$ and let $\overline{\Psi^k}$ and $\overline{\Upsilon^k}$ form a matching pair of hierarchical k-sets, which we assume to be chained (according to Definition 6.13).*

a) We say that a 0-site $\Upsilon^0_{\mu_{\langle k,0\rangle}} \in \mathcal{M}(\overline{\Upsilon^k})$ is chained to $\overline{\Psi^k}$, if there exists a 0-site $\Psi^0_{\mu_{\langle k,0\rangle}} \in \mathcal{M}(\overline{\Psi^k})$, and a restricted open path from a nearest neighbor (from above) of $\Psi^0_{\mu_{\langle k,0\rangle}}$ to a nearest neighbor of $\Upsilon^0_{\mu_{\langle k,0\rangle}}$ from below.

b) We say that the r-segment $\Upsilon^r_{\mu_{\langle k,r\rangle}} \in \mathcal{M}(\overline{\Upsilon^k})$, $0 < r < \ell-1$, is chained to $\overline{\Psi^k}$ if it contains an $(r-1)$-segment $\Upsilon^{r-1}_{\mu_{\langle k,r-1\rangle}} \in \mathcal{M}(\overline{\Upsilon^k})$ which is chained to $\overline{\Psi^k}$.

Remark. A 0-path as in a) above will be open, oriented, and will lie entirely in the tunnel $T(\Upsilon^k, \Psi^k)$ when $k \geq \ell$, and for $k = \ell-1$ it will lie in $Z(\Upsilon^{\ell-1}, \Psi^{\ell-1})$. [9]

Definition 6.15. (Chained k-sites) *Let $\mathcal{C} \in \mathbf{C}_\ell$ with level ℓ and mass m. For $k \in \{\ell-1, \dots, m-1\}$, two k-sites S^k and $\widehat{S}^k$ that form a matching pair with respect to $B(\mathcal{C})$ are said to be chained through $B(\mathcal{C})$ if the corresponding hierarchical k-sets $\overline{\Psi^k}(S^k)$ and $\overline{\Upsilon^k}(\widehat{S}^k)$ are chained through $B(\mathcal{C})$, as defined above.*

Notation. The event of two hierarchical k-sets $\overline{\Psi^k}$, $\overline{\Upsilon^k}$, or analogously two k-sites S^k, $\widehat{S}^k$ being chained through $B(\mathcal{C})$ is denoted by

$$\overline{\Psi^k} \underset{B(\mathcal{C})}{\leftrightsquigarrow} \overline{\Upsilon^k} \tag{6.5}$$

and respectively

$$S^k \underset{B(\mathcal{C})}{\leftrightsquigarrow} \widehat{S}^k. \tag{6.6}$$

7 Proof of (5.29). Proof of Theorem 5.15

This section is dedicated to the conclusion of the proof of Theorem 5.15. Following the outline presented in Section 5.3, it remains to verify (5.29), i.e. that property (b_{m+1}) follows from $(a_j), (b_j), (c_j), (d_j)$ for all $j \leq m$ and (a_{m+1}), where $m \geq 1$. This is indeed the most delicate step, and we need a careful analysis of the bad layers. The ingredients started being developed in the last section, and the procedure will be completed now with the help of Proposition 7.1, which includes a more detailed description that leads to property (b_m). Its proof will also be done by induction.

Notation. Let $\varkappa = \varkappa' + 2$ with $\varkappa'$ as in Lemma 5.12. We recursively define for all $m \geq 1$:

$$\begin{aligned} p_{0,m} &:= p_B^m p_G^{\varkappa(m-1)}, \\ p_{j,m} &:= (1-(1-p_{j-1,m})^J) p_j^{\varkappa(m-j-1)}, \qquad 1 \leq j \leq m-1, \\ p_{m,m} &:= 1-(1-p_{m-1,m})^J, \end{aligned} \tag{7.1}$$

where $J = \lfloor \widehat{\rho} \frac{c}{48} L^{1/2} \rfloor$, $\widehat{\rho} = \rho - 1/2$ as in Definition 6.13, $p_j = 1 - q_j$, $q_j = q_0^{j+1}$, $p_0 = p_G, q_0 = 1 - p_G$, as in Theorem 5.15.

For $m \geq 1$ and $p_{j,m}$ given as above, we set:

$(b_m)'$ For every block $\mathcal{C} \in \mathbf{C}_\ell$ of mass m and level ℓ, every $j \in \{\ell-1, \dots, m-1\}$ and every choice of hierarchical j-sets $\overline{\Psi^j}, \overline{\Upsilon^j}$ that form a matching pair with respect to the bad layer $B(\mathcal{C})$, one has

$$P^\gamma(\overline{\Psi^j} \underset{B(\mathcal{C})}{\leftrightsquigarrow} \overline{\Upsilon^j}) \geq p_{j,m}. \tag{7.2}$$

[9] Defined analogously to $T(S, \widehat{S})$ and $Z(S, \widehat{S})$.

For $m \geq 2$:

$(b_m)''$ For every $B(\mathcal{C})$, j and $\overline{\Psi^j}, \overline{\Upsilon^j}$ as in $(b_m)'$, and every $s \in \{0, \ldots, j-1\}$, the conditional distribution (under P^γ) of the number of $\Upsilon^s_{\langle \mu_{\langle j,s+1 \rangle}, i \rangle} \in \mathcal{M}(\overline{\Upsilon^j})$ that are chained to $\overline{\Psi^j}$, given that $\Upsilon^{s+1}_{\mu_{\langle j,s+1 \rangle}}$ is chained to $\overline{\Psi^j}$, is stochastically larger than $F_{p_{s,m}}$, where F_p denotes the distribution of a Binomial random variable with J trials and success probability p, conditioned to have at least one success. That is,

$$|\{i \colon \Upsilon^s_{\langle \mu_{\langle j,s+1 \rangle}, i \rangle} \in \mathcal{M}(\overline{\Upsilon^j}) \colon \Upsilon^s_{\langle \mu_{\langle j,s+1 \rangle}, i \rangle} \text{chained to } \overline{\Psi^j}\}| \Big| \left[\Upsilon^{s+1}_{\mu_{\langle j,s+1 \rangle}} \text{ chained to } \overline{\Psi^j} \right] \succeq F_{p_{s,m}}, \tag{7.3}$$

with $\succeq$ standing for stochastically larger in the usual sense.

Proposition 7.1. *Under the same conditions of Theorem 5.15, and for any $k \geq 1$, the validity of properties (a_m) and (d_m) for all $m \leq k$ implies the validity of properties $(b_m)'$ for all $1 \leq m \leq k+1$ and $(b_m)''$ for all $2 \leq m \leq k+1$.*

Proof. We first observe that properties (a_1) and (d_1) are indeed true, that $(b_1)'$ follows directly from the definitions and that $(b_2)'$ and $(b_2)''$ are simple to verify from (a_1) and (d_1). We may also see this as a brutal simplification of the inductive argument in this proof considering the simplicity of a block $\mathcal{C} \in \mathbf{C}_1$ with $m(\mathcal{C}) = 2$.

Induction step. It suffices to show that if $k \geq 1$ is given, if $(a_j), (d_j)$ hold for all $1 \leq j \leq k$, $1 \leq m \leq k$ and $(b_m)'$ and $(b_m)''$ hold, then also $(b_{m+1})'$ and $(b_{m+1})''$ hold.

We will first establish $(b_{m+1})'$. Let $\mathcal{C} \in \mathbf{C}_\ell$ be a block of mass $m+1$ and level ℓ. Throughout the proof we consider its descending decomposition representation and construct a class of particularly chosen hierarchical sets that will play a role in the induction.

Construction for the induction step.

Let $(\overline{\Psi^m}, \overline{\Upsilon^m})$ form a matching pair with respect to the bad layer $B(\mathcal{C})$, and let $\{\widetilde{m}_s\}_{s=1}^v$ denote the itinerary of the descending decomposition of $\mathcal{C}$, with $\{\widetilde{\mathcal{C}}_s\}_{s=1}^v$ its corresponding blocks, and $\{B(\widetilde{\mathcal{C}}_s)\}_{s=1}^v$ the corresponding bad layers. Recall that (Lemma 6.4) the interval between any two consecutive blocks $\widetilde{\mathcal{C}}_s$ and $\widetilde{\mathcal{C}}_{s+1}$ is always porous media of level $\widetilde{m}_{s+1}$.

