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The full spectrum of random walks on complete finite d-ary trees*

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Abstract

In the present paper, we determine the full spectrum of the simple random walk on finite, complete d-ary trees. We also find an eigenbasis for the transition matrix. As an application, we apply our results to get a lower bound for the interchange process on complete, finite d-ary trees, which we conjecture to be sharp.

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1 Introduction

Finding the spectrum of a transition matrix is a very popular subject in graph theory and Markov chain theory. There are only a few techniques known to describe the exact spectrum of a Markov chain, and they usually work under very specific conditions, such as when the Markov chain is a random walk on a finite group, generated only by a conjugacy class [12]. Most well-known examples where a transition matrix has been diagonalized usually rely on combination of advanced representation theory, Fourier analysis, and combinatorial arguments [3, 5, 6, 7, 13, 15, 17, 22]. But even in most of these cases, there is no description of what an eigenbasis of the transition matrix would look like, which in general is needed as well in order to understand the transition matrix.

In this work, we present the full spectrum of the simple random walk on complete, finite d-ary trees and a corresponding eigenbasis, and we use this information to produce a lower bound for the interchange process on the trees, which we conjecture is sharp. Consider the complete, d-ary tree $\mathcal{T}_h$ of height h, which has $n := 1 + d + \cdots + d^h = \frac{d^{h+1}-1}{d-1}$ vertices. We study the simple random walk on $\mathcal{T}_h$, whose transition matrix is denoted by Q_h, according to which when we are at the root we stay fixed with probability $1/(d+1)$,

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or we move to a child with probability $1/(d+1)$ each. When we are at a leaf, we stay fixed with probability $d/(d+1)$ otherwise we move to the unique parent with probability $1/(d+1)$. For any other node, we choose one of the neighbors with probability $1/(d+1)$.

This is a well studied Markov chain. Rojo [24] and later Rojo and Soto [23] linked the spectrum of the Laplacian of the complete binary tree to the spectrum of suitable submatrices. Aldous [1] proved that the cover time is asymptotic to $2h^2d^{h+1}\log h/(h-1)$. The order of the spectral gap and the mixing time of this Markov chain have been widely known for a long time. In fact, the random walk on $\mathcal{T}_h$ is one of the most famous examples of a random walk not exhibiting cutoff (see Example 18.6 of [20]). In fact, Basu, Hermon and Peres [4] have introduced a criterion for cutoff for random walks on trees which the simple random walk on complete, finite d-ary trees fails to satisfy. However, finding the exact value of the spectral gap has been an open question for years, let alone finding the entire spectrum and an eigenbasis of the transition matrix Q_h.

We denote by ρ the root of $\mathcal{T}_h$, by $V(\mathcal{T}_h)$ the vertex set of $\mathcal{T}_h$, and by $E(\mathcal{T}_h)$ the set of edges of $\mathcal{T}_h$. Let $\ell : V(\mathcal{T}_h) \to [0,\dots,h]$ denote the distance from the root. For every node v, let $\mathcal{T}^v$ be the complete d-ary subtree rooted at v, namely consisting of v and all vertices of $V(\mathcal{T}_h)$ that are descendants of v in $\mathcal{T}_h$. Let $\mathcal{T}_i^v$ be the complete d-ary subtree of $\mathcal{T}^v$ rooted at the i-th child of v.

The next theorem includes the first result of this paper, presenting the eigenvalues and an eigenbasis of Q_h.

Theorem 1.1. *(a) Q_h is diagonalizable with 1 being an eigenvalue with multiplicity 1. Every other eigenvalue $\lambda \neq 1$ of Q_h is of the form*

$$\lambda = \frac{d}{d+1}\left(x + \frac{1}{xd}\right), \tag{1.1}$$

where $x \neq \pm\frac{1}{\sqrt{d}}$ is a solution of one of the following $h+1$ equations:

$$d^{h+1}x^{2h+2} = 1 \tag{1.2}$$

and

$$d^{k+2}x^{2k+4} - d^{k+2}x^{2k+3} + dx - 1 = 0, \quad \text{for } 0 \le k \le h-1. \tag{1.3}$$

Conversely, each solution $x \neq \pm\frac{1}{\sqrt{d}}$ of these equations corresponds to an eigenvalue λ according to (1.1)*. For each of these equations, if x is a solution then so is $\frac{1}{xd}$. Both x and $\frac{1}{xd}$ correspond to the same λ. The correspondence between x and λ is* 2*-to-*1*.*

(b) For each solution $x \neq \pm\frac{1}{\sqrt{d}}$ of (1.2)*, an eigenvector f_λ with respect to λ is given by the formula*

$$f_\lambda(v) = \frac{dx^2 - x}{dx^2 - 1}x^i + \frac{x-1}{dx^2-1}\frac{1}{d^i x^i} \quad \text{for every } v \text{ with } \ell(v) = i,\ 0 \le i \le h. \tag{1.4}$$

For each $0 \le k \le h-1$, each solution $x \neq \pm\frac{1}{\sqrt{d}}$ of (1.3)*, each $v \in V(\mathcal{T}_h)$ such that $\ell(v) = h-1-k$, and each $j \in [1,\dots,d-1]$, an eigenvector $f_{v,j,j+1}$ with respect to λ is given by the formula*

$$f_{v,j,j+1}(w) = \begin{cases} \frac{dx^{i+2}}{dx^2-1} - \frac{1}{(dx^2-1)d^i x^i} & \text{for } w \in \mathcal{T}_j^v, \text{ where } i = \ell(w) - h + k, \\ -\frac{dx^{i+2}}{dx^2-1} + \frac{1}{(dx^2-1)d^i x^i} & \text{for } w \in \mathcal{T}_{j+1}^v, \text{ where } i = \ell(w) - h + k, \\ 0, \text{ otherwise.} \end{cases} \tag{1.5}$$

(c) The collection of these eigenvectors together with the all-1 vector form an eigenbasis of Q_h.

In Lemma 2.1 and Figure 1, we describe and illustrate the eigenvectors in more detail.

The idea behind the proof is to consider appropriate projections of the random walk. For example, let X_t be the state of the random walk at time t and let Y_t be the distance of X_t from the root. Then Y_t is a Markov chain on $[0, h]$, whose eigenvalues are also eigenvalues of Q_h. Also, the eigenvectors of Y_t lift to give the eigenvectors presented in (1.4). This computation is not going to give us the full spectrum, however.

For example, in the case of the binary tree, another type of projection to consider is as follows. We consider the process W_t, which is equal to $-Y_t$ if $X_t \in \mathcal{T}_1^\rho$ and equal to Y_t otherwise. The second largest eigenvalue can be derived by this new process, while the eigenvectors are of the form presented in (1.5). The reason why this is the right process to study is hidden in the mixing time of the random walk on $\mathcal{T}_h$. A coupling argument roughly says that we have to wait until X_t reaches the root ρ. The first time that X_t hits ρ is captured by W_t, since W_t is a Markov chain on $[-h, h]$, where the bias is towards the ends and away from zero. The projections that we consider form birth and death processes, whose spectral and mixing properties have been studied by Chen and Saloff-Coste [9, 10, 11] and Ding, Lubetzky, and Peres [14]. To capture the entire spectrum, our method is to find in each eigenspace a well-structured eigenvector, which occurs by considering an appropriate projection, which gives rise to a birth and death process.

