Electronic Journal of Probability

Electron. J. Probab. **27** (2022), article no. 58, 1–61.
ISSN: 1083-6489 https://doi.org/10.1214/22-EJP772

Fluctuations of transverse increments in two-dimensional first passage percolation

Ujan Gangopadhyay*

Abstract

We consider a model of first passage percolation (FPP) where the nearest-neighbor edges of the standard two-dimensional Euclidean lattice are equipped with random variables. These variables are i.i.d. nonnegative, continuous, and have a finite moment generating function in a neighborhood of 0. We derive consequences about transverse increments of passage times, assuming the model satisfies certain properties. Approximately, the assumed properties are the following: We assume that the standard deviation of the passage time on scale r is of some order $\sigma(r)$, and $\{\sigma(r), r > 0\}$ grows approximately as a power of r. Also, the tails of the passage time distributions for distance r satisfy an exponential bound on a scale $\sigma(r)$ uniformly over r. In addition, the boundary of the limit shape in a neighborhood of some fixed direction θ has a uniform quadratic curvature. By transverse increment we mean the difference between passage times from the origin to a pair of points which are approximately at the direction θ and the direction between the pair of points is the direction of the tangent to the boundary of the limit shape at the direction θ. The main consequence derived is the following. If $\sigma(r)$ varies as r^{χ} for some $\chi > 0$, and ξ is such that $\chi = 2\xi - 1$, then the fluctuation of the transverse increment of passage time between a pair of points situated at distance r from each other is of the order of $r^{\chi/\xi}$.

Keywords: first passage percolation; transverse increments; wandering exponent; fluctuation exponent.
MSC2020 subject classifications: Primary 60K35, Secondary 82B43.
Submitted to EJP on May 5, 2021, final version accepted on March 30, 2022.
Supersedes arXiv:2011.14686.

1 Introduction

In this paper, we investigate the transverse increments of passage times in the classical model of first passage percolation (FPP) on $\mathbb{Z}^2$, which was introduced in [19].

*National University of Singapore. E-mail: ujangangopadhyay@gmail.com

1.1 A brief description of the model

Let $\mathtt{E}(\mathbb{Z}^2)$ be the set of nearest-neighbor edges in $\mathbb{Z}^2$. On $\mathtt{E}(\mathbb{Z}^2)$ we consider a collection of random variables $\mathbb{T} := \{ \tau_e : e \in \mathtt{E}(\mathbb{Z}^2) \}$, which are called *edge-weights.* We assume certain properties of the edge-weights. We categorize these assumptions as being basic or technical. We use the basic assumptions throughout the paper. We use the technical assumptions more selectively.

The basic assumptions: We assume that the edge-weights are i.i.d., nonnegative, and continuous. In addition, there exists $C > 0$ such that $\mathbb{E}[\exp(C\tau_e)] < \infty$.

Using the edge-weights, we define the *passage time* of a self-avoiding lattice path γ, denoted by $T(\gamma)$, as the sum of the edge-weights of all the edges on the path γ, i.e.,

$$T(\gamma) := \sum_{\gamma \text{ contains } e} \tau_e \ .$$

In the above definition, we adopt the convention that *a path is a continuous, piece-wise constant curve in* $\mathbb{R}^2$ *which traces the edges of the integer lattice.* Next, we define the passage time between two points $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\mathbb{Z}^2$ as

$$T(\boldsymbol{u}, \boldsymbol{v}) := \inf\{ T(\gamma) : \gamma \text{ is a path joining } \boldsymbol{u} \text{ and } \boldsymbol{v} \} \ .$$

It follows from the above definition that T is a random pseudo-metric on $\mathbb{Z}^2$. In [34] it was shown, assuming only the i.i.d. and nonnegative assumptions on the edge-weights, the infimum in the definition of T is attained for some paths, i.e., the infimum is a minimum. Since we have assumed that the edge-weights are also continuous, it follows that there is only one such minimizing path almost surely. We call this path *the geodesic* between $\boldsymbol{u}$ and $\boldsymbol{v}$, and denote it by $\Gamma(\boldsymbol{u}, \boldsymbol{v})$. Since the edge-weights have finite expectation, the passage times $T(\boldsymbol{u}, \boldsymbol{v})$, for all $\boldsymbol{u}$, $\boldsymbol{v}$, also have finite expectation. Therefore,

$$h(\boldsymbol{u}) := \mathbb{E}[T(\mathbf{0}, \boldsymbol{u})]$$

is well-defined. From the triangle inequality of T it follows that h is subadditive, i.e., for any $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{Z}^2$ we have

$$h(\boldsymbol{u} + \boldsymbol{v}) \le h(\boldsymbol{u}) + h(\boldsymbol{v}) \ .$$

The subadditive ergodic theorem in [25] implies that for any $\boldsymbol{u} \in \mathbb{Z}^2$ the following limits exist almost surely and in L^1:

$$g(\boldsymbol{u}) := \lim_{n \to \infty} \frac{T(\mathbf{0}, n\boldsymbol{u})}{n} = \lim_{n \to \infty} \frac{h(n\boldsymbol{u})}{n} = \inf_{n > 0} \frac{h(n\boldsymbol{u})}{n} \ .$$

The domain of g can be extended to $\mathbb{Q}^2$ by taking limit along appropriate subsequences in the above definition. By extending the domain in this way, g becomes a norm on $\mathbb{Q}^2$. Therefore, the domain of g can be further extended to $\mathbb{R}^2$. The unit ball in the norm g is

$$\mathcal{B} := \left\{ \boldsymbol{x} \in \mathbb{R}^2 : g(\boldsymbol{x}) \le 1 \right\} ,$$

This is called the *limit shape.* The *wet region* at time t is defined as

$$\mathcal{B}(t) := \left\{ \boldsymbol{x} \in \mathbb{Z}^2 : T(\mathbf{0}, \boldsymbol{x}) \le t \right\} .$$

The shape theorem in [13] implies, under conditions milder than our basic assumptions, $\mathcal{B}(t)$ approaches $\mathcal{B}$ in an appropriate sense as $t \to \infty$. In addition, $\mathcal{B}$ is compact, convex, has a nonempty interior, and has all the symmetries of the lattice.

Notation 1.1. *Here, we define passage times between points in* $\mathbb{R}^2$*. For* $\boldsymbol{x} \in \mathbb{R}^2$*, let* $\lfloor \boldsymbol{x} \rfloor$ *be the down-left corner of the unit square containing* $\boldsymbol{x}$ *in* $\mathbb{Z}^2$*. For* $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^2$*, let*

$$T(\boldsymbol{x}, \boldsymbol{y}) \coloneqq T(\lfloor \boldsymbol{x} \rfloor, \lfloor \boldsymbol{y} \rfloor) \, .$$

Similarly, by $\Gamma(\boldsymbol{x}, \boldsymbol{y})$ *we mean the geodesic* $\Gamma(\lfloor \boldsymbol{x} \rfloor, \lfloor \boldsymbol{y} \rfloor)$*. Furthermore, for* $\boldsymbol{x} \in \mathbb{R}^2$*, let*

$$h(\boldsymbol{x}) \coloneqq h(\lfloor \boldsymbol{x} \rfloor) \, .$$

Remark 1.2. Throughout the paper, we denote by $C, C_0, C_1, C_2, \ldots$ constants that depend only on the distribution of the edge-weights. We restart numbering of C_is in each proof. Often we break the proof of a theorem in propositions and claims. In these situations, we do not restart numbering the constants in the proof of the propositions and claims. Also, when we use a result which has been proved before, we use "tilde-version" of the variables and the parameters.

Remark 1.3. Extending definition of T from $\mathbb{Z}^2$ to $\mathbb{R}^2$ yields a minor technical issue. Although $g(\boldsymbol{x}) \leq h(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{Z}^2$, this may not be true for $\boldsymbol{x} \in \mathbb{R}^2$. Instead, we have, for some constant $C_1 > 0$ and for all $\boldsymbol{x} \in \mathbb{R}^2$

$$g(\boldsymbol{x}) - C_1 \leq h(\boldsymbol{x}) \, .$$

Similarly, we have, for some constant $C_2 > 0$ and for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^2$

$$h(\boldsymbol{x} + \boldsymbol{y}) - C_2 \leq h(\boldsymbol{x}) + h(\boldsymbol{y}) \, .$$

1.2 Heuristics of the main results

It is common in the literature, for instance, in the works [29], [30], [27], [14], [15], to assume specific unproved properties of the limit shape. Often properties such as differentiability, and curvature, either locally or globally, which eliminates the possibility of facets or corners. These properties are believed to be valid under our assumptions, but there is no proof yet. We also make similar assumptions.

Suppose the boundary of the limit shape is differentiable at a direction θ, and θ^t is the corresponding tangential direction. By *transverse increments* we mean differences of the form $T(\mathbf{0}, \boldsymbol{x}) - T(\mathbf{0}, \boldsymbol{y})$ where $\boldsymbol{x}$ has direction θ, $\boldsymbol{x} - \boldsymbol{y}$ has direction θ^t. Heuristically we can say what the order of the fluctuations of transverse increments should be. For this, we need the scaling exponents χ and ξ.

It is believed that for FPP on Euclidean lattices of any dimension, that there exists an exponent χ, called the *'fluctuation exponent,'* such that $T(\mathbf{0}, \boldsymbol{v}) - h(\boldsymbol{v})$ is of the order of $\|\boldsymbol{v}\|^\chi$ ($\|\cdot\|$ is the Euclidean norm.) Also, it is believed that there exists an exponent ξ, called the *'wandering exponent,'* such that the geodesic $\Gamma(\mathbf{0}, \boldsymbol{v})$ wanders $\|\boldsymbol{v}\|^\xi$ distance on average from the line joining $\mathbf{0}$ and $\boldsymbol{v}$. The two exponents are related by the equation $\chi = 2\xi - 1$ which has been proved in [12] assuming χ and ξ exist in a certain sense. In dimension $d = 2$ it is believed that $\chi = 1/3$ and $\xi = 2/3$. In $d = 3$ it is believed that χ is approximately $1/4$, and in higher dimensions there does not seem to be a consensus even among physicists about values of χ and ξ, see for example [28], [26], [17], [24], [5]. In the exactly solvable models of two-dimensional last passage percolation, it has been proved that $\chi = 1/3$ and $\xi = 2/3$, see [20], [21], [7].

If one assumes the existence of these exponents in some appropriate sense, then fluctuations of the transverse increment $T(\mathbf{0}, \boldsymbol{x}) - T(\mathbf{0}, \boldsymbol{y})$ should be of the order of $\|\boldsymbol{x} - \boldsymbol{y}\|^{\chi/\xi}$. The heuristic of this is the following, see Figure 1. We expect that the geodesics $\Gamma(\mathbf{0}, \boldsymbol{x})$ and $\Gamma(\mathbf{0}, \boldsymbol{y})$ stay disjoint after starting from $\boldsymbol{x}$ and $\boldsymbol{y}$ respectively for a distance of the order of $\|\boldsymbol{x} - \boldsymbol{y}\|^{1/\xi}$. Then these two branches should contribute approximately independently $\|\boldsymbol{x} - \boldsymbol{y}\|^{\chi/\xi}$ to the fluctuation. The right scale of the coalescence time as above has been proved in [9] for the exactly solvable model of two-dimensional last passage percolation.

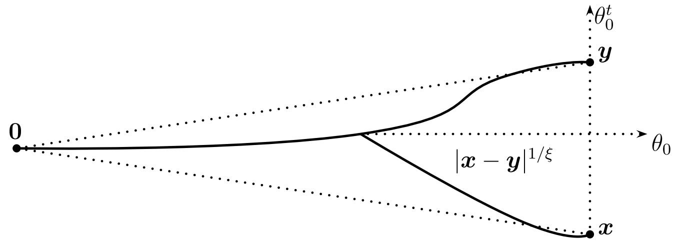

Figure 1: Illustration for the heuristic of the exponent χ/ξ: directions of $\boldsymbol{x}$ and $\boldsymbol{y}$ are approximately θ_0; direction of $\boldsymbol{y}-\boldsymbol{x}$ is θ_0^t; the two geodesics $\Gamma(\mathbf{0},\boldsymbol{x})$ and $\Gamma(\mathbf{0},\boldsymbol{y})$ are expected to coalesce approximately at distance $\|\boldsymbol{x}-\boldsymbol{y}\|^{1/\xi}$ in $-\theta_0$ direction when traced starting from $\boldsymbol{x}$ and $\boldsymbol{y}$ respectively.

One reason for studying the fluctuations of transverse increments is the following. In $d=2$, it is believed that the transverse increments behave like increments of Brownian motion, that is, the increments are approximately uncorrelated. If this is true, then the exponent for fluctuation of transverse increments should be $1/2$ so that $\chi/\xi=1/2$. This with $\chi=2\xi-1$ would imply $\chi=1/3$ and $\xi=2/3$.

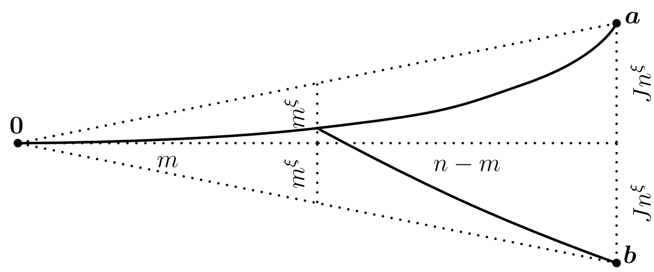

Figure 2: Illustration for the heuristic of the exponent $2\chi/(1-\xi)$: directions of $\boldsymbol{a}$ and $\boldsymbol{b}$ are approximately θ_0; direction of $\boldsymbol{a}-\boldsymbol{b}$ is θ_0^t; distances of $\boldsymbol{a}$ and $\boldsymbol{b}$ from origin are approximately n; distance between $\boldsymbol{a}$ and $\boldsymbol{b}$ is $2Jn^\xi$; the two geodesics $\Gamma(\mathbf{0},\boldsymbol{a})$ and $\Gamma(\mathbf{0},\boldsymbol{b})$ are expected to branch apart at approximately distance m from the origin where $m^\xi/m=Jn^\xi/n$.

As an application of the upper bound on fluctuations of transverse increments we get an upper bound on long-range correlations. By long-range correlation we mean the correlation between $T(\mathbf{0},\boldsymbol{a})$ and $T(\mathbf{0},\boldsymbol{b})$ where $\boldsymbol{a}$ and $\boldsymbol{b}$ are points approximately in the same direction from the origin and distance between $\boldsymbol{a}$ and $\boldsymbol{b}$ is large compared to typical wanderings of the geodesics $\Gamma(\mathbf{0},\boldsymbol{a})$ and $\Gamma(\mathbf{0},\boldsymbol{b})$. Heuristically we can say the following about this correlation, see Figure 2. Assuming χ and ξ exist, typical wandering of the geodesics $\Gamma(\mathbf{0},\boldsymbol{a})$ and $\Gamma(\mathbf{0},\boldsymbol{b})$ is of the order of n^ξ where n is the distance of the points $\boldsymbol{a}$, $\boldsymbol{b}$ from the origin. Suppose the distance between $\boldsymbol{a}$ and $\boldsymbol{b}$ is Jn^ξ for some large J. Then $\Gamma(\mathbf{0},\boldsymbol{a})$ and $\Gamma(\mathbf{0},\boldsymbol{b})$ are expected to branch apart at a distance m from the origin such that the distance between the rays joining $\mathbf{0}$ to $\boldsymbol{a}$ and $\mathbf{0}$ to $\boldsymbol{b}$ at distance m from the origin is of the order of the typical wandering of the geodesics of at distance m from origin. So we have approximately $m^\xi/m=Jn^\xi/n$. Hence $m=nJ^{-(1-\xi)^{-1}}$. Then the covariance between $T(\mathbf{0}, \boldsymbol{a})$ and $T(\mathbf{0}, \boldsymbol{b})$ is expected to be of the order of $m^{2\chi}$. So the correlation should be of the order of $J^{-2\chi/(1-\xi)}$.

1.3 Advanced assumptions

Along with our basic assumptions we make the following assumptions. Similar assumptions have been used in [3], [4], and [18]. We assume that there exists $\sigma : (0, \infty) \to (0, \infty)$ such that the following hold.

Assumption 1.4. *There exist positive constants C_1, C_2, such that for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^2$ and all $t > 0$, we have*

$$\mathbb{P}(|T(\boldsymbol{x}, \boldsymbol{y}) - h(\boldsymbol{x} - \boldsymbol{y})| \geq t\sigma(\|\boldsymbol{x} - \boldsymbol{y}\|)) \leq C_1 \exp(-C_2 t) . \tag{A1}$$

Assumption 1.5. *There exist constants $\mathfrak{p} > 0$, $\mathfrak{q} > 0$, $\alpha \in (0,1)$, $\beta \in (0,1)$, $\alpha \leq \beta$, such that for all $x > y > 0$ we have*

$$\mathfrak{p} \left(\frac{x}{y}\right)^{\alpha} \leq \frac{\sigma(x)}{\sigma(y)} \leq \mathfrak{q} \left(\frac{x}{y}\right)^{\beta} . \tag{A2}$$

Assumption 1.6. *There exist positive constants ϵ, C, such that for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^2$ with $\|\boldsymbol{x} - \boldsymbol{y}\| \geq C$, we have*

$$\begin{aligned} &\mathbb{P}(T(\boldsymbol{x}, \boldsymbol{y}) \leq h(\boldsymbol{x} - \boldsymbol{y}) - \epsilon\sigma(\|\boldsymbol{x} - \boldsymbol{y}\|)) \geq \epsilon , \\ &\mathbb{P}(T(\boldsymbol{x}, \boldsymbol{y}) \geq h(\boldsymbol{x} - \boldsymbol{y}) + \epsilon\sigma(\|\boldsymbol{x} - \boldsymbol{y}\|)) \geq \epsilon . \end{aligned} \tag{A3}$$

Remark 1.7. Assumption 1.4 is known to hold for various sub-optimal σ. First, it was shown in [23] that (A1) holds for $\sigma(x) = x^{1/2}$. In [31] it was shown that (A1) holds for $\sigma(x) = x^{1/2}$ and with t^2 in place of t in the right-hand side. In [11] it was shown that (A1) holds for $\sigma(x) = x^{1/2}/(\log_+ x)^{1/2}$ when the edge-weights take the value a with probability $1/2$ and the value b with probability $1/2$ for some $b > a > 0$. In [10] it was shown that (A1) holds for the scale $\sigma(x) = x^{1/2}/(\log_+ x)^{1/2}$ but for a much larger family of distributions called “nearly-gamma” distributions. In [15] it was shown that (A1) holds for the scale $\sigma(x) = x^{1/2}/(\log_+ x)^{1/2}$ under very mild conditions on the distribution of the edge weights.

Remark 1.8. In this paper we are considering two-dimensional percolation. Thus, Assumptions 1.4-1.6 are expected to hold for some $\sigma(x)$ of the order $x^{1/3}$.

Remark 1.9. By Remark 1.1 of [3], we assume without loss of generality that σ is monotonically increasing and continuous.

Remark 1.10. Under Assumption 1.4, Assumption 1.6 is equivalent to saying that there exist positive constants C_1, C_2, such that for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^2$ with $\|\boldsymbol{x} - \boldsymbol{y}\| \geq C_1$, we have

$$\operatorname{Var}(T(\boldsymbol{x}, \boldsymbol{y})) \geq C_2 \sigma^2(\|\boldsymbol{x} - \boldsymbol{y}\|) .$$

Remark 1.11. The assumption $\beta < 1$ is natural because, from results in [23], the passage times are known to satisfy exponential concentration with scaling exponent $1/2$, which shows that χ must be $\leq 1/2$ under any reasonable definition.

Remark 1.12. The assumption $\alpha > 0$ is natural because, for a certain definition of χ and ξ it was shown in [33] that $\chi \geq (1 - (d-1)\xi)/2$ in d-dimensions, which in two dimensions coupled with $\chi \geq 2\xi - 1$ and $\chi \leq 1/2$ yields $\chi \geq 1/8$.

Notation 1.13. *For any direction θ the unit vector in direction θ is denoted by $\boldsymbol{e}_\theta$. By abuse of notation, we denote the standard unit vectors in $\mathbb{R}^2$ by $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$.*

Definition 1.14. *We say that a direction θ_0 is of* type I *if there exist constants $C > 0$, $\delta_1 > 0$, $\delta_2 > 0$ such that the following holds: the limit shape boundary $\partial\mathcal{B}$ is differentiable in the sector $(\theta_0 - \delta_1, \theta_0 + \delta_1)$; for $|\delta| \le \delta_2$ and $\theta \in (\theta_0 - \delta_1, \theta_0 + \delta_1)$, we have*

$$g(\boldsymbol{e}_\theta + \delta \boldsymbol{e}_{\theta^t}) - g(\boldsymbol{e}_\theta) \ge C\delta^2 ,$$

where θ^t is the direction of the tangent to $\partial\mathcal{B}$ at the point in direction θ.

Remark 1.15. We take the direction of the tangents in counter-clockwise direction around the limit shape boundary.

Remark 1.16. An alternative formulation of type I direction is the following: there exist constants $C > 0$, $\delta_1 > 0$ such that $\partial\mathcal{B}$ is differentiable in the sector $(\theta_0 - \delta_1, \theta_0 + \delta_1)$; for $\theta \in (\theta_0 - \delta_1, \theta_0 + \delta_1)$ and all $\delta \in \mathbb{R}$ we have

$$g(\boldsymbol{e}_\theta + \delta \boldsymbol{e}_{\theta^t}) - g(\boldsymbol{e}_\theta) \ge C \min\{ |\delta|, \delta^2 \} .$$

Definition 1.17. *We say that a direction θ_0 is of* type II *if there exist constants $C > 0$, $\delta_1 > 0$, $\delta_2 > 0$ such that the following holds: the limit shape boundary $\partial\mathcal{B}$ is differentiable in the sector $(\theta_0 - \delta_1, \theta_0 + \delta_1)$; for $|\delta| \le \delta_2$ and $\theta \in (\theta_0 - \delta_1, \theta_0 + \delta_1)$ we have*

$$g(\boldsymbol{e}_\theta + \delta \boldsymbol{e}_{\theta^t}) - g(\boldsymbol{e}_\theta) \le C\delta^2 , \tag{1.1}$$

where, as before, θ^t is the direction of the tangent to $\partial\mathcal{B}$ at the point in direction θ.

Remark 1.18. In a neighborhood of a type I direction the limit shape boundary cannot have a facet. Similarly in a neighborhood of a type II direction the limit shape boundary cannot have a corner.

Remark 1.19. As observed in Remark 1.2 of [3], the condition in (1.1) can be alternatively stated as follows. If $\boldsymbol{u}_\theta$ is the point on $\partial\mathcal{B}$ in direction θ, then in a neighborhood of $\boldsymbol{u}_\theta$, the boundary is squeezed between the tangent at $\boldsymbol{u}_\theta$ and a parabola tangent to $\partial\mathcal{B}$ at $\boldsymbol{u}_\theta$. This implies that the direction of the tangent grows at most linearly in a neighborhood of θ. So, if θ_0 is a direction of type II, then there exist constants $C > 0$, $\delta_1 > 0$, $\delta_2 > 0$ such that the following holds: $\partial\mathcal{B}$ is differentiable in the sector $(\theta_0 - \delta_1, \theta_0 + \delta_1)$; for $\theta_1, \theta_2 \in (\theta_0 - \delta_1, \theta_0 + \delta_1)$ with $|\theta_1 - \theta_2| \le \delta_2$, we have $|\theta_1^t - \theta_2^t| \le C|\theta_1 - \theta_2|$.

Remark 1.20. If θ_0 is of type I then in a neighborhood of θ_0 all directions are uniformly of type I in the sense that the condition in Definition 1.14 holds with same parameters C, δ_1, δ_2 for all directions in this neighborhood of θ_0. The same can be said about directions of type II.

1.4 Main results

Notation 1.21. *Given two linearly independent directions θ_1, θ_2 we define the projections $\pi^1_{\theta_1,\theta_2}$ and $\pi^2_{\theta_1,\theta_2}$ so that for any $\boldsymbol{v}$ we have*

$$\boldsymbol{v} \;=\; \pi^1_{\theta_1,\theta_2}(\boldsymbol{v})\, \boldsymbol{e}_{\theta_1} \;+\; \pi^2_{\theta_1,\theta_2}(\boldsymbol{v})\, \boldsymbol{e}_{\theta_2} .$$

Notation 1.22. *For $n > 0$ let*

$$\Delta(n) \coloneqq (n\sigma(n))^{1/2} .$$

Our first main result is the following.

Theorem 1.23. *Let θ_0 be a direction of both type I and II. For $n > 0$, $L > 0$, define*

$$\begin{aligned}
\mathcal{I}(n,L) &\coloneqq \Big\{ \boldsymbol{x} \in \mathbb{R}^2 \colon \pi^1_{\theta_0,\theta_0^t}(\boldsymbol{x}) = n ,\; 0 \le \pi^2_{\theta_0,\theta_0^t}(\boldsymbol{x}) \le L \Big\} ,\\
\mathcal{D}(n,L) &\coloneqq \max\{ |T(\boldsymbol{0},\boldsymbol{x}) - T(\boldsymbol{0},\boldsymbol{y})| \colon \boldsymbol{x},\boldsymbol{y} \in \mathcal{I}(n,L) \} ,\\
\mathcal{I}'(n,L) &\coloneqq \Big\{ \boldsymbol{x} \in \mathbb{R}^2 \colon \pi^1_{\theta_0,\theta_0^t}(\boldsymbol{x}) = n ,\; -L \le \pi^2_{\theta_0,\theta_0^t}(\boldsymbol{x}) \le 0 \Big\} ,\\
\mathcal{D}'(n,L) &\coloneqq \max\{ |T(\boldsymbol{0},\boldsymbol{x}) - T(\boldsymbol{0},\boldsymbol{y})| \colon \boldsymbol{x},\boldsymbol{y} \in \mathcal{I}'(n,L) \} .
\end{aligned}$$

Fix $\eta \in (0,1]$. Then, under the Assumptions 1.4 and 1.5, there exist positive constants C_1, C_2, L_0, n_0, t_0, such that for $L \geq L_0$, $n \geq n_0$, $t \geq t_0$, $L \leq \Delta(n)$, we have

$$\mathbb{P}\big(\mathcal{D}(n,L) \geq t(\log L)^\eta \sigma\left(\Delta^{-1}(L)\right)\big) \leq C_1 \exp(-C_2 t(\log L)^\eta) \ ,$$
$$\mathbb{P}\big(\mathcal{D}'(n,L) \geq t(\log L)^\eta \sigma\left(\Delta^{-1}(L)\right)\big) \leq C_1 \exp(-C_2 t(\log L)^\eta) \ .$$

We prove this in Section 5. The following theorem is our lower bound on the fluctuations of transverse increments. In this theorem, we show that the standard deviation of the transverse increment between a pair of points at a distance L is at least of the order of $\sigma\left(\Delta^{-1}(L)\right)$ with a correction factor smaller than any power of L.

Theorem 1.24. *Let θ_0 be a direction of both type I and II. Fix $\nu \in (1/2,1)$. Then, under the Assumptions 1.4, 1.5, and 1.6, there exist positive constants L_0, n_0, such that for $L \geq L_0$, $n \geq n_0$, $L \leq \Delta(n)$, we have*

$$\mathrm{Var}\Big(T(\mathbf{0}, n\boldsymbol{e}_{\theta_0}) - T(\mathbf{0}, n\boldsymbol{e}_{\theta_0} + L\boldsymbol{e}_{\theta_0^t})\Big) \geq \exp(-(\log L)^\nu)\sigma^2\big(\Delta^{-1}(L)\big) \ .$$

Same bound holds for variance of $T(\mathbf{0}, n\boldsymbol{e}_{\theta_0}) - T(\mathbf{0}, n\boldsymbol{e}_{\theta_0} - L\boldsymbol{e}_{\theta_0^t})$.

We prove this in Section 6. As a corollary of Theorems 1.23 and 1.24 we get the following result. It shows that if we assume χ and ξ exist in a certain sense, then χ/ξ is the correct scaling exponent for the fluctuations of the transverse increments.

Corollary 1.25. *Suppose there exists $\chi > 0$ such that*

$$\lim_{x\to\infty} \frac{\log \sigma(x)}{\log x} = \chi \ ,$$

and let

$$\xi \coloneqq \frac{1+\chi}{2} = \lim_{x\to\infty} \frac{\log \Delta(x)}{\log x} \ .$$

Let θ_0 be a direction of both type I and II. Then, under the Assumptions 1.4, 1.5, and 1.6, there exist functions f_1, f_2, f_3, which converge to 0 at ∞, and positive constants C_1, C_2, C_3, n_0, L_0, t_0, such that for $n \geq n_0$, $L \geq L_0$, $t \geq t_0$, $L \leq n^{\xi + f_1(n)}$, we have

$$\mathbb{P}\Big(\Big|T(\mathbf{0}, n\boldsymbol{e}_{\theta_0}) - T(\mathbf{0}, n\boldsymbol{e}_{\theta_0} + L\boldsymbol{e}_{\theta_0^t})\Big| \geq tL^{\chi/\xi + f_2(L)}\Big) \leq C_1 \exp(-C_2 t) \ ,$$
$$\mathrm{Var}\Big(T(\mathbf{0}, n\boldsymbol{e}_{\theta_0}) - T(\mathbf{0}, n\boldsymbol{e}_{\theta_0} + L\boldsymbol{e}_{\theta_0^t})\Big) \geq C_3 L^{2\chi/\xi + f_3(L)} \ .$$

Same bounds hold for $T(\mathbf{0}, n\boldsymbol{e}_{\theta_0}) - T(\mathbf{0}, n\boldsymbol{e}_{\theta_0} - L\boldsymbol{e}_{\theta_0^t})$.

Proof. Fix $\eta \in (0,1)$ and $\nu \in (1/2,1)$. Define f_1, f_2, f_3 such that for all $x > 1$,

$$x^{\xi + f_1(x)} = \Delta(x) \ ,$$
$$x^{\chi/\xi + f_2(x)} = \sigma(\Delta^{-1}(x))(\log x)^\eta \ ,$$
$$x^{2\chi/\xi + f_3(x)} = e^{-(\log x)^{\nu_0}} \sigma^2(x) \ .$$

Then f_1, f_2, f_3 converge to 0 at ∞ and the result readily follows from Theorems 1.23 and 1.24. ☐

Notation 1.26. *Define*

$$f(n) \coloneqq \frac{\Delta(n)(\log n)^{1/2}}{n} \quad \textit{and} \quad f^{-1}(y) \coloneqq \sup\{\, x \colon f(x) \geq y \,\} \ .$$

Remark 1.27. Since $\beta < 1$ by (A2), and because we have assumed σ is monotonically increasing and continuous, we get that f is continuous and goes to 0 at ∞. Therefore, f^{-1} is continuous, monotonically decreasing, and converges to 0 at ∞.

Now we state the result on upper bound of long-range correlations.

Theorem 1.28. *Let θ_0 be a direction of both type I and II. Recall β and $\mathfrak{q}$ from* (A2)*. Fix $\delta \in (0,(1-\beta)/2)$. Then, under the Assumptions 1.4 and 1.5, there exist positive constants C, J_0, n_0, such that for $n \geq n_0$, $J \in [\mathfrak{q}^{1/2}J_0, n^{\delta}]$, we have*

$$\begin{aligned}&\mathrm{Cov}\Big(T(\mathbf{0}, n\boldsymbol{e}_{\theta_0} - J\Delta(n)(\log n)^{1/2}\boldsymbol{e}_{\theta_0^t}), T(\mathbf{0}, n\boldsymbol{e}_{\theta_0} + J\Delta(n)(\log n)^{1/2}\boldsymbol{e}_{\theta_0^t})\Big)\\ &\leq C\sigma^2\left(f^{-1}\left(\frac{J}{J_0}f(n)\right)\right)\log n \ .\end{aligned}$$

The next corollary shows how we get the exponent $2\chi/(1-\xi)$ under further regularity assumptions on σ.

Corollary 1.29. *Suppose $\sigma(n) = n^{\chi}L(n)$, where L is a slowly varying function. Let $\xi := (1+\chi)/2$. Fix $\delta_1 \in (0,(1-\beta)/2)$. Let θ_0 be a direction of both type I and II. Then, under the Assumptions 1.4 and 1.5, there exist positive constants C, J_0, such that the following holds: for any $\delta_2 > 0$ there exists $n_0 > 0$ such that for $n \geq n_0$ and $J \in [\mathfrak{q}^{1/2}J_0, n^{\delta_1}]$, we have*

$$\begin{aligned}&\mathrm{Corr}\Big(T(\mathbf{0}, n\boldsymbol{e}_{\theta_0} - J\Delta(n)(\log n)^{1/2}\boldsymbol{e}_{\theta_0^t}), T(\mathbf{0}, n\boldsymbol{e}_{\theta_0} + J\Delta(n)(\log n)^{1/2}\boldsymbol{e}_{\theta_0^t})\Big)\\ &\leq CJ^{-2\chi/(1-\xi)+\delta_2}\log n \ .\end{aligned}$$

Proof. From Theorem 1.28 we get positive constants C_1, J_0, n_0, such that for $n \geq n_0$, $J \in [\mathfrak{q}^{1/2}J_0, n^{\delta_1}]$, we have

$$\mathrm{Cov}\,(\,T(\mathbf{0},\boldsymbol{a}), T(\mathbf{0},\boldsymbol{b})\,) \leq C_1\sigma^2(m)\log n \ , \tag{1.2}$$

where

$$\begin{aligned}\boldsymbol{a} &:= n\boldsymbol{e}_{\theta_0} + J\Delta(n)(\log n)^{1/2}\boldsymbol{e}_{\theta_0^t} \ ,\\ \boldsymbol{b} &:= n\boldsymbol{e}_{\theta_0} - J\Delta(n)(\log n)^{1/2}\boldsymbol{e}_{\theta_0^t} \ ,\\ m &:= f^{-1}(Jf(n)/J_0) \ .\end{aligned}$$

Using $J \leq n^{\delta_1}$, $\delta_1 < (1-\beta)/2$, and (A2), we get $\|\boldsymbol{a}\| \leq C_2 n$, $\|\boldsymbol{b}\| \leq C_2$. Hence, using (1.2) and (A2), we get

$$\mathrm{Corr}(T(\mathbf{0},\boldsymbol{a}), T(\mathbf{0},\boldsymbol{b})) \leq C_3\frac{\sigma^2(m)}{\sigma^2(n)}\log n \ . \tag{1.3}$$

From $m = f^{-1}(Jf(n)/J_0)$ we get

$$J_0\frac{\Delta(m)(\log m)^{1/2}}{m} = J\frac{\Delta(n)(\log n)^{1/2}}{n} \ .$$

Therefore, using $J \in [\mathfrak{q}^{1/2}J_0, n_1^{\delta}]$, $\delta_1 < (1-\beta)/2$, and (A2), we get $m \leq n$ and $\log m \geq C_4\log n$. Fix an $\delta_2 > 0$ and let $\delta_3 > 0$ be such that

$$\frac{2\chi}{1-\xi} - \delta_2 \leq \frac{2\chi - 2\delta_3}{1-\xi+\delta_3/2} \ . \tag{1.4}$$

Since L is slowly-varying, by possibly increasing n_0 based on δ_2, we get

$$\frac{L(n)}{L(m)} \geq \left(\frac{n}{m}\right)^{-\delta_3} \ . \tag{1.5}$$

Therefore, using $\Delta(n)(\log n)^{1/2} = n^{\xi} L(n)^{1/2} (\log n)^{1/2}$ we get

$$\left(\frac{n}{m}\right)^{1-\xi} = \frac{J}{J_0}\left(\frac{L(n)\log n}{L(m)\log m}\right)^{1/2} \geq \frac{J}{J_0}\left(\frac{n}{m}\right)^{-\delta_3/2} .$$

Combining this with (1.5) and (1.4), we get

$$\frac{\sigma^2(m)}{\sigma^2(n)} = \left(\frac{m}{n}\right)^{2\chi}\left(\frac{L(m)}{L(n)}\right)^2 \leq \left(\frac{m}{n}\right)^{2\chi - 2\delta_3} \leq \left(\frac{J_0}{J}\right)^{\frac{2\chi-2\delta_3}{1-\xi+\delta_3/2}} \leq J_0^{\frac{2\chi}{1-\xi}} J^{-\frac{2\chi}{1-\xi}+\delta_2} .$$

Combining this with (1.3) completes the proof of Corollary 1.29. □

2 Wandering of geodesics

In this section we establish some upper bounds on the wandering of geodesics. Lemma 2.2 provides a preliminary bound on the wandering of geodesics. The proof of Lemma 2.2 follows from Proposition 5.8 of [22] under our basic assumptions. Lemma 2.2 has been shown to hold under milder assumptions in Theorem 6.2 of [6].

Notation 2.1. *For any set $A \subset \mathbb{R}^2$ let*

$$\mathrm{Diam}(A) \coloneqq \sup\{\, \|\boldsymbol{x} - \boldsymbol{y}\| \colon \boldsymbol{x}, \boldsymbol{y} \in A\,\} .$$

Lemma 2.2. *There exist positive constants C_1, C_2, C_3, such that the following holds. If $\|\boldsymbol{u} - \boldsymbol{v}\| \geq C_1$ for some $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^2$, then*

$$\mathbb{P}(\mathrm{Diam}(\Gamma(\boldsymbol{u}, \boldsymbol{v})) \geq C_2\|\boldsymbol{u} - \boldsymbol{v}\|) \leq \exp(-C_3\|\boldsymbol{u} - \boldsymbol{v}\|) .$$

Utilizing the curvature of the limit shape we get a refined bound on the wandering of the geodesics. The curvature of the limit shape is utilized in the following manner. Consider two points in $\mathbb{R}^2$. The shortest path between these two points in the g-norm is, of course, the line joining them. When the geodesic between these two points wander transversely too far from the line joining them, the extra distance covered by the geodesic in the g-norm can be thought of as a cost for excessive wandering. Therefore, a lower bound of this cost yields an upper bound of the wandering of the geodesic. A lower bound on this cost can be obtained from a lower bound of the curvature of the limit shape. This is stated in Lemma 2.3 which is essentially same as Lemma 2.3 of [3]. Thus we skip the proof of Lemma 2.3.

Lemma 2.3. *Let θ_0 be a direction of type I. Then there exist positive constants C and δ such that for $n > 0$, k, l, d, satisfying $|l|/n \leq \delta$, we have*

$$g(\boldsymbol{u}) + g(\boldsymbol{a} - \boldsymbol{u}) - g(\boldsymbol{a}) \geq C \min\left\{|d|\,,\ \frac{d^2}{n}\right\} ,$$

where $\boldsymbol{a} \coloneqq n\boldsymbol{e}_{\theta_0} + l\boldsymbol{e}_{\theta_0^t}$ and $\boldsymbol{u} \coloneqq k\boldsymbol{e}_{\theta_0} + (l\frac{k}{n} + d)\boldsymbol{e}_{\theta_0^t}$.

Geodesics cannot wander too much transversely because the cost associated with the g-norm becomes difficult to be compensated by the fluctuations of passage times. Thus, bounds on the fluctuations $T(\boldsymbol{0}, \boldsymbol{x}) - g(\boldsymbol{x})$, combined with Lemma 2.3 yields upper bounds on transverse wanderings of geodesics. By Assumption 1.4 we know $T(\boldsymbol{0}, \boldsymbol{x}) - h(\boldsymbol{x})$ satisfies exponential concentration in the scale $\sigma(\|\boldsymbol{x}\|)$ uniformly over $\boldsymbol{x}$. So we need a bound on the differences $h(\boldsymbol{x}) - g(\boldsymbol{x})$. These differences are known as *nonrandom fluctuations* in the literature. A general method of bounding the nonrandom fluctuations was developed in [1, 2]. There it was shown, using exponential concentration of $T(\boldsymbol{0}, \boldsymbol{x}) - h(\boldsymbol{x})$ on the scale of $\|\boldsymbol{x}\|^{1/2}$ from [23], that $h(\boldsymbol{x}) - g(\boldsymbol{x})$ is at most of the order of $\|\boldsymbol{x}\|^{1/2}\log\|\boldsymbol{x}\|$.

