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Limit theorems for trawl processes*

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Abstract

In this work we derive limit theorems for trawl processes. First, we study the asymptotic behavior of the partial sums of the discretized trawl process $(X_{i\Delta_n})_{i=0}^{\lfloor nt \rfloor -1}$, under the assumption that as $n \uparrow \infty$, $\Delta_n \downarrow 0$ and $n\Delta_n \to \mu \in [0, +\infty]$. Second, we prove a general result on functional convergence in distribution of trawl processes. As an application of this result, we show that a trawl process whose Lévy measure tends to infinity converges in distribution, under suitable rescaling, to a Gaussian moving average process.

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1 Introduction

In this paper, we study probabilistic limit theorems for a class of stationary infinitely divisible stochastic processes called *trawl processes*, which were introduced for the first time in 2011 by Barndorff-Nielsen [2]. By construction, a trawl process allows for both a very flexible autocorrelation structure and the possibility of generating any kind of marginal distribution within the class of infinitely divisible distributions. Often

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the marginal distribution is chosen among infinitely divisible distributions on positive integers, with a view to applying the process as a model of serially correlated temporal count data, although in general such an assumption is not necessary.

Barndorff-Nielsen et al. [5] provide the first systematic study of trawl processes, investigating their probabilistic properties and analyzing volatility modulation within this framework. A distinctive feature of the class of trawl processes is that it allows to model independently its correlation structure from its marginal distribution. More specifically, if L denotes a homogeneous Lévy basis on $\mathbb{R}^2$ (see Section 2), and $A = \{(r, y) : r \leq 0, 0 \leq y \leq a(-r)\}$ where $a : \mathbb{R}^+ \to \mathbb{R}^+$ is a non-increasing integrable function, then $X_t := L(A_t),\ \ t \in \mathbb{R}$, where $A_t := A + (t, 0)$, is termed as trawl process. Under this framework, L determines the distribution of X_t while the autocorrelation function of X is described by the so-called trawl function a via the relation

$$\rho_X(h) = \frac{\int_h^\infty a(s)\mathrm{d}s}{\int_0^\infty a(s)\mathrm{d}s},\ \ h \geq 0.$$

This expression reveals that if high-frequency observations of X, say $(X_{\Delta_n k})_{k=0,1,\ldots,n}$, are available, then inference on a can be done through the sample autocovariance function which, as it is well known, consists of a quadratic and a linear functional of X. Motivated by this, during the first part of this paper we focus on the linear component of the sample autocovariance function, i.e. we study the asymptotic behavior of the (centered) partial sums

$$\mathbf{S}^n = \left(\sum_{k=0}^{\lfloor nt \rfloor -1} (X_{\Delta_n k} - \mathbb{E}(X_{\Delta_n k})) \right)_{t \geq 0},$$

where $(\Delta_n)_{n\in\mathbb{N}}$ is a sequence of non-negative constants such that $\Delta_n \downarrow 0$ and $n\Delta_n \to \mu \in [0, +\infty]$ as $n \uparrow \infty$. The limiting behavior of this functional depends on the value of μ. Thus, we divide our analysis in three different scenarios, namely $0 < \mu < \infty$, $\mu = 0$, and $\mu = +\infty$. When $0 < \mu < \infty$ we obtain that the above functional becomes a Riemann sum and thus we derive a functional convergence in probability to $\int_0^{t\mu}(X_s - \mathbb{E}(X_s))ds$. In the case $\mu = 0$, it turns out that the behavior of $\mathbf{S}^n$ depends on the increments of X around 0. Based on this, we show that $\mathbf{S}^n$, after centering and properly rescaling, converges stably to certain stochastic integral driven by a Lévy process. Lastly, when $\mu = +\infty$ the limit depends on whether the trawl process X has short or long memory. Under short memory, we show that, when properly scaled, $\mathbf{S}^n$ converges to a Brownian motion. In contrast, when X exhibits long memory we have to further distinguish whether the Gaussian component of the trawl process is present or not. If the Gaussian component is present, then $\mathbf{S}^n$ under proper scaling converges towards a fractional Brownian motion with Hurst parameter $H > 1/2$. Interestingly, if the Gaussian component is absent, the limit is no longer Gaussian and the rate of convergence of $\mathbf{S}^n$ is governed by the Blumenthal-Getoor index of the trawl process. We note that these findings agree with those obtained by Grahovac et al. [13] on superpositions of Ornstein-Uhlenbeck type processes.

Our second main result is a general functional limit theorem for trawl processes in terms of the characteristic triplets of their Lévy seeds. As an application of the general result, we establish a link between trawl processes and stationary Gaussian processes. In particular, we show that the sequence of scaled trawl processes converges in distribution, as their Lévy measures tend to infinity, to a limiting process which admits a Gaussian moving average representation. Note that this result gives an insight on the modeling set-up introduced by Márquez and Schmiegel [18]. In this work the authors proposed to use a (conditionally) Gaussian moving average to describe the main component of a turbulent velocity field while assuming that the energy dissipation can be represented by a trawl process. What is remarkable about this model, and very much in line with our results, is that these two physical quantities are function of each other, i.e. the energy dissipation is a function of the velocity field and vice versa.

Since article [5] appeared, there has been an increasing interest in trawl processes, covering a wide range of issues ranging from applications to theoretical investigations, and for the convenience of the reader we provide here a brief review of the recent literature on these processes.

Prior to the present paper, several limit theorems for trawl processes have been derived. Doukhan et al. [10] characterize a class of discrete time stationary trawl processes and study the functional limits of their partial sums. Grahovac et al. [14] investigate the intermittency property of trawl process, while Paulauskas [20] investigates trawl processes (and general linear processes) with tapered innovations. Additionally, Talarczyk and Treszczotko [28] study limit theorems for integrated trawl processes with symmetric Lévy bases.

In a more applied realm, Noven et al. [19] develop a latent trawl process model for extreme values and apply it to environmental time series. This work is extended by Courgeau and Veraart [9], who derive an asymptotic theory for inference on the latent trawl model for extreme values. Further work in the direction of extreme values has been done by Bacro et al. [1], who propose hierarchical space-time modelling of asymptotically independent exceedances based on a space-time extension of the trawl process and apply their model to precipitation data. In finance, Shephard and Yang [27] and Veraart [31] adapt the trawl process to provide a coherent statistical model of high-frequency data, the latter considering multivariate trawl processes, while the suitability of trawl processes for the modeling of high-frequency data is further corroborated by the results of Rossi and Santucci de Magistris [23].

With regards to estimation methodology for trawl processes, in addition to the aforementioned works [1, 5, 9, 19, 27], Doukhan [11] introduces spectral estimation for non-linear long range dependent discrete time trawl processes, and Shephard and Yang [26] develop likelihood inference for exponential-trawl processes.

The paper is structured as follows. Section 2 lays out the notation used throughout the paper and discusses some essential preliminaries. In Sections 3 and 4 we formulate the main results of the paper, concerning the asymptotics of partials sums of trawl processes and the convergence of a sequence of trawl processes to a Gaussian moving average, respectively. For the sake of ease of exposition, we defer the proofs of these results to the end of the paper, namely to Section 5.

2 Preliminaries

In this section, we introduce the basic notations and recall several basic results and concepts that will be used throughout this paper.

2.1 Functions of regular variation

A function $g : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be *regularly varying at* ∞ with index $\alpha \in \mathbb{R}$ if as $t \to \infty$

$$\frac{g(tx)}{g(t)} \to x^{-\alpha}, \ \forall\, x > 0.$$

In this case we will write $g \in \mathrm{RV}_\alpha^\infty$. If we replace $t \to \infty$ by $t \to 0^+$ in the previous equation, then g is called *regularly varying at* 0 and in this case we denote this as $g \in \mathrm{RV}_\alpha^0$. If, in the previous definitions, $\alpha = 0$, then we will refer to g as *slowly varying*.

It is well known that if $g \in \mathrm{RV}_0^{\infty}$, then as $x \to \infty$

$$g(x)x^{\varepsilon} \to \begin{cases} +\infty & \text{if } \varepsilon > 0; \\ 0 & \text{if } \varepsilon < 0. \end{cases}$$

One of the key results for functions of regular variation is the *Karamata's Theorem* (KT for short) which states that, if $g \in \mathrm{RV}_{\alpha}^{\infty}$ and locally bounded in $[x_0, +\infty)$, then the following limit results hold:

1. If $\rho \geq \alpha - 1$, then
$$\frac{1}{x^{\rho+1}g(x)} \int_{x_0}^{x} g(s)s^{\rho}\mathrm{d}s \to \frac{1}{\rho - \alpha + 1}, \;\; x \to \infty,$$
2. For every $\rho < \alpha - 1$, we have that
$$\frac{1}{x^{\rho+1}g(x)} \int_{x}^{\infty} g(s)s^{\rho}\mathrm{d}s \to \frac{1}{\alpha - 1 - \rho}, \;\; x \to \infty.$$

For a complete exposition on the basic properties of functions of regular variation we refer the reader to [7].

2.2 Stable convergence

For the rest of this paper, $(\Omega, \mathcal{F}, \mathbb{P})$ will denote a complete probability space. The notations $\xrightarrow{\mathbb{P}}$ and $\xrightarrow{d}$ stand, respectively, for convergence in probability and distribution of random vectors (r.v.'s for short). If $X_n/Y_n \xrightarrow{\mathbb{P}} 0$ when $n \to \infty$, we write $X_n = \mathrm{o}_{\mathbb{P}}(Y_n)$. Given a sub-$\sigma$-field $\mathcal{G} \subseteq \mathcal{F}$ and a sequence of r.v.'s $(\xi_n)_{n\geq 1}$ on $(\Omega, \mathcal{F}, \mathbb{P})$, by *$\mathcal{G}$-stably convergence in distribution* of ξ_n towards a random vector (r.v. for short) ξ (in symbols $\xi_n \xrightarrow{\mathcal{G}\text{-}d} \xi$), we mean that, conditioned on any non-null event in $\mathcal{G}$, $\xi_n \xrightarrow{d} \xi$. In this framework, if $(X_t^n)_{t\in\mathbb{R}, n\in\mathbb{N}}$ is a family of stochastic processes, we will write $X^n \xrightarrow{\mathcal{G}\text{-}fd} X$ if the finite-dimensional distributions (f.d.d. for short) of X^n converge $\mathcal{G}$-stably toward the f.d.d. of X. We further write $X^n \overset{\mathcal{G}\text{-}\mathcal{D}[0,T]}{\Longrightarrow} X$, if X^n converges to X in the Skorohod topology and $X^n \xrightarrow{\mathcal{G}\text{-}fd} X$. We refer the reader to [15] for a concise exposition of stable convergence.

2.3 Lévy bases and infinite divisibility

Let η be a measure on $\mathcal{B}(\mathbb{R}^d)$, the Borel sets on $\mathbb{R}^d$, and let $\mathcal{B}_b^{\eta}(\mathbb{R}^d) := \{A \in \mathcal{B}(\mathbb{R}^d) : \eta(A) < \infty\}$. The family $L = \{L(A) : A \in \mathcal{B}_b^{\eta}(\mathbb{R}^d)\}$ of real-valued r.v.'s will be called a *Lévy basis* if it is an infinitely divisible (ID for short) independently scattered random measure, that is, L is σ-additive almost surely and such that for any $A, B \in \mathcal{B}_b^{\eta}(\mathbb{R}^d)$, $L(A)$ and $L(B)$ are ID r.v.'s that are independent whenever $A \cap B = \emptyset$. The cumulant of a r.v. ξ, in case it exists, will be denoted by $\mathcal{C}(z;\xi) := \log \mathbb{E}(e^{iz\xi})$. We will say that L is *separable* with *control measure* η, if

$$\mathcal{C}(z; L(A)) = \eta(A)\psi(z), \;\; A \in \mathcal{B}_b^{\eta}(\mathbb{R}^d), z \in \mathbb{R},$$

where

$$\psi(z) := i\gamma z - \frac{1}{2}b^2z^2 + \int_{\mathbb{R}\setminus\{0\}} (e^{izx} - 1 - izx\mathbf{1}_{|x|\leq 1})\nu(\mathrm{d}x), \;\; z \in \mathbb{R}, \tag{2.1}$$

with $\gamma \in \mathbb{R}$, $b \geq 0$ and ν is a Lévy measure, i.e. $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}\setminus\{0\}} (1 \wedge |x|^2)\nu(\mathrm{d}x) < \infty$. When $\eta = Leb$, in which Leb represents the Lebesgue measure on $\mathbb{R}^d$, L is called *homogeneous*. The ID r.v. associated to the characteristic triplet (γ, b, ν) is called the *Lévy seed* of L and will be denoted by L'. As usual, (γ, b, ν) will be called the characteristic triplet of L and ψ its characteristic exponent. The *Blumenthal-Getoor index* of an ID distribution with triplet (γ, b, ν), is defined and denoted as

$$\beta_\nu := \inf \left\{ \beta > 0 : \int_{|x| \leq 1} |x|^\beta \nu(\mathrm{d}x) < \infty \right\}.$$

Within this framework, we will also refer to β_ν as the Bluementhal-Getoor index of a homogeneous Lévy basis with characteristic triplet (γ, b, ν). In this paper, the sigma field generated by L is denoted by $\mathcal{F}_L$.

For any Lévy measure ν, we associate the functions $\nu^\pm : (0, \infty) \to \mathbb{R}^+$, defined as $\nu^+(x) := \nu(x, \infty)$ and $\nu^-(x) := \nu(-\infty, -x)$. Let $K_+ + K_- > 0$ and $0 < \beta < 2$. A separable Lévy basis is called strictly β-stable with parameters $(\beta, K_+, K_-, \gamma)$ if its Lévy seed is distributed according to a strictly β-stable distribution, that is, the characteristic triplet of L' has no Gaussian component ($b = 0$), its Lévy measure satisfies

$$\frac{\nu(\mathrm{d}x)}{\mathrm{d}x} = K_+ |x|^{-1-\beta} \mathbf{1}_{\{x>0\}} + K_- |x|^{-1-\beta} \mathbf{1}_{\{x<0\}},$$

and either $\gamma = (K_- - K_+)/|\beta - 1|$ when $\beta \neq 1$, or γ arbitrary with $K_+ = K_-$ in the case of $\beta = 1$. The characteristic exponent of a strictly β-stable with parameters $(\beta, K_+, K_-, \gamma)$ admits the representation for every $z \in \mathbb{R}$

$$\psi(z; \beta, K_+, K_-, \gamma) := \begin{cases} -\sigma |z|^\beta (1 - i\rho \mathrm{sign}(z) \tan(\pi\beta/2)) & \text{if } \beta \neq 1; \\ -K_+ \pi |z| + i\gamma z & \text{if } \beta = 1, \end{cases} \tag{2.2}$$

where

$$\sigma := \Gamma(-\beta) \cos(\pi\beta/2)(K_+ + K_-), \quad \text{and} \quad \rho := \frac{K_+ - K_-}{K_+ + K_-}.$$

2.4 Trawl processes

Let L be a homogeneous Lévy basis on $\mathbb{R}^2$ with characteristic triplet (γ, b, ν). In addition, let $a : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-increasing integrable function and put

$$A = \{(r, y) : r \leq 0, 0 \leq y \leq a(-r)\}.$$

The process defined by

$$X_t := L(A_t), \;\; t \in \mathbb{R}, \tag{2.3}$$

where $A_t := A + (t, 0)$, is termed as a *trawl process*. From now on, we will refer to A and a, as the trawl set and the trawl function, respectively. It is well known that X is strictly stationary and, in the case when L is square integrable, its auto-covariance function is given by

$$\mathit{\Gamma}_X(h) := Var(L') \int_{|h|}^\infty a(u)\mathrm{d}u, \;\; h \in \mathbb{R}. \tag{2.4}$$

Moreover, $\mathit{\Gamma}_X$ uniquely characterizes a. More precisely, if L is square integrable, and X and $\tilde{X}$ are two trawls processes associated to L with trawls functions a and $\tilde{a}$, respectively, then $a = \tilde{a}$ a.e. if and only if

$$\mathit{\Gamma}_X = \mathit{\Gamma}_{\tilde{X}}.$$

For a detailed exposition on the basic properties of trawl processes we refer to [5] and [4].

3 Limit theorems for partial sums of trawl processes

In this section, we focus on the limit theorems for the partial sums of $(X_{i\Delta_n})_{i=0}^{n-1}$ under the assumption that as $n \uparrow \infty$, $\Delta_n \downarrow 0$ and $n\Delta_n \to \mu \in [0, +\infty]$. More specifically, we study the asymptotic behavior of the process $\mathbf{S}^n = (S_{[nt]}^{\Delta_n})_{t\geq 0}$, where

$$S_m^{\Delta} := \sum_{k=0}^{m-1} (X_{\Delta k} - \mathbb{E}(X_{\Delta k})), \;\; m \in \mathbb{N}, \Delta > 0,$$

with X as in (2.3). Note that we will always assume that the associated Lévy basis L has characteristic triplet (γ, b, ν) with $\mathbb{E}(|L'|) < \infty$, and that a is continuous in $[0, \infty)$. Furthermore, for the sake of exposition all of our proofs are presented in Section 5.

3.1 Main results

In this section, we state our main results concerning $\mathbf{S}^n$. As expected, the rate of convergence will depend entirely on the sampling scheme, which is in turn represented by μ. In what follows we will use the notation

$$\tilde{X}_t := X_t - \mathbb{E}(X_t), \;\; t \in \mathbb{R}.$$

3.1.1 The case $0 < \mu < \infty$

Let us start by assuming that $n\Delta_n \to \mu \in (0, \infty)$. In this situation the points

$$t_i = i\Delta_n, \;\; i = 0, \dots, [nt] - 1,$$

form a partition of $[0, t\mu]$. Consequently, $\Delta_n \mathbf{S}_t^n$ becomes a Riemann sum for the mapping $s \mapsto \tilde{X}_s$. Based on this observation, the following result is not surprising.

Proposition 3.1. *Suppose that* $\mathbb{E}(|L'|^2) < \infty$ *and* $\Delta_n n \to \mu \in (0, \infty)$*. Then for every* $V > 0$

$$\sup_{0\leq t\leq V} \left| \Delta_n \mathbf{S}_t^n - \int_0^{t\mu} \tilde{X}_s \mathrm{d}s \right| \xrightarrow{\mathbb{P}} 0.$$

3.1.2 The case $\mu = 0$

Let us now turn our attention to the case when $\mu = 0$. Intuitively, when this occurs, one should expect that

$$X_{\Delta_n n} \approx X_{\Delta_n i} \approx X_0, \;\; i = 0, 1, \dots, n-1,$$

for n large, which suggests that

$$\frac{1}{n} S_n^{\Delta_n} \approx \tilde{X}_0.$$

This turns out to be true as the following result shows.