Our analysis has immediate applications to card shuffling, namely the interchange process on $\mathcal{T}_h$, and to the exclusion process. We enumerate the nodes in $V(\mathcal{T}_h)$ and we assign cards to the nodes. At time zero, card i is assigned to node i. The interchange process on $\mathcal{T}_h$ chooses an edge uniformly at random and then flips a fair coin. If heads, interchange the cards on the ends of e; if tails, stay fixed. A configuration of the deck corresponds to an element of the symmetric group, which we denote by S_n where we recall that n is the number of vertices of $\mathcal{T}_h$.

Let $g \in S_n$. Let P be the transition matrix of the interchange process on the complete, finite d-ary tree $\mathcal{T}_h$ and let $P_{id}^t(g)$ be the probability that we are at g after t steps, given that we start at the identity. We define the total variation distance between P_{id}^t and the uniform measure U to be

$$d(t) = \frac{1}{2} \sum_{x \in S_n} \left| P_{id}^t(x) - \frac{1}{n!} \right|.$$

A celebrated result concerning the interchange process was the proof of Aldous conjecture [8, Theorem 1.1], which states that the spectral gap of P is the same as the spectral gap of the Markov chain that the ace of spades performs. Adjusting our computations, we now get the following result.

Theorem 1.2. *Consider the interchange process on the complete d-ary tree of depth h. We have that*

(a) the spectral gap of the transition matrix is $\frac{(d-1)^2}{2(n-1)[(d-1)n+1]} + O\left(\frac{d \log_d n}{n^3}\right)$,

(b) and if $t = \frac{1}{d-1} n^2 \log n - \frac{1}{d-1} n^2 \log\left(\frac{1}{\varepsilon}\right) + O\left(n^2\right)$, then

$$d(t) \geq 1 - \varepsilon,$$

where ε is any positive constant. Here, the implicit constants do not depend on d, h, n.

The order for the mixing time of the interchange process on $\mathcal{T}_h$ has been known to be of the form $C(d)n^2 \log n$ (see Corollary 8 of [2] and Theorem 1.4a of [18]). However, no explicit bounds on constant $C(d)$ have been stated. We conjecture that the lower bound in part (b) of Theorem 1.2 is sharp and that the interchange process on $\mathcal{T}_h$ exhibits cutoff at $\frac{1}{d-1}n^2 \log n$. This would imply that this process mixes much faster than the interchange process on the path, another card shuffle that uses $n-1$ transpositions, which Lacoin [19] recently proved exhibits cutoff at $\frac{1}{2\pi^2}n^3 \log n$.

We can get lower bounds for the mixing time of another well studied process, the exclusion process on the complete d-ary tree. This is a famous interacting particle system process, according to which at time zero, $k \leq n/2$ nodes of the tree are occupied by indistinguishable particles. At time t, we pick an edge uniformly at random and we flip the two ends. Similar computations to the ones of the proof of Theorem 1.2 give that if $t = \frac{1}{d-1}n^2 \log k - \frac{1}{d-1}n^2 \log\left(\frac{1}{\varepsilon}\right) + o\left(n^2 \log k\right)$, then

$$d(t) \geq 1 - \varepsilon,$$

where $\varepsilon > 0$ is a constant. Combining Oliveira's result [21] with Theorem 1.2 (b), we get that the order of the mixing time of the exclusion process on the complete d-ary tree is $n^2 \log k$.

As potential open questions, we suggest trying to find the spectrum or just the exact value of the spectral gap for the simple random on finite Galton-Watson trees or for the frog model as presented in [16].

2 The spectrum of Q_h

This section is devoted to the proof of Theorem 1.1, which presents the full spectrum of Q_h and the corresponding eigenvectors.

Let λ be an eigenvalue of Q_h and let $E(\lambda)$ be the corresponding eigenspace. We first show that there exists an eigenvector in $E(\lambda)$ that has the form described in Theorem 1.1 (b).

Lemma 2.1. *The eigenspace $E(\lambda)$ contains an eigenvector f that has one of the following forms.*

(a) [Completely symmetric] $f(v) = f(w)$ for every $v, w \in V(\mathcal{T}_h)$ such that $\ell(v) = \ell(w)$. In this case we call f completely symmetric for $\mathcal{T}_h$.

(b) [Pseudo anti-symmetric] There is a node v and $i, j \in \{1, \ldots, d\}$ such that $f(w) = 0$ for every $w \notin V(\mathcal{T}_i^v \cup \mathcal{T}_j^v)$, $f|_{\mathcal{T}_i^v}$ and $f|_{\mathcal{T}_j^v}$ are completely symmetric, and $f|_{\mathcal{T}_i^v} = -f|_{\mathcal{T}_j^v}$. We call such f pseudo anti-symmetric.

The following illustrations explain what the described eigenvectors look like for binary trees.

Proof. Assume that $E(\lambda)$ does not contain a completely symmetric eigenvector. Let f be a nonzero element of $E(\lambda)$. Since f is not completely symmetric, there exist vertices of the same level at which f takes different values. Let v be a vertex with the largest $\ell(v)$ such that there are at least two of its children, say the i-th and j-th children, at which f has different values. For example, if there are two leaves u and w at which $f(u) \neq f(w)$ that have the same parent v' then we simply take v to be v'.

By the choice of v, $f|_{\mathcal{T}_k^v}$ is completely symmetric for all $k \in [d]$. Indeed, let u be the k-th child of v. We have $\mathcal{T}_k^v = \mathcal{T}^u$. By the choice of v, f takes the same value at all children of u. Let u_1, u_2 be two arbitrary children of u. Again by the choice of v, f takes

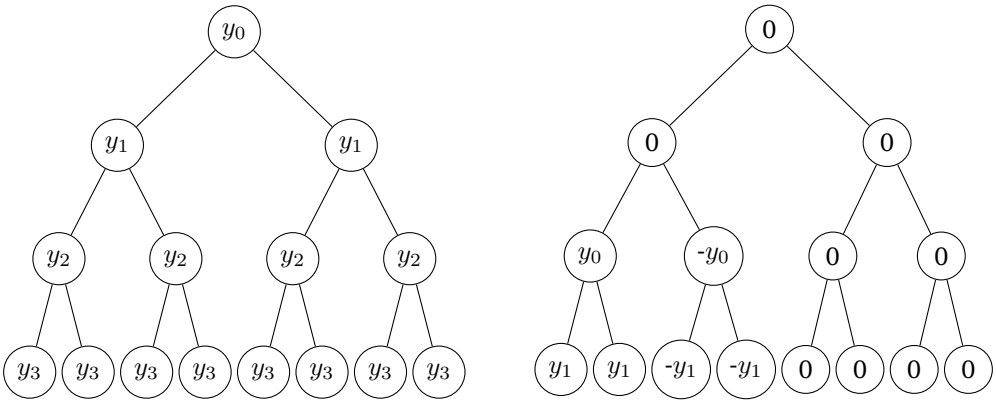

Figure 1: Completely symmetric eigenvectors (left) and pseudo anti-symmetric eigenvectors (right)

the same value, denoted by f_1, at all children of u_1, and the same value, denoted by f_2, at all children of u_2. Since f is an eigenvector of Q_h,

$$\lambda f(u_1) = \frac{d}{d+1} f_1 + \frac{1}{d+1} f(u) \quad \text{and} \quad \lambda f(u_2) = \frac{d}{d+1} f_2 + \frac{1}{d+1} f(u).$$

Since $f(u_1) = f(u_2)$, $f_1 = f_2$. Thus, f takes the same value at all grandchildren of u. Repeating this argument shows that $f|_{\mathcal{T}^u}$ is completely symmetric.