In our case, Alexander's method can be used without any significant alteration to yield a bound of the order of $\sigma(\|\boldsymbol{x}\|)\log\|\boldsymbol{x}\|$. We state this without proof in Proposition 2.6. Improvements to the logarithmic bound have been made in [32], [16], [4] in related models, which we briefly discuss in Section 4. In our model, we improve the logarithmic bound to $\sigma(\|\boldsymbol{x}\|)(\log\|\boldsymbol{x}\|)^\eta$ for arbitrary small $\eta > 0$ in Section 4. This improvement is necessary to prove the lower bound result Theorem 1.24. To state the bound on the nonrandom fluctuations we use the notion of *'general approximation property'* from [2].

Notation 2.4. *Let Φ be the set of functions from $(0,\infty)$ to $[0,\infty)$ such that for every $\phi \in \Phi$ there exists some $C \geq 0$ such that $\phi(x) > 0$ for $x > C$ and $\inf_{x>y>C} \phi(x)/\phi(y) > 0$. For $\eta \in (0,1]$, define $\phi_\eta(k) = 0$ for $k \leq 1$, and for $k > 1$*

$$\phi_\eta(k) := k^{-\alpha}\sigma(k)(\log k)^\eta \ .$$

Also define $\widehat{\phi}(k) = 0$ for $k \leq 2$, and for $k > 2$

$$\widehat{\phi}(k) := k^{-\alpha}\sigma(k)\log\log k \ .$$

So $\widehat{\phi}$ and ϕ_η for all $\eta \in (0,1]$ belong to Φ.

Definition 2.5. *For $\nu \geq 0$ and $\phi \in \Phi$ we say that h satisfies the* general approximation property *with exponent ν and correction factor ϕ in a sector of directions $\mathcal{S}$ if there exist positive constants C and M such that for all $\boldsymbol{x} \in \mathbb{R}^2$ with $\|\boldsymbol{x}\| \geq M$ and having direction in the sector $\mathcal{S}$ we have*

$$h(\boldsymbol{x}) \leq g(\boldsymbol{x}) + C\|\boldsymbol{x}\|^\nu \phi(\|\boldsymbol{x}\|) \ .$$

When we want to specify the relevant constants, we say h satisfies $\mathrm{GAP}(\nu,\phi,M,C)$ in sector $\mathcal{S}$.

We often refer to functions ϕ as in the above definition as correction factors. In [2], the class of correction factors consisted of non-decreasing functions and the general approximation property was not restricted to any particular set of directions. In our setup the class of correction factors is extended, and we also pay attention to the sector of directions. These are some minor modifications we need in our setup. As we mentioned before, we get the following result in our context by following Alexander's method.

Proposition 2.6. *Under the Assumptions 1.4 and 1.5, there exist positive constants C and M such that h satisfies $\mathrm{GAP}(\alpha,\phi_1,M,C)$ in all directions, i.e., for all $\boldsymbol{x} \in \mathbb{R}^2$ satisfying $\|\boldsymbol{x}\| \geq M$, we have*

$$h(\boldsymbol{x}) \leq g(\boldsymbol{x}) + C\sigma(\|\boldsymbol{x}\|)\log\|\boldsymbol{x}\| \ .$$

Let us now introduce a notation to measure wandering of geodesics.

Notation 2.7. *Suppose θ_0 is a direction where the boundary of the limit shape is differentiable. Let θ_0^t be the direction of the tangent. Let $\boldsymbol{u}$, $\boldsymbol{v}$ be points in $\mathbb{R}^2$ with $\pi^1_{\theta_0,\theta_0^t}(\boldsymbol{v}-\boldsymbol{u}) \neq 0$. For $\boldsymbol{w} \in \mathbb{R}^2$ define*

$$\mathfrak{C}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) := \pi^2_{\theta_0,\theta_0^t}(\boldsymbol{w}-\boldsymbol{u}) - \pi^1_{\theta_0,\theta_0^t}(\boldsymbol{w}-\boldsymbol{u})\frac{\pi^2_{\theta_0,\theta_0^t}(\boldsymbol{v}-\boldsymbol{u})}{\pi^1_{\theta_0,\theta_0^t}(\boldsymbol{v}-\boldsymbol{u})} \ .$$

For $k \in \mathbb{R}$ define

$$\mathcal{W}(\boldsymbol{u},\boldsymbol{v},k,\theta_0) := \max\left\{ |\mathfrak{C}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})| \colon \boldsymbol{w} \in \Gamma(\boldsymbol{u},\boldsymbol{v}) \ , \ \pi^1_{\theta_0,\theta_0^t}(\boldsymbol{w}-\boldsymbol{u}) = k \right\} .$$

Thus $\mathcal{W}(\boldsymbol{u},\boldsymbol{v},k,\theta_0)$ is the maximum wandering of the geodesic $\Gamma(\boldsymbol{u},\boldsymbol{v})$, in $\pm\theta_0^t$ directions, from the line joining $\boldsymbol{u}$ and $\boldsymbol{v}$, when the geodesic is at a distance k from $\boldsymbol{u}$ in θ_0 direction.

The parameter k is continuous, it can take any real value. In the above definition, $\boldsymbol{w}$ is not necessarily a lattice point. But, there is no issue with measurability because if θ_0^t is one of the axial directions then we can consider only lattice points for $\boldsymbol{w}$, and of θ_0^t is not one of the axial directions then the set of points $\boldsymbol{w} \in \mathbb{R}^2$ satisfying $\boldsymbol{w} \in \Gamma(\boldsymbol{u}, \boldsymbol{v})$ and $\pi^1_{\theta_0,\theta_0^t}(\boldsymbol{w} - \boldsymbol{u}) = k$ is countable.

For $k \in \mathbb{R}$ define

$$\overline{\mathcal{W}}(\boldsymbol{u}, \boldsymbol{v}, k, \theta_0) := \max\Big\{ |\mathfrak{C}(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})| \colon \boldsymbol{w} \in \Gamma(\boldsymbol{u}, \boldsymbol{v}) \cap \mathbb{Z}^2, \pi^1_{\theta_0,\theta_0^t}(\boldsymbol{w} - \boldsymbol{u}) \in [k, k+1) \Big\} .$$

Thus, $\overline{\mathcal{W}}(\boldsymbol{u}, \boldsymbol{v}, k, \theta_0)$ measures maximum wandering in a cylinder, whereas $\mathcal{W}(\boldsymbol{u}, \boldsymbol{v}, k, \theta_0)$ measures maximum wandering along a line. In the above definition $\boldsymbol{w}$ is a lattice point (points with both coordinates integer valued).

The following relation holds between $\mathcal{W}$ and $\overline{\mathcal{W}}$:

$$\mathcal{W}(\boldsymbol{u}, \boldsymbol{v}, k, \theta_0) \le \overline{\mathcal{W}}(\boldsymbol{u}, \boldsymbol{v}, k, \theta_0) + 1 . \tag{2.1}$$

In Theorem 2.8, we consider ‘global’ wandering of geodesics. We consider a point $\boldsymbol{a}$ approximately at distance n and direction θ_0 from $\mathbf{0}$. We consider wandering of $\Gamma(\mathbf{0}, \boldsymbol{a})$ at distance k from $\mathbf{0}$ at direction θ_0 i.e., we consider $\mathcal{W}(\mathbf{0}, \boldsymbol{a}, k, \theta_0)$. Here k is arbitrary. We show that $\mathcal{W}(\mathbf{0}, \boldsymbol{a}, k, \theta_0)$ is at most of the order of $\Delta(n)$ with some logarithmic correction factors. This bound is sub-optimal for k bigger than a multiple of n, because in this case we get a better bound using Lemma 2.2.

In Theorem 2.10 we consider ‘local’ wandering of geodesics. Again we consider a point $\boldsymbol{a}$ approximately at distance n and direction θ_0 from $\mathbf{0}$. We consider wandering of $\Gamma(\mathbf{0}, \boldsymbol{a})$ at distance k from $\mathbf{0}$ at direction θ_0 i.e., we consider $\mathcal{W}(\mathbf{0}, \boldsymbol{a}, k, \theta_0)$. We show that $\mathcal{W}(\mathbf{0}, \boldsymbol{a}, k, \theta_0)$ is at most of the order of $\Delta(k)$. Thus we get a better bound than Theorem 2.8 when k is of smaller order than n.

In Theorems 2.5 and 2.7 of [8], and in Theorem 3 of [9] similar bound on wandering of geodesics has been proved in the integrable model of last passage percolation. There the results are sharper in the sense that there are no logarithmic correction factors involved. This is because, in the integrable model of last passage percolation exact asymptotics of the distribution of the passage times are known.

Theorem 2.8. *Let θ_0 be a direction of type I. Suppose h satisfies GAP with exponent α and correction factor ϕ_η for some $\eta \in (0, 1]$ in a sector $(\theta_0 - \delta, \theta_0 + \delta)$. Then, under Assumptions 1.4 and 1.5, there exist positive constants C_1, C_2, δ_1, δ_2, n_0, t_0, such that for $n \ge n_0$, $t \ge t_0$, $t\Delta(n)(\log n)^{\eta/2} \le n\delta_1$, $|l| \le n\delta_2$, $k \in (-\infty, \infty)$, we have*

$$\mathbb{P}\Big(\overline{\mathcal{W}}\Big(\mathbf{0}, n\boldsymbol{e}_{\theta_0} + l\boldsymbol{e}_{\theta_0^t}, k, \theta_0\Big) \ge t\Delta(n)(\log n)^{\eta/2}\Big) \le C_1 \exp\big(-C_2 t^2 (\log n)^\eta\big) .$$

The same bound holds for $\mathcal{W}\Big(\mathbf{0}, n\boldsymbol{e}_{\theta_0} + l\boldsymbol{e}_{\theta_0^t}, k, \theta_0\Big)$.

Proof. Due to symmetry of the lattice, without loss of generality we assume $\theta_0 \in [0, \pi/4]$. Fix $\delta_1 > 0$, $\delta_2 > 0$, to be assumed appropriately small whenever required. Fix $n_0 > 0$, $t_0 > 0$, to be assumed appropriately large whenever required. Consider n, t, l satisfying $n \ge n_0$, $t \ge t_0$, $t\Delta(n)(\log n)^{\eta/2} \le n\delta_1$, $|l| \le n\delta_2$. Let $\boldsymbol{a} := n\boldsymbol{e}_{\theta_0} + l\boldsymbol{e}_{\theta_0^t}$. We will focus on $\overline{\mathcal{W}}(\mathbf{0}, \boldsymbol{a}, k, \theta_0)$. The bound on $\mathcal{W}(\mathbf{0}, \boldsymbol{a}, k, \theta_0)$ will follow from (2.1) with slight adjustment to the various constants.

Assuming $\delta_2 < 1$ we get $\|\boldsymbol{a}\| \le 2n$. Therefore, by Lemma 2.2, the geodesic $\Gamma(\mathbf{0}, \boldsymbol{a})$ stays inside a square of side length $C_1 n$ centered at the $\mathbf{0}$, with probability at least $1 - \exp(-C_2 n)$. Using $t\Delta(n)(\log n)^{\eta/2} \le n\delta_1$, assuming δ_1 is small enough, and using (A2), we get $t^2(\log n)^\eta \le C_3 n/\sigma(n) \le C_4 n^{1-\alpha}$. Hence, the probability bound in the statement is trivial for $|k| \ge C_5 n$. So let us consider k satisfying $|k| \le C_5 n$.

We split the probability under consideration as

$$\mathbb{P}\Big(\overline{\mathcal{W}}(\mathbf{0},\boldsymbol{a},k,\theta_0) \geq t\Delta(n)(\log n)^{\eta/2}\Big)$$
$$\leq \mathbb{P}\big(\overline{\mathcal{W}}(\mathbf{0},\boldsymbol{a},k,\theta_0) \geq n\big) + \mathbb{P}\Big(t\Delta(n)(\log n)^{\eta/2} \leq \overline{\mathcal{W}}(\mathbf{0},\boldsymbol{a},k,\theta_0) \leq n\Big) . \tag{2.2}$$

For any point $\boldsymbol{u}$ on $\Gamma(\mathbf{0},\boldsymbol{a})$ we have

$$\begin{aligned}0 &= T(\mathbf{0},\boldsymbol{u}) + T(\boldsymbol{u},\boldsymbol{a}) - T(\mathbf{0},\boldsymbol{a})\\ &= (T(\mathbf{0},\boldsymbol{u}) - h(\boldsymbol{u})) + (T(\boldsymbol{u},\boldsymbol{a}) - h(\boldsymbol{a}-\boldsymbol{u})) - (T(\mathbf{0},\boldsymbol{a}) - h(\boldsymbol{a}))\\ &\quad + (h(\boldsymbol{u}) - g(\boldsymbol{u})) + (h(\boldsymbol{a}-\boldsymbol{u}) - g(\boldsymbol{a}-\boldsymbol{u})) - (h(\boldsymbol{a}) - g(\boldsymbol{a}))\\ &\quad + (g(\boldsymbol{u}) + g(\boldsymbol{a}-\boldsymbol{u}) - g(\boldsymbol{a})) .\end{aligned}$$

Therefore

$$\begin{aligned}&|T(\mathbf{0},\boldsymbol{u}) - h(\boldsymbol{u})| + |T(\boldsymbol{u},\boldsymbol{a}) - h(\boldsymbol{a}-\boldsymbol{u})| + |T(\mathbf{0},\boldsymbol{a}) - h(\boldsymbol{a})|\\ &\geq (h(\boldsymbol{u}) - g(\boldsymbol{u})) + (h(\boldsymbol{a}-\boldsymbol{u}) - g(\boldsymbol{a}-\boldsymbol{u})) - (h(\boldsymbol{a}) - g(\boldsymbol{a})) + (g(\boldsymbol{u}) + g(\boldsymbol{a}-\boldsymbol{u}) - g(\boldsymbol{a}))\\ &\geq (g(\boldsymbol{u}) + g(\boldsymbol{a}-\boldsymbol{u}) - g(\boldsymbol{a})) - (h(\boldsymbol{a}) - g(\boldsymbol{a})) .\end{aligned} \tag{2.3}$$

Let V be the set of lattice points $\boldsymbol{u}$ (points with both coordinates integer) with $\pi^1_{\theta_0,\theta_0^t}(\boldsymbol{u}) \in [k,k+1)$. Define a function $d : V \to \mathbb{R}$ as follows. For $\boldsymbol{u} \in V$ let

$$d(\boldsymbol{u}) \coloneqq \pi^2_{\theta_0,\theta_0^t}(\boldsymbol{u}) - \pi^1_{\theta_0,\theta_0^t}(\boldsymbol{u})\frac{l}{n} .$$

Let $V_1 \subset V$ be the set of points $\boldsymbol{u} \in V$ with $|d(\boldsymbol{u})| \geq n$. Let V_2 be the set of points $\boldsymbol{u} \in V$ with $t\Delta(n)(\log n)^{\eta/2} \leq |d(\boldsymbol{u})| \leq n$. Thus, if $\overline{\mathcal{W}}(\mathbf{0},\boldsymbol{a},k,\theta_0) \geq n$, then $\Gamma(\mathbf{0},\boldsymbol{a})$ goes through a point $\boldsymbol{u} \in V_1$. Assuming δ_2 is small enough and using Lemma 2.3 we get

$$g(\boldsymbol{u}) + g(\boldsymbol{a}-\boldsymbol{u}) - g(\boldsymbol{a}) \geq C_6|d(\boldsymbol{u})| . \tag{2.4}$$

Using h satisfies GAP with exponent α and correction factor ϕ_η in a neighborhood of θ_0 and assuming δ_2 is small enough we get

$$h(\boldsymbol{a}) - g(\boldsymbol{a}) \leq C_7\sigma(n)(\log n)^\eta . \tag{2.5}$$

Using $|d(\boldsymbol{u})| \geq n$, $|l|/n \leq \delta_2 < 1$, and $|k| \leq C_5 n$, we get

$$\max\{\|\boldsymbol{u}-\boldsymbol{a}\|, \|\boldsymbol{u}\|, \|\boldsymbol{a}\|\} \leq C_8|d(\boldsymbol{u})| .$$

Therefore, using (A1) and (A2), we get for all $t' > 0$

$$\begin{aligned}&\mathbb{P}(\max\{|T(\mathbf{0},\boldsymbol{u}) - h(\boldsymbol{u})|, |T(\boldsymbol{u},\boldsymbol{a}) - h(\boldsymbol{a}-\boldsymbol{u})|, |T(\mathbf{0},\boldsymbol{a}) - h(\boldsymbol{a})|\} \geq t'\sigma(|d(\boldsymbol{u})|))\\ &\leq C_9\exp(-C_{10}t') .\end{aligned} \tag{2.6}$$

Combining (2.3)-(2.6) and using (A2), we get

$$\begin{aligned}&\mathbb{P}\big(\overline{\mathcal{W}}(\mathbf{0},\boldsymbol{a},k,\theta_0) \geq n\big)\\ &\leq \sum_{\boldsymbol{u}\in V_1} C_{11}\exp(-C_{12}(C_6|d(\boldsymbol{u})| - C_7\sigma(n)(\log n)^\eta)/\sigma(|d(\boldsymbol{u})|))\\ &\leq \sum_{\boldsymbol{u}\in V_1} C_{13}\exp(-C_{14}|d(\boldsymbol{u})|/\sigma(|d(\boldsymbol{u})|))\\ &\leq C_{15}\exp(-C_{16}n/\sigma(n)) .\end{aligned} \tag{2.7}$$

If $\overline{\mathcal{W}}(\mathbf{0}, \boldsymbol{a}, k, \theta_0) \in [t\Delta(n)(\log n)^{\eta/2}, n]$, then $\Gamma(\mathbf{0}, \boldsymbol{a})$ goes through a point $\boldsymbol{u} \in V_2$. Assuming δ_2 is small enough and using Lemma 2.3 we get

$$g(\boldsymbol{u}) + g(\boldsymbol{a} - \boldsymbol{u}) - g(\boldsymbol{a}) \geq C_{17} \frac{|d(\boldsymbol{u})|^2}{n} \geq C_{17} t^2 \sigma(n)(\log n)^{\eta} . \tag{2.8}$$

Since $|d(\boldsymbol{u})| \leq n$, we have

$$\max\{\|\boldsymbol{u} - \boldsymbol{a}\|, \|\boldsymbol{u}\|, \|\boldsymbol{a}\|\} \leq C_{18} n .$$

Hence, using Assumptions 1.4 and 1.5 we get, for all $t' > 0$

$$\begin{aligned}&\mathbb{P}(\max\{|T(\mathbf{0}, \boldsymbol{u}) - h(\boldsymbol{u})|, |T(\boldsymbol{u}, \boldsymbol{a}) - h(\boldsymbol{a} - \boldsymbol{u})|, |T(\mathbf{0}, \boldsymbol{a}) - h(\boldsymbol{a})|\} \geq t'\sigma(n)) \\ &\leq C_{19} \exp(-C_{20} t') .\end{aligned}$$

Using this with (2.3), (2.5), (2.8) we get

$$\begin{aligned}&\mathbb{P}\Big(\overline{\mathcal{W}}(\mathbf{0}, \boldsymbol{a}, k, \theta_0) \in \Big[t\Delta(n)(\log n)^{\eta/2}, n\Big]\Big) \\ &\leq \sum_{\boldsymbol{u} \in V_2} C_{21} \exp\left(-C_{22}\left(C_{17}\frac{|d(\boldsymbol{u})|^2}{n} - C_7 \sigma(n)(\log n)^{\eta}\right)/\sigma(n)\right) \\ &\leq C_{23} \exp\big(-C_{24} t^2 (\log n)^{\eta}\big) . \end{aligned} \tag{2.9}$$

Assuming δ_1 is small enough and combining (2.7), (2.9), (2.2) we get

$$\mathbb{P}\Big(\overline{\mathcal{W}}(\mathbf{0}, \boldsymbol{a}, k, \theta_0) \geq t\Delta(n)(\log n)^{\eta/2}\Big) \leq C_{25} \exp\big(-C_{26} t^2 (\log n)^{\eta}\big) .$$

This completes the proof of Theorem 2.8. □

Corollary 2.9. *Let θ_0 be a direction of type I. Then, under Assumptions 1.4 and 1.5, there exist positive constants C_1, C_2, δ_1, δ_2, n_0, t_0, such that for $n \geq n_0$, $t \geq t_0$, $t\Delta(n)(\log n)^{1/2} \leq n\delta_1$, $|l| \leq n\delta_2$, we have*

$$\mathbb{P}\left(\max_{-\infty < k < \infty} \mathcal{W}\left(\mathbf{0}, n\boldsymbol{e}_{\theta_0} + l\boldsymbol{e}_{\theta_0^t}, k, \theta_0\right) \geq t\Delta(n)(\log n)^{1/2}\right) \leq C_1 \exp\left(-C_2 t^2 \log n\right) .$$

Here, the maximum over all real numbers k. But there is no issue with measurability since we can only consider those values of k for which there exist a lattice point $\boldsymbol{u}$ such that $\pi^1_{\theta_0, \theta_0^t}(\boldsymbol{u}) = k$.

Proof. Due to symmetry of the lattice, without loss of generality we assume $\theta_0 \in [0, \pi/4]$. Fix $\delta_1 > 0$, $\delta_2 > 0$, to be assumed appropriately small whenever required. Fix $n_0 > 0$, $t_0 > 0$, to be assumed appropriately large whenever required. Consider n, t, and l, satisfying $n \geq n_0$, $t \geq t_0$, $t\Delta(n)(\log n)^{1/2} \leq n\delta_1$, $|l| \leq n\delta_2$. Let $\boldsymbol{a} \coloneqq n\boldsymbol{e}_{\theta_0} + l\boldsymbol{e}_{\theta_0^t}$. Since $\max_{k \in \mathbb{R}} \mathcal{W}(\mathbf{0}, \boldsymbol{a}, k, \theta_0)$ is equal to $\max_{k \in \mathbb{Z}} \overline{\mathcal{W}}(\mathbf{0}, \boldsymbol{a}, k, \theta_0)$, we work with $\overline{\mathcal{W}}$ instead of $\mathcal{W}$.

Assuming $\delta_2 < 1$ we get $\|\boldsymbol{a}\| \leq 2n$, so that by Lemma 2.2, the geodesic $\Gamma(\mathbf{0}, \boldsymbol{a})$ stays inside a square of side length $C_1 n$ around $\mathbf{0}$ with probability at least $1 - \exp(-C_2 n)$. On this event $|\pi^1_{\theta_0, \theta_0^t}(\boldsymbol{u})| \leq C_3 n$ for all $\boldsymbol{u}$ in $\Gamma(\mathbf{0}, \boldsymbol{a})$. Assuming $\delta_1 < 1$ and using (A2) we get $t^2 \log n \leq n/\sigma(n) \leq C_4 n^{1-\alpha}$. Thus

$$\begin{aligned}&\mathbb{P}\left(\max_{|k| \geq C_3 n,\, k \in \mathbb{Z}} \overline{\mathcal{W}}(\mathbf{0}, \boldsymbol{a}, k, \theta_0) \geq t\Delta(n)(\log n)^{1/2}\right) \\ &\leq \exp(-C_2 n) \leq C_5 \exp\big(-C_6 t^2 \log n\big) . \end{aligned} \tag{2.10}$$

Using Proposition 2.6, Theorem 2.8, and a union bound, we get

$$\mathbb{P}\left(\max_{|k|\le C_3 n\,,\,k\in\mathbb{Z}} \overline{\mathcal{W}}\left(\mathbf{0},\boldsymbol{a},k,\theta_0\right) \ge t\Delta(n)(\log n)^{1/2}\right) \le C_7 \exp\bigl(-C_8 t^2 \log n\bigr)\ . \tag{2.11}$$

Combining (2.10) and (2.11) completes the proof of Corollary 2.9. □

In Theorem 2.8 and Corollary 2.9 we deal with global wandering of geodesics. We have shown transverse wandering of the geodesic between two points roughly at a distance n is at most of the order of $\Delta(n)$ up to some logarithmic correction factor. In Theorem 2.10 we deal with local wandering of geodesics. We consider two points roughly at distance n. We consider wandering of the geodesic between them at a distance $k \le n$ from one of the end points. We show that this wandering is at most of the order of $\Delta(k)$ with some logarithmic correction factor.

Theorem 2.10. *Let θ_0 be a direction of type I. Suppose h satisfies GAP with exponent α and correction factor ϕ_η for some $\eta \in (0,1]$ in a neighborhood of θ_0. Then, under the Assumptions 1.4 and 1.5, there exist positive constants C_1, C_2, δ_1, δ_2, k_0, n_0, t_0, such that for $n \ge n_0$, $t \ge t_0$, $k \ge k_0$, $k < n$, $t\Delta(k)(\log k)^{\eta/2} \le k\delta_1$, $|l| \le n\delta_2$, we have*

$$\mathbb{P}\Big(\mathcal{W}\left(\mathbf{0}, n\boldsymbol{e}_{\theta_0} + l\boldsymbol{e}_{\theta_0^t}, k, \theta_0\right) \ge t\Delta(k)(\log k)^{\eta/2}\Big) \le C_1 \exp\left(-C_2 t^2 (\log k)^{\eta}\right)\ .$$

Proof. Due to symmetry of the lattice, without loss of generality we assume $\theta_0 \in [0,\pi/4]$. Fix $\delta_1 > 0$, $\delta_2 > 0$, to be assumed appropriately small whenever required. Fix $k_0 > 0$, $n_0 > 0$, $t_0 > 0$, to be assumed appropriately large whenever required. Consider k, n, l, t satisfying $n \ge n_0$, $t \ge t_0$, $k \ge k_0$, $t\Delta(k)(\log k)^{\eta/2} \le k\delta_1$, $|l| \le n\delta_2$. Let $\boldsymbol{a} \coloneqq n\boldsymbol{e}_{\theta_0} + l\boldsymbol{e}_{\theta_0^t}$.

Construction of sequences of points $(\boldsymbol{a}_p)_{p=0}^m$ and $(\boldsymbol{b}_p)_{p=0}^m$: We construct two sequences of points $(\boldsymbol{a}_p)_{p=0}^{m+1}$ and $(\boldsymbol{b}_p)_{p=0}^{m+1}$ in the following manner.

(i) Fix a parameter ζ satisfying $1 < \zeta < 2/(1+\beta)$. Define another parameter $\epsilon \coloneqq 1 - \zeta(1+\beta)/2$. Fix parameters $\lambda > 1$, $\lambda' > 0$. Later we choose λ so that (2.14) holds, and we choose λ' based on λ so that (2.12) holds.

(ii) Let $m \ge 0$ be such that $\lambda^m k < n \le \lambda^{m+1} k$.

(iii) Define both $\boldsymbol{a}_{m+1}$ and $\boldsymbol{b}_{m+1}$ to be the point $\boldsymbol{a}$.

(iv) To define $(\boldsymbol{a}_p)_{p=0}^m$ and $(\boldsymbol{b}_p)_{p=0}^m$ first we fix the $\pi^1_{\theta_0,\theta_0^t}$ values of these points. For each $0 \le p \le m$ let

$$\pi^1_{\theta_0,\theta_0^t}(\boldsymbol{a}_p) = \pi^1_{\theta_0,\theta_0^t}(\boldsymbol{b}_p) = \lambda^p k\ .$$

(v) Now we fix the $\pi^2_{\theta_0,\theta_0^t}$ values of the points $(\boldsymbol{a}_p)_{p=0}^m$ and $(\boldsymbol{b}_p)_{p=0}^m$. We have already set $\boldsymbol{a}_{m+1} = \boldsymbol{b}_{m+1} = \boldsymbol{a}$ so that $\pi^2_{\theta_0,\theta_0^t}(\boldsymbol{a}_{m+1}) = \pi^2_{\theta_0,\theta_0^t}(\boldsymbol{b}_{m+1}) = l$. Now we define $\pi^2_{\theta_0,\theta_0^t}(\boldsymbol{a}_m),\dots,\pi^2_{\theta_0,\theta_0^t}(\boldsymbol{a}_0)$ and $\pi^2_{\theta_0,\theta_0^t}(\boldsymbol{b}_m),\dots,\pi^2_{\theta_0,\theta_0^t}(\boldsymbol{b}_0)$ recursively. For each $0 \le p \le m$ let

$$\pi^2_{\theta_0,\theta_0^t}(\boldsymbol{a}_p) = \frac{\pi^1_{\theta_0,\theta_0^t}(\boldsymbol{a}_p)}{\pi^1_{\theta_0,\theta_0^t}(\boldsymbol{a}_{p+1})}\pi^2_{\theta_0,\theta_0^t}(\boldsymbol{a}_{p+1}) + \lambda' t\Delta(\lambda^{\zeta p}k)(\log(\lambda^{\zeta p}k))^{\eta/2}\ ,$$

$$\pi^2_{\theta_0,\theta_0^t}(\boldsymbol{b}_p) = \frac{\pi^1_{\theta_0,\theta_0^t}(\boldsymbol{b}_p)}{\pi^1_{\theta_0,\theta_0^t}(\boldsymbol{b}_{p+1})}\pi^2_{\theta_0,\theta_0^t}(\boldsymbol{b}_{p+1}) - \lambda' t\Delta(\lambda^{\zeta p}k)(\log(\lambda^{\zeta p}k))^{\eta/2}\ .$$

Recall that we have assumed $\theta_0 \in [0,\pi/4]$. Also by our convention of taking tangents in the counter-clockwise direction we have $\theta_0^t > 0$. This means, $\boldsymbol{a}_p$ is above the line joining $\mathbf{0}$ and $\boldsymbol{a}_{p+1}$. And the distance in θ_0^t direction of the point $\boldsymbol{a}_p$ and the line joining $\mathbf{0}$ and $\boldsymbol{a}_{p+1}$ is $\lambda' t\Delta(\lambda^{\zeta p}k)(\log(\lambda^{\zeta p}k))^{\eta/2}$. Similarly $\boldsymbol{b}_p$ is below the line joining $\mathbf{0}$ and $\boldsymbol{b}_{p+1}$. And the distance in θ_0^t direction of the point $\boldsymbol{b}_p$ and the line joining $\mathbf{0}$ and $\boldsymbol{b}_{p+1}$ is $\lambda' t\Delta(\lambda^{\zeta p}k)(\log(\lambda^{\zeta p}k))^{\eta/2}$.

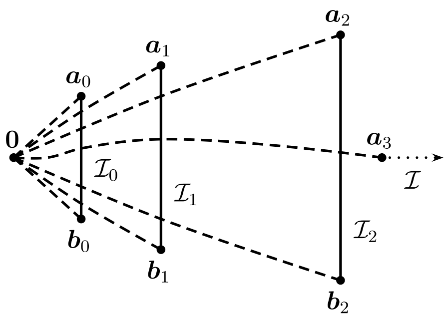

Figure 3: Rough sketch for $m = 2$: $\boldsymbol{a}_{m+1} = \boldsymbol{b}_{m+1} = \boldsymbol{a}$; for $0 \le p \le m$ the segment joining $\boldsymbol{a}_p$ and $\boldsymbol{b}_p$ is $\mathcal{I}_p$, and extending $\mathcal{I}_p$ we get the line $\overline{\mathcal{I}}_p$; for all $0 \le p \le m$ if the geodesics $\Gamma(\mathbf{0}, \boldsymbol{a}_p)$ and $\Gamma(\mathbf{0}, \boldsymbol{b}_p)$ do not wander excessively then for all $0 \le p \le m$ the geodesic $\Gamma(\mathbf{0}, \boldsymbol{a})$ intersects $\mathcal{I}_p$ whenever it intersects $\overline{\mathcal{I}}_p$.

Defining $\left(\overline{\mathcal{I}}_p\right)_{p=0}^m$, $\left(\mathcal{I}_p\right)_{p=0}^m$, $\mathcal{I}$: For $0 \le p \le m$, let

$$\overline{\mathcal{I}}_p \coloneqq \left\{ \boldsymbol{x} \in \mathbb{R}^2 \colon \pi^1_{\theta_0, \theta_0^t}(\boldsymbol{x}) = \lambda^p k \right\} .$$

Thus $\boldsymbol{a}_p$, $\boldsymbol{b}_p$ are on this line. Let $\mathcal{I}_p$ be the segment joining $\boldsymbol{a}_p$ and $\boldsymbol{b}_p$. Define the half-line

$$\mathcal{I} \coloneqq \left\{ \boldsymbol{x} \in \mathbb{R}^2 \colon \pi^1_{\theta_0, \theta_0^t}(\boldsymbol{x}) \ge n,\ \frac{\pi^2_{\theta_0, \theta_0^t}(\boldsymbol{x})}{\pi^1_{\theta_0, \theta_0^t}(\boldsymbol{x})} = \frac{l}{n} \right\} .$$

Strategy of the proof: We want to establish a lower bound of the probability of the event $\mathcal{W}(\mathbf{0}, \boldsymbol{a}, k, \theta_0) \le t\Delta(k)(\log k)^{\eta/2}$. We will choose λ' (depending on λ) in a such a way that the event $\mathcal{W}(\mathbf{0}, \boldsymbol{a}, k, \theta_0) \le t\Delta(k)(\log k)^{\eta/2}$ includes the event that whenever $\Gamma(\mathbf{0}, \boldsymbol{a})$ intersects $\overline{\mathcal{I}}_0$, it intersects $\mathcal{I}_0$. So we will establish a lower bound of this event. This event happens if all of the events listed below happen:

(i) For all $0 \le p \le m$, the geodesic $\Gamma(\mathbf{0}, \boldsymbol{a}_{p+1})$ intersects $\mathcal{I}_p$ whenever it intersects $\overline{\mathcal{I}}_p$.

(ii) For all $0 \le p \le m$, the geodesic $\Gamma(\mathbf{0}, \boldsymbol{a}_p)$ does not intersect $\mathcal{I}$.

(iii) For all $0 \le p \le m$, the geodesic $\Gamma(\mathbf{0}, \boldsymbol{b}_{p+1})$ intersects $\mathcal{I}_p$ whenever it intersects $\overline{\mathcal{I}}_p$.

(iv) For all $0 \le p \le m$, the geodesic $\Gamma(\mathbf{0}, \boldsymbol{b}_p)$ does not intersect $\mathcal{I}$.

If these events happen, then $\Gamma(\mathbf{0}, \boldsymbol{a})$ intersects $\mathcal{I}_p$ whenever it intersects $\overline{\mathcal{I}}_p$, for each $0 \le p \le m$. In particular, $\Gamma(\mathbf{0}, \boldsymbol{a})$ intersects $\mathcal{I}_0$ whenever it intersects $\overline{\mathcal{I}}_0$. Here we crucially use the uniqueness of geodesics. All the geodesics $\Gamma(\mathbf{0}, \boldsymbol{a}_p)$ and $\Gamma(\mathbf{0}, \boldsymbol{b}_p)$ for $0 \le p \le m$ after starting from $\mathbf{0}$ cannot touch or intersect each other, Figure 3 shows the situation for $m = 2$. Now we will choose λ' and establish lower bounds on the probability of the events listed above.

Choosing λ': For any $0 \le p \le q \le m$, using (A2) we get

$$\frac{\lambda^{-q}\Delta\big(\lambda^{\zeta q}k\big)\big(\log\big(\lambda^{\zeta q}k\big)\big)^{\eta/2}}{\lambda^{-p}\Delta(\lambda^{\zeta p}k)(\log(\lambda^{\zeta p}k))^{\eta/2}} \le C_1\lambda^{-\epsilon(q-p)}\left(\frac{q\zeta\log\lambda+\log k}{p\zeta\log\lambda+\log k}\right)^{\eta/2}$$
$$\le C_2\lambda^{-\epsilon(q-p)}\left(1+\frac{(q-p)\zeta\log\lambda}{p\zeta\log\lambda+\log k}\right)^{\eta/2} \le C_3\lambda^{-\epsilon(q-p)}\left(1+(q-p)\log\lambda\right)^{\eta/2} .$$

Therefore, we can choose λ', depending on λ, such that

$$\sum_{q=p}^{m}\lambda^{-(q-p)}\Delta\big(\lambda^{\zeta q}k\big)\big(\log\big(\lambda^{\zeta q}k\big)\big)^{\eta/2} \le \frac{1}{\lambda'}\Delta\big(\lambda^{\zeta p}k\big)\big(\log\big(\lambda^{\zeta p}k\big)\big)^{\eta/2}. \tag{2.12}$$

Therefore, for $0 \le p \le m$

$$\pi^2_{\theta_0,\theta_0^t}(\boldsymbol{a}_p) = \lambda^p k\frac{l}{n} + \lambda' t\sum_{q=p}^{m}\lambda^{-(q-p)}\Delta\big(\lambda^{\zeta q}k\big)\big(\log\big(\lambda^{\zeta q}k\big)\big)^{\eta/2}$$
$$\le \lambda^p k\frac{l}{n} + t\Delta\big(\lambda^{\zeta p}k\big)\big(\log\big(\lambda^{\zeta p}k\big)\big)^{\eta/2} ,$$
$$\pi^2_{\theta_0,\theta_0^t}(\boldsymbol{b}_p) = \lambda^p k\frac{l}{n} - \lambda' t\sum_{q=p}^{m}\lambda^{-(q-p)}\Delta\big(\lambda^{\zeta q}k\big)\big(\log\big(\lambda^{\zeta q}k\big)\big)^{\eta/2}$$
$$\ge \lambda^p k\frac{l}{n} - t\Delta\big(\lambda^{\zeta p}k\big)\big(\log\big(\lambda^{\zeta p}k\big)\big)^{\eta/2} .$$

Therefore, the event $\mathcal{W}(\mathbf{0},\boldsymbol{a},k,\theta_0) \le t\Delta(k)(\log k)^{\eta/2}$ includes the event that whenever $\Gamma(\mathbf{0},\boldsymbol{a})$ intersects $\overline{\mathcal{I}}_0$, it intersects $\mathcal{I}_0$.

Defining the events $(\mathcal{E}^1_p)_{p=0}^m$, $(\mathcal{E}^2_p)_{p=0}^m$: For $0 \le p \le m$ let

$$\mathcal{E}^1_p := \left\{\mathcal{W}(\mathbf{0},\boldsymbol{a}_{p+1},\lambda^p k,\theta_0) \le \lambda' t\Delta\big(\lambda^{\zeta p}k\big)\big(\log\big(\lambda^{\zeta p}k\big)\big)^{\eta/2}\right\},$$
$$\mathcal{E}^2_p := \left\{\mathcal{W}(\mathbf{0},\boldsymbol{b}_{p+1},\lambda^p k,\theta_0) \le \lambda' t\Delta\big(\lambda^{\zeta p}k\big)\big(\log\big(\lambda^{\zeta p}k\big)\big)^{\eta/2}\right\}.$$

Recall that $\mathbb{T}$ denotes the overall edge-weight configuration. Thus, for each $0 \le p \le m$, if $\mathbb{T} \in \mathcal{E}^1_p \cap \mathcal{E}^2_p$, then the geodesics $\Gamma(\mathbf{0},\boldsymbol{a}_{p+1})$ and $\Gamma(\mathbf{0},\boldsymbol{b}_{p+1})$ intersect $\mathcal{I}_p$ whenever they intersect $\overline{\mathcal{I}}_p$.

Defining the events $(\mathcal{E}^3_p)_{p=0}^m$, $(\mathcal{E}^4_p)_{p=0}^m$: Let

$$\mathcal{E}^3_m := \left\{\max_{n'\ge n}\mathcal{W}(\mathbf{0},\boldsymbol{a}_m,n',\theta_0) \le \lambda' t\Delta\big(\lambda^{\zeta m}k\big)\big(\log\big(\lambda^{\zeta m}k\big)\big)^{\eta/2}\right\},$$
$$\mathcal{E}^4_m := \left\{\max_{n'\ge n}\mathcal{W}(\mathbf{0},\boldsymbol{b}_m,n',\theta_0) \le \lambda' t\Delta\big(\lambda^{\zeta m}k\big)\big(\log\big(\lambda^{\zeta m}k\big)\big)^{\eta/2}\right\}.$$

Let us assume $\delta_2 < 1/2$ so that $\|\boldsymbol{a}\| > n/2$. Define for $0 \le p < m$

$$\mathcal{E}^3_p := \{\operatorname{Diam}(\Gamma(\mathbf{0},\boldsymbol{a}_p)) \le n/2\},$$
$$\mathcal{E}^4_p := \{\operatorname{Diam}(\Gamma(\mathbf{0},\boldsymbol{b}_p)) \le n/2\}.$$

So, for $0 \le p \le m$, if $\mathbb{T} \in \mathcal{E}^3_p$, then $\Gamma(\mathbf{0},\boldsymbol{a}_p)$ does not intersect $\mathcal{I}$, and if $\mathbb{T} \in \mathcal{E}^4_p$, then $\Gamma(\mathbf{0},\boldsymbol{b}_p)$ does not intersect $\mathcal{I}$.