An *entrance set* $\overline{\Psi^m}(s)$, $s = 2, \ldots, v$, will be a suitable hierarchical m-set located at the 0-layer just below $B(\widetilde{\mathcal{C}}_s)$, and an *exit set* $\overline{\Upsilon^m}(s)$, $s = 1, \ldots, v$, a suitable hierarchical m-set located at the 0-layer just above $B(\widetilde{\mathcal{C}}_s)$ for $s = 1, \ldots, v-1$, with $\overline{\Upsilon^m}(v)$ located at the 0-layer just above $B(\widetilde{\mathcal{C}}_v)$, or at its last 0-layer, according to $g_v < \max(\mathcal{C}) - 1$ or $g_v = \max(\mathcal{C}) - 1$ (Lemma 6.4).

Large segments of exit and entrance sets. For each $s = 1, \ldots, v-1$, the m-segment $\Upsilon^m(s)$ and all j-segments $\Upsilon^j_{\mu_{\langle m,j \rangle}}(s)$, $\widetilde{m}_{s+1} \leq j < m$, in $\overline{\Upsilon^m}(s)$, are obtained by taking the corresponding segments Υ^m and $\Upsilon^j_{\mu_{\langle m,j \rangle}}$ and projecting them vertically on the 0-layer located just above $B(\widetilde{\mathcal{C}}_s)$, and then by taking as $\Upsilon^j_{\mu_{\langle m,j \rangle}}(s)$ a j-segment which intersects this projection: when there are two such j-segments, to avoid ambiguities we take the one which intersects the left half of the projection. For $s = v$ the only difference is that when $g_v = \max(\mathcal{C}) - 1$ the segments will be located at the last 0-layer of $B(\widetilde{\mathcal{C}}_v)$.

For the entrance sets $\overline{\Psi^m}(s)$ with $s = 2, \ldots, v$ we proceed in the same way: the m-segment $\Psi^m(s)$ and all j-segments $\Psi^j_{\mu_{\langle m,j \rangle}}(s)$, $\widetilde{m}_s \leq j < m$, in $\overline{\Psi^m}(s)$, are obtained by taking the corresponding segments Υ^m, and $\Upsilon^j_{\mu_{\langle m,j \rangle}}$ and projecting them vertically on the 0-layer located just below $B(\widetilde{\mathcal{C}}_s)$, with the same selection rule as above in case there are two such j-segments.

Construction of the exit sets $\overline{\Upsilon^m}(s)$. Consider first the case $1 \leq s < v$. To continue the construction of the j-segments at scales smaller than $\widetilde{m}_{s+1}$, we consider, for each already defined $\widetilde{m}_{s+1}$-segment of this collection, the reversed $\widetilde{m}_{s+1}$-site for which this segment is the last 0-layer, i.e. $\widehat{S}^{\widetilde{m}_{s+1}}$ such that $\Upsilon(\widehat{S}^{\widetilde{m}_{s+1}}) = \Upsilon^{\widetilde{m}_{s+1}}_{\mu_{\langle m,\widetilde{m}_{s+1}\rangle}}(s)$, and check if this site has c-(reverse) dense kernel. If the answer is affirmative, we take $\Upsilon^{j}_{\mu_{\langle m,j\rangle}}(1),\ j = 0,\dots,\widetilde{m}_{s+1}-1$ as the bottom 0-layers of the reverse dense kernel sites of $\widehat{S}^{\widetilde{m}_{s+1}}$, or in other words all the scales from $\widetilde{m}_{s+1}$ down to zero correspond to $\overline{\Upsilon^{\widetilde{m}_{s+1}}}(\widehat{S}^{\widetilde{m}_{s+1}})$. These are called "compatible" segments. For those sites $\widehat{S}^{\widetilde{m}_{s+1}}$ that do not have c-(reverse) dense kernel, we select the $\Upsilon^{j}_{\mu_{\langle m,j\rangle}}(s),\ j = 0,\dots,\widetilde{m}_{s+1}-1$ in an arbitrary way among the correspondent sub-segments of bottom 0-layers of the site. We call such choice of segments "incompatible" with the process. Only compatible segments will play a role in the construction.

In the case $s = v$ we make essentially the same construction, as if $\widetilde{m}_{s+1} = 0$, with the difference that when $\max(\mathcal{C}) = \max(\widetilde{\mathcal{C}}_v) + 1$ we locate the hierarchical set at the last 0-layer of $B(\widetilde{\mathcal{C}}_v)$.

Observe that the construction of $\overline{\Upsilon^m}(s)$ and the compatibility of its segments depend on Γ and on the occupation variables in between $B(\widetilde{\mathcal{C}}_s)$ and $B(\widetilde{\mathcal{C}}_{s+1})$.

Step 1, part 1. We check if at least one among the pairs of hierarchical $(m-1)$-sets $\overline{\Psi^{m-1}_{\mu_{\langle m,m-1\rangle}}}$ and $\overline{\Upsilon^{m-1}_{\mu_{\langle m,m-1\rangle}}}(1)$ is chained through $B(\widetilde{\mathcal{C}}_1)$. If so, we move to the next item; otherwise we stop the procedure and say that $\overline{\Psi^m}$ and $\overline{\Upsilon^m}$ are not chained through $B(\mathcal{C})$.

Step 1, part 2. (*Zooming*) For each pair of hierarchical $(m-1)$-sets $\overline{\Psi^{m-1}_{\mu_{\langle m,m-1\rangle}}}$ and $\overline{\Upsilon^{m-1}_{\mu_{\langle m,m-1\rangle}}}(1)$ chained through $B(\widetilde{\mathcal{C}}_1)$ we select all multi-indices $\mu_{\langle m,\widetilde{m}_2\rangle}$ and the corresponding $\widetilde{m}_2$-segments $\Upsilon^{\widetilde{m}_2}_{\mu_{\langle m,\widetilde{m}_2\rangle}}(1)$, which are compatible and from which there exists an open oriented 0-level path that connects to $\Psi^{m-1}_{\mu_{\langle m,m-1\rangle}}$ through $B(\widetilde{\mathcal{C}}_1)$.

Step 1, part 3. (*Transfer*) For the $\widetilde{m}_2$-segments $\Upsilon^{\widetilde{m}_2}_{\mu_{\langle m,\widetilde{m}_2\rangle}}(1)$ selected in the previous item, we first check whether

(i) the corresponding forward site $S(\widehat{S}^{\widetilde{m}_2}_{\mathbf{x}})$ is c-(forward) passable.

If the answer is positive, it implies that at least one of the seeds of $Q_l(S(\widehat{S}^{\widetilde{m}_2}_{\mathbf{x}}))$ or $Q_r(S(\widehat{S}^{\widetilde{m}_2}_{\mathbf{x}}))$ is also connected to Ψ^{m-1} (we may call it "active"). This gives us a way of completing the construction of the hierarchical set $\overline{\Psi^m}(2)$ at scales smaller than $\widetilde{m}_2$:

Construction of the entrance set $\overline{\Psi^m}(2)$. Take $S^{\widetilde{m}_2}_{\mathbf{x}'}$ such that $\Psi(S^{\widetilde{m}_2}_{\mathbf{x}'}) = \Psi^{\widetilde{m}_2}_{\mu_{\langle m,\widetilde{m}_2\rangle}}(2)$, and check whether

(ii) there exists an oriented passable $\widetilde{m}_2$-path starting from the $\widetilde{m}_2$-site which is s-passable from the active seed of $S(\widehat{S}^{\widetilde{m}_2}_{\mathbf{x}})$ to $S^{\widetilde{m}_2}_{\mathbf{x}'}$, and entirely contained in $\mathcal{Z}(S(\widehat{S}^{\widetilde{m}_2}_{\mathbf{x}}), S^{\widetilde{m}_2}_{\mathbf{x}'})$.

