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Fluctuation around the circular law for random matrices with real entries

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Abstract

We extend our recent result [22] on the central limit theorem for the linear eigenvalue statistics of non-Hermitian matrices X with independent, identically distributed *complex* entries to the *real* symmetry class. We find that the expectation and variance substantially differ from their complex counterparts, reflecting (i) the special spectral symmetry of real matrices onto the real axis; and (ii) the fact that real i.i.d. matrices have many real eigenvalues. Our result generalizes the previously known special cases where either the test function is analytic [49] or the first four moments of the matrix elements match the real Gaussian [59, 44]. The key element of the proof is the analysis of several weakly dependent Dyson Brownian motions (DBMs). The conceptual novelty of the real case compared with [22] is that the correlation structure of the stochastic differentials in each individual DBM is non-trivial, potentially even jeopardising its well-posedness.

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1 Introduction

We consider an ensemble of $n \times n$ random matrices X with *real* i.i.d. entries of zero mean and variance $1/n$; the corresponding model with *complex* entries has been studied in [22]. According to the *circular law* [6, 58, 38] (see also [11]), the density of the

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eigenvalues $\{\sigma_i\}_{i=1}^n$ of X converges to the uniform distribution on the unit disk. Our main result is that the fluctuation of their linear statistics is Gaussian, i.e.

$$L_n(f) := \sum_{i=1}^n f(\sigma_i) - \mathbf{E}\sum_{i=1}^n f(\sigma_i) \sim \mathcal{N}(0, V_f) \tag{1.1}$$

converges, as $n \to \infty$, to a centred normal distribution for regular test functions f with at least $2+\delta$ derivatives. We compute the variance V_f and the next-order deviation of the expectation $\mathbf{E}\sum_{i=1}^n f(\sigma_i)$ from the value $\frac{n}{\pi}\int_{|z|\le 1} f(z)$ given by the circular law. As in the complex case, both quantities depend on the fourth cumulant of the single entry distribution of X, but in the real case they also incorporate the spectral symmetry of X onto the real axis. Moreover, the expectation carries additional terms, some of them are concentrated around the real axis; a by-product of the approximately $\sqrt{n}$ real eigenvalues of X. For the Ginibre (Gaussian) case they may be computed from the explicit density [27, 26], but for general distributions they were not known before. As expected, the spectral symmetry essentially enhances V_f by a factor of two compared with the complex case but this effect is modified by an additional term involving the fourth cumulant. Previous works considered either the case of analytic test functions f [48, 49] or the (approximately) Gaussian case, i.e. when X is the real Ginibre ensemble or at least the first four moments of the matrix elements of X match the Ginibre ensemble [59, 44]. In both cases some terms in the unified formulas for the expectation and the variance vanish and thus the combined effect of the spectral symmetry, the eigenvalues on the real axis, and the role of the fourth cumulant was not detectable in these works. We remark that a CLT for polynomial statistics of only the real eigenvalues for real Ginibre matrices was proven in [56].

In [53] the limiting random field $L(f) := \lim_{n\to\infty} L_n(f)$ for complex Ginibre matrices has been identified as a projection of the *Gaussian free field (GFF)* [55]. We extended this interpretation [22] to general complex i.i.d. matrices with non-negative fourth cumulant and obtained a rank-one perturbation of the projected GFF. As a consequence of the CLT in the present paper, we find that in the real case the limiting random field is a version of the same GFF, symmetrised with respect to the real axis, reflecting the fact that complex eigenvalues of real matrices come in pairs of complex conjugates.

In general, proving CLTs for the real symmetry class is considerably harder than for the complex one. The techniques based upon the first four moment matching [59, 44] are insensitive to the symmetry class, hence these results are obtained in parallel for both real and complex ensembles. Beyond this method, however, most results on CLT for non-Hermitian matrices were restricted to the complex case [25, 33, 47, 51, 52, 54], see the introduction of [22] for a detailed history, as well as for references to the analogous CLT problem for Hermitian ensembles and log-gases. The special role that the real axis plays in the spectrum of the real case substantially complicates even the explicit formulas for the Ginibre ensemble both for the density [26] as well as for the k-point correlation functions [37, 12, 42]. Besides the complexity of the explicit formulas, there are several conceptual reasons why the real case is more involved. We now explain them since they directly motivated the new ideas in this paper compared with [22].

In [22] we started with Girko's formula [38] in the form given in [59] that relates the eigenvalues of X with resolvents of a family of $2n\times 2n$ Hermitian matrices

$$H^z := \begin{pmatrix} 0 & X-z \\ X^*-\overline{z} & 0 \end{pmatrix} \tag{1.2}$$

parametrized by $z \in \mathbf{C}$. For any smooth, compactly supported test function f we have

$$\sum_{i=1}^n f(\sigma_i) = -\frac{1}{4\pi}\int_{\mathbf{C}} \Delta f(z) \int_0^\infty \Im \operatorname{Tr} G^z(i\eta)\,\mathrm{d}\eta\,\mathrm{d}^2 z, \tag{1.3}$$

where $G^z(w) := (H^z - w)^{-1}$ is the resolvent of H^z. We therefore needed to understand the resolvent $G^z(i\eta)$ along the imaginary axis on all scales $\eta \in (0, \infty)$.

The main contribution to (1.3) comes from the $\eta \sim 1$ *macroscopic* regime, which is handled by proving a multi-dimensional CLT for resolvents with several z and η parameters and computing their expectation and covariance by cumulant expansion. The local laws along the imaginary axis from [2, 3] serve as a basic input (in the current work, however, we need to extend them for spectral parameters w away from the imaginary axis). The core of the argument in the real case is similar to the complex case in [22], however several additional terms have to be computed due to the difference between the real and complex cumulants. By explicit calculations, these additional terms break the rotational symmetry in the z parameter and, unlike in the complex case, the answer is not a function of $|z|$ any more. The *mesoscopic* regime $n^{-1} \ll \eta \ll 1$ is treated together with the macroscopic one; the fact that only the $\eta \sim 1$ regime contributes to (1.3) is revealed *a posteriori* after these calculations.