Proposition 3.2. *Suppose that* $\mathbb{E}(|L'|) < \infty$ *and* a *is continuously differentiable in a neighbourhood of* 0*. If* $\Delta_n n \to 0$ *as* $n \to \infty$*, then*

$$\frac{1}{n} \mathbf{S}_t^n \xrightarrow{\mathbb{P}} t\tilde{X}_0, \;\; t \geq 0.$$

Next, we proceed to derive second order asymptotics for $\mathbf{S}^n$ when $\mu = 0$. Following the previously discussed heuristic argument, one should expect that, for large n,

$$\frac{1}{n} \mathbf{S}_t^n - t\tilde{X}_0 \approx t(X_{\Delta_n n} - X_0).$$

Therefore, in this case, the asymptotic result is determined by the behaviour of the increments of X. Before presenting our results in this framework, we introduce our working assumption, which reads as follows:

Assumption 3.3. *There exists a constant* $0 < \beta < 2$ *such that (see Section 2)* $\nu^{\pm}(x) \sim \tilde{K}_{\pm} x^{-\beta}$ *as* $x \to 0^+$ *with* $\tilde{K}_+ + \tilde{K}_- > 0$*. Furthermore, if* $\beta = 1$ *assume in addition that* $\tilde{K}_+ = \tilde{K}_-$ *and* $\mathrm{PV}\int_{-1}^{1} x\nu(\mathrm{d}x)$*, the Cauchy principal value, exists.*

Theorem 3.4. *Let the assumptions of Proposition 3.2 hold and set* $Z_t^n := \left(\frac{[nt]}{n} X_0 - \frac{1}{n}\mathbf{S}_t^n\right)$*. Then the following holds.*

i. *If* $b > 0$*, then*

$$\frac{1}{\sqrt{n\Delta_n}} Z_t^n \overset{\mathcal{F}\text{-}fd}{\longrightarrow} \sigma \int_0^t (t-s)\mathrm{d}B_s, \;\; t \geq 0,$$

where B *is a Brownian motion which can be chosen independent of* L*, and* $\sigma^2 = 2b^2 a(0)$*.*

iii. *Suppose that* $b = 0$ *and Assumption 3.3 holds. Then, as* $n \to \infty$*,*

$$\frac{1}{(n\Delta_n)^{1/\beta}} Z_t^n \overset{\mathcal{F}\text{-}fd}{\longrightarrow} \int_0^t (t-s)\mathrm{d}Y_s, \;\; t \geq 0,$$

where Y *is a symmetric* β*-stable Lévy processes with* $K_+ = \beta a(0)(\tilde{K}_+ + \tilde{K}_-)$*, which is in addition independent of* L*.*

3.1.3 The case $\mu = +\infty$

Suppose now that $n\Delta_n \to \mu = +\infty$ as $n \uparrow \infty$ and $\Delta_n \downarrow 0$. In order to get some intuition of what one should expect in this situation, firstly let $\Delta_n = \Delta$, i.e. the space between observations if fixed. Obviously, $\Delta_n n \to +\infty$ and the process $(X_{\Delta n})_{n \geq 1}$ is strictly stationary. In this situation $\mathbf{S}^n$ becomes the partial sums of a discrete-time stationary process. In view of this, when properly scaled, $\mathbf{S}^n$ typically converges to either a Brownian motion or a fractional Brownian motion (fBm for short), depending whether $(X_{\Delta n})_{n \geq 1}$ has short memory or long memory, respectively. It turned out that in our general setup, the former result still holds, while in the latter $\mathbf{S}^n$ will converge to a fBm only when L has a Gaussian component. Before presenting our results for this sampling scheme, we introduce our working assumptions.

Assumption 3.5 (SM)**.** *There is* $p_0 > 2$ *such that* $\mathbb{E}(|L'|^{p_0}) < \infty$ *and* $a(s) = \mathrm{O}(s^{-p_0})$ *as* $s \uparrow +\infty$*.*

Assumption 3.6 (LM)**.** *Assume that* $\mathbb{E}(|L'|^2) < \infty$*, and that there is a strictly positive continuous function* $a' \in \mathrm{RV}_{\alpha+1}^{\infty}$*, with* $1 < \alpha < 2$*, such that*

$$a(s) = \int_s^{\infty} a'(y)\mathrm{d}y, \;\; s \geq 0.$$

Assumption 3.7 (LM')**.** *Assumption 3.6 holds and for some* $c_a > 0$*,* $a'(y) \sim c_a y^{-\alpha-1}$ *as* $y \to \infty$*.*

Our first result concerns the short memory case:

Theorem 3.8. *Suppose that* $\mu = +\infty$*, and that Assumption 3.5 is fulfilled. Put*

$$\mathcal{G}^X = \sigma\bigg(\bigcup_{k\geq 1}\bigcap_{N \geq k} \sigma(X_0, X_{\Delta_N}, \ldots, X_{k\Delta_N})\bigg).$$

Then, as $n \uparrow \infty$

$$\sqrt{\frac{\Delta_n}{n}}\mathbf{S}^n \overset{\mathcal{G}^X-\mathcal{D}[0,1]}{\Longrightarrow} \sigma_a B,$$

where $\sigma_a^2 = Var(L')\int_{\mathbb{R}} a(s)\mathrm{d}s$ *and* B *is a Brownian motion independent of* $\mathcal{G}^X$*.*

Remark 3.9. By the independent scatteredness property of L, the limiting process appearing in Theorem 3.8 is not only independent of $\mathcal{G}^X$, but also of

$$\sigma\left(L(D) : D \cap \bigcup_{t\geq 0} A_t = \emptyset\right).$$

Furthermore, in view that the array of σ-fields

$$\mathcal{F}^X_{j,n} := \sigma(X_0, X_{\Delta_n}, \dots, X_{j\Delta_n}), \quad j = 0, 1, \dots, n-1,$$

is "almost nested", we conjecture that

$$\mathcal{G}^X = \sigma(X_t, t \geq 0).$$

The asymptotic behaviour drastically changes when X has long memory. We will now present two theorems distinguishing the cases of presence and absence of the Gaussian component in L.

Theorem 3.10. *Let Assumption 3.6 hold. Suppose that $b > 0$, $\mathbb{E}(|L'|^2) < \infty$, and $\mu = +\infty$. Then as $n \uparrow \infty$*

$$\frac{1}{n\sqrt{a(n\Delta_n)n\Delta_n}}\mathbf{S}^n \overset{\mathcal{D}[0,1]}{\Longrightarrow} \sigma_\alpha B^H, \quad n \to \infty,$$

where $\sigma^2_\alpha = \frac{2b^2}{(\alpha-1)(2-\alpha)(3-\alpha)}$, B^H is a fBm of index $H = \frac{3-\alpha}{2} > 1/2$.

In the absence of the Gaussian component, the limit is no longer Gaussian and the rate of convergence for $\mathbf{S}^n$ varies according to the behaviour of β_ν, the Blumenthal-Getoor index of L. More precisely, we have the following theorem.

Theorem 3.11. *Let Assumption 3.6 hold. Suppose that $b = 0$, $\mathbb{E}\left(|L'|^2\right) < \infty$, and that $\mu = +\infty$. The following holds:*

i. *If $\beta_\nu < \alpha$, let Assumption 3.7 hold. Then, as $n \uparrow \infty$*

$$\frac{\Delta_n}{(c_a n\Delta_n)^{\frac{1}{\alpha}}}\mathbf{S}^n \overset{fd}{\to} Y,$$

where Y is a strictly α-stable Lévy process satisfying that (see (2.2))

$$\mathcal{C}(z; Y_1) = \psi(z; \alpha, K_{+,\alpha}, K_{+,\alpha}, \tilde{\gamma}),$$

with $K_{+,\alpha} = \int_0^\infty x^\alpha \nu(\mathrm{d}x)$ and $K_{-,\alpha} = \int_{-\infty}^0 |x|^\alpha \nu(\mathrm{d}x)$.

ii. *When $2 > \beta_\nu > \alpha > 1$, further assume that for some $\tilde{K}_+ + \tilde{K}_- > 0$, $\nu^\pm(x) \sim \tilde{K}_\pm x^{-\beta_\nu}$ as $x \to 0^+$. Then, as $n \to \infty$,*

$$\frac{1}{n\left(a(n\Delta_n)n\Delta_n\right)^{1/\beta_\nu}}\mathbf{S}^n_1 \overset{d}{\to} \xi,$$

where ξ is strictly β_ν-stable, such that

$$\mathcal{C}(z; \xi) = \psi(z; \beta_\nu, K_{+,\alpha,\beta_\nu}, K_{-,\alpha,\beta_\nu}, \tilde{\gamma}),$$

in which $K_{\pm,\alpha,\beta_\nu} = \varrho_\alpha \beta_\nu \tilde{K}_\pm$ and

$$\varrho_\alpha = \frac{1}{\alpha-1} + \alpha\int_0^1 (1-s)s^{\beta_\nu-(\alpha+1)}\mathrm{d}s + 2\int_0^1 s^{\beta_\nu-\alpha}\mathrm{d}s.$$

Remark 3.12. Here, the notation $\nu^{\pm}(x) \sim \tilde{K}_{\pm} x^{-\beta}$ means that $x^{\beta}\nu^{\pm}(x) \to \tilde{K}_{\pm}$ when $x \downarrow 0$. Moreover, this property only concerns the behaviour of the Lévy measure of L around zero. Hence, one can simultaneously have that this condition is satisfied and that the second moment of L is finite. An example of such an infinitely divisible distribution is the normal inverse Gaussian distribution (see [25]).

Before presenting our last result for this section we would like to comment why, to the best of our knowledge, Theorems 3.8–3.11 cannot (in general) be deduced from existing results in the literature. First, (fractional) Donsker-type theorems are typically stated for a fixed discrete-time stationary process, while in our case, $(X_{\Delta_n n})_{n\geq 0}$ is a sequence of processes. Another alternative, which is more specific to our framework (infill and long span asymptotics), would be to apply the results obtained in [12] for sequences of discrete-time moving average processes. However, as pointed out in [10], in general trawl processes cannot be represented as a moving average process (see also [2] and references therein). Finally, in the Gaussian case, i.e. when $\nu = 0$, one could try to use the continuous version of the Breuer-Major theorem (see for instance [8] and references therein) to derive limit theorems of the non-linear functional

$$\mathbf{S}_G^n(t) = \sum_{k=0}^{\lfloor nt \rfloor -1} (G(X_{\Delta_n k}) - \mathbb{E}(G(X_{\Delta_n k})),\ t \geq 0,$$

via the sequence

$$Z_t^n = \int_0^{n\Delta_n t} [G(X_s) - \mathbb{E}(G(X_s))]\mathrm{d}s,\ t \geq 0,$$

in which G is a measurable function. Since for $G(x) = x$, it holds that

$$\frac{1}{\sqrt{\Delta_n n}} \mathbb{E}\left(|Z_t^n - \Delta_n \mathbf{S}_G^n(t)|\right) \leq \mathrm{O}(\Delta_n\sqrt{n}) + \mathrm{o}(1).$$

We conclude that Z_t^n is asymptotically equivalent to $\mathbf{S}_G^n(t)$ whenever $\Delta_n\sqrt{n} \to 0$. The latter condition is clearly stronger than the one imposed in our main results. We would like to emphasize that it is not clear whether the previous bound is optimal or not. Therefore, future research should examine the situations in which the asymptotic behaviour of $\mathbf{S}_G^n$ can be described by Z^n.

Most of the estimates used in our proofs rely heavily on the square integrability of L. Thus, it is natural to consider the situation in which this condition does not hold anymore. The following result gives a partial answer to this question.

Theorem 3.13. *Let Assumption 3.6 hold. Suppose that L is strictly β-stable with parameters $(K_+, K_-, \beta, \hat{\gamma})$ and that $\mu = +\infty$. Then the following holds:*

i. *If $1 < \beta < \alpha$, then*

$$\frac{\Delta_n}{(n\Delta_n)^{1/\beta}} \mathbf{S}^n \overset{fd}{\to} Y,\ \ n \to \infty,$$

where Y is a strictly β-stable Lévy process satisfying that (see (2.2))

$$\mathcal{C}(z; Y_1) = \mathcal{C}(z; Y_1) = \psi(z; \beta, \varrho_a K_{+,a}, \varrho_a K, \hat{\gamma}),$$

where $\varrho_a = \int_0^{\infty} s^{\beta} a'(s)\mathrm{d}s$.

ii. *If $2 > \beta > \alpha$, then the conclusion in Theorem 3.11 ii. remains valid.*

4 Functional convergence of trawl processes

4.1 General limit theorem

In this section, we study the convergence in distribution of a sequence of trawl processes to an infinitely divisible process in the Skorohod space $\mathcal{D}[0,\infty)$. Note that the convergence of the finite-dimensional distributions of trawl processes in general can be characterized using existing limit theorems for infinitely divisible distributions, e.g., Lemma 15.15 in [17], so our focus is primarily on tightness.

To ensure the tightness of the sequence of trawl processes with a fixed trawl set A, we need a technical assumption that is formulated in terms of the set $\tilde{B}_{t,s,r} := A_s \setminus A_t \setminus A_r$, for $r \le s \le t$. An illustration of the set $\tilde{B}_{t,s,r}$, among some other relevant sets, is given in Fig. 1 in Section 4.7.

Assumption 4.1 (Behaviour of trawl sets). *We assume that a is monotone, and that given any $r,s,t \in \mathbb{R}$ such that $r \le s \le t$ we have $Leb(A_t \setminus A_s) \le C(t-s)^{\frac{1}{2}+\frac{\epsilon}{2}}$ and $Leb(\tilde{B}_{t,s,r}) \le C'(t-r)^{1+\epsilon}$, where $C, C' \in (0,\infty)$ and $\epsilon > 0$.*

Remark 4.2. When a is monotone, we can write

$$Leb(\tilde{B}_{t,s,r}) = \int_r^s (a(s-p) - a(t-p))\mathrm{d}p.$$

Remark 4.3. We stress that Assumption 4.1 is indeed only needed for tightness in the proof of Theorem 4.7 below, and the convergence of the finite-dimensional distributions does not rely on it.

We will now look at two examples of a that satisfy Assumption 4.1 and one that does not.

Example 4.4 (Exponential). For $p \ge 0$ consider $a(p) := Ce^{-p}$ with $C > 0$, then by the mean value theorem we have that

$$Leb(\tilde{B}_{t,s,r}) = C\int_r^s (e^{p-s} - e^{p-t})\mathrm{d}p \le C\int_r^s (t-s)e^{p-s}\mathrm{d}p \le C(s-r)(t-s) \le C(t-r)^2,$$

and

$$Leb(A_t \setminus A_s) = C\int_s^t e^{p-t}\mathrm{d}p \le C(t-s).$$

Example 4.5 (Lipschitz functions). Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a Lipschitz monotone function with Lipschitz constant $M > 0$. Consider $a(p) := Cf(p)$ with $C > 0$, then by the Lipschitz condition we obtain that

$$\begin{aligned} Leb(\tilde{B}_{t,s,r}) &= C\int_r^s (f(s-p) - f(t-p))\mathrm{d}p \\ &\le CM\int_r^s (t-s)\mathrm{d}p \le CM(t-r)^2. \end{aligned}$$

and

$$Leb(A_t \setminus A_s) = C\int_s^t f(t-p)\mathrm{d}p \le f(0)C(t-s).$$

Example 4.6 (Unbounded trawl). Suppose that $a(p) = p^{-\frac{1}{2}+\frac{\epsilon}{2}}$,$p \in (0,p_0)$, for some $p_0 > 0$ and $\epsilon \in (0,\frac{1}{2})$. Then for $r,s,t \in (0,p_0)$ such that $r \le s \le t$,

$$\begin{aligned} Leb(\tilde{B}_{t,s,r}) &= \int_r^s \left[(s-p)^{-\frac{1}{2}+\frac{\epsilon}{2}} - (t-p)^{-\frac{1}{2}+\frac{\epsilon}{2}}\right]\mathrm{d}p \\ &= C\left[(s-r)^{\frac{1}{2}+\frac{\epsilon}{2}} - (t-r)^{\frac{1}{2}+\frac{\epsilon}{2}} + (t-s)^{\frac{1}{2}+\frac{\epsilon}{2}}\right], \end{aligned}$$

and notice that when $s = (t+r)/2$ (namely $t-s = s-r$), we have, by denoting $x := t-s$,

$$x^{\frac{1}{2}+\frac{\epsilon}{2}} - (2x)^{\frac{1}{2}+\frac{\epsilon}{2}} + x^{\frac{1}{2}+\frac{\epsilon}{2}} = \left(2 - 2^{\frac{1}{2}+\frac{\epsilon}{2}}\right) x^{\frac{1}{2}+\frac{\epsilon}{2}} = C' x^{\frac{1}{2}+\frac{\epsilon}{2}},$$

which does **not** satisfies the desired condition of Assumption 4.1.

Let us now rewrite (2.1) as follows:

$$\psi(z) = i\kappa z + \int_{\mathbb{R}} \left(e^{izx} - 1 - \frac{izx}{1+x^2}\right) \frac{1+x^2}{x^2} \tilde{\nu}(\mathrm{d}x), \quad z \in \mathbb{R},$$

where

$$\tilde{\nu}(\mathrm{d}x) := b^2 \delta_0(\mathrm{d}x) + \frac{x^2}{1+x^2}\nu(\mathrm{d}x), \quad \text{and} \quad \kappa := \gamma + \int_{\mathbb{R}\setminus\{0\}} \left(\frac{x}{1+x^2} - x\mathbf{1}_{|x|\leq 1}\right)\nu(\mathrm{d}x), \quad (4.1)$$

and where the integrand in (4.1) is defined as $-\frac{z^2}{2}$ when $x=0$ to attain continuity (see page 295 in [17]). In the general limit theorem we use the following notation: For any $c \in \mathbb{R}$ let

$$\tilde{\nu}^{(c)}(\mathrm{d}x) := (b^{(c)})^2 \delta_0(\mathrm{d}x) + \frac{x^2}{1+x^2}\nu^{(c)}(\mathrm{d}x), \quad \text{and}$$
$$\kappa^{(c)} := c\gamma^{(c)} + \int_{\mathbb{R}\setminus\{0\}} \left(\frac{x}{1+x^2} - x\mathbf{1}_{|x|\leq 1}\right)\nu^{(c)}(\mathrm{d}x),$$

where $\gamma^{(c)} := c\gamma + \int_{\mathbb{R}} cx(\mathbf{1}_{|cx|\leq 1} - \mathbf{1}_{|x|\leq 1})\nu(\mathrm{d}x)$, $b^{(c)} := cb$, and $\nu^{(c)}(B) = \nu(c^{-1}B) = \nu(\{x : cx \in B\})$. In particular, if $(\tilde{\nu}, \kappa)$ is the characteristic couple (resp. $(\gamma^{(c)}, b^{(c)}, \nu^{(c)})$ is the characteristic triplet) of an ID random variable X then $(\tilde{\nu}^{(c)}, \kappa^{(c)})$ is the characteristic couple (resp. $(\gamma^{(c)}, b^{(c)}, \nu^{(c)})$ is the characteristic triplet) of cX.