If the answer to (i) and (ii) is positive we say that $\overline{\Upsilon^{\widetilde{m}_2}_{\mu_{\langle m,\widetilde{m}_2\rangle}}}(1)$ and $\overline{\Psi^{\widetilde{m}_2}_{\mu_{\langle m,\widetilde{m}_2\rangle}}}(2)$ are *active*. Otherwise the procedure of building connection from $\Upsilon^{\widetilde{m}_2}_{\mu_{\langle m,\widetilde{m}_2\rangle}}(1)$ is stopped. This completes Step 1.

Remark 7.2. Notice that $\overline{\Psi^m}(2)$ lies just below $B(\widetilde{\mathcal{C}}_2)$, but a positive answer to (i) and (ii) above, besides guaranteeing the connection of $\Psi^{\widetilde{m}_2}(2)$ to the corresponding $\Upsilon^{\widetilde{m}_2}(1)$ (and therefore to Ψ^{m-1} by force of the previous sub-step) also gives connection by open oriented path of 0-sites to suitable sites at the top 0-layer of $B(\widetilde{\mathcal{C}}_2)$ (according to the definition of passability at the scale $\widetilde{m}_2$), which then implies the existence of an open path to a 0-site in $B(\widetilde{\mathcal{C}}_2)$ which is nearest neighbor of a corresponding $\widetilde{m}_2$-segment $\Upsilon^{\widetilde{m}_2}_{\mu_{\langle m,\widetilde{m}_2\rangle}}(2)$. The first part does not depend on the occupation variables in $B(\widetilde{\mathcal{C}}_2)$, and one might find convenient to think of the event in (ii) as the intersection of these two conditions involving disjoint sets of 0-sites.

Step $s,\ 1 < s \le v$. Having determined the active $\overline{\Upsilon^{\widetilde{m}_s}_{\mu_{\langle m,\widetilde{m}_s\rangle}}}(s-1)$ and $\overline{\Psi^{\widetilde{m}_s}_{\mu_{\langle m,\widetilde{m}_s\rangle}}}(s)$, the process continues only from the compatible corresponding sub-segments $\Upsilon^{\widetilde{m}_{s+1}}_{\mu_{\langle m,\widetilde{m}_{s+1}\rangle}}(s)$.

Sub-case $s < v$. The construction repeats what was done above for $s = 1$:

- We check if $\overline{\Psi^{\widetilde{m}_s}_{\mu_{\langle m,\widetilde{m}_s\rangle}}}(s)$ and $\overline{\Upsilon^{\widetilde{m}_s}_{\mu_{\langle m,\widetilde{m}_s\rangle}}}(s)$ are chained through $B(\widetilde{\mathcal{C}}_s)$;
- For each pair of hierarchical $(\widetilde{m}_s-1)$-sets $\overline{\Psi^{\widetilde{m}_s-1}_{\mu_{\langle m,\widetilde{m}_s-1\rangle}}}(s)$ and $\overline{\Upsilon^{\widetilde{m}_s-1}_{\mu_{\langle m,\widetilde{m}_s-1\rangle}}}(s)$ chained through $B(\widetilde{\mathcal{C}}_s)$, we select all multi-indices $\mu_{\langle m,\widetilde{m}_{s+1}\rangle}$ and corresponding $\widetilde{m}_{s+1}$-segments $\Upsilon^{\widetilde{m}_{s+1}}_{\mu_{\langle m,\widetilde{m}_{s+1}\rangle}}(s)$ which are compatible and for which there exists a 0-level path (open, oriented) connecting them to $\Psi^{\widetilde{m}_s-1}_{\mu_{\langle \widetilde{m}_s,\widetilde{m}_s-1\rangle}}$ through $B(\widetilde{\mathcal{C}}_s)$.
- item 2, called transfer, and the construction of $\overline{\Psi^m}(s+1)$ both follow the same procedure as when $s=1$, replacing $\widetilde{m}_2$ by $\widetilde{m}_{s+1}$. We then say that $\overline{\Upsilon^{\widetilde{m}_{s+1}}_{\mu_{\langle m,\widetilde{m}_{s+1}\rangle}}}(s)$ and $\overline{\Psi^{\widetilde{m}_{s+1}}_{\mu_{\langle m,\widetilde{m}_{s+1}\rangle}}}(s+1)$ are active if the analogue of the previous (i)–(ii) both hold.

Sub-case $s=v$. This splits into two situations:

a) $\max(\mathcal{C}) > \max(\widetilde{\mathcal{C}}_v)+1$. In this case we act as if $\widetilde{m}_{s+1}=0$, i.e. we first of all perform zooming by selecting all active elements down to 0 level, and repeat the transfer procedure.

b) $\max(\mathcal{C}) = \max(\widetilde{\mathcal{C}}_v)+1$. In this case we act as if $\widetilde{m}_{s+1}=0$, i.e. we first of all perform the zooming by selecting all active elements down to 0 level, however the transfer procedure reduces to connecting over the last bad line of $B(\mathcal{C})$.

Estimates needed for the induction step.

At this point we recall Lemma 5.12, which will be repeatedly used below.

Remark 7.3. Lemma 5.12 is used in the part of the procedure called transfer above. It will be used at the various scales $j\le m$, with $\tilde p = p_j$, and n of order $\sqrt{L}$ which we may assume large enough so that the estimate applies.

Let $(\overline{\Psi^m},\overline{\Upsilon^m})$ be a matching pair with respect to the $B(\mathcal{C})$ under consideration. By the induction assumption $(b_m)'$, we have that for each pair of indices $\mu_{\langle m,m-1\rangle}$

$$P\left(\overline{\Upsilon^{m-1}_{\mu_{\langle m,m-1\rangle}}}(1) \underset{B(\widetilde{\mathcal{C}}_1)}{\leftrightsquigarrow} \overline{\Psi^{m-1}_{\mu_{\langle m,m-1\rangle}}}\right) \ge p_{m-1,m}. \tag{7.4}$$

For a fixed family of hierarchical $(m-1)$-sets, the events in (7.4) are (conditionally) independent, so that the distribution of the number of chained pairs, given that at least one of them is chained, is stochastically larger than $F_{p_{m-1,m}}$.

On the other hand, from the induction assumption $(b_m)''$ we have that for each $0\le j<m-1$ and each pair $\mu_{\langle m,j+1\rangle}$

$$|\{i\colon \overline{\Upsilon^{j}_{\langle\mu_{\langle m,j+1\rangle},i\rangle}}(1) \underset{B(\widetilde{\mathcal{C}}_1)}{\leftrightsquigarrow} \overline{\Psi^{m-1}_{\mu_{\langle m,m-1\rangle}}}\}| \Big| [\overline{\Upsilon^{j+1}_{\mu_{\langle m,j+1\rangle}}}(1) \underset{B(\widetilde{\mathcal{C}}_1)}{\leftrightsquigarrow} \overline{\Psi^{m-1}_{\mu_{\langle m,m-1\rangle}}}] \succeq F_{p_{j,m}}, \tag{7.5}$$

i.e. conditioned on $\overline{\Upsilon^{j+1}_{\mu_{\langle m,j+1\rangle}}}(1)$ being chained to $\overline{\Psi^{m-1}_{\mu_{\langle m,m-1\rangle}}}$, the number of indices i so that $\overline{\Upsilon^{j}_{\langle\mu_{\langle m,j+1\rangle},i\rangle}}(1)$ is chained to $\overline{\Psi^{m-1}_{\mu_{\langle m,m-1\rangle}}}$ is stochastically larger than $F_{p_{j,m}}$. We shall use (7.5) for j going down to $j=\widetilde{m}_2$.