Bounding probability of the events $(\mathcal{E}_p^1)_{p=0}^m$, $(\mathcal{E}_p^2)_{p=0}^m$, $\mathcal{E}_m^3$, $\mathcal{E}_m^4$: Fix $0 \le p \le m$ and consider the event $\mathcal{E}_p^1$. We use Theorem 2.8 to bound $\mathbb{P}(\,(\mathcal{E}_p^1)^c\,)$. We use the following parameters:

$$\tilde{\eta} := \eta\,, \quad \tilde{n} := \pi^1_{\theta_0,\theta_0^t}(\boldsymbol{a}_{p+1})\,, \quad \tilde{l} := \pi^2_{\theta_0,\theta_0^t}(\boldsymbol{a}_{p+1})\,, \quad \tilde{t} := \lambda' t \frac{\Delta\big(\lambda^{\zeta p}k\big)\big(\log\big(\lambda^{\zeta p}k\big)\big)^{\eta/2}}{\Delta(\tilde{n})(\log \tilde{n})^{\eta/2}}\,.$$

Recall from Remark 1.2 our convention of using tilde on parameters. So we use Theorem 2.8 with $\tilde{\eta}$ in place of η, $\tilde{n}$ in place n, and so on. We need to verify $\tilde{n} \ge \tilde{n}_0$, $\tilde{t} \ge \tilde{t}_0$, $\tilde{t}\Delta(\tilde{n})(\log \tilde{n})^{\tilde{\eta}/2} \le \tilde{n}\delta_1$, $|\tilde{l}| \le \tilde{n}\tilde{\delta}_2$. The condition $\tilde{n} \ge \tilde{n}_0$ holds by taking k_0 large enough because $\tilde{n} \ge k \ge k_0$. Using $\tilde{n} \le \lambda^{p+1}k$ and (A2) we get $\tilde{t} \ge C_4 t$. So $\tilde{t} \ge \tilde{t}_0$ holds by choosing t_0 large enough. Using $\tilde{n} \ge \lambda^p k$, (A2), $t\Delta(k)(\log k)^{\eta/2} \le k\delta_1$, and assuming δ_1 is small enough, we get

$$\frac{1}{\tilde{n}}\tilde{t}\Delta(\tilde{n})(\log \tilde{n})^{\eta/2} \le C_5 \frac{1}{\lambda^p k} t\Delta(\lambda^p k)(\log \lambda^p k)^{\eta/2} \le C_6 \frac{1}{k} t\Delta(k)(\log k)^{\eta/2} \le C_7\delta_1 \le \tilde{\delta}_1\,.$$

If $p = m$, then $|\tilde{l}|/\tilde{n} = |l|/n \le \delta_2 \le \tilde{\delta}_2$. For $p < m$, using $t\Delta(k)(\log k)^{\eta/2} \le k\delta_1$, $|l| \le n\delta_2$, (A2), and assuming δ_1, δ_2 are small enough, we get

$$\begin{aligned}
\frac{|\tilde{l}|}{\tilde{n}} &= \frac{|\pi^2_{\theta_0,\theta_0^t}(\boldsymbol{a}_{p+1})|}{\pi^1_{\theta_0,\theta_0^t}(\boldsymbol{a}_{p+1})} \\
&\le \frac{1}{\lambda^{p+1}k}\left(\lambda^{p+1}k\frac{|l|}{n} + t\Delta\Big(\lambda^{\zeta(p+1)}k\Big)\Big(\log\Big(\lambda^{\zeta(p+1)}k\Big)\Big)^{\eta/2}\right) \\
&\le \delta_2 + \delta_1 \frac{\Delta\big(\lambda^{\zeta(p+1)}k\big)\big(\log\big(\lambda^{\zeta(p+1)}k\big)\big)^{\eta/2}}{\lambda^p\Delta(k)(\log k)^{\eta/2}} \\
&\le \delta_2 + C_8\delta_1\lambda^{-\epsilon(p+1)}\left(1 + \frac{(p+1)\zeta\log\lambda}{\log k}\right)^{\eta/2} \\
&\le \delta_2 + C_9\delta_1 \\
&\le \tilde{\delta}_2\,.
\end{aligned}$$

So the conditions for applying Theorem 2.8 hold. Using $\tilde{n} \le \lambda^{p+1}k$ and (A2) we get

$$\tilde{t}^2(\log \tilde{n})^{\eta} \ge C_{10}t^2\lambda^{p\epsilon}(\log k)^{\eta}\,.$$

Therefore, applying Theorem 2.8 we get

$$\mathbb{P}\Big(\big(\mathcal{E}_p^1\big)^c\Big) \le C_{11}\exp\big(-C_{12}t^2\lambda^{p\epsilon}(\log k)^{\eta}\big)\,. \tag{2.13}$$

Similar bounds hold for $\mathcal{E}_p^2$ for $0 \le p \le m$, $\mathcal{E}_m^3$, $\mathcal{E}_m^4$.

Bounding probability of $\mathcal{E}_p^3$ and $\mathcal{E}_p^4$ for $0 \le p < m$: Fix $0 \le p < m$. From the verification of $|\tilde{l}|/|\tilde{n}| \le \tilde{\delta}_2$ we get $\|\boldsymbol{a}_p\| \le 2\lambda^p k$ and $\|\boldsymbol{b}_p\| \le 2\lambda^p k$. We have $n > \lambda^m k$. Thus $n/2 \ge (\lambda/4)\|\boldsymbol{a}_p\|$ and $n/2 \ge (\lambda/4)\|\boldsymbol{a}_p\|$. Thus, assuming λ large enough, we get for $i = 3, 4$ and $0 \le p < m$, using Lemma 2.2,

$$\mathbb{P}\Big(\big(\mathcal{E}_p^i\big)^c\Big) \le C_{13}\exp(-C_{14}\lambda^p k)\,. \tag{2.14}$$

Combining (2.13) and (2.14) we get

$$\mathbb{P}\Big(\mathcal{W}\left(\mathbf{0}, \boldsymbol{a}, k, \theta_0\right) \ge t\Delta(k)(\log k)^{\eta/2}\Big) \le \mathbb{P}\Big(\cup_{i=1}^4 \cup_{p=0}^m \big(\mathcal{E}_p^i\big)^c\Big) \le C_{15}\exp\big(-C_{16}t^2(\log k)^{\eta}\big)\,.$$

This completes the proof of Theorem 2.10. □

As a corollary we can deal with wandering of a geodesic within a fixed distance from one of the endpoints of the geodesic. In the next result we consider geodesics with one endpoint at the origin, and consider wandering of the geodesic in a neighborhood of the origin.

Corollary 2.11. *Let θ_0 be a direction of type I. Then, under the Assumptions 1.4 and 1.5, there exist positive constants C_1, C_2, δ_1, δ_2, k_0, n_0, t_0, such that for $k \geq k_0$, $n \geq n_0$, $t \geq t_0$, $t\Delta(k)(\log k)^{1/2} \leq k\delta_1$, $|l| \leq n\delta_2$, we have*

$$\mathbb{P}\left(\max_{k' \leq k} \mathcal{W}\left(\mathbf{0}, n\boldsymbol{e}_{\theta_0} + l\boldsymbol{e}_{\theta_0^t}, k', \theta_0\right) \geq t\Delta(k)(\log k)^{1/2}\right) \leq C_1 \exp\left(-C_2 t^2 \log k\right) .$$

Here k' takes real values. There is no issue with measurability since we only need to consider $k' = k$ and those values of $k' < k$ for which there exists some lattice point $\boldsymbol{u}$ such that $\pi^1_{\theta_0,\theta_0^t}(\boldsymbol{u}) = k'$.

Proof. Due to symmetry of the lattice, without loss of generality we assume $\theta_0 \in [0, \pi/4]$. Fix $\delta_1 > 0$, $\delta_2 > 0$, to be assumed appropriately small whenever required. Fix $k_0 > 0$, $n_0 > 0$, $t_0 > 0$, to be assumed large enough whenever required. Let $\boldsymbol{a} := n\boldsymbol{e}_{\theta_0} + l\boldsymbol{e}_{\theta_0^t}$. Define

$$\mathcal{I} := \left\{ \boldsymbol{v} \in \mathbb{R}^2 \colon \pi^1_{\theta_0,\theta_0^t}(\boldsymbol{v}) = k, \left|\pi^2_{\theta_0,\theta_0^t}(\boldsymbol{v}) - k\frac{l}{n}\right| \leq \frac{t}{2}\Delta(k)(\log k)^{1/2} \right\} .$$

The event $\max_{k' \leq k} \mathcal{W}(\mathbf{0}, \boldsymbol{a}, k', \theta_0) \geq t\Delta(k)(\log k)^{1/2}$ can happen in two ways: either we have $\mathcal{W}(\mathbf{0}, \boldsymbol{a}, k, \theta_0) \geq (t/2)\Delta(k)(\log k)^{1/2}$, or $\Gamma(\mathbf{0}, \boldsymbol{a})$ passes through some $\boldsymbol{v} \in \mathcal{I}$ and $\max_{k'} \mathcal{W}(\mathbf{0}, \boldsymbol{v}, k', \theta_0) \geq (t/2)\Delta(k)(\log k)^{1/2}$. In the first case, by Theorem 2.10 and Proposition 2.6, we get

$$\mathbb{P}\left(\mathcal{W}(\mathbf{0}, \boldsymbol{a}, k, \theta_0) \geq \frac{t}{2}\Delta(k)(\log k)^{1/2}\right) \leq C_1 \exp\left(-C_2 t^2 \log k\right) . \tag{2.15}$$

For the second case, consider $\boldsymbol{v} \in \mathcal{I}$. We apply Corollary 2.9 with

$$\tilde{\theta_0} := \theta_0 , \quad \tilde{n} := k , \quad \tilde{l} := \pi^2_{\theta_0,\theta_0^t}(\boldsymbol{v}) , \quad \tilde{t} := t .$$

(Recall from Remark 1.2 our convention of using tilde on parameters.) Then

$$\frac{|\tilde{l}|}{\tilde{n}} \leq \frac{|l|}{n} + \frac{1}{k}\frac{t}{2}\Delta(k)(\log k)^{1/2} \leq \delta_1 + \frac{\delta_2}{2} .$$

Also

$$\frac{1}{\tilde{n}}\tilde{t}\Delta(\tilde{n})(\log \tilde{n})^{1/2} \leq \frac{1}{k}t\Delta(k)(\log k)^{1/2} \leq \delta_2 .$$

Therefore, assuming δ_1, δ_2 are small enough, we get

$$\mathbb{P}\left(\max_{k'} \mathcal{W}(\mathbf{0}, \boldsymbol{v}, k', \theta_0) \geq \frac{t}{2}\Delta(k)(\log k)^{1/2}\right) \leq C_3 \exp\left(-C_4 t^2 \log k\right) .$$

Recall that $\mathcal{I}$ is of length $t\Delta(k)(\log k)^{1/2}$. We need to take a union bound over $\boldsymbol{v} \in \mathcal{I}$. Although there are uncountable many such $\boldsymbol{v}$, number of lattice points $\lfloor \boldsymbol{v} \rfloor$ (recall Notation 1.1) where $\boldsymbol{v} \in \mathcal{I}$ is of the order of $t\Delta(k)(\log k)^{1/2}$. Thus, by a union bound we get

$$\mathbb{P}\left(\max_{k'} \mathcal{W}(\mathbf{0}, \boldsymbol{v}, k', \theta_0) \geq t\Delta(k)(\log k)^{1/2} \text{ for some } \boldsymbol{v} \in \mathcal{I}\right)$$
$$\leq C_5 \exp\left(-C_6 t^2 \log k\right) . \tag{2.16}$$

Combining (2.15), (2.16) completes the proof of Corollary 2.11. □

Remark 2.12. Recall from Notation 1.1 that geodesic between points $\boldsymbol{u}$, $\boldsymbol{v}$ which are not necessarily lattice points is defined as the geodesic between $\lfloor \boldsymbol{u} \rfloor$ and $\lfloor \boldsymbol{v} \rfloor$. In Theorem 2.8, Corollary 2.9, Theorem 2.10, and Corollary 2.11, we are dealing with geodesics having one endpoint $\mathbf{0}$ and we are measuring the wandering at a distance from $\mathbf{0}$. Later while applying these results we may have a point of $\mathbb{R}^2$ in place of $\mathbf{0}$. This does not cause any major complication, i.e., bounds that hold for wandering of $\Gamma(\mathbf{0}, \boldsymbol{u})$ also hold for wandering of $\Gamma(\boldsymbol{v}, \boldsymbol{u}+\boldsymbol{v})$, where $\boldsymbol{u}$ and $\boldsymbol{v}$ are not necessarily lattice points.

3 Preliminary upper bound of the transverse increments

In this section, our principal objective is to prove Theorem 3.2. This is a special case of Theorem 1.23 which is our main upper bound on the transverse increments. In Section 4, we use Theorem 3.2 to prove Theorem 4.4 which is a refinement of the bound on nonrandom fluctuations of Proposition 2.6. We use this refinement to prove Theorem 1.23 in Section 5.

We need the following result on curvature of the boundary of the limit shape. We skip the proof because the result is essentially same as Lemma 2.7 of [3].

Lemma 3.1. *Let θ_0 be a direction of type II. Then there exist positive constants C, δ_1, δ_2, such that for d, $k > 0$, L, satisfying $|d| \le k\delta_1$, $|L| \le k\delta_2$, we have*

$$\left| g(k\boldsymbol{e}_{\theta_0} + (d+L)\boldsymbol{e}_{\theta_0^t}) - g(k\boldsymbol{e}_{\theta_0} + d\boldsymbol{e}_{\theta_0^t}) \right| \le C \left(\frac{L^2}{k} + \frac{|d| \cdot |L|}{k} \right) .$$

The preliminary upper bound of the transverse increments is the following.

Theorem 3.2. *Let θ_0 be a direction of both type I and II. For $n > 0$, $L > 0$, let $\mathcal{I}(n,L)$ and $\mathcal{D}(n,L)$ be as defined in Theorem 1.23. Then, under Assumptions 1.4 and 1.5, there exist positive constants C_1, C_2, L_0, n_0, t_0, such that for $L \ge L_0$, $n \ge n_0$, $t \ge t_0$, $L \le \Delta(n)$, we have*

$$\mathbb{P}\big(\mathcal{D}(n,L) \ge t \log L \sigma\big(\Delta^{-1}(L)\big)\big) \le C_1 \exp(-C_2 t \log L) .$$

The same bound holds for $\mathcal{D}'(n,L)$.

Proof. Due to the symmetry of the lattice, without loss of generality we assume $\theta_0 \in [0, \pi/4]$. Fix $L_0 > 0$, $n_0 > 0$, $t_0 > 0$. We assume L_0, n_0, t_0 are large whenever required. Consider $L \ge L_0$, $n \ge n_0$, $t \ge t_0$, $L \le \Delta(n)$. We focus on $\mathcal{D}(n,L)$. The bound on $\mathcal{D}'(n,L)$ can be proved similarly. Based on the values of t we consider two cases. Because for suitably large values of t we are in a large deviation regime, and the proof is straightforward.

Case I: Suppose

$$t \ge 4\,\mu\,L\left(\sigma\big(\Delta^{-1}(L)\big)\log L\right)^{-1} ,$$

where μ is the expected passage edge-weight. Since $\mathcal{I}(n,L)$ has width L, the set of lattice points $\{\lfloor \boldsymbol{x} \rfloor : \boldsymbol{x} \in \mathcal{I}(n,L)\}$ (see Notation 1.1) can be joined by a lattice path of at most $\lceil 2L \rceil$ edges. Hence $\mathcal{D}(n,L) \le X_1 + \cdots + X_{\lceil 2L \rceil}$, where X_i's are i.i.d. random variables which have the same distribution as that of the edge-weights. Therefore,

$$\begin{aligned}
&\mathbb{P}\big(\mathcal{D}(n,L) \ge t\,(\log L)\,\sigma(\Delta^{-1}(L))\big) \\
\le\ &\mathbb{P}\big(X_1 + \cdots + X_{\lceil 2L \rceil} \ge t\,(\log L)\,\sigma(\Delta^{-1}(L)\big) \\
\le\ &C_1 \exp\big(-C_2\,t\,(\log L)\,\sigma(\Delta^{-1}(L))\big) \\
\le\ &C_3 \exp(-C_4\,t\,\log L) .
\end{aligned}$$

This concludes the proof in this case.

Case II: Suppose

$$t \le 4\,\mu\,L\left(\sigma\big(\Delta^{-1}(L)\big)\log L\right)^{-1} . \tag{3.1}$$

Define

$$\begin{aligned} J &:= \left[-t^{1/2}L(\log\Delta^{-1}(L))^{1/2}, \left(1-\frac{\Delta^{-1}(L)}{n}\right)L + t^{1/2}L\left(\log\Delta^{-1}(L)\right)^{1/2}\right] , \\ \mathcal{I}^* &:= \left\{ \boldsymbol{x}\in\mathbb{R}^2 \colon \pi^1_{\theta_0,\theta_0^t}(\boldsymbol{x}) = n-\Delta^{-1}(L)\,,\, \pi^2_{\theta_0,\theta_0^t}(\boldsymbol{x})\in J \right\} , \\ \mathcal{E} &:= \{\,\Gamma(\boldsymbol{0},\boldsymbol{u}) \text{ intersects } \mathcal{I}^* \text{ for all } \boldsymbol{u}\in\mathcal{I}(n,L)\,\} . \end{aligned} \tag{3.2}$$

Observe that, if $\mathbb{T}\notin\mathcal{E}$, then $\mathcal{W}\left(\boldsymbol{u},\boldsymbol{0},\Delta^{-1}(L),-\theta_0\right) \ge t^{1/2}L(\log\Delta^{-1}(L))^{1/2}$ for some $\boldsymbol{u}\in\mathcal{I}(n,L)$. Consider a point $\boldsymbol{u}\in\mathcal{I}(n,L)$. We want to apply Theorem 2.10 with the variables

$$\begin{aligned} &\tilde{k} := \Delta^{-1}(L)\,,\ \ \tilde{l} := \pi^2_{\theta_0,\theta_0^t}(\boldsymbol{u})\,,\ \ \tilde{n} := \pi^2_{\theta_0,\theta_0^t}(\boldsymbol{u}) = n\,, \\ &\tilde{t} := t^{1/2}\,, \qquad \tilde{\theta_0} := -\theta_0\,, \qquad \tilde{\eta} := 1\,. \end{aligned}$$

(Recall from Remark 1.2 our convention about using parameters with tilde, essentially we want to use Theorem 2.10 with $\tilde{k}$ in place of k and so on.) We need to verify that these variables satisfy the conditions of Theorem 2.10. The point $\boldsymbol{u}$ is not necessarily a lattice point. But this issue has been addressed in Remark 2.12. The conditions on $\tilde{\theta_0}$ hold because by assumption θ_0 is of type I (so that, by symmetry $-\theta_0$ is also of type I), and by Proposition 2.6 h satisfies GAP with correction factor ϕ_1 in all directions. Now, we need to verify the conditions $\tilde{k}\ge\tilde{k_0}$, $\tilde{n}\ge\tilde{n_0}$, $\tilde{t}\ge\tilde{t_0}$, $\tilde{t}\Delta(\tilde{k})(\log\tilde{k})^{1/2}\le\tilde{k}\tilde{\delta_1}$, $|\tilde{l}|\le\tilde{n}\tilde{\delta_2}$. Assuming n_0, L_0, t_0 are large enough, we get $\tilde{k}\ge\tilde{k_0}$, $\tilde{n}\ge\tilde{n_0}$, $\tilde{t}\ge\tilde{t_0}$. From $\boldsymbol{u}\in\mathcal{I}(n,L)$ we get $|\pi^2_{\theta_0,\theta_0^t}(u)|\le L$. Therefore, using $L\le\Delta(n)$, (A2), and assuming n_0 is large enough, we get

$$\frac{|\tilde{l}|}{\tilde{n}} \le \frac{L}{n} \le \frac{\Delta(n)}{n} \le n_0^{-(1-\beta)/2} \le \tilde{\delta_2}\,.$$

Using (3.1), (A2), and assuming L_0 is large enough, we get

$$\begin{aligned} \frac{1}{\tilde{k}}\tilde{t}\Delta(\tilde{k})(\log\tilde{k})^{1/2} = \frac{t^{1/2}L\left(\log\Delta^{-1}(L)\right)^{1/2}}{\Delta^{-1}(L)} &\le C_5\frac{L^{3/2}(\log\Delta^{-1}(L))^{1/2}}{\Delta^{-1}(L)(\sigma(\Delta^{-1}(L))^{1/2}(\log L)^{1/2}} \\ &\le C_6\frac{L^{1/2}}{(\Delta^{-1}(L))^{1/2}} \le C_7 L^{-\frac{(1-\beta)}{2(1+\beta)}} \le C_8 L_0^{-\frac{(1-\beta)}{2(1+\beta)}} \le \tilde{\delta_1}\,. \end{aligned}$$

Therefore, all the conditions for applying Theorem 2.10 are satisfied, and we get

$$\mathbb{P}\Big(\mathcal{W}\left(\boldsymbol{u},\boldsymbol{0},\Delta^{-1}(L),-\theta_0\right) \ge t^{1/2}L(\log\Delta^{-1}(L))^{1/2}\Big) \le C_9\exp(-C_{10}t\log L)\,.$$

Therefore, taking a union bound over $\{\,\lfloor\boldsymbol{u}\rfloor \colon \boldsymbol{u}\in\mathcal{I}(n,L)\,\}$ (recall Notation 1.1) we get

$$\mathbb{P}(\mathcal{E}^c) \le C_{11}\exp(-C_{12}t\log L)\,. \tag{3.3}$$

So in order to complete the proof we consider $\mathbb{T}\in\mathcal{E}$.

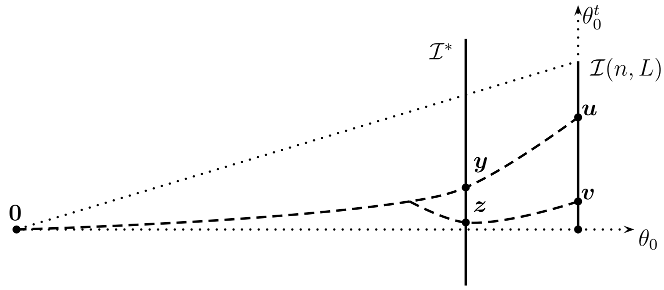

Figure 4: Illustration for Theorem 3.2 under case II: distance of $\mathcal{I}(n,L)$ from $\mathbf{0}$ in θ_0 direction is n, width of $\mathcal{I}(n,L)$ in θ_0^t direction is L, distance of $\mathcal{I}^*$ from $\mathbf{0}$ is $n-\Delta^{-1}L$, if $\mathbb{T}\in\mathcal{E}$ then geodesics from $\mathbf{0}$ to points in $\mathcal{I}(n,L)$ passes through $\mathcal{I}^*$.

Consider two points $\boldsymbol{u}$ and $\boldsymbol{v}$ on $\mathcal{I}(n,L)$. Since $\mathbb{T}\in\mathcal{E}$, there exist points $\boldsymbol{y}$ and $\boldsymbol{z}$ on $\mathcal{I}^*$ such that the geodesic $\Gamma(\mathbf{0},\boldsymbol{u})$ passes through $\boldsymbol{y}$, and the geodesic $\Gamma(\mathbf{0},\boldsymbol{v})$ passes through $\boldsymbol{z}$. Then

$$T(\mathbf{0},\boldsymbol{u})-T(\mathbf{0},\boldsymbol{v})\le(T(\mathbf{0},\boldsymbol{z})+T(\boldsymbol{z},\boldsymbol{u}))-(T(\mathbf{0},\boldsymbol{z})+T(\boldsymbol{z},\boldsymbol{v}))=T(\boldsymbol{z},\boldsymbol{u})-T(\boldsymbol{z},\boldsymbol{v})\ .$$

Similarly, we get the opposite inequality with $\boldsymbol{y}$ in place of $\boldsymbol{z}$. Therefore,

$$|T(\mathbf{0},\boldsymbol{u})-T(\mathbf{0},\boldsymbol{v})|\le\max_{\boldsymbol{x}\in\mathcal{I}^*}|T(\boldsymbol{x},\boldsymbol{u})-T(\boldsymbol{x},\boldsymbol{v})|\ . \tag{3.4}$$

Fix an $\boldsymbol{x}\in\mathcal{I}^*$. Then

$$\begin{aligned}|T(\boldsymbol{x},\boldsymbol{u})-T(\boldsymbol{x},\boldsymbol{v})|&\le|T(\boldsymbol{x},\boldsymbol{u})-h(\boldsymbol{u}-\boldsymbol{x})|+|T(\boldsymbol{x},\boldsymbol{v})-h(\boldsymbol{v}-\boldsymbol{x})|\\&\quad+|h(\boldsymbol{u}-\boldsymbol{x})-g(\boldsymbol{v}-\boldsymbol{x})|+|h(\boldsymbol{v}-\boldsymbol{x})-g(\boldsymbol{v}-\boldsymbol{x})|+|g(\boldsymbol{u}-\boldsymbol{x})-g(\boldsymbol{v}-\boldsymbol{x})|\ .\end{aligned} \tag{3.5}$$

Since $\boldsymbol{u}$, $\boldsymbol{v}$ are in $\mathcal{I}(n,L)$, we have

$$\pi^2_{\theta_0,\theta_0^t}(\boldsymbol{u}-\boldsymbol{v})\le L\ . \tag{3.6}$$

From (3.2) we get

$$\pi^1_{\theta_0,\theta_0^t}(\boldsymbol{u}-\boldsymbol{x})=\Delta^{-1}(L)\ , \tag{3.7}$$

and

$$\left|\pi^2_{\theta_0,\theta_0^t}(\boldsymbol{u}-\boldsymbol{x})\right|\le C_{13}t^{1/2}L(\log\Delta^{-1}(L))^{1/2}\ . \tag{3.8}$$

Combining (3.6), (3.7), (3.1) we get

$$\|\boldsymbol{u}-\boldsymbol{x}\|\le C_{14}\Delta^{-1}(L)\ . \tag{3.9}$$

Hence, by Proposition 2.6, and using $\log L$ and $\log\Delta^{-1}(L)$ are of the same order, we get

$$|h(\boldsymbol{u}-\boldsymbol{x})-g(\boldsymbol{u}-\boldsymbol{x})|\le C_{15}\sigma(\Delta^{-1}(L))\log L\ . \tag{3.10}$$

Similarly, (3.7)-(3.10) hold for $\boldsymbol{u}$ replaced with $\boldsymbol{v}$. By Lemma 3.1 and using (3.6), (3.7), (3.8), we get

$$|g(\boldsymbol{u}-\boldsymbol{x})-g(\boldsymbol{v}-\boldsymbol{x})|\le C_{16}t^{1/2}\frac{L^2}{\Delta^{-1}(L)}\log\Delta^{-1}(L)\le C_{17}t^{1/2}\sigma(\Delta^{-1}(L))\log L\ . \tag{3.11}$$

Using (3.10) and the same for $\boldsymbol{u}$ replaced with $\boldsymbol{v}$, (3.11), and (3.5), we get

$$\begin{aligned}&\mathbb{P}\big(|T(\boldsymbol{x},\boldsymbol{u})-T(\boldsymbol{x},\boldsymbol{v})|\ge t\sigma(\Delta^{-1}(L))\log L\big)\\ \le\ &\mathbb{P}\big(|T(\boldsymbol{x},\boldsymbol{u})-h(\boldsymbol{u}-\boldsymbol{x})|\ge C_{18}t\sigma(\Delta^{-1}(L))\log L\big)\\ &+\mathbb{P}\big(|T(\boldsymbol{x},\boldsymbol{v})-h(\boldsymbol{v}-\boldsymbol{x})|\ge C_{18}t\sigma(\Delta^{-1}(L))\log L\big)\ .\end{aligned}$$

Therefore, using (3.9) and the same for $\boldsymbol{u}$ replaced by $\boldsymbol{v}$, and using (A1) we get

$$\mathbb{P}\big(|T(\boldsymbol{x},\boldsymbol{u})-T(\boldsymbol{x},\boldsymbol{v})|\ge t\sigma(\Delta^{-1}(L))\log L\big)\le C_{19}\exp(-C_{20}t\log L)\ . \tag{3.12}$$

The number of choices of lattice points corresponding to $\boldsymbol{x}$, $\boldsymbol{u}$, $\boldsymbol{v}$ i.e., number of triplets $(\lfloor\boldsymbol{x}\rfloor,\lfloor\boldsymbol{u}\rfloor,\lfloor\boldsymbol{v}\rfloor)$ (see Notation 1.1) is at most $C_{21}tL^3(\log L)^{1/2}$. Using (3.4), (3.12), and a union bound, we get

$$\mathbb{P}\big(\mathcal{D}(n,L)\ge t\sigma(\Delta^{-1}(L))\log L\ \text{ and }\ \mathbb{T}\in\mathcal{E}\big)\le C_{22}\exp(-C_{23}t\log L)\ .$$

This concludes the proof because we already found in (3.3) that $\mathcal{E}^c$ has appropriately small probability. □

We need the following variation of the last result. In the last result we chose a direction of both type I and type II, and considered the transverse increment over a segment with one endpoint having that chosen direction. In the next result, we consider transverse increment over a segment whose endpoints have direction in a neighborhood of a fixed direction which is of both type I and type II.

Corollary 3.3. *Let θ_0 be a direction of both type I and II. For $n>0$, $L>0$, d, let*

$$\begin{aligned}\mathcal{I}(n,L,d)&:=\Big\{\,\boldsymbol{x}\in\mathbb{R}^2:\pi^1_{\theta_0,\theta_0^t}(\boldsymbol{x})=n\,,\ d\le\pi^2_{\theta_0,\theta_0^t}(\boldsymbol{x})\le d+L\,\Big\}\ ,\\ \mathcal{D}(n,L,d)&:=\max\{\,|T(\mathbf{0},\boldsymbol{x})-T(\mathbf{0},\boldsymbol{y})|:\boldsymbol{x},\boldsymbol{y}\in\mathcal{I}(n,L,d)\,\}\ .\end{aligned}$$

Then, under the Assumptions 1.4 and 1.5, there exist positive constants δ_1, δ_2 C_1, C_2, C_3, L_0, n_0, t_0, such that for $L\ge L_0$, $n\ge n_0$, $t\ge t_0$, $|d|\le n\delta_1$, $L\le\delta_2\Delta(n)$, we have

$$\mathbb{P}\left(\mathcal{D}(n,L,d)\ge C_3L\frac{|d|}{n}+t\sigma(\Delta^{-1}(L))\log L\right)\le C_1\exp(-C_2t\log L)\ .$$

Proof. Due to symmetry of the lattice, without loss of generality we assume $\theta_0\in[0,\pi/4]$. Fix $\delta_1>0$, $\delta_2>0$, to be assumed appropriately small whenever required. Fix $n_0>0$, $L_0>0$, $t_0>0$, to be assumed appropriately large whenever required. Consider n, L, t, d such that $L\ge L_0$, $n\ge n_0$, $|d|\le n\delta_1$, $L\le\delta_2\Delta(n)$. Let $\boldsymbol{u}:=n\boldsymbol{e}_{\theta_0}+d\boldsymbol{e}_{\theta_0^t}$, $\boldsymbol{v}:=n\boldsymbol{e}_{\theta_0}+(d+L)\boldsymbol{e}_{\theta_0^t}$. Let θ_1, θ_2 be the directions of $\boldsymbol{u}$ and $\boldsymbol{v}$ respectively. Let $\boldsymbol{w}$ be the projection of $\boldsymbol{u}$ on the line joining $\mathbf{0}$ and $\boldsymbol{v}$ in direction θ_1^t which exists assuming δ_1 is small enough. Let $\mathcal{I}^*$ be the segment joining $\boldsymbol{u}$ and $\boldsymbol{w}$.

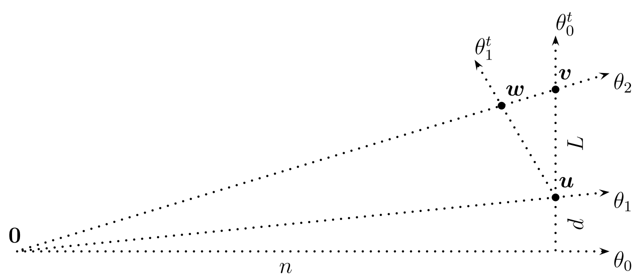

Figure 5: Illustration for Corollary 3.3. The segment joining $\boldsymbol{u}$ and $\boldsymbol{v}$ is $\mathcal{I}(n,L,d)$. The segment joining $\boldsymbol{u}$ and $\boldsymbol{w}$ is $\mathcal{I}^*$.

Let

$$\mathcal{D}^* := \max\{\, |T(\mathbf{0},\boldsymbol{x}) - T(\mathbf{0},\boldsymbol{y})| \colon \boldsymbol{x},\boldsymbol{y} \in \mathcal{I}^* \,\}\ .$$

Let $|\mathcal{I}^*|$ be the length of $\mathcal{I}^*$. To bound $\mathcal{D}^*$ we use Theorem 3.2 with the following variables

$$\tilde{\theta_0} = \theta_1\,, \quad \tilde{n} := \|\boldsymbol{u}\|\,, \quad \tilde{L} := |\mathcal{I}^*|, \quad \tilde{t} = \frac{t}{2}, \quad \tilde{\eta} = 1\ .$$

Recall Remark 1.20. Since θ_0 is of both type I and type II, assuming δ_1 is small enough, we get all possible values of θ_1 are uniformly of both type I and type II i.e., they satisfy the curvature conditions with same constants. Thus the condition on θ_1 for applying Theorem 3.2 holds. Assuming δ_1 is small enough, we get

$$C_1 L \le |\mathcal{I}^*| \le C_2 L\,, \quad \text{and} \quad C_3 n \le \tilde{n} \le C_4 n$$

(because if $\delta_1 \to 0$ then we have $d \to 0$, $\|\boldsymbol{w} - \boldsymbol{v}\| \to 0$, and $|\mathcal{I}^*| \to L$). Therefore, $\tilde{L} \ge \tilde{L_0}$, $\tilde{n} \ge \tilde{n_0}$ hold assuming L_0 and n_0 are large enough. Also from $L \le \delta_2\Delta(n)$ we get $\tilde{L} \le \Delta(\tilde{n})$. Hence all the conditions for applying Theorem 3.2 hold and we get

$$\mathbb{P}\left(\mathcal{D}^* \ge \frac{t}{2}\sigma(\Delta^{-1}(L))\log L\right) \le C_5 \exp(-C_6 t \log L)\ . \tag{3.13}$$

Now let us consider the difference $|\mathcal{D}^* - \mathcal{D}(n,L,d)|$. Considering the triangle with vertices $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$, we get

$$\|\boldsymbol{v} - \boldsymbol{w}\| = L\frac{|\sin(\theta_1^t - \theta_0^t)|}{|\sin(\theta_1^t - \theta_2)|}\ . \tag{3.14}$$

Assuming δ_1 and δ_2 are small enough, we get

$$\left|\sin\left(\theta_1^t - \theta_2\right)\right| \ge C_7\left|\sin\left(\theta_0^t - \theta_0\right)\right|\,, \tag{3.15}$$

and

$$\begin{aligned}\left|\sin\left(\theta^t - \theta_0^t\right)\right| &\le C_8\left|\theta_1^t - \theta_0^t\right| \le C_9|\theta_1 - \theta_0| \\ &\le C_{10}|\sin(\theta_1 - \theta_0)| \le C_{11}\frac{|d|}{n}\left|\sin\left(\theta_0^t - \theta\right)\right| \le C_{12}\frac{|d|}{n}\left|\sin\left(\theta_0^t - \theta_0\right)\right|\,, \end{aligned} \tag{3.16}$$

where the second inequality holds by Remark 1.19. Combining (3.14)-(3.16) we get

$$\|\boldsymbol{v} - \boldsymbol{w}\| \le C_{13} L\frac{|d|}{n}\ .$$

Therefore, if $\boldsymbol{x}$ is a point on $\mathcal{I}(n,L,d)$ and $\boldsymbol{y}$ is its projection on $\mathcal{I}^*$ in direction θ_2, then $\|\boldsymbol{x} - \boldsymbol{y}\| \le C_{14}L|d|/n$. Since h is subadditive and therefore sublinear, we get $h(\boldsymbol{x} - \boldsymbol{y}) \le C_{15}L|d|/n$. Assuming $\delta_1 < 1$, taking L_0 large enough, and using (A2), we get $|d|/n \le \delta_1 < 1 \le \Delta^{-1}(L)/L$, so that $L|d|/n \le \Delta^{-1}(L)$. Therefore, using (A1) we get for all $t' > 0$

$$\mathbb{P}\left(T(\boldsymbol{x},\boldsymbol{y}) \ge C_{15}L\frac{|d|}{n} + t'\sigma(\Delta^{-1}(L))\right) \le C_{16}\exp(-C_{17}t')\ .$$

Using $|\mathcal{I}^*| \le C_2 L$, $|\mathcal{I}(n,L,d)| = L$, and using a union bound, we get

$$\mathbb{P}\left(|\mathcal{D}^* - \mathcal{D}(n,L)| \ge C_{15}L\frac{|d|}{n} + t'\sigma(\Delta^{-1}(L))\right) \le C_{18}L^2\exp(-C_{19}t')\ .$$

Therefore, taking $t' = (t/2)\log L$, and assuming t_0 and L_0 are large enough, we get

$$\mathbb{P}\left(|\mathcal{D}^* - \mathcal{D}(n,L)| \ge C_{15}L\frac{|d|}{n} + \frac{t}{2}\sigma(\Delta^{-1}(L))\log L\right) \le C_{20}\exp(-C_{21}t\log L)\ . \tag{3.17}$$

Combining (3.13) and (3.17) completes the proof. □

4 Refined upper bound on nonrandom fluctuations

Theorem 3.2 of the previous section provides a preliminary upper bound of the fluctuations of the transverse increments. To prove the refined bound of Theorem 1.23, we need to reduce the correction factor in Proposition 2.6 from $\log$ to fixed but arbitrary small power of $\log$. Related results are known in the literature. In [32] it has been shown that $(\log x)^{1/2}$ is a valid correction factor in FPP on Cayley Graphs on integer lattices. In [16] it has been shown that any iterate of $\log$ is a valid correction factor in a spherically symmetric model of FPP. In [4] an upper bound without any correction factor has been shown in a spherically symmetric model of FPP. In our case, to reduce the correction factor, we use a modified version of the procedure of [2]. We now introduce the concept of convex hull approximation property from [2].

Notation 4.1. *Let $\mathcal{S}_0$ be the set of directions where the boundary of $\mathcal{B}$ is differentiable. Consider $\boldsymbol{x} \in \mathbb{R}^2$ with direction in $\mathcal{S}_0$. Let $H_{\boldsymbol{x}}$ be the tangent to $\partial g(\boldsymbol{x})\mathcal{B}$ at $\boldsymbol{x}$. Let $H^0_{\boldsymbol{x}}$ be the line through $\mathbf{0}$ parallel to $H_{\boldsymbol{x}}$. Let $g_{\boldsymbol{x}}$ be the unique linear functional on $\mathbb{R}^2$ satisfying $g_{\boldsymbol{x}}(\boldsymbol{y}) = 0$ for all $y \in H^0_{\boldsymbol{x}}$, and $g_{\boldsymbol{x}}(\boldsymbol{x}) = g(\boldsymbol{x})$. Recall Φ from Notation 2.4. Define for $\phi \in \Phi$, $\nu \geq 0$, $C > 0$, $K > 0$,*

$$Q_{\boldsymbol{x}}(\nu, \phi, C, K) := \left\{ \boldsymbol{y} \in \mathbb{Z}^2 \colon \|\boldsymbol{y}\| \leq K\|\boldsymbol{x}\| \, , \, g_{\boldsymbol{x}}(\boldsymbol{y}) \leq g(\boldsymbol{x}) \, , \, h(\boldsymbol{y}) \leq g_{\boldsymbol{x}}(\boldsymbol{y}) + C\|\boldsymbol{x}\|^{\nu} \phi(\|\boldsymbol{x}\|) \right\} .$$

Definition 4.2. *We say that h satisfies the convex hull approximation property (CHAP) with exponent $\nu \geq 0$ and correction factor $\phi \in \Phi$ in a set of directions $\mathcal{S} \subset \mathcal{S}_0$, if there exist constants $M > 0$, $C > 0$, $K > 0$, $a > 1$ such that $\boldsymbol{x}/\Upsilon \in$ Co$(Q_{\boldsymbol{x}}(\nu, \phi, C, K))$ for some $\Upsilon \in [1, a]$, for all $\boldsymbol{x} \in \mathbb{Q}^2$ with $\|\boldsymbol{x}\| \geq M$ and direction of $\boldsymbol{x}$ in $\mathcal{S}$, where Co denotes the convex hull. When we want to specify the specific constants, we say h satisfies $\mathrm{CHAP}(\nu, \phi, M, C, K, a)$ in sector $\mathcal{S}$.*

The procedure in [2] in our terminology as follows. The objective of [2] was to prove GAP with exponent α and correction factor ϕ_1. To achieve this, first, it is shown that CHAP with exponent α and correction factor ϕ_1 holds. This is done in an iterative way. GAP with exponent 1 and correction factor ϕ_1 holds trivially because h is sublinear. Then the exponent of GAP is reduced from 1 to α iteratively using CHAP with exponent α. Here we are not concerned about the exponent α.