The scale $\eta \lesssim n^{-1}$ in (1.3) requires a very different treatment since local laws are not applicable any more and individual eigenvalues $0 \le \lambda_1^z \le \lambda_2^z \ldots$ of H^z near zero substantially influence the fluctuation of $G^z(i\eta)$ (since H^z has a symmetric spectrum, we consider only positive eigenvalues). The main insight of [22] was that it is sufficient to establish that the small eigenvalues, say, λ_1^z and $\lambda_1^{z'}$, are asymptotically independent if z and z' are relatively far away, say $|z - z'| \ge n^{-1/100}$. This was achieved by exploiting the fast local equilibration mechanism of the *Dyson Brownian motion (DBM)*, which is the stochastic flow of eigenvalues $\boldsymbol{\lambda}^z(t) := \{\lambda_i^z(t)\}$ generated by adding a time-dependent Gaussian (Ginibre) component. The initial condition of this flow was chosen carefully to almost reproduce X after a properly tuned short time. We needed to follow the evolution of $\boldsymbol{\lambda}^z(t)$ for different z parameters simultaneously. These flows are correlated since they are driven by the same random source. We thus needed to study a family of DBMs, parametrized by z, with correlated driving Brownian motions. The correlation structure is given by the *overlap* of the eigenfunctions of H^z and $H^{z'}$. We could show that this overlap is small, hence the Brownian motions are essentially independent, if z and z' are far away. This step required to develop a new type of local law for *products* of resolvent, e.g. for $\operatorname{Tr} G^z(\mathrm{i}\eta)G^{z'}(\mathrm{i}\eta')$ with $\eta, \eta' \sim n^{-1+\epsilon}$. Finally, we trailed the joint evolution of $\boldsymbol{\lambda}^z(t)$ and $\boldsymbol{\lambda}^{z'}(t)$ by their independent Ginibre counterparts, showing that they themselves are asymptotically independent.

We follow the same strategy in the current paper for the real case, but we immediately face with the basic question: how do the low lying eigenvalues of H^z, equivalently the small singular values of $X - z$, behave? We do not need to compute their joint distribution, but we need to approximate them with an appropriate Ginibre ensemble. For *complex* X in [22] the approximating Ginibre ensemble was naturally complex. For *real* X there seem to be two possibilities. The key insight of our current analysis is that the small singular values of $X - z$ behave as those of a *complex* Ginibre matrix even though X is *real*, as long as z is genuinely complex (Theorem 2.8). In particular, we prove that the least singular value of $X - z$ belongs to the complex universality class. Moreover, we prove that the small singular values of $X - z_1$ and the ones of $X - z_2$ are asymptotically independent as long as z_1 and z_2 are far from each other.

To explain the origin of this apparent mismatch, we will derive the DBM

$$\mathrm{d}\lambda_i^z = \frac{\mathrm{d}b_i^z}{\sqrt{n}} + \frac{1}{2n} \sum_{j \ne i} \frac{1 + \Lambda_{ij}^z}{\lambda_i^z - \lambda_j^z} \, \mathrm{d}t + \ldots \tag{1.4}$$

for $\boldsymbol{\lambda}^z(t)$, ignoring some additional terms with negative indices coming from the spectral symmetry of H^z (see (7.14) and (B.15) for the precise equation). The correlations of the driving Brownian motions are given by

$$\mathbf{E}\,\mathrm{d}b_i^z\,\mathrm{d}b_j^{z'} = \frac{1}{2}\big[\Theta_{ij}^{z,z'} + \Theta_{ij}^{z,\overline{z}'}\big]\,\mathrm{d}t \tag{1.5}$$

with overlaps Θ, Λ defined as

$$\Theta_{ij}^{z,z'} := 4\Re\big[\langle \boldsymbol{u}_j^{z'}, \boldsymbol{u}_i^z\rangle\langle \boldsymbol{v}_i^z, \boldsymbol{v}_j^{z'}\rangle\big], \qquad \Lambda_{ij}^z := \Theta_{ij}^{z,\overline{z}}, \tag{1.6}$$

where $(\boldsymbol{u}_i^z, \boldsymbol{v}_i^z) \in \mathbf{C}^{2n}$ is the (normalized) eigenvector of H^z corresponding to the eigenvalue λ_i^z. Note that $\Theta_{ij}^{z,z} = \delta_{i,j}$, and for $j \neq i$ we have that $\Lambda_{ij}^z \approx 0$. Moreover, if z is very close to the real axis, then the eigenvectors of H^z are essentially real and $\Lambda_{ii}^z = \Theta_{ii}^{z,\overline{z}} \approx \Theta_{ii}^{z,z} = 1$. With $z = z'$, this leads to (1.4) being essentially a *real* DBM with $\beta = 1$. (We recall that the parameter $\beta = 1, 2$, customarily indicating the real or complex symmetry class of a random matrix, also expresses the ratio of the coefficient of the repulsion to the strength of the diffusion in the DBM setup.) However, if z and $\overline{z}$ are far away, i.e. z is away from the real axis, then we can show that the overlap $\Lambda^z = \Theta^{z,\overline{z}}$ is small, hence $\Lambda_{ij}^z \approx 0$ for all i, j, including $i = j$. Thus the variance of the driving Brownian motions in (1.5) with $z = z'$ is reduced by a factor of two, rendering (1.4) essentially a *complex* DBM with $\beta = 2$.

The appearance of Λ^z in (1.4) and the second term $\Theta^{z,\overline{z}'}$ in (1.5) is specific to the real symmetry class; they were not present in the complex case [22]. They have three main effects for our analysis. First, they change the symmetry class of the DBM (1.4) as we just explained. Second, due to the symmetry relation $\lambda_{-1}^z = -\lambda_1^z$ and $b_{-1}^z = -b_1^z$, the strength of the level repulsion between λ_1^z and λ_{-1}^z in (1.4) is already critically small even for $\Lambda^z = 0$, see e.g. [18, Appendix A], hence the well-posedness of (1.4) does not follow from standard results on DBM. Third, $\Theta^{z,\overline{z}}$ renders the driving Brownian motions $\boldsymbol{b}^z = \{b_i^z\}$ correlated for different indices i even for the *same* z, since Λ_{ij}^z in general is nonzero. In fact, the vector $\boldsymbol{b}^z$ is even not Gaussian, hence strictly speaking it is only a multidimensional martingale but not a Brownian motion in general. In contrast, $\Theta_{ij}^{z,z} = \delta_{i,j}$ and only the overlaps $\Theta_{ij}^{z,z'}$ for *different* $z \neq z'$ are nontrivial. Thus in the complex case [22], lacking the term $\Theta^{z,\overline{z}}$ in (1.5), the DBM (1.4) for any fixed z was the conventional DBM with independent Brownian motions and parameter $\beta = 2$ (c.f. [22, Eq. (7.15)]) and only the DBMs for *different* z's were mildly correlated. In the real case the correlations are already present within (1.4) for the *same* z due to $\Lambda^z = \Theta^{z,\overline{z}} \neq 0$.