Assume $\widetilde{m}_2\ge 1$, i.e. $v\ge 2$. For each index $\mu_{\langle m,\widetilde{m}_2\rangle}$ which yields a chained set at all steps from $m-1$ down to $\widetilde{m}_2$ one now checks if the $\widetilde{m}_2$-set $\overline{\Upsilon^{\widetilde{m}_2}_{\mu_{\langle m,\widetilde{m}_2\rangle}}}$ is "compatible" and if the conditions (i) and (ii) described in the previous construction hold. We use the induction assumption, which guarantees the validity of conditions $(a_i)-(d_i)$ for all $i\le m$. Applying this and Lemma 5.12, we get for each such index $\mu_{\langle m,\widetilde{m}_2\rangle}$:

$$P\left(\overline{\Upsilon^{\widetilde{m}_2}_{\mu_{\langle m,\widetilde{m}_2\rangle}}}(1) \text{ and } \overline{\Psi^{\widetilde{m}_2}_{\mu_{\langle m,\widetilde{m}_2\rangle}}}(2) \text{ are ``active''}\right) \ge p^{\varkappa}_{\widetilde{m}_2} \tag{7.6}$$

where $\varkappa=\varkappa'+2$ as defined just before (7.1) (the $+2$ appears since we need to check that the starting $\widetilde{m}_2$-site at the bottom has reverse c-dense kernel (compatible), and is forward c-passable). Using (a_m) we then get that for all the previous indices $\mu_{\langle m,\widetilde{m}_2\rangle}$ as above (for them we have $\overline{\Upsilon^{\widetilde{m}_2}_{\mu_{\langle m,\widetilde{m}_2\rangle}}}(1)$ is chained to $\overline{\Psi^{m-1}_{\mu_{\langle m,m-1\rangle}}}$):

$$P\left(\exists i\colon \overline{\Upsilon^{\widetilde{m}_2-1}_{\mu_{\langle\langle m,\widetilde{m}_2\rangle,i\rangle}}}(2) \underset{B(\widetilde{\mathcal{C}}_2)}{\leftrightsquigarrow} \overline{\Psi^{\widetilde{m}_2-1}_{\mu_{\langle\langle m,\widetilde{m}_2\rangle,i\rangle}}}(2)\right) \geq p_{\widetilde{m}_2,\widetilde{m}_2}. \tag{7.7}$$

The event on the l.h.s. of (7.7) we naturally denote as

$$\left[\overline{\Upsilon^{\widetilde{m}_2}_{\mu_{\langle m,\widetilde{m}_2\rangle}}}(2) \underset{B(\widetilde{\mathcal{C}}_2)}{\leftrightsquigarrow} \overline{\Psi^{\widetilde{m}_2}_{\mu_{\langle m,\widetilde{m}_2\rangle}}}(2)\right]$$

and by the induction assumption we can write, analogously to (7.5), for each $j < \widetilde{m}_2 - 1$:

$$|\{i\colon \overline{\Upsilon^{j}_{\langle\mu_{\langle m,j+1\rangle},i\rangle}}(1) \underset{B(\widetilde{\mathcal{C}}_2)}{\leftrightsquigarrow} \overline{\Psi^{\widetilde{m}_2-1}_{\mu_{\langle m,\widetilde{m}_2-1\rangle}}}(2)\}| \Big| \left[\overline{\Upsilon^{j+1}_{\mu_{\langle m,j+1\rangle}}}(2) \underset{B(\widetilde{\mathcal{C}}_2)}{\leftrightsquigarrow} \overline{\Psi^{\widetilde{m}_2-1}_{\mu_{\langle m,\widetilde{m}_2-1\rangle}}}(2)\right] \succeq F_{p_{j,\widetilde{m}_2}}. \tag{7.8}$$

It is simple to check that $F_p \succeq F_{\tilde{p}}$ when $1 \geq p \geq \tilde{p} > 0$, and we may therefore replace $p_{j,\widetilde{m}_2}$ by $p_{j,m}$ on the r.h.s. of (7.8):

$$|\{i\colon \overline{\Upsilon^{j}_{\langle\mu_{\langle m,j+1\rangle},i\rangle}}(1) \underset{B(\widetilde{\mathcal{C}}_2)}{\leftrightsquigarrow} \overline{\Psi^{\widetilde{m}_2-1}_{\mu_{\langle m,\widetilde{m}_2-1\rangle}}}(2)\}| \Big| \left[\overline{\Upsilon^{j+1}_{\mu_{\langle m,j+1\rangle}}}(2) \underset{B(\widetilde{\mathcal{C}}_2)}{\leftrightsquigarrow} \overline{\Psi^{\widetilde{m}_2-1}_{\mu_{\langle m,\widetilde{m}_2-1\rangle}}}(2)\right] \succeq F_{p_{j,m}}. \tag{7.9}$$

Again we shall use (7.9) for all j down to $\widetilde{m}_3$.

Continuing for $s \leq v-1$ we extend the lower bounds for the probability of an active $\overline{\Upsilon^{\widetilde{m}_{s+1}}_{\mu_{\langle m,\widetilde{m}_{s+1}\rangle}}}(s)$ given the indices $\mu_{\langle m,\widetilde{m}_{s+1}\rangle}$ yielded “active” hierarchical sets in the previous steps.

The construction at the final step $s = v$ is slightly different as remarked above, and we consider two cases: a) $\max(\mathcal{C}) > \max(\widetilde{\mathcal{C}}_v) + 1$; b) $\max(\mathcal{C}) = \max(\widetilde{\mathcal{C}}_v) + 1$.

In both cases we proceed as before as if $\widetilde{m}_{s+1} = 0$, so that we use the analogue of (7.8) all the way down to $j = 0$. The only difference is that in case a) we again have a transfer operation, and we once more use Lemma 5.12, this time at scale 0, but in a space without bad layers and of vertical length at least L. In case b) we do not have the transfer operation, and the hierarchical set $\overline{\Upsilon^m}(v)$ stays on the last bad layer of $B_{\widetilde{\mathcal{C}}_v}$.

In both cases, the final step to connect each final $\Upsilon^0_{\mu_{\langle m,0\rangle}}(v)$ to the matching $\Upsilon^0_{\mu_{\langle m,0\rangle}}$ has probability bounded from below by $p_G^{\varkappa} p_B$.

Computing the probability. Verification of $(b_{m+1})'$. It is useful to establish a comparison with the following simple auxiliary scheme. Consider the following system of boxes: a unique $(m+1)$-box (or box of scale $m+1$) contains J m-boxes, each of them containing J boxes of scale $m-1$, and so on down to scale 1: each 1-box contains J boxes of scale 0, thought as points.

Definition 7.4. *Checking procedure:*

(a) Each 0*-box is “good” with probability* $p_G^{m\varkappa} p_B^{m+1}$*, all independently.*

(b) For each $k = 1, \ldots, m-1$ *a* k*-box is “good” if:*

- *it contains at least one “good”* $(k-1)$*-box;*
- *it is “approved” at* k*-step, which happens with probability* $p_k^{\varkappa(m-k)}$ *independently of everything else.*

(c) For $k = m, m+1$ *a* k*-box is “good” if it contains at least one “good”* $(k-1)$*-box.*

With all “approvals” taken independently, and independent of the initial assignments (good/ not good), it is straightforward to see that for each $k = 0, \ldots, m+1$, each k-box will be “good” with probability $p_{k,m+1}$.

Of course we could think of the previous procedure in two stages:

Stage 1

(a) Each 0-box is “pre-good” with probability $p_G^{(m-1)\varkappa} p_B^m$.

(b) For each $k = 1, \ldots, m-1$, a k-box is “pre-good” if:

- it contains at least one “pre-good” $(k-1)$-box;
- it is “pre-approved” at k-step, which happens with probability $p_k^{\varkappa(m-k-1)}$ independently of everything else.

(c) For $k = m, m+1$ a k-box is “pre-good” if it contains at least one “pre-good” $(k-1)$-box.

Stage 2 Each “pre-good” 0-box is “tested” again with probability $p_G^{\varkappa} p_B$; if successful, it is declared “good”. In increasing order each k-box ($k = 1, \dots m-1$) is “tested” again with probability $p_k^{\varkappa}$, all “tests” being independently; if test is successful and if it contains at least one “good” $(k-1)$-box, it is then declared “good”. For $k = m, m+1$, a k-box is declared “good” if it contains at least one “good” $(k-1)$-box.

After taking into account the estimates obtained with the procedure based on the itinerary of the descending decomposition of the block $\mathcal{C}$ with mass $m+1$ and level ℓ, we see that it is comparable (dominates, in the sense of stochastic order) with the previous “auxiliary scheme” with two stages: the first corresponds to the estimates provided by (7.5), (7.9) (at all steps $s = 1, \dots, v$), and the “testing at stage 2” comes from the “transfer” part, with the difference that the “test” with probability $p_j^{\varkappa}$ takes place only at $j = \widetilde{m}_{s+1}$, for $s = 1, \dots, v$ along the itinerary (recall $\widetilde{m}_{v+1} = 0$). At the scales which do not appear in the itinerary, the “test” is automatically successful with probability one.