Theorem 4.7. *Let $(X_t^{(n)})_{n\in\mathbb{N}}$ be a sequence of trawl process with characteristics (γ_n, b_n, ν_n) or equivalently $(\tilde{\nu}_n, \kappa_n)$, $n \in \mathbb{N}$. Let Y_t be a trawl process with trawl seed having characteristics (γ, b, ν) or equivalently $(\tilde{\nu}, \kappa)$. Let $(r_n)_{n\in\mathbb{N}}$ be a sequence of real valued constants. Assume that Assumption 4.1 holds. Then,*

$$\left\{r_n X_t^{(n)}\right\}_{t\in[0,\infty)} \overset{\mathcal{D}[0,\infty)}{\Longrightarrow} \{Y_t\}_{t\in[0,\infty)}, \quad \text{as } n \to \infty,$$

holds if and only if $\tilde{\nu}_n^{(r_n)} \overset{weak}{\to} \tilde{\nu}$ and $\kappa_n^{(r_n)} \to \kappa$ as $n \to \infty$.

4.2 Convergence to a Gaussian moving average

As the limit Y in Theorem 4.7 is a stationary, infinitely divisible process, it may be possible to express it in simpler form as a causal moving average

$$Y_t = \int_{-\infty}^{t} g(t-s)\mathrm{d}L_s, \quad t \in \mathbb{R}, \quad (4.2)$$

with some kernel function $g : \mathbb{R} \to \mathbb{R}$ and a Lévy process L_t. If this is possible, it is subsequently interesting to study how g relates to the trawl function a, for example from a modelling point of view. Imagine that we would like to approximate a moving average with a particular kernel by a sequence of trawl processes. How can we choose the right a?. In other words, how do we choose the right sequence of trawl processes?. We restrict ourselves here to the Gaussian case where L_t reduces to a Brownian motion, as in this situation we are able to give a rather complete result. In the non-Gaussian case, difficulties may arise with the existence of representation (4.2), especially if Y is not square integrable (see Section 5 in [2] for further discussion).

Theorem 4.8. *Let $a(h) = -\frac{d}{dh}\int_0^\infty g(s)g(h+s)\mathrm{d}s$ for every $h \geq 0$, where $g \in L^2(\mathbb{R})$, and let $X_t^{(n)}$ be the associated trawl process with characteristics (γ_n, b_n, ν_n) for any $n \in \mathbb{N}$. Assume further that Assumption 4.1 holds. Then the convergence*

$$\left\{r_n X_t^{(n)}\right\}_{t\in[0,\infty)} \overset{\mathcal{D}[0,\infty)}{\Longrightarrow} \left\{\int_{-\infty}^t g(t-s)\mathrm{d}B_s\right\}_{t\in[0,\infty)}, \quad \text{as } n \to \infty,$$

where B is a one-dimensional Brownian motion, holds if and only if $\tilde{\nu}_n^{(r_n)} \overset{weak}{\to} \delta_0$ and $\kappa_n^{(r_n)} \to 0$ as $n \to \infty$.

Remark 4.9. Observe that, in the case when g is differentiable and positive monotone (resp. bounded), then, by the monotone (resp. bounded) convergence theorem, we have $\frac{\mathrm{d}}{\mathrm{d}h}\int_0^\infty g(s)g(h+s)\mathrm{d}s = \int_0^\infty g(s)g'(h+s)\mathrm{d}s$.

It is worth stressing that Theorems 4.7 and 4.8 apply to the centered process $r_n(X_t^{(n)} - \mathbb{E}[X_t^{(n)}])$ as well. In the following remark we present some conditions on the characteristic triplet of the converging centered trawl process. These conditions are stronger than those in Theorem 4.8, but they are useful in practice (as we show in the subsequent example).

Remark 4.10. Consider the following conditions: $\int_{\mathbb{R}} r_n^2 x^2 \nu_n(\mathrm{d}x) \to 1$, $\int_{\mathbb{R}} r_n^3 |x^3| \nu_n(\mathrm{d}x) \to 0$, and $r_n^2 b_n^2 \to 0$ as $n \to \infty$. Then, we have that

$$\begin{aligned}
&\log \mathbb{E}\Bigg(\exp\left(i\sum_{j=1}^k z_j r_n\left(X_{t_j}^{(n)} - \mathbb{E}[X_{t_j}^{(n)}]\right)\right)\Bigg)\\
&\quad= \int_{\mathbb{R}}\int_{\mathbb{R}}\Bigg[-\frac{1}{2}r_n^2 b_n^2\Bigg(\sum_{j=1}^k z_j \mathbf{I}_{A_{t_j}}(y,s)\Bigg)^2\\
&\qquad\qquad + \int_{\mathbb{R}}\Bigg(e^{ir_n x\sum_{j=1}^k z_j \mathbf{I}_{A_{t_j}}(y,s)} - 1 - i\mathbf{I}_{[-1,1]}(x) r_n x \sum_{j=1}^k z_j \mathbf{I}_{A_{t_j}}(y,s)\Bigg)\nu_n(\mathrm{d}x)\\
&\qquad\qquad - i\int_{|x|>1} x r_n \sum_{j=1}^k z_j \mathbf{I}_{A_{t_j}}(y,s)\nu_n(dx)\Bigg]\mathrm{d}s\mathrm{d}y\\
&\quad= -\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{1}{2}r_n^2 b_n^2\Bigg(\sum_{j=1}^k z_j \mathbf{I}_{A_{t_j}}(y,s)\Bigg)^2\mathrm{d}s\mathrm{d}y\\
&\qquad - \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{1}{2}r_n^2 x^2\Bigg(\sum_{j=1}^k z_j \mathbf{I}_{A_{t_j}}(y,s)\Bigg)^2\nu_n(\mathrm{d}x)\mathrm{d}s\mathrm{d}y\\
&\qquad + \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\Bigg(e^{ir_n x\sum_{j=1}^k z_j \mathbf{I}_{A_{t_j}}(y,s)} - 1 - ir_n x\sum_{j=1}^k z_j \mathbf{I}_{A_{t_j}}(y,s)\\
&\qquad\qquad + \frac{1}{2}r_n^2 x^2\Bigg(\sum_{j=1}^k z_j \mathbf{I}_{A_{t_j}}(y,s)\Bigg)^2\Bigg)\nu_n(\mathrm{d}x)\mathrm{d}s\mathrm{d}y.
\end{aligned}$$

It is possible to see that the last term is bounded by $C\int_{\mathbb{R}} r_n^3|x|^3\nu_n(\mathrm{d}x)$, where $C > 0$. Moreover, we have that

$$\begin{aligned}
\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{1}{2}\Bigg(\sum_{j=1}^k z_j \mathbf{I}_{A_{t_j}}(y,s)\Bigg)^2 r_n^2 x^2\nu_n(\mathrm{d}x)\mathrm{d}s\mathrm{d}y &= \frac{1}{2}\sum_{j,l=1}^k z_j z_l Leb(A_{t_j}\cap A_{t_l})\int_{\mathbb{R}} r_n^2 x^2 \nu_n(\mathrm{d}x)\\
&\to \frac{1}{2}\sum_{j,l=1}^k z_j z_l Leb(A_{t_j}\cap A_{t_l}), \qquad \text{as } n\to\infty.
\end{aligned}$$

This implies the convergence of the finite-dimensional distributions. For tightness, the same arguments as the ones used in the proof of Theorem 4.7 apply.

Example 4.11 (The Poisson case)**.** In this example we show that the assumptions on the characteristics of Theorem 4.8 (see in particular Remark 4.10) are satisfied in the case $X_t^{(n)} = L^{(n)}(A_t) \sim \text{Poisson}(\lambda^{(n)} Leb(A))$ for all $t \in [0,\infty)$, where $\lambda^{(n)}$ is the intensity parameter (or equivalently, $L'^{(n)} \sim \text{Poisson}(\lambda^{(n)})$). In particular, we have that

$$\mathcal{C}(z; L'^{(n)}) = \lambda^{(n)} \left(e^{iz} - 1\right).$$

Let $r_n = \frac{1}{\sqrt{n}}$. In order to satisfies the assumptions we have to impose that $\lambda^{(n)} = n + o(n)$ (*e.g.* $\lambda^{(n)} = n + bn^{\gamma}$ for $b \in \mathbb{R}$ and $\gamma < 1$). Indeed,

$$\int_{\mathbb{R}} \frac{x^2}{n} \nu^{(n)}(\mathrm{d}x) \to 1 \Leftrightarrow \frac{\lambda^{(n)}}{n} \to 1, \quad \text{and} \quad \int_{\mathbb{R}} \frac{|x^3|}{n\sqrt{n}} \nu^{(n)}(\mathrm{d}x) \to 0 \Leftrightarrow \frac{\lambda^{(n)}}{n\sqrt{n}} \to 0, \quad \text{as } n \to \infty.$$

Concerning the trawl function, we can take $a(p) = Ce^{-p}$, which satisfies Assumption 4.1 (see Example 4.4).

Example 4.12 (A kernel g that satisfies Assumption 4.1)**.** Let $g : \mathbb{R}_+ \to \mathbb{R}_+$ be integrable, monotonically decreasing and second order differentiable with $g''(x) > 0$, $\forall x \in \mathbb{R}_+$, and let $C > 0$. Then $a(h) = -C \int_0^\infty g(s) g'(h+s) \mathrm{d}s$ satisfies the assumptions of Theorem 4.8. Indeed, it is possible to see that a is positive (since g' is negative), monotonically decreasing (since g' is monotonically increasing), and satisfies Assumption 4.1 thanks to Example 4.5; indeed

$$\sup_{p \geq 0} \frac{1}{C} |a'(p)| = \frac{1}{C} \sup_{p \geq 0} \int_0^\infty g(s) g''(p+s) \mathrm{d}s \leq \frac{g''(0)}{C} \int_0^\infty g(s) \mathrm{d}s < \infty,$$

where we further used that g'' is monotonically decreasing.

5 Proofs

Throughout all our proofs, the non-random positive constants will be denoted by the generic symbol $C > 0$, and they may change from line to line. Additionally, for simplicity and without loss of generality, we may and do assume that $\mathbb{E}(L') = 0$ and $Var(L') = 1$ in such a way that $\Gamma_X(h) = \int_h^\infty a(s) \mathrm{d}s$, for $h \geq 0$. We note that below we will use the notation $T_n = n\Delta_n$.

5.1 Technical lemmas

We start by analysing the variance of S_m^Δ.

Lemma 5.1. *Suppose that* $\mathbb{E}(|L'|^2) < \infty$. *Then*

$$Var(S_m^\Delta) = \frac{2}{\Delta^2} \int_0^{m\Delta} \int_0^r \Gamma_X(s) \mathrm{d}s \mathrm{d}r + \mathrm{O}(m), \quad m \in \mathbb{N}, \Delta > 0. \tag{5.1}$$

Furthermore, if $\Delta m \to \mu \in [0, +\infty]$ *as* $\Delta \downarrow 0$ *and* $m \uparrow \infty$*, then we have the following result.*

i. *If* $\mu = 0$*, then*

$$\frac{1}{m^2} Var(S_m^\Delta) \to \Gamma_X(0).$$

ii. *If* $0 < \mu < \infty$*, then*

$$\Delta^2 Var(S_m^\Delta) \to 2 \int_0^\beta \int_0^r \Gamma_X(s) \mathrm{d}s \mathrm{d}r.$$

iii. *If* $\mu = +\infty$*, then*

- **a)** *If* $\int_0^\infty \int_r^\infty a(s)\mathrm{d}s\mathrm{d}r < \infty$*, then* $Var(S_m^\Delta) \sim \int_{\mathbb{R}} \Gamma_X(s)\mathrm{d}s\frac{m}{\Delta}$*, as* $\Delta \downarrow 0$ *and* $m \uparrow \infty$*.*
- **b)** *If* $\int_0^\infty \int_r^\infty a(s)\mathrm{d}s\mathrm{d}r = +\infty$ *assume in addition that* $a \in \mathrm{RV}_\alpha^\infty$ *with* $1 < \alpha < 2$*. Then,* $Var(S_m^\Delta) \sim c_\alpha Var(L')a(m\Delta)m^3\Delta$*,* $c_\alpha = \frac{2}{(\alpha-1)(2-\alpha)(3-\alpha)}$*.*

Proof. We have that

$$Var(S_m^\Delta) = m\Gamma_X(0) + 2\sum_{i=1}^{m-1}\sum_{j=1}^{i}\Gamma_X(j\Delta). \tag{5.2}$$

Now,

$$R(m,\Delta) := \frac{1}{\Delta^2}\left\{\Delta^2\sum_{i=1}^{m-1}\sum_{j=1}^{i}\Gamma_X(j\Delta) - \int_0^{m\Delta}\int_0^r \Gamma_X(s)\mathrm{d}s\mathrm{d}r\right\}$$

$$= R_1(m,\Delta) + R_2(m,\Delta) + R_3(m,\Delta) + R_4(m,\Delta),$$

where

$$R_1(m,\Delta) := \frac{1}{\Delta^2}\sum_{i=1}^{m-1}\sum_{j=1}^{i}\int_{i\Delta}^{(i+1)\Delta}\int_{j\Delta}^{(j+1)\Delta}\left[\Gamma_X(j\Delta) - \Gamma_X(s)\right]\mathrm{d}s\mathrm{d}r;$$

$$R_2(m,\Delta) := -\frac{1}{\Delta^2}\int_0^\Delta\int_0^r \Gamma_X(s)\mathrm{d}s\mathrm{d}r;$$

$$R_3(m,\Delta) := \frac{1}{\Delta^2}\sum_{i=1}^{m-1}\int_{i\Delta}^{(i+1)\Delta}\int_r^{(i+1)\Delta}\Gamma_X(s)\mathrm{d}s\mathrm{d}r;$$

$$R_4(m,\Delta) := -\frac{(m-1)}{\Delta}\int_0^\Delta \Gamma_X(s)\mathrm{d}s.$$

From (2.4), Γ_X is non-increasing on $\mathbb{R}^+$, with derivative $-Var(L')a$. Therefore

$$\begin{aligned}|R_1(m,\Delta)| &\leq C\frac{1}{\Delta}\sum_{i=1}^{m-1}\sum_{j=1}^{i}\int_{i\Delta}^{(i+1)\Delta}\int_{j\Delta}^{(j+1)\Delta} a(j\Delta)\mathrm{d}s\mathrm{d}r\\ &= C\frac{1}{\Delta}\sum_{i=1}^{m-1}\sum_{j=1}^{i}\int_{i\Delta}^{(i+1)\Delta}\int_{(j-1)\Delta}^{j\Delta} a(j\Delta)\mathrm{d}s\mathrm{d}r\\ &\leq C\frac{1}{\Delta}\sum_{i=1}^{m-1}\sum_{j=1}^{i}\int_{i\Delta}^{(i+1)\Delta}\int_{(j-1)\Delta}^{j\Delta} a(s)\mathrm{d}s\mathrm{d}r\\ &\leq C\frac{1}{\Delta}\int_0^{m\Delta}\int_0^r a(s)\mathrm{d}s\mathrm{d}r.\end{aligned}$$

In a similar way, we obtain that

$$|R_2(m,\Delta)| + |R_3(m,\Delta)| \leq \frac{1}{\Delta}\int_0^{\Delta m}\Gamma_X(r)\mathrm{d}r.$$

The results above imply that

$$R(m,\Delta) \leq Cm\Gamma_X(0).$$

This estimate together with (5.2) give (5.1).

Now assume that $\Delta m \to \beta \in [0,+\infty]$. i., ii. and part a) of iii. follow immediately by (5.1) and the Dominated Convergence Theorem. Therefore, for the rest of the proof, we will assume that $\Delta m \to +\infty$ and that $a \in \mathrm{RV}_{\alpha}^{\infty}$ in which $1<\alpha<2$. By KT we get that

$$\frac{2}{\Delta^2}\int_0^{m\Delta}\int_0^r \Gamma_X(s)\mathrm{d}s\mathrm{d}r \sim c_\alpha a(m\Delta)m^3\Delta, \text{ as } \Delta m \to +\infty.$$

Since $a \in \mathrm{RV}_{\alpha}^{\infty}$, it admits the representation $a(x)=x^{-\alpha}l(x)$, with l a slowly varying function at ∞. Thus,

$$a(m\Delta)m^2\Delta = (m\Delta)^{2-\alpha}l(m\Delta)\Delta^{-1} \to +\infty,$$

where we have used that, for any slowly varying function, $l(x)x^\rho \to +\infty$ as $x \uparrow \infty$ whenever $\rho>0$. Consequently, by (5.1), we deduce that

$$\frac{1}{a(m\Delta)m^3\Delta}\left| Var(S_m^\Delta) - \frac{2}{\Delta}\int_0^{m\Delta}\int_0^r \Gamma_X(s)\mathrm{d}s\mathrm{d}r\right| \to 0, \text{ as } n\to\infty,$$

which completes the proof. □

Next, we find a very useful decomposition for S_m^Δ. For any $\Delta>0$, let

$$\mathcal{P}_A^\Delta(i,j) := \{(r,s) : a(t_{j+1}-s) < r \le a(t_j - s), t_{i-1} < s \le t_i\},$$

where $t_i = t_i(\Delta) = i\Delta$ with the convention that $t_{-1}=-\infty$. It is clear that $\mathcal{P}_A^\Delta(i,j) \cap \mathcal{P}_A^\Delta(i',j') = \emptyset$ whenever either $i \neq i'$ or $j \neq j'$ for $i=0,\dots,m-1$ and $j \geq i$. Moreover,

$$Leb\left\{A_{k\Delta} \setminus \bigcup_{i=0}^{k}\bigcup_{j=k}^{\infty}\mathcal{P}_A^\Delta(i,j)\right\} = 0, \tag{5.3}$$

and

$$Leb(\mathcal{P}_A^\Delta(i,i+j)) = \begin{cases} \int_{t_j}^{t_{j+1}} a(s)\mathrm{d}s & \text{if } i=0, j\geq 0; \\ \int_{t_j}^{t_{j+1}}[a(s)-a(s+\Delta)]\mathrm{d}s & \text{if } i=1,\dots,m-1, j<m-i. \end{cases} \tag{5.4}$$

Based on these observations, the following result is obvious.