We note that Dyson Brownian motions with nontrivial coefficients in the repulsion term have already been investigated in [17] (see also [19]) in the context of spectral universality of addition of random matrices twisted by Haar unitaries, however the driving Brownian motions were independent. The issue of well-posedness, nevertheless, has already emerged in [17] when the more critical orthogonal group ($\beta = 1$) was considered. The corresponding part of our analysis partly relies on techniques developed in [17]. We have already treated the dependence of Brownian motions for different z's in [22] for the complex case; but the more general dependence structure characteristic to the real case is a new challenge that the current work resolves.

Notations and conventions

We introduce some notations we use throughout the paper. For integers $k \in \mathbf{N}$ we use $[k] := \{1, \dots, k\}$. We write $\mathbf{H}$ for the upper half-plane $\mathbf{H} := \{z \in \mathbf{C} \mid \Im z > 0\}$, $\mathbf{D} \subset \mathbf{C}$ for the open unit disk, and we use the notation $\mathrm{d}^2 z := 2^{-1}\mathrm{i}(\mathrm{d}z \wedge \mathrm{d}\overline{z})$ for the two dimensional volume form on $\mathbf{C}$. For positive quantities f, g we write $f \lesssim g$ and $f \sim g$ if $f \le Cg$ and $cg \le f \le Cg$, respectively, for some constants $c, C > 0$ which depend only on the *model parameters* appearing in (2.1). For any two positive real numbers $\omega_*, \omega^* \in \mathbf{R}_+$, by $\omega_* \ll \omega^*$ we denote that $\omega_* \le c\omega^*$ for some sufficiently small constant $0 < c \le 1/1000$. We denote vectors by bold-faced lower case Roman letters $\boldsymbol{x}, \boldsymbol{y}, \ldots, \in \mathbf{C}^k$, for some $k \in \mathbf{N}$, and use the notation $\mathrm{d}\boldsymbol{x} := \mathrm{d}x_1 \ldots \mathrm{d}x_k$. Vector and matrix norms, $\|\boldsymbol{x}\|$ and $\|A\|$, indicate the usual Euclidean norm and the corresponding induced matrix norm. For any $k \times k$ matrix A we set $\langle A \rangle := k^{-1} \operatorname{Tr} A$ to denote the normalized trace of A. Moreover, for vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{C}^k$ and matrices $A, B \in \mathbf{C}^{k \times k}$ we define

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle := \sum \overline{x_i} y_i, \qquad \langle A, B \rangle := \langle A^* B \rangle = \frac{1}{k} \operatorname{Tr} A^* B.$$

We will use the concept of "event with very high probability" meaning that for any fixed $D > 0$ the probability of the event is bigger than $1 - n^{-D}$ if $n \geq n_0(D)$. Moreover, we use the convention that $\xi > 0$ denotes an arbitrary small exponent which is independent of n.

2 Main results

We consider *real i.i.d. matrices* X, i.e. $n \times n$ matrices whose entries are independent and identically distributed as $x_{ab} \stackrel{\mathrm{d}}{=} n^{-1/2}\chi$ for some real random variable χ, satisfying the following:

Assumption 2.1. *We assume that* $\mathbf{E}\,\chi = 0$ *and* $\mathbf{E}\,\chi^2 = 1$. *In addition we assume the existence of high moments, i.e. that there exist constants* $C_p > 0$, *for any* $p \in \mathbf{N}$, *such that*

$$\mathbf{E}|\chi|^p \leq C_p. \tag{2.1}$$

The *circular law* [6, 7, 11, 39, 10, 36, 38, 50, 58] asserts that the empirical distribution of eigenvalues $\{\sigma_i\}_{i=1}^n$ of a complex i.i.d. matrix X converges to the uniform distribution on the unit disk $\mathbf{D}$, i.e.

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^n f(\sigma_i) = \frac{1}{\pi} \int_{\mathbf{D}} f(z)\, \mathrm{d}^2 z, \tag{2.2}$$

with very high probability for any continuous bounded function f. Our main result is a central limit theorem for the centred *linear statistics*

$$L_n(f) := \sum_{i=1}^n f(\sigma_i) - \mathbf{E} \sum_{i=1}^n f(\sigma_i) \tag{2.3}$$

for general real i.i.d. matrices and generic test functions f, complementing the recent central limit theorem [22] for the linear statistics of *complex i.i.d. matrices*. This CLT, formulated in Theorem 2.2, and its proof have two corollaries of independent interest that are formulated in Section 2.1 and Section 2.2.

In order to state the result we introduce some notations. For any function h defined on the boundary of the unit disk $\partial\mathbf{D}$ we define its Fourier transform as

$$\widehat{h}(k) = \frac{1}{2\pi} \int_0^{2\pi} h(e^{\mathrm{i}\theta}) e^{-\mathrm{i}\theta k}\, \mathrm{d}\theta, \qquad k \in \mathbf{Z}. \tag{2.4}$$

For $f, g \in H^{2+\delta}(\Omega)$ for some domain $\Omega \supset \overline{\mathbf{D}}$ we define

$$\begin{aligned} \langle g, f \rangle_{\dot{H}^{1/2}(\partial\mathbf{D})} &:= \sum_{k\in\mathbf{Z}} |k| \overline{\widehat{g}(k)} \widehat{f}(k), \qquad \|f\|^2_{\dot{H}^{1/2}(\partial\mathbf{D})} := \langle f, f \rangle_{\dot{H}^{1/2}(\partial\mathbf{D})}, \\ \langle g, f \rangle_{H_0^1(\mathbf{D})} &:= \langle \nabla g, \nabla f \rangle_{L^2(\mathbf{D})}, \qquad \|f\|^2_{H_0^1(\mathbf{D})} := \langle f, f \rangle_{H_0^1(\mathbf{D})}, \end{aligned} \tag{2.5}$$

where, in a slight abuse of notation, we identified f and g with their restrictions to $\partial\mathbf{D}$. We use the convention that f is extended to $\mathbf{C}$ by setting it equal to zero on Ω^c. Finally, we introduce the projection

$$(P_{\mathrm{sym}} f)(z) := \frac{f(z) + f(\overline{z})}{2}. \tag{2.6}$$

which maps functions on the complex plane to their symmetrisation with respect to the real axis.