Verification of $(b_{m+1})''$. The scheme used to define when a matching pair of $j+1$-sets is chained, by taking at each step J separated matching j-sets then yields (conditional) independence (at each step), and allows to easily conclude $(b_{m+1})''$ from $(b_{m+1})'$.

This concludes the proof of Proposition 7.1. □

To conclude the proof of Theorem 5.15 as stated in Section 5, i.e. for p_G close enough to 1, it remains essentially to show that by taking L large one can compare the numbers $p_{m-1,m}$ given by (7.1) with p_m defined immediately after (7.1) for all m. This will allow to conclude the induction step for (b_m) given by (5.23). The details are given below.

Conclusion of the proof of Theorem 5.15.

It remains to prove (5.29). Let us assume $m \geq 2$ and the validity of $(a_j), (b_j), (c_j), (d_j)$ for all $j \leq m-1$ and (a_m) as well.

Taking into account what has been proven earlier in this section, it remains to verify that

$$8N(1 - p_{m-1,m})^{\rho \frac{c}{12} \frac{L}{N}} \leq q_m, \text{ for all } m \geq 2, \tag{7.10}$$

where N is given by (5.21), and $p_{m-1,m}$, q_m, p_m are as in (7.1) and the line that follows it.

For this, and since L will be taken large it suffices to obtain

$$p_{m,m} \geq p_m, \quad \forall m \geq 2. \tag{7.11}$$

Let

$$\Theta = \prod_{k=0}^{\infty} p_k > 0,$$

which is an increasing function of $p_G = p_0$, as also $\rho = \rho(p_G)$.

We recall the interpretation of $p_{m,m}$ given in Definition 7.4 (with $m+1$ now replaced by m), and proceed with a similar checking procedure, leaving the p_B^m-probability for the final step of the 0-boxes, i.e. with the trivial observation that if one has t (a fixed integer) independent Bernoulli random variables with probability of success given by $p\tilde{p}$, then the probability of no success is bounded from above by

$$(1-\tilde{p})^{\lfloor tp/2 \rfloor} + e^{-tI_p(p/2)}$$

where $I_p(x) = x\log(x/p) + (1-x)\log((1-x)/(1-p))$, for $x \in (0,1)$, is the Cramér transform. (This follows at once from the decomposition of the Bernoulli essays into two independent ones, of probabilities $\tilde{p}$ and p respectively, and Cramér Theorem for the second one.)

At all steps i from 0 to $m-2$ each i-box is tested independently of anything else with probability $p_i^{\varkappa(m-i-1)}$, and at the end the 0-box has to be approved with probability $\tilde{p} = p_B^m$.[10] Using Cramér Theorem we can then estimate from above the probability that the m-box is not "good", by splitting it into cases: (a) for each i, the number of tested $i-1$-boxes which are successful is not smaller then half of its expected number; (b) the event in (a) fails at some step i. Thus,

$$1 - p_{m,m} \le (1-p_B^m)^{4(J/2)^m \prod_{i=0}^{m-2} p_i^{\varkappa(m-i-1)}} + \sum_{i=1}^{m-1} e^{-4(J/2)^{i+1} \prod_{j=2}^{i} p_{m-j}^{\varkappa(j-1)} f(p_{m-i-1}^{i\varkappa})} \tag{7.12}$$

with

$$f(p) = I_p(p/2) = (1-\frac{p}{2})\log(\frac{2-p}{1-p}) - \log 2. \tag{7.13}$$

It follows at once that L_0 large can be taken so that for all $m \ge 2$, and all $L \ge L_0(p_G, p_B)$,

$$(1-p_B^m)^{4(J/2)^m \prod_{i=0}^{m-2} p_i^{\varkappa(m-i-1)}} \le (1-p_B^m)^{4(J/2)^m (\Theta^\varkappa)^{m-1}} \le \frac{1}{2} q_m.$$

For the second term in (7.12), we split it into two pieces. For the piece corresponding to large values of i we use

$$\sum_{i=\lfloor m/2 \rfloor}^{m-1} e^{-2(J/2)^{i+1} \prod_{j=2}^{i} p_{m-j}^{\varkappa(j-1)} f(p_{m-i-1}^{i\varkappa})} \le \frac{m}{2} \exp\left\{-2(\frac{J}{2})^{m/2} \Theta^{\varkappa m} f(p_G^{\varkappa(m-1)})\right\}$$

which we can bound from above by $\frac{1}{4}q_m$ for all $m \ge 2$, provided $L \ge L_0'$ similarly as above. It remains to estimate

$$\sum_{i=1}^{\lfloor m/2 \rfloor - 1} e^{-4(J/2)^i \prod_{j=2}^{i} p_{m-j}^{\varkappa(j-1)} f(p_{m-i-1}^{i\varkappa})}.$$

Since we may assume (by taking L large) that $J\Theta^\varkappa > 2$, this last term is bounded from above by

$$(\frac{m}{2} - 1)\exp\{-2J\Theta^\varkappa f(p_{m/2}^{\varkappa m/2})\} \le \frac{m}{2}\exp\{-4 f(p_{m/2}^{\varkappa m/2})\}.$$

To have this bounded from above by $\frac{1}{4}q_m$ we need $4f(p_{m/2}^{\varkappa m/2}) > (m+1)\log q_0^{-1} + \log(2m)$, and a simple analysis of f given by (7.13) shows this is the case provided $q_0 = 1 - p_G$ is chosen sufficiently small. Indeed, writing for convenience $q_0 = e^{-y}$, it remains to check

$$-\log(1 - p_{m/2}^{\varkappa m/2}) \ge \frac{1}{4}(\log(4m) + (m+1)y) \tag{7.14}$$

[10] the m box and its $m-1$ boxes are not tested, according to (7.1)

Assuming $\varkappa m/2$ is an integer (small modification otherwise)

$$\begin{aligned}1-p_{m/2}^{\varkappa m/2} =& 1-\sum_{i=0}^{\varkappa m/2}\binom{\varkappa m/2}{i}(-1)^i e^{-i(m/2+1)y}\\ =& \sum_{i=1}^{\varkappa m/2}\binom{\varkappa m/2}{i}(-1)^{i+1}e^{-i(m/2+1)y}\\ \le& \sum_{i=1}^{\varkappa m/2}\binom{\varkappa m/2}{i}e^{-i(m/2+1)y}\\ \le& 2^{\varkappa m/2}e^{-(m/2+1)y}\end{aligned}$$

Thus for all such m

$$-\log(1-p_{m/2}^{\varkappa m/2}) \ge -\frac{\varkappa m}{2}\log 2 + (\frac{m}{2}+1)y \ge \frac{1}{4}(m+1)y,$$

provided $\frac{\varkappa}{2}\ln 2 < y$, which holds for p_G sufficiently close to 1. □

8 Extension to $p_G > p_c$

Theorem 8.1. *Let $p_B > 0$ and $p_G > p_c$. Then there exists $L_0(p_G, p_B)$ finite so that for all $L \ge L_0(p_G, p_B)$,* (5.26) *holds for all γ which is L-spaced.*

Proof. It suffices to prove an extension of Theorem 5.15 applicable to $p_G > p_c, p_B > 0$.

Some modifications of the scheme described in the previous sections are needed. The main point is a modification of the renormalized lattice at scale 1 in such a way that for passability of good S^1 sites we already have a probability that is larger than the previous p^*. [11] Having this we can use essentially the same argument as in the proof of Theorem 5.15. We now explain the main points, though omitting full details:

• The parameters ρ and c in Assumption 5.1 are suitably modified: we replace ρ by $\rho' \in (0, \theta(p_G))$ and we keep (ii), recalling that this implies a smaller value for c' instead of the value that we had fixed. (The change of c is not so important as a smaller but fixed positive value does not create any problem, and the same c' could be used at all scales. But we may as well use c' for the first scale and the previous c for all other scales.)