In contrast to [2], here we want to change the correction factor of GAP from ϕ_1 to ϕ_η for some small $\eta > 0$ while keeping the exponent α unchanged. We do this in two steps. In the first step, we prove that CHAP holds with exponent α and correction factor $\widehat{\phi}$ (recall $\widehat{\phi}$ from Notation 2.4). From Proposition 2.6 we get h satisfies GAP with exponent α and correction factor ϕ_1. Using this, we reduce the correction factor of GAP from ϕ_1 to ϕ_η, which is the second step. The result on CHAP with exponent α and correction factor $\widehat{\phi}$ is the following.

Theorem 4.3. *Let θ_0 be a direction of both type I and type II. Then, under Assumptions 1.4 and 1.5, there exists $\delta > 0$ such that CHAP holds for h with exponent α and correction factor $\widehat{\phi}$ in the sector $(\theta_0 - \delta, \theta_0 + \delta)$.*

After carrying out the second step we get the refined upper bound on the nonrandom fluctuations stated below.

Theorem 4.4. *Let θ_0 be a direction of both type I and type II. Fix $\eta \in (0, 1]$. Then, under Assumptions 1.4 and 1.5, there exist constants $C > 0$, $M > 0$, $\delta > 0$ such that for all $\|\boldsymbol{x}\| \geq M$ with direction of $\boldsymbol{x}$ in $(\theta_0 - \delta, \theta_0 + \delta)$, we have*

$$h(\boldsymbol{x}) \leq g(\boldsymbol{x}) + C\sigma(\|\boldsymbol{x}\|)(\log\|\boldsymbol{x}\|)^{\eta} \ ,$$

i.e., $\mathrm{GAP}(\alpha, \phi_\eta, C, M)$ holds in the sector $(\theta_0 - \delta, \theta_0 + \delta)$.

In the first subsection below we prove Theorem 4.3, then in the second subsection we prove Theorem 4.4 using Theorem 4.3.

4.1 Proof of Theorem 4.3

Due to the symmetry of the lattice, without loss of generality we assume $\theta_0 \in [0, \pi/4]$. Let us now choose a parameter $\delta > 0$ which is fixed throughout the proof of Theorem 4.3. Since θ_0 is of both type I and II, we choose $\delta > 0$ such that $\partial\mathcal{B}$ is differentiable in $(\theta_0 - 2\delta, \theta_0 + 2\delta)$ and there exists $\delta_1 > 0$ such that for all $\theta \in (\theta_0 - 2\delta, \theta_0 + 2\delta)$ and $|\delta_2| \le \delta_1$

$$C_1\delta_2^2 \le g(\boldsymbol{e}_\theta + \delta_2 \boldsymbol{e}_{\theta^t}) - g(\boldsymbol{e}_\theta) \le C_2\delta_2^2 \,. \tag{4.1}$$

So all $\theta \in (\theta_0 - \delta, \theta_0 + \delta)$ are of both type I and type II with same constants. This allows us to use results which hold in type I or type II directions with same constants for all $\theta \in (\theta_0 - \delta, \theta_0 + \delta)$.

We extract a sufficient condition from [2] for the Theorem 4.3 to hold. We state the condition as Proposition 4.5. Since it is essentially proved in [2], we do not prove it here. To state the condition we need the concept of skeletons of paths defined below.

Construction of fine skeletons: For $\boldsymbol{x} \in \mathbb{R}^2$, $n > 0$, $\lambda > 0$, $K > 0$, the *fine* $Q_{\boldsymbol{x}}(\alpha, \widehat{\phi}, \lambda, K)$*-skeleton* of a self-avoiding path γ from $\mathbf{0}$ to $n\boldsymbol{x}$ is the sequence of marked points $\boldsymbol{v}_0, \dots, \boldsymbol{v}_m$ on γ constructed as follows. Let $\boldsymbol{v}_0 := \mathbf{0}$, and given $\boldsymbol{v}_i$, let $\boldsymbol{v}'_{i+1}$ be the first point (if any) in γ such that $\boldsymbol{v}'_{i+1} - \boldsymbol{v}_i \notin Q_{\boldsymbol{x}}(\alpha, \widehat{\phi}, \lambda, K)$; then let $\boldsymbol{v}_{i+1}$ be the last lattice point in γ before $\boldsymbol{v}'_{i+1}$ if $\boldsymbol{v}'_{i+1}$ exists; otherwise let $\boldsymbol{v}_{i+1} = \lfloor n\boldsymbol{x} \rfloor$ and end the construction.

Proposition 4.5. *Consider an infinite sequence of i.i.d. copies of the passage-time configuration $(\hat{T}^i)_{i=0}^\infty$ on the lattice. Suppose for some positive constants λ_1, λ_2, λ_3, we have*

$$\begin{aligned} &\mathbb{P}\Bigg(\sum_{i=0}^{m-1} \Big[h(\boldsymbol{v}_i - \boldsymbol{v}_i + 1) - \hat{T}^i(\boldsymbol{v}_i, \boldsymbol{v}_{i+1}) \Big] \ge \frac{\lambda_1}{16} m\sigma(\|\boldsymbol{x}\|) \log\log\|\boldsymbol{x}\| \\ &\qquad \text{for some } m \ge 1 \text{ and some } Q_{\boldsymbol{x}}(\alpha, \widehat{\phi}, \lambda_1, 5)\text{-skeleton } (\boldsymbol{v}_j)_{j=0}^m \\ &\qquad \text{of a path from } \mathbf{0} \text{ to } n\boldsymbol{x} \text{ for some } n \Bigg) \\ &\le \exp\left(-\lambda_2 \log\log\|\boldsymbol{x}\|\right) \end{aligned} \tag{4.2}$$

for all $\boldsymbol{x}$ with $\|\boldsymbol{x}\| \ge \lambda_3$ and direction of $\boldsymbol{x} \in (\theta_0 - \delta, \theta_0 + \delta)$. Then there exists $M > 0$ such that h satisfies $\mathrm{CHAP}\Big(\alpha, \widehat{\phi}, M, \lambda_1, 4, 2\Big)$ in the sector $(\theta_0 - \delta, \theta_0 + \delta)$.

In order to verify (4.2) we need the concept of 'coarse skeletons.'

Construction of coarse skeletons: Consider $\boldsymbol{x} \in \mathbb{R}^2$ with direction $\theta \in (\theta_0 - \delta, \theta_0 + \delta)$. Define

$$\ell_{\boldsymbol{x}} := \frac{\|\boldsymbol{x}\|}{(\log\|\boldsymbol{x}\|)^{2/\alpha}} \,. \tag{4.3}$$

Also, for $i, j \in \mathbb{Z}$, define

$$B_{ij} := \left\{ \boldsymbol{y} \in \mathbb{R}^2 \colon \pi^1_{\theta,\theta^t}(\boldsymbol{y}) \in [i\ell_{\boldsymbol{x}}, (i+1)\ell_{\boldsymbol{x}}) \,, \pi^2_{\theta,\theta^t}(\boldsymbol{y}) \in [j\Delta(\ell_{\boldsymbol{x}}), (j+1)\Delta(\ell_{\boldsymbol{x}})) \right\} , \tag{4.4}$$

where α is defined in Assumption 1.5. So B_{ij} is a parallelogram with side lengths $\ell_{\boldsymbol{x}}$ and $\Delta(\ell_{\boldsymbol{x}})$, and these parallelograms cover the whole plane. Given $\boldsymbol{v} \in B_{ij}$, let

$$G_{\boldsymbol{x}}(\boldsymbol{v}) := \lfloor i\ell_{\boldsymbol{x}}\boldsymbol{e}_\theta + j\Delta(\ell_{\boldsymbol{x}})\boldsymbol{e}_{\theta^t} \rfloor , \quad F_{\boldsymbol{x}}(\boldsymbol{v}) := \lfloor (i+1)\ell_{\boldsymbol{x}}\boldsymbol{e}_\theta + j\Delta(\ell_{\boldsymbol{x}})\boldsymbol{e}_{\theta^t} \rfloor \,.$$

Recall that we have assumed that $\theta_0 \in [0, \pi/4]$. Thus $\theta \in (-\delta, \pi/4 + \delta)$. Also recall from Remark 1.15 that we take tangent in counterclockwise direction. Thus we can assume $\theta^t > 0$. So, $F_{\boldsymbol{x}}(\boldsymbol{v})$ is the lattice point corresponding to down-right corner of the parallelogram B_{ij} containing $\boldsymbol{v}$ and $G_{\boldsymbol{x}}(\boldsymbol{v})$ is the down-left corner, see Figure 6. Suppose $(\boldsymbol{v}_i)_{i=0}^m$ is a fine $Q_{\boldsymbol{x}}(\alpha, \widehat{\phi}, \lambda, 5)$-skeleton of some path for some $\lambda > 0$. Then its *coarse skeleton* $(\boldsymbol{w}_j)_{j=0}^{2m-1}$ is defined as follows. For $0 \le i \le m-1$, let $\boldsymbol{w}_{2i} \coloneqq F_{\boldsymbol{x}}(\boldsymbol{v}_i)$ and $\boldsymbol{w}_{2i+1} \coloneqq G_{\boldsymbol{x}}(\boldsymbol{v}_{i+1})$.

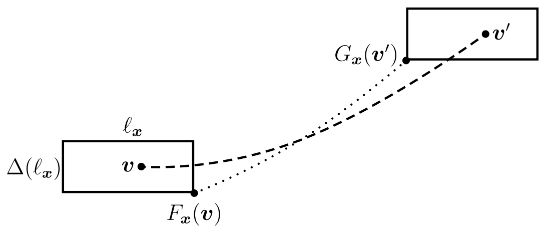

Figure 6: Construction of the coarse skeleton: for every pair of consecutive points $\boldsymbol{v}$, $\boldsymbol{v}'$ in a fine skeleton of a path, we have $F_{\boldsymbol{x}}(\boldsymbol{v})$ and $G_{\boldsymbol{x}}(\boldsymbol{v}')$ as consecutive points in the coarse skeleton of the path.

Remark 4.6. If for some n, $(\boldsymbol{v}_i)_{i=0}^m$ is a fine $Q_{\boldsymbol{x}}(\alpha, \widehat{\phi}, \lambda, 5)$-skeleton of a path from $\mathbf{0}$ to $n\boldsymbol{x}$ and $(\boldsymbol{w}_j)_{j=0}^{2m-1}$ is the corresponding coarse skeleton, then $\|\boldsymbol{v}_i - \boldsymbol{v}_{i+1}\| \le 5\|\boldsymbol{x}\|$ and $\|\boldsymbol{w}_{2i-1} - \boldsymbol{w}_{2i}\| \le 6\|\boldsymbol{x}\|$ for large enough $\|\boldsymbol{x}\|$.

We state two propositions which in combination establishes (4.2). We define a few constants first. Let C_3, C_4, C_5, C_6, C_7, be positive constants such that for all $\boldsymbol{x} \in \mathbb{R}^2$ with $\|\boldsymbol{x}\| \ge C_3$, and for all $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^2$ with $C_4 \le \|\boldsymbol{u} - \boldsymbol{v}\| \le 6\|\boldsymbol{x}\|$, we have

$$\mathbb{P}(|T(\boldsymbol{u}, \boldsymbol{v}) - h(\boldsymbol{u} - \boldsymbol{v})| \ge t\sigma(\|\boldsymbol{x}\|)) \le C_5 \exp(-C_6 t) \;, \tag{4.5}$$

and

$$C_7 \coloneqq 256 \cdot \alpha^{-1} \cdot C_6^{-1} \cdot (1 + C_5) \;. \tag{4.6}$$

For all $\boldsymbol{x} \in \mathbb{R}^2$, we use the shorthand notation

$$Q_{\boldsymbol{x}} \coloneqq Q_{\boldsymbol{x}}\left(\alpha, \widehat{\phi}, C_7, 5\right) .$$

Proposition 4.7. *Under the assumptions of Theorem 4.3, there exist positive constants C_8, C_9 such that for $\boldsymbol{x}$ with direction in $(\theta_0 - \delta, \theta_0 + \delta)$ and $\|\boldsymbol{x}\| \ge C_8$, and for all $m \ge 1$, we have*

$$\mathbb{P}\left(\sum_{i=0}^{m-1}\left[h(\boldsymbol{w}_{2i} - \boldsymbol{w}_{2i+1}) - \hat{T}^i(\boldsymbol{w}_{2i}, \boldsymbol{w}_{2i+1})\right] > \frac{C_7}{32} m\sigma\left(\|\boldsymbol{x}\|\right) \log\log\|\boldsymbol{x}\| \right.$$
$$\left. \text{for some coarse } Q_{\boldsymbol{x}}\text{-skeleton } (\boldsymbol{w}_j)_{j=0}^{2m} \text{ of a path from } \mathbf{0} \text{ to } n\boldsymbol{x} \text{ for some } n\right)$$
$$\le \exp(-mC_9 \log\log\|\boldsymbol{x}\|) \;.$$

Proposition 4.8. *Under the assumptions of Theorem 4.3, there exist positive constants C_{10}, C_{11} such that for $\boldsymbol{x}$ with direction in $(\theta_0 - \delta, \theta_0 + \delta)$ and $\|\boldsymbol{x}\| \ge C_{10}$, and for all $m \ge 1$,* *we have*

$$\mathbb{P}\Bigg(\sum_{i=0}^{m-1}\Big[h(\boldsymbol{v}_i-\boldsymbol{v}_{i+1})-\hat{T}^i(\boldsymbol{v}_i,\boldsymbol{v}_{i+1})-h(\boldsymbol{w}_{2i}-\boldsymbol{w}_{2i+1})+\hat{T}^i(\boldsymbol{w}_{2i},\boldsymbol{w}_{2i+1})\Big] \\ \geq \frac{C_7}{32}m\sigma(\|\boldsymbol{x}\|)\log\log\|\boldsymbol{x}\| \text{ for some fine } Q_{\boldsymbol{x}}\text{-skeleton } (\boldsymbol{v}_j)_{j=0}^m \text{ and the} \\ \text{corresponding coarse skeleton } (\boldsymbol{w}_j)_{j=0}^{2m} \text{ of a path from } \mathbf{0} \text{ to } n\boldsymbol{x} \text{ for some } n\Bigg) \\ \leq \exp(-mC_{11}\log\log\|\boldsymbol{x}\|) \ .$$

So (4.2) holds by Propositions 4.8 and 4.7. Hence, to complete the proof of Theorem 4.3, we only need to prove these two propositions. We need the following lemma first.

Lemma 4.9. *Under the assumptions of Theorem 4.3, there exist positive constants C_{12}, C_{13} such that for $\boldsymbol{x}$ with direction in $(\theta_0-\delta,\theta_0+\delta)$ and $\|\boldsymbol{x}\|\geq C_{12}$ we have the following.*

(i) For all $\boldsymbol{y}\in Q_{\boldsymbol{x}}$, $|\pi^2_{\theta,\theta^t}(\boldsymbol{y})|\leq C_{13}\Delta(\|\boldsymbol{x}\|)(\log\log\|\boldsymbol{x}\|)^{1/2}$, where θ is the direction of $\boldsymbol{x}$.

(ii) The number of coarse $Q_{\boldsymbol{x}}$-skeletons of length $2m+1$ is at most $(\log\|\boldsymbol{x}\|)^{4m/\alpha}$, where α is defined in (A2)*.*

Proof. Fix $\boldsymbol{x}$ with direction θ in $(\theta_0-\delta,\theta_0+\delta)$. We assume that $\|\boldsymbol{x}\|$ is large enough whenever required.

Proof of (i): Consider $\boldsymbol{y}\in Q_{\boldsymbol{x}}$. Then $g(\boldsymbol{y})\leq h(\boldsymbol{y})\leq g_{\boldsymbol{x}}(\boldsymbol{y})+C_7\sigma(\|\boldsymbol{x}\|)\log\log\|\boldsymbol{x}\|$, so

$$g(\boldsymbol{y})-g_{\boldsymbol{x}}(\boldsymbol{y})\leq C_7\sigma(\|\boldsymbol{x}\|)\log\log\|\boldsymbol{x}\| \ . \tag{4.7}$$

Consider three cases.

Case I: Suppose $\pi^1_{\theta,\theta^t}(\boldsymbol{y})>0$ and $|\pi^2_{\theta,\theta^t}(\boldsymbol{y})|\leq\delta_1\pi^1_{\theta,\theta^t}(\boldsymbol{y})$, where δ_1 is defined in (4.1). From (4.1) we get

$$g(\boldsymbol{y})-g_{\boldsymbol{x}}(\boldsymbol{y})=g(\pi^1_{\theta,\theta^t}(\boldsymbol{y})\boldsymbol{e}_\theta+\pi^2_{\theta,\theta^t}(\boldsymbol{y})\boldsymbol{e}_{\theta^t})-g(\pi^1_{\theta,\theta^t}(\boldsymbol{y})\boldsymbol{e}_\theta)\geq C_{14}\frac{\pi^2_{\theta,\theta^t}(\boldsymbol{y})^2}{\pi^1_{\theta,\theta^t}(\boldsymbol{y})} \ . \tag{4.8}$$

Since $\boldsymbol{y}\in Q_{\boldsymbol{x}}$, we have

$$\pi^1_{\theta,\theta^t}(\boldsymbol{y})\leq C_{15}\|\boldsymbol{y}\|\leq C_{16}\|\boldsymbol{x}\| \ .$$

This with (4.8) and (4.7) implies

$$|\pi^2_{\theta,\theta^t}(\boldsymbol{y})|\leq C_{17}\Delta(\|\boldsymbol{x}\|)(\log\log\|\boldsymbol{x}\|)^{1/2} \ .$$

Case II: Now suppose $\pi^1_{\theta,\theta^t}(\boldsymbol{y})>0$ and $|\pi^2_{\theta,\theta^t}(\boldsymbol{y})|\geq\delta_1\pi^1_{\theta,\theta^t}(\boldsymbol{y})$. Let us consider $\pi^2_{\theta,\theta^t}(\boldsymbol{y})>0$, the case $\pi^2_{\theta,\theta^t}(\boldsymbol{y})<0$ is similar. Using convexity of g and (4.1) we get

$$\begin{aligned} & g(\boldsymbol{y})-g_{\boldsymbol{x}}(\boldsymbol{y}) \\ &= g(\pi^1_{\theta,\theta^t}(\boldsymbol{y})\boldsymbol{e}_\theta+\pi^2_{\theta,\theta^t}(\boldsymbol{y})\boldsymbol{e}_{\theta^t})-g(\pi^1_{\theta,\theta^t}(\boldsymbol{y})\boldsymbol{e}_\theta) \\ &\geq \frac{g(\pi^1_{\theta,\theta^t}(\boldsymbol{y})\boldsymbol{e}_\theta+\delta_1\pi^1_{\theta,\theta^t}(\boldsymbol{y})\boldsymbol{e}_{\theta^t})-g(\pi^1_{\theta,\theta^t}(\boldsymbol{y})\boldsymbol{e}_\theta)}{\delta_1\pi^1_{\theta,\theta^t}(\boldsymbol{y})/\pi^2_{\theta,\theta^t}(\boldsymbol{y})} \\ &= \pi^2_{\theta,\theta^t}(\boldsymbol{y})\delta_1^{-1}(g(\boldsymbol{e}_\theta+\delta_1\boldsymbol{e}_{\theta^t})-g(\boldsymbol{e}_\theta)) \\ &\geq C_{18}\pi^2_{\theta,\theta^t}(\boldsymbol{y}) \ . \end{aligned}$$

Hence using (4.7) and (A2) we get

$$\pi^2_{\theta,\theta^t}(\boldsymbol{y}) \le C_{19}\sigma(\|\boldsymbol{x}\|)\log\log\|\boldsymbol{x}\| \le C_{20}\Delta(\|\boldsymbol{x}\|)(\log\log\|\boldsymbol{x}\|)^{1/2} .$$

Case III: Suppose $\pi^1_{\theta,\theta^t}(\boldsymbol{y}) < 0$. Then $g_{\boldsymbol{x}}(\boldsymbol{y}) < 0$. Using $\boldsymbol{y} \in Q_{\boldsymbol{x}}$, (4.7), and (A2) we get

$$\pi^2_{\theta,\theta^t}(\boldsymbol{y}) \le C_{21}\|\boldsymbol{y}\| \le C_{22}g(\boldsymbol{y}) \le C_{23}\sigma(\|\boldsymbol{x}\|)\log\log\|\boldsymbol{x}\| \le C_{24}\Delta(\|\boldsymbol{x}\|)(\log\log\|\boldsymbol{x}\|)^{1/2} .$$

Proof of (ii): Given any $\boldsymbol{v}$ let $Q_{\boldsymbol{x}}(\boldsymbol{v})$ denote the translate of $Q_{\boldsymbol{x}}$ by $\boldsymbol{v}$. Suppose part of a coarse skeleton is given as $(\boldsymbol{w}_0, \dots, \boldsymbol{w}_{2i})$. We find an upper bound on number of possibilities of $(\boldsymbol{w}_{2i+1}, \boldsymbol{w}_{2i+2})$. Consider a fine skeleton $(\boldsymbol{v}_0, \dots, \boldsymbol{v}_i)$ corresponding to $(\boldsymbol{w}_0, \dots, \boldsymbol{w}_{2i})$. Since $(\boldsymbol{w}_{2i-1}, \boldsymbol{w}_{2i})$ is fixed, all choices of $\boldsymbol{v}_i$ lie in the same $B_{i_0j_0}$. Consider the union of $Q_{\boldsymbol{x}}(\boldsymbol{v})$ over all $\boldsymbol{v} \in B_{i_0j_0}$. Each $Q_{\boldsymbol{x}}(\boldsymbol{v})$ is contained in a parallelogram of length $C_{25}\|\boldsymbol{x}\|$ in θ direction and length $C_{26}\Delta(\|\boldsymbol{x}\|)(\log\log\|\boldsymbol{x}\|)^{1/2}$ in θ^t direction. Hence, the union of all such $Q_{\boldsymbol{x}}(\boldsymbol{v})$ as $\boldsymbol{v}$ varies in a parallelogram $B_{i_0j_0}$ is contained in a parallelogram of length $C_{27}\|\boldsymbol{x}\|$ in θ direction and length $C_{28}\Delta(\|\boldsymbol{x}\|)(\log\log\|\boldsymbol{x}\|)^{1/2}$ in θ^t direction (using (i)). From (4.3) we have

$$\frac{\Delta(\|\boldsymbol{x}\|)(\log\log\|\boldsymbol{x}\|)^{1/2}}{\Delta(\ell_{\boldsymbol{x}})} \le C_{29}(\log\|\boldsymbol{x}\|)^{(1+\beta)/\alpha}(\log\log\|\boldsymbol{x}\|)^{1/2} ,$$

and,

$$\frac{\|\boldsymbol{x}\|}{\ell_{\boldsymbol{x}}} = (\log\|\boldsymbol{x}\|)^{2/\alpha} .$$

Therefore, using $\beta < 1$, the number of parallelograms B_{ij} that cover the aforementioned union is at most $(\log\|\boldsymbol{x}\|)^{4/\alpha}$. Hence, the number of choices of $(\boldsymbol{w}_{2i+1}, \boldsymbol{w}_{2i+2})$ is at most $(\log\|\boldsymbol{x}\|)^{4/\alpha}$. Iterating this from $i = 0$ to m, we get the result. □

4.1.1 Proof of Proposition 4.7

Fix a point $\boldsymbol{x} \in \mathbb{R}^2$ with direction $\theta \in (\theta_0 - \delta, \theta_0 + \delta)$. We assume $\|\boldsymbol{x}\|$ is large enough whenever required. Fix a coarse $Q_{\boldsymbol{x}}$-skeleton $(\boldsymbol{w}_j)_{j=0}^{2m}$ for some $m \ge 1$. By Remark 4.6 and equation (4.5) we get

$$\mathbb{P}\Big(h(\boldsymbol{w}_{2i} - \boldsymbol{w}_{2i+1}) - \hat{T}^i(\boldsymbol{w}_{2i}, \boldsymbol{w}_{2i+1}) \ge t\sigma(\|\boldsymbol{x}\|)\Big) \le C_5\exp(-C_6 t) .$$

For $C_{30} \coloneqq C_6/(1 + C_5)$ we get

$$\mathbb{E}\bigg[\exp\bigg(C_{30}\Big(h(\boldsymbol{w}_{2i} - \boldsymbol{w}_{2i+1}) - \hat{T}^i(\boldsymbol{w}_{2i}, \boldsymbol{w}_{2i+1})\Big)^+/\sigma(\|\boldsymbol{x}\|)\bigg)\bigg] \le 2 .$$

Using the independence of $\hat{T}^i$s we get

$$\mathbb{E}\left[\exp\left(C_{30}\sum_{i=0}^{m-1}\Big(h(\boldsymbol{w}_{2i} - \boldsymbol{w}_{2i+1}) - \hat{T}^i(\boldsymbol{w}_{2i}, \boldsymbol{w}_{2i+1})\Big)^+/\sigma(\|\boldsymbol{x}\|)\right)\right] \le 2^m .$$

Hence, for all $t > 0$, we have

$$\begin{aligned}&\mathbb{P}\left(\sum_{i=0}^{m-1} h(\boldsymbol{w}_{2i} - \boldsymbol{w}_{2i+1}) - \hat{T}^i(\boldsymbol{w}_{2i}, \boldsymbol{w}_{2i+1}) > tm\sigma(\|\boldsymbol{x}\|)\log\log\|\boldsymbol{x}\|\right)\\ &\le 2^m\exp(-C_{30}\, m\, t\, \log\log\|\boldsymbol{x}\|) .\end{aligned}$$

From Lemma 4.9 we get that the number of coarse skeletons of length $2m+1$ is at most $(\log\|\boldsymbol{x}\|)^{m(4/\alpha)}$. Therefore, by (4.6) we get

$$\mathbb{P}\Bigg(\sum_{i=0}^{m-1}\Big[h(\boldsymbol{w}_{2i}-\boldsymbol{w}_{2i+1})-\hat{T}^i(\boldsymbol{w}_{2i},\boldsymbol{w}_{2i+1})\Big] > \frac{C_7}{32}m\sigma(\|\boldsymbol{x}\|)\log\log\|\boldsymbol{x}\| \\ \text{for some coarse } Q_{\boldsymbol{x}}\text{-skeleton } (\boldsymbol{w}_j)_{j=0}^{2m} \text{ of a path from } \mathbf{0} \text{ to } n\boldsymbol{x} \text{ for some } n\Bigg)$$

$$\leq 2^m(\log\|\boldsymbol{x}\|)^{m(4/\alpha)}\exp\left(-C_{30}\frac{C_7}{32}m\log\log\|\boldsymbol{x}\|\right)$$

$$\leq \exp(-mC_{31}\log\log\|\boldsymbol{x}\|)\ .$$

This completes the proof of Proposition 4.7.

4.1.2 Proof of Proposition 4.8

Fix a point $\boldsymbol{x}\in\mathbb{R}^2$ with direction $\theta\in(\theta_0-\delta,\theta_0+\delta)$. We assume $\|\boldsymbol{x}\|$ is large enough whenever required. Fix a coarse $Q_{\boldsymbol{x}}$-skeleton $(\boldsymbol{w}_j)_{j=0}^{2m-1}$ of a path from $\mathbf{0}$ to $n\boldsymbol{x}$ for some $n\geq 1$. For $0\leq i\leq m-1$, define the set

$$V_i := \left\{\,(\boldsymbol{v},\boldsymbol{v}')\in\mathbb{Z}^2\times\mathbb{Z}^2\colon F_{\boldsymbol{x}}(\boldsymbol{v})=\boldsymbol{w}_{2i}\,,\, G_{\boldsymbol{x}}(\boldsymbol{v}')=\boldsymbol{w}_{2i+1}\,,\,\boldsymbol{v}'-\boldsymbol{v}\in Q_{\boldsymbol{x}}\,\right\}\,,$$

and also define

$$X_i := \max_{(\boldsymbol{v},\boldsymbol{v}')\in V_i}\frac{h(\boldsymbol{v}-\boldsymbol{v}')-\hat{T}^i(\boldsymbol{v},\boldsymbol{v}')-h(\boldsymbol{w}_{2i}-\boldsymbol{w}_{2i+1})+\hat{T}^i(\boldsymbol{w}_{2i},\boldsymbol{w}_{2i+1})}{\sigma(\|\boldsymbol{x}\|)}\ .$$

By Remark 4.6, we have $\|\boldsymbol{v}-\boldsymbol{v}'\|\leq 5\|\boldsymbol{x}\|$ for all $(\boldsymbol{v},\boldsymbol{v}')\in V_i$ and $0\leq i\leq m-1$. Therefore, the number of elements in V_i for any $0\leq i\leq m-1$ is at most $C_{32}\|\boldsymbol{x}\|^4$. Hence, for all $t>0$ we have

$$\mathbb{P}(X_i\geq t)\leq C_{33}\|\boldsymbol{x}\|^4\exp(-C_{34}t)\ . \tag{4.9}$$

If we show that for some constant $C_{35}>0$

$$\mathbb{P}\left(\sum_{i=0}^{m-1}X_i\geq\frac{C_7}{32}m\log\log\|\boldsymbol{x}\|\right)\leq\exp(-C_{35}m\log\|\boldsymbol{x}\|)\ , \tag{4.10}$$

then Proposition 4.8 follows using Lemma 4.9. Therefore, we prove (4.10) now. Let

$$C_{36} := 8C_{34}^{-1}\ , \tag{4.11}$$

and let N_0, N_1 be positive integers such that

$$2^{N_0}<\frac{C_7}{96}\log\log\|\boldsymbol{x}\|\leq 2^{N_0+1}\ , \tag{4.12}$$

$$2^{N_1-1}<C_{36}\log\|\boldsymbol{x}\|\leq 2^{N_1}\ . \tag{4.13}$$

Then

$$\mathbb{P}\left(\sum_{i=0}^{m-1}X_i\geq\frac{C_7}{32}m\log\log\|\boldsymbol{x}\|\right)\leq\ \mathbb{P}\left(\sum_{i=0}^{m-1}X_i\mathbb{1}(2^{N_0}\leq X_i<2^{N_1})\geq\frac{C_7}{96}m\log\log\|\boldsymbol{x}\|\right) \\ +\mathbb{P}\left(\sum_{i=0}^{m-1}X_i\mathbb{1}(X_i\geq 2^{N_1})\geq\frac{C_7}{96}m\log\log\|\boldsymbol{x}\|\right)\ . \tag{4.14}$$

For the second term in the right-hand side of (4.14) we have

$$\begin{aligned}
&\mathbb{P}\left(\sum_{i=0}^{m-1} X_i \mathbb{1}(X_i \geq 2^{N_1}) \geq \frac{C_7}{96} m \log\log\|\boldsymbol{x}\|\right) \\
&\leq \mathbb{P}\left(\sum_{q=N_1}^{\infty}\sum_{i=0}^{m-1} 2^{q+1}\mathbb{1}(2^{q+1} > X_i \geq 2^q) \geq \frac{C_7}{96} m \log\log\|\boldsymbol{x}\| \sum_{q=N_1}^{\infty} 2^{-(q-N_1+1)}\right) \\
&\leq \sum_{q=N_1}^{\infty} \mathbb{P}\left(\sum_{i=0}^{m-1} \mathbb{1}(X_i \geq 2^q) \geq \frac{C_7}{96} m \log\log\|\boldsymbol{x}\| 2^{N_1-2(q+1)}\right) \\
&\leq \sum_{q=N_1}^{\infty} \exp(-mI(a_1(q)|b_1(q))) \ , \qquad (4.15)
\end{aligned}$$

where

$$a_1(q) \coloneqq \frac{C_7}{96} m \log\log\|\boldsymbol{x}\| 2^{N_1-2(q+1)}\ , \quad b_1(q) \coloneqq \max_{0\leq i\leq m-1} \mathbb{P}(X_i \geq 2^q)\ ,$$

and I is the large deviation rate function for Bernoulli random variables:

$$I(x|y) \coloneqq x\log\frac{x}{y} + (1-x)\log\frac{1-x}{1-y}\ . \qquad (4.16)$$

Using (4.9), (4.11), and (4.13), we get for $q \geq N_1$

$$b_1(q) \leq C_{33}\|\boldsymbol{x}\|^4 \exp(-C_{34}2^q) \leq \exp(-C_{37}2^q)\ .$$

So $b_1(q)$ is much smaller than $a_1(q)$. Therefore, using (4.16) for $I(a_1(q)|b_1(q))$ and expanding the log terms we see that the term $a_1(q)\log(b_1(q)^{-1})$ dominates the others. Hence

$$I(a_1(q)|b_1(q)) \geq C_{38} a_1(q) \log(b_1(q)^{-1}) \geq C_{39} \log\log\|\boldsymbol{x}\| \log\|\boldsymbol{x}\| 2^{-q}\ .$$

Therefore, continuing from (4.15) and using (4.9) we get

$$\mathbb{P}\left(\sum_{i=0}^{m-1} X_i \mathbb{1}(X_i \geq 2^{N_1}) \geq \frac{C_7}{96} m \log\log\|\boldsymbol{x}\|\right) \leq \exp(-C_{40} m \log\|\boldsymbol{x}\|)\ . \qquad (4.17)$$

For the first term in the right-hand side of (4.14) we have

$$\begin{aligned}
&\mathbb{P}\left(\sum_{i=0}^{m-1} X_i \mathbb{1}(2^{N_0} \leq X_i < 2^{N_1}) \geq \frac{C_7}{96} m \log\log\|\boldsymbol{x}\|\right) \\
&\leq \mathbb{P}\left(\sum_{i=0}^{m-1}\sum_{q=N_0}^{N_1-1} 2^{q+1}\mathbb{1}(X_i \geq 2^q) \geq \frac{C_7}{96} m \log\log\|\boldsymbol{x}\|\right) \\
&\leq \sum_{q=N_0}^{N_1-1} \mathbb{P}\left(\sum_{i=0}^{m-1} \mathbb{1}(X_i \geq 2^q) \geq \frac{C_7}{96} m (\log\log\|\boldsymbol{x}\|) 2^{-(q+1)}(N_1-N_0)^{-1}\right) \\
&\leq \sum_{q=N_0}^{N_1-1} \exp(-mI(a_2(q)|b_2(q))) \ , \qquad (4.18)
\end{aligned}$$

where

$$a_2(q) \coloneqq \frac{C_7}{96} (\log\log\|\boldsymbol{x}\|) 2^{-(q+1)}(N_1-N_0)^{-1}\ , \quad b_2(q) \coloneqq \max_{0\leq i\leq m-1} \mathbb{P}(X_i \geq 2^q)\ .$$

We use the following claim to derive a bound of $b_2(q)$. The proof of the claim is presented later.

Claim 4.10. *For $0 \le i \le m-1$, $(\boldsymbol{v}, \boldsymbol{v}') \in V_i$, and $t \in [2^{N_0}, 2^{N_1 - 1}]$, we have*

$$\begin{aligned}&\mathbb{P}\Big(h(\boldsymbol{v}-\boldsymbol{v}') - \hat{T}^i(\boldsymbol{v}, \boldsymbol{v}') - h(\boldsymbol{w}_{2i} - \boldsymbol{w}_{2i+1}) + \hat{T}^i(\boldsymbol{w}_{2i}, \boldsymbol{w}_{2i+1})) \ge t\sigma(\|\boldsymbol{x}\|)\Big) \\ &\le \exp(-C_{41} t \log\|\boldsymbol{x}\|) \; .\end{aligned}$$

By this claim and using that the number of elements in V_i is at most $C_{32}\|\boldsymbol{x}\|^4$ we get, for $N_0 \le q \le N_1 - 1$

$$b_2(q) = \max_{0 \le i \le m-1} \mathbb{P}(X_i \ge 2^q) \le C_{42}\|\boldsymbol{x}\|^4 \exp(-C_{43} 2^q \log\|\boldsymbol{x}\|) \le \exp(-C_{44} 2^q \log\|\boldsymbol{x}\|) \; .$$

From (4.13) we get $C_{45}2^{-q} \le a_2(q) \le C_{46}2^{-q}$. Therefore, using (4.16) for $I(a_2(q)|b_2(q))$, and expanding the log terms, we see that the term $a_2(q)\log(b_2(q)^{-1})$ dominates the others. Hence

$$I(a_2(q)|b_2(q)) \ge C_{47} a_2(q) \log(b_2(q)^{-1}) \ge C_{48} \log\|\boldsymbol{x}\| \; .$$

Therefore, continuing from (4.18) we get

$$\mathbb{P}\left(\sum_{i=0}^{m-1} X_i \mathbb{1}(2^{N_0} \le X_i < 2^{N_1}) \ge \frac{C_7}{96} m \log\log\|\boldsymbol{x}\|\right) \le \exp(-mC_{49}\log\|\boldsymbol{x}\|) \; .$$

Combining this with (4.17) we get (4.10). Therefore, to complete the proof of Proposition 4.8 we only need to prove Claim 4.10.

Proof of Claim 4.10. Fix $0 \le i \le m-1$ and $(\boldsymbol{v}, \boldsymbol{v}') \in V_i$. Define

$$\mathcal{D}(\boldsymbol{v}, \boldsymbol{v}') \coloneqq h(\boldsymbol{v} - \boldsymbol{v}') - \hat{T}^i(\boldsymbol{v}, \boldsymbol{v}') - h(\boldsymbol{w}_{2i} - \boldsymbol{w}_{2i+1}) + \hat{T}^i(\boldsymbol{w}_{2i}, \boldsymbol{w}_{2i+1}) \; .$$

Define

$$\ell_0 \coloneqq \frac{\|\boldsymbol{x}\|}{(\log\|\boldsymbol{x}\|)^{1/\alpha}} \; .$$

We consider two cases.

Case I: Suppose $\pi^1_{\theta,\theta^t}(\boldsymbol{v}' - \boldsymbol{v}) \le \ell_0$. This includes the case $\pi^1_{\theta,\theta^t}(\boldsymbol{v}' - \boldsymbol{v}) < 0$. Using Lemma 4.9, $|\pi^2_{\theta,\theta^t}(\boldsymbol{v} - \boldsymbol{w}_{2i})|$ and $|\pi^2_{\theta,\theta^t}(\boldsymbol{v}' - \boldsymbol{w}_{2i+1})|$ are at most $C_{50}\Delta(\boldsymbol{x})\log\log\|x\|$ which is smaller than ℓ_0. Since $F_{\boldsymbol{x}}(\boldsymbol{v}) = \boldsymbol{w}_{2i}$ and $G_{\boldsymbol{x}}(\boldsymbol{v}') = \boldsymbol{w}_{2i+1}$, $|\pi^1_{\theta,\theta^t}(\boldsymbol{v} - \boldsymbol{w}_{2i})|$ and $|\pi^1_{\theta,\theta^t}(\boldsymbol{v}' - \boldsymbol{w}_{2i+1})|$ are at most $\ell_{\boldsymbol{x}}$ which is smaller than ℓ_0. Hence $\|\boldsymbol{v} - \boldsymbol{w}_{2i}\|$ and $\|\boldsymbol{v}' - \boldsymbol{w}_{2i+1}\|$ are at most of the order of ℓ_0. Therefore, using (A1) and (A2), we get

$$\mathbb{P}(\mathcal{D}(\boldsymbol{v}, \boldsymbol{v}') \ge t\sigma(\|\boldsymbol{x}\|)) \le C_{51}\exp(-C_{52} t\sigma(\|\boldsymbol{x}\|)/\sigma(\|\ell_0\|)) \le \exp(-C_{53} t \log\|\boldsymbol{x}\|) \; .$$

So the claim is proved in this case.