Lemma 5.2. *Let* $\chi_{i,j}^\Delta := L(\mathcal{P}_A^\Delta(i,j)) - \mathbb{E}(L(\mathcal{P}_A^\Delta(i,j)))$. *Then, almost surely*

$$S_m^\Delta = \frac{1}{\Delta}\sum_{i=0}^{m-1}\sum_{j=i}^{m-1} t_{j-i+1}\chi_{i,j}^\Delta + \frac{1}{\Delta}\sum_{i=0}^{m}(t_m - t_i)\zeta_{i,m}^\Delta \tag{5.5}$$

$$= S_m^{\Delta,1} + S_m^{\Delta,2} + S_m^{\Delta,3} + S_m^{\Delta,4}, \tag{5.6}$$

where $\zeta_{i,m}^\Delta := \sum_{j=m}^\infty \chi_{i,j}^\Delta$ *and*

$$S_m^{\Delta,1} := \frac{1}{\Delta}\sum_{i=1}^{m-1}\sum_{j=1}^{m-i} t_j\chi_{i,j+i-1}^\Delta; \;\; S_m^{\Delta,2} := \frac{1}{\Delta}\sum_{i=1}^{m}(t_m-t_i)\zeta_{i,m}^\Delta;$$

$$S_m^{\Delta,3} := \frac{1}{\Delta}\sum_{j=0}^{m-1} t_{j+1}\chi_{0,j}^\Delta; \;\; S_m^{\Delta,4} := \frac{t_m}{\Delta}\zeta_{0,m}^\Delta.$$

When $\beta=+\infty$, it turns out that in the short memory case $S_m^{\Delta,1}$ dominates the asymptotic behaviour.

Lemma 5.3. *Let $m_n \in \mathbb{N}$ be such that $m_n \uparrow \infty$, $\Delta_n m_n \to \infty$ and $\Delta_n \downarrow 0$, as $n \to \infty$. Suppose that $\mathbb{E}(|L'|^2) < \infty$ and that $\int_0^\infty \int_r^\infty a(s)\mathrm{d}s\mathrm{d}r < \infty$. Then*

$$S_{m_n}^{\Delta_n} = S_{m_n}^{\Delta_n,1} + \mathrm{o}_{\mathbb{P}}\left(\sqrt{\frac{m_n}{\Delta_n}}\right).$$

The proof of Lemma 5.3 heavily relies on the next property.

Lemma 5.4. *Let $f \geq 0$ be an integrable continuous function such that $\int_0^\infty \int_x^\infty f(s)\mathrm{d}s\mathrm{d}x < \infty$. Then, as $x \to +\infty$*

$$x \int_x^\infty f(s)\mathrm{d}s \to 0, \text{ and } \frac{1}{x}\int_0^x s^2 f(s)\mathrm{d}s \to 0.$$

Proof. If $f \equiv 0$ a.e. the result is trivial, so assume that $f > 0$. For $x \geq 0$, put $F(x) := \int_x^\infty f(s)\mathrm{d}s$. Integration by parts gives that

$$\int_0^x F(s)\mathrm{d}s = xF(x) + \int_0^x sf(s)\mathrm{d}s.$$

In view that $f > 0$ and $\int_0^\infty F(s)ds < \infty$, the Dominated Convergence Theorem guarantees that

$$0 \leq \lim_{x\to\infty} \int_0^x sf(s)\mathrm{d}s = \int_0^\infty sf(s)\mathrm{d}s \leq \int_0^\infty F(s)\mathrm{d}s < \infty.$$

This shows in particular that the following limit exists

$$\infty > \ell = \lim_{x\to\infty} xF(x) = \int_0^\infty F(s)\mathrm{d}s - \int_0^\infty sf(s)\mathrm{d}s \geq 0.$$

Observe that if $\ell > 0$, then, as $x \to +\infty$, $1/\left(x\int_x^\infty f(s)ds\right) \to 1/\ell$. Thus, if $\ell > 0$, we could find $x_0 > 0$ such that for all $x > x_0$

$$\frac{1}{x} < C\int_x^\infty f(s)\mathrm{d}s,$$

which contradicts that $\int_0^\infty \int_x^\infty f(s)\mathrm{d}s\mathrm{d}x < \infty$. Hence, $\ell = 0$ as required.

To show the last part, observe first that when $\int_0^\infty \int_y^\infty \int_x^\infty f(s)\mathrm{d}s\mathrm{d}x\mathrm{d}y < \infty$, an analogous argument as above shows that

$$0 \leq \int_0^\infty sF(s)\mathrm{d}s \leq \int_0^\infty \int_s^\infty F(x)\mathrm{d}x\mathrm{d}s < \infty.$$

Therefore, from the first part of the proof, as $x \to +\infty$

$$\frac{1}{x}\int_0^x s^2 f(s)ds = -xF(x) + \frac{2}{x}\int_0^x F(s)sds \to 0.$$

Now suppose that $\int_0^\infty \int_y^\infty \int_x^\infty f(s)\mathrm{d}s\mathrm{d}x\mathrm{d}y = +\infty$ and put $\bar{F}(x) := \int_0^x F(s)\mathrm{d}s$. Clearly $\int_0^x \bar{F}(s)\mathrm{d}s \to +\infty$ as $x \to +\infty$, and, for all $x \geq 0$,

$$\frac{1}{x}\int_0^x s^2 f(s)ds = -xF(x) + 2\left[\bar{F}(x) - \frac{1}{x}\int_0^x \bar{F}(s)ds\right]. \tag{5.7}$$

Moreover, by L'Hospital's Rule and the continuity of f we have that

$$\frac{1}{x}\int_0^x \bar{F}(s)\mathrm{d}s \to \int_0^\infty F(s)\mathrm{d}s,$$

which applied to (5.7) concludes the proof. □

Proof of Lemma 5.3. Since L is independently scattered, we get by (5.4) that, for any $m \in \mathbb{N}$ and $\Delta > 0$,

$$
\begin{aligned}
Var(S_m^{\Delta,2}) &= \frac{1}{\Delta^2}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}(t_m-t_i)^2 a(t_m-s)\mathrm{d}s;\\
Var(S_m^{\Delta,3}) &= \frac{1}{\Delta^2}\int_0^{\Delta m}s^2a(s)\mathrm{d}s+\frac{2}{\Delta^2}\sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}}\int_s^{t_{j+1}}r\mathrm{d}ra(s)\mathrm{d}s;\\
Var(S_m^{\Delta,4}) &= m^2\int_{\Delta m}^{\infty}a(s)ds.
\end{aligned}
$$

Moreover, in view that the trawl function is non-negative, continuous and such that $\int_0^\infty\int_x^\infty a(s)\mathrm{d}s\mathrm{d}x<\infty$, Lemma 5.4 can be applied in order to obtain that

$$\frac{\Delta_n}{m_n}Var(S_{m_n}^{\Delta_n,4}) = \Delta_n m_n\int_{\Delta_n m_n}^{\infty}a(s)ds \to 0.$$

We proceed now to show that for every $m \in \mathbb{N}$ and $\Delta > 0$

$$\left|Var(S_m^{\Delta,2})-\frac{1}{\Delta^2}\int_0^{\Delta m}s^2a(s)\mathrm{d}s\right|\leq C\frac{1}{\Delta}\int_0^{\Delta m}sa(s)\mathrm{d}s+\mathrm{O}(1); \tag{5.8}$$

$$\left|Var(S_m^{\Delta,3})-\frac{1}{\Delta^2}\int_0^{\Delta m}s^2a(s)\mathrm{d}s\right|\leq C\frac{1}{\Delta}\int_0^{\Delta m}sa(s)\mathrm{d}s+\mathrm{O}(1). \tag{5.9}$$

Let $R(m,\Delta)=\sum_{i=1}^m\int_{t_{i-1}}^{t_i}(t_m-t_i)^2a(t_m-s)\mathrm{d}s$ and $R'(m,\Delta)=\sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}}\int_s^{t_{j+1}}r\mathrm{d}ra(s)\mathrm{d}s$. Then,

$$
\begin{aligned}
\left|R(m,\Delta)-\int_0^{\Delta m}(t_m-s)^2a(t_{m-1}-s)\mathrm{d}s\right| &\leq C\Delta\sum_{i=1}^m\int_{t_{i-1}}^{t_i}(t_m-t_i)a(t_m-s)\mathrm{d}s\\
&\leq C\Delta\int_0^{\Delta m}sa(s)\mathrm{d}s+C\Delta^2\int_0^{\Delta m}a(s)\mathrm{d}s,
\end{aligned}
$$

which is exactly (5.8). In a similar way, we see that

$$
\begin{aligned}
|R'(m,\Delta)| &\leq 2\Delta\sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}}t_{j+1}a(s)\mathrm{d}s\\
&\leq 2\Delta\int_0^{\Delta m}sa(s)\mathrm{d}s+2\Delta^2\int_0^{\Delta m}a(s)\mathrm{d}s.
\end{aligned}
$$

Relation (5.9) is obtained easily from this. Finally, note that from (5.8), (5.9) and Lemma 5.4, it follows that for $l=2,3$

$$\frac{\Delta_n}{m_n}Var(S_{m_n}^{\Delta_n,l})=\frac{1}{m_n\Delta_n}\int_0^{\Delta_n m_n}s^2a(s)\mathrm{d}s+\mathrm{o}(1)\to 0,\ \ n\to\infty,$$

completing the proof. □

We proceed now to find some estimates for the characteristic function of $S_m^{\Delta,l}$, for $l=1,2,3$. For doing this, the following result is essential and its proof follows the lines of the proof of Proposition 3.6 in [22] as well as the well-known inequality

$$\left|e^{izx}-1\right|\leq 2\left(|zx|\,\mathbf{1}_{|zx|\leq 1}+\mathbf{1}_{|zx|>1}\right).$$

Lemma 5.5. *Let ψ be the characteristic exponent of an ID distribution with mean 0. Then ψ is continuously differentiable and there is a constant $C>0$ depending only on (γ,b,ν) such that*

$$|\psi(z)| \leq b^2 |z|^2 + C\int_{\mathbb{R}} (1\wedge |xz|^2)\nu(\mathrm{d}x), \;\; z\in\mathbb{R}; \tag{5.10}$$

$$|\psi'(z)| \leq b^2 |z| + C\int_{\mathbb{R}} (1\wedge |xz|)\, |x|\, \nu(\mathrm{d}x), \;\; z\in\mathbb{R}. \tag{5.11}$$

Lemma 5.6. *Suppose that $\mathbb{E}\left(|L'|^2\right)<\infty$ and let*

$$I_m^{\Delta,1}(z) := \int_0^{\Delta m} (\Delta m - s)\psi\left(\frac{s}{\Delta}z\right)\left[\frac{a(s)-a(s+\Delta)}{\Delta}\right]\mathrm{d}s;$$
$$I_m^{\Delta,2}(z) := \int_0^{\Delta m} \psi\left(\frac{s}{\Delta}z\right) a(s)\mathrm{d}s.$$

Then the following estimates hold

$$\begin{aligned}\left|\mathcal{C}\left(z;S_m^{\Delta,1}\right) - I_m^{\Delta,1}(z)\right| \leq & C\frac{|z|^2}{\Delta}\int_0^{\Delta m} (\Delta m - s)(s+\Delta)\mathrm{d}\,|a|\,(s)\\ &+ C\Delta\int_0^{\Delta m}\left|\psi\left(\frac{s}{\Delta}z\right)\right|\mathrm{d}\,|a|\,(s)\\ &+ C\,|z|^2\int_0^{\Delta m}(s+\Delta)\mathrm{d}\,|a|\,(s);\end{aligned}$$

$$\left|\mathcal{C}\left(z;S_m^{\Delta,2}\right) - I_m^{\Delta,2}(z)\right| + \left|\mathcal{C}\left(z;S_m^{\Delta,3}\right) - I_m^{\Delta,2}(z)\right| \leq C\frac{|z|^2}{\Delta}\int_0^{\Delta m}(s+\Delta)a(s)\mathrm{d}s.$$

Proof. Recall that we are assuming that L is centered. By the independent scatteredness property of L, we have

$$\begin{aligned}\mathcal{C}\left(z;S_m^{\Delta,1}\right) =& \sum_{j=1}^{m-1}\int_{t_{j-1}}^{t_j}(t_m - s)\psi\left(\frac{t_j}{\Delta}z\right)\left[\frac{a(s)-a(s+\Delta)}{\Delta}\right]\mathrm{d}s\\ &+\sum_{j=1}^{m-1}\int_{t_{j-1}}^{t_j}(s-t_j)\psi\left(\frac{t_j}{\Delta}z\right)\left[\frac{a(s)-a(s+\Delta)}{\Delta}\right]\mathrm{d}s;\end{aligned} \tag{5.12}$$

$$\mathcal{C}\left(z;S_m^{\Delta,2}\right) = \sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}\psi\left(\frac{t_m-t_i}{\Delta}z\right)a(t_m-s)\mathrm{d}s; \tag{5.13}$$

$$\mathcal{C}\left(z;S_m^{\Delta,3}\right) = \sum_{j=1}^{m-1}\int_{t_j}^{t_{j+1}}\psi\left(\frac{t_{j+1}}{\Delta}z\right)a(s)\mathrm{d}s. \tag{5.14}$$

Since

$$\int_{t_{j-1}}^{t_j}[a(s)-a(s+\Delta)]\,\mathrm{d}s = \int_{t_{j-1}}^{t_j}(s-t_{j-1})\mathrm{d}\,|a|\,(s) + \int_{t_j}^{t_{j+1}}(t_{j+1}-s)\mathrm{d}\,|a|\,(s),\;\; j\geq 0,$$

the claimed estimates are easily obtained by noting that from Lemma 5.5 and the Mean Value Theorem

$$\left|\psi\left(\frac{t_j}{\Delta}z\right)-\psi\left(\frac{s}{\Delta}z\right)\right| \leq C\frac{|z|^2}{\Delta}(s+\Delta),\;\; t_{j-1}\leq s\leq t_j,$$

while, for $t_{i-1}\leq s\leq t_i$,

$$\left|\psi\left(\frac{t_m-t_i}{\Delta}z\right)-\psi\left(\frac{t_m-s}{\Delta}z\right)\right| \leq C\frac{|z|^2}{\Delta}(t_m-s).$$ □

5.2 Proof of Propositions 3.1 and 3.2

Proof of Proposition 3.1. For simplicity, we will assume that $\mu = 1$. Following the reasoning in Section 3 in [3], we can always find a measurable modification of X, so without loss of generality we may and do assume that X is measurable and almost surely $\int_0^t X_s^2 \mathrm{d}s < \infty$, for all $t \geq 0$. Thus, using the well-known bound $\left(\sum_{i=1}^d |x_i|\right)^2 \leq d \sum_{i=1}^d |x_i|^2$ and Jensen's inequality, we see that for any $V > 0$ and $t \leq V$

$$\begin{aligned}\left|\Delta_n \mathbf{S}_t^n - \int_0^{[nt]\Delta_n} X_s \mathrm{d}s\right|^2 &\leq V n \Delta_n \sum_{i=0}^{[nt]-1} \int_{t_i}^{t_{i+1}} |X_{t_i} - X_s|^2 \mathrm{d}s \\ &\leq C \sum_{i=0}^{[nV]-1} \int_{t_i}^{t_{i+1}} |X_{t_i} - X_s|^2 \mathrm{d}s,\end{aligned}$$

where we have used that $n\Delta_n$ is bounded. From this estimate we deduce that as $n \to \infty$

$$\begin{aligned}\mathbb{E}\left(\sup_{0 \leq t \leq T} \left|\Delta_n \mathbf{S}_t^n - \int_0^{[nt]\Delta_n} X_s \mathrm{d}s\right|^2\right) &\leq C \sum_{i=0}^{[nV]-1} \int_{t_i}^{t_{i+1}} \int_0^{s-t_i} a(r) \mathrm{d}r \mathrm{d}s \\ &\leq C a(0)([nV]\,\Delta_n)\Delta_n \to 0.\end{aligned}$$

The result now follows by observing that

$$\left|\int_0^t X_s \mathrm{d}s - \int_0^{[nt]\Delta_n} X_s \mathrm{d}s\right|^2 \leq |t - [nt]\,\Delta_n| \int_0^V X_s^2 \mathrm{d}s,$$

and using the fact that $|t - [nt]\,\Delta_n| \leq \Delta_n + V\,|1 - \Delta_n n|$. □

Proof of Proposition 3.2. Plainly, from (5.3)

$$\frac{[nt]}{n} X_0 - \frac{1}{n} S_{[nt]}^{\Delta_n,4} = \frac{[nt]}{n} \sum_{j=0}^{[nt]-1} \chi_{0,j}^{\Delta_n}, \tag{5.15}$$

which in view of (5.4) implies that

$$\mathcal{C}\left(z; \left(\frac{1}{n} S_{[nt]}^{\Delta_n,4} - \frac{[nt]}{n} X_0\right)\right) = \psi\left(\frac{[nt]}{n} z\right) \int_0^{[nt]\Delta_n} a(s) \mathrm{d}s \to 0.$$

Therefore, thanks to (5.5), we only need to check that, for $l = 1, 2, 3$, $\frac{1}{n} S_n^{\Delta_n,l} \xrightarrow{\mathbb{P}} 0$. To see this, observe that from equations (5.12)-(5.14) and the continuity of ψ

$$\left|\mathcal{C}\left(z; \frac{1}{n} S_n^{\Delta_n,1}\right)\right| \leq C t_n \int_0^{t_n} \left|\frac{a(s) - a(s + \Delta_n)}{\Delta_n}\right| \mathrm{d}s \leq C t_n^2 \to 0,$$

and

$$\left|\mathcal{C}\left(z; \frac{1}{n} S_n^{\Delta_n,2}\right) + \mathcal{C}\left(z; \frac{1}{n} S_n^{\Delta_n,3}\right)\right| \leq C \int_0^{t_n} a(s) \mathrm{d}s \to 0,$$

where we have further used that a is continuously differentiable in a neighborhood of 0. This completes the proof. □

5.3 Proof of Theorem 3.4

Our proof in this case relies heavily on the asymptotic behaviour of the Lévy measure of L around 0. It is worth noting that, if L is deterministic, then almost surely $Z_t^n \equiv 0$, so by the Lévy-Itô decomposition of Lévy bases (see [21]), in our proof we will always assume that $\gamma = \int_{|x|\leq 1} x\nu(\mathrm{d}x)$ or $\gamma = 0$, depending whether $\int_{\mathbb{R}}(1 \wedge |x|)\nu(\mathrm{d}x) < \infty$ or not. In this situation, under Assumption 3.3, Theorem 2 in [16] establishes that as $\varepsilon \to 0$

$$\varepsilon\psi(\varepsilon^{-1/\beta}z) \to \psi_\beta(z) = \begin{cases} -\frac{1}{2}b^2z^2 & \text{if } b > 0 \text{ and } \beta = 2, \\ \psi(z;\beta,\tilde{K}_+\beta,\tilde{K}_-\beta,\tilde{\gamma}) & \text{under iii. and } 0 < \beta < 2, \end{cases} \tag{5.16}$$

where $\psi(\cdot;\beta,K_+\beta,K_-\beta,\tilde{\gamma})$ as in (2.2). Note that the convergence takes place uniformly on compacts. The proof is divided into several steps: In the first step, we show that $S_n^{\Delta_n,1} = \mathrm{o}_{\mathbb{P}}(n{T_n}^{1/\beta})$. In the second step, we argue that L can be assumed to be strictly β-stable. Finally, we show that **i.** and **ii.** hold.