• We now complete the modifications of the blocks $\mathcal{H}_v^1$ (and $\hat{\mathcal{H}}_v^1$) and the 1-sites $S^1_{(u,v)}$. Given $p_G > p_c$, we may take K large enough so that the probability of an infinite oriented path in the homogeneous percolation model P_{p_G} starting with K sites is at least $1-(1-p^*)/4$. With such a K, we change the definition of the approximate endpoints in the good type one blocks $\mathcal{H}_v^1$, replacing 3 by $K+1$. This is not a problem as we may assume that our scale parameter L is large enough. We then enlarge the size of the 0-seeds $Q^{(0)}$: it keeps the triangular shape but has K sites at its top line. To simplify we say that the seed is open is all its sites are open. Of course this will have a much smaller probability, but it is used only at scale 1.

• Given these modifications, we may also chose L large enough so that conditions (c_1) and (d_1) become satisfied when at scale 1 with ρ in Assumption 5.1 replaced by $\rho' \in (0, \theta(p_G))$.

• Increasing L if needed $L \ge L_0(p_G, p_B)$, one can check that the conditional probability of S^1 being $s-$ passable given $Q^{(0)}$ is larger p^*, where the notion of passability at the level 1 includes two enlarged 0-seeds on the top left and top right parts of S^1.

[11] Or even larger than the previous lower bound for p_1, i.e. $1-(1-p^*)^2$.

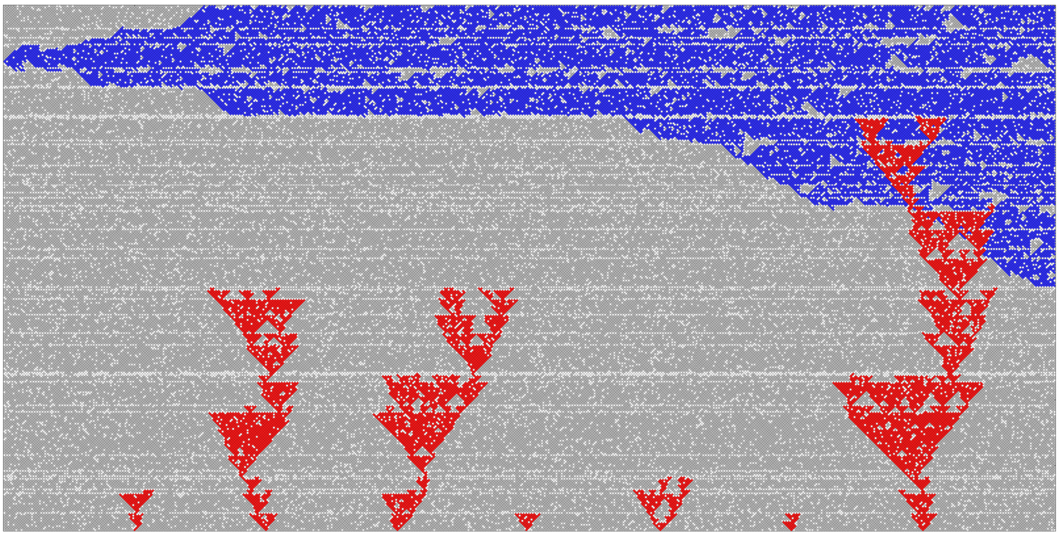

Figure 7: A simulation on a finite grid of $\widetilde{\mathbb{Z}}^2_+$, with $p_G = 0.88$, $p_B = 0.22$, $\delta = 0.125$. In red, the model studied in this paper. In blue, with orientation along the layers, a discrete analogue of the contact process in [5].

• The blocks $\mathcal{H}^k_v$ and $\hat{\mathcal{H}}^k_v$ and renormalized sites $S^k_{(u,v)}$ for $k \geq 2$ and renormalization scheme keep the same definition as before. In particular, the k-seeds, contain only three passable sites of scales $1 \leq j \leq k-1$.

• At this point the only non-trivial modification involves the induction step for (b_m) done in Sections 6 and 7. Going down to level 1 we follow the same procedure as before. The key change is in the definition of a pair of good (type 2) 1-sites, say S^1 and $\widehat{S}^1$, being chained through a bad layer $B(\mathcal{C})$, where $\mathcal{C} \in \mathbf{C}_1$ has mass $m \geq 2$ and level 1, and in the corresponding probability estimate of such event. Assume that two sites S^1 and $\widehat{S}^1$ have s-dense kernel (here the kernel coincide with the site) and respectively reverse $\hat{c}$-dense kernel. To concatenate open 0-sites in the cluster within S^1 to some open 0-site in the reverse cluster within $\widehat{S}^1$, we of course need to act differently from the case of large p_G since the 0-sites of the top line of S^1 have no reason to be straight below one of those of the reverse cluster in bottom line of $\widehat{S}^1$, as the density can now be arbitrary small. We just consider a number (order $\sqrt{L}$) of separated such sites in the central part of the top line of S^1 and examine those for which we have an open path within its zone (a rectangle of width of order $\sqrt{L}$ through $B(\mathcal{C})$, so that separated sites have disjoint zones) that crosses the bad layer and reaches the first line of $\widehat{S}^1$. This is done in the same fashion as before, recalling Lemma 5.12 and Remark 5.13. All we need is that from one of these points we have an open oriented path that crosses the bad layer within its zone and connects to the reverse cluster in $\widehat{S}^1$ anywhere within $\widehat{S}^1$, as we may again use the planarity. From this we see that the probability estimates are compatible with those that we had in Section 7 and the proof extends. □

A comment. It would be interesting to be able to say something about the shape of the cluster. Within the current approach, this would involve being able to let the parameter $c = c_k$ grow with the scale k. It is conceivable that one may indeed be able to pursue this.

A Appendix

Proof of Lemma 5.9. We shall only need the statement if $\mathcal{A}$ is an interval of a integers, and therefore we shall prove (5.14) only in this case. However [8] (p. 1029) proves that this is the worst case, i.e., that if (5.14) holds for $\mathcal{A}$ an interval, then it holds in general. ([8] discusses bond percolation, but a small modification of his argument works for site percolation.)

Now let $\mathcal{A} = \{0, 2, \dots, 2(a-1)\} \times \{0\}$ and let $\mathcal{F}$ be the collection of sites $(x, y) \in \widetilde{\mathbb{Z}}^2_+$ for which there exists an open path from $\mathcal{A}$ to (x, y) (with (x, y) itself also open). Then

$$1 - P_p(\text{there is an open path from } \mathcal{A} \text{ to } \infty \big| \text{ all of } \mathcal{A} \text{ is open}) \leq P_p\left(\bigcup_F \{\mathcal{F} = F\}\right), \tag{A.1}$$

where the union runs over all finite connected subsets F of $\widetilde{\mathbb{Z}}^2_+$ which contain all of $\mathcal{A}$. We bound the right hand side of (A.1) by the usual contour method, as we explain now. As in Section 10 of [8] or [17], let D be the diamond $\{(x, y) : |x| + |y| \leq 1\} \subset \mathbb{R}^2$. For F a finite connected subset of $\widetilde{\mathbb{Z}}^2_+$ which contains $\mathcal{A}$, we define $\widetilde{F} = F + D$ and $\Gamma(F) =$ the topological boundary of the infinite component of $\mathbb{R}^2 \setminus \widetilde{F}$. Then $\Gamma(F)$ is made up of edges of the lattice $\mathbb{Z}^2_{odd} := \{(x, y) \in \mathbb{Z}^2\ x + y \text{ is odd}\}$ and it separates $\mathcal{A} \subset F$ from infinity. Suppose that $\mathcal{F} = F$ occurs and that e is an edge between two vertices of $\{(x, y) \in \mathbb{Z}^2 : x + y \text{ is even}\}$ which crosses one of the sides of one of the diamonds $v + D, v \in F$. In fact we must then have that one endpoint of e equals v and the other endpoint, w say, lies in the unbounded component of $\mathbb{R}^2 \setminus \widetilde{F}$. There are then two possibilities. Either

$$w \text{ lies below } v, \tag{A.2}$$

so that a path on $\widetilde{\mathbb{Z}}^2_+$ is prevented from going from v to w by the orientation of $\widetilde{\mathbb{Z}}^2_+$. Or,

$$w \text{ lies above } v, \tag{A.3}$$

in which case w must be closed (otherwise w would belong to F, since an open path to v can be continued by going along e from v to w). It follows from this argument that the event $\cup_F \{\mathcal{F} = F\}$ is contained in the event that there exists some contour Γ made up of sides of the diamonds $u + D, u \in \widetilde{\mathbb{Z}}^2_+$, which separates $\mathcal{A}$ from infinity, and which has the following property: if the edge $\{v, w\}$ crosses one of the sides which make up Γ and $v \in$ interior (Γ) and $w \in$ exterior $(\Gamma) \cap \widetilde{\mathbb{Z}}^2_+$ and (A.3) holds, then w is closed. Consequently, the right hand side of (A.1) is bounded by

$$\sum_\Gamma P_p(\text{each } w \in \widetilde{\mathbb{Z}}^2_+ \text{ as above for which (A.3) holds is vacant}). \tag{A.4}$$