Theorem 2.2 (Central Limit Theorem for linear statistics)**.** *Let X be a real $n\times n$ i.i.d. matrix satisfying Assumption 2.1 with eigenvalues $\{\sigma_i\}_{i=1}^n$, and denote the fourth* cumulant[1] *of χ by $\kappa_4 := \mathbf{E}\,\chi^4 - 3$. Fix $\delta>0$, let $\Omega\subset\mathbf{C}$ be open and such that $\overline{\mathbf{D}}\subset\Omega$. Then, for complex-valued test functions $f\in H^{2+\delta}(\Omega)$, the centred linear statistics $L_n(f)$, defined in* (2.3)*, converge*

$$L_n(f)\Longrightarrow L(f),$$

to complex Gaussian random variables $L(f)$ with expectation $\mathbf{E}\,L(f)=0$ and variance $\mathbf{E}|L(f)|^2 = C(f,f) =: V_f$ and $\mathbf{E}\,L(f)^2 = C(\overline{f},f)$, where

$$\begin{aligned}C(g,f) &:= \frac{1}{2\pi}\langle \nabla P_{\mathrm{sym}} g, \nabla P_{\mathrm{sym}} f\rangle_{L^2(\mathbf{D})} + \langle P_{\mathrm{sym}} g, P_{\mathrm{sym}} f\rangle_{\dot H^{1/2}(\partial\mathbf{D})}\\ &\quad + \kappa_4\left(\frac{1}{\pi}\int_{\mathbf{D}}\overline{g(z)}\,\mathrm{d}^2 z - \frac{1}{2\pi}\int_0^{2\pi}\overline{g(e^{\mathrm{i}\theta})}\,\mathrm{d}\theta\right)\left(\frac{1}{\pi}\int_{\mathbf{D}} f(z)\,\mathrm{d}^2 z - \frac{1}{2\pi}\int_0^{2\pi} f(e^{\mathrm{i}\theta})\,\mathrm{d}\theta\right).\end{aligned} \tag{2.7}$$

For the k-th moments we have an effective convergence rate of

$$\mathbf{E}\,L_n(f)^k\overline{L_n(f)}^l = \mathbf{E}\,L(f)^k\overline{L(f)}^l + \mathcal{O}\Big(n^{-c(k+l)}\Big)$$

for some constant $c(k+l)>0$. Moreover, the expectation in (2.3) *is given by*

$$\begin{aligned}\mathbf{E}\sum_{i=1}^n f(\sigma_i) &= E(f) + \mathcal{O}\big(n^{-c}\big)\\ E(f) &:= \frac{n}{\pi}\int_{\mathbf{D}} f(z)\,\mathrm{d}^2 z + \frac{1}{4\pi}\int_{\mathbf{D}}\frac{f(\Re z)-f(z)}{(\Im z)^2}\,\mathrm{d}^2 z - \frac{\kappa_4}{\pi}\int_{\mathbf{D}} f(z)(2|z|^2-1)\,\mathrm{d}^2 z \\ &\quad - \frac{1}{2\pi}\int_0^{2\pi} f(e^{\mathrm{i}\theta})\,\mathrm{d}\theta + \frac{1}{2\pi}\int_{-1}^{1}\frac{f(x)}{\sqrt{1-x^2}}\,\mathrm{d}x + \frac{f(1)+f(-1)}{4}\end{aligned} \tag{2.8}$$

for some small constant $c>0$.

Remark 2.3.

(i) Both expectation $E(f)$ and covariance $C(g,f)$ only depend on the symmetrised functions $P_{\mathrm{sym}}f$ and $P_{\mathrm{sym}}g$. Indeed, $E(f)=E(P_{\mathrm{sym}}f)$, and the coefficient of κ_4 in (2.7) can also be written as an integral over $P_{\mathrm{sym}}f$ and $P_{\mathrm{sym}}g$.

(ii) By polarisation, a multivariate central limit theorem as in [22, Corollary 2.4] follows immediately and any mixed k-th moments have an effective convergence rate of order $n^{-c(k)}$.

(iii) The variance $V_f = \mathbf{E}|L(f)|^2$ in Theorem 2.2 is strictly positive whenever f is not constant on the unit disk (see [22, Remark 2.3]).

Remark 2.4 (Comparison with [44] and [49])**.**

(i) The central limit theorem [44, Theorem 2] is a special case of Theorem 2.2. Indeed, [44, Theorem 2] implies that for real i.i.d. matrices with entries matching the real Ginibre ensemble to the fourth moment, and real-valued smooth test functions

[1] Note that in the real case the fourth cumulant is given by $\kappa_4 = \kappa(\chi,\chi,\chi,\chi) = \mathbf{E}\,\chi^4 - 3$, while in the complex case [22] the relevant fourth cumulant was given by $\kappa(\chi,\chi,\overline{\chi},\overline{\chi}) = \mathbf{E}|\chi|^4 - 2$.

f compactly supported within the upper half of the unit disk $L_n(f)$ converge to a real Gaussian of variance

$$\frac{1}{4\pi}\langle \nabla f, \nabla f\rangle_{L^2(\mathbf{D})} = \frac{1}{2\pi}\langle \nabla P_{\mathrm{sym}} f, \nabla P_{\mathrm{sym}} f\rangle_{L^2(\mathbf{D})}, \tag{2.9}$$

where we used that $z \mapsto f(z)$ and $z \mapsto f(\overline{z})$ are assumed to have disjoint support. Due to the moment matching assumption, $\kappa_4 = 0$ in the setting of [44].

(ii) The central limit theorem [49, Corollary 2.6] is also a special case of Theorem 2.2. Indeed, [49, Corollary 2.6] implies that for real i.i.d. matrices and test functions f which are analytic in a neighbourhood of the unit disk and satisfy $P_{\mathrm{sym}} f \colon \overline{\mathbf{D}} \to \mathbf{R}$ the linear statistics $L_n(f)$ converge to a Gaussian of variance

$$\begin{aligned}\frac{1}{\pi}\int_{\mathbf{D}} |\partial_z f(z)|^2 \,\mathrm{d}^2 z &= \frac{1}{4\pi}\langle \nabla f, \nabla f\rangle_{L^2(\mathbf{D})} + \frac{1}{2}\langle f, f\rangle_{\dot{H}^{1/2}(\partial\mathbf{D})}\\ &= \frac{1}{2\pi}\langle \nabla P_{\mathrm{sym}} f, \nabla P_{\mathrm{sym}} f\rangle_{L^2(\mathbf{D})} + \langle P_{\mathrm{sym}} f, P_{\mathrm{sym}} f\rangle_{\dot{H}^{1/2}(\partial\mathbf{D})}.\end{aligned}$$

Here in the first step we used the analyticity of f (see [22, Eq. (2.11)]), and in the second step we used that $\langle (\nabla f)(z), (\nabla f(\bar{\cdot}))(z)\rangle = 0$ and that $\widehat{f}(k) = 0$ for $k < 0$ while $\widehat{f(\bar{\cdot})}(k) = 0$ for $k > 0$ by analyticity. We thus arrived at (2.7), since the coefficient of κ_4 in (2.7) vanishes also by analyticity of f in the setting of [49].