Step 1: $S_n^{\Delta_n,1} = \mathrm{o}_{\mathbb{P}}(n(n\Delta_n)^{1/\beta})$. Assume that (5.16) holds and set

$$\begin{aligned} A_n'(z) := &\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}(T_n - s)\psi_\beta\left(\frac{t_j}{{T_n}^{1+1/\beta}}z\right)\left[\frac{a(s)-a(s+\Delta_n)}{\Delta_n}\right]\mathrm{d}s \\ &+ \sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}(s - t_j)\psi_\beta\left(\frac{t_j}{{T_n}^{1+1/\beta}}z\right)\left[\frac{a(s)-a(s+\Delta_n)}{\Delta_n}\right]\mathrm{d}s. \end{aligned}$$

The C^1-property of a and the fact that $0 \leq t_j/T_n \leq 1$ lead us to

$$\begin{aligned} &\left|\mathcal{C}\left(z;\frac{1}{nT_n^{1/\beta}}S_n^{\Delta_n,2}\right) - A_n'(z)\right| \\ &\qquad\leq C\left(\sup_{|u|\leq|z|}T_n\left|\psi_\beta\left(\frac{u}{{T_n}^{1/\beta}}\right) - \psi\left(\frac{u}{{T_n}^{1/\beta}}\right)\right|\right)(T_n+\Delta_n) \to 0. \end{aligned}$$

Now, the strict stability and the continuity of ψ_β give us that

$$|A_n'(z)| \leq C(T_n + \Delta_n) \to 0, \ \ n \to \infty,$$

where we once again used the fact that a is continuously differentiable in a neighbourhood of 0. This is enough for the negligibility of $\frac{1}{n(n\Delta_n)^{1/\beta}}S_n^{\Delta_n,1}$. □

Step 2: An approximation. In this step we assume that (5.16) holds for some $0 < \beta \leq 2$. From (5.5), (5.15) and the previous step, we have that

$$Z_t^n = U_t^n - \hat{U}_t^n + \mathrm{o}_{\mathbb{P}}(T_n^{1/\beta}), \ t \geq 0, \tag{5.17}$$

where $U_t^n := \sum_{j=0}^{[nt]-1}\left(\frac{[nt]}{n} - \frac{j+1}{n}\right)\chi_{0,j}^{\Delta_n}$ and $\hat{U}_t^n := \frac{1}{n}S_{[nt]}^{\Delta_n,2}$. Furthermore, $(U_t^{n,\beta},{\hat{U}_t}^{n,\beta})_{t\geq 0}$ are defined as $(U_t^n,{\hat{U}_t}^n)_{t\geq 0}$ when we replace L by a homogeneous strictly β-stable distribution whose seed has characteristic exponent given by ψ_β. Note that U^n and $\hat{U}^n$ ($U^{n,\beta}$ and $\hat{U}^{n,\beta}$) are independent. We are going to show that the f.d.d. of $(U^n,\hat{U}^n)$ are asymptotically equivalent to those of $(U^{n,\beta},\hat{U}^{n,\beta})$. Indeed, fix $q \in \mathbb{N}$, $\lambda_1,\ldots,\lambda_q \in \mathbb{R}$ and $0 = u_0 < u_1 < \cdots < u_q$ and note that

$$\begin{aligned} \sum_{l=1}^{q}\lambda_l U_{u_l}^n &= \sum_{j=0}^{[nu_q]-1}\theta_{j,n}\chi_{0,j}^{\Delta_n}, \\ \sum_{l=1}^{q}\lambda_l \hat{U}_{u_l}^n &= \sum_{j=0}^{[nu_q]-1}\sum_{k=1}^{q}\left(\zeta_{i,[nu_k]}^{\Delta_n} - \zeta_{i,[nu_{k+1}]}^{\Delta_n}\right)\hat{\theta}_{i,k,n}, \end{aligned}$$

where $\zeta^{\Delta_n}_{i,[nu_{q+1}]} := 0$, and

$$\theta_{j,n} := \sum_{l=1}^{q}\sum_{m=l}^{q}\lambda_m \mathbf{1}_{[nu_{l-1}]\le j<[nu_l]}\left(\frac{[nu_m]}{n}-\frac{j+1}{n}\right),$$
$$\hat{\theta}_{i,k,n} := \sum_{m=1}^{k}\sum_{l=m}^{k}\lambda_l \mathbf{1}_{[nu_{m-1}]\le i<[nu_m]}\left(\frac{[nu_l]}{n}-\frac{i}{n}\right).$$

Whence, from (5.4) and (5.16), as $n\to\infty$,

$$\left|\mathcal{C}\left(z;\frac{1}{T_n^{1/\beta}}\sum_{l=1}^{q}\lambda_l U^n_{u_l}\right)-\mathcal{C}\left(z;\frac{1}{T_n^{1/\beta}}\sum_{l=1}^{q}\lambda_l U^{n,\beta}_{u_l}\right)\right|$$
$$\le C\sup_{|u|\le|C_\lambda z|}T_n\left|\psi_\beta\left(\frac{u}{T_n{}^{1/\beta}}\right)-\psi\left(\frac{u}{T_n{}^{1/\beta}}\right)\right|\to 0,$$

$$\left|\mathcal{C}\left(z;\frac{1}{T_n^{1/\beta}}\sum_{l=1}^{q}\lambda_l \hat{U}^n_{u_l}\right)-\mathcal{C}\left(z;\frac{1}{T_n^{1/\beta}}\sum_{l=1}^{q}\lambda_l \hat{U}^{n,\beta}_{u_l}\right)\right|$$
$$\le C\sup_{|u|\le|C_\lambda z|}T_n\left|\psi_\beta\left(\frac{u}{T_n{}^{1/\beta}}\right)-\psi\left(\frac{u}{T_n{}^{1/\beta}}\right)\right|\to 0,$$

where $C_\lambda := 2u_q\sum_{l=1}^{q}\sum_{m=l}^{q}|\lambda_m|\ge|\theta_{j,n}|+\left|\hat{\theta}_{i,k,n}\right|$, as claimed. □

Step 3: Proof of i. and ii. We start by showing that the f.d.d. distributions of $\frac{1}{\sqrt{n\Delta_n}}Z^n$ converge to those stated in the theorem. In the last part we show that the convergence in distribution can be strengthened to stable convergence.

Assume that $b>0$. In this case, by virtue of Step 2, we may and do assume that $\gamma=0$ and $\nu\equiv 0$. Accordingly, U^n and $\hat{U}^n$ are two independent centered Gaussian processes satisfying (5.17). Therefore, the convergence in **i.** is achieved whenever

$$\frac{1}{T_n}\mathbb{E}(U^n_tU^n_u)\to\sigma^2\int_0^{t\wedge u}(t-r)(u-r)\mathrm{d}r;\ \frac{1}{T_n}\mathbb{E}(\hat{U}^n_t\hat{U}^n_u)\to\sigma^2\int_0^{t\wedge u}(t-r)(u-r)\mathrm{d}r. \quad (5.18)$$

To see that this is the case, take $t\ge u\ge 0$. Then

$$\frac{1}{T_n}\mathbb{E}\left(U^n_tU^n_u\right)=b^2\sum_{j=0}^{[nu]-1}\int_{j/n}^{(j+1)/n}\left(\frac{[nt]}{n}-\frac{j+1}{n}\right)\left(\frac{[nu]}{n}-\frac{j+1}{n}\right)a(T_ns)\mathrm{d}s,$$
$$\frac{1}{T_n}\mathbb{E}(\hat{U}^n_t\hat{U}^n_u)=b^2\sum_{i=1}^{[nu]-1}\int_{(i-1)/n}^{i/n}\left(\frac{[nt]}{n}-\frac{i}{n}\right)\left(\frac{[nu]}{n}-\frac{i}{n}\right)a\left[T_n\left(\frac{[nt]}{n}-s\right)\right]\mathrm{d}s.$$

The results in (5.18) follow from the Dominated Convergence Theorem.

Suppose now that $b=0$ and Assumption 3.3 holds. Therefore, by the previous step, we may and do assume that L is strictly stable with characteristic exponent ψ_β, with $0<\beta<2$. Therefore, using the notation of Step 2, the strict stability of ψ_β results in

$$\mathcal{C}\left(z;\frac{1}{T_n^{1/\beta}}\sum_{l=1}^{q}\lambda_lU^n_{u_l}\right)=\sum_{l=1}^{q}\sum_{j=[nu_{l-1}]}^{[nu_l]-1}\int_{j/n}^{(j+1)/n}\psi_\beta\left(\sum_{m=l}^{q}\lambda_m\left(\frac{[nu_m]}{n}-\frac{j+1}{n}\right)z\right)a(T_ns)\mathrm{d}s$$
$$=\sum_{l=1}^{q}\int_{[nu_{l-1}]/n}^{[nu_l]/n}\psi_\beta\left(\sum_{m=l}^{q}\lambda_m\left(\frac{[nu_m]}{n}-\frac{[sn]+1}{n}\right)z\right)a(T_ns)\mathrm{d}s$$
$$\to a(0)\sum_{l=1}^{q}\int_{u_{l-1}}^{u_l}\psi_\beta\left(\sum_{m=l}^{q}\lambda_m\left(u_m-s\right)z\right)\mathrm{d}s,\ \ n\to\infty.$$

Similarly, as $n \to \infty$

$$\begin{aligned}
&\mathcal{C}\left(z; \frac{1}{T_n^{1/\beta}} \sum_{l=1}^{q} \lambda_l \hat{U}_{u_l}^n\right) \\
&\quad = \sum_{x=1}^{q} \sum_{i=1\vee[nu_{x-1}]}^{[nu_x]-1} \int_{(i-1)/n}^{i/n} \psi_\beta \left(\sum_{l=x}^{q} \lambda_m \left(\frac{[nu_l]}{n} - \frac{i}{n}\right) z\right) a\left[T_n\left(\frac{[nu_q]}{n} - s\right)\right] \mathrm{d}s \\
&\quad\quad + \sum_{x=1}^{q} \sum_{i=1\vee[nu_{x-1}]}^{[nu_x]-1} \sum_{k=x}^{q-1} \int_{(i-1)/n}^{i/n} \psi_\beta \left(\sum_{l=x}^{k} \lambda_m \left(\frac{[nu_l]}{n} - \frac{i}{n}\right) z\right) \\
&\quad\quad \times \left(a\left[T_n\left(\frac{[nu_k]}{n} - s\right)\right] - a\left[T_n\left(\frac{[nu_{k+1}]}{n} - s\right)\right]\right) \mathrm{d}s \\
&\quad \to a(0) \sum_{x=1}^{q} \int_{u_{x-1}}^{u_x} \psi_\beta \left(\sum_{l=x}^{q} \lambda_m (u_l - s) z\right) \mathrm{d}s.
\end{aligned}$$

Since U^n and $\hat{U}^n$ are independent, the previous relations show the desired weak convergence of $\frac{1}{(n\Delta_n)^{1/\beta}} Z^n$. Therefore, in order to conclude the proof, it remains to verify that the convergence also takes place stably and the limit is independent of L. Let B be a bounded Borel set and H the limiting process stated in the theorem. Since for every $n \in \mathbb{N}$, Z^n is $\mathcal{F}_L$-measurable, thanks to Theorem 3.2 in [15], it is sufficient to show that

$$\left(\{(n\Delta_n)^{-1/\beta} Z_{u_l}\}_{l=1}^{q}, L(B)\right) \to \left(\{H_{u_l}\}_{l=1}^{q}, L(B)\right), \tag{5.19}$$

and that for all $z_1, \ldots, z_{q+1} \in \mathbb{R}$.

$$\mathcal{C}\left((z_1, \ldots, z_{q+1}); (\{H_{u_l}\}_{l=1}^{q}, L(B))\right) = \mathcal{C}\left((z_1, \ldots, z_q); \{H_{u_l}\}_{l=1}^{q}\right) + \mathcal{C}\left(z_{q+1}; L(B)\right). \tag{5.20}$$

Set

$$B_n = \bigcup_{j=0}^{[u_q n]} \mathcal{P}_A^{\Delta_n}(0, j) \cup \bigcup_{i=1}^{[u_q n]} \bigcup_{j \geq i} \mathcal{P}_A^{\Delta_n}(i, j).$$

Then by (5.4), $Leb(B \cap B_n) \leq 2a(0)[u_q n]\Delta_n \to 0$, meaning that $L(B \cap B_n) \xrightarrow{\mathbb{P}} 0$. Relations (5.19) and (5.20) are easily obtained by decomposing $L(B) = L(B \cap B_n) + L(B \backslash B_n)$, the preceeding observation, and an application of Slutsky's Theorem. □

5.4 Proof of Theorem 3.8

Here we show the validity of Theorem 3.8. The proof will be divided into three steps. We first show the convergence of the finite-dimensional distributions. Secondly, we verify that our sequence is tight. We conclude by proving that the convergence is also stable. Therefore, for the rest of this subsection we assume that Assumption 3.5 holds. We finally emphasize that thanks to the Lévy-Itô decomposition of Lévy bases (see [21]) and Lemma 5.1, we may and do assume that L has no Gaussian component, i.e. $b = 0$.

Step 1: Convergence of the f.d.d. Fix $r \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_r \in \mathbb{R}$ and $0 = u_0 < u_1 < \cdots < u_r$. We start by noting that from Lemma 5.1, we have that

$$\sqrt{\frac{\Delta_n}{n}} \sum_{q=1}^{r} \lambda_q \mathbf{S}_{t_q}^n = \sqrt{\frac{\Delta_n}{n}} \sum_{q=1}^{r} \lambda_q S_{[nt_q]}^{\Delta_n,1} + \mathrm{o}_{\mathbb{P}}(1), \quad n \in \mathbb{N}.$$

Thus, in order show the convergence of the finite-dimensional distributions, it is enough to verify that

$$\sqrt{\frac{\Delta_n}{n}}\sum_{q=1}^{r}\lambda_q S^{\Delta_n,1}_{[nt_q]}\xrightarrow{d}\sigma_a\left(\sum_{q,q'=1}^{r}\lambda_q\lambda_{q'}t_q\wedge t_{q'}\right)^{1/2}N(0,1). \tag{5.21}$$

Since

$$\sqrt{\frac{\Delta_n}{n}}\sum_{q=1}^{r}\lambda_q S^{\Delta_n,1}_{[nt_q]}=\frac{1}{\sqrt{T_n}}\sum_{i=1}^{n-1}\xi_{i,n},$$

where the array

$$\xi_{i,n}=\sum_{j=i}^{n-1}\Delta_n(j-i+1)d_{n,j}\chi^{\Delta}_{i,j},\ \ d_{n,j}:=\sum_{q=1}^{r}\lambda_q\mathbf{1}_{j<[nt_q]}, \tag{5.22}$$

is centered and row-wise independent, (5.21) will be achieved whenever

$$\frac{1}{T_n}\sum_{i=1}^{n-1}\mathbb{E}\left(|\xi_{i,n}|^2\right)\to\sigma_a^2\sum_{q,q'=1}^{r}\lambda_q\lambda_{q'}t_q\wedge t_{q'},\ \ n\to\infty, \tag{5.23}$$

as well as the Lyapunov condition is satisfied, i.e. for some $p>2$

$$I_{n,p,1}:=\sum_{i=1}^{n-1}\mathbb{E}\left(|\xi_{i,n}|^p\right)=\mathrm{o}\left((T_n)^{p/2}\right),\ \text{ as } n\to\infty. \tag{5.24}$$

In view that, for all $t>u$, $S^{\Delta_n,1}_{[nt]}=S^{\Delta_n,1}_{[nu]}+\sum_{j=[nu]}^{[nt]-1}\sum_{i=1}^{j}(j-i+1)\chi^{\Delta_n}_{i,j}$, we conclude from the proof of Lemma 5.3 that

$$\mathbb{E}\left(S^{\Delta_n,1}_{[nt]}S^{\Delta_n,1}_{[nu]}\right)=\mathrm{Var}\left(S^{\Delta_n,1}_{[nu]}\right)=\mathrm{Var}\left(\mathbf{S}^n_u\right)+\mathrm{o}(n/\Delta_n),\ \ t>u.$$