It is shown in [17] and [8] that the number of w for which (A.3) holds is at least $|\Gamma|/2$, where $|\Gamma|$ denotes the number of edges in Γ. Moreover, as one traverses the line $\{x = y + 2i\}$, starting at $(2i, 0) \in \mathcal{A}$ and increasing x (and y), the first vertex $w \in \widetilde{\mathbb{Z}}^2_+$ in the unbounded component of $\mathbb{R}^2 \setminus \Gamma$ which one meets has to be closed. Since this holds for every $0 \leq i \leq a - 1$, the number of w for which (A.3) holds is at least a. In fact, there have to be at least $a + 1$ such vertices w, because the first vertex w on the line $x = -y$ which lies in the unbounded component of $\mathbb{R}^2 \setminus F$ also satisfies (A.3), but does not lie on any of the lines $x = y + 2i$. It follows that the term in (A.4) for a specific Γ is at most $(1-p)^{(|\Gamma|/2) \vee (a+1)}$. Moreover, the number of possible Γ with $|\Gamma| = n$ is at most 3^{n-1}, because each possible Γ which separates $\mathcal{A}$ from infinity must contain the lower left edge of the diamond $(0, 0) + D$, centered at the origin. It follows that (A.4), and hence also the right hand side of (A.1) is bounded by

$$\sum_{n=1}^{\infty} 3^{n-1}(1-p)^{(n/2) \vee (a+1)}.$$

The lemma follows. $\square$

Proof of Lemma 5.10. In general, and in particular in the definitions (5.16) and (5.18) of ν_n, an open path has to have its initial point and its endpoint open. For the sake of the proof of the present lemma we shall call a path open if all its vertices other than its initial point are open. Until the last three sentences of the proof we allow its initial point to be open or closed.

Clearly $\nu_n(\alpha, \beta, \mathcal{A}, \eta)$ is increasing in $\mathcal{A}$, so that it suffices to prove (5.18) for $\mathcal{A} =$ the origin. We shall restrict ourselves to $p \geq \tilde{p}$ as in Lemma 5.9. By obvious monotonicity we then have $\theta(p) \geq \theta(\tilde{p}) > 0$. In addition it is immediate from the definition (5.17) that $s(p) \leq 1$. In fact

$$r_n \leq n \text{ and } \ell_n \leq n \text{ for all } n. \tag{A.5}$$

Thus, it holds

$$\theta(p) \geq \theta(\tilde{p}) > 0 \text{ and } 0 < s(\tilde{p}) \leq s(p) \leq 1 \tag{A.6}$$

for the p which we are considering.

Now let $\varepsilon > 0$ and $\eta > 0$ be given. We define

$$m = m(n, \eta) = \left\lfloor \frac{\eta}{8} n \right\rfloor, k_0 = k_0(\eta) = \left\lfloor \frac{n}{m} \right\rfloor - 1, m' = n - k_0 m. \tag{A.7}$$

Finally, we choose ε_1 such that

$$0 < \varepsilon_1 \leq \frac{\varepsilon}{2k_0}. \tag{A.8}$$

and then $\bar{p} = \bar{p}(\varepsilon, \eta) < 1$ so that $\bar{p} \geq \tilde{p} \vee (1 - \varepsilon/2)$ and

$$\theta(\bar{p}) := P_{\bar{p}}(\Omega_0) \geq 1 - \varepsilon_1. \tag{A.9}$$

Such a $\bar{p} < 1$ exists by (5.14).

First we observe that (5.17) implies that for every $p \geq \bar{p}$ and $\eta_1 > 0$ there exists a constant $c_6 = c_6(\varepsilon_1, \eta_1, p)$ such that

$$\begin{aligned} &P_p\left(\{|r_t - ts(p)| > c_6 + \eta_1 t \text{ or } |\ell_t + ts(p)| > c_6 + \eta_1 t \text{ for some } t \in \mathbb{Z}_+\}\right) \\ \leq\; &P_p\left(\{|r_t - ts(p)| > c_6 + \eta_1 t \text{ or } |\ell_t + ts(p)| > c_6 + \eta_1 t \text{ for some } t \in \mathbb{Z}_+\} \cap \Omega_0\right) + P_p(\Omega_0^c) \\ \leq\; &2\varepsilon_1. \end{aligned} \tag{A.10}$$

We observe next that if Ω_0 occurs, then r_t and ℓ_t are well defined for all t. Furthermore, for any m there must exist open paths $\pi_\ell = \pi_\ell(\cdot)$ and $\pi_r = \pi_r(\cdot)$ from the origin to (ℓ_m, m) and to (r_m, m), respectively, and these paths must lie in $[-m, m] \times [0, m]$ (see (A.5)). Next let $x \in \mathbb{Z}$ with $x + m$ even be such that $-\ell_m \leq x \leq r_m$. Consider the open paths starting at (x, m) going *downwards*, that is against the orientation on $\widetilde{\mathbb{Z}}_+^2$ assumed so far. Assume that for a given $x \in [-\ell_m, r_m]$ there exists an infinite downward open path, $\widetilde{\pi}_x$ say, starting at (x, m). Since this path starts between $(-\ell_m, m)$ and (r_m, m), it must hit $\pi_l \cup \pi_r$. Furthermore, the path $\widetilde{\pi}_x$ necessarily stays in $[x - m, x + m] \times [0, m]$ up till time m. For the sake of argument, let $\widetilde{\pi}_x$ first intersect π_ℓ in a point (y, q) with $0 \leq q \leq m$. Then the piece of π_ℓ from the origin to (y, q), followed by the piece of $\widetilde{\pi}_x$, traversed in the forward direction, from (y, q) to (x, m) forms an open oriented path from the origin to (x, m). A similar argument applies if $\widetilde{\pi}_x$ hits π_r. Thus, if there exists a downward infinite open path from (x, m), then there exists an open path from the origin to (x, m). By the estimates on the locations of π_ℓ, π_r and of $\widetilde{\pi}_x$ which we have just given, this path must be contained in $[-2m, 2m] \times [0, m]$.