Consider the vector g obtained from f by switching its values on $\mathcal{T}_i^v$ and $\mathcal{T}_j^v$. More specifically, $g|_{\mathcal{T}_i^v} = f|_{\mathcal{T}_j^v}$, $g|_{\mathcal{T}_j^v} = f|_{\mathcal{T}_i^v}$, and $g = f$ elsewhere.

By the symmetry of the tree and the matrix Q_h, g also belongs to $E(\lambda)$. So is $f - g$, which we denote by h. Observe that h is an eigenvector that is 0 everywhere except on $\mathcal{T}_i^v \cup \mathcal{T}_j^v$ and $h|_{\mathcal{T}_i^v} = f|_{\mathcal{T}_i^v} - f|_{\mathcal{T}_j^v} = -h|_{\mathcal{T}_j^v}$. Moreover, h is completely symmetric when restricted to $\mathcal{T}_i^v$ and $\mathcal{T}_j^v$ because both f and g are, as seen above. Thus, $h \in E(\lambda)$ and is pseudo anti-symmetric. □

2.1 Completely symmetric eigenvectors

In this section, we describe completely symmetric eigenvectors. We show that the completely symmetric eigenvectors of Q_h are given by the formula (1.4) and correspond to λ and x satisfying (1.1) and (1.2) as in Theorem 1.1.

Since a completely symmetric eigenvector of Q_h has the same value at every node of the same level (see Figure 2), we can project it onto the path $[0, h]$ and obtain an eigenvector of the corresponding random walk on the path.

Lemma 2.2. *There are exactly $h + 1$ linearly independent completely symmetric eigenvectors of Q_h.*

Proof. Each symmetric eigenvector of Q_h corresponds one-to-one to an eigenvector of the following projection onto the path $[0, h]$ with transition matrix R_h:

- $R_h(0,1) = \frac{d}{d+1}$, $R_h(0,0) = \frac{1}{d+1}$,
- $R_h(l, l-1) = \frac{1}{d+1}$, $R_h(l, l+1) = \frac{d}{d+1}$ for all $1 \le l \le h-1$,
- $R_h(h, h-1) = \frac{1}{d+1}$, $R_h(h,h) = \frac{d}{d+1}$.

Since R_h is a reversible transition matrix with stationary distribution

$$\pi := [1, d, d^2, \ldots, d^h],$$

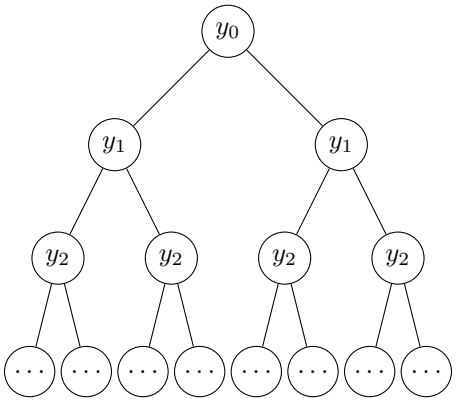

Figure 2: Completely symmetric eigenvectors

the matrix $A := D^{1/2}R_hD^{-1/2}$ is symmetric where D is the diagonal matrix with $D(x,x) = \pi(x)$. Therefore, A is diagonalizable and so is R_h. In other words, R_h has $h+1$ independent real eigenvectors. This implies that Q_h has $h+1$ linearly independent completely symmetric eigenvectors. □

Lemma 2.3. *The matrix R_h has 1 as an eigenvalue with multiplicity 1. Each of the remaining h eigenvalues $\lambda \neq 1$ of R_h is of the form*

$$\lambda = \frac{d}{d+1}\left(x + \frac{1}{xd}\right)$$

where $x \neq \pm\frac{1}{\sqrt{d}}$ is a non-real solution of the equation

$$d^{h+1}x^{2h+2} = 1.$$

This equation has exactly $2h$ such solutions. If x is a solution, so is $\frac{1}{xd}$. There is a 2-to-1 correspondence between x and λ. An eigenvector $y = (y_0, y_1, \ldots, y_h)$ of R_h with respect to λ is given by

$$y_i = \frac{dx^2 - x}{dx^2 - 1}x^i + \frac{x-1}{dx^2-1}\frac{1}{d^ix^i} \quad \text{for every } 0 \leq i \leq h.$$

The vector $f : \mathcal{T}_h \to \mathbb{R}$ that takes value y_i at all nodes of depth i is an eigenvector of Q_h with respect to λ.

Proof. Let λ be an eigenvalue of R_h and $y = (y_0, y_1, \ldots, y_h)$ be an eigenvector corresponding to λ. We have

(R1) $\frac{1}{d+1}y_0 + \frac{d}{d+1}y_1 = \lambda y_0$,

(R2) $\frac{1}{d+1}y_{i-1} + \frac{d}{d+1}y_{i+1} = \lambda y_i$ for all $1 \leq i \leq h-1$,

(R3) $\frac{1}{d+1}y_{h-1} + \frac{d}{d+1}y_h = \lambda y_h$.

Since y is not the zero vector, the above equations imply that $y_0 \neq 0$. Without loss of generality, we assume $y_0 = 1$.