Remark 2.5 (Comparison with the complex case)**.** We remark that the limiting variance in the case of complex i.i.d. matrices, as studied in [22], is generally different from the real case. In the complex case $L_n(f)$ converges to a complex Gaussian with variance

$$\begin{aligned}V_f^{(\mathbf{C})} &= V_f^{(\mathbf{C},1)} + \kappa_4 V_f^{(\mathbf{C},2)},\\ V_f^{(\mathbf{C},1)} &:= \frac{1}{4\pi}\|\nabla f\|^2_{L^2(\mathbf{D})} + \frac{1}{2}\|f\|^2_{\dot{H}^{1/2}(\partial(\mathbf{D}))}, \quad V_f^{(\mathbf{C},2)} := |\langle f\rangle_{\mathbf{D}} - \langle f\rangle_{\partial\mathbf{D}}|^2,\end{aligned}$$

where $\langle\cdot\rangle_{\mathbf{D}}$ denotes the averaging over $\mathbf{D}$ as in (2.7). In contrast, in the real case the limiting variance is given by

$$V_f^{(\mathbf{R})} = 2V^{(\mathbf{C},1)}_{P_{\mathrm{sym}} f} + \kappa_4 V_f^{(\mathbf{C},2)}.$$

Thus the variances agree exactly in the case of analytic test functions by (2.9) and $V_f^{(\mathbf{C},2)} = 0$, while e.g. in the case of symmetric test functions, $f = P_{\mathrm{sym}} f$ and vanishing fourth cumulant $\kappa_4 = 0$ the real variance is twice as big as the complex one, $V_f^{(\mathbf{R})} = 2V_f^{(\mathbf{C})}$.

Remark 2.6 (Real correction to the expected circular law)**.** In [26, Theorem 6.2] Edelman computed the density of genuinely complex eigenvalues of the real Ginibre ensemble to be

$$\rho_n(x+\mathrm{i}y) := \sqrt{\frac{2n}{\pi}}|y|e^{2ny^2}\,\mathrm{erfc}(\sqrt{2n}|y|)\frac{\Gamma(n-1, n(x^2+y^2))}{\Gamma(n-1)} \tag{2.10}$$

in terms of the *upper incomplete Gamma function* $\Gamma(s,x)$. Using the large n asymptotics uniform in $z = x + \mathrm{i}y$ for the incomplete Gamma function [61, Eq. (2.2)] we obtain

$$\rho_n(z) \approx \sqrt{\frac{2n}{\pi}}|\Im z|e^{2n(\Im z)^2}\,\mathrm{erfc}(\sqrt{2n}|\Im z|)\,\mathrm{erfc}\Big(\mathrm{sgn}(|z|-1)\sqrt{n(|z|^2 - 1 - 2\log|z|)}\Big),$$

which, using asymptotics of the error function for any fixed $|z| < 1$,

$$\sqrt{\frac{2n}{\pi}}|\Im z|e^{2n(\Im z)^2}\,\mathrm{erfc}(\sqrt{2n}|\Im z|) \approx \frac{1}{2\pi} - \frac{1}{8n\pi(\Im z)^2},$$

gives that

$$\rho_n(z) = \frac{1}{\pi} - \frac{1}{4\pi n}\frac{1}{(\Im z)^2} + \mathcal{O}(n^{-1}),$$

in agreement with the second term in the rhs. of (2.8) accounting for the n^{-1}-correction to the circular law away from the real axis.

The situation very close to the real axis is much more subtle. The density of the real Ginibre eigenvalues is explicitly known [27, Corollary 4.3] and it is asymptotically uniform on $[-1,1]$, see [27, Corollary 4.5], giving a singular correction of mass of order $n^{-1/2}$ to the circular law. However, the abundance of real eigenvalues is balanced by the sparsity of genuinely complex eigenvalues in a narrow strip around the real axis — a consequence of the factor $|y|$ in (2.10). Since these two effects of order $n^{-1/2}$ cancel each other on the scale of our test functions f, they are not directly visible in (2.8). Instead we obtain a smaller order correction of order n^{-1} specific to the real axis, in form of the second, the penultimate and the ultimate term in (2.8).

Remark 2.7 (Special case: Polynomial test functions)**.** We remark that in [35, 57] exact n-dependent formulae for $\mathbf{E}\operatorname{Tr} X^k = \mathbf{E}\sum_i \sigma_i^k$ and real Ginibre X have been obtained. Translated into our scaling it follows from [35, Corollary 4] that

$$\mathbf{E}\operatorname{Tr} X^k = \begin{cases} 1, & k \text{ even}, \\ 0, & k \text{ odd}, \end{cases} + \mathcal{O}_k(1) \tag{2.11}$$

for integers $k \geq 1$, as $n \to \infty$ (note that the trace is unnormalised). The asymptotics (2.11) are consistent with (2.8) since

$$\int_{\mathbf{D}} z^k \, \mathrm{d}^2 z = 0, \quad \int_{-1}^{1} (e^{\mathrm{i}\theta})^k \, \mathrm{d}\theta = 0, \quad \frac{1^k + (-1)^k}{4} = \begin{cases} \frac{1}{2}, & k \text{ even}, \\ 0, & k \text{ odd}, \end{cases}$$

and

$$\frac{1}{4\pi}\int_{\mathbf{D}} \frac{(\Re z)^k - z^k}{(\Im z)^2} \, \mathrm{d}^2 z = \begin{cases} \frac{1}{2} - 2^{-k}\binom{k-1}{k/2}, & k \text{ even}, \\ 0, & k \text{ odd}, \end{cases}$$

$$\frac{1}{2\pi}\int_{-1}^{1} \frac{x^k}{\sqrt{1-x^2}} \, \mathrm{d}x = \begin{cases} 2^{-k}\binom{k-1}{k/2}, & k \text{ even}, \\ 0, & k \text{ odd}. \end{cases}$$

2.1 Connection to the Gaussian free field

It has been observed in [53] that for complex Ginibre matrices the limiting random field $L(f)$ can be viewed as a projection of the *Gaussian free field (GFF)* [55]. In [22, Section 2.1] we extended this interpretation to general complex i.i.d. matrices with $\kappa_4 \geq 0$ and provided an interpretation as a rank-one perturbation of the projected GFF. The real case yields the symmetrised version of the same GFF with respect to the real axis, reflecting the fact that the complex eigenvalues of real matrices come in pairs of complex conjugates. We keep the explanation brief due to the similarity to [22, Section 2.1].