Let us write J_x for the indicator function of the event that there is an open path contained in $[-2m, 2m] \times [0, m]$ from the origin to (x, m). Also, let $\widetilde{I}_x$ and I_x be the indicator functions of the events that there exists an infinite open backwards path from (x,m), respectively an infinite open forwards path from $(x,0)$, which stays in $[-2m,2m]$ during $[0,m]$. The preceding argument shows that for any $M \ge 0$, on the event

$$\mathcal{E}(M,m) := \{-\ell_m \le -2M \le 2M \le r_m\}, \tag{A.11}$$

it holds

$$\begin{aligned}\widetilde{\nu}_m(M,\mathbf{0}) := &\text{ number of points } (x,m) \text{ with } x \in [-2M,2M], \\ &x+m \text{ even, for which there exists an open path from } \mathbf{0} \\ &\text{to } (x,m) \text{ which stays inside } [-2m,2m]\times[0,m] \\ &\ge \textstyle\sum_{\substack{-2M\le x\le 2M\\ x+m \text{ even}}} J_x \ge \sum_{\substack{-2M\le x\le 2M\\ x+m \text{ even}}} \widetilde{I}_x.\end{aligned} \tag{A.12}$$

Now, the monotonicity of $s(\cdot)$ and (A.6) and (5.17) imply that for each fixed $M>0$ there exists an $m_1 = m_1(\varepsilon_1, M)$ such that for $m \ge m_1$ and $p \ge \bar{p}$

$$P_p\{\mathcal{E}(M,m) \text{ fails}\} \le P_{\bar{p}}\{\mathcal{E}(M,m) \text{ fails}\} \le \varepsilon_1. \tag{A.13}$$

Further, the joint distribution of the $\widetilde{I}_x,\ x+m$ even is the same as the joint distribution of the $I_x,\ x$ even. Also, if

$$M + c_6 \le \frac{m}{2} \text{ and } |x| \le \frac{m}{2}, \tag{A.14}$$

then

$$\begin{aligned}I_x \ge K_x := I[&\text{there exists an open path } \pi \text{ from } (x,0) \text{ to infinity so that} \\ &\pi(t) \in [x-M-c_6-t, x+M+c_6+t] \text{ for all } t \ge 0].\end{aligned}$$

By the ergodic theorem (see (5.2) for θ)

$$\liminf_{M\to\infty} \frac{1}{M} \sum_{\substack{x\in[-2M,2M],\\ x \text{ even}}} I_x \ge \lim_{M\to\infty} \frac{1}{M} \sum_{\substack{x\in[-2M,2M],\\ x \text{ even}}} K_x \ge 2\theta(p) \text{ a.s. } [P_p]. \tag{A.15}$$

Thus there exists an $M_0 = M_0(\varepsilon_1)$ such that for all $p \in [\bar{p}, 1)$

$$\begin{aligned}&P_p\left(\widetilde{I}_x = 1 \text{ for some } x \in [-2M_0, 2M_0]\right) \\ &\ge P_{\bar{p}}\left(\tfrac{1}{M_0}\textstyle\sum_{\substack{x\in[-2M_0,2M_0]\\ x+m \text{ even}}} \widetilde{I}_x \ge (2-\varepsilon_1)\theta\right) \ge 1-\varepsilon_1.\end{aligned} \tag{A.16}$$

We take $m_2 = m_2(\varepsilon_1)$ such that $m_2/2 \ge 2M_0 + c_6$. Then (A.14) with M_0 for M holds true for any $|x| \le 2M_0, m \ge m_2$. We now apply (A.13) and (A.16) to obtain for all $m \ge m_2, p \ge p_3$

$$\begin{aligned}P_p(&\text{there is at least one } x \in [-2M_0, 2M_0] \text{ with an open path from} \\ &\mathbf{0} \text{ to } (x,m) \text{ which is contained in } [-2m,2m]\times[0,m]) \\ &\ge 1-2\varepsilon_1.\end{aligned}$$

In other words, if we first determine the state of all vertices (x,y) with $0 \le y \le m$, we will find with probability $1-2\varepsilon_1$ at least one vertex (x_1,m) with $x_1 \in [-2M_0, 2M_0]$, x_1+m even and with an open connection from $\mathbf{0}$ to (x_1,m) which stays in $[-2m,2m]\times[0,m]$. On the event that such an x exists, let x_1 be the smallest x in $[-2M_0,2M_0]$ with these properties. We can then repeat the argument (after a shift by (x_1,m)), to find that with a further conditional probability of at least $1-2\varepsilon_1$, there exists an $x_2 \in [x_1 -$

$2M_0, x_1 + 2M_0] \subset [-4M_0, 4M_0]$ with an open path from (x_1, m) to $(x_2, 2m)$ which stays in $[x_1 - 2m, x_1 + 2m] \subset [-4m, 4m]$ during $[m, 2m]$. Concatenation of the open path from $\mathbf{0}$ to (x_1, m) and the path from (x_1, m) to $(x_2, 2m)$ gives an open path from $\mathbf{0}$ to $(x_2, 2m)$ which stays in $[-4m, 4m]$ during $[0, 2m]$. Similarly, we find by repeating the argument k_0 times that there is a probability of at least $(1 - 2\varepsilon_1)^{k_0}$ that $\mathbf{0}$ is connected by an open path which stays in $[-2k_0m, 2k_0m] \times [0, k_0m]$ to a vertex (x_{k_0}, k_0m) with $|x_{k_0}| \le 2k_0M_0$.

We need to concatenate paths once more. This time we replace m by $m' \in [m, 2m]$ (see (A.7)) and the sum over $-2M_0 \le x \le 2M_0$ in (A.15) by the sum over $\alpha m' \le x \le \beta m'$ for some fixed $-s(p) \le \alpha \le \beta \le s(p)$. In essentially the same way as before we conclude that for $p \ge \bar{p}$ and $m \ge m_3 = m_3(\varepsilon_1, p)$ (suitable)

$$\begin{gathered} P_p\big(\text{for all } -s(p) \le \alpha \le \beta \le s(p) \text{ there are at least } [\theta(p)(\beta - \alpha)/2 - \varepsilon_1]m' \\ \text{values of } x \text{ with } \alpha m' \le x \le \beta m', x + m' \text{ even, for which there} \\ \text{is an open path from } \mathbf{0} \text{ to } (x, m') \text{ which stays inside} \\ [-2m', 2m'] \times [0, m'] \text{ during } [0, m']\big) \\ \ge 1 - 2\varepsilon_1. \end{gathered}$$

If x_{k_0} as described above exists, then there is a conditional probability, given the state of all vertices $(x, y) \in \widetilde{\mathbb{Z}}^2_+$ with $y \le k_0m$, of at least $(1 - 2\varepsilon_1)$ that (x_{k_0}, k_0m) is connected to at least

$$[\theta(\beta - \alpha)/2 - \varepsilon_1]m' \ge [\theta(\beta - \alpha)/2 - \varepsilon_1]m \ge [\theta(\beta - \alpha)/2 - \varepsilon_1]\left\lfloor \frac{\eta}{8}n \right\rfloor$$

vertices $(x', km + m') = (x', n)$ in $[\alpha m' - 2k_0m, \beta m' + 2k_0m] \times \{n\}$ by open paths which stay in

$$[-2k_0M_0 - 2m', 2k_0M_0 + 2m'] \times [0, n] \text{ during } [0, k_0m + m'] = [0, n].$$

But by (A.7) there exists some $n_0 = n_0(\varepsilon, \eta)$ such that for $n \ge n_0$ it holds $m \ge m_1 \vee m_2 \vee m_3$ and

$$2k_0M_0 + 2m' \le 2k_0M_0 + 4m \le 2\frac{n}{m}M_0 + 4\frac{\eta}{8}n \le \eta n,$$

so that the constructed paths stay in $[-\eta n, \eta n] \times [0, n]$, as is required for them to be counted in ν_n. Also, by our choice of ε_1 in (A.8)

$$(1 - 2\varepsilon_1)^{k_0+1} \ge 1 - 2(k_0 + 1)\varepsilon_1 \ge 1 - \varepsilon/2.$$

We had to concatenate $k_0 + 1$ paths, each of which existed with a conditional probability of at least $1 - 2\varepsilon_1$, given the previously chosen paths. Thus the whole construction works with a probability of at least $(1 - 2\varepsilon_1)^{k_0+1} \ge 1 - \varepsilon/2$. This proves (5.18) when $\mathcal{A} = \mathbf{0}$. As pointed out before this proves the lemma if we do not insist that the starting point of an open path is open. However, if we revert to our previous convention that an open path must have an open initial and final point, then our construction of open paths from $\mathbf{0}$ to the horizontal line $\{y = n\}$ is valid only on the event $\{\mathbf{0} \text{ is open}\}$. We therefore have to discard the event $\{\mathbf{0} \text{ is closed}\}$. Correspondingly, the probability of finding the required open paths is at least $(1 - \varepsilon_1)^{k_0+1} - (1 - p) \ge 1 - \varepsilon/2 - \varepsilon/2 = 1 - \varepsilon$ (recall that $p \ge \bar{p} \ge 1 - \varepsilon/2$; see the line before (A.9)). $\square$

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