Case II: Suppose

$$\pi^1_{\theta,\theta^t}(\boldsymbol{v}' - \boldsymbol{v}) \ge \ell_0 \; . \tag{4.19}$$

Let

$$\begin{aligned}\mathcal{D}_1(\boldsymbol{v}, \boldsymbol{v}') &\coloneqq h(\boldsymbol{v} - \boldsymbol{v}') - T(\boldsymbol{v}, \boldsymbol{v}') - h(\boldsymbol{v} - \boldsymbol{w}_{2i+1}) + T(\boldsymbol{v}, \boldsymbol{w}_{2i+1}) \; , \\ \mathcal{D}_2(\boldsymbol{v}, \boldsymbol{v}') &\coloneqq h(\boldsymbol{v} - \boldsymbol{w}_{2i+1}) - T(\boldsymbol{v}, \boldsymbol{w}_{2i+1}) - h(\boldsymbol{w}_{2i} - \boldsymbol{w}_{2i+1}) + T(\boldsymbol{w}_{2i}, \boldsymbol{w}_{2i+1}) \; .\end{aligned}$$

Therefore, $\mathcal{D}(\boldsymbol{v}, \boldsymbol{v}') = \mathcal{D}_1(\boldsymbol{v}, \boldsymbol{v}') + \mathcal{D}_2(\boldsymbol{v}, \boldsymbol{v}')$. Hence

$$\mathbb{P}(\mathcal{D}(\boldsymbol{v}, \boldsymbol{v}') \ge t\sigma(\|\boldsymbol{x}\|)) \le \mathbb{P}\left(\mathcal{D}_1(\boldsymbol{v}, \boldsymbol{v}') \ge \frac{t}{2}\sigma(\|\boldsymbol{x}\|)\right) + \mathbb{P}\left(\mathcal{D}_2(\boldsymbol{v}, \boldsymbol{v}') \ge \frac{t}{2}\sigma(\|\boldsymbol{x}\|)\right) . \tag{4.20}$$

We only consider the first term on the right-hand side, the second term can be dealt with similarly. Suppose i_0 and j_0 are such that $\boldsymbol{v}' \in B_{i_0 j_0}$, where recall that B_{ij} is defined in (4.4). Let

$$J \coloneqq \left[j_0\Delta(\ell_{\boldsymbol{x}}) - t^{1/2}\Delta(\ell_{\boldsymbol{x}})(\log \ell_{\boldsymbol{x}})^{1/2}, (j_0+1)\Delta(\ell_{\boldsymbol{x}}) + t^{1/2}\Delta(\ell_{\boldsymbol{x}})(\log \ell_{\boldsymbol{x}})^{1/2} \right],$$
$$R(t) \coloneqq \left\{ \boldsymbol{y} \in \mathbb{R}^2 \colon \pi^1_{\theta,\theta^t}(\boldsymbol{y}) = i_0\ell_{\boldsymbol{x}} \,,\, \pi^2_{\theta,\theta^t}(\boldsymbol{y}) \in J \right\} .$$

So $R(t)$ is an extension of a side of the parallelogram $B_{i_0 j_0}$. Define the event

$$\mathcal{E}(t) \coloneqq \{ \Gamma(\boldsymbol{v},\boldsymbol{v}') \text{ intersects } R(t) \} .$$

Since $\boldsymbol{v}' \in B_{i_0 j_0}$, the distance of the segment $R(t)$ from $\boldsymbol{v}'$ in $-\theta$ direction is less than $\ell_{\boldsymbol{x}}$. Therefore, if $\mathbb{T} \notin \mathcal{E}(t)$, then $\mathcal{W}(\boldsymbol{v}',\boldsymbol{v},k,-\theta) \geq t^{1/2}\Delta(\ell_{\boldsymbol{x}})(\log \ell_{\boldsymbol{x}})^{1/2}$ for some $k \leq \ell_{\boldsymbol{x}}$. Hence, to bound the probability of $\mathbb{T} \notin \mathcal{E}(t)$ we use Corollary 2.11 with

$$\tilde{\theta_0} \coloneqq -\theta\,, \quad \tilde{n} \coloneqq \pi^1_{\theta,\theta^t}(\boldsymbol{v}'-\boldsymbol{v})\,, \quad \tilde{l} \coloneqq \pi^2_{\theta,\theta^t}(\boldsymbol{v}-\boldsymbol{v}')\,, \quad \tilde{k} \coloneqq \ell_{\boldsymbol{x}}\,, \quad \tilde{t} \coloneqq t^{1/2} .$$

(Recall from Remark 1.2 our convention of using tilde on parameters.) We now verify the conditions of Corollary 2.11. Recall that by our choice of δ from (4.1) θ is a direction of type I, hence so is $-\theta$. Using (4.19) we get $\tilde{n} \geq \ell_0 \geq \tilde{n_0}$. By Lemma 4.9 we have $|\tilde{l}| \leq C_{54}\Delta(\|x\|)\log\log\|\boldsymbol{x}\|$. Hence, $|\tilde{l}| \leq \tilde{n}\tilde{\delta_2}$, as required. Using $t \leq 2^{N_1-1}$ and (4.13), we get $t^{1/2}\Delta(\tilde{k})(\log\tilde{k}) \leq \tilde{k}\tilde{\delta_3}$, as required. Thus, all the conditions for applying Corollary 2.11 hold, and we get

$$\mathbb{P}(\,\mathcal{E}(t)^c\,) \leq \exp(-C_{55}t\log\ell_{\boldsymbol{x}}) \leq \exp(-C_{56}t\log\|\boldsymbol{x}\|) . \tag{4.21}$$

Let $R \coloneqq R(2^{N_1-1})$. For any $\boldsymbol{y}$ in R let

$$\mathcal{D}_1'(\boldsymbol{y}) \coloneqq h(\boldsymbol{v}-\boldsymbol{y}) - T(\boldsymbol{v},\boldsymbol{y}) - h(\boldsymbol{v}-\boldsymbol{w}_{2i+1}) + T(\boldsymbol{v},\boldsymbol{w}_{2i+1})\,,$$
$$\mathcal{D}_2'(\boldsymbol{y}) \coloneqq h(\boldsymbol{y}-\boldsymbol{v}') - T(\boldsymbol{y},\boldsymbol{v}') .$$

If $\Gamma(\boldsymbol{v},\boldsymbol{v}')$ passes through $\boldsymbol{y}$ then $\mathcal{D}_1(\boldsymbol{v},\boldsymbol{v}') \leq \mathcal{D}_1'(\boldsymbol{y}) + \mathcal{D}_2'(\boldsymbol{y})$. Hence

$$\mathbb{P}\left(\mathcal{D}_1(\boldsymbol{v},\boldsymbol{v}') \geq \frac{t}{2}\sigma(\|\boldsymbol{x}\|)\right) \leq \mathbb{P}\left(\max_{\boldsymbol{y}\in R}\mathcal{D}_1'(\boldsymbol{y}) \geq \frac{t}{4}\sigma(\|\boldsymbol{x}\|)\right) + \mathbb{P}\left(\max_{\boldsymbol{y}\in R}\mathcal{D}_2'(\boldsymbol{y}) \geq \frac{t}{4}\sigma(\|\boldsymbol{x}\|)\right) + \mathbb{P}(\,\mathcal{E}(t)^c\,) . \tag{4.22}$$

Let us consider the first term in the right-hand side first. We use Corollary 3.3 with

$$\tilde{\theta_0} \coloneqq \theta\,,$$
$$\tilde{n} \coloneqq i_0\ell_{\boldsymbol{x}} - \pi^1_{\theta,\theta^t}(\boldsymbol{v})\,,$$
$$\tilde{L} \coloneqq \Delta(\ell_{\boldsymbol{x}})(1 + 2^{(N_1-1)/2+1}(\log\ell_{\boldsymbol{x}})^{1/2})\,,$$
$$\tilde{d} \coloneqq \Delta(\ell_{\boldsymbol{x}})(j_0 - 2^{(N_1-1)/2}(\log\ell_{\boldsymbol{x}})^{1/2}) - \pi^2_{\theta,\theta^t}(\boldsymbol{v}) .$$

(Recall from Remark 1.2 our convention of using tilde on parameters.) We now verify the conditions of Corollary 3.3. By our choice of δ in (4.1), θ is of both type I and type II. Since $\tilde{n}$ is the distance of R from $\boldsymbol{v}$ in direction $-\theta$, we have

$$\tilde{n} \geq \ell_0 - \ell_{\boldsymbol{x}} \geq C_{57}\ell_0 \geq \tilde{n_0} .$$

Using (4.13), (A2), and $\tilde{n} \geq C_{57}\ell_0$ from above, we get

$$\tilde{L} \leq C_{58}\Delta(\ell_{\boldsymbol{x}})\log\|\boldsymbol{x}\| \leq C_{59}\Delta(\ell_0)(\log\|\boldsymbol{x}\|)^{1-(1+\beta)/(2\alpha)} \leq \tilde{\delta_2}\Delta(\tilde{n}) . \tag{4.23}$$

By Lemma 4.9 we have

$$|\pi^2_{\theta,\theta^t}(\boldsymbol{v}'-\boldsymbol{v})| \le C_{60}\Delta(\boldsymbol{x})(\log\log\|\boldsymbol{x}\|)^{1/2} \ .$$

Since $\boldsymbol{v}' \in B_{i_0 j_0}$, we have

$$|\pi^2_{\theta,\theta^t}(\boldsymbol{v}') - j_0\Delta(\ell_{\boldsymbol{x}})| \le \Delta(\ell_{\boldsymbol{x}}) \ .$$

Therefore, using (A2) and $\tilde{n} \ge C_{57}\ell_0$, we get

$$\begin{aligned} |\tilde{d}| &\le C_{61}\Delta(\|\boldsymbol{x}\|)(\log\log\|\boldsymbol{x}\|)^{1/2} + 2^{N_1/2}\Delta(\ell_{\boldsymbol{x}})(\log \ell_{\boldsymbol{x}})^{1/2} \\ &\le C_{62}\Delta(\|\boldsymbol{x}\|)(\log\log\|\boldsymbol{x}\|)^{1/2} \le \tilde{\delta_1}\tilde{n} \ . \end{aligned} \tag{4.24}$$

By Corollary 3.3 we get for $\tilde{t} \ge \tilde{t_0}$

$$\mathbb{P}\left(\max_{\boldsymbol{y}\in R}\mathcal{D}_1'(\boldsymbol{y}) \ge C_{63}\tilde{L}\frac{|\tilde{d}|}{\tilde{n}} + \tilde{t}\sigma(\Delta^{-1}(\tilde{L}))\log\Delta^{-1}(\tilde{L})\right) \le C_{64}\exp\Big(-C_{65}\tilde{t}\log\tilde{L}\Big) \ . \tag{4.25}$$

Let $\tilde{t}$ be such that

$$C_{66}\tilde{L}\frac{|\tilde{d}|}{\tilde{n}} + \tilde{t}\sigma\Big(\Delta^{-1}\Big(\tilde{L}\Big)\Big)\log\Delta^{-1}(\tilde{L}) = \frac{t}{4}\sigma(\|\boldsymbol{x}\|) \ . \tag{4.26}$$

We need to verify $\tilde{t} \ge \tilde{t_0}$. Using (4.23) and (A2), we have

$$\tilde{L} \le C_{58}\Delta(\ell_{\boldsymbol{x}})\log\|\boldsymbol{x}\| \le C_{67}\Delta(\|\boldsymbol{x}\|)(\log\|\boldsymbol{x}\|)^{-1/\alpha} \ . \tag{4.27}$$

Using this with (4.24) we get

$$\tilde{L}\frac{|\tilde{d}|}{\tilde{n}} \le C_{68}\sigma(\|\boldsymbol{x}\|)(\log\log\|\boldsymbol{x}\|)^{1/2}(\log\|\boldsymbol{x}\|)^{-1/\alpha} \ . \tag{4.28}$$

Using $t \ge 2^{N_0}$, (4.12), (4.26), (4.28), and (4.27), we get

$$\begin{aligned} \tilde{t} &\ge C_{69}t\frac{\sigma(\|\boldsymbol{x}\|)}{\sigma(\Delta^{-1}(\tilde{L}))\log\Delta^{-1}(\tilde{L})} \ge C_{70}t\left(\frac{\|\boldsymbol{x}\|}{\Delta^{-1}(\tilde{L})}\right)^{\alpha}(\log\|\boldsymbol{x}\|)^{-1} \\ &\ge C_{71}t\left(\frac{\Delta(\|\boldsymbol{x}\|)}{\tilde{L}}\right)^{2\alpha/(1+\beta)}(\log\|\boldsymbol{x}\|)^{-1} \ge C_{72}t(\log\|\boldsymbol{x}\|)^{(1-\beta)/(1+\beta)} \ge \tilde{t_0} \ . \end{aligned}$$

Therefore, from (4.25) and (4.26), we get

$$\begin{aligned} \mathbb{P}\left(\max_{\boldsymbol{y}\in R}\mathcal{D}_1'(\boldsymbol{y}) \ge \frac{t}{4}\sigma(\|\boldsymbol{x}\|)\right) &\le C_{73}\exp\Big(-C_{74}t(\log\|\boldsymbol{x}\|)^{2/(1+\beta)}\Big) \\ &\le C_{75}\exp(-C_{76}t(\log\|\boldsymbol{x}\|)) \ . \end{aligned} \tag{4.29}$$

Now we consider the second term in the right-hand side of (4.22). By (4.27) width of R is less than $\Delta(\boldsymbol{x})$. Distance of R from $\boldsymbol{v}'$ in θ direction is less than $\ell_{\boldsymbol{x}}$. So $\|\boldsymbol{y}-\boldsymbol{v}'\| \le C_{77}\ell_{\boldsymbol{x}}$ for all $\boldsymbol{y} \in R$. Thus, using (A1), (A2), and a union bound, we get

$$\begin{aligned} \mathbb{P}\left(\max_{\boldsymbol{y}\in R}\mathcal{D}_2'(\boldsymbol{y}) \ge \frac{t}{4}\sigma(\|\boldsymbol{x}\|)\right) &\le C_{78}\|\boldsymbol{x}\|\exp(-C_{79}t\sigma(\|\boldsymbol{x}\|)/\sigma(\ell_{\boldsymbol{x}})) \\ &\le C_{80}\exp\Big(-C_{81}t(\log\|\boldsymbol{x}\|)^2\Big) \ . \end{aligned}$$

Using this in (4.22) together with (4.21) and (4.29) we get appropriate bound for the first term in the right-hand side of (4.20). The second term can be dealt with similarly. This completes the proof of Claim 4.10. □

This also completes the proof of Proposition 4.8.

4.2 Proof of Theorem 4.4

Due to the symmetry of the lattice, without loss of generality we assume $\theta_0 \in [0, \pi/4]$. We also assume that $\eta < 1$ because for $\eta = 1$ the result is same as Proposition 2.6.

Let n be a positive integer such that $(1-\alpha)^n \le \eta/2$. Define for $0 \le m \le n$ and $k \ge 3$

$$\psi_m(k) := k^{-\alpha}\sigma(k)(\log k)^{(1-\alpha)^m}(\log\log k)^{1-(1-\alpha)^m} .$$

Because $\psi_n(k) \le \phi_\eta(k)$ for large enough k, to prove Theorem 4.4 it is enough to show that h satisfies GAP with exponent α and correction factor ψ_n.

By assumptions of Theorem 4.4 θ_0 is a direction of both type I and II. Therefore, by Theorem 4.3, there exist positive constants δ, C_c, M_c, K, a, such that h satisfies $\mathrm{CHAP}\left(\alpha, \widehat{\phi}, M_c, C_c, K, a\right)$ in the sector of directions $(\theta_0 - \delta, \theta_0 + \delta)$. Define $\mathcal{S}_0 := [0, 2\pi]$, and for $1 \le m \le n$ define

$$\mathcal{S}_m := \left[\theta_0 - \delta\frac{n-m+1}{n}, \theta_0 + \delta\frac{n-m+1}{n}\right] .$$

We show that h satisfies GAP with exponent α and correction factor ψ_n in $\mathcal{S}_n$.

By Proposition 2.6, h satisfies GAP with exponent α and correction factor $\phi_1 = \psi_0$ in all directions. Hence, there exist constants $C_g > 0$ and $M_g > 0$ such that for $\|\boldsymbol{x}\| \ge M_g$ we have

$$h(\boldsymbol{x}) \le g(\boldsymbol{x}) + C_g\sigma\left(\|\boldsymbol{x}\|\right)\log\|\boldsymbol{x}\| . \tag{4.30}$$

We use an inductive argument. Fix $0 \le m < n$. Suppose h satisfies GAP with exponent α, correction factor ψ_m, in the sector $\mathcal{S}_m$, with constants C and M. We show that h satisfies GAP with exponent α, correction factor ψ_{m+1}, in the sector $\mathcal{S}_{m+1}$, with constant C and M. This establishes that h satisfies GAP with exponent α, correction factor ψ_n in the sector $\mathcal{S}_n$. The constants C and M need to remain unchanged. We will see that if C and M are chosen large enough then the inductive step works. We assume without loss of generality $K > 1$, $M > 3$, $M > M_c$, $M > M_g$. Also we assume M is large enough, independent of m, so that $\psi_{m+1}(\|\boldsymbol{x}\|) \ge 1$ for all $\|\boldsymbol{x}\| \ge M$, which is possible by (A2). Since h has sublinear growth, there exists constant $r > 0$ such that for all $\boldsymbol{x}$ we have $h(\boldsymbol{x}) \le r\|\boldsymbol{x}\|$. Let $\nu := (1-\beta)/4$, $c_0 := 3C_c$, $c_1 := C_c a$, $c_2 := 3K$, $c_3 := (c_1 + c_0 + C_g c_2(\alpha\nu)^{-1})\mathfrak{p}^{-1}$, $c_4 := c_2^\alpha \mathfrak{p}^{-1}$,

$$c_5 := c_3^\alpha c_4^{1-\alpha}\left(\left(\frac{\alpha}{1-\alpha}\right)^{1-\alpha} + \left(\frac{1-\alpha}{\alpha}\right)^{\alpha}\right) ,$$

and $c_6 := (1-\alpha)c_3(\alpha c_4)^{-1}$. We start the inductive step now. Consider $\boldsymbol{x}$ with direction $\theta \in \mathcal{S}_{m+1}$ and $\|\boldsymbol{x}\| \ge M$. We need to show

$$h(\boldsymbol{x}) \le g(\boldsymbol{x}) + C\|\boldsymbol{x}\|^\alpha \psi_{m+1}(\|\boldsymbol{x}\|) . \tag{4.31}$$

We are free to choose C and M large enough, independent of m. In various steps we assume C is large enough depending on M, and M is chosen to be large enough without depending on C.

Bounding $h(\boldsymbol{x})$ when $\|\boldsymbol{x}\|$ is small: Suppose $\|\boldsymbol{x}\| \le c_2 M$. Assuming $C \ge rc_2 M$, we get

$$h(\boldsymbol{x}) \le r\|\boldsymbol{x}\| \le rc_2 M \le C \le C\|\boldsymbol{x}\|^\alpha \psi_{m+1}(\boldsymbol{x}) \le g(\boldsymbol{x}) + C\|\boldsymbol{x}\|^\alpha \psi^m(\|\boldsymbol{x}\|) .$$

Thus (4.31) is verified.

Defining x^*, x_{L} and x_{S} when $\|x\|$ is large: Suppose

$$\|\boldsymbol{x}\| \geq c_2 M.$$

Take $q \in [c_2, \|\boldsymbol{x}\|/M] \cap \mathbb{Q}$. Then $\|\boldsymbol{x}/q\| \geq M \geq M_c$. Applying $\mathrm{CHAP}(\alpha, \widehat{\phi}, M_c, C_c, K, a)$ to $\boldsymbol{x}/q$ we get

$$\boldsymbol{x}/q = \sum_{i=1}^{3} \Upsilon_{qi} \boldsymbol{y}_{qi} \text{ with } \Upsilon_{qi} \geq 0\,, \quad \sum_{i=1}^{3} \Upsilon_{qi} \in [1, a] \text{ and } \boldsymbol{y}_{qi} \in Q_{\boldsymbol{x}/q}\Big(\alpha, \widehat{\phi}, C_c, K\Big)\,. \tag{4.32}$$

Let

$$L(q) := \Big\{ 1 \leq i \leq 3 \colon \left\|\boldsymbol{y}_{qi}\right\| \geq \|\boldsymbol{x}/q\|^{1-\nu} \Big\}$$

and

$$\boldsymbol{x}^* := \sum_{i=1}^{3} \lfloor q \Upsilon_{qi} \rfloor \boldsymbol{y}_{qi}\,, \quad \boldsymbol{x}_{\mathsf{L}} := \sum_{i \in L(q)} \gamma_{qi} \boldsymbol{y}_{qi}\,, \quad \boldsymbol{x}_{\mathsf{S}} := \sum_{i \notin L(q)} \gamma_{qi} \boldsymbol{y}_{qi}\,, \tag{4.33}$$

where

$$\gamma_{qi} := q\Upsilon_{qi} - \lfloor q \Upsilon_{qi} \rfloor \in [0, 1)\,.$$

Therefore,

$$\boldsymbol{x} = \boldsymbol{x}^* + \boldsymbol{x}_{\mathsf{L}} + \boldsymbol{x}_{\mathsf{S}}\,.$$

Direction of x_{L}: Consider $i \in L(q)$. Then

$$\left\|\boldsymbol{y}_{qi}\right\| \geq \|\boldsymbol{x}/q\|^{1-\nu}\,. \tag{4.34}$$

Using Lemma 4.9, $\boldsymbol{y}_{qi} \in Q_{\boldsymbol{x}/q}(\alpha, \widehat{\phi}, C_c, K)$, $\|\boldsymbol{x}/q\| \geq M$, assuming M is large enough, and (A2), we get

$$|\pi^2_{\theta,\theta^t}(\boldsymbol{y}_{qi})| \leq C_1 \Delta(\boldsymbol{x}/q)(\log\log\|\boldsymbol{x}/q\|)^{1/2} \leq C_2 \|\boldsymbol{x}/q\|^{(1+\beta)/2} (\log\log\|\boldsymbol{x}/q\|)^{1/2}\,.$$

Therefore, using (4.34), (A2), and $1 - \nu = 1 - (1-\beta)/4 > (1+\beta)/2$, we get

$$|\pi^1_{\theta,\theta^t}(\boldsymbol{y}_{qi})| \geq \left\|\boldsymbol{y}_{qi}\right\| - |\pi^2_{\theta,\theta^t}(\boldsymbol{y}_{qi})| \geq C_3 \|\boldsymbol{x}/q\|^{1-\nu}\,,$$

and, further, using $1 - \nu - (1+\beta)/2 = (1-\beta)/4$ we get

$$\frac{|\pi^2_{\theta,\theta^t}(\boldsymbol{y}_{qi})|}{|\pi^1_{\theta,\theta^t}(\boldsymbol{y}_{qi})|} \leq C_4 \frac{\|\boldsymbol{x}/q\|^{(1+\beta)/2}(\log\log\|\boldsymbol{x}/q\|)^{1/2}}{\|\boldsymbol{x}/q\|^{1-\nu}} \leq C_5 \frac{(\log\log M)^{1/2}}{M^{(1-\beta)/4}}\,.$$

Since $\theta \in \mathcal{S}_{m+1}$, assuming M is large enough, we get direction of $\boldsymbol{y}_{qi}$ is in $\mathcal{S}_m$. This implies $\boldsymbol{x}_{\mathsf{L}}$ has direction in $\mathcal{S}_m$.

Bounding $h(x^*)$: Using subadditivity of h, (4.33), (4.32), and (A2), we get

$$\begin{aligned} h(\boldsymbol{x}^*) \leq \sum_{i=1}^{3} \lfloor q\Upsilon_{qi} \rfloor h(\boldsymbol{y}_{qi}) &\leq \sum_{i=1}^{3} \lfloor q\Upsilon_{qi} \rfloor \Big[g_{\boldsymbol{x}}(\boldsymbol{y}_{qi}) + C_c \|\boldsymbol{x}/q\|^{\alpha} \widehat{\phi}(\|\boldsymbol{x}/q\|) \Big] \\ &\leq g_{\boldsymbol{x}}(\boldsymbol{x}^*) + c_1 \mathfrak{p}^{-1} q^{1-\alpha} \|\boldsymbol{x}\|^{\alpha} \widehat{\phi}(\|\boldsymbol{x}\|)\,. \end{aligned} \tag{4.35}$$

Bounding $h(x_\mathsf{S})$ when $\|x_\mathsf{S}\|$ is large: Suppose $\|\boldsymbol{x}_\mathsf{S}\| \geq M_g$. Using $q \geq c_2 = 3K > 3$, $\|\boldsymbol{x}/q\| \geq M \geq 1$, and (4.33) we get

$$\|\boldsymbol{x}_\mathsf{S}\| = \left\| \sum_{i \notin L(q)} \gamma_{qi} \boldsymbol{y}_{qi} \right\| \leq \sum_{i \notin L(q)} \gamma_{qi} \|\boldsymbol{y}_{qi}\| \leq 3\|\boldsymbol{x}/q\|^{1-\nu} \leq 3\|\boldsymbol{x}/q\| \leq \|\boldsymbol{x}\| \ .$$

Using this and $\log\|\boldsymbol{x}\| \leq \|\boldsymbol{x}\|^{\alpha\nu}(\alpha\nu)^{-1}$ we get

$$\begin{aligned} \frac{\sigma(\|\boldsymbol{x}_\mathsf{S}\|)\log\|\boldsymbol{x}_\mathsf{S}\|}{\sigma(\|\boldsymbol{x}\|)\log\log\|\boldsymbol{x}\|} &\leq \mathfrak{p}^{-1}\left(\frac{\|\boldsymbol{x}_\mathsf{S}\|}{\|\boldsymbol{x}\|}\right)^{\alpha} \frac{\log\|\boldsymbol{x}\|}{\log\log\|\boldsymbol{x}\|} \\ &\leq \mathfrak{p}^{-1}\frac{c_2^{\alpha} q^{\alpha\nu}\log\|\boldsymbol{x}\|}{q^{\alpha}\|\boldsymbol{x}\|^{\alpha\nu}\log\log\|\boldsymbol{x}\|} \leq \mathfrak{p}^{-1}c_2^{\alpha}(\alpha\nu)^{-1}q^{-\alpha(1-\nu)} \ . \end{aligned}$$

Since $\|\boldsymbol{x}_\mathsf{S}\| \geq M_g$, using (4.30) we get

$$\begin{aligned} & h(\boldsymbol{x}_\mathsf{S}) \\ &\leq g(\boldsymbol{x}_\mathsf{S}) + C_g\|\boldsymbol{x}_\mathsf{S}\|^{\alpha}\phi_1(\|\boldsymbol{x}_\mathsf{S}\|) \\ &\leq \sum_{i \notin L(q)} \gamma_{qi} g(\boldsymbol{y}_{qi}) + C_g\sigma(\|\boldsymbol{x}_\mathsf{S}\|)\log\|\boldsymbol{x}_\mathsf{S}\| \\ &\leq g_{\boldsymbol{x}}(\boldsymbol{x}_\mathsf{S}) + \sum_{i \notin L(q)} \gamma_{qi}\left[g(\boldsymbol{y}_{qi}) - g_{\boldsymbol{x}}(\boldsymbol{y}_{qi})\right] + C_g\sigma(\|\boldsymbol{x}_\mathsf{S}\|)\log\|\boldsymbol{x}_\mathsf{S}\| \\ &\leq g_{\boldsymbol{x}}(\boldsymbol{x}_\mathsf{S}) + \sum_{i \notin L(q)} \gamma_{qi}\left[g(\boldsymbol{y}_{qi}) - g_{\boldsymbol{x}}(\boldsymbol{y}_{qi})\right] + C_g\mathfrak{p}^{-1}c_2^{\alpha}(\alpha\nu)^{-1}q^{-\alpha(1-\nu)}\sigma(\|\boldsymbol{x}\|)\log\log\|\boldsymbol{x}\| \\ &\leq g_{\boldsymbol{x}}(\boldsymbol{x}_\mathsf{S}) + \sum_{i \notin L(q)} \gamma_{qi}\left[g(\boldsymbol{y}_{qi}) - g_{\boldsymbol{x}}(\boldsymbol{y}_{qi})\right] + C_g\mathfrak{p}^{-1}c_2^{\alpha}(\alpha\nu)^{-1}q^{1-\alpha}\sigma(\|\boldsymbol{x}\|)\log\log\|\boldsymbol{x}\| \ . \end{aligned} \tag{4.36}$$

Bounding $h(x_\mathsf{S})$ when $\|x_\mathsf{S}\|$ is small: Here we consider the case $\|\boldsymbol{x}_\mathsf{S}\| \leq M_g$. Since $\boldsymbol{y}_{qi} \in Q_{\boldsymbol{x}/q}(\alpha, \phi_2, C_c, K)$,

$$0 \leq h(\boldsymbol{y}_{qi}) \leq g_{\boldsymbol{x}}(\boldsymbol{y}_{qi}) + C_c q^{-\alpha}\|\boldsymbol{x}\|^{\alpha}\widehat{\phi}(\|\boldsymbol{x}/q\|) \ .$$

Therefore

$$g_{\boldsymbol{x}}(\boldsymbol{y}_{qi}) \geq -C_c q^{-\alpha}\|\boldsymbol{x}\|^{\alpha}\widehat{\phi}(\|\boldsymbol{x}/q\|) \ .$$

So, letting $I(q) \coloneqq \left\{i \leq 3 : g_{\boldsymbol{x}}(\boldsymbol{y}_{qi}) < 0\right\}$, and using definition of c_0, and (A2), we have

$$\begin{aligned} g_{\boldsymbol{x}}(\boldsymbol{x}_\mathsf{S}) = \sum_{i \notin L(q)} \gamma_{qi} g_{\boldsymbol{x}}(\boldsymbol{y}_{qi}) &\geq -\sum_{i \in I(q)} \left|g_{\boldsymbol{x}}(\boldsymbol{y}_{qi})\right| \\ &\geq -c_0 q^{-\alpha}\|\boldsymbol{x}\|^{\alpha}\widehat{\phi}(\|\boldsymbol{x}/q\|) \geq -\mathfrak{p}^{-1}c_0 q^{-\alpha}\|\boldsymbol{x}\|^{\alpha}\widehat{\phi}(\|\boldsymbol{x}\|) \ . \end{aligned}$$

Therefore, using $\|\boldsymbol{x}_\mathsf{S}\| \leq M_g$ and $h(\boldsymbol{x}_\mathsf{S}) \leq r\|\boldsymbol{x}_\mathsf{S}\|$

$$h(\boldsymbol{x}_\mathsf{S}) \leq rM_g \leq g_{\boldsymbol{x}}(\boldsymbol{x}_\mathsf{S}) + \mathfrak{p}^{-1}c_0 q^{-\alpha}\|\boldsymbol{x}\|^{\alpha}\widehat{\phi}(\|\boldsymbol{x}\|) + rM_g \ . \tag{4.37}$$

Bounding $h(x_\mathsf{L})$ when $\|x_\mathsf{L}\|$ is large: Suppose $\|\boldsymbol{x}_\mathsf{L}\| \geq M$. Using $q \geq c_2 \geq 1$, $\|\boldsymbol{x}/q\| \geq M \geq 1$, (4.33), and $\|\boldsymbol{y}_{qi}\| \leq K\|\boldsymbol{x}/q\|$, we get

$$\|\boldsymbol{x}_\mathsf{L}\| = \left\| \sum_{i \in L(q)} \gamma_{qi}\boldsymbol{y}_{qi} \right\| \leq \sum_{i \in L(q)} \gamma_{qi}\|\boldsymbol{y}_{qi}\| \leq c_2\|\boldsymbol{x}/q\| \leq \|\boldsymbol{x}\| \ .$$

Using this and applying $\mathrm{GAP}(\alpha, \psi_m, M, C)$, which holds by the induction hypothesis for m, we get

$$\begin{aligned} h(\boldsymbol{x}_{\mathsf{L}}) &\leq g(\boldsymbol{x}_{\mathsf{L}}) + C\|\boldsymbol{x}_{\mathsf{L}}\|^{\alpha}\psi_m\left(\|\boldsymbol{x}_{\mathsf{L}}\|\right) \\ &\leq \sum_{i \in L(q)} \gamma_{qi} g(\boldsymbol{y}_{qi}) + C\mathfrak{p}^{-1}c_2^{\alpha}q^{-\alpha}\|\boldsymbol{x}\|^{\alpha}\psi_m\left(\|\boldsymbol{x}\|\right) \\ &\leq g_{\boldsymbol{x}}(\boldsymbol{x}_{\mathsf{L}}) + \sum_{i \in L(q)} \gamma_{qi}\left[g(\boldsymbol{y}_{qi}) - g_{\boldsymbol{x}}(\boldsymbol{y}_{qi})\right] + C\mathfrak{p}^{-1}c_2^{\alpha}q^{-\alpha}\|\boldsymbol{x}\|^{\alpha}\psi_m(\|\boldsymbol{x}\|) \,. \end{aligned} \tag{4.38}$$

Bounding $h(\boldsymbol{x}_{\mathsf{L}})$ when $\|\boldsymbol{x}_{\mathsf{L}}\|$ is small: Suppose $\|\boldsymbol{x}_{\mathsf{L}}\| \leq M$. Then by similar calculations that lead to (4.37) we get

$$h(\boldsymbol{x}_{\mathsf{L}}) \leq rM \leq g_{\boldsymbol{x}}(\boldsymbol{x}_{\mathsf{L}}) + \mathfrak{p}^{-1}c_0q^{-\alpha}\|\boldsymbol{x}\|^{\alpha}\widehat{\phi}(\|\boldsymbol{x}\|) + rM \,. \tag{4.39}$$

Overall bound on $h(\boldsymbol{x})$: Using $\boldsymbol{y}_{qi} \in Q_{\boldsymbol{x}/q}(\alpha, \widehat{\phi}, C_c, K)$, definition of c_0, and (A2), we get

$$\begin{aligned} \sum_{i=1}^{3} \gamma_{qi}\left[g(\boldsymbol{y}_{qi}) - g_{\boldsymbol{x}}(\boldsymbol{y}_{qi})\right] &\leq \sum_{i=1}^{3} \gamma_{qi}\left[h(\boldsymbol{y}_{qi}) - g_{\boldsymbol{x}}(\boldsymbol{y}_{qi})\right] \\ &\leq c_0\|\boldsymbol{x}/q\|^{\alpha}\widehat{\phi}(\|\boldsymbol{x}/q\|) \leq c_0\mathfrak{p}^{-1}q^{-\alpha}\|\boldsymbol{x}\|^{\alpha}\widehat{\phi}(\|\boldsymbol{x}\|) \,. \end{aligned}$$

Combining this with (4.35)-(4.39), we get

$$\begin{aligned} &h(\boldsymbol{x}) \\ &\leq h(\boldsymbol{x}^*) + h(\boldsymbol{x}_{\mathsf{L}}) + h(\boldsymbol{x}_{\mathsf{S}}) \\ &\leq g(\boldsymbol{x}) + c_3q^{1-\alpha}\sigma(\|\boldsymbol{x}\|)\log\log\|\boldsymbol{x}\| \\ &\quad + Cc_4q^{-\alpha}\sigma(\|\boldsymbol{x}\|)(\log\|\boldsymbol{x}\|)^{(1-\alpha)^m}(\log\log\|\boldsymbol{x}\|)^{1-(1-\alpha)^m} + rM + rM_g \,. \end{aligned} \tag{4.40}$$

Optimizing over q: The optimal q that minimizes the right-hand side (4.40) is

$$q_0 \coloneqq C\frac{\alpha c_4}{(1-\alpha)c_3}\left(\frac{\log\|\boldsymbol{x}\|}{\log\log\|\boldsymbol{x}\|}\right)^{(1-\alpha)^m} \,.$$

Plugging in $q = q_0$ in (4.40) we see that if C is large enough depending on r, M, M_g, then

$$\begin{aligned} h(\boldsymbol{x}) &\leq g(\boldsymbol{x}) + c_5C^{1-\alpha}\sigma(\|\boldsymbol{x}\|)(\log\|\boldsymbol{x}\|)^{(1-\alpha)^{m+1}}(\log\log\|\boldsymbol{x}\|)^{1-(1-\alpha)^{m+1}} + rM + rM_g \\ &\leq g(\boldsymbol{x}) + C\sigma(\|\boldsymbol{x}\|)(\log\|\boldsymbol{x}\|)^{(1-\alpha)^{m+1}}(\log\log\|\boldsymbol{x}\|)^{1-(1-\alpha)^{m+1}} \,. \end{aligned}$$

Thus we get (4.31) provided we prove q_0 is feasible.

Feasibility of q_0: We need to verify that $q_0 \in [c_2, \|\boldsymbol{x}\|/M]$. We get $q_0 \geq c_2$ using $\|\boldsymbol{x}\| \geq M$, $m \leq n$, choosing $C > 1$, and assuming M is large enough. Suppose $\|\boldsymbol{x}\| < q_0M$ so that we have

$$CM \geq c_6\|\boldsymbol{x}\|\left(\frac{\log\|\boldsymbol{x}\|}{\log\log\|\boldsymbol{x}\|}\right)^{-(1-\alpha)^m} \,. \tag{4.41}$$

This gives an upper bound on $\|\boldsymbol{x}\|$. So q_0 is not feasible when $\|\boldsymbol{x}\|$ is too small. But we can prove (4.31) in a different way. Consider two cases. *Case I:* Suppose $\boldsymbol{x}$ is such that

$$C \geq C_g\left(\frac{\log\|\boldsymbol{x}\|}{\log\log\|\boldsymbol{x}\|}\right)^{1-(1-\alpha)^{m+1}} \,.$$

Then from (4.30) we get

$$h(\boldsymbol{x}) \le g(\boldsymbol{x}) + C\sigma(\|\boldsymbol{x}\|)(\log\|\boldsymbol{x}\|)^{(1-\alpha)^{m+1}}(\log\log\|\boldsymbol{x}\|)^{1-(1-\alpha)^{m+1}} .$$

Thus (4.31) is verified. *Case II:* Suppose $\boldsymbol{x}$ is such that

$$C \le C_g \left(\frac{\log\|\boldsymbol{x}\|}{\log\log\|\boldsymbol{x}\|}\right)^{1-(1-\alpha)^{m+1}} .$$

Combining with (4.41) we get

$$C_g M \left(\frac{\log\|\boldsymbol{x}\|}{\log\log\|\boldsymbol{x}\|}\right)^{1-(1-\alpha)^{m+1}} \ge c_6\|\boldsymbol{x}\| \left(\frac{\log\|\boldsymbol{x}\|}{\log\log\|\boldsymbol{x}\|}\right)^{-(1-\alpha)^m} .$$

So

$$\frac{C_g M}{c_6} \ge \|\boldsymbol{x}\| \left(\frac{\log\|\boldsymbol{x}\|}{\log\log\|\boldsymbol{x}\|}\right)^{-(1-(1-\alpha)^{m+1}+(1-\alpha)^m)} \ge \|\boldsymbol{x}\| \left(\frac{\log\|\boldsymbol{x}\|}{\log\log\|\boldsymbol{x}\|}\right)^{-2} .$$

Therefore, $M \ge F(\|\boldsymbol{x}\|)$, where $F : [3,\infty) \to (0,\infty)$ is $F(k) := c_6 C_g^{-1} k(\log k/\log\log k)^{-2}$. Observe that F is strictly increasing. Therefore, taking $C \ge rF^{-1}(M)$, we get

$$h(\boldsymbol{x}) \le r\|\boldsymbol{x}\| \le rF^{-1}(M) \le C \le g(\boldsymbol{x}) + C\psi_{m+1}(\|\boldsymbol{x}\|) .$$

Thus (4.31) is verified.

This concludes the inductive step and proves Theorem 4.4.