The Gaussian free field on $\mathbf{C}$ is a *Gaussian Hilbert space* of random variables $h(f)$ indexed by functions in the Sobolev space $f \in H_0^1(\mathbf{C})$ such that the map $f \mapsto h(f)$ is linear and

$$\mathbf{E}\, h(f) = 0, \quad \mathbf{E}\, \overline{h(f)} h(g) = \langle f, g \rangle_{H_0^1(\mathbf{C})} = \langle \nabla f, \nabla g \rangle_{L^2(\mathbf{C})}. \tag{2.12}$$

The Sobolev space $H_0^1(\mathbf{C}) = \overline{C_0^\infty(\mathbf{C})}^{\|\cdot\|_{H_0^1(\mathbf{C})}}$ can be orthogonally decomposed into

$$H_0^1(\mathbf{D}) \oplus H_0^1(\overline{\mathbf{D}}^c) \oplus H_0^1(\mathbf{D} \cup \overline{\mathbf{D}}^c)^\perp,$$

i.e. the H_0^1-closure of smooth functions which are compactly supported in $\mathbf{D}$ or $\overline{\mathbf{D}}^c$, and their orthogonal complement $H_0^1((\partial\mathbf{D})^c)^\perp$, the closed subspace of functions analytic outside of $\partial\mathbf{D}$ (see e.g. [55, Thm. 2.17]). With the orthogonal projection P onto the first and third of these subspaces,

$$P := P_{H_0^1(\mathbf{D})} + P_{H_0^1((\partial\mathbf{D})^c)^\perp},$$

we have (see [22, Eq. (2.13)])

$$\|Pf\|^2_{H_0^1(\mathbf{C})} = \|f\|^2_{H_0^1(\mathbf{D})} + 2\pi\|f\|^2_{\dot{H}^{1/2}(\partial\mathbf{D})}. \tag{2.13}$$

If $\kappa_4 \geq 0$, then L can be interpreted as

$$L = \frac{1}{\sqrt{2\pi}} PP_{\mathrm{sym}} h + \sqrt{\kappa_4}\Big(\langle\cdot\rangle_{\mathbf{D}} - \langle\cdot\rangle_{\partial\mathbf{D}}\Big)\Xi, \tag{2.14}$$

where Ξ is a standard real Gaussian, independent of h, and the projection of h is to be interpreted by duality, i.e. $(PP_{\mathrm{sym}}h)(f) := h(PP_{\mathrm{sym}}f)$, cf. [22, Eq. (2.15)]. Indeed,

$$\mathbf{E}\left|\frac{1}{\sqrt{2\pi}} h(PP_{\mathrm{sym}}f) + \sqrt{\kappa_4}(\langle f\rangle_{\mathbf{D}} - \langle f\rangle_{\partial\mathbf{D}})\Xi\right|^2 = C(f,f),$$

as a consequence of (2.12) and (2.13).

2.2 Universality of the local singular value statistics of $X - z$ close to zero

As a by-product of our analysis we obtain the universality of the small singular values of $X - z$, and prove that (up to a rescaling) their distribution asymptotically agrees with the singular value distribution of a *complex* Ginibre matrix $\widetilde{X}$ if $z \notin \mathbf{R}$, even though X is a *real* i.i.d. matrix. In the following by $\{\lambda_i^z\}_{i\in[n]}$ we denote the singular values of $X - z$ in increasing order.

It is natural to express universality in terms of the k-point correlation functions $p_{k,z}^{(n)}$ which are defined implicitly by

$$\mathbf{E}\binom{n}{k}^{-1} \sum_{\{i_1,\dots,i_k\}\subset[n]} f(\lambda_{i_1}^z,\dots,\lambda_{i_k}^z) = \int_{\mathbf{R}^k} f(\boldsymbol{x}) p_{k,z}^{(n)}(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}, \tag{2.15}$$

for test functions f. The summation in (2.15) is over all the subsets of k distinct integers from $[n]$. Denote by $p_k^{(\infty,\mathbf{C})}$ the scaling limit of the k-point correlation function $p_k^{(n,\mathbf{C})}$ of the singular values of a complex $n\times n$ Ginibre matrix $\widetilde{X}$. See e.g. [34, Eqs. (2.3)–(2.4)] or [9, Eq. (1.3)] for the explicit expression of $p_k^{(\infty,\mathbf{C})}$.

Theorem 2.8 (Universality of small singular values of $X - z$)**.** *Fix $z \in \mathbf{C}$ with $|\Im z| \sim 1$, and $|z| \leq 1-\epsilon$, for some small fixed $\epsilon > 0$. Let X be an i.i.d. matrix with real entries satisfying Assumption 2.1, and denote by ρ^z the self consistent density of states of the singular values of $X - z$ (see* (3.3) *later). Then for any $k \in \mathbf{N}$, and for any compactly supported test function $F \in C_c^1(\mathbf{R}^k)$, it holds*

$$\int_{\mathbf{R}^k} F(\boldsymbol{x})\left[\rho^z(0)^{-k} p_{k,z}^{(n)}\left(\frac{\boldsymbol{x}}{n\rho^z(0)}\right) - p_k^{(\infty,\mathbf{C})}(\boldsymbol{x})\right]\mathrm{d}\boldsymbol{x} = \mathcal{O}\left(n^{-c(k)}\right), \tag{2.16}$$

where $c(k) > 0$ is a small constant only depending on k. The implicit constant in $\mathcal{O}(\cdot)$ may depend on k, $\|F\|_{C^1}$, and C_p from (2.1)*.*

Remark 2.9. Theorem 2.8 states that the local statistics of the singular values of $X - z$ close to zero, for $|\Im z| \sim 1$, asymptotically agree with the ones of a complex Ginibre matrix $\widetilde{X}$, even if the entries of X are real i.i.d. random variables. It is expected that the same result holds for all (possibly n-dependent) z as long as $|\Im z| \gg n^{-1/2}$, while in the opposite regime $|\Im z| \ll n^{-1/2}$ the local statistics of the *real* Ginibre prevails with an interpolating family of new statistics which emerges for $|\Im z| \sim n^{-1/2}$.