(5.23) follows now easily from this and Lemma 5.1. In order to check (5.24) first observe that thanks to Assumption 3.5, $\int_{\mathbb{R}}|\Gamma_X(s)|\,\mathrm{d}s<\infty$, and that for any $0\leq p<p_0$ the measure

$$\mu_{p,a}(\mathrm{ds})=\mathbf{1}_{s\geq 0}s^p\mathrm{d}\,|a|\,(s),$$

is finite. Now, fix $p_0\wedge 3>p>2$. By Rosenthal's inequality

$$\mathbb{E}\left(|\xi_{j,n}|^p\right)\leq C\left\{\sum_{j=1}^{n-i}|t_j|^p\,\mathbb{E}\left(\left|\chi^{\Delta_n}_{i,j+i-1}\right|^p\right)+\left[\sum_{j=1}^{n-i}|t_j|^2\,\mathbb{E}\left(\left|\chi^{\Delta_n}_{i,j+i-1}\right|^2\right)\right]^{p/2}\right\},$$

where we further used that $d_{n,j}$ is uniformly bounded. Moreover, from (5.4), we deduce that the Lévy measure of $\chi^{\Delta_n}_{i,j+i-1}$ is given by

$$\nu_{\chi^{\Delta_n}_{i,j+i-1}}(\cdot)=\int_{t_{j-1}}^{t_j}\left[a(s)-a(s+\Delta_n)\right]\mathrm{d}s\nu(\cdot),\ j=1,\dots,n-i.$$

Therefore, from Corollary 1.2.7. in [30], there is a constant $C>0$ only depending on p and $\nu(\cdot)$, such that

$$\mathbb{E}\left(\left|\chi^{\Delta_n}_{i,j+i-1}\right|^p\right)\leq C\max\left\{\int_{t_{j-1}}^{t_j}\left[a(s)-a(s+\Delta_n)\right]\mathrm{d}s,\left(\int_{t_{j-1}}^{t_j}\left[a(s)-a(s+\Delta_n)\right]\mathrm{d}s\right)^{p/2}\right\}.$$

Hence,

$$I_{n,p,1} \leq C(I^{(1)}_{n,p,1} + I^{(2)}_{n,p,1} + I^{(3)}_{n,p,1}), \tag{5.25}$$

where we let

$$I^{(1)}_{n,p,1} := T_n \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} |t_j|^p \,\mathrm{d}\,|a|\,(s),$$

$$I^{(2)}_{n,p,1} := \Delta_n^{p/2} n \sum_{j=1}^{n} \left(\int_{t_{j-1}}^{t_j} |t_j|^2 \,\mathrm{d}\,|a|\,(s) \right)^{p/2},$$

$$I^{(3)}_{n,p,1} := \Delta_n^{p/2} n \left(\sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} |t_j|^2 \,\mathrm{d}\,|a|\,(s) \right)^{p/2}.$$

Thus, (5.24) is obtained whenever $I^{(1)}_{n,p,1} + I^{(2)}_{n,p,1} + I^{(3)}_{n,p,1} = \mathrm{o}\left((T_n)^{p/2}\right)$. Observe that, for any $j = 1, \ldots, n$, $t_{j-1} \leq \zeta \leq t_j$ and $p_0 \wedge 3 > q \geq 2$, it holds that

$$\int_{t_{j-1}}^{t_j} |\zeta^q - s^q| \,\mathrm{d}\,|a|\,(s) \leq C(\Delta_n \mu_{q-1,a}(t_{j-1}, t_j] + \Delta_n^2 \mu_{q-2,a}(t_{j-1}, t_j] + \Delta_n^q \mu_{0,a}(t_{j-1}, t_j]). \tag{5.26}$$

Using the previous property and the fact that $\mu_{p,a}(\mathbb{R}) < \infty$, one easily deduces that, for any $p_0 \wedge 3 > p > 2$,

$$\begin{aligned} \frac{1}{(T_n)^{p/2}} \left| I^{(1)}_{n,p,1} \right| &\leq C \frac{1}{(T_n)^{p/2-1}} \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} |t_j|^p \,\mathrm{d}\,|a|\,(s) \\ &\leq C \frac{1}{(T_n)^{p/2-1}} \to 0. \end{aligned} \tag{5.27}$$

Similarly,

$$\frac{1}{(T_n)^{p/2}} \left| I^{(3)}_{n,p,1} \right| \leq C \frac{1}{n^{p/2-1}} \to 0.$$

Finally, by Jensen's inequality,

$$\frac{1}{(T_n)^{p/2}} \left| I^{(2)}_{n,p,1} \right| \leq C \frac{1}{n^{p/2-1}} \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} |t_j|^p \,\mathrm{d}\,|a|\,(s) \to 0,$$

which concludes the argument for (5.21). □

Step 2: Tightness. Using similar arguments as in the proof of Lemma 2.1 in [29] and Lemma 5.1, we deduce that $\sqrt{\frac{\Delta_n}{n}}\mathbf{S}^n$ is tight if for any sequence $m_n \in \mathbb{N}$, such that $m_n \uparrow \infty$, $\Delta_n m_n \to \infty$ and $\Delta_n \downarrow 0$, as $n \to \infty$, it holds that

$$\mathbb{E}\left(\left|S^{\Delta_n}_{m_n}\right|^p\right) = \mathrm{O}\left(\left(\frac{m_n}{\Delta_n}\right)^{p/2}\right), \tag{5.28}$$

for some $p > 2$. We proceed now to show that (5.28) is fulfilled. By (5.5), Lemma 5.1 and Rosenthal's inequality, we have that

$$\left(\frac{\Delta_n}{m_n}\right)^{p/2} \mathbb{E}\left(\left|S^{\Delta_n}_{m_n}\right|^p\right) \leq C\left\{I_{n,p,1} + I_{n,p,2} + I_{n,p,3} + I_{n,p,4}\right\} + \mathrm{O}(1),$$

where $I_{n,p,1}$ is as in step 1 but we replace n by m_n, and

$$I_{n,p,2} = \frac{1}{(m_n\Delta_n)^{p/2}} \sum_{i=1}^{m_n-1} (t_{m_n} - t_i)^p \mathbb{E}\left(\left|\chi_{i,m_n-1}^{\Delta_n}\right|^p\right),$$
$$I_{n,p,3} = \frac{1}{(m_n\Delta_n)^{p/2}} \sum_{i=1}^{m_n-1} (t_{j+1})^p \mathbb{E}\left(\left|\chi_{0,j}^{\Delta_n}\right|^p\right),$$
$$I_{n,p,4} = (\Delta_n m_n)^{p/2}\, \mathbb{E}\left(\left|\chi_{0,m_n-1}^{\Delta_n}\right|^p\right).$$

We have already seen that $I_{n,p,1} \to 0$ as $n \to \infty$. On the other hand, invoking once again Corollary 1.2.7 in [30] and using (5.4), we get

$$\mathbb{E}\left(\left|\chi_{i,m_n-1}^{\Delta_n}\right|^p\right) \le C \max\left\{\int_{t_{i-1}}^{t_i} a(t_{m_n-1} - s)\mathrm{d}s, \left(\int_{t_{i-1}}^{t_i} a(t_{m_n-1} - s)\mathrm{d}s\right)^{p/2}\right\},$$
$$\mathbb{E}\left(\left|\chi_{0,j}^{\Delta_n}\right|^p\right) \le C \max\left\{\int_{t_{j-1}}^{t_j} a(s)\mathrm{d}s, \left(\int_{t_{i-1}}^{t_i} a(s)\mathrm{d}s\right)^{p/2}\right\},$$
$$\mathbb{E}\left(\left|\chi_{0,m_n-1}^{\Delta_n}\right|^p\right) \le C \max\left\{\int_{t_{m_n-1}}^{\infty} a(s)\mathrm{d}s, \left(\int_{t_{m_n-1}}^{\infty} a(s)\mathrm{d}s\right)^{p/2}\right\}.$$

Similarly as in Step 1, we deduce that for $p_0 \wedge 3 > p > 2$ and n large enough, the following estimates are valid

$$I_{n,p,2} \le C \frac{1}{(m_n\Delta_n)^{p/2}} \sum_{i=1}^{m_n-1} (t_{m_n} - t_i)^p \int_{t_{i-1}}^{t_i} a(t_{m-1} - s)\mathrm{d}s,$$
$$I_{n,p,3} \le C \frac{1}{(m_n\Delta_n)^{p/2}} \sum_{i=1}^{m_n-1} (t_{j+1})^p \int_{t_{j-1}}^{t_j} a(s)\mathrm{d}s,$$
$$I_{n,p,4} \le C\,(\Delta_n m_n)^{p/2} \int_{\Delta_n(m_n-1)}^{\infty} a(s)\mathrm{d}s.$$

Assumption 3.5 now asserts that, as $n \to \infty$,

$$I_{p,4} \le C(\Delta_n m_n)^{p/2 - p_0 + 1} \to 0,$$

because $2 < p < p_0 < 2(p_0 - 1)$. Furthermore, analogous arguments used to establish (5.8) and (5.9), can be applied in order to get that

$$I_{p,2} + I_{p,3} = C \frac{1}{(\Delta_n m_n)^{p/2}} \int_0^{\Delta_n m_n} s^p a(s)\mathrm{d}s + \mathrm{o}(1) \to 0,$$

which implies (5.28). □

Step 3: Stability. From Step 2, Proposition 3.9 in [15] and its subsequent remark, the stable convergence in $\mathcal{D}([0,1])$ will be obtained if (5.21) can be strengthened to $\mathcal{G}^X$-stable convergence. Consider the filtration

$$\mathcal{F}_i^n := \sigma\left(\left\{L(\mathcal{P}_A^{\Delta_n}(k,j))\right\}_{j \ge k} : k = 0, \dots, i\right), \;\; i = 0, \dots, n-1, \; n \in \mathbb{N}. \tag{5.29}$$

From Step 1

$$\sqrt{\frac{\Delta_n}{n}} \sum_{q=1}^{r} \lambda_q \mathbf{S}_{t_q}^n = \frac{1}{\sqrt{T_n}} \sum_{i=1}^{n-1} \xi_{i,n} + \mathrm{o}_{\mathbb{P}}(1), \;\; n \in \mathbb{N},$$

where $\xi_{i,n}$, defined in (5.22), is $\mathcal{F}_i^n$-measurable and independent of $\mathcal{F}_{i-1}^n$, for all $i = 1, \ldots, n-1$. Consequently, thanks to (5.23), (5.24) and Theorem 6.1 in [15], (5.21) can be strengthened to $\mathcal{G}$-stable convergence, where

$$\mathcal{G} := \sigma\left(\bigcup_{n\geq 1}\bigcap_{N\geq n} \mathcal{F}_n^N\right),$$

but in view that

$$\sigma(X_0, X_{\Delta_n}, \ldots, X_{i\Delta_n}) \subseteq \mathcal{F}_i^n, \ \ i = 0, 1, \ldots, n-1,$$

it follows immediately that $\mathcal{G}^X \subseteq \mathcal{G}$, which concludes the proof. □

5.5 Proof of Theorems 3.10-3.13

In this subsection, we justify the statements of Theorems 3.8-3.11. In what follows (γ, b, ν) and ψ will denote respectively, the characteristic triplet and exponent of L. We note that, for the sake of exposition, the proof of each theorem is given in a corresponding subsubsection.

5.5.1 The case $b > 0$

For every $n \in \mathbb{N}$, we will let $r_n = \frac{1}{n\sqrt{a(n\Delta_n)n\Delta_n}}$. Observe that the same argument used in step 2 in the proof of Theorem 3.8 together with Lemma 5.1, give us automatically that $r_n\mathbf{S}^n$ is tight in $\mathcal{D}([0,1])$ if $\mathbb{E}\left(|L'|^2\right) < \infty$. Therefore, we only need to show that within the framework of Assumption 3.6, the finite-dimensional distributions of $r_n\mathbf{S}^n$ converges to those of the fBm with index $H = \frac{3-\alpha}{2}$.

Proof of Theorem 3.10. Let us start by noting that from the proof of Theorem 3.10 **ii.** below, and the Lévy-Itô decomposition of Lévy bases, it follows that the non-Gaussian component of $r_n\mathbf{S}^n$ is negligible. Consequently, we may and do assume that $r_n\mathbf{S}^n$ is centered and Gaussian. Moreover, it satisfies that and such that for all $1 \geq t \geq u \geq 0$

$$\mathbb{E}(\mathbf{S}_t^n\mathbf{S}_u^n) = \frac{1}{2}\left\{\mathrm{Var}(\mathbf{S}_t^n) + \mathrm{Var}(\mathbf{S}_u^n) - \mathrm{Var}(\mathbf{S}_{t-u}^n)\right\},$$

where we have used that X is stationary. The convergence of the finite-dimensional distributions of $r_n\mathbf{S}^n$ follows now by Lemma 5.1. □

5.5.2 The case $b = 0$

In this part, unless otherwise said, we will always assume that $b = 0$. Observe that, under the assumptions of Theorem 3.11 ii., (5.16) is once again valid. We recall for the convenience of the reader that we are also assuming that L has mean zero. Finally, we would like to stress that in view that (5.33) below holds whenever L has mean zero, then relation (5.34) as well as (5.36)-(5.38) remain valid if we replace β_ν by 2. It follows from this that

$$\frac{1}{n\sqrt{a(n\Delta_n)n\Delta_n}}\mathbf{S}_t^n \xrightarrow{\mathbb{P}} 0, \ \ n \to \infty.$$

We now proceed to present a proof for Theorem 3.11.

Proof of Theorem 3.11. The proof is organized as follow: First, based on our assumption, we derive some preliminary estimates. Secondly, by using (5.5), we approximate the characteristic function of $\mathbf{S}_t^n$ by means of $I_n^{\Delta,1}(\cdot)$ and $I_n^{\Delta,2}(\cdot)$, where the latter are as in

Lemma 5.6. We conclude by applying such approximation to obtain the desired result. For the rest of the proof, we will use the notation $T_n = n\Delta_n$, $r_n = a(T_n)T_n$, as well as

$$B_n = \begin{cases} \Delta_n/(c_a T_n)^{\frac{1}{\alpha}} & \text{if } \beta_\nu < \alpha; \\ 1/nr_n^{1/\beta_\nu} & \text{if } 2 > \beta_\nu > \alpha. \end{cases}$$

Preliminary estimates: First, Assumptions 3.6 and 3.7 allow us to invoke the so-called Potter's bounds (see Theorem 1.5.6 in [7]). Such a result provides the existence of a positive constant only depending on $\varepsilon > 0$, such that, for all $0 \le r \le 1$,

$$\frac{a(T_n s)}{a(T_n)} \le C\left(s^{-\alpha-\varepsilon} \vee s^{-\alpha+\varepsilon}\right), \text{ and } \frac{a'(r\Delta_n + T_n s)}{a'(T_n)} \le C\left(s^{-(\alpha+1)-\varepsilon} \vee s^{-(\alpha+1)+\varepsilon}\right), \quad s > 0. \tag{5.30}$$

From Lemma 5.5 and the square integrability of L', we have that, for every $2 \ge \theta > \beta_\nu$,

$$|\psi(zy)| \le C\left(|y|^2 \wedge |y|^\theta\right), \quad z, y \in \mathbb{R}. \tag{5.31}$$

On the other, using that $\mathbb{E}\left(|L'|^2\right) < \infty$, it follows that as $x \to \infty$, $\nu^{\pm}(x) = \mathrm{O}(x^\theta)$, for all $\theta \le 2$. Thus, if $\nu^{\pm}(x) \sim \tilde{K}_{\pm} x^{-\beta_\nu}$ as $x \to 0^+$, for some constants $\tilde{K}_+ + \tilde{K}_- > 0$, then the càdlàg function $\ell(x) := \bar{v}(x)x^{\beta_\nu} := [v^+(x) + v^-(x)]\, x^{\beta_\nu}$ is uniformly bounded in $[0,\infty)$. Consequently, by Lemma 5.5, for any $y > 0$, $z \in \mathbb{R}$

$$|\psi(zy)| \le C(|z| \vee 1)^2 \int_0^1 \bar{v}(x/y) x \mathrm{d}x \le C y^{\beta_\nu}. \tag{5.32}$$

Estimating the characteristic function: In this part we will only assume that $a \in \mathrm{RV}_\alpha^\infty$. Recall that from (5.5), the following decomposition holds

$$\mathcal{C}(z; S_n^{\Delta_n}) = \sum_{k=1}^4 \mathcal{C}(z; S_n^{\Delta_n,k}), \quad n \in \mathbb{N}, z \in \mathbb{R}.$$

From Lemmas 5.6 and 5.4 as well as KT, for $l = 2, 3$

$$\begin{aligned} \mathcal{E}_1^n(z) := \left|\mathcal{C}\left(z; S_n^{\Delta_n,1}\right) - I_n^{\Delta,1}(z)\right| \le & C|z|^2 n\mathrm{O}(1) \\ & + C\Delta_n \int_0^{T_n} \left|\psi\left(\frac{s}{\Delta} z\right)\right| \mathrm{d}\,|a|\,(s), \\ \mathcal{E}_l^n(z) := \left|\mathcal{C}\left(z; S_n^{\Delta_n,l}\right) - I_n^{\Delta,2}(z)\right| \le & C\frac{|z|^2}{\Delta_n}\left(\mathrm{O}(T_n^2 a(T_n)) + \mathrm{O}(\Delta_n)\right). \end{aligned} \tag{5.33}$$

Suppose first that $\beta_\nu > \alpha$. If (5.32) holds, then

$$\begin{aligned} \mathcal{E}_1^n(B_n z) \le & C\frac{\Delta_n}{r_n^{2/\beta_\nu} T_n}\mathrm{O}(1) + \frac{C}{n}\frac{1}{T_n^{\beta_\nu} a(T_n)} \int_0^{T_n} s^{\beta_\nu} \mathrm{d}\,|a|\,(s), \\ \mathcal{E}_l^n(B_n z) \le & C\frac{\Delta_n}{r_n^{2/\beta_\nu} T_n^2}\left(\mathrm{O}(T_n^2 a(T_n)) + \mathrm{O}(\Delta_n)\right). \end{aligned}$$

Moreover, by KT

$$\int_0^{T_n} s^{\beta_\nu} \mathrm{d}\,|a|\,(s) = \mathrm{O}(T_n^{\beta_\nu} a(T_n)).$$

We deduce from this and (5.33) that

$$\begin{aligned} \mathcal{E}_1^n(B_n z) = & C\frac{\Delta_n}{r_n^{2/\beta_\nu} T_n} + \mathrm{o}(1) \to 0, \\ \mathcal{E}_l^n(B_n z) = & \frac{\Delta_n}{r_n^{\frac{2-\beta_\nu}{\beta_\nu}} T_n}\mathrm{O}(1) + \frac{\Delta_n^2}{r_n^{2/\beta_\nu} T_n^2}\mathrm{O}(1) \to 0, \end{aligned} \tag{5.34}$$

where we further used that $a \in \mathrm{RV}_\alpha^\infty$ as well as the fact that $\beta_\nu + 2(1-\alpha) > 0$ and $(1-\alpha)(2-\beta_\nu) + \beta_\nu > 0$. Suppose now that $\beta_\nu < \alpha$. If (5.31) holds, we obtain from (5.33) that, for all $2 > \alpha > \theta > \beta_\nu$,

$$\mathcal{E}_1^n\left[B_n z\right] \leq C\Delta_n \int_0^{T_n} \left(\left|s/T_n^{\frac{1}{\alpha}}\right|^2 \wedge \left|s/T_n^{\frac{1}{\alpha}}\right|^\theta\right) \mathrm{d}\left|a\right|(s) + \mathrm{o}(1),$$
$$\mathcal{E}_l^n\left[B_n z\right] \leq C\Delta_n a(T_n) T_n^{\frac{2(\alpha-1)}{\alpha}} \mathrm{O}(1).$$

Using that $2(\alpha-1) - \alpha < 0$, results in $\mathcal{E}_l^n\left[B_n z\right]$ vanishing as $n \to \infty$, for $l = 2, 3$. Now, KT implies that

$$\int_0^{T_n} \left(\left|s/T_n^{\frac{1}{\alpha}}\right|^2 \wedge \left|s/T_n^{\frac{1}{\alpha}}\right|^\theta\right) \mathrm{d}\left|a\right|(s) = \int_0^{T_n^{\frac{1}{\alpha}}} \left(s/T_n^{\frac{1}{\alpha}}\right)^2 \mathrm{d}\left|a\right|(s) + \int_{T_n^{\frac{1}{\alpha}}}^{T_n} \left(s/T_n^{\frac{1}{\alpha}}\right)^\theta \mathrm{d}\left|a\right|(s)$$
$$= \mathrm{O}(a(T_n^{\frac{1}{\alpha}})) + \mathrm{O}(T_n^{-\frac{\theta}{\alpha}}).$$

We conclude from this that $\mathcal{E}_1^n\left[B_n z\right] \to 0$ as $n \to \infty$.