Let x_1, x_2 be the solutions to the characteristic equation of (R2):

$$\frac{1}{d+1} - \lambda x + \frac{d}{d+1}x^2 = 0$$

or equivalently

$$dx^2 - (d+1)\lambda x + 1 = 0. \tag{2.1}$$

By (2.1), we have

$$x_1 x_2 = \frac{1}{d}$$

and

$$\lambda = \frac{d}{d+1}(x_1 + x_2) = \frac{d}{d+1}\left(x_1 + \frac{1}{x_1 d}\right). \tag{2.2}$$

If $x_1 \neq x_2$ then we can write $y_0 = \alpha_1 - \alpha_2$, $y_1 = \alpha_1 x_1 - \alpha_2 x_2$ for some α_1, α_2. We show that for all $0 \leq i \leq h$,

$$y_i = \alpha_1 x_1^i - \alpha_2 x_2^i. \tag{2.3}$$

Indeed, assuming that (2.3) holds for $y_0, \ldots, y_i$ for some $1 \leq i \leq h-1$ then by (2.1),

$$\begin{aligned} \lambda y_i - \frac{1}{d+1} y_{i-1} &= \alpha_1 x_1^{i-1}\left(\lambda x_1 - \frac{1}{d+1}\right) - \alpha_2 x_2^{i-1}\left(\lambda x_2 - \frac{1}{d+1}\right) \\ &= \frac{d}{d+1}\left(\alpha_1 x_1^{i+1} - \alpha_2 x_2^{i+1}\right). \end{aligned}$$

Thus, by (R2),

$$\frac{d}{d+1} y_{i+1} = \frac{d}{d+1}\alpha_1 x_1^{i+1} - \frac{d}{d+1}\alpha_2 x_2^{i+1}$$

and so

$$y_{i+1} = \alpha_1 x_1^{i+1} - \alpha_2 x_2^{i+1}.$$

Thus, (2.3) also holds for y_{i+1} and hence, for all $y_0, \ldots, y_h$.

Similarly, by (R3), we get

$$\begin{aligned} \frac{d}{d+1} y_h &= \lambda y_h - \frac{1}{d+1} y_{h-1} = \alpha_1 x_1^{h-1}\left(\lambda x_1 - \frac{1}{d+1}\right) - \alpha_2 x_2^{h-1}\left(\lambda x_2 - \frac{1}{d+1}\right) \\ &= \frac{d}{d+1}\left(\alpha_1 x_1^{h+1} - \alpha_2 x_2^{h+1}\right). \end{aligned}$$

Thus,

$$\alpha_1 x_1^{h+1} - \alpha_2 x_2^{h+1} = \alpha_1 x_1^h - \alpha_2 x_2^h \tag{2.4}$$

as they are both equal to y_h.

By (2.3), (R1) becomes

$$d(\alpha_1 x_1 - \alpha_2 x_2) = \left(x_1 d + \frac{1}{x_1} - 1\right)(\alpha_1 - \alpha_2). \tag{2.5}$$

For simplicity, we write $\alpha = \alpha_1$ and $x = x_1$. Recall that $y_0 = 1$ and so by (2.3) for $i = 0$, we get

$$\alpha_2 = \alpha - 1.$$

Equation (2.5) becomes

$$d\alpha x - \frac{\alpha - 1}{x} = dx + \frac{1}{x} - 1$$

which gives

$$\alpha_1 = \alpha = \frac{dx^2 - x}{dx^2 - 1} \quad \text{and} \quad \alpha_2 = \alpha - 1 = \frac{1 - x}{dx^2 - 1}. \tag{2.6}$$

Plugging (2.6) into (2.4) and taking into account $x_2 = \frac{1}{xd}$, we get

$$(dx - 1)(x - 1)(d^{h+1} x^{2h+2} - 1) = 0.$$

If $x = 1$ then $\alpha_2 = \alpha - 1 = 0$ by (2.6). And so, $y = \alpha(1, \dots, 1)$ which is an eigenvector of the eigenvalue 1. Since $\lambda \neq 1$, $x \neq 1$. If $x = \frac{1}{d}$ then $x_2 = \frac{1}{xd} = 1$. By the symmetry of x_1 and x_2, this also corresponds to $\lambda = 1$ which is not the case.

Thus, x satisfies

$$d^{h+1}x^{2h+2} - 1 = 0.$$

This equation has $2h$ non-real solutions and 2 real solutions $\pm\frac{1}{\sqrt{d}}$. For each non-real solution x_1, observe that $x_2 := \frac{1}{dx_1}$ is also a non-real solution. Note that $x_1 \neq x_2$ and by setting λ and y as in (2.2) and (2.3) with α_1 and α_2 as in (2.6), one can check that y is indeed an eigenvector corresponding to λ. Thus, these $2h$ non-real solutions correspond to exactly h eigenvalues $\lambda \neq 1$ of R_n. Since R_n has exactly $h + 1$ eigenvalues, these are all. □

2.2 Pseudo anti-symmetric eigenvectors

In this section, we describe the pseudo anti-symmetric eigenvectors. We show that the pseudo anti-symmetric eigenvectors of Q_h are given by the formula (1.5) and correspond to λ and x satisfying (1.1) and (1.3) as in Theorem 1.1.

Consider a pseudo anti-symmetric eigenvector f with node v and indices i, j as described in Lemma 2.1 (see Figure 1). Let $k = h - \ell(v) - 1 \in [0, h-1]$. As in Figure 1 and Figure 3 below, let $y = (y_0, y_1, \dots, y_k)$ where y_0 is the value of f at the i-th child of v, which is denoted by u, y_1 is the value of f at the children of u and so on. With these notations, we also write f as $f_{y,v,i,j}$. Observe that y is an eigenvector of the following matrix S_k:

- $S_k(0,1) = \frac{d}{d+1}$,
- $S_k(l, l-1) = \frac{1}{d+1}$, $S_k(l, l+1) = \frac{d}{d+1}$ for all $1 \leq l \leq k-1$,
- $S_k(k, k-1) = \frac{1}{d+1}$, $S_k(k,k) = \frac{d}{d+1}$.

Conversely, for any eigenvector y of S_k, for any node v at depth $h - k - 1$ and for any choice of $i, j \in [1, d]$ with $i \neq j$, we can lift it to a pseudo anti-symmetric eigenvector $f_{y,v,i,j}$.

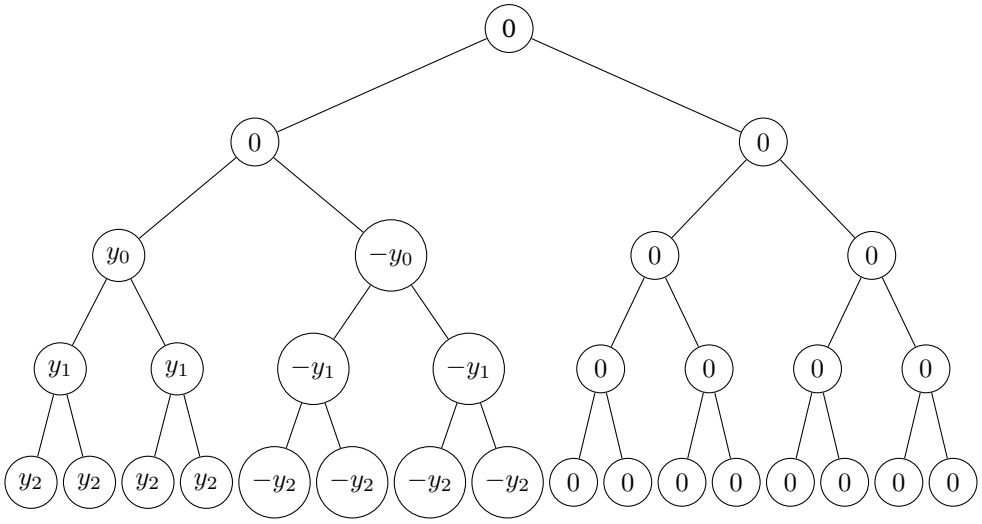

Figure 3: Pseudo anti-symmetric eigenvectors

Lemma 2.4. *For each $k \in [0, h-1]$, S_k has $k+1$ eigenvectors. For each eigenvector y of S_k and for each v with $\ell(v) = h-k-1$, there are $d-1$ linearly independent pseudo anti-symmetric eigenvectors of Q_h of the form $f_{y,v,i,j}$.*

Proof. Since S_k differs from R_k only at the $(0,0)$ entry, it also satisfies the equation $\pi(x)S_k(x,y) = \pi(y)S_k(y,x)$ where $\pi = [1, d, d^2, \ldots, d^k]$. Thus, like R_k, the matrix DS_kD^{-1} is symmetric where D is the diagonal matrix with $D(x,x) = \pi(x)^{1/2}$. By symmetry, DS_kD^{-1} has $k+1$ eigenvalues and so does S_k.