Besides the universality of small singular values of $X - z$, our methods also allow us to conclude the asymptotic independence of the small singular values of $X - z_1$ and those of $X - z_2$ for generic z_1, z_2. More precisely, similarly to (2.15), we define the correlation function $p^{(n)}_{k_1,z_1;k_2,z_2}$ for the singular values of $X - z_1$ and $X - z_2$ implicitly by

$$\begin{aligned} \mathbf{E}\binom{n}{k_1}^{-1}\binom{n}{k_2}^{-1} \sum_{\substack{\{i_1,\dots,i_{k_1}\}\subset[n]\\ \{j_1,\dots,j_{k_2}\}\subset[n]}} f(\boldsymbol{\lambda}^{z_1}_{\boldsymbol{i}}, \boldsymbol{\lambda}^{z_2}_{\boldsymbol{j}}) \\ = \int_{\mathbf{R}^{k_1}} \mathrm{d}\boldsymbol{x}_1 \int_{\mathbf{R}^{k_2}} \mathrm{d}\boldsymbol{x}_2\, f(\boldsymbol{x}_1, \boldsymbol{x}_2) p^{(n)}_{k_1,z_1;k_2,z_2}(\boldsymbol{x}_1, \boldsymbol{x}_2), \end{aligned} \tag{2.17}$$

for any test function f, and any $k_1, k_2 \in \mathbf{N}$, where we used the notations $\boldsymbol{\lambda}^{z_1}_{\boldsymbol{i}} := (\lambda^{z_1}_{i_1}, \dots, \lambda^{z_1}_{i_{k_1}})$ and $\boldsymbol{\lambda}^{z_2}_{\boldsymbol{j}} := (\lambda^{z_2}_{j_1}, \dots, \lambda^{z_2}_{j_{k_2}})$.

Theorem 2.10 (Asymptotic independence of small singular values of $X - z_1, X - z_2$)**.** *Let $z_1, z_2 \in \mathbf{C}$ be as z in Theorem 2.8, and assume that $|z_1 - z_2|, |z_1 - \overline{z}_2| \sim 1$. Let X be an i.i.d. matrix with real entries satisfying Assumption 2.1, then for any $k_1, k_2 \in \mathbf{N}$, and for any compactly supported test function $F \in C^1_c(\mathbf{R}^k)$, with $k = k_1 + k_2$, using the notation $\boldsymbol{x} = (\boldsymbol{x}_1, \boldsymbol{x}_2)$, with $\boldsymbol{x}_l \in \mathbf{R}^{k_l}$, it holds*

$$\int_{\mathbf{R}^k} F(\boldsymbol{x}) \left[\frac{p^{(n)}_{k_1,z_1;k_2,z_2}\left(\frac{\boldsymbol{x}_1}{n\rho^{z_1}}, \frac{\boldsymbol{x}_2}{n\rho^{z_2}}\right)}{(\rho^{z_1})^{k_1}(\rho^{z_2})^{k_2}} - p^{(\infty,\mathbf{C})}_{k_1}(\boldsymbol{x}_1) p^{(\infty,\mathbf{C})}_{k_2}(\boldsymbol{x}_2) \right] \mathrm{d}\boldsymbol{x} = \mathcal{O}\left(n^{-c(k)}\right), \tag{2.18}$$

where $\rho^{z_l} = \rho^{z_l}(0)$, and $c(k) > 0$ is a small constant only depending on k. The implicit constant in $\mathcal{O}(\cdot)$ may depend on k, $\|F\|_{C^1}$, and C_p from (2.1).

Remark 2.11. We stated Theorem 2.8 for two different z_1, z_2 for notational simplicity. The analogous result holds for any finitely many $z_1, \dots, z_q$ such that $|z_l - z_m|, |z_l - \overline{z}_m| \sim 1$, with $l, m \in [q]$.

3 Proof strategy

The proof of Theorem 2.2 follows a similar strategy as the proof of [22, Thm. 2.2] with several major changes. We use Girko's formula to relate the eigenvalues of X to the resolvent of the $2n \times 2n$ matrix

$$H^z := \begin{pmatrix} 0 & X - z \\ (X - z)^* & 0 \end{pmatrix}, \tag{3.1}$$

the so called *Hermitisation* of $X - z$. We denote the eigenvalues of H^z, which come in pairs symmetric with respect to zero, by $\{\lambda^z_{\pm i}\}_{i\in[n]}$. The *local law*, see Theorem 3.1 below, asserts that the resolvent $G(w) = G^z(w) := (H^z - w)^{-1}$ of H^z with $\eta = \Im w \neq 0$ becomes approximately deterministic, as $n \to \infty$. Its limit is expressed via the unique solution of the scalar equation

$$-\frac{1}{m^z} = w + m^z - \frac{|z|^2}{w + m^z}, \quad \eta \Im m^z(w) > 0, \quad \eta = \Im w \neq 0, \tag{3.2}$$

which is a special case of the *matrix Dyson equation* (MDE), see e.g. [1] and (5.1) later. Note that on the imaginary axis $m^z(\mathrm{i}\eta) = \mathrm{i}\Im m^z(\mathrm{i}\eta)$. We define the *self-consistent density* *of states* of H^z and its extension to the upper half-plane by

$$\rho^z(E) := \rho^z(E + \mathrm{i}0), \qquad \rho^z(w) := \frac{1}{\pi}\Im m^z(w). \tag{3.3}$$

In terms of m^z the deterministic approximation to G^z is given by the $2n \times 2n$ block matrix

$$M^z(w) := \begin{pmatrix} m^z(w) & -zu^z(w) \\ -\overline{z}u^z(w) & m^z(w) \end{pmatrix}, \quad u^z(w) := \frac{m^z(w)}{w + m^z(w)}, \tag{3.4}$$

where each block is understood to be a scalar multiple of the $n \times n$ identity matrix. We note that m, u, M are uniformly bounded in z, w, i.e.

$$\|M^z(w)\| + |m^z(w)| \lesssim 1, \quad |u^z(w)| \le |m^z(w)|^2 + |u^z(w)|^2|z|^2 < 1, \tag{3.5}$$

see e.g. [22, Eqs. (3.3)–(3.5)].

The *local law* for $G^z(w)$ in its full averaged and isotropic form has been obtained for $w \in \mathrm{i}\mathbf{R}$ in [2] for the bulk regime $|1 - |z|| \ge \epsilon$ and in [3] for the edge regime $|1 - |z|| < \epsilon$. In fact, in the companion paper [22] on the complex CLT the local law for w on the imaginary axis was sufficient. For the real CLT, however, we need its extension to general spectral parameters w in the bulk $|1 - |z|| \ge \epsilon$ case that we state below. We remark that tracial and entry-wise form of the local law in Theorem 3.1 has already been established in [16, Theorem 3.4].

Theorem 3.1 (Optimal local law for G)**.** *For any $\epsilon > 0$ and $z \in \mathbf{C}$ with $|1 - |z|| \ge \epsilon$ the resolvent G^z at $w \in \mathbf{H}$ with $\eta = \Im w$ is very well approximated by the deterministic matrix M^z in the sense that*

$$\begin{aligned} |\langle (G^z(w) - M^z(w))A\rangle| &\le \frac{C_\epsilon \|A\| n^\xi}{n\eta}, \\ |\langle \boldsymbol{x}, (G^z(w) - M^z(w))\boldsymbol{y}\rangle| &\le C_\epsilon \|\boldsymbol{x}\|\|\boldsymbol{y}\| n^\xi \Big(\frac{1}{\sqrt{n\eta}} + \frac{1}{n\eta}\Big), \end{aligned} \tag{3.6}$$

with very high probability for some $C_\epsilon \le \epsilon^{-100}$, uniformly for $\eta \ge n^{-100}$, $|1 - |z|| \ge \epsilon$, and for any deterministic matrices A and vectors $\boldsymbol{x}, \boldsymbol{y}$, and $\xi > 0$.