5 Upper bound of the transverse increments

In this section, we prove Theorem 1.23, which is our main result on upper bound of the transverse increments. Let θ_0 be a direction of both type I and II. Due to symmetry of the lattice, without loss of generality we assume $\theta_0 \in [0, \pi/4]$. Fix $L_0 > 0$, $n_0 > 0$, $t_0 > 0$, to be assumed large enough whenever required. Consider n, L, t, satisfying $n \ge n_0$, $L \ge L_0$, $t \ge t_0$, and $L \le \Delta(n)$. We establish the bound on $\mathcal{D}(n, L)$, the bound on $\mathcal{D}'(n, L)$ can be established in a similar manner. If $t \ge 4\mu L(\sigma(\Delta^{-1}(L))((\log\Delta^{-1}(L))^\eta)^{-1})$, where μ is the expected edge-weight, then we are in a large-deviation regime, and the proof is similar to Case I of Theorem 3.2. Therefore, let us assume

$$t \le 4\,\mu\,L\left(\sigma\big(\Delta^{-1}(L)\big)\big(\log\Delta^{-1}(L)\big)^\eta\right)^{-1} . \tag{5.1}$$

Define an interval J and a segment $\mathcal{I}^*$ as

$$J := \left[-t^{1/2}L\big(\log\Delta^{-1}(L)\big)^{\eta/2} , \left(1 - \frac{\Delta^{-1}(L)}{n}\right)L + t^{1/2}L\big(\log\Delta^{-1}(L)\big)^{\eta/2}\right] ,$$

$$\mathcal{I}^* := \left\{ \boldsymbol{x} \in \mathbb{R}^2 : \pi^1_{\theta_0,\theta_0^t}(\boldsymbol{x}) = n - \Delta^{-1}(L) , \pi^2_{\theta_0,\theta_0^t}(\boldsymbol{x}) \in J \right\} .$$

Let

$$M := (1+\beta)/(2\alpha) , \quad N_1 := \lfloor(\log L)^M\rfloor , \quad N_2 := \lfloor t^{1/2}(\log L)^{M+\eta/2}\rfloor . \tag{5.2}$$

Divide the segment $\mathcal{I}(n, L)$ in N_1 segments of equal length: $\mathcal{I}_1, \ldots, \mathcal{I}_{N_1}$, with endpoints $\boldsymbol{a}_0, \ldots, \boldsymbol{a}_{N_1}$, as shown in Figure 7. Divide the segment $\mathcal{I}^*$ in N_2 segments of equal length: $\mathcal{I}^*_1, \ldots, \mathcal{I}^*_{N_2}$, with endpoints $\boldsymbol{b}_0, \ldots, \boldsymbol{b}_{N_2}$, as shown in Figure 7. By (A2), $\log L$ is of the same order as $\log\Delta^{-1}(L)$, i.e.,

$$C_1 \log L \le \log\Delta^{-1}(L) \le C_2 \log L . \tag{5.3}$$

Therefore, length of the segments $\mathcal{I}_i$ and $\mathcal{I}_j^*$ are bounded as

$$C_3 L(\log L)^{-M} \le \pi^2_{\theta_0,\theta_0^t}(\boldsymbol{a}_{i-1}-\boldsymbol{a}_i) \le C_4 L(\log L)^{-M} \ , \tag{5.4}$$

$$C_5 L(\log L)^{-M} \le \pi^2_{\theta_0,\theta_0^t}(\boldsymbol{b}_{j-1}-\boldsymbol{b}_j) \le C_6 L(\log L)^{-M} \ . \tag{5.5}$$

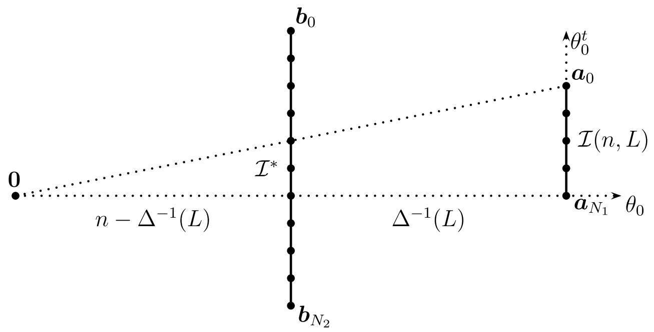

Figure 7: Setup of Theorem 1.23: width of the portion of $\mathcal{I}^*$ between the dotted lines is $L(1-\Delta^{-1}(L)/n)$; width of the portion of $\mathcal{I}^*$ above the line joining $\mathbf{0}$ and $\boldsymbol{a}_0$ is $t^{1/2}L\left(\log\Delta^{-1}(L)\right)^{\eta/2}$; width of the portion of $\mathcal{I}^*$ below the line joining $\mathbf{0}$ and $\boldsymbol{a}_{N_1}$ is also $t^{1/2}L(\log\Delta^{-1}(L))^{\eta/2}$.

Define the event

$$\mathcal{E} \coloneqq \{\, \Gamma(\mathbf{0},\boldsymbol{a}_i) \text{ passes through } \mathcal{I}^* \text{ for all } 0 \le i \le N_1 \,\} \ .$$

Define

$$\begin{aligned}
D^{[1]} &\coloneqq \max\{\, |T(\mathbf{0},\boldsymbol{a}_{i_1})-T(\mathbf{0},\boldsymbol{a}_{i_2})| \colon 0 \le i_1 < i_2 \le N_1 \,\} \ ,\\
D^{[2]} &\coloneqq \max\{\, |T(\boldsymbol{x},\boldsymbol{a}_{i_1})-T(\boldsymbol{x},\boldsymbol{a}_{i_2})| \colon 0 \le i_1 < i_2 \le N_1 \, , \, \boldsymbol{x} \in \mathcal{I}^* \,\} \ ,\\
D^{[3]} &\coloneqq \max\{\, |T(\boldsymbol{b}_j,\boldsymbol{a}_{i_1})-T(\boldsymbol{b}_j,\boldsymbol{a}_{i_2})| \colon 0 \le i_1 < i_2 \le N_1 \, , \, 0 \le j \le N_2 \,\} \ ,\\
D^{[4]} &\coloneqq \max\left\{\, \left|T(\boldsymbol{b},\boldsymbol{a}_i)-T(\boldsymbol{b}',\boldsymbol{a}_i)\right| \colon \boldsymbol{b},\boldsymbol{b}' \in \mathcal{I}_j^* \, , \, 0 \le j \le N_2 \, , \, 0 \le i \le N_1 \,\right\} \ ,\\
D_i &\coloneqq \max\{\, |T(\mathbf{0},\boldsymbol{x})-T(\mathbf{0},\boldsymbol{y})| \colon \boldsymbol{x},\boldsymbol{y} \in \mathcal{I}_i \,\} \, , \text{ for } 0 \le i \le N_1 \ .
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathcal{D}(n,L) &\le D^{[1]} + \max_{0 \le i \le N_1} D_i \ ,\\
D^{[2]} &\le D^{[3]} + 2D^{[4]} \ .
\end{aligned} \tag{5.6}$$

We claim that if $\mathbb{T} \in \mathcal{E}$ then

$$D^{[1]} \le D^{[2]} \ . \tag{5.7}$$

To prove this, we take $0 \le i_1 < i_2 \le N_1$. If $\mathbb{T} \in \mathcal{E}$, then there exist points $\boldsymbol{y}$ and $\boldsymbol{z}$ in $\mathcal{I}^*$ such that $\Gamma(\mathbf{0},\boldsymbol{a}_{i_1})$ passes through $\boldsymbol{y}$, and $\Gamma(\mathbf{0},\boldsymbol{a}_{i_2})$ passes through $\boldsymbol{z}$. Therefore

$$T(\mathbf{0},\boldsymbol{a}_{i_1})-T(\mathbf{0},\boldsymbol{a}_{i_2}) \le T(\mathbf{0},\boldsymbol{z})+T(\boldsymbol{z},\boldsymbol{a}_{i_1})-T(\mathbf{0},\boldsymbol{z})-T(\boldsymbol{z},\boldsymbol{a}_{i_2}) = T(\boldsymbol{z},\boldsymbol{a}_{i_1})-T(\boldsymbol{z},\boldsymbol{a}_{i_2}) \ .$$

Similarly,

$$T(\mathbf{0},\boldsymbol{a}_{i_2})-T(\mathbf{0},\boldsymbol{a}_{i_1}) \le T(\boldsymbol{y},\boldsymbol{a}_{i_2})-T(\boldsymbol{y},\boldsymbol{a}_{i_1}) \ .$$

Therefore
$$|T(\mathbf{0},\boldsymbol{a}_{i_1})-T(\mathbf{0},\boldsymbol{a}_{i_2})|\le \max_{\boldsymbol{x}\in\mathcal{I}^*}|T(\boldsymbol{x},\boldsymbol{a}_{i_1})-T(\boldsymbol{x},\boldsymbol{a}_{i_2})|\ .$$

Thus, (5.7) is proved by taking maximum over values of i_1, i_2. Combining (5.6) and (5.7) we get that on the event $\mathcal{E}$,

$$\mathcal{D}(n,L)\le D^{[3]}+2D^{[4]}+\max_{0\le i\le N_1}D_i\ .$$

Therefore

$$\mathbb{P}\big(\mathcal{D}(n,L)\ge t\sigma(\Delta^{-1}(L))(\log L)^\eta\big)\le \mathbb{P}(\,\mathcal{E}^c\,)+\mathbb{P}\left(D^{[3]}\ge \frac{t}{4}\sigma(\Delta^{-1}(L))(\log L)^\eta\right)$$
$$+\,\mathbb{P}\left(D^{[4]}\ge \frac{t}{4}\sigma(\Delta^{-1}(L))(\log L)^\eta\right)+\mathbb{P}\left(\max_{0\le i\le N_1}D_i\ge \frac{t}{4}\sigma(\Delta^{-1}(L))(\log L)^\eta\right).\quad (5.8)$$

First, we show that $\mathbb{P}(\,\mathcal{E}^c\,)$ is small. If $\mathbb{T}\notin\mathcal{E}$, then for some i, $\Gamma(\mathbf{0},\boldsymbol{a}_i)$ wanders more than $t^{1/2}L(\log\Delta^{-1}(L))^{\eta/2}$ in $\pm\theta_0^t$ directions when it is at a distance $\Delta^{-1}(L)$ from $\boldsymbol{a}_i$ in $-\theta_0$ direction. Since θ_0 is a direction of both type I and type II, by Theorem 4.4 h satisfies GAP with exponent α and correction factor ϕ_η in a neighborhood of θ_0. Thus, applying Theorem 2.10 with the variables

$$\tilde{\theta_0}\coloneqq-\theta_0\,,\qquad \tilde{\eta}\coloneqq\eta\,,\qquad\qquad \tilde{n}\coloneqq n\,,$$
$$\tilde{k}\coloneqq\Delta^{-1}(L)\,,\ \ \tilde{l}\coloneqq\pi^2_{\theta_0,\theta_0^t}(\boldsymbol{a}_i)\,,\ \ \tilde{t}\coloneqq t^{1/2}\ ,$$

(recall from Remark 1.2 our convention of using tilde on parameters) and using (5.3), we get

$$\mathbb{P}\Big(\mathcal{W}\left(\boldsymbol{a}_i,\mathbf{0},\Delta^{-1}(L),-\theta_0\right)\ge t^{1/2}L(\log\Delta^{-1}(L))^{\eta/2}\Big)\le C_7\exp(-C_8t(\log L)^\eta)\ ,\qquad (5.9)$$

provided $\tilde{n}\ge\tilde{n_0}$, $\tilde{t}\ge\tilde{t_0}$, $\tilde{k}\ge\tilde{k_0}$, $\tilde{t}\Delta(\tilde{k})(\log\tilde{k})^{\tilde{\eta}/2}\le\tilde{k}\tilde{\delta_1}$, and $\tilde{l}\le\tilde{n}\tilde{\delta_2}$. We verify these conditions now. Taking n_0, L_0, t_0 large enough we get $\tilde{n}\ge\tilde{n_0}$, $\tilde{t}\ge\tilde{t_0}$, and $\tilde{k}\ge\tilde{k_0}$. Using (5.1) and (A2) we get

$$\frac{\tilde{t}\Delta(\tilde{k})(\log\tilde{k})^{\eta/2}}{\tilde{k}}=\frac{t^{1/2}L(\log\Delta^{-1}(L))^{\eta/2}}{\Delta^{-1}(L)}\le C_9\frac{L^{1/2}}{(\Delta^{-1}(L))^{1/2}}$$
$$\le C_{10}L^{-(1-\beta)/2}\le C_{11}L_0^{-(1-\beta)/2}\le\tilde{\delta_1}\ .\quad (5.10)$$

Using $L\le\Delta(n)$ and (A2), we get

$$\frac{|\tilde{l}|}{\tilde{n}}\le\frac{L}{n}\le\frac{\Delta(n)}{n}\le C_{12}n_0^{-(1-\beta)/2}\le\tilde{\delta_2}\ .$$

Thus all the conditions for (5.9) to hold are true. From (5.9) taking a union bound over i values we get
$$\mathbb{P}(\,\mathcal{E}^c\,)\le C_{13}\exp(-C_{14}t(\log L)^\eta)\ .\qquad (5.11)$$

Now we show that the second term in the right-hand side of (5.8) is small. Take $\boldsymbol{x}$ in $\mathcal{I}^*$ and $\boldsymbol{u}$, $\boldsymbol{v}$ in $\mathcal{I}(n,L)$. Then

$$|T(\boldsymbol{x},\boldsymbol{u})-T(\boldsymbol{x},\boldsymbol{v})|\le|T(\boldsymbol{x},\boldsymbol{u})-h(\boldsymbol{u}-\boldsymbol{x})|+|T(\boldsymbol{x},\boldsymbol{v})-h(\boldsymbol{v}-\boldsymbol{x})|$$
$$+\,|h(\boldsymbol{u}-\boldsymbol{x})-g(\boldsymbol{u}-\boldsymbol{x})|+|h(\boldsymbol{v}-\boldsymbol{x})-g(\boldsymbol{v}-\boldsymbol{x})|$$
$$+\,|g(\boldsymbol{u}-\boldsymbol{x})-g(\boldsymbol{v}-\boldsymbol{x})|\ .$$

From the definition of $\mathcal{I}^*$ it follows

$$\pi^1_{\theta_0,\theta_0^t}(\boldsymbol{u}-\boldsymbol{x}) = \Delta^{-1}(L), \quad |\pi^2_{\theta_0,\theta_0^t}(\boldsymbol{u}-\boldsymbol{x})| \le C_{15} t^{1/2} L (\log \Delta^{-1}(L))^{\eta/2} . \tag{5.12}$$

By same calculation as in (5.10) we get that the direction of $\boldsymbol{u}-\boldsymbol{x}$ can be made arbitrarily close to θ_0 by choosing L_0 large enough. So by Theorem 4.4 and equation (5.3) we get

$$|h(\boldsymbol{u}-\boldsymbol{x}) - g(\boldsymbol{u}-\boldsymbol{x})| \le C_{16}\sigma(\Delta^{-1}(L))(\log L)^{\eta} . \tag{5.13}$$

Same holds true for $\boldsymbol{v}$ replaced by $\boldsymbol{u}$. Using (5.12), Lemma 3.1, and (5.3), we get

$$|g(\boldsymbol{u}-\boldsymbol{x}) - g(\boldsymbol{v}-\boldsymbol{x})| \le C_{17} t^{1/2} \frac{L^2(\log(\Delta^{-1}(L)))^{\eta}}{\Delta^{-1}(L)} \le C_{18} t^{1/2} \sigma(\Delta^{-1}(L))(\log L)^{\eta} . \tag{5.14}$$

Using (5.13), (5.14), and (A1) we get

$$\mathbb{P}\left(|T(\boldsymbol{x},\boldsymbol{u}) - T(\boldsymbol{x},\boldsymbol{v})| \ge \frac{t}{4}\sigma(\Delta^{-1}(L))(\log L)^{\eta} \right) \le C_{19}\exp(-C_{20} t (\log L)^{\eta}) . \tag{5.15}$$

Equation 5.15 is true for fixed $\boldsymbol{x}$, $\boldsymbol{u}$ and $\boldsymbol{v}$. Thus we get for all $0 \le i_1 < i_2 \le N_1$ and $0 \le j \le N_2$

$$\mathbb{P}\left(|T(\boldsymbol{b}_j,\boldsymbol{a}_{i_1}) - T(\boldsymbol{b}_j,\boldsymbol{a}_{i_2})| \ge \frac{t}{4}\sigma(\Delta^{-1}(L))(\log L)^{\eta} \right) \le C_{21}\exp(-C_{22} t (\log L)^{\eta}) . \tag{5.16}$$

By (5.2), the number of triplets (i_1, i_2, j) is less than $C_{23} t^{1/2} (\log L)^{3M+\eta/2}$. Therefore, from (5.16) we get by a union bound

$$\mathbb{P}\left(D^{[3]} \ge \frac{t}{4}\sigma(\Delta^{-1}(L))(\log L)^{\eta} \right) \le C_{24}\exp(-C_{25} t (\log L)^{\eta}) . \tag{5.17}$$

Now let us consider the third term in the right-hand side of (5.8). Fix $0 \le i \le N_1$ and $1 \le j \le N_2$. Applying Corollary 3.3 with the variables

$$\tilde{\theta_0} \coloneqq -\theta_0 \,,\ \tilde{n} \coloneqq \Delta^{-1}(L) \,,\ \tilde{L} \coloneqq |\pi^2_{\theta_0,\theta_0^t}(\boldsymbol{b}_{j-1}-\boldsymbol{b}_j)| \,,\ \tilde{d} \coloneqq \pi^2_{\theta_0,\theta_0^t}(\boldsymbol{b}_j - \boldsymbol{a}_i) \,,$$

(recall from Remark 1.2 our convention of using tilde on parameters) we get for all $\tilde{t} \ge \tilde{t_0}$

$$\begin{aligned} &\mathbb{P}\left(\max_{\boldsymbol{b},\boldsymbol{b}' \in \mathcal{I}_j^*} \big| T(\boldsymbol{b},\boldsymbol{a}_i) - T(\boldsymbol{b}',\boldsymbol{a}_i) \big| \ge C_{26} \tilde{L} \frac{|\tilde{d}|}{\tilde{n}} + \tilde{t}\sigma(\Delta^{-1}(\tilde{L})) \log \Delta^{-1}(\tilde{L}) \right) \\ &\le C_{27}\exp\Big(-C_{28} t \log \tilde{L} \Big) , \end{aligned} \tag{5.18}$$

provided the following conditions are satisfied: $|\tilde{d}| \le \tilde{\delta_1}\tilde{n}$, $\tilde{L} \le \tilde{\delta_2}\Delta(\tilde{n})$, $\tilde{n} \ge \tilde{n_0}$, $\tilde{L} \ge \tilde{L_0}$. Let us now verify these conditions. From definition of $\mathcal{I}^*$ we get

$$|\tilde{d}| \le C_{29} t^{1/2} L \big(\log \Delta^{-1}(L)\big)^{\eta/2} ,$$

so that by calculations similar to (5.10) we get $|\tilde{d}| \le \tilde{\delta_1}\tilde{n}$. Combining this with (5.5) we get

$$\tilde{L}\frac{|\tilde{d}|}{\tilde{n}} \le C_{30} t^{1/2} \sigma(\Delta^{-1}(L))(\log L)^{-M} \tag{5.19}$$

Let $\tilde{t}$ be such that

$$C_{26}\tilde{L}\frac{|\tilde{d}|}{\tilde{n}} + \tilde{t}\sigma(\Delta^{-1}(\tilde{L}))\log \Delta^{-1}(\tilde{L}) = \frac{t}{4}\sigma(\Delta^{-1}(L))(\log L)^{\eta}. \tag{5.20}$$

Using (5.19), (A2), lower bound on $\tilde{L}$ from (5.5), value of M from (5.2), and (5.3), we get

$$\tilde{t} \geq C_{31} t \frac{\sigma(\Delta^{-1}(L))(\log L)^{\eta}}{\sigma(\Delta^{-1}(\tilde{L}))\log \Delta^{-1}(\tilde{L})} \geq C_{32} t (\log L)^{2\alpha M/(1+\beta)+\eta-1}$$
$$\geq C_{33} t (\log L)^{\eta} \geq C_{33} t_0 (\log L_0)^{\eta} .$$

So we have $\tilde{t} \geq \tilde{t_0}$, assuming t_0 and L_0 are large enough. Therefore, all the conditions for (5.18) are satisfied. Combining (5.18) with (5.20), we get

$$\mathbb{P}\left(\max_{\boldsymbol{b},\boldsymbol{b}' \in \mathcal{I}_j^*} \left| T(\boldsymbol{b}, \boldsymbol{a}_i) - T(\boldsymbol{b}', \boldsymbol{a}_i) \right| \geq \frac{t}{4} \sigma(\Delta^{-1}(L))(\log L)^{\eta} \right) \leq C_{34} \exp\left(-C_{35} t (\log L)^{1+\eta}\right) ,$$

The number of choices of i and j is at most $C_{36} t^{1/2} (\log L)^{2M+\eta/2}$. Hence using a union bound we get

$$\mathbb{P}\left(D^{[4]} \geq \frac{t}{4} \sigma(\Delta^{-1}(L))(\log L)^{\eta} \right) \leq C_{37} \exp\left(-C_{38} t (\log L)^{1+\eta}\right) . \tag{5.21}$$

Now we are going to consider the fourth term in the right-hand side of (5.8). Fix an i. To bound D_i we apply Corollary 3.3 with the following variables:

$$\tilde{n} \coloneqq n \, , \; \tilde{L} \coloneqq |\pi^2_{\theta_0, \theta_0^t}(\boldsymbol{a}_{i-1} - \boldsymbol{a}_i)| \, , \; \tilde{d} \coloneqq \pi^2_{\theta_0, \theta_0^t}(\boldsymbol{a}_i) \, .$$

By Corollary 3.3 we have for all $\tilde{t} \geq \tilde{t_0}$

$$\mathbb{P}\left(D_i \geq C_{39} \tilde{L} \frac{|\tilde{d}|}{\tilde{n}} + \tilde{t} \sigma(\tilde{k}) \log \tilde{k} \right) \leq C_{40} \exp\left(-C_{41} t \log \tilde{k}\right) , \tag{5.22}$$

provided the following conditions are satisfied: $|\tilde{d}| \leq \tilde{\delta_1} \tilde{n}$, $\tilde{L} \leq \tilde{\delta_2} \Delta(\tilde{n})$, $\tilde{n} \geq \tilde{n_0}$, $\tilde{L} \geq \tilde{L_0}$. From definition of $\mathcal{I}(n, L)$ we have $|\tilde{d}| \leq L$. Therefore,

$$\frac{|\tilde{d}|}{\tilde{n}} \leq \frac{L}{n} \leq \frac{\Delta(n)}{n} \leq C_{42} n_0^{-(1-\beta)/2} \leq \tilde{\delta_1} \, .$$

From (5.4) we get $\tilde{L} \leq L$. Further using $L \leq \Delta(n)$ we get

$$\tilde{L} \leq L \leq \Delta(n) \leq \tilde{\delta_2} \tilde{n} \, .$$

Also, using the bound on $\tilde{L}$ from (5.4) we get

$$\tilde{L} \frac{|\tilde{d}|}{\tilde{n}} \leq C_{43} L (\log L)^{-M} \frac{L}{n} \leq C_{44} \frac{L^2}{\Delta^{-1}(L)} (\log L)^{-M} \leq C_{45} \sigma(\Delta^{-1}(L))(\log L)^{-M} \, . \tag{5.23}$$

Let $\tilde{t}$ be such that

$$C_{39} \tilde{L} \frac{|\tilde{d}|}{\tilde{n}} + \tilde{t} \sigma(\Delta^{-1}(\tilde{L})) \log \Delta^{-1}(\tilde{L}) = \frac{t}{4} \sigma(\Delta^{-1}(L))(\log L)^{\eta} \, . \tag{5.24}$$

Therefore, using (5.23), bound on $\tilde{L}$ from (5.4), and value of M from (5.2), we get

$$\tilde{t} \geq C_{46} t \frac{\sigma(\Delta^{-1}(L))(\log L)^{\eta}}{\sigma(\Delta^{-1}(\tilde{L}) \log \Delta^{-1}(\tilde{L})} \geq C_{47} t (\log L)^{2\alpha M/(1+\beta)+\eta-1}$$
$$\geq C_{48} t (\log L)^{\eta} \geq C_{48} t_0 (\log L_0)^{\eta} \, .$$

So $\tilde{t} \geq \tilde{t_0}$ assuming t_0 and L_0 are large enough. Combining this with (5.22) and (5.24) we get

$$\mathbb{P}\Big(D_i \geq \frac{t}{4}\sigma(\Delta^{-1}(L))(\log L)^\eta\Big) \leq C_{49}\exp\big(-C_{50}t(\log L)^{1+\eta}\big) \ .$$

Using a union bound over values of i we get

$$\mathbb{P}\Big(\max_i D_i \geq t\sigma(\Delta^{-1}(L))(\log L)^\eta\Big) \leq C_{51}\exp\big(-C_{52}t(\log L)^{1+\eta}\big) \ . \tag{5.25}$$

Combining (5.11), (5.17), (5.21), (5.25), and (5.8), completes the proof of Theorem 1.23.

6 Lower bound on the variance of the transverse increments

In this section, we prove Theorem 1.24. In accordance with the statement of Theorem 1.24, we consider a direction θ_0 which is of both type I and type II. Due to the symmetry of the lattice, without loss of generality we assume $\theta_0 \in [0, \pi/4]$. We also fix a constant $\nu \in (1/2, 1)$ as in the statement of Theorem 1.24. In addition, we fix a constant $\eta \in (1/2, 1)$. Consider $n > 0$ and $L > 0$ satisfying $L \leq \Delta(n)$. Define k such that

$$L = \Delta(k)(\log k)^\eta \ . \tag{6.1}$$

Therefore, a lower limit of k yields lower limits of both n and L. Hence, we will state results which hold for large enough k, tacitly assuming n and L are also large enough so that the two relations $L \leq \Delta(n)$ and (6.1) hold. We will establish the lower bound on the variance of $T(\mathbf{0}, n\boldsymbol{e}_{\theta_0}) - T(\mathbf{0}, n\boldsymbol{e}_{\theta_0} + L\boldsymbol{e}_{\theta_0^t})$. The lower bound on the variance of $T(\mathbf{0}, n\boldsymbol{e}_{\theta_0}) - T(\mathbf{0}, n\boldsymbol{e}_{\theta_0} - L\boldsymbol{e}_{\theta_0^t})$ can be established in a similar manner. Let us introduce some more notations. We will use these throughout this section.

Notation 6.1. *Let $\boldsymbol{a} \coloneqq n\boldsymbol{e}_{\theta_0}$, $\boldsymbol{b} \coloneqq n\boldsymbol{e}_{\theta_0} + L\boldsymbol{e}_{\theta_0^t}$,*

$$\begin{aligned} h^* &\coloneqq \max\{\, h(\boldsymbol{x} - \boldsymbol{a}) \colon \boldsymbol{x} \in \mathbb{Z}^2 \,,\, \|\boldsymbol{x} - \boldsymbol{a}\| \leq k \,\} \ , \\ H &\coloneqq \{\, \boldsymbol{x} \in \mathbb{Z}^2 \colon h(\boldsymbol{x} - \boldsymbol{a}) \leq h^* \,\} \ , \\ \tau &\coloneqq \min\{\, T(\mathbf{0}, \boldsymbol{x}) \colon \boldsymbol{x} \in H \,\} \ , \\ F &\coloneqq \{\, \boldsymbol{x} \in \mathbb{Z}^2 \colon T(\mathbf{0}, \boldsymbol{x}) \leq \tau \,\} \ , \\ \partial H &\coloneqq \{\, \boldsymbol{x} \in H \colon \boldsymbol{x} \pm \boldsymbol{e}_i \in H^c \text{ for some } i = 1, 2 \,\} \ . \end{aligned}$$

So the set $H \subset \mathbb{Z}^2$ is the smallest h-ball around the point $\boldsymbol{a}$ which contains within itself the Euclidean ball (in $\mathbb{Z}^2$) of radius k around $\boldsymbol{a}$; the set ∂H is the vertex-boundary of H i.e., the set of vertices in H which are also adjacent to some vertex outside H; τ is the time required to reach H from the origin $\mathbf{0}$; the set F is the set of vertices that can be reached by time τ from $\mathbf{0}$ i.e., F is the wet region $\mathcal{B}(\tau)$. We emphasize the fact that H, F, ∂H are subsets of $\mathbb{Z}^2$ i.e., they contains lattice points.

We also define $\mathtt{E} \subset \mathtt{E}(\mathbb{Z}^2)$ to be the set of nearest-neighbor edges which have at least one endpoint in F, and we let $\mathcal{F}$ be the sigma-field generated by τ, F, and $\{\, \tau_e \colon e \in \mathtt{E} \,\}$.

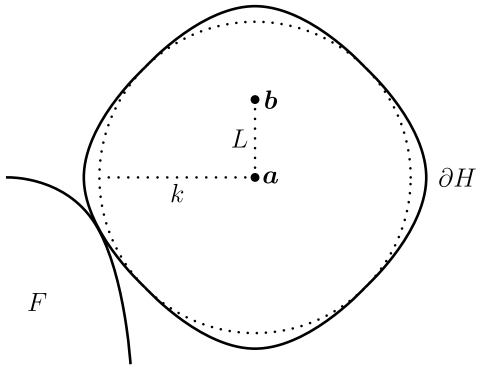

Figure 8: Illustration for $\boldsymbol{a}$, $\boldsymbol{b}$, H, F, ∂H: direction of $\boldsymbol{a}$ is θ_0, direction of $\boldsymbol{b}-\boldsymbol{a}$ is θ_0^t. We want to prove a lower bound of the variance of the transverse increment $T(\mathbf{0},\boldsymbol{a})-T(\mathbf{0},\boldsymbol{b})$.

Remark 6.2. Since h is sublinear and g is a norm, we have for $\|\boldsymbol{u}\| \geq C_1$,

$$C_2\|\boldsymbol{u}\| \leq h(\boldsymbol{u}) \leq C_3\|\boldsymbol{u}\| \ . \tag{6.2}$$

Therefore

$$C_4 k \leq h^* \leq C_5 k \ .$$

Every $\boldsymbol{y} \in \partial H$ has an adjacent vertex that does not belong to H. Therefore, for all $\boldsymbol{y} \in \partial H$ we have

$$h^* \geq h(\boldsymbol{y}-\boldsymbol{a}) \geq h^* - C_6 \ . \tag{6.3}$$

Combining (6.2)-(6.3) we get that for all $\boldsymbol{y} \in \partial H$,

$$C_7 k \leq \|\boldsymbol{y}-\boldsymbol{a}\| \leq C_8 k \ . \tag{6.4}$$

Therefore, H can be inscribed in a square whose sides are of the order of k around $\boldsymbol{a}$.

Remark 6.3. Using (A2), (6.1), and $L \leq \Delta(n)$, we get that for any $\delta > 0$, $k \leq \delta n$ for large enough k depending on δ. Therefore, using Remark 6.2, we get that $\mathbf{0}$ lies outside H. In this case, the region F touches the region H i.e., $F \cap H \subset \partial H$. Since the edge-weights are continuous, F touches H at only one point (almost surely). So we assume that k is large enough such that the origin $\mathbf{0}$ lies outside H.

Now we state three propositions, and then we will complete the proof of Theorem 1.24 using these propositions. We prove these propositions separately in later subsections.

Proposition 6.4. *Under the assumptions of Theorem 1.24, there exist constants $C_9 > 0$, $\epsilon_1 > 0$, such that for large enough k*

$$\mathbb{P}(\mathbb{P}(T(\mathbf{0},\boldsymbol{a}) \leq h^* + \tau - \epsilon_1\sigma(k)|\mathcal{F}) \leq \epsilon_1) \leq \exp\left(-k^{C_9}\right) \ .$$

By definition of τ (see Notation 6.1), $T(\mathbf{0},\boldsymbol{a})-\tau$ is an upper bound of the time it takes for $\Gamma(\mathbf{0},\boldsymbol{a})$ to exit H after starting from $\boldsymbol{a}$. Recall from (6.3) that h^* is approximately the expected passage time from $\boldsymbol{a}$ to any point on ∂H. Also recall from (6.4) that points of ∂H are at a distance of the order of k from $\boldsymbol{a}$. Therefore, Proposition 6.4 implies that, for most $\mathcal{F}$ (i.e., with probability $> 1-\exp\left(-k^{C_9}\right)$ the following happens given $\mathcal{F}$: there is a nonnegligible probability (i.e., with probability $> 1-\epsilon_1$) that the time taken for $\Gamma(\mathbf{0},\boldsymbol{a})$ to exit H starting from $\boldsymbol{a}$ is less than h^* by a fraction of $\sigma(k)$ i.e., the geodesic $\Gamma(\mathbf{0},\boldsymbol{a})$ is faster than usual before exiting H starting from $\boldsymbol{a}$ with nonnegligible probability.

Proposition 6.5. *Under the assumptions of Theorem 1.24, there exist constants* $\nu_1 \in (1/2, \nu)$, $\epsilon_2 > 0$, *such that for large enough* k

$$\mathbb{P}(T(\mathbf{0}, \boldsymbol{a}) \geq h^* + \tau + \epsilon_2 \sigma(k)) \geq \exp(-(\log k)^{\nu_1}) \ .$$

Proposition 6.5 implies that, roughly speaking, there is a nonnegligible probability (that is $\geq \exp(-(\log k)^{\nu_1})$) that the time taken by the geodesic $\Gamma(\mathbf{0}, \boldsymbol{a})$ to exit H starting from $\boldsymbol{a}$ is greater than h^* by a fraction of $\sigma(k)$. That is, the geodesic $\Gamma(\mathbf{0}, \boldsymbol{a})$ is slower than usual before exiting H starting from $\boldsymbol{a}$ with some nonnegligible probability.

Clearly Propositions 6.4 and 6.5 will yield a lower bound on the variance of $T(\mathbf{0}, \boldsymbol{a})$ given $\mathcal{F}$. Proposition 6.6 bounds the covariance of $T(\mathbf{0}, \boldsymbol{a})$ and $T(\mathbf{0}, \boldsymbol{b})$ given $\mathcal{F}$. Thus we will get a lower bound on the variance of $T(\mathbf{0}, \boldsymbol{a}) - T(\mathbf{0}, \boldsymbol{b})$ given $\mathcal{F}$.

Proposition 6.6. *Under the assumptions of Theorem 1.24, there exist a constant* $C_{10} > 0$, *such that for large enough* k

$$0 \leq \mathbb{E}[\mathrm{Cov}(T(\mathbf{0}, \boldsymbol{a}), T(\mathbf{0}, \boldsymbol{b})\mathcal{F})] \leq C_{10} \ .$$

Now we complete the proof of Theorem 1.24 using Propositions 6.4-6.6. Let $\epsilon_3 \coloneqq \min\{\epsilon_1, \epsilon_2\}$. Expanding the expectation of the conditional variance given $\mathcal{F}$, we get

$$\begin{aligned} &\mathrm{Var}(T(\mathbf{0}, \boldsymbol{a}) - T(\mathbf{0}, \boldsymbol{b})) \\ \geq\ &\mathbb{E}[\mathrm{Var}(T(\mathbf{0}, \boldsymbol{a})|\mathcal{F})] + \mathbb{E}[\mathrm{Var}(T(\mathbf{0}, \boldsymbol{b})|\mathcal{F})] - 2\,\mathbb{E}[\mathrm{Cov}(T(\mathbf{0}, \boldsymbol{a}), T(\mathbf{0}, \boldsymbol{b})|\mathcal{F})] \ . \end{aligned} \tag{6.5}$$

For any random variable X, $\mathrm{Var}(X) = \mathbb{E}\left(X - X'\right)^2/2$, where X' is another random variable with the same distribution as X and is independent of X. Therefore, for any random variable X and for any $a > b$,

$$\mathrm{Var}(X) \geq \frac{1}{2}(a - b)^2\, \mathbb{P}(X \geq a)\, \mathbb{P}(X \leq b) \ .$$

Thus

$$\begin{aligned} &\mathrm{Var}(T(\mathbf{0}, \boldsymbol{a})|\mathcal{F}) \\ &\geq C_{11}\sigma^2(k)\, \mathbb{P}(T(\mathbf{0}, \boldsymbol{a}) \geq h^* + \tau + \epsilon_3\sigma(k)|\mathcal{F})\, \mathbb{P}(T(\mathbf{0}, \boldsymbol{a}) \leq h^* + \tau - \epsilon_3\sigma(k)|\mathcal{F}) \ . \end{aligned} \tag{6.6}$$

As a shorthand notation let us use

$$\begin{aligned} X &\coloneqq \mathbb{P}(T(\mathbf{0}, \boldsymbol{a}) \geq h^* + \tau + \epsilon_3\sigma(k)|\mathcal{F}) \cdot \mathbb{P}(T(\mathbf{0}, \boldsymbol{a}) \leq h^* + \tau - \epsilon_3\sigma(k)|\mathcal{F}) \ , \\ Y &\coloneqq \mathbb{P}(T(\mathbf{0}, \boldsymbol{a}) \geq h^* + \tau + \epsilon_3\sigma(k)|\mathcal{F}) \cdot \epsilon_3 \ . \end{aligned}$$

Proposition 6.5 implies that

$$\mathbb{E}[Y] \geq \epsilon_3 \cdot \exp(-(\log k)^{\nu_1}) \ . \tag{6.7}$$

Furthermore, using Proposition 6.4 and $0 \leq X, Y \leq 1$, we get

$$\mathbb{E}\big[(Y - X)^+\big] \leq \mathbb{P}(Y \geq X) \leq \exp\!\big(-k^{C_9}\big) \ .$$

Therefore, using (6.7) and the inequality $\mathbb{E}[X] \geq \mathbb{E}[Y] - \mathbb{E}[(Y - X)^+]$, we get

$$\mathbb{E}[X] \geq \epsilon_3 \cdot \exp(-(\log k)^{\nu_1}) - \exp\!\big(-k^{C_9}\big) \geq C_{12} \exp(-(\log k)^{\nu_1}) \ .$$

Combining this with (6.6), we get

$$\mathbb{E}[\mathrm{Var}(T(\mathbf{0}, \boldsymbol{a})|\mathcal{F})] \geq C_{13}\sigma^2(k) \exp(-(\log k)^{\nu_1}) \ .$$

Therefore, by (6.5), Proposition 6.6, and (A2), we get

$$\mathrm{Var}(T(\mathbf{0},\boldsymbol{a})-T(\mathbf{0},\boldsymbol{b}))\geq C_{14}\sigma^2(k)\exp(-(\log k)^{\nu_1})\ .$$

Therefore, using (6.1), $\nu>\nu_1$, and (A2), we get

$$\mathrm{Var}(T(\mathbf{0},\boldsymbol{a})-T(\mathbf{0},\boldsymbol{b}))\geq \sigma^2\big(\Delta^{-1}(L)\big)\exp(-(\log L)^{\nu})\ .$$

This completes the proof of Theorem 1.24 using Propositions 6.4-6.6. Now we prove these propositions.

6.1 Proof of Proposition 6.4

Recall the definitions of $\mathcal{F}$, τ, and $\mathtt{E}$ from Notation 6.1. Conditioned on $\mathcal{F}$, consider i.i.d. random variables $\{\,\tau'_e\colon e\in\mathtt{E}\,\}$ each having distribution of the original edge-weights. For a path γ let

$$T'(\gamma)\coloneqq\sum_{\substack{\gamma\text{ contains }e\\ \text{and }e\in\mathtt{E}}}\tau'_e+\sum_{\substack{\gamma\text{ contains }e\\ \text{and }e\in\mathtt{E}^c}}\tau_e\ .$$

For any two points $\boldsymbol{y},\boldsymbol{z}\in\mathbb{R}^2$, let

$$T'(\boldsymbol{y},\boldsymbol{z})\coloneqq\inf\{\,T'(\gamma)\colon\gamma\text{ is a path from }\boldsymbol{y}\text{ to }\boldsymbol{z}\,\}\ .$$

Therefore, the conditional distribution of all the passage times $\{\,T'(\boldsymbol{y},\boldsymbol{z})\colon\boldsymbol{y},\boldsymbol{z}\in\mathbb{R}^2\,\}$ given $\mathcal{F}$ is same as the unconditional distribution of all the passage times $\{\,T(\boldsymbol{y},\boldsymbol{z})\colon\boldsymbol{y},\boldsymbol{z}\in\mathbb{R}^2\,\}$. For all $\boldsymbol{y},\boldsymbol{z}\in\mathbb{R}^2$ let $\Gamma'(\boldsymbol{y},\boldsymbol{z})$ be the geodesic corresponding to $T'(\boldsymbol{y},\boldsymbol{z})$. Let $\boldsymbol{x}$ be the point where F touches H. Let $\boldsymbol{u}$ be the first point belonging to F when the geodesic $\Gamma'(\boldsymbol{a},\boldsymbol{x})$ is traced starting from $\boldsymbol{a}$, see Figure 9.