For each eigenvector y of S_k, we create $d-1$ linearly independent vectors $f_{y,v,i,i+1}$ for $1 \le i \le d-1$. It is clear that any $f_{y,v,i,j}$ can be written as a linear combination of these vectors. This completes the proof. □

We now describe the eigenvectors of S_k.

Lemma 2.5. *Each of the $k+1$ eigenvalues λ of S_k is of the form*

$$\lambda = \frac{d}{d+1}\left(x + \frac{1}{dx}\right)$$

where $x \neq \pm\frac{1}{\sqrt{d}}$ is a solution of the equation

$$d^{k+2}x^{2k+4} - d^{k+2}x^{2k+3} + dx - 1 = 0.$$

This equation has $2k+2$ solutions that differ from $\frac{1}{\sqrt{d}}$. If x is a solution, so is $\frac{1}{dx}$. There is a 2-to-1 correspondence between x and λ. An eigenvector $y = (y_0, y_1, \ldots, y_k)$ of S_k with respect to λ is given by

$$y_i = \frac{dx^{i+2}}{dx^2-1} - \frac{1}{(dx^2-1)d^ix^i} \quad \text{for every } 0 \le i \le k.$$

Proof. Let λ be an eigenvalue of S_k and $y = (y_0, y_1, \ldots, y_k)$ be an eigenvector corresponding to λ. We have

(S1) $\frac{d}{d+1}y_1 = \lambda y_0$,

(S2) $\frac{1}{d+1}y_{i-1} + \frac{d}{d+1}y_{i+1} = \lambda y_i$ for all $1 \le i \le k-1$,

(S3) $\frac{1}{d+1}y_{k-1} + \frac{d}{d+1}y_k = \lambda y_k$.

As before, we let x_1, x_2 be the solutions to the equation

$$\frac{1}{d+1} - \lambda x + \frac{d}{d+1}x^2 = 0.$$

By exactly the same argument as in the proof of Lemma 2.3, we derive by setting $y_0 = 1$ that

$$y_i = \alpha_1 x_1^i - \alpha_2 x_2^i$$

where

$$\alpha_1 = \frac{dx^2}{dx^2-1} \quad \text{and} \quad \alpha_2 = \frac{1}{dx^2-1}$$

and x_1 and x_2 satisfy

$$d^{k+2}x^{2k+4} - d^{k+2}x^{2k+3} + dx - 1 = 0 \tag{2.7}$$

Note that, $x = \pm\frac{1}{\sqrt{d}}$ are solutions of (2.7). The remaining $2k+2$ solutions split into pairs $(x, \frac{1}{dx})$ of distinct components. For each of these pairs, let $x_1 := x$ and $x_2 := \frac{1}{dx}$. We have $x_1 \neq x_2$ and by setting λ and y as in (2.2) and (2.3) with $\alpha_1 = \frac{dx^2}{dx^2-1}$ and $\alpha_2 = \frac{1}{dx^2-1}$, one can check that y is indeed an eigenvector corresponding to λ. Thus, these $2k+2$ solutions correspond to exactly $k+1$ eigenvalues λ of S_k. Since S_k has exactly $k+1$ eigenvalues, these are all of them. □

2.3 Proof of Theorem 1.1

The following lemma shows that we can retrieve all eigenvectors of Q_h from completely symmetric and pseudo anti-symmetric eigenvectors. Let $\mathcal{A}_{S_k}$ be the eigenbasis of S_k as described in Lemma 2.5 and $\mathcal{B}$ be a collection of $h+1$ independent completely symmetric eigenvectors of Q_h as in Lemma 2.3. Let

$$\mathcal{A} := \{f_{y,v,i,i+1}, v \in V(\mathcal{T}_{h-1}), y \in \mathcal{A}_{S_{h-\ell(v)-1}}, i \in [d-1]\}.$$

Lemma 2.6. *The collection $\mathcal{A} \cup \mathcal{B}$ is an eigenbasis for Q_h.*

Assuming Lemma 2.6, we now put everything together to complete the proof of Theorem 1.1.

Proof of Theorem 1.1. The first part of the theorem follows from Lemmas 2.3 and 2.5. As seen in Lemma 2.3, the set $\mathcal{B}$ in Lemma 2.6 consists of eigenvectors as in (1.4) and the all-1 vector. By Lemmas 2.4 and 2.5, the set $\mathcal{A}$ consists of eigenvectors as in (1.5). That gives the second part. Finally, the third part follows from Lemma 2.6. □

Before proving Lemma 2.6, we make the following simple observation. For a rooted-tree T that is not necessarily regular, recall that a vector $f : T \to \mathbb{R}$ is said to be *completely symmetric* if $f(u) = f(v)$ for all pairs of vertices u, v at the same level. A vector f is said to be *energy-preserving* if for all levels l,

$$\sum_{v \in T : \ell(v) = l} f(v) = 0.$$

Observation 2.7. For any rooted-tree T and any vector $f : T \to \mathbb{R}$, if f is both energy-preserving and completely symmetric then it is the zero vector.

Proof of Lemma 2.6. First of all, we check that their number is equal to n. By Lemmas 2.2 and 2.4, the total number of vectors is

$$h + 1 + \sum_{k=0}^{h-1} (k+1)(d-1)d^{h-k-1}$$

where d^{h-k-1} is the number of nodes v of depth $h-k-1$. By algebraic manipulation, this number is exactly $\frac{d^{h+1}-1}{d-1} = n$.

We now prove that the vectors considered are linearly independent. Assume that there exist coefficients $c_{y,v,i}$ and c_g such that

$$\sum c_{y,v,i} f_{y,v,i,i+1} + \sum_{g \in \mathcal{B}} c_g g = 0$$

where the first sum runs over all $v \in V(\mathcal{T}_{h-1}), y \in \mathcal{A}_{S_{h-\ell(v)-1}}$ and $i \in [d-1]$. We need to show that $c_{y,v,i}$ and c_g are all 0.