Convergence in i.: So far we have shown that, if $a \in \mathrm{RV}_\alpha^\infty$ and either $\beta_\nu > \alpha$ and (5.32) or $\beta_\nu < \alpha$ and (5.31) hold, then for $l = 2, 3$

$$\mathcal{C}(z; B_n S_n^{\Delta_n,1}) = I_n^{\Delta,1}\left(B_n z\right) + \mathrm{o}(1);$$
$$\mathcal{C}(z; B_n S_n^{\Delta_n,l}) = I_n^{\Delta,2}\left(B_n z\right) + \mathrm{o}(1).$$

We proceed to verify that $S_n^{\Delta_n,l}$ is negligible for $l = 2, 3, 4$, whenever $\beta_\nu < \alpha$ and $a \in \mathrm{RV}_\alpha^\infty$. Indeed, from (5.31), for all $z \neq 0$ and $2 > \alpha > \theta > \beta_\nu$, we get due to KT that

$$\left|\mathcal{C}\left(z; B_n S_n^{\Delta_n,4}\right)\right| = \left|\psi\left(z T_n^{\frac{\alpha-1}{\alpha}}\right) \int_{T_n}^\infty a(s)\mathrm{d}s\right| \leq C T_n^{\frac{\theta(\alpha-1)}{\alpha}} \int_{T_n}^\infty a(s)\mathrm{d}s \to 0.$$

In a similar way, we deduce from (5.30) and KT that for $(3-\alpha) \wedge (\alpha - \theta) > \varepsilon > 0$

$$\left|I_n^{\Delta,2}\left(B_n z\right)\right| \leq C T_n^{\frac{1}{\alpha}} \int_0^{T_n^{\frac{\alpha-1}{\alpha}}} \left(s^2 \wedge s^\theta\right) a(T_n^{\frac{1}{\alpha}} s)\mathrm{d}s \tag{5.35}$$
$$= \mathrm{O}(T_n^{\frac{1}{\alpha}} a(T_n^{\frac{1}{\alpha}})) + T_n^{\frac{1}{\alpha}} a(T_n^{\frac{1}{\alpha}}) \int_1^{T_n^{\frac{\alpha-1}{\alpha}}} s^{\theta-\alpha+\varepsilon}\mathrm{d}s.$$

If $\beta_\nu < \alpha - 1$, we can further choose $\beta_\nu < \theta < \alpha - 1$ and $0 < \varepsilon < \alpha - 1 - \theta$ in such a way $\int_1^\infty s^{\theta-\alpha+\varepsilon}\mathrm{d}s < \infty$. Otherwise, we have

$$\int_1^{T_n^{\frac{\alpha-1}{\alpha}}} s^{\theta-\alpha+\varepsilon}\mathrm{d}s = \mathrm{O}(T_n^{\frac{\alpha-1}{\alpha}(\theta-(\alpha-1)+\varepsilon)}).$$

Either way, $T_n^{\frac{1}{\alpha}} a(T_n^{\frac{1}{\alpha}}) \int_1^{T_n^{\frac{\alpha-1}{\alpha}}} s^{\theta-\alpha+\varepsilon}\mathrm{d}s \to 0$ whenever $a \in \mathrm{RV}_\alpha^\infty$, which implies immediately that $I_n^{\Delta,2}\left(B_n z\right) \to 0$. Using the previous reasoning, we conclude that, if $a \in \mathrm{RV}_\alpha^\infty$, then

$$B_n \mathbf{S}_t^n = B_n S_{[nt]}^{\Delta_n,1} + \mathrm{o}_{\mathbb{P}}(1).$$

Since $(S_{[nt]}^{\Delta_n,1})_{t\geq 0}$ has independent increments, it only remains to verify that, for all $1 \geq t > u \geq 0$, as $n \to \infty$,

$$\mathcal{C}\left(z; B_n\left[S_{[nt]}^{\Delta_n,1} - S_{[nu]}^{\Delta_n,1}\right]\right) \to \mathcal{C}(z; Y_t - Y_u),$$

where Y is as in the theorem. For simplicity, we will only consider the case when $t=1$, $u=0$ and $c_a=1$. Suppose now that $z\neq 0$, then by doing the change of variables $x=(s\,|z|)/T_n^{\frac{1}{\alpha}}$ and invoking Assumption 3.7, we see that

$$I_n^{\Delta,1}(B_n z) = \frac{a'(T_n^{\frac{1}{\alpha}})T_n^{\frac{\alpha+1}{\alpha}}}{|z|}\int_0^1\int_0^{T_n^{\frac{\alpha-1}{\alpha}}|z|}\left[1-\frac{x}{|z|\,T_n^{\frac{\alpha-1}{\alpha}}}\right]\psi(\mathrm{sign}(z)x)\frac{a'\left(r\Delta_n+T_n^{\frac{1}{\alpha}}\frac{x}{|z|}\right)}{a'(T_n^{\frac{1}{\alpha}})}\mathrm{d}x\mathrm{d}r.$$

From (5.30) and (5.31), we deduce that for $2>\alpha>\theta>\beta_\nu$ and any $\varepsilon>0$

$$\left|\left[1-\frac{x}{zT^{\frac{\alpha-1}{\alpha}}}\right]\psi(\mathrm{sign}(z)x)\frac{a'\left(r\Delta_n+T_n^{\frac{1}{\alpha}}\frac{x}{z}\right)}{a'(T_n^{\frac{1}{\alpha}})}\right|\mathbf{1}_{x\leq T^{\frac{\alpha-1}{\alpha}}z}$$
$$\leq C\left(|x|^2\wedge|x|^\theta\right)\left(|x|^{-(\alpha+1)-\varepsilon}\vee|x|^{-(\alpha+1)+\varepsilon}\right).$$

Choosing $0<\varepsilon<(2-\alpha)\wedge(\alpha-\theta)$ allows us to apply the Dominated Convergence Theorem in order to get that

$$I_n^{\Delta,1}(B_n z)\to|z|^\alpha\int_0^\infty\psi(\mathrm{sign}(z)x)x^{-(\alpha+1)}\mathrm{d}x.$$

The result now follows from Fubini's Theorem and the relation (see Lemma 14.11 in [24])

$$\int_0^\infty(e^{\pm ir}-1\pm ir)r^{-\varpi-1}\mathrm{d}r=\Gamma(-\varpi)e^{\mp i\frac{\pi\varpi}{2}},\quad 1<\varpi<2.$$

Convergence of ii.: We conclude the proof by showing that ii. holds, so for the rest of the proof we will assume that $\nu^{\pm}(x)\sim\tilde{K}_{\pm}x^{-\beta_\nu}$ as $x\to 0^+$ and that $2>\beta_\nu>\alpha$. From the first and second part of the proof, it is enough to show that in this situation

$$I_n^{\Delta,1}(B_n z)\to\alpha\int_0^1(1-s)s^{\beta_\nu-(\alpha+1)}\mathrm{d}s\psi_{\beta_\nu}(z);\tag{5.36}$$

$$I_n^{\Delta,2}(B_n z)\to\int_0^1 s^{\beta-\alpha}\mathrm{d}s\psi_{\beta_\nu}(z),\ l=2,3;\tag{5.37}$$

$$\mathcal{C}\left(z;B_nS_n^{\Delta_n,4}\right)\to\frac{1}{\alpha-1}\psi_{\beta_\nu}(z).\tag{5.38}$$

Since $r_n\to 0$, we deduce from KT and (5.16) that as $n\to\infty$

$$\mathcal{C}\left(z;B_nS_n^{\Delta_n,4}\right)=r_n\psi\left(\frac{z}{r_n^{1/\beta_\nu}}\right)\frac{1}{r_n}\int_{T_n}^\infty a(s)\mathrm{d}s\to\frac{1}{\alpha-1}\psi_{\beta_\nu}(z),\ \ z\in\mathbb{R}.$$

On the other hand, by doing the change of variables $x=s/T_n$, we get that

$$I_n^{\Delta,1}(B_n z)=\frac{T_na'(T_n)}{a(T_n)}\int_0^1\int_0^1(1-s)r_n\psi\left(\frac{zs}{r_n^{1/\beta_\nu}}\right)\frac{a'(T_n(r/n+s))}{a'(T_n)}\mathrm{d}s\mathrm{d}r,$$
$$I_n^{\Delta,2}(B_n z)=\int_0^1 r_n\psi\left(\frac{zs}{r_n^{1/\beta_\nu}}\right)\frac{a(T_ns)}{a(T_n)}\mathrm{d}s.$$

From (5.16) and the fact that $a'\in\mathrm{RV}_{\alpha+1}^\infty$, it follows that for $0<r,s\leq 1$ as $n\to\infty$

$$\frac{T_na'(T_n)}{a(T_n)}r_n\psi\left(\frac{zs}{r_n^{1/\beta_\nu}}\right)\frac{a'(T_n(r/n+s))}{a'(T_n)}\to\alpha\psi_{\beta_\nu}(zs)s^{-(\alpha+1)},$$
$$r_n\psi\left(\frac{zs}{r_n^{1/\beta_\nu}}\right)\frac{a(T_ns)}{a(T_n)}\to\psi_{\beta_\nu}(zs)s^{-\alpha}.$$

Moreover, from (5.30) and (5.32), we infer that

$$\left| r_n \psi\left(\frac{zs}{r_n^{1/\beta_\nu}}\right) \frac{a'(T_n(r/n+s))}{a'(T_n)} \right| \leq C s^{\beta_\nu-(\alpha+1)-\varepsilon},$$
$$\left| r_n \psi\left(\frac{zs}{r_n^{1/\beta_\nu}}\right) \frac{a(T_n s)}{a(T_n)} \right| \leq C s^{\beta_\nu-\alpha-\varepsilon}.$$

Therefore, (5.36) and (5.37) now follow by letting $\beta_\nu - \alpha > \varepsilon > 0$ and by applying the Dominated Convergence Theorem. □

5.6 Proof of Theorem 3.13

Proof of Theorem 3.13. We will only show that **i.** holds, since the proof of **ii.** is identical to the proof of **ii.** in Theorem 3.11. Observe that, by the strict stability of L, (5.33) and KT for all $z \in \mathbb{R}$ and $l = 2, 3, 4$

$$\mathcal{C}\left(z; \frac{\Delta_n}{T_n^{1/\beta}} S_n^{\Delta_n,l}\right) \to 0, \quad n \to \infty.$$

This implies that $\frac{\Delta_n}{T_n^{1/\beta}} S_{[tn]}^{\Delta_n} = \frac{\Delta_n}{T_n^{1/\beta}} S_{[tn]}^{\Delta_n,1} + o_{\mathbb{P}}(1)$. Therefore, as in the proof of Theorem 3.11, we only need to show that, for all $1 \geq t > u \geq 0$, as $n \to \infty$,

$$\mathcal{C}\left(z; B_n\left[S_{[nt]}^{\Delta_n,1} - S_{[nu]}^{\Delta_n,1}\right]\right) \to \mathcal{C}(z; Y_t - Y_u).$$

As before, we will only consider the case when $u = 0$. Now, from Assumption 3.7, $\int_0^\infty s^\beta a'(s)\mathrm{d}s < \infty$ and

$$\mathcal{C}\left(z; \frac{\Delta_n}{T_n^{1/\beta}} S_{[tn]}^{\Delta_n,1}\right) = \psi(z) \int_0^1 \int_0^{[nt]\Delta} \left(\frac{[nt]}{n} - \frac{s}{T_n}\right) \left([s/\Delta_n]\,\Delta_n\right)^\beta a'(r\Delta_n + s)\mathrm{d}s\mathrm{d}r$$
$$+ \psi(z) \frac{1}{T_n} \int_0^1 \sum_{j=1}^{[nt]-1} \int_{t_j}^{t_{j+1}} (s - t_j)\, t_j^\beta a'(r\Delta_n + s)\mathrm{d}s\mathrm{d}r,$$

in which

$$\left| \frac{1}{T_n} \int_0^1 \sum_{j=1}^{[nt]-1} \int_{t_j}^{t_{j+1}} (s - t_j)\, t_j^\beta a'(r\Delta_n + s)\mathrm{d}s\mathrm{d}r \right| \leq \frac{1}{n} \int_0^\infty s^\beta a'(s)\mathrm{d}s \to 0,$$

as well as for $t_j \leq s \leq t_{j+1}$ and $0 \leq r \leq 1$

$$\left| \left(\frac{[nt]}{n} - \frac{s}{T_n}\right) \left([s/\Delta_n]\,\Delta_n\right)^\beta a'(r\Delta_n + s)\mathrm{d}s\mathrm{d}r \right| \leq s^\beta a'(r\Delta_n + s).$$

Hence, the Generalized Dominated Convergence Theorem asserts that

$$\mathcal{C}\left(z; \frac{\Delta_n}{T_n^{1/\beta}} S_{[tn]}^{\Delta_n,1}\right) \to t\psi(z) \int_0^\infty s^\beta a'(s)\mathrm{d}s,$$

as required. □

5.7 Proof of Theorem 4.7

Proof of Theorem 4.7. In this proof without loss of generality we consider the case $r_n > 0$, for every $n \in \mathbb{N}$. Assume first that $\tilde{\nu}_n^{(r_n)} \overset{weak}{\to} \tilde{\nu}$ and $\kappa_n^{(r_n)} \to \kappa$ as $n \to \infty$. Thanks

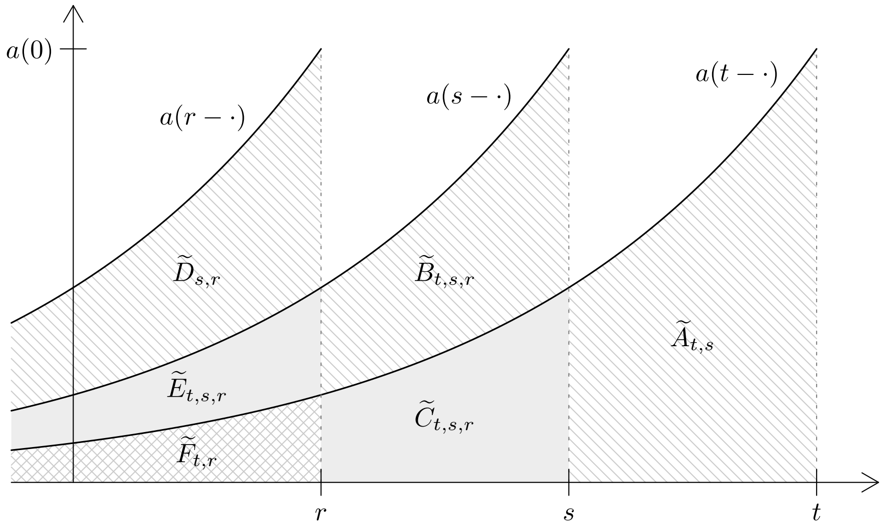

Figure 1: Separation of overlapping trawl sets into disjoint sets.

to Theorem 16.7 in [6] it is sufficient to prove the statement in the space $\mathcal{D}[0,T]$ for any finite $T>0$.

We first prove that the finite-dimensional distributions convergence. For given $t_0<t_1<\cdots<t_m$ and every $i,j=0,\dots,m$ with $j\geq i$, define

$$\mathcal{P}_A(i,j):=\{(r,s):a(t_{j+1}-s)<r\leq a(t_j-s),t_{i-1}<s\leq t_i\},$$

with the convention that $t_{-1}=-\infty$ and $t_{m+1}=\infty$. Using that for all $k=0,1,\dots,m$,

$$Leb\bigg(A_{t_k}\setminus\bigcup_{i=0}^{k}\bigcup_{j=k}^{m}\mathcal{P}_A(i,j)\bigg)=0,$$

it follows that for any $z_0,\dots,z_m\in\mathbb{R}$, it holds that

$$\begin{aligned}\log\mathbb{E}\bigg(\exp\bigg(i\sum_{k=0}^{m}z_jr_nX_{t_k}^{(n)}\bigg)\bigg)&=\sum_{i=0}^{m}\sum_{j=i}^{m}\log\mathbb{E}\bigg(\exp\bigg(i\tilde{z}_{i,j}r_nL^{(n)}(\mathcal{P}_A(i,j))\bigg)\bigg)\\&=\sum_{i=0}^{m}\sum_{j=i}^{m}Leb(\mathcal{P}_A(i,j))\mathcal{C}(\tilde{z}_{i,j};r_nL'_n),\end{aligned}$$

where $\tilde{z}_{i,j}=\sum_{k=i}^{j}z_k$. Then, by Lemma 15.15 in [17] and by the convergences $\tilde{\nu}_n^{(r_n)}\overset{weak}{\to}\tilde{\nu}$ and $\kappa_n^{(r_n)}\to\kappa$ as $n\to\infty$, we obtain that

$$\mathcal{C}(\tilde{z}_{i,j};r_nL'_n)\to\mathcal{C}(\tilde{z}_{i,j};L'_Y),\quad\text{as } n\to\infty,$$

where L'_Y is the trawl seed of the trawl process Y_t. Thus, we conclude that the finite-dimensional distributions of $r_nX^{(n)}$ converge to those of Y.