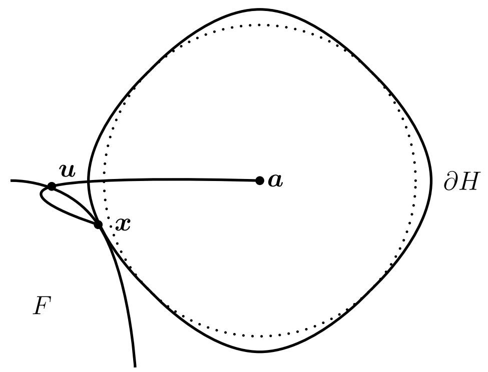

Figure 9: Setup for Proposition 6.4: given a realization of the edge-weights on the whole lattice, we take another configuration on $\mathtt{E}$, which is the set of edges having at least one endpoint in F. The geodesic $\Gamma'(\boldsymbol{a},\boldsymbol{x})$ is then constructed in the environment where we have the new edge-weight configuration on $\mathtt{E}$ and the original realization of edge-weights in $\mathtt{E}^c$.

Since $\boldsymbol{u}\in F$, we have $T(\mathbf{0},\boldsymbol{u})\leq\tau$. Let $\boldsymbol{w}$ be the lattice point on $\Gamma'(\boldsymbol{a},\boldsymbol{x})$ preceding $\boldsymbol{u}$ when $\Gamma'(\boldsymbol{a},\boldsymbol{x})$ is traced starting from $\boldsymbol{a}$. Then $\Gamma'(\boldsymbol{a},\boldsymbol{w})$ consists of edges only in $\mathtt{E}^c$, and hence $T(\boldsymbol{a},\boldsymbol{w})\leq T'(\boldsymbol{a},\boldsymbol{w})$. Therefore,

$$T(\boldsymbol{a},\boldsymbol{u})\leq T(\boldsymbol{a},\boldsymbol{w})+T(\boldsymbol{w},\boldsymbol{u})\leq T'(\boldsymbol{a},\boldsymbol{w})+T(\boldsymbol{w},\boldsymbol{u})\ .$$

For any $\boldsymbol{v} \in \mathbb{Z}^2$ let

$$d(\boldsymbol{v}) := \max\{\, T(\boldsymbol{v}, \boldsymbol{v} \pm \boldsymbol{e}_i) \colon i = 1, 2 \,\} \,.$$

So $T(\boldsymbol{w}, \boldsymbol{u}) \le d(\boldsymbol{u})$, and hence $T(\boldsymbol{u}, \boldsymbol{a}) \le d(\boldsymbol{u}) + T'(\boldsymbol{u}, \boldsymbol{a})$. Therefore,

$$T(\boldsymbol{0}, \boldsymbol{a}) \le T(\boldsymbol{0}, \boldsymbol{u}) + T(\boldsymbol{u}, \boldsymbol{a}) \le \tau + d(\boldsymbol{u}) + T'(\boldsymbol{u}, \boldsymbol{a}) \le \tau + d(\boldsymbol{u}) + T'(\boldsymbol{x}, \boldsymbol{a}) \,. \tag{6.8}$$

By (A3) we get there exists $\epsilon_4 > 0$ such that

$$\mathbb{P}(T'(\boldsymbol{x}, \boldsymbol{a}) \le h(\boldsymbol{x} - \boldsymbol{a}) - \epsilon_4 \sigma(\|\boldsymbol{x} - \boldsymbol{a}\|) | \mathcal{F}) \ge \epsilon_4 \,. \tag{6.9}$$

Since F touches H at $\boldsymbol{x}$, we have $\boldsymbol{x} \in \partial H$. Hence, $h(\boldsymbol{x} - \boldsymbol{a}) \le h^*$ and $\|\boldsymbol{x} - \boldsymbol{a}\| \ge k$. Therefore, by (A2), (6.8), and (6.9), we get for some $\epsilon_5 > 0$

$$\begin{aligned} \epsilon_5 \le\ & \mathbb{P}(T'(\boldsymbol{x}, \boldsymbol{a}) \le h^* - \epsilon_5 \sigma(k) | \mathcal{F}) \\ \le\ & \mathbb{P}\Big(T(\boldsymbol{0}, \boldsymbol{a}) \le h^* + \tau - \frac{\epsilon_5}{2} \sigma(k) | \mathcal{F}\Big) + \mathbb{P}\Big(d(\boldsymbol{u}) \ge \frac{\epsilon_5}{2} \sigma(k) | \mathcal{F}\Big) \,. \end{aligned}$$

Therefore, taking $\epsilon_6 := \epsilon_5/2$, we get

$$\begin{aligned} & \mathbb{P}(\mathbb{P}(T(\boldsymbol{0}, \boldsymbol{a}) \le h^* + \tau - \epsilon_6 \sigma(k) | \mathcal{F}) \le \epsilon_6) \\ \le\ & \mathbb{P}(\mathbb{P}(d(\boldsymbol{u}) \ge \epsilon_6 \sigma(k) | \mathcal{F}) \ge \epsilon_6) \\ \le\ & \epsilon_6^{-1}\, \mathbb{P}(d(\boldsymbol{u}) \ge \epsilon_6 \sigma(k)) \,. \end{aligned} \tag{6.10}$$

Since $\boldsymbol{x} \in \partial H$, by Remark 6.2 we get $\|\boldsymbol{x} - \boldsymbol{a}\| \le C_{15} k$. Since $\boldsymbol{u}$ lies on the geodesic $\Gamma'(\boldsymbol{a}, \boldsymbol{x})$, by Lemma 2.2 we get

$$\mathbb{P}(\|\boldsymbol{u} - \boldsymbol{a}\| \ge C_{16} k | \mathcal{F}) \le \exp(-C_{17} k) \,.$$

Therefore,

$$\mathbb{P}(\|\boldsymbol{u} - \boldsymbol{a}\| \ge C_{16} k) \le \exp(-C_{17} k) \,. \tag{6.11}$$

Since edge-weights have exponential moments, for any $\boldsymbol{v}$ and all $t > 0$ we have

$$\mathbb{P}(d(\boldsymbol{v}) \ge t) \le C_{18} \exp(-C_{19} t) \,. \tag{6.12}$$

Therefore, using (6.11) and (6.12), we get

$$\mathbb{P}(d(\boldsymbol{u}) \ge \epsilon_6 \sigma(k)) \le C_{20} k^2 \exp(-C_{21} \sigma(k)) \,.$$

Combining this with (6.10) and using (A2) we get

$$\mathbb{P}(\mathbb{P}(T(\boldsymbol{0}, \boldsymbol{a}) \le h^* + \tau - \epsilon_6 \sigma(k) | \mathcal{F}) \le \epsilon_6) \le \epsilon_6^{-1} C_{20} k^2 \exp(-C_{21} \sigma(k)) \le \exp\big(-k^{C_9}\big) \,.$$

This completes the proof of Proposition 6.4.

6.2 Proof of Proposition 6.5

We begin with an outline of the proof. Recall from Notation 6.1 that τ is the passage time from the origin to H, and h^* is the maximum average passage time from $\boldsymbol{a}$ to any point on ∂H. By definition, H contains an Euclidean ball of size k around $\boldsymbol{a}$, and by Remark 6.2 H is contained in a ball of radius of the order of k around $\boldsymbol{a}$. Thus, $\sigma(k)$ is the order of fluctuation of passage times from $\boldsymbol{a}$ to any point on ∂H. Therefore, to prove Proposition 6.5, it suffices to show that the time it takes for $\Gamma(\boldsymbol{0}, \boldsymbol{a})$ to reach ∂H after starting from $\boldsymbol{a}$ is slower than h^* by a fraction of $\sigma(k)$ with a non-negligible probability. Here, by a non-negligible probability we mean probability at least $\exp(-(\log k)^{\epsilon})$ for some $\epsilon \in (1/2, \nu)$. To prove this, we define H^*, a subset of ∂H, such that with high probability $\Gamma(\mathbf{0}, \boldsymbol{a})$ does not intersect $\partial H \setminus H^*$. Then, we show that passage times from $\boldsymbol{a}$ to the points of H^* can be uniformly slow with non-negligible probability. To achieve this, we further define G^*, a polygonal line which is roughly a discrete approximation of a sector of a g-ball around $\boldsymbol{a}$. We show that passage times from $\boldsymbol{a}$ to points on G^* are uniformly slow with non-negligible probability, and passage time from G^* to H^* are sufficiently small, so that passage times from $\boldsymbol{a}$ to points of H^* are also slow. Moreover, G^* is constructed in a way so that $\Gamma(\mathbf{0}, \boldsymbol{a})$, when traced from starting from $\boldsymbol{a}$, intersect G^* before H^* with high probability.

Let us now begin the proof formally. By Remark 6.2, the maximum distance in the direction $-\theta_0$ from $\boldsymbol{a}$ of a point in ∂H is at most of the order of k, i.e.,

$$\max_{\boldsymbol{x} \in \partial H} \pi^1_{\theta_0, \theta_0^t}(\boldsymbol{a} - \boldsymbol{x}) \le C_{22} k \; . \tag{6.13}$$

We do not need to use any absolute value in the above equation because we have assumed $\theta_0 \in [0, \pi/4]$. Using Corollary 2.11 and (A2), we have for all $\boldsymbol{x}$ on $\Gamma(\mathbf{0}, \boldsymbol{a})$ with $\pi^1_{\theta_0, \theta_0^t}(\boldsymbol{a} - \boldsymbol{x}) \le C_{22} k$

$$\left| \pi^2_{\theta_0, \theta_0^t}(\boldsymbol{a} - \boldsymbol{x}) \right| \le C_{23} \Delta(k) (\log k)^{1/2} \; ,$$

with probability at least $1 - e^{-C_{24} \log k}$. Thus, defining the event

$$\mathcal{E} \coloneqq \Big\{ \Gamma(\mathbf{0}, \boldsymbol{a}) \text{ does not wander more than } C_{23} \Delta(k) (\log k)^{1/2} \text{ in } \pm \theta_0^t \text{ directions} \\ \text{before exiting } H \text{ when traced starting from } \boldsymbol{a} \Big\} \; ,$$

we get

$$\mathbb{P}(\, \mathcal{E} \,) \ge 1 - \exp(-C_{24} \log k) \; . \tag{6.14}$$

This motivates us to define H^* as the portion of ∂H, facing towards the origin i.e., towards direction $-\theta_0$ from $\boldsymbol{a}$, having width $2C_{23}\Delta(k)(\log k)^{1/2}$ in θ_0^t direction, i.e.,

$$H^* \coloneqq \left\{ \boldsymbol{x} \in \partial H \colon \left| \pi^2_{\theta_0, \theta_0^t}(\boldsymbol{a} - \boldsymbol{x}) \right| \le C_{23} \Delta(k) (\log k)^{1/2} \, , \, \pi^1_{\theta_0, \theta_0^t}(\boldsymbol{a} - \boldsymbol{x}) \ge 0 \right\} . \tag{6.15}$$

Thus, (6.14) implies that the geodesic $\Gamma(\mathbf{0}, \boldsymbol{a})$ does not pass through any point in the set $\partial H \setminus H^*$ with probability at least $1 - \exp(-C_{24} \log k)$. Now we establish a bound on the width of H^*.

Lemma 6.7. *Under the assumptions of Theorem 1.24 we have for large enough* k

$$\min_{\boldsymbol{x} \in \partial H} \pi^1_{\theta_0, \theta_0^t}(\boldsymbol{a} - \boldsymbol{x}) \ge C_{25} k \; , \tag{6.16}$$

and

$$\max_{\boldsymbol{x} \in H^*} \pi^1_{\theta_0, \theta_0^t}(\boldsymbol{a} - \boldsymbol{x}) - \min_{\boldsymbol{x} \in H^*} \pi^1_{\theta_0, \theta_0^t}(\boldsymbol{a} - \boldsymbol{x}) \le C_{26} \Delta(k) (\log k)^{1/2} \; . \tag{6.17}$$

Proof. Consider $\boldsymbol{x} \in H^*$. Recall the definition of $g_{\boldsymbol{a}}$ from Notation 4.1. Then, using that g is a norm, we get

$$\begin{aligned} g(\boldsymbol{a} - \boldsymbol{x}) &= g(\pi^1_{\theta_0, \theta_0^t}(\boldsymbol{a} - \boldsymbol{x}) \boldsymbol{e}_{\theta_0} + \pi^2_{\theta_0, \theta_0^t}(\boldsymbol{a} - \boldsymbol{x}) \boldsymbol{e}_{\theta_0^t}) \\ &\le g(\pi^1_{\theta_0, \theta_0^t}(\boldsymbol{a} - \boldsymbol{x}) \boldsymbol{e}_{\theta_0}) + g(\pi^2_{\theta_0, \theta_0^t}(\boldsymbol{a} - \boldsymbol{x}) \boldsymbol{e}_{\theta_0^t}) \\ &= g_{\boldsymbol{a}}(\boldsymbol{a} - \boldsymbol{x}) + g(\pi^2_{\theta_0, \theta_0^t}(\boldsymbol{a} - \boldsymbol{x}) \boldsymbol{e}_{\theta_0^t}) \\ &\le g_{\boldsymbol{a}}(\boldsymbol{a} - \boldsymbol{x}) + C_{27} |\pi^2_{\theta_0, \theta_0^t}(\boldsymbol{a} - \boldsymbol{x}) \boldsymbol{e}_{\theta_0^t}| \\ &= g_{\boldsymbol{a}}(\boldsymbol{a} - \boldsymbol{x}) + C_{27} |\pi^2_{\theta_0, \theta_0^t}(\boldsymbol{a} - \boldsymbol{x})| \; . \end{aligned}$$

Therefore, using (6.15), we get

$$g_{\boldsymbol{a}}(\boldsymbol{a}-\boldsymbol{x}) \geq g(\boldsymbol{a}-\boldsymbol{x}) - C_{27}|\pi^2_{\theta_0,\theta_0^t}(\boldsymbol{a}-\boldsymbol{x})| \geq g(\boldsymbol{a}-\boldsymbol{x}) - C_{28}\Delta(k)(\log k)^{1/2} \ . \tag{6.18}$$

Using Proposition 2.6, (6.3), (6.4), and (A2), we get

$$g(\boldsymbol{a}-\boldsymbol{x}) \geq h^* - C_{29}\sigma(k)\log k \ . \tag{6.19}$$

Combining (6.18) and (6.19) we get

$$g_{\boldsymbol{a}}(\boldsymbol{a}-\boldsymbol{x}) \geq h^* - C_{30}\Delta(k)(\log k)^{1/2} \ . \tag{6.20}$$

Using (6.3) and Remark 1.3, we get

$$g_{\boldsymbol{a}}(\boldsymbol{a}-\boldsymbol{x}) \leq g(\boldsymbol{a}-\boldsymbol{x}) \leq h(\boldsymbol{a}-\boldsymbol{x}) + C_{31} \leq h^* + C_{31} \ . \tag{6.21}$$

Combining (6.20) and (6.21) we get

$$\max_{\boldsymbol{x}\in H^*} g_{\boldsymbol{a}}(\boldsymbol{a}-\boldsymbol{x}) - \min_{\boldsymbol{x}\in H^*} g_{\boldsymbol{a}}(\boldsymbol{a}-\boldsymbol{x}) \leq C_{32}\Delta(k)(\log k)^{1/2} \ .$$

This establishes (6.17) because $g_{\boldsymbol{a}}(\boldsymbol{a}-\boldsymbol{x})$ is proportional to $\pi^1_{\theta_0,\theta_0^t}(\boldsymbol{a}-\boldsymbol{x})$. Combining (6.13) and (6.17) we get (6.16). □

Construction and properties of G^*: We denote the vertices of G^* by $(\boldsymbol{b}_i)_{i=-N_2}^{N_1}$. For $-N_2 \leq i < N_1$ we denote by G_i the segment joining $\boldsymbol{b}_i$ and $\boldsymbol{b}_{i+1}$. For each $i \neq 0$, we denote the direction of $\boldsymbol{a}-\boldsymbol{b}_i$ by θ_i. We will define $\boldsymbol{b}_0$ in such a way that $\boldsymbol{a}-\boldsymbol{b}_0$ has direction θ_0. So for each i the direction of $\boldsymbol{a}-\boldsymbol{b}_i$ is θ_i. Recall that we have assumed $\theta_0 \in [0,\pi/4]$. Thus, by our convention of orienting tangents in the counter-clockwise direction (see Remark 1.15) we have $\theta_0^t > 0$. We construct G^* satisfying the following properties:

- **(1)** $\boldsymbol{b}_0$ is situated on the line joining $\mathbf{0}$ and $\boldsymbol{a}$;
- **(2)** the points $\{\boldsymbol{b}_i : 0 < i \leq N_1\}$ are above the line joining $\mathbf{0}$ and $\boldsymbol{a}$;
- **(3)** the points $\{\boldsymbol{b}_i : -N_2 \leq i < 0\}$ are below the line joining $\mathbf{0}$ and $\boldsymbol{a}$;
- **(4)** θ_is are in a small neighborhood of θ_0, say $[\theta_0-\delta,\theta_0+\delta]$, so that θ_i^t exists for all i;
- **(5)** for $0 \leq i < N_1$, direction of $\boldsymbol{b}_{i+1}-\boldsymbol{b}_i$ is θ_i^t;
- **(6)** for $-N_2 < i \leq 0$ direction of $\boldsymbol{b}_{i-1}-\boldsymbol{b}_i$ is $-\theta_i^t$;
- **(7)** width of G^* in $\pm\theta_0^t$ direction is same as that of H^*;
- **(8)** total number of segments of G^* is of the order of $(\log k)^{\nu_2}$ for some $\nu_2 > 0$;
- **(9)** length of the segments G_i for $-N_2 < i < N_1 - 1$ are same, denoted by ℓ; the segments G_{-N_2} and G_{N_1-1} have lengths $\leq \ell$.

Let us now proceed with the construction. Using Remark 1.19, we choose a $\delta > 0$ such that the limit shape boundary is differentiable in the sector $[\theta_0-\delta,\theta_0+\delta]$, and for θ belonging to this sector, we have

$$\left|\theta^t - \theta_0^t\right| \leq C_{33}|\theta-\theta_0| \ . \tag{6.22}$$

Let ν_1, ν_2 be constants such that

$$\frac{1}{2} < \nu_2 < \nu_1 < \nu \ . \tag{6.23}$$

Let

$$\ell \coloneqq \frac{\Delta(k)(\log k)^{1/2}}{(\log k)^{\nu_2}} \ . \tag{6.24}$$

The point $\boldsymbol{b}_0$ is defined by the conditions:

$$\pi^2_{\theta_0,\theta_0^t}(\boldsymbol{b}_0) = 0\,, \quad \pi^1_{\theta_0,\theta_0^t}(\boldsymbol{b}_0) = \max_{y\in H^*} \pi^1_{\theta_0,\theta_0^t}(\boldsymbol{y}) \ . \tag{6.25}$$

We construct $(\boldsymbol{b}_i)_{i=0}^{N_1}$ inductively as follows. Suppose for some $j \geq 0$ we have defined $(\boldsymbol{b}_0, \ldots, \boldsymbol{b}_j)$. Further assume that $(\boldsymbol{b}_0, \ldots, \boldsymbol{b}_j)$ satisfies the conditions:

- **(i)** $|\theta_i - \theta_0| \leq \delta$ for all $0 \leq i \leq j$;
- **(ii)** $\pi^2_{\theta_0,\theta_0^t}(\boldsymbol{b}_j) < C_{23}\Delta(k)(\log k)^{1/2}$;
- **(iii)** direction of $\boldsymbol{b}_{i+1} - \boldsymbol{b}_i$ is θ_i^t for all $0 \leq i < j$.

Recall C_{23} is the constant used to define H^* in (6.15). Due to convexity of $\partial\mathcal{B}$ and due to our convention of orienting tangents in the counter-clockwise direction, we have $\pi^1_{\theta_0,\theta_0^t}(\boldsymbol{b}_{i+1} - \boldsymbol{b}_i) \geq 0$ and $\pi^2_{\theta_0,\theta_0^t}(\boldsymbol{b}_{i+1} - \boldsymbol{b}_i) \geq 0$ for all $0 \leq i < j$. We construct $\boldsymbol{b}_{j+1}$ as follows. Let $\boldsymbol{b}'_j$ be the point such that direction of $\boldsymbol{b}'_j - \boldsymbol{b}_j$ is θ_j^t, $\left\|\boldsymbol{b}'_j - \boldsymbol{b}_j\right\| = \ell$. We define $\boldsymbol{b}_{j+1}$ to be $\boldsymbol{b}'_j$, if $\pi^2_{\theta_0,\theta_0^t}(\boldsymbol{b}'_j) \leq C_{23}\Delta(k)(\log k)^{1/2}$. Otherwise, we take $\boldsymbol{b}_{j+1}$ to be the point $\boldsymbol{b}''_j$ on the line joining $\boldsymbol{b}_j$ and $\boldsymbol{b}'_j$ which satisfies $\pi^2_{\theta_0,\theta_0^t}(\boldsymbol{b}''_j) = C_{23}\Delta(k)(\log k)^{1/2}$, and end the construction. In (ii) above, the inequality is strong, because if we have an equality, then we do not proceed with the construction. To establish that this construction is well-defined, it suffices to show that $|\theta_{j+1} - \theta_0| \leq \delta$, where θ_{j+1} is the direction of $\boldsymbol{a} - \boldsymbol{b}_{j+1}$. Assuming δ is small enough and using (6.22) we get for all $0 \leq i \leq j$

$$\pi^1_{\theta_0,\theta_0^t}(\boldsymbol{b}_{i+1} - \boldsymbol{b}_i) = \|\boldsymbol{b}_{i+1} - \boldsymbol{b}_i\| \frac{|\sin(\theta_i^t - \theta_0^t)|}{|\sin(\theta_0^t - \theta_0)|} \leq C_{34}\delta\|\boldsymbol{b}_{i+1} - \boldsymbol{b}_i\| \,, \tag{6.26}$$

and

$$\pi^2_{\theta_0,\theta_0^t}(\boldsymbol{b}_{i+1} - \boldsymbol{b}_i) = \|\boldsymbol{b}_{i+1} - \boldsymbol{b}_i\| \frac{|\sin(\theta_i^t - \theta_0)|}{|\sin(\theta_0^t - \theta_0)|} \geq C_{35}\|\boldsymbol{b}_{i+1} - \boldsymbol{b}_i\| \ . \tag{6.27}$$

By construction we have

$$\pi^2_{\theta_0,\theta_0^t}(\boldsymbol{b}_{j+1} - \boldsymbol{a}) = \pi^2_{\theta_0,\theta_0^t}(\boldsymbol{b}_{j+1} - \boldsymbol{b}_0) = \pi^2_{\theta_0,\theta_0^t}(\boldsymbol{b}_{j+1}) \leq C_{23}\Delta(k)(\log k)^{1/2} \ . \tag{6.28}$$

Taking sum over $0 \leq i \leq j$ in (6.26) and (6.27), and using (6.28) we get

$$\pi^1_{\theta_0,\theta_0^t}(\boldsymbol{b}_{j+1} - \boldsymbol{b}_0) \leq C_{36}\delta\pi^2_{\theta_0,\theta_0^t}(\boldsymbol{b}_{j+1} - \boldsymbol{b}_0) \leq C_{37}\delta\Delta(k)(\log k)^{1/2} \ . \tag{6.29}$$

Therefore, using (6.16) and (6.25), we get

$$\pi^1_{\theta_0,\theta_0^t}(\boldsymbol{a} - \boldsymbol{b}_{j+1}) \geq C_{38}k \ . \tag{6.30}$$

Combining this with (6.28) and using (A2) we get $|\theta_{j+1} - \theta_0| \leq \delta$ for large enough k. This shows that the construction is well-defined. In a similar way we construct $\{\, \boldsymbol{b}_i \colon -N_2 \leq i < 0 \,\}$. We require $\pi^2_{\theta_0,\theta_0^t}(\boldsymbol{b}_i)$ decreases from 0 to $-C_{23}\Delta(k)(\log k)^{1/2}$ as i runs from 0 to $-N_2$. Equations (6.28) and (6.30) also yield for all $\boldsymbol{x}$ in the part of G^* joining $\boldsymbol{b}_0$ and $\boldsymbol{b}_{N_1}$

$$C_{39}k \leq \|\boldsymbol{a} - \boldsymbol{x}\| \leq C_{40}k \ , \tag{6.31}$$

but the same holds for all $\boldsymbol{x}$ in the part joining $\boldsymbol{b}_0$ and $\boldsymbol{b}_{-N_2}$. By (6.27), (6.28), and (6.24) we get $N_1 \leq C_{41}(\log k)^{\nu_2}$, and the same is true for N_2. Hence the total number of sides of G^* is bounded as

$$N \coloneqq N_1 + N_2 \leq C_{42}(\log k)^{\nu_2} \ . \tag{6.32}$$

By (6.29), width of the part of G^* joining $\boldsymbol{b}_0$ and $\boldsymbol{b}_{N_1}$ is at most $C_{43}\Delta(k)(\log k)^{1/2}$ in θ_0 direction, and the same holds for the part joining $\boldsymbol{b}_0$ and $\boldsymbol{b}_{-N_2}$. Therefore, for all $\boldsymbol{x}, \boldsymbol{y} \in G^*$

$$|\pi^1_{\theta_0,\theta_0^t}(\boldsymbol{x}-\boldsymbol{y})| \le C_{44}\Delta(k)(\log k)^{1/2} \ . \tag{6.33}$$

This ends the discussion on construction and properties of G^*. We now state two propositions and complete the proof of Proposition 6.5 using them. Then we proceed to prove these propositions.

Proposition 6.8. *Under the assumptions of Theorem 1.24, there exists $\epsilon_7 > 0$ such that for all large enough k we have*

$$\mathbb{P}(T(\boldsymbol{x},\boldsymbol{a}) \ge h(\boldsymbol{x}-\boldsymbol{a}) + \epsilon_7\sigma(k) \text{ for all } \boldsymbol{x} \in G^*) \ge \exp(-C_{45}(\log k)^{\nu_2}) \ .$$

Proposition 6.9. *Under the assumptions of Theorem 1.24, we have*

$$\mathbb{P}\Big(|T(\boldsymbol{x},\boldsymbol{y}) - h(\boldsymbol{x}-\boldsymbol{y})| \ge \frac{\epsilon_7}{2}\sigma(k) \text{ for some } \boldsymbol{x} \in H^*, \boldsymbol{y} \in G^*\Big) \le \exp\big(-k^{C_{46}}\big) \ ,$$

where ϵ_7 is the constant from Proposition 6.8.

Let us now complete proof of Proposition 6.5 using these propositions. Define the event

$$\mathcal{E}_1 \coloneqq \{\, \Gamma(\mathbf{0},\boldsymbol{a}), \text{ when traced from } \boldsymbol{a} \text{ to } \mathbf{0}, \text{ first intersects } G^* \text{ and then } H^* \,\} \ .$$

Since $\mathcal{E}_1 \subset \mathcal{E}$, we have using (6.14)

$$\mathbb{P}(\, \mathcal{E}_1^c \,) \le \exp(-C_{24}\log k) \ . \tag{6.34}$$

Define the events

$$\mathcal{E}_2 \coloneqq \{\, T(\boldsymbol{x},\boldsymbol{a}) \ge h(\boldsymbol{x}-\boldsymbol{a}) + \epsilon_8\sigma(k) \text{ for all } \boldsymbol{x} \in G^* \,\} \ ,$$

and

$$\mathcal{E}_3 \coloneqq \Big\{\, |T(\boldsymbol{x},\boldsymbol{y}) - h(\boldsymbol{x}-\boldsymbol{y})| \le \frac{\epsilon_8}{2}\sigma(k) \text{ for all } \boldsymbol{x} \in H^* \ , \ \boldsymbol{y} \in G^* \,\Big\} \ .$$

Using (6.34), Proposition 6.8, and Proposition 6.9, and (6.23), we get

$$\mathbb{P}(\, \mathcal{E}_1 \cap \mathcal{E}_2 \cap \mathcal{E}_3 \,) \ge \exp(-(\log k)^{\nu_1}) \ .$$

So let us suppose $\mathbb{T} \in \mathcal{E}_1 \cap \mathcal{E}_2 \cap \mathcal{E}_3$.

Since $\mathbb{T} \in \mathcal{E}_1$, there exist points $\boldsymbol{w}^* \in G^*$ and $\boldsymbol{v}^* \in H^*$, both situated on the geodesic $\Gamma(\mathbf{0},\boldsymbol{a})$, such that

$$T(\boldsymbol{a},\boldsymbol{v}^*) = T(\boldsymbol{a},\boldsymbol{w}^*) + T(\boldsymbol{w}^*,\boldsymbol{v}^*) \ , \tag{6.35}$$

see Figure 10. Since $\boldsymbol{v}^* \in H^* \subset \partial H$, we have

$$T(\mathbf{0},\boldsymbol{v}^*) \ge \tau \ . \tag{6.36}$$

Combining (6.35) and (6.36) we get

$$T(\mathbf{0},\boldsymbol{a}) = T(\mathbf{0},\boldsymbol{v}^*) + T(\boldsymbol{v}^*,\boldsymbol{a}) \ge \tau + T(\boldsymbol{a},\boldsymbol{w}^*) + T(\boldsymbol{w}^*,\boldsymbol{v}^*) \ . \tag{6.37}$$

From $\mathbb{T} \in \mathcal{E}_2 \cap \mathcal{E}_3$ we get

$$\begin{aligned} T(\boldsymbol{a},\boldsymbol{w}^*) + T(\boldsymbol{w}^*,\boldsymbol{v}^*) &\ge h(\boldsymbol{w}^*-\boldsymbol{a}) + \epsilon_8\sigma(k) + h(\boldsymbol{v}^*-\boldsymbol{w}^*) - \frac{\epsilon_8}{2}\sigma(k) \\ &\ge h(\boldsymbol{v}^*-\boldsymbol{a}) + \frac{\epsilon_8}{2}\sigma(k) - C_{47} \ , \end{aligned}$$

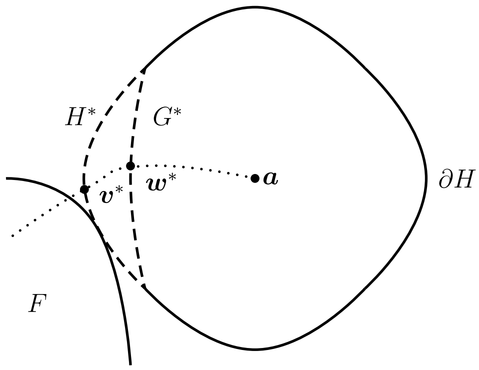

Figure 10: Setup of Proposition 6.5: we show that (i) the geodesic $\Gamma(\mathbf{0}, \boldsymbol{a})$ when traced starting from $\boldsymbol{a}$ intersects first G^* and then H^* with high probability; (ii) passage time from $\boldsymbol{a}$ to points in G^* can be large with non-negligible probability; (iii) passage time between points in G^* and points in H^* are not too small with high probability. So passage time from $\boldsymbol{a}$ to G^* can be large with non-negligible probability.

where we get the extra constant at the end by Remark 1.3. Since $\boldsymbol{v}^*$ is on ∂H, from (6.3) we have

$$h(\boldsymbol{v}^* - \boldsymbol{a}) \geq h^* - C_{48} \ . \tag{6.38}$$

For large enough k we have

$$\frac{\epsilon_8}{2}\sigma(k) - C_{47} - C_{48} \geq \frac{\epsilon_8}{4}\sigma(k) \ . \tag{6.39}$$

Therefore, combining (6.37)-(6.39) we get

$$T(\mathbf{0}, \boldsymbol{a}) \geq \tau + h^* + \frac{\epsilon_8}{4}\sigma(k) \ .$$

Letting $\epsilon_2 \coloneqq \epsilon_8/4$ completes proof of Proposition 6.5. Let us now prove Propositions 6.8 and 6.9.

6.2.1 Proof of Proposition 6.8

Recall G_i is the segment of G^* joining $\boldsymbol{b}_i$ and $\boldsymbol{b}_{i+1}$ for all $-N_2 \leq i < N_1$. For all i, let $\boldsymbol{a}_i$ be the point on G_i which has the maximum expected passage time from $\boldsymbol{a}$. By (A3), there exists $\epsilon_8 > 0$ such that for each i,

$$\mathbb{P}(T(\boldsymbol{a}, \boldsymbol{a}_i) \geq h(\boldsymbol{a}_i - \boldsymbol{a}) + \epsilon_8\sigma(\|\boldsymbol{a}_i - \boldsymbol{a}\|)) \geq \epsilon_8 \ .$$

Therefore, using (6.31) and (A2) we get, for some $\epsilon_9 > 0$ and for all i,

$$\mathbb{P}(T(\boldsymbol{a}, \boldsymbol{a}_i) \geq h(\boldsymbol{a}_i - \boldsymbol{a}) + \epsilon_9\sigma(k)) \geq \epsilon_9 \ . \tag{6.40}$$

Define for each i

$$D_i \coloneqq \max\{\, |T(\boldsymbol{a}, \boldsymbol{x}) - T(\boldsymbol{a}, \boldsymbol{y})| \colon \boldsymbol{x}, \boldsymbol{y} \in G_i \,\} \ .$$

Recall $\nu_2 > 1/2$ from (6.23). Choose $\nu_3 > 0$ such that

$$\nu_3 < \frac{2\alpha}{(1+\beta)}(\nu_2 - 1/2) \ . \tag{6.41}$$

Recall from Remark 1.2 our convention of using tilde on parameters. To bound D_i for $i < 0$ we use Theorem 1.23 with the variables

$$\tilde{\eta} \coloneqq \nu_3 \,, \quad \tilde{\theta_0} \coloneqq -\theta_{i+1} \,, \quad \tilde{L} \coloneqq \ell \,, \quad \tilde{n} \coloneqq \|\boldsymbol{b}_{i+1} - \boldsymbol{a}\| \,.$$

To bound D_i for $i \geq 0$ we use Theorem 1.23 with the variables

$$\tilde{\eta} \coloneqq \nu_3 \,, \quad \tilde{\theta_0} \coloneqq -\theta_i \,, \quad \tilde{L} \coloneqq \ell \,, \quad \tilde{n} \coloneqq \|\boldsymbol{b}_i - \boldsymbol{a}\| \,.$$

Using (6.24), (6.31), and (A2), we get $\tilde{L} \geq \tilde{L_0}$, $\tilde{n} \geq \tilde{n_0}$, $\tilde{L} \leq \Delta(\tilde{n})$, as required. Using Theorem 1.23 and (A2), we get for large enough t

$$\mathbb{P}\big(D_i \geq t \,(\log k)^{\nu_3} \,\sigma\big(\Delta^{-1}(\ell)\big)\big) \leq C_{49} \exp(-C_{50} t \,(\log k)^{\nu_3}) \,. \tag{6.42}$$

Using (6.24) and (A2), we get

$$\frac{\sigma(k)}{\sigma(\Delta^{-1}(\ell))(\log k)^{\nu_3}} \geq C_{51} (\log k)^{-\nu_3 + (\nu_2 - 1/2)(2\alpha)/(1+\beta)} \,.$$

By (6.41), this can be made arbitrarily large by choosing k large. Therefore, in (6.42) we can choose

$$t = \frac{\epsilon_9}{2} \frac{\sigma(k)}{\sigma(\Delta^{-1}(\ell))(\log k)^{\nu_3}} \,,$$

and we get for large enough k

$$\mathbb{P}\Big(D_i \geq \frac{\epsilon_9}{2} \sigma(k)\Big) \leq \frac{\epsilon_9}{2} \,.$$

Combining this with (6.40) we get for all i

$$\mathbb{P}\Big(T(\boldsymbol{a}, \boldsymbol{x}) \geq h(\boldsymbol{x} - \boldsymbol{a}) + \frac{\epsilon_9}{2} \sigma(k) \text{ for all } \boldsymbol{x} \in G_i\Big) \geq \frac{\epsilon_9}{2} \,.$$

Since for each i, $\inf\{\, T(\boldsymbol{a}, \boldsymbol{x}) - h(\boldsymbol{x} - \boldsymbol{a}) \colon \boldsymbol{x} \in G_i \,\}$ is an increasing function of the edge-weight configuration, by the FKG inequality we get

$$\mathbb{P}\Big(T(\boldsymbol{a}, \boldsymbol{x}) \geq h(\boldsymbol{x} - \boldsymbol{a}) + \frac{\epsilon_9}{2} \sigma(k) \text{ for all } \boldsymbol{x} \in G^*\Big) \geq \Big(\frac{\epsilon_9}{2}\Big)^N \,,$$

where recall from (6.32) that N is the total number of segments in G^*. Using (6.32) we get

$$\mathbb{P}\Big(T(\boldsymbol{a}, \boldsymbol{x}) \geq h(\boldsymbol{x} - \boldsymbol{a}) + \frac{\epsilon_9}{2} \sigma(k) \text{ for all } \boldsymbol{x} \in G^*\Big) \geq \exp(-C_{52} (\log k)^{\nu_2}) \,.$$

This completes the proof of Proposition 6.8.

6.2.2 Proof of Proposition 6.9

By (6.17), width of H^* in the direction θ_0 is $2C_{23}\Delta(k)(\log k)^{1/2}$. By (6.33), width of G^* in the direction θ_0 is $C_{53}\Delta(k)(\log k)^{1/2}$. By construction of G^* we have

$$\min_{\boldsymbol{x} \in G^*} \pi^1_{\theta_0, \theta_0^t}(\boldsymbol{x}) = \max_{\boldsymbol{y} \in H^*} \pi^1_{\theta_0, \theta_0^t}(\boldsymbol{y}) \,.$$

Both G^* and H^* are centered around the line joining $\mathbf{0}$ and $\boldsymbol{a}$, and both go up to distance $C_{23}\Delta(k)(\log k)^{1/2}$ in the directions θ_0^t and $-\theta_0^t$. Therefore, for all $\boldsymbol{x} \in G^*$ and $\boldsymbol{y} \in H^*$

$$\|\boldsymbol{x} - \boldsymbol{y}\| \leq C_{54} \Delta(k)(\log k)^{1/2} \,.$$

Using (A2), the number of pairs $(\lfloor \boldsymbol{x} \rceil, \boldsymbol{y})$ (recall Notation 1.1) where $\boldsymbol{x} \in G^*$ and $\boldsymbol{y} \in H^*$ is at most $C_{55}k^2$. Hence, using (A1), (A2), and a union bound, we get

$$\begin{aligned}
&\mathbb{P}\Big(|T(\boldsymbol{x},\boldsymbol{y}) - h(\boldsymbol{x}-\boldsymbol{y})| \geq \frac{\epsilon_7}{2}\sigma(k) \text{ for some } \boldsymbol{x} \in G^* \text{ and } \boldsymbol{y} \in H^*\Big) \\
&\leq C_{56}k^2 \exp\left(-C_{57}\frac{\sigma(k)\log k}{\sigma(\Delta(k)(\log k)^{1/2})}\right) \\
&\leq \exp\big(-k^{C_{58}}\big) .
\end{aligned}$$

This completes the proof of Proposition 6.9.