Since pseudo anti-symmetric vectors are energy-preserving on $\mathcal{T}_h$, the sum

$$\sum_{g \in \mathcal{B}} c_g g = -\sum c_{y,v,i} f_{y,v,i,i+1}$$

is both completely symmetric and energy-preserving. Thus, by Observation 2.7,

$$\sum c_{y,v,i} f_{y,v,i,i+1} = \sum_{g \in \mathcal{B}} c_g g = 0. \tag{2.8}$$

By the independence of vectors in $\mathcal{B}$, we conclude that $c_g = 0$ for all $g \in \mathcal{B}$.

To complete the proof of the lemma, we now prove by induction on the vertices of $v \in V(\mathcal{T}_{h-1})$ and $i \in [d-1]$ that $c_{y,v,i} = 0$ for all $y \in \mathcal{A}_{S_{h-\ell(v)-1}}$. For this induction, we use the natural ordering of pairs (v,i) as follows.

$$\{(v,i) < (v',i')\} \quad \text{if and only if} \quad \{\ell(v) < \ell(v')\} \text{ or } \{\ell(v) = \ell(v') \text{ and } i < i'\}.$$

For the base case, which is $v := \rho$ and $i := 1$, from (2.8), we have

$$F_{\rho,1} := \sum_{y \in \mathcal{A}_{S_{h-1}}} c_{y,\rho,1} f_{y,\rho,1,2} = -\sum c_{y,u,j} f_{y,u,j,j+1}$$

where the second sum runs over all $u \in V(\mathcal{T}_{h-1})$ and $j \in [d-1]$ with $(\rho,1) < (u,j)$ and all $y \in \mathcal{A}_{S_{h-\ell(v)-1}}$. When restricting on the subtree $\mathcal{T}_1^\rho$, $F_{\rho,1}$ is a completely symmetric vector because all of the $f_{y,\rho,1,2}$ are completely symmetric. Likewise, $F_{\rho,1}$ is energy-preserving on $\mathcal{T}_1^\rho$, because of the vectors $f_{y,u,j,j+1}$. By Observation 2.7, $F_{\rho,1} = 0$ on $\mathcal{T}_1^\rho$. Since the $f_{y,\rho,1,2}$ are only supported on $\mathcal{T}_1^\rho \cup \mathcal{T}_2^\rho$ and $f_{y,\rho,1,2}|_{\mathcal{T}_1^\rho} = -f_{y,\rho,1,2}|_{\mathcal{T}_2^\rho}$, we have $F_{\rho,1}|_{\mathcal{T}_1^\rho} = -F_{\rho,1}|_{\mathcal{T}_2^\rho}$. Therefore, $F_{\rho,1} = 0$ on $\mathcal{T}_2^\rho$ and thus on $\mathcal{T}_h$. So,

$$\sum_{y \in \mathcal{A}_{S_{h-1}}} c_{y,\rho,1} f_{y,\rho,1,2} = 0.$$

By the independence of vectors in $\mathcal{A}_{S_{h-1}}$, we conclude that $c_{y,\rho,1} = 0$ for all $y \in \mathcal{A}_{S_{h-1}}$, establishing the base case.

For the induction step, assume that for some (v,i), it is proven that $c_{y,w,k} = 0$ for all $(w,k) < (v,i)$ and $y \in \mathcal{A}_{S_{h-\ell(w)-1}}$. We now show that $c_{y,v,i} = 0$ for all $y \in \mathcal{A}_{S_{h-\ell(v)-1}}$. By this assumption, the left-most side in (2.8) reduces to

$$\sum c_{y,u,j} f_{y,u,j,j+1} = 0 \tag{2.9}$$

where the sum runs over all $(u,j) \geq (v,i)$. Our argument now is similar to the base case. From (2.9), we have

$$F_{v,i} := \sum_{y \in \mathcal{A}_{S_{h-\ell(v)-1}}} c_{y,v,i} f_{y,v,i,i+1} = -\sum_{y,(v,i)<(u,j)} c_{y,u,j} f_{y,u,j,j+1}.$$

Similarly to the base case, when restricting on the subtree $\mathcal{T}_i^v$, $F_{v,i}$ is both completely symmetric and energy-preserving on $\mathcal{T}_j^v$. By Observation 2.7, $F_{v,i} = 0$ on $\mathcal{T}_j^v$. This leads to $F_{v,i} = 0$ on $\mathcal{T}_{i+1}^v$ and thus $F_{v,i} = 0$ on $\mathcal{T}_h$. So,

$$\sum_{y \in \mathcal{A}_{S_{h-\ell(v)-1}}} c_{y,v,i} f_{y,v,i,i+1} = 0.$$

By the independence of vectors in $\mathcal{A}_{S_{h-\ell(v)-1}}$, we conclude that $c_{y,v,i} = 0$ for all $y \in \mathcal{A}_{S_{h-\ell(v)-1}}$, establishing the induction step and thus finishing the proof. □

3 Proof of Theorem 1.2

In this section, we prove Theorem 1.2, which gives the spectral gap for the interchange process on T_h and a lower bound for its mixing time.

3.1 Proof of Theorem 1.2(a)

Consider the interchange process on $\mathcal{T}_h$. Let Q'_h be the transition matrix of the ace of spades. In other words, Q'_h is the transition matrix of any fixed card on the tree. By Aldous' spectral gap conjecture proved in [8, Theorem 1.1], the spectral gap of the

interchange process on the complete d-ary tree of depth h is the same as the spectral gap of Q'_h. We note that

$$Q'_h = \frac{2n-d-3}{2(n-1)} I_n + \frac{d+1}{2(n-1)} Q_h. \tag{3.1}$$

And therefore, the spectral gap of Q'_h is $\frac{d+1}{2(n-1)}$ times the spectral gap of Q_h.

Thus Theorem 1.2 (a) is deduced from the following.

Lemma 3.1. *For sufficiently large h, the spectral gap of Q_h is equal $1-\lambda_2$ where λ_2 is the second largest eigenvalue of Q_h. Moreover,*

$$\lambda_2 = 1 - \frac{(d-1)^2}{(d+1)\cdot d^{h+1}} + O\left(\frac{\log_d n}{n^2}\right) = 1 - \frac{(d-1)^2}{(d+1)\cdot[(d-1)n+1]} + O\left(\frac{\log_d n}{n^2}\right). \tag{3.2}$$

To prove Lemma 3.1, we use Theorem 1.1. Let λ be an eigenvalue of Q_h and x be a solution of

$$dx^2 - (d+1)\lambda x + 1 = 0 \tag{3.3}$$

which we have encountered in (2.1).

If $\lambda^2 \geq \frac{4d}{(d+1)^2}$ then this equation has two real solutions both of which have the same sign as λ.

In Equation (1.2), we have $d^{h+1}x^{2h+2} = 1$ which only has nonreal solutions except $x = \pm\frac{1}{\sqrt{d}}$. Combine this observation with Theorem 1.1, each eigenvalue $\lambda^2 \geq \frac{4d}{(d+1)^2}$ is given by Equation (1.1) for some $x \neq \pm\frac{1}{\sqrt{d}}$ satisfying

$$d^{k+1}x^{2k+2} - d^{k+1}x^{2k+1} + dx - 1 = 0, \tag{3.4}$$

for $k \in [1,h]$. This is simply Equation (1.3) (with k being shifted for notational convenience).