We now prove tightness. First, observe that (see Fig. 1)

$$L^n(A_t)-L^n(A_s)=L^n(\tilde{A}_{t,s})-L^n(\tilde{B}_{t,s,r}\cup\tilde{E}_{t,s,r})=L^n(\tilde{A}_{t,s})-L^n(\tilde{B}_{t,s,r})-L^n(\tilde{E}_{t,s,r}),$$

that

$$L^n(A_s)-L^n(A_r)=L^n(\tilde{B}_{t,s,r}\cup\tilde{C}_{t,s,r})-L^n(\tilde{D}_{t,s,r})=L^n(\tilde{B}_{t,s,r})+L^n(\tilde{C}_{t,s,r})-L^n(\tilde{D}_{t,s,r}),$$

and that $L^n(\tilde{A}_{t,s})$, $L^n(\tilde{B}_{t,s,r})$, $L^n(\tilde{E}_{t,s,r})$, $L^n(\tilde{C}_{t,s,r})$, $L^n(\tilde{D}_{t,s,r})$ are independent of each other. Then, we have

$$\begin{aligned}
&\mathbb{P}(|L^n(A_t) - L^n(A_s)| \wedge |L^n(A_s) - L^n(A_r)| \geq \lambda) \\
&= \mathbb{P}\Big(|L^n(\tilde{A}_{t,s}) - L^n(\tilde{B}_{t,s,r}) - L^n(\tilde{E}_{t,s,r})| \\
&\qquad \wedge |L^n(\tilde{B}_{t,s,r}) + L^n(\tilde{C}_{t,s,r}) - L^n(\tilde{D}_{t,s,r})| \geq \lambda\Big) \\
&\leq \mathbb{P}\Big(\big(|L^n(\tilde{A}_{t,s})| + |L^n(\tilde{B}_{t,s,r})| + |L^n(\tilde{E}_{t,s,r})|\big) \\
&\qquad \wedge \big(|L^n(\tilde{B}_{t,s,r})| + |L^n(\tilde{C}_{t,s,r})| + |L^n(\tilde{D}_{t,s,r})|\big) \geq \lambda\Big) \\
&= \mathbb{P}\left(|L^n(\tilde{B}_{t,s,r})| + \min\big(|L^n(\tilde{A}_{t,s})| + |L^n(\tilde{E}_{t,s,r})|, |L^n(\tilde{C}_{t,s,r})| + |L^n(\tilde{D}_{t,s,r})|\big) \geq \lambda\right) \\
&\leq \mathbb{P}\left(|L^n(\tilde{B}_{t,s,r})| \geq \frac{\lambda}{2}\right) \\
&\quad + \mathbb{P}\left(\min\big(|L^n(\tilde{A}_{t,s})| + |L^n(\tilde{E}_{t,s,r})|, |L^n(\tilde{C}_{t,s,r})| + |L^n(\tilde{D}_{t,s,r})|\big) \geq \frac{\lambda}{2}\right) \\
&= \mathbb{P}\left(|L^n(\tilde{B}_{t,s,r})| \geq \frac{\lambda}{2}\right) \\
&\quad + \mathbb{P}\left(|L^n(\tilde{A}_{t,s})| + |L^n(\tilde{E}_{t,s,r})| \geq \frac{\lambda}{2}\right)\mathbb{P}\left(|L^n(\tilde{C}_{t,s,r})| + |L^n(\tilde{D}_{t,s,r})| \geq \frac{\lambda}{2}\right) \\
&\leq \mathbb{P}\left(|L^n(\tilde{B}_{t,s,r})| \geq \frac{\lambda}{2}\right) + \left[\mathbb{P}\left(|L^n(\tilde{A}_{t,s})| \geq \frac{\lambda}{4}\right) + \mathbb{P}\left(|L^n(\tilde{E}_{t,s,r})| \geq \frac{\lambda}{4}\right)\right] \\
&\quad \times \left[\mathbb{P}\left(|L^n(\tilde{C}_{t,s,r})| \geq \frac{\lambda}{4}\right) + \mathbb{P}\left(|L^n(\tilde{D}_{t,s,r})| \geq \frac{\lambda}{4}\right)\right] \qquad (5.39)
\end{aligned}$$

where we used that given two independent real valued random variables Z and V we have that $\mathbb{P}(\min(Z,V) \geq \lambda) = \mathbb{P}(Z \geq \lambda)\mathbb{P}(V \geq \lambda)$.

Now, consider the following inequality which holds for any real valued random variable ξ (see eq. (3.34) in [28] and see also [17])

$$\mathbb{P}(|\xi| > \lambda) \leq \lambda \int_{-2/\lambda}^{2/\lambda} (1 - \mathbb{E}[\exp(i\theta\xi)])\mathrm{d}\theta.$$

Moreover, consider that

$$e^b - e^{ia+b} = e^b(1 - e^{2i\frac{a}{2}}) = -e^b e^{i\frac{a}{2}}(e^{i\frac{a}{2}} - e^{-i\frac{a}{2}}) = -2ie^b e^{i\frac{a}{2}} \sin\left(\frac{a}{2}\right),$$

from which we get that $|e^b - e^{ia+b}| = 2e^b|\sin\left(\frac{a}{2}\right)|$, and consider that $|1 - e^{ia+b}| \leq |1 - e^b| + |e^b - e^{ia+b}|$, for every $a, b \in \mathbb{R}$. Observe also that $2|\sin(x/2)| \leq 10(1 - e^{-|x|})$, and that $1 - e^{-x} \leq x$ (and so that $1 - e^{-|x|} \leq |x|$), for $x \in \mathbb{R}$.

Let us use these inequalities for $\mathbb{P}\left(r_n|L^n(\tilde{B}_{t,s,r})| \geq \frac{\lambda}{2}\right)$, that is let $\phi(\theta)$ be the characteristic exponent of $r_n L^n(\tilde{B}_{t,s,r})$ then

$$\begin{aligned}
&\mathbb{P}\left(r_n|L^n(\tilde{B}_{t,s,r})| \geq \frac{\lambda}{2}\right) \\
&\leq \frac{\lambda}{2}\int_{-4/\lambda}^{4/\lambda}(1 - e^{\phi(\theta)})\mathrm{d}\theta = \frac{\lambda}{2}\int_{-4/\lambda}^{4/\lambda} 1 - e^{\mathrm{Re}(\phi(\theta)) + i\mathrm{Im}(\phi(\theta))}\mathrm{d}\theta \\
&\leq \frac{\lambda}{2}\int_{-4/\lambda}^{4/\lambda}(|1 - e^{\mathrm{Re}(\phi(\theta))}| + |e^{\mathrm{Re}(\phi(\theta))} - e^{\mathrm{Re}(\phi(\theta)) + i\mathrm{Im}(\phi(\theta))}|)\mathrm{d}\theta \qquad (5.40) \\
&\leq 5\lambda \int_{-4/\lambda}^{4/\lambda} \mathrm{Leb}(\tilde{B}_{t,s,r})\left(|\theta\gamma_n^{(r_n)}| + \frac{1}{2}\theta^2(b_n^{(r_n)})^2 + \int_{\mathbb{R}}\left(1 \wedge x^2\right)\nu_n^{(r_n)}(\mathrm{d}x)\right)\mathrm{d}\theta.
\end{aligned}$$

where for the first addend in (5.40) we used that

$$\mathrm{Re}(\phi(\theta)) = \mathrm{Leb}(\tilde{B}_{t,s,r}) \left(-\frac{1}{2}\theta^2 (b_n^{(r_n)})^2 + \int_{\mathbb{R}} (\cos(\theta x) - 1)\nu_n^{(r_n)}(\mathrm{d}x) \right) \leq 0,$$

that

$$\left| \mathrm{Re}\Big(\int_{\mathbb{R}} (e^{izx} - 1 - izx\mathbf{1}_{|x|\leq 1})\nu(\mathrm{d}x) \Big) \right| \leq \int_{\mathbb{R}} \Big(1 \wedge x^2\Big) \nu_n^{(r_n)}(\mathrm{d}x),$$

(and similarly for the imaginary part) and so that

$$\begin{aligned} |1 - e^{\mathrm{Re}(\phi(\theta))}| = 1 - e^{\mathrm{Re}(\phi(\theta))} &\leq -\mathrm{Re}(\phi(\theta)) \\ &\leq \mathrm{Leb}(\tilde{B}_{t,s,r}) \left(\frac{1}{2}\theta^2 (b_n^{(r_n)})^2 + \int_{\mathbb{R}} \Big(1 \wedge x^2\Big) \nu_n^{(r_n)}(\mathrm{d}x) \right) \end{aligned}$$

and for the second addend in (5.40) we used that $e^{\mathrm{Re}(\phi(\theta))} \leq 1$ and so that

$$\begin{aligned} |e^{\mathrm{Re}(\phi(\theta))} - e^{\mathrm{Re}(\phi(\theta)) + i\mathrm{Im}(\phi(\theta))}| &\leq 10|\mathrm{Im}(\phi(\theta))| \\ &\leq 10\mathrm{Leb}(\tilde{B}_{t,s,r}) \left(|\theta\gamma_n^{(r_n)}| + \int_{\mathbb{R}} \Big(1 \wedge x^2\Big) \nu_n^{(r_n)}(\mathrm{d}x) \right). \end{aligned}$$

By the previously shown convergence $r_n L_n' \stackrel{d}{\to} L_Y'$ as $n \to \infty$ and by Theorem 8.7 in [24] we have that $\int_{\mathbb{R}} \Big(1 \wedge x^2\Big) \nu_n^{(r_n)}(\mathrm{d}x) \to \int_{\mathbb{R}} \Big(1 \wedge x^2\Big) \nu(\mathrm{d}x)$, $\gamma_n^{(r_n)} \to \gamma$, and

$$\lim_{\varepsilon \downarrow 0} \limsup_{n\to\infty} |(b_n^{(r_n)})^2 + \int_{|x|<\varepsilon} x^2 \nu_n^{(r_n)}(\mathrm{d}x) - b^2| = 0.$$

Hence, we have that $|\gamma_n^{(r_n)}|$, $(b_n^{(r_n)})^2$, and $\int_{\mathbb{R}} \left(1 \wedge x^2\right) \nu_n^{(r_n)}(dx)$ are uniformly bounded and thus we have

$$\begin{aligned} \mathbb{P}\left(r_n |L^n(\tilde{B}_{t,s,r})| \geq \frac{\lambda}{2} \right) &\leq \mathrm{Leb}(\tilde{B}_{t,s,r}) 5\lambda \int_{-4/\lambda}^{4/\lambda} \left(C_0|\theta| + C_1\theta^2 + C_2 \right) \mathrm{d}\theta \\ &= \mathrm{Leb}(\tilde{B}_{t,s,r}) 10 \Big(\frac{8C_0}{\lambda} + \frac{64C_1}{3\lambda^2} + 4C_2 \Big). \end{aligned}$$

To prove tightness we use Theorem 13.5 in [6] which requires to obtain the bound: for $r \leq s \leq t$, $n \geq 1$ and $\lambda > 0$,

$$\mathbb{P}\left(r_n \left(|L^n(A_t) - L^n(A_s)| \wedge |L^n(A_s) - L^n(A_r)| \right) \geq \lambda \right) \leq \frac{C}{\lambda^\beta}(t-r)^{1+\varepsilon},$$

where $C > 0$, $\beta \geq 0$ and $\varepsilon > 0$. It is possible to see by its proof and by Theorems 13.2 and 13.2 in [6] (see also eq. (12.31) and (12.32) therein) that it is sufficient to take $\lambda \in (0, 1)$. Thus, in the following we consider only the case of $\lambda \in (0, 1)$. Then, we have that

$$\mathrm{Leb}(\tilde{B}_{t,s,r}) 10 \Big(\frac{8C_0}{\lambda} + \frac{64C_1}{3\lambda^2} + 4C_2 \Big) \leq \mathrm{Leb}(\tilde{B}_{t,s,r}) \frac{C_3}{\lambda^2}.$$

From Assumption 4.1 it is possible to see that $\mathrm{Leb}(\tilde{B}_{t,s,r})$ is bounded by $(t-r)^{1+\varepsilon}$ for some $\varepsilon > 0$. Thus, we have

$$\mathbb{P}\left(r_n |L^n(\tilde{B}_{t,s,r})| \geq \frac{\lambda}{2} \right) \leq \frac{C_3}{\lambda^2}(t-r)^{1+\varepsilon}.$$

For the other summands of (5.39) we have that

$$\begin{aligned} \mathbb{P}\left(r_n |L^n(\tilde{A}_{t,s})| \geq \frac{\lambda}{4} \right) &\leq \mathrm{Leb}(\tilde{A}_{t,s}) \frac{5\lambda}{2} \int_{-8/\lambda}^{8/\lambda} \left(C_0|\theta| + C_1\theta^2 + C_2 \right) \mathrm{d}\theta \\ &= \mathrm{Leb}(\tilde{A}_{t,s}) 10 \big(\frac{16C_0}{\lambda} + \frac{256C_1}{3\lambda^2} + 4C_2 \big) \leq \frac{C_4}{\lambda^2}(t-s)^{\frac{1}{2}+\frac{\varepsilon}{2}} \end{aligned}$$

and similarly for the probabilities $\mathbb{P}\left(r_n|L^n(\tilde{E}_{t,s,r})| \geq \frac{\lambda}{4}\right)$, $\mathbb{P}\left(r_n|L^n(\tilde{C}_{t,s,r})| \geq \frac{\lambda}{4}\right)$, and $\mathbb{P}\left(r_n|L^n(\tilde{D}_{t,s,r})| \geq \frac{\lambda}{4}\right)$. Then, we have that

$$\mathbb{P}\left(r_n|L^n(\tilde{A}_{t,s})| \geq \frac{\lambda}{4}\right)\mathbb{P}\left(r_n|L^n(\tilde{C}_{t,s,r})| \geq \frac{\lambda}{4}\right) \\ \leq \frac{C_4^2}{\lambda^4}(t-s)^{\frac{1}{2}+\frac{\varepsilon}{2}}(s-r)^{\frac{1}{2}+\frac{\varepsilon}{2}} \leq \frac{C_4^2}{\lambda^4}(t-r)^{\frac{1}{2}+\frac{\varepsilon}{2}}(t-r)^{\frac{1}{2}+\frac{\varepsilon}{2}} = \frac{C_4^2}{\lambda^4}(t-r)^{1+\varepsilon}.$$

Since $\lambda < 1$ then we obtain that

$$\mathbb{P}(r_n(|L^n(A_t) - L^n(A_s)| \wedge |L^n(A_s) - L^n(A_r)|) \geq \lambda) \leq \frac{C_5}{\lambda^4}(t-r)^{1+\varepsilon}.$$

Thus, we obtain the desired bound. Since for any $t \in \mathbb{R}$ we have that $Y_t - Y_s \Rightarrow 0$ as $s \to t$, also condition (13.12) in Theorem 13.5 in [6] is satisfied and so we obtain the stated convergence in distribution.

Assume now that $\{r_n X_t^{(n)}\}_{t\in[0,T]} \overset{\mathcal{D}[0,T]}{\Longrightarrow} \{Y_t\}_{t\in[0,T]}$, as $n \to \infty$. Since coordinate projections are continuous functionals of the sample path, by the continuous mapping theorem we obtain that $\{r_n X_t^{(n)}\}_{t\in[0,T]} \overset{fd}{\to} \{Y_t\}_{t\in[0,T]}$, as $n \to \infty$. This implies that

$$\mathcal{C}(z; r_n L_n') \to \mathcal{C}(z; L_Y'), \quad \text{as } n \to \infty,$$

for every $z \in \mathbb{R}$, which by Lemma 15.15 in [17] implies that $\tilde{\nu}_n^{(r_n)} \overset{weak}{\to} \tilde{\nu}$ and $\kappa_n^{(r_n)} \to \kappa$ as $n \to \infty$. □

5.8 Proof of Theorem 4.8

Proof of Theorem 4.8. From the same arguments as the ones used in the proof of Theorem 4.7 we obtain that

$$\log \mathbb{E}\left(\exp\left(i\sum_{k=0}^{m} z_j r_n X_{t_k}^{(n)}\right)\right) \to \frac{1}{2}\sum_{j,l=0}^{m} z_j z_l Leb(A_{t_j} \cap A_{t_l}), \qquad \text{as } n \to \infty.$$

Since $Leb(A_{t_j} \cap A_{t_l}) = Leb(A_{\max(t_j,t_l)-\min(t_j,t_l)} \cap A_0) = \int_0^\infty a(\max(t_j,t_l) - \min(t_j,t_l) + s)\mathrm{d}s$, we need to show that $\int_0^\infty a(\max(t_j,t_l) - \min(t_j,t_l) + s)\mathrm{d}s = \int_{-\infty}^{\min(t_j,t_l)} g(t_j - s)g(t_l - s)\mathrm{d}s$, for every $j, l = 0, \ldots, m$. Therefore, in general terms, we have the following condition to satisfy

$$\int_0^\infty a(h+s)ds = \int_{-\infty}^0 g(h-s)g(-s)\mathrm{d}s.$$

Notice that

$$\int_0^\infty a(h+s)ds = \int_h^\infty a(s)\mathrm{d}s \quad \text{and} \quad \int_{-\infty}^0 g(h-s)g(-s)\mathrm{d}s = \int_0^\infty g(h+s)g(s)\mathrm{d}s,$$

so by the fundamental theorem of calculus we need to satisfy

$$-a(h) = \frac{\mathrm{d}}{\mathrm{d}h}\int_h^\infty a(s)\mathrm{d}s = \frac{\mathrm{d}}{\mathrm{d}h}\int_0^\infty g(h+s)g(s)\mathrm{d}s,$$

which is indeed satisfied by assumption. Thus, we obtain the finite-dimensional distributions convergence. The tightness and the necessity of the conditions $\tilde{\nu}_n^{(r_n)} \overset{weak}{\to} \delta_0$ and $\kappa_n^{(r_n)} \to 0$ as $n \to \infty$ is proved using the same arguments as the ones used in the proof of Theorem 4.7. □

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