6.3 Proof of Proposition 6.6

The passage times $T(\mathbf{0},\boldsymbol{a})$ and $T(\mathbf{0},\boldsymbol{b})$ are increasing functions of the edge-weight configuration $\mathbb{T}$. Therefore, using the FKG inequality and taking expectation we get

$$\mathbb{E}[\mathrm{Cov}(T(\mathbf{0},\boldsymbol{a}), T(\mathbf{0},\boldsymbol{b})|\mathcal{F})] \geq 0 .$$

Define two collections of paths

$$\begin{aligned}
\Pi(\mathbf{0},\boldsymbol{a}) \coloneqq \Big\{ &\gamma : \gamma \text{ is a path from } \mathbf{0} \text{ to } \boldsymbol{a}\text{, and for all lattice points} \\
&\boldsymbol{y} \text{ in } \gamma \text{ outside } F \text{ we have } \left|\pi^2_{\theta_0,\theta_0^t}(\boldsymbol{y}-\boldsymbol{a})\right| < \frac{1}{2}\Delta(k)(\log k)^{\eta} \Big\} , \\
\Pi(\mathbf{0},\boldsymbol{b}) \coloneqq \Big\{ &\gamma : \gamma \text{ is a path from } \mathbf{0} \text{ to } \boldsymbol{b}\text{, and for all lattice points} \\
&\boldsymbol{y} \text{ in } \gamma \text{ outside } F \text{ we have } \left|\pi^2_{\theta_0,\theta_0^t}(\boldsymbol{y}-\boldsymbol{b})\right| < \frac{1}{2}\Delta(k)(\log k)^{\eta} \Big\} .
\end{aligned}$$

Since $|\pi^2_{\theta_0,\theta_0^t}(\boldsymbol{a}-\boldsymbol{b})| = L = \Delta(k)(\log k)^{\eta}$, a path in $\Pi(\mathbf{0},\boldsymbol{a})$ does not touch a path in $\Pi(\mathbf{0},\boldsymbol{b})$ outside F. Let

$$T'(\mathbf{0},\boldsymbol{a}) \coloneqq \min_{\gamma \in \Pi(\mathbf{0},\boldsymbol{a})} T(\gamma) , \quad T'(\mathbf{0},\boldsymbol{b}) \coloneqq \min_{\gamma \in \Pi(\mathbf{0},\boldsymbol{b})} T(\gamma) .$$

Since paths in $\Pi(\mathbf{0},\boldsymbol{a})$ do not intersect paths in $\Pi(\mathbf{0},\boldsymbol{b})$ outside F, $T'(\mathbf{0},\boldsymbol{a})$ and $T'(\mathbf{0},\boldsymbol{b})$ are independent conditioned on $\mathcal{F}$.

Let $\Gamma(\boldsymbol{a},F)$ be the geodesic from $\boldsymbol{a}$ to F i.e., the path with minimum passage time from $\boldsymbol{a}$ to a point in F. Let $T(\boldsymbol{a},F) \coloneqq T(\Gamma(\boldsymbol{a},F))$. Since F touches H, we have

$$T(\boldsymbol{a},F) \leq \max_{\boldsymbol{y}\in H} T(\boldsymbol{a},\boldsymbol{y}) .$$

By Remark 6.2, H is contained in a square of side length $C_{59}k$. Therefore, by Lemma 2.2 we get that there exist positive constants C_{60} and C_{61} such that

$$\mathbb{P}(T(\boldsymbol{a},F) \geq C_{60}k) \leq \exp(-C_{61}k) . \tag{6.43}$$

Let

$$T^t(\mathbf{0},\boldsymbol{a}) \coloneqq \min\{\, T'(\mathbf{0},\boldsymbol{a}), \tau + C_{60}k \,\} , \quad T^t(\mathbf{0},\boldsymbol{b}) \coloneqq \min\{\, T'(\mathbf{0},\boldsymbol{b}), \tau + C_{60}k \,\} .$$

Because $T'(\mathbf{0},\boldsymbol{a})$ and $T'(\mathbf{0},\boldsymbol{b})$ are independent conditioned on $\mathcal{F}$, $T^t(\mathbf{0},\boldsymbol{a})$ and $T^t(\mathbf{0},\boldsymbol{b})$ are also independent conditioned on $\mathcal{F}$. Therefore

$$\begin{aligned}
&\mathbb{E}[\mathrm{Cov}(T(\mathbf{0},\boldsymbol{a}),T(\mathbf{0},\boldsymbol{b})|\mathcal{F})]\\
=\ &\mathbb{E}\big[\mathrm{Cov}\big(T(\mathbf{0},\boldsymbol{a})-T^t(\mathbf{0},\boldsymbol{a}),T(\mathbf{0},\boldsymbol{b})-\tau|\mathcal{F}\big)\big]\\
&+\mathbb{E}\big[\mathrm{Cov}\big(T^t(\mathbf{0},\boldsymbol{a})-\tau,T(\mathbf{0},\boldsymbol{b})-T^t(\mathbf{0},\boldsymbol{b})|\mathcal{F}\big)\big]\\
\le\ &\mathbb{E}\Big[\Big(\mathbb{E}\Big(\big(T(\mathbf{0},\boldsymbol{a})-T^t(\mathbf{0},\boldsymbol{a})\big)^2|\mathcal{F}\Big)\Big)^{1/2}\Big(\mathbb{E}\Big((T(\mathbf{0},\boldsymbol{b})-\tau)^2|\mathcal{F}\Big)\Big)^{1/2}\Big]\\
&+\mathbb{E}\Big[\Big(\mathbb{E}\Big(\big(T(\mathbf{0},\boldsymbol{b})-T^t(\mathbf{0},\boldsymbol{b})\big)^2|\mathcal{F}\Big)\Big)^{1/2}\Big(\mathbb{E}\Big(\big(T^t(\mathbf{0},\boldsymbol{a})-\tau\big)^2|\mathcal{F}\Big)\Big)^{1/2}\Big]\\
\le\ &\Big(\mathbb{E}\Big[\big(T(\mathbf{0},\boldsymbol{a})-T^t(\mathbf{0},\boldsymbol{a})\big)^2\Big]\Big)^{1/2}\Big(\mathbb{E}\Big[\big(T(\mathbf{0},\boldsymbol{b})-\tau\big)^2\Big]\Big)^{1/2}\\
&+\Big(\mathbb{E}\Big[\big(T(\mathbf{0},\boldsymbol{b})-T^t(\mathbf{0},\boldsymbol{b})\big)^2\Big]\Big)^{1/2}\Big(\mathbb{E}\Big[\big(T^t(\mathbf{0},\boldsymbol{a})-\tau\big)^2\Big]\Big)^{1/2}\ . \qquad (6.44)
\end{aligned}$$

Let $\boldsymbol{x}$ be the point where F touches H. Therefore $T(\mathbf{0},\boldsymbol{x})=\tau\le T(\mathbf{0},\boldsymbol{a})$. Hence $0\le T(\mathbf{0},\boldsymbol{a})-\tau\le T(\boldsymbol{a},\boldsymbol{x})$. Using that H is contained in a box of size $C_{59}k$ we get

$$\mathbb{E}\left[(T(\mathbf{0},\boldsymbol{a})-\tau)^4\right]\le\mathbb{E}\left[T(\boldsymbol{a},\boldsymbol{x})^4\right]\le C_{62}k^4\ . \qquad (6.45)$$

Since $T'(\mathbf{0},\boldsymbol{a})$ is the minimum passage time restricted to some paths from $\mathbf{0}$ to $\boldsymbol{a}$, we have $T'(\mathbf{0},\boldsymbol{a})\ge T(\mathbf{0},\boldsymbol{a})$. Therefore $T^t(\mathbf{0},\boldsymbol{a})\ge\tau$. Therefore

$$\left|T^t(\mathbf{0},\boldsymbol{a})-\tau\right|\le C_{63}k\ . \qquad (6.46)$$

Combining (6.45) and (6.46) we get

$$\begin{aligned}
&\Big(\mathbb{E}\big(T(\mathbf{0},\boldsymbol{a})-T^t(\mathbf{0},\boldsymbol{a})\big)^2\Big)^{1/2}\\
\le\ &\Big(\mathbb{E}\big(T(\mathbf{0},\boldsymbol{a})-T^t(\mathbf{0},\boldsymbol{a})\big)^4\Big)^{1/4}\,\mathbb{P}\big(T(\mathbf{0},\boldsymbol{a})\ne T^t(\mathbf{0},\boldsymbol{a})\big)^{1/4}\\
\le\ &\Big[\Big(\mathbb{E}(T(\mathbf{0},\boldsymbol{a})-\tau)^4\Big)^{1/4}+\Big(\mathbb{E}\big(T^t(\mathbf{0},\boldsymbol{a})-\tau\big)^4\Big)^{1/4}\Big]\,\mathbb{P}\big(T(\mathbf{0},\boldsymbol{a})\ne T^t(\mathbf{0},\boldsymbol{a})\big)^{1/4}\\
\le\ &C_{64}k\,\mathbb{P}\big(T(\mathbf{0},\boldsymbol{a})\ne T^t(\mathbf{0},\boldsymbol{a})\big)^{1/4}\ . \qquad (6.47)
\end{aligned}$$

Similarly we have

$$\mathbb{E}\left(T(\mathbf{0},\boldsymbol{b})-\tau\right)^2\le C_{65}k^2\ . \qquad (6.48)$$

and

$$\Big(\mathbb{E}\left(T(\mathbf{0},\boldsymbol{b})-T^t(\mathbf{0},\boldsymbol{b})\right)^2\Big)^{1/2}\le C_{66}k\,\mathbb{P}\left(T(\mathbf{0},\boldsymbol{b})\ne T^t(\mathbf{0},\boldsymbol{b})\right)^{1/4}\ . \qquad (6.49)$$

Therefore, combining (6.44), (6.47)-(6.49) we get

$$\begin{aligned}
&\mathbb{E}[\mathrm{Cov}(T(\mathbf{0},\boldsymbol{a}),T(\mathbf{0},\boldsymbol{b})|\mathcal{F})]\\
\le\ &C_{67}k^2\Big(\mathbb{P}\big(T(\mathbf{0},\boldsymbol{a})\ne T^t(\mathbf{0},\boldsymbol{a})\big)^{1/4}+\mathbb{P}\big(T(\mathbf{0},\boldsymbol{b})\ne T^t(\mathbf{0},\boldsymbol{b})\big)^{1/4}\Big)\ . \qquad (6.50)
\end{aligned}$$

Let us consider the term $\mathbb{P}(T(\mathbf{0},\boldsymbol{a})\ne T^t(\mathbf{0},\boldsymbol{a}))$, the other one can be dealt similarly. Depending on C_{60} there exist positive constants C_{68} and C_{69} such that for large enough k we have

$$\mathbb{P}(\min\{\,T(\boldsymbol{a},\boldsymbol{y})\colon\|\boldsymbol{y}-\boldsymbol{a}\|=C_{68}k\,\}\le C_{60}k)\le\exp(-C_{69}k)\ . \qquad (6.51)$$

Define the events

$$\mathcal{E}_1 := \{ T(\boldsymbol{a}, F) \le C_{60} k \} ,$$
$$\mathcal{E}_2 := \{ \min\{ T(\boldsymbol{a}, \boldsymbol{y}) \colon \|\boldsymbol{y} - \boldsymbol{a}\| = C_{68} k \} > C_{60} k \} ,$$
$$\mathcal{E}_3 := \{ \mathrm{Diam}(\Gamma(\boldsymbol{a}, F)) \le 2C_{68} k \} .$$

Therefore, $\mathcal{E}_1$ and $\mathcal{E}_2$ implies $\mathcal{E}_3$. Recall Notation 2.7. Let

$$\mathcal{E}_4 := \left\{ \max_{k' \le 2C_{68} k} \mathcal{W}(\boldsymbol{a}, \boldsymbol{0}, k', -\theta_0) < \frac{1}{2} \Delta(k) (\log k)^{\eta} \right\} .$$

Using Corollary 2.11 and $\eta > 1/2$, we get

$$\mathbb{P}(\mathcal{E}_4^c) \le \exp\big(-C_{70} (\log k)^{2\eta} \big) . \tag{6.52}$$

If $\mathbb{T} \in \mathcal{E}_1 \cap \mathcal{E}_3 \cap \mathcal{E}_4$ then $T(\boldsymbol{0}, \boldsymbol{a}) = T^t(\boldsymbol{0}, \boldsymbol{a})$. Therefore, combining (6.43), (6.51), and (6.52), we get

$$\mathbb{P}\big(T(\boldsymbol{0}, \boldsymbol{a}) \ne T^t(\boldsymbol{0}, a) \big) \le \exp\big(-C_{71} (\log k)^{2\eta} \big) .$$

Similar bound holds $\mathbb{P}(T(\boldsymbol{0}, \boldsymbol{b}) \ne T^t(\boldsymbol{0}, \boldsymbol{b}))$. Therefore, using (6.50) and $\eta > 1/2$, we get

$$\mathbb{E}[\mathrm{Cov}(T(\boldsymbol{0}, \boldsymbol{a}), T(\boldsymbol{0}, \boldsymbol{b}) | \mathcal{F})] \le C_{72}$$

for any $C_{72} > 0$, provided k is large enough. This completes the proof of Proposition 6.6.

7 Upper bound of the long-range correlations

In this section, our objective is to prove Theorem 1.28. Due to symmetry of the lattice, without loss of generality we assume $\theta_0 \in [0, \pi/4]$. Fix $J_0 > 0$, $n_0 > 0$ to be assumed large enough whenever required. Consider $n \ge n_0$, $J \in [\mathfrak{q}^{1/2} J_0, n^{\delta}]$. Let $m := f^{-1}(J f(n)/J_0)$, so that

$$J_0 \frac{\Delta(m)(\log m)^{1/2}}{m} = J \frac{\Delta(n)(\log n)^{1/2}}{n} . \tag{7.1}$$

Using $J \ge \mathfrak{q}^{1/2} J_0$ and (A2) we get $m \le n$. Using $\delta < (1-\beta)/2$, $J \le n^{\delta}$, (A2), and assuming n_0 is large enough, we get

$$\log m \ge C_1 \log n . \tag{7.2}$$

As a shorthand notation let us use

$$\boldsymbol{a} := n \boldsymbol{e}_{\theta_0} + J \Delta(n) (\log n)^{1/2} \boldsymbol{e}_{\theta_0^t} ,$$
$$\boldsymbol{b} := n \boldsymbol{e}_{\theta_0} - J \Delta(n) (\log n)^{1/2} \boldsymbol{e}_{\theta_0^t} .$$

Using $\delta < (1 - \beta)/2$ and $J \le n^{\delta}$, we get $\|\boldsymbol{a}\|$ and $\|\boldsymbol{b}\|$ are at most $C_2 n$. Let

$$H := \left\{ \boldsymbol{x} \in \mathbb{R}^2 \colon \pi^1_{\theta_0, \theta_0^t}(\boldsymbol{x}) \ge m \right\} .$$

Let $\mathcal{F}$ be the sigma-field generated by all the edge-weights τ_e such that both endpoints of the edge e are in H. Our objective is to establish upper bound of $\mathrm{Cov}(T(\boldsymbol{0}, \boldsymbol{a}), T(\boldsymbol{0}, \boldsymbol{b}))$. We split the covariance in expectation of conditional covariances given $\mathcal{F}$ and covariance of conditional expectations given $\mathcal{F}$, and establish upper bound of them separately.

Proposition 7.1. *Assuming J_0 and n_0 are large enough, we have*

$$|\mathrm{Cov}(\mathbb{E}(T(\boldsymbol{0}, \boldsymbol{a}) | \mathcal{F}), \mathbb{E}(T(\boldsymbol{0}, \boldsymbol{b}) | \mathcal{F}))| \le C_3 . \tag{7.3}$$

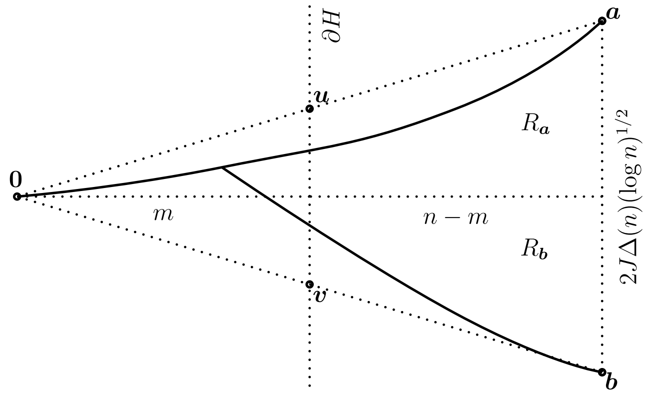

Figure 11: Setup of Proposition 7.1: distance between $\boldsymbol{u}$ and $\boldsymbol{v}$ is $2J\Delta(m)(\log m)^{1/2}$; H is the region to the right of ∂H; $R_{\boldsymbol{a}}$ is the subset of H above the line in direction θ_0, $R_{\boldsymbol{b}}$ is the region below the line; with high probability $\Gamma(\mathbf{0},\boldsymbol{a})$ stays in $R_{\boldsymbol{a}}$ while it is in H, $\Gamma(\mathbf{0},\boldsymbol{b})$ stays in $R_{\boldsymbol{b}}$ while it is in H.

Proof. Define two regions

$$R_{\boldsymbol{a}} \coloneqq \left\{ \boldsymbol{x}\in\mathbb{R}^2 \colon \pi^1_{\theta_0,\theta_0^t}(\boldsymbol{x}) \ge m\,,\ \pi^2_{\theta_0,\theta_0^t}(\boldsymbol{x}) > 0 \right\}\,,$$
$$R_{\boldsymbol{b}} \coloneqq \left\{ \boldsymbol{x}\in\mathbb{R}^2 \colon \pi^1_{\theta_0,\theta_0^t}(\boldsymbol{x}) \ge m\,,\ \pi^2_{\theta_0,\theta_0^t}(\boldsymbol{x}) < 0 \right\}\,.$$

Define the event

$$\mathcal{E}_1 \coloneqq \{\, \Gamma(\mathbf{0},\boldsymbol{a}) \text{ stays inside } R_{\boldsymbol{a}} \text{ while it is in the region } H \,\}\,,$$

i.e., for all $\boldsymbol{u}\in\Gamma(\mathbf{0},\boldsymbol{a})$ with $\pi^1_{\theta_0,\theta_0^t}(\boldsymbol{u}) \ge m$ we have $\pi^2_{\theta_0,\theta_0^t}(\boldsymbol{u}) > 0$. Similarly, define the event

$$\mathcal{E}_2 \coloneqq \{\, \Gamma(\mathbf{0},\boldsymbol{b}) \text{ stays inside } R_{\boldsymbol{b}} \text{ while it is in the region } H \,\}\,.$$

Define a set of points $\mathcal{V}$ as follows. Recall that we have assumed $\theta_0\in[0,\pi/4]$. If $\theta_0=0$ then let $\mathcal{V}$ be the points $k\boldsymbol{e}_1$ with $k\in[m,n]\cap\mathbb{Z}$ (recall $\boldsymbol{e}_1$ is the positive x-axis direction). If $\theta_0\neq 0$, then let $\mathcal{V}$ be the set of points $\boldsymbol{u}$ on the edges of the integer lattice grid (so that $\boldsymbol{u}$ has at least one integer coordinate) satisfying $\pi^1_{\theta_0,\theta_0^t}(\boldsymbol{u})\in[m,n]$ and $\pi^2_{\theta_0,\theta_0^t}=0$. Thus, in both cases, the number of points in $\mathcal{V}$ is bounded by $C_4 n$. Let $\mathcal{K}$ be the set of values of $\pi^1_{\theta_0,\theta_0^t}(\boldsymbol{u})$ where $\boldsymbol{u}\in\mathcal{V}$. Therefore, if $\mathbb{T}\notin\mathcal{E}_1$, then we have the following three cases:

- **(i)** either $\mathcal{W}(\mathbf{0},\boldsymbol{a},m,\theta_0) \ge J_0\Delta(m)(\log m)^{1/2}$; or,
- **(ii)** $\mathcal{W}(\mathbf{0},\boldsymbol{a},k,\theta_0) \ge J_0(k/m)\Delta(m)(\log m)^{1/2}$ for some $k\in\mathcal{K}$; or,
- **(iii)** $\mathcal{W}(\mathbf{0},\boldsymbol{a},k,\theta_0) \ge J\Delta(n)(\log n)^{1/2}$ for some $k\ge n$.

Using Corollary 2.11 and (7.2) we get

$$\mathbb{P}\Big(\mathcal{W}(\mathbf{0},\boldsymbol{a},m,\theta_0) \ge J_0\Delta(m)\,(\log m)^{1/2}\Big) \le C_5\exp\big(-C_6 J_0^2\log n\big)\,. \tag{7.4}$$

By (A2) and (7.2) we get for $k\in[m,n]$

$$J_0(k/m)\Delta(m)(\log m)^{1/2} \ge J_0 C_7\Delta(k)(\log k)^{1/2}\,.$$

Therefore, using Corollary 2.11 and (7.2) we get for each $k \in \mathcal{K}$

$$\mathbb{P}\Big(\mathcal{W}(\mathbf{0}, \boldsymbol{a}, k, \theta_0) \geq J_0 \frac{k}{m} \Delta(m) (\log m)^{1/2}\Big) \leq C_8 \exp\big(-C_9 J_0^2 \log n\big) . \tag{7.5}$$

Assuming J_0 is large enough, we can take a union bound over $k \in \mathcal{K}$. Then we get

$$\begin{aligned} &\mathbb{P}\Big(\mathcal{W}(\mathbf{0}, \boldsymbol{a}, k, \theta_0) \geq J_0 \frac{k}{m} \Delta(m) (\log m)^{1/2} \text{ for some } k \in \mathcal{K}\Big) \\ &\leq C_{10} \exp\big(-C_{11} J_0^2 \log n\big) . \end{aligned} \tag{7.6}$$

Using Corollary 2.9 we get

$$\begin{aligned} &\mathbb{P}\Big(\mathcal{W}(\mathbf{0}, \boldsymbol{a}, k, \theta_0) \geq J_0 \Delta(n) (\log n)^{1/2} \text{ for some } k \geq n\Big) \\ &\leq C_{12} \exp\left(-C_{13} J_0^2 \log n\right) . \end{aligned} \tag{7.7}$$

Combining (7.4), (7.6), and (7.7) we get

$$\mathbb{P}(\mathcal{E}_1^c) \leq C_{14} \exp\big(-C_{15} J_0^2 \log n\big) . \tag{7.8}$$

The same holds for the event $\mathcal{E}_2$.

Let $\hat{T}(\mathbf{0}, \boldsymbol{a})$ be the minimum passage time among of all paths from $\mathbf{0}$ to $\boldsymbol{a}$ which stays in $R_{\boldsymbol{a}}$ when in H. Similarly define $\hat{T}(\mathbf{0}, \boldsymbol{b})$. Then $\mathbb{E}(\hat{T}(\mathbf{0}, \boldsymbol{a})|\mathcal{F})$ and $\mathbb{E}(\hat{T}(\mathbf{0}, \boldsymbol{b})|\mathcal{F})$ are independent because $R_{\boldsymbol{a}}$ and $R_{\boldsymbol{b}}$ are disjoint. If $\mathbb{T} \in \mathcal{E}_1$, then $T(\mathbf{0}, \boldsymbol{a}) = \hat{T}(\mathbf{0}, \boldsymbol{a})$, and if $\mathbb{T} \in \mathcal{E}_2$, then $T(\mathbf{0}, \boldsymbol{b}) = \hat{T}(\mathbf{0}, \boldsymbol{b})$. Using $\|\boldsymbol{a}\| \leq C_2 n$, $\|\boldsymbol{b}\| \leq C_2 n$, (7.8), and the same bound for $\mathcal{E}_2$, we get

$$\begin{aligned} &\mathrm{Cov}(\mathbb{E}(T(\mathbf{0}, \boldsymbol{a})|\mathcal{F}), \mathbb{E}(T(\mathbf{0}, \boldsymbol{b})|\mathcal{F})) \\ &\leq \left(\mathbb{E}\Big(T(\mathbf{0}, \boldsymbol{a}) - \hat{T}(\mathbf{0}, \boldsymbol{a})\Big)^2\right)^{1/2} \big(\mathbb{E}\, T(\mathbf{0}, \boldsymbol{b})^2\big)^{1/2} \\ &\quad + \left(\mathbb{E}\Big(T(\mathbf{0}, \boldsymbol{b}) - \hat{T}(\mathbf{0}, \boldsymbol{b})\Big)^2\right)^{1/2} \big(\mathbb{E}\, T(\mathbf{0}, \boldsymbol{a})^2\big)^{1/2} \\ &\leq \mathbb{P}(\mathcal{E}_1^c)^{1/2} \left(\mathbb{E}\Big(T(\mathbf{0}, \boldsymbol{a}) - \hat{T}(\mathbf{0}, \boldsymbol{a})\Big)^4\right)^{1/4} \big(\mathbb{E}\, T(\mathbf{0}, \boldsymbol{b})^2\big)^{1/2} \\ &\quad + \mathbb{P}(\mathcal{E}_2^c)^{1/2} \left(\mathbb{E}\Big(T(\mathbf{0}, \boldsymbol{b}) - \hat{T}(\mathbf{0}, \boldsymbol{b})\Big)^4\right)^{1/4} \big(\mathbb{E}\, T(\mathbf{0}, \boldsymbol{b})^2\big)^{1/2} \\ &\leq C_{16} n^2 \exp\big(-C_{17} J_0^2 \log n\big) \\ &\leq C_{18} . \end{aligned}$$

This completes the proof of Proposition 7.1. □

Now we consider the expected conditional covariance. We have

$$\mathbb{E}[\mathrm{Cov}(T(\mathbf{0}, \boldsymbol{a}), T(\mathbf{0}, \boldsymbol{b})|\mathcal{F})] \leq (\mathbb{E}[\mathrm{Var}(T(\mathbf{0}, \boldsymbol{a})|\mathcal{F})])^{1/2} (\mathbb{E}[\mathrm{Var}(T(\mathbf{0}, \boldsymbol{b})|\mathcal{F})])^{1/2} . \tag{7.9}$$

We establish an upper bound of $\mathbb{E}[\mathrm{Var}(T(\mathbf{0}, \boldsymbol{a})|\mathcal{F})]$. A similar upper bound also holds for $\mathbb{E}[\mathrm{Var}(T(\mathbf{0}, \boldsymbol{a})|\mathcal{F})]$. Consider an independent edge-weight configuration on the edges which have at least one endpoint not in the half-space H. Let $T'(\mathbf{0}, \boldsymbol{a})$ be the passage time from $\mathbf{0}$ to $\boldsymbol{a}$ on the new configuration. Let $\Gamma'(\mathbf{0},\boldsymbol{a})$ be the corresponding geodesic. Then $T(\mathbf{0},\boldsymbol{a})$ and $T'(\mathbf{0},\boldsymbol{a})$ are independent given $\mathcal{F}$. Therefore

$$\begin{aligned}&\mathbb{E}\left[\mathrm{Var}\left(T(\mathbf{0},\boldsymbol{a})|\mathcal{F}\right)\right]\\ =\ &\mathbb{E}\left[\mathbb{E}\left(\frac{1}{2}\left(T(\mathbf{0},\boldsymbol{a})-T'(\mathbf{0},\boldsymbol{a})\right)^2|\mathcal{F}\right)\right]\\ =\ &\frac{1}{2}\,\mathbb{E}\left[\left(T(\mathbf{0},\boldsymbol{a})-T'(\mathbf{0},\boldsymbol{a})\right)^2\right]\ .\end{aligned} \tag{7.10}$$

Proposition 7.2. *Assuming J_0 and n_0 are large enough, we have*

$$\mathbb{E}\left[\left(T(\mathbf{0},\boldsymbol{a})-T'(\mathbf{0},\boldsymbol{a})\right)^2\right]\le C_{19}\sigma^2(m)\log n\ . \tag{7.11}$$

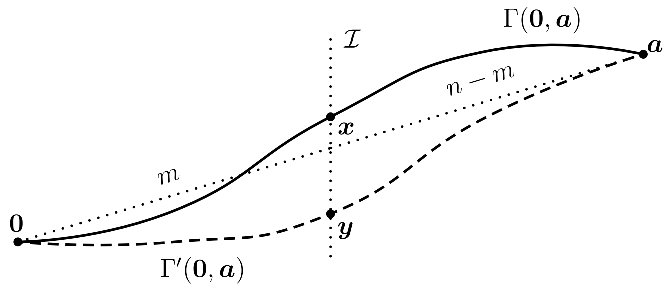

Figure 12: Setup of Proposition 7.2: the segment $\mathcal{I}$ is a part of ∂H, see Figure 11 for the location of ∂H; the geodesic $\Gamma'(\mathbf{0},\boldsymbol{a})$ is constructed by taking a new configuration in the left-side of ∂H. With high probability, both geodesics $\Gamma(\mathbf{0},\boldsymbol{a})$ and $\Gamma'(\mathbf{0},\boldsymbol{a})$ intersect $\mathcal{I}$ when they intersect ∂H.

Proof. Consider the line segment

$$\mathcal{I}\coloneqq\left\{\boldsymbol{x}\in\mathbb{R}^2\colon \pi^1_{\theta_0,\theta_0^t}(\boldsymbol{x})=m\,,\,0\le\pi^2_{\theta_0,\theta_0^t}(\boldsymbol{x})\le 2J_0\Delta(m)(\log m)^{1/2}\right\}.$$

Define the event $\mathcal{E}_3$

$$\mathcal{E}_3\coloneqq\{\,\Gamma(\mathbf{0},\boldsymbol{a})\text{ and }\Gamma'(\mathbf{0},\boldsymbol{a})\text{ pass through }\mathcal{I}\,\}\ .$$

If $\mathbb{T}\notin\mathcal{E}_3$ then both $\Gamma(\mathbf{0},\boldsymbol{a})$ and $\Gamma'(\mathbf{0},\boldsymbol{a})$ wander more than $J_0\Delta(m)(\log m)^{1/2}$ in $\pm\theta_0^t$ directions when they are at distance m in θ_0 direction from $\mathbf{0}$. So, using Theorem 2.8 and $\log m\ge C_1\log n$, we get

$$\mathbb{P}\left(\,\mathcal{E}_3^c\,\right)\le C_{20}\exp\left(-C_{21}J_0^2\log n\right)\ .$$

Therefore, using $\|\boldsymbol{a}\|\le C_2 n$ and assuming J_0 is large enough, we get

$$\mathbb{E}\Big[\left(T(\mathbf{0},\boldsymbol{a})-T'(\mathbf{0},\boldsymbol{a})\right)^2\mathbb{1}(\mathcal{E}_3^c)\Big]\le C_{22}\ . \tag{7.12}$$

If we have $\mathbb{T}\in\mathcal{E}_3$, $\Gamma(\mathbf{0},\boldsymbol{a})$ passes through $\boldsymbol{x}\in\mathcal{I}$, and $\Gamma'(\mathbf{0},\boldsymbol{a})$ passes through $\boldsymbol{y}\in\mathcal{I}$, then

$$T(\mathbf{0},\boldsymbol{x})-T'(0,\boldsymbol{x})\le T(\mathbf{0},\boldsymbol{a})-T'(\mathbf{0},\boldsymbol{a})\le T(\mathbf{0},\boldsymbol{y})-T'(\mathbf{0},\boldsymbol{y})\ .$$

Therefore

$$\mathbb{E}\Big[\left(T(\mathbf{0},\boldsymbol{a})-T'(\mathbf{0},\boldsymbol{a})\right)^2\mathbb{1}(\mathcal{E}_3)\Big]\le\mathbb{E}\Big[\max_{\boldsymbol{z}\in\mathcal{I}}\left(T(\mathbf{0},\boldsymbol{z})-T'(\mathbf{0},\boldsymbol{z})\right)^2\Big]\ . \tag{7.13}$$

For every $\boldsymbol{z} \in \mathcal{I}$, $T(\mathbf{0}, \boldsymbol{z})$ and $T'(\mathbf{0}, \boldsymbol{z})$ have the same mean. Using (7.1), $J \le n^{\delta}$, $\delta \le (1-\beta)/2$, and (A2), we get $\|\boldsymbol{z}\| \le C_{23} m$ for all $\boldsymbol{z} \in \mathcal{I}$. Therefore, by (A1) and (A2), for all $t > 0$

$$\mathbb{P}\left(\max_{\boldsymbol{z}\in\mathcal{I}} |T(\mathbf{0},\boldsymbol{z}) - T'(\mathbf{0},\boldsymbol{z})| \ge t\sigma(m)\right) \le C_{24} m \exp(-C_{25} t) \ .$$

Therefore,

$$\mathbb{E}\left[\max_{\boldsymbol{z}\in\mathcal{I}} (T(\mathbf{0},\boldsymbol{z}) - T'(\mathbf{0},\boldsymbol{z}))^2\right] \le C_{26}\sigma^2(m)\log m \ .$$

Combining this with (7.12), (7.13), and using $m \le n$ proves Proposition 7.2. □

Combining (7.10) and (7.11) we get

$$\mathbb{E}[\mathrm{Var}(T(\mathbf{0},\boldsymbol{a})|\mathcal{F})] \le C_{27}\sigma^2(m)\log n \ .$$

By symmetry, the same statement holds if we replace $\boldsymbol{a}$ by $\boldsymbol{b}$. Therefore by (7.9) we get

$$\mathbb{E}[\mathrm{Cov}(T(\mathbf{0},\boldsymbol{a}), T(\mathbf{0},\boldsymbol{b})|\mathcal{F})] \le C_{28}\sigma^2(m)\log n \ . \tag{7.14}$$

Therefore, by (A2), the bound on the covariance of the conditional expectations in (7.3) is negligible compared to the bound on the expectation of the conditional covariance in (7.14). Thus, combining (7.3) and (7.14), proves Theorem 1.28.

References

[1] K. S. Alexander, *A note on some rates of convergence in first-passage percolation*, Ann. Appl. Probab. **3** (1993), no. 1, 81–90. MR1202516

[2] K. S. Alexander, *Approximation of subadditive functions and convergence rates in limiting-shape results*, Ann. Probab. **25** (1997), no. 1, 30–55. MR1428498

[3] K. S. Alexander, *Geodesics, bigeodesics, and coalescence in first passage percolation in general dimension*, 2020, preprint available at https://arxiv.org/abs/2001.08736.

[4] K. S. Alexander, *Uniform fluctuation and wandering bounds in first passage percolation*, 2020, preprint available at https://arxiv.org/abs/2011.07223.

[5] S. G. Alves, T. J. Oliveira, and S. C. Ferreira, *Universality of fluctuations in the kardar-parisi-zhang class in high dimensions and its upper critical dimension*, Phys. Rev. E **90** (2014), 020103.

[6] A. Auffinger, M. Damron, and J. Hanson, *Rate of convergence of the mean for sub-additive ergodic sequences*, Adv. Math. **285** (2015), 138–181. MR3406498

[7] M. Balázs, E. Cator, and T. Seppäläinen, *Cube root fluctuations for the corner growth model associated to the exclusion process*, Electron. J. Probab. **11** (2006), no. 42, 1094–1132. MR2268539

[8] R. Basu, C. Hoffman, and A. Sly, *Nonexistence of bigeodesics in planar exponential last passage percolation*, Comm. Math. Phys. **389** (2022), no. 1, 1–30. MR4365136

[9] R. Basu, S. Sarkar, and A. Sly, *Coalescence of geodesics in exactly solvable models of last passage percolation*, J. Math. Phys. **60** (2019), no. 9, 093301, 22. MR4002528

[10] M. Benaïm and R. Rossignol, *Exponential concentration for first passage percolation through modified Poincaré inequalities*, Ann. Inst. Henri Poincaré Probab. Stat. **44** (2008), no. 3, 544–573. MR2451057

[11] I. Benjamini, G. Kalai, and O. Schramm, *First passage percolation has sublinear distance variance*, Ann. Probab. **31** (2003), no. 4, 1970–1978. MR2016607

[12] S. Chatterjee, *The universal relation between scaling exponents in first-passage percolation*, Ann. of Math. (2) **177** (2013), no. 2, 663–697. MR3010809

[13] J. T. Cox and R. Durrett, *Some limit theorems for percolation processes with necessary and sufficient conditions*, Ann. Probab. **9** (1981), no. 4, 583–603. MR0624685

[14] M. Damron and J. Hanson, *Busemann functions and infinite geodesics in two-dimensional first-passage percolation*, Comm. Math. Phys. **325** (2014), no. 3, 917–963. MR3152744

[15] M. Damron and J. Hanson, *Bigeodesics in first-passage percolation*, Comm. Math. Phys. **349** (2017), no. 2, 753–776. MR3595369

[16] M. Damron and X. Wang, *Entropy reduction in Euclidean first-passage percolation*, Electron. J. Probab. **21** (2016), Paper No. 65, 23. MR3580031

[17] H. C. Fogedby, *Kardar-parisi-zhang equation in the weak noise limit: Pattern formation and upper critical dimension*, Phys. Rev. E **73** (2006), 031104. MR2231344

[18] S. Ganguly and M. Hegde, *Optimal tail exponents in general last passage percolation via bootstrapping & geodesic geometry*, 2020, preprint available at https://arxiv.org/abs/2007.03594.

[19] J. M. Hammersley and D. J. A. Welsh, *First-passage percolation, subadditive processes, stochastic networks, and generalized renewal theory*, Proc. Internat. Res. Semin., Statist. Lab., Univ. California, Berkeley, Calif, Springer-Verlag, New York, 1965, pp. 61–110. MR0198576

[20] K. Johansson, *Shape fluctuations and random matrices*, Comm. Math. Phys. **209** (2000), no. 2, 437–476. MR1737991

[21] K. Johansson, *Transversal fluctuations for increasing subsequences on the plane*, Probab. Theory Related Fields **116** (2000), no. 4, 445–456. MR1757595

[22] H. Kesten, *Aspects of first passage percolation*, École d'été de probabilités de Saint-Flour, XIV—1984, Lecture Notes in Math., vol. 1180, Springer, Berlin, 1986, pp. 125–264. MR876084

[23] H. Kesten, *On the speed of convergence in first-passage percolation*, Ann. Appl. Probab. **3** (1993), no. 2, 296–338. MR1221154

[24] S.-W. Kim and J. M. Kim, *A restricted solid-on-solid model in higher dimensions*, Journal of Statistical Mechanics: Theory and Experiment **2014** (2014), no. 7, P07005. MR3249152

[25] J. F. C. Kingman, *The ergodic theory of subadditive stochastic processes*, J. Roy. Statist. Soc. Ser. B **30** (1968), 499–510. MR0254907

[26] P. Le Doussal and K. J. Wiese, *Two-loop functional renormalization for elastic manifolds pinned by disorder in n dimensions*, Physical Review E **72** (2005), no. 3.

[27] C. Licea and C. M. Newman, *Geodesics in two-dimensional first-passage percolation*, Ann. Probab. **24** (1996), no. 1, 399–410. MR1387641

[28] E. Marinari, A. Pagnani, G. Parisi, and Z. Rácz, *Width distributions and the upper critical dimension of kardar-parisi-zhang interfaces*, Physical Review E **65** (2002), no. 2.

[29] C. M. Newman, *A surface view of first-passage percolation*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994), Birkhäuser, Basel, 1995, pp. 1017–1023. MR1404001

[30] C. M. Newman and M. S. T. Piza, *Divergence of shape fluctuations in two dimensions*, Ann. Probab. **23** (1995), no. 3, 977–1005. MR1349159

[31] M. Talagrand, *Concentration of measure and isoperimetric inequalities in product spaces*, Inst. Hautes Études Sci. Publ. Math. (1995), no. 81, 73–205. MR1361756

[32] R. Tessera, *Speed of convergence in first passage percolation and geodesicity of the average distance*, Ann. Inst. Henri Poincaré Probab. Stat. **54** (2018), no. 2, 569–586. MR3795059

[33] J. Wehr and M. Aizenman, *Fluctuations of extensive functions of quenched random couplings*, J. Statist. Phys. **60** (1990), no. 3-4, 287–306. MR1069633

[34] J. C. Wierman and W. Reh, *On conjectures in first passage percolation theory*, Ann. Probability **6** (1978), no. 3, 388–397. MR0478390

Electronic Journal of Probability
Electronic Communications in Probability

Advantages of publishing in EJP-ECP

- Very high standards
- Free for authors, free for readers
- Quick publication (no backlog)
- Secure publication (LOCKSS[1])
- Easy interface (EJMS[2])

Economical model of EJP-ECP

- Non profit, sponsored by IMS[3], BS[4] , ProjectEuclid[5]
- Purely electronic

Help keep the journal free and vigorous

- Donate to the IMS open access fund[6] (click here to donate!)
- Submit your best articles to EJP-ECP
- Choose EJP-ECP over for-profit journals

[1]LOCKSS: Lots of Copies Keep Stuff Safe http://www.lockss.org/
[2]EJMS: Electronic Journal Management System http://www.vtex.lt/en/ejms.html
[3]IMS: Institute of Mathematical Statistics http://www.imstat.org/
[4]BS: Bernoulli Society http://www.bernoulli-society.org/
[5]Project Euclid: https://projecteuclid.org/
[6]IMS Open Access Fund: http://www.imstat.org/publications/open.htm