We show the following

Lemma 3.2. *(a) For all $k \in [1,h]$, Equation (3.4) has no solutions in $\left(-\infty, -\frac{1}{\sqrt{d}}\right)$. There are no eigenvalues of Q_h less than $-\sqrt{\frac{4d}{(d+1)^2}}$.*

(b) There exists a constant $h_0 > 0$ such that for all $k \geq h_0$, the largest solution x of (3.4) satisfies

$$1 - \frac{a}{d^{k+1}} < x < 1 - \frac{d-1}{d^{k+1}} \quad \text{where} \quad a = d - 1 + \frac{2(d-1)^2(k+1)}{d^{k+1}}. \tag{3.5}$$

Furthermore, for $k = h$, the eigenvalue that corresponds to this x satisfies

$$\left|\lambda - \left(1 - \frac{(d-1)^2}{(d+1)\cdot d^{h+1}}\right)\right| = O\left(\frac{\log_d n}{n^2}\right). \tag{3.6}$$

Assuming Lemma 3.2, we conclude that for sufficiently large h, the largest x that satisfies one of the equations (3.4) for some k in $[1,h]$ satisfies

$$1 - \frac{a}{d^{h+1}} < x < 1 - \frac{d-1}{d^{h+1}} \quad \text{where} \quad a = d - 1 + \frac{2(d-1)^2(h+1)}{d^{h+1}}.$$

Since the right-hand side of (1.1) is increasing in x for $x \geq \frac{1}{\sqrt{d}}$, the second largest eigenvalue λ_2 of Q_h corresponds to such x and so it satisfies (3.6), proving (3.2). By the first part of Lemma 3.2, there are no eigenvalues of Q_h whose absolute value is larger than λ_2. This proves Lemma 3.1.

Proof of Lemma 3.2. Let $f(x) = d^{k+1}x^{2k+2} - d^{k+1}x^{2k+1} + dx - 1$.

To prove part (a), for all $x < -\frac{1}{\sqrt{d}}$, we have

$$d^{k+1}x^{2k+2} > 1 \quad \text{and} \quad -d^{k+1}x^{2k+1} > -dx$$

and so f has no roots in $\left(-\infty, -\frac{1}{\sqrt{d}}\right)$. Assume that there were an eigenvalue $\lambda < -\sqrt{\frac{4d}{(d+1)^2}}$. In this case, $\lambda^2 \geq \frac{4d}{(d+1)^2}$ and so Equation (3.3) has two negative solutions $x_1 < x_2$ with $x_1 x_2 = \frac{1}{d}$. We conclude that $x_1 < -\frac{1}{\sqrt{d}}$. This is a contradiction because x_1 satisfies (3.4) for some k while for all k the function f has no roots less than $-\frac{1}{\sqrt{d}}$.

To prove part (b), for all $x \geq \frac{2k+1}{2k+2}$, we have

$$f'(x) = d^{k+1}x^{2k}\left((2k+2)x - (2k+1)\right) + d > 0.$$

Thus, f is increasing on the interval $[1 - \frac{1}{2k+2}, \infty)$ which contains $[1 - \frac{a}{d^{k+1}}, 1 - \frac{d-1}{d^{k+1}}]$ for sufficiently large k. Thus, to prove (3.5), it suffices to show that

$$f\left(1 - \frac{a}{d^{k+1}}\right) < 0 < f\left(1 - \frac{d-1}{d^{k+1}}\right) \tag{3.7}$$

for sufficiently large k. Indeed,

$$\begin{aligned} f\left(1 - \frac{a}{d^{k+1}}\right) &= d - 1 - a\left(1 - \frac{a}{d^{k+1}}\right)^{2k+1} - \frac{a}{d^k} \\ &< d - 1 - a\left(1 - \frac{a}{d^{k+1}}\right)^{2k+1} < d - 1 - a\left(1 - \frac{a(2k+1)}{d^{k+1}}\right). \end{aligned}$$

Using $a = d - 1 + \frac{2(d-1)^2(k+1)}{d^{k+1}}$ and standard manipulation gives that

$$f\left(1 - \frac{a}{d^{k+1}}\right) < -\frac{2(d-1)^2(k+1)}{d^{k+1}} + \frac{a^2(2k+1)}{d^{k+1}} \leq 0,$$

for sufficiently large k. Thus the first inequality of (3.7) holds. For the second inequality, we have

$$\begin{aligned} f\left(1 - \frac{d-1}{d^{k+1}}\right) &= d - 1 - (d-1)\left(1 - \frac{d-1}{d^{k+1}}\right)^{2k+1} - \frac{d-1}{d^k} \\ &\geq d - 1 - (d-1)\left(1 - \frac{3(d-1)}{d^{k+1}}\right) - \frac{d-1}{d^k} > 0, \end{aligned}$$

establishing (3.7).

We have shown that there exists a solution $x = 1 - \alpha$ where $\frac{d-1}{d^{k+1}} \leq \alpha \leq \frac{a}{d^{k+1}}$. Let λ be the eigenvalue corresponding to x as in (2.2). We have

$$\frac{d+1}{d}\lambda = 1 - \alpha + \frac{1}{d(1-\alpha)} \in \left(1 - \alpha + \frac{1}{d}(1+\alpha), 1 - \alpha + \frac{1}{d}(1 + \alpha + 2\alpha^2)\right).$$

In other words,

$$\frac{d+1}{d}\lambda \in \left(\frac{d+1}{d} - \frac{d-1}{d}\alpha, \frac{d+1}{d} - \frac{d-1}{d}\alpha + \frac{2}{d}\alpha^2\right).$$

Using the bounds $\frac{d-1}{d^{k+1}} \leq \alpha \leq \frac{a}{d^{k+1}}$, we obtain

$$\lambda - \left(1 - \frac{(d-1)^2}{(d+1)\cdot d^{k+1}}\right) \leq \frac{2}{d+1}\alpha^2 \leq \frac{2a^2}{(d+1)\cdot d^{2k+2}} \leq \frac{2}{(d+1)\cdot d^{2k+1}}$$

and

$$\lambda - \left(1 - \frac{(d-1)^2}{(d+1)\cdot d^{k+1}}\right) \geq -\frac{d-1}{d+1}\alpha + \frac{(d-1)^2}{(d+1)\cdot d^{k+1}} \geq -\frac{2(d-1)^3(k+1)}{(d+1)\cdot d^{2k+2}} \geq -\frac{2(k+1)}{d^{2k}}.$$

Thus, for $k = h$,

$$\left|\lambda - \left(1 - \frac{(d-1)^2}{(d+1)\cdot d^{h+1}}\right)\right| \leq \frac{2(h+1)}{d^{2h}}.$$

These bounds together with the equation $n = \frac{d^{h+1}-1}{d-1} \in (d^h, 2d^h)$ give part (b). □

3.2 Proof of Theorem 1.2 (b)

For the proof of the lower bound, we use Wilson's lemma.

Lemma 3.3 (Lemma 5, [25])**.** *Let ε and R be positive numbers and $0 < \gamma < 2 - \sqrt{2}$. Let $F : X \to \mathbb{R}$ be a function on the state space X of a Markov chain (C_t) such that*

$$\mathbb{E}[F(C_{t+1})|C_t] = (1-\gamma)F(C_t), \quad \mathbb{E}\Big[[F(C_{t+1}) - F(C_t)]^2 \,|C_t\Big] \leq R,$$

and

$$t \leq \frac{\log \max_{x\in X} F(x) + \frac{1}{2}\log(\gamma\varepsilon/(4R))}{-\log(1-\gamma)}.$$

Then the total variation distance from stationarity at time t is at least $1 - \varepsilon$.

Proof of Theorem 1.2 (b). Let $0 < x < 1$ be a solution of (1.3) (for $k = h$) satisfying (3.5) and λ be the eigenvalue of Q_h corresponding to x. In particular,

$$\lambda = \frac{d}{d+1}\left(x + \frac{1}{dx}\right).$$

Let $f : \mathcal{T}_h \to \mathbb{R}$ be an eigenvector of Q_h corresponding to λ. As in the proof of Lemma 2.5, we can choose f as follows.

$$f(v) = \begin{cases} 0 & \text{if } v \notin T_1^\rho \cup T_2^\rho, \\ dx^{\ell(v)+2} - \frac{1}{d^{\ell(v)-1}x^{\ell(v)-2}}, & \text{if } v \in T_1^\rho, \\ -dx^{\ell(v)+2} + \frac{1}{d^{\ell(v)-1}x^{\ell(v)-2}}, & \text{if } v \in T_2^\rho. \end{cases} \tag{3.8}$$

We now consider the interchange process on the d-ary tree $\mathcal{T}_h$. Fix an arbitrary enumeration of the vertices of $\mathcal{T}_h$ by $1, 2, \ldots, n$. Let $\sigma \in S_n$. We define $F(\sigma) = \sum_{v=1}^n f(v)f(\sigma(v))$. Then, we have that

$$\mathbb{E}[F(\sigma_{t+1})|\sigma_t] = \frac{1}{n-1}\sum_e \mathbb{E}[F(\sigma_{t+1})|\sigma_t, e],$$

where the sum runs over all $n-1$ edges e of the tree and the conditioning on the right is conditioning on the edge e being chosen. So,

$$\begin{aligned} \mathbb{E}[F(\sigma_{t+1})|\sigma_t] &= \frac{1}{n-1}\sum_e \sum_{v=1}^n f(v)\mathbb{E}\left[f(\sigma_{t+1}(v))|\sigma_t, e\right] \\ &= \frac{1}{n-1}\sum_{v=1}^n f(v)\sum_e \mathbb{E}\left[f(\sigma_{t+1}(v))|\sigma_t, e\right]. \end{aligned}$$

By direct computation, we obtain

$$\sum_e \mathbb{E}\left(f(\sigma_{t+1}(v))|\sigma_t, e\right) = \left(n - \frac{d}{2} - \frac{3}{2} + \frac{\lambda(d+1)}{2}\right) f(\sigma_t(v)).$$

So, we have

$$\mathbb{E}[F(\sigma_{t+1})|\sigma_t)] = \frac{n - \frac{d}{2} - \frac{3}{2} + \frac{\lambda(d+1)}{2}}{n-1} F(\sigma_t).$$

For n sufficiently big, we have that

$$F(id) = \sum_{v=1}^{n} f^2(v) \geq 2d^{h-1} \left(dx^{h+2} - \frac{1}{d^{h-1}x^{h-2}} \right)^2 \geq \frac{dn}{2},$$

where the first inequality arises from keeping only the leaves of $T_1^\rho \cup T_2^\rho$ and the last inequality follows from Lemma 3.2 (b).

Finally, we consider what happens if we change a configuration σ_t by transposing an edge e, which connects two vertices u and v. We have that

$$|F(\sigma_{t+1}) - F(\sigma_t)| = |(f(u) - f(v))(f(\sigma(u)) - f(\sigma(v)))|.$$

For all vertices w, we have by definition of f,

$$|f(w)| \leq d$$

and by Lemma 3.2 (b), assuming wlog that $\ell(u) = \ell(v) + 1 =: l + 1$,

$$|f(v) - f(u)| \leq dx^{l+2}(1-x) + \frac{1}{d^{l-1}x^{l-2}} \left(1 - \frac{1}{dx}\right) \leq \frac{1}{d^{h-1}} + \frac{1}{d^{l-1}} = \frac{2}{d^{l-1}}.$$

Thus, $|(f(u) - f(v))(f(\sigma(u)) - f(\sigma(v)))|^2 \leq \frac{16}{d^{2l-4}}$. The definition of f given by (3.8) gives that the left-hand side is 0 if neither u nor v belongs to $T_1^\rho \cup T_2^\rho$. And so,

$$\mathbb{E}_e \left((F(\sigma_{t+1}) - F(\sigma_t))^2 \, |\sigma_t \right) \leq \frac{1}{n-1} \sum_{l=0}^{h} 2d^l \frac{16}{d^{2l-4}} \leq \frac{64d^4}{n-1},$$

where $2d^l$ is the number of edges e that connect levels l and $l+1$ of $T_\rho^1 \cup T_\rho^2$.

So we can take $R = \frac{64d^4}{n-1}$.

Set $\lambda' = \frac{n - \frac{d}{2} - \frac{3}{2} + \frac{\lambda(d+1)}{2}}{n-1}$. Using Wilson's lemma, we have that if

$$t \leq t_0 := \frac{\log(F(id)) + \frac{1}{2}\log((1-\lambda')\varepsilon/(4R))}{-\log(\lambda')}$$

then the total variation distance is at least $1 - \varepsilon$. By Lemma 3.1,

$$\lambda = 1 - \frac{(d-1)^2}{(d+1) \cdot d^{h+1}} + O\left(\frac{\log_d n}{n^2}\right),$$

which gives

$$1 - \lambda' = \frac{(d+1)(1-\lambda)}{2(n-1)} = \frac{(d-1)^2}{2(n-1)d^{h+1}} + O\left(\frac{d\log_d n}{n^3}\right).$$

We get

$$\begin{aligned} t_0 &= \frac{\frac{\log n}{2} + \frac{\log \varepsilon}{2} + O(\log d)}{\frac{(d-1)^2}{2(n-1)d^{h+1}} + O\left(\frac{d\log_d n}{n^3}\right)} = \left(\frac{\log n}{2} + \frac{\log \varepsilon}{2} + O(\log d)\right)\left(\frac{2n^2}{d-1} + O\left(n\log_d n\right)\right) \\ &= \frac{n^2\left(\log n + \log \varepsilon\right)}{d-1} + O\left(n^2\right). \end{aligned}$$

This completes the proof. □

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