Remark 3.2 (Cusp fluctuation averaging)**.** For $w \in \mathrm{i}\mathbf{R}$ we may choose $C_\epsilon = 1$ by [3, Theorem 5.2] which takes into account the *cusp fluctuation averaging effect*. Since it is not necessary for the present work we refrain from adapting this technique for general w and rather present a conceptually simpler proof resulting in the ϵ-dependent bounds (3.6).

As in [22] we express the linear statistics (1.1) of eigenvalues σ_i of X through the resolvent G^z via Girko's Hermitisation formula (1.3)

$$\begin{aligned} L_n(f) &= \frac{1}{4\pi}\int_{\mathbf{C}} \Delta f(z)\Big[\log|\det(H^z - \mathrm{i}T)| - \mathbf{E}\log|\det(H^z - \mathrm{i}T)|\Big]\,\mathrm{d}^2 z \\ &\quad - \frac{n}{2\pi\mathrm{i}}\int_{\mathbf{C}} \Delta f(z)\left[\left(\int_0^{\eta_0} + \int_{\eta_0}^{\eta_c} + \int_{\eta_c}^{T}\right)\big[\langle G^z(\mathrm{i}\eta) - \mathbf{E}\,G^z(\mathrm{i}\eta)\rangle\big]\,\mathrm{d}\eta\right]\mathrm{d}^2 z \\ &=: J_T + I_0^{\eta_0} + I_{\eta_0}^{\eta_c} + I_{\eta_c}^{T}, \end{aligned} \tag{3.7}$$

for $\eta_0 = n^{-1-\delta_0}$, $\eta_c = n^{-1+\delta_1}$, and $T = n^{100}$, where J_T in (3.7) corresponds to the rhs. of the first line in (3.7) whilst $I_0^{\eta_0}, I_{\eta_0}^{\eta_c}, I_{\eta_c}^{T}$ correspond to the three different η-integrals in the second line of (3.7). Here we used that by spectral symmetry of H^z it follows that $\langle G^z(\mathrm{i}\eta)\rangle \in \mathrm{i}\mathbf{R}$ and therefore $\Im\langle G^z(\mathrm{i}\eta)\rangle = \langle G^z(\mathrm{i}\eta)\rangle/\mathrm{i}$ in order to obtain (3.7) from (1.3). The regime J_T can be trivially estimated by [22, Lemma 4.3], while the regime $I_0^{\eta_0}$ can be controlled using [60, Thm. 3.2] as in [22, Lemma 4.4] (see [22, Remark 4.5] for an alternative proof). Both contributions are negligible. For the main term $I_{\eta_c}^{T}$ we prove the following *resolvent CLT*.

Proposition 3.3 (CLT for resolvents). *Let $\epsilon > 0$, $\eta_1, \dots, \eta_p > 0$, and $z_1, \dots, z_p \in \mathbf{C}$ be such that for any $i \neq j$, $\min\{\eta_i, \eta_j\} \geq n^{\epsilon-1}|z_i - z_j|^{-2}$. Then for any $\xi > 0$ the traces of the resolvents $G_i = G^{z_i}(\mathrm{i}\eta_i)$ satisfy an asymptotic Wick theorem*

$$\begin{aligned} \mathbf{E}\prod_{i\in[p]}\langle G_i - \mathbf{E}\, G_i\rangle &= \sum_{P\in \mathrm{Pairings}([p])}\prod_{\{i,j\}\in P}\mathbf{E}\langle G_i - \mathbf{E}\, G_i\rangle\langle G_j - \mathbf{E}\, G_j\rangle + \mathcal{O}\left(\Psi\right) \\ &= \frac{1}{n^p}\sum_{P\in \mathrm{Pairings}([p])}\prod_{\{i,j\}\in P}\frac{\widehat{V}_{i,j} + \kappa_4 U_i U_j}{2} + \mathcal{O}(\Psi), \end{aligned} \tag{3.8}$$

where

$$\Psi := \frac{n^\xi}{(n\eta_*)^{1/2}}\frac{1}{\min_{i\neq j}|z_i - z_j|^4}\prod_{i\in[p]}\Big(\frac{1}{|1-|z_i||} + \frac{1}{(\Im z_i)^2}\Big)\frac{1}{n\eta_i}, \qquad \eta_* := \min_i \eta_i, \tag{3.9}$$

and $\widehat{V}_{i,j} = \widehat{V}(z_i, z_j, \eta_i, \eta_j)$ and $U_i = U(z_i, \eta_i)$ are defined as

$$\begin{aligned} \widehat{V}(z_i, z_j, \eta_i, \eta_j) &:= V(z_i, z_j, \eta_i, \eta_j) + V(z_i, \overline{z_j}, \eta_i, \eta_j) \\ V(z_i, z_j, \eta_i, \eta_j) &:= \frac{1}{2}\partial_{\eta_i}\partial_{\eta_j}\log\big[1 + (u_i u_j |z_i||z_j|)^2 - m_i^2 m_j^2 - 2u_i u_j \Re z_i \overline{z_j}\big], \\ U(z_i, \eta_i) &:= \frac{\mathrm{i}}{\sqrt{2}}\partial_{\eta_i} m_i^2, \end{aligned} \tag{3.10}$$

with $m_i = m^{z_i}(\mathrm{i}\eta_i)$ and $u_i = u^{z_i}(\mathrm{i}\eta_i)$ from (3.2)–(3.4).

Moreover, the expectation of the normalised trace of $G = G_i$ is given by

$$\mathbf{E}\langle G\rangle = \langle M\rangle + \mathcal{E} + \mathcal{O}\left(\Big(\frac{1}{|1-|z||} + \frac{1}{|\Im z|^2}\Big)\Big(\frac{1}{n^{3/2}(1+\eta)} + \frac{1}{(n\eta)^2}\Big)\right), \tag{3.11}$$

where

$$\mathcal{E} := -\frac{\mathrm{i}\kappa_4}{4n}\partial_\eta(m^4) + \frac{\mathrm{i}}{4n}\partial_\eta \log\Big(1 - u^2 + 2u^3|z|^2 - u^2(z^2 + \overline{z}^2)\Big). \tag{3.12}$$

Proposition 3.3 is the real analogue of [22, Prop. 3.3]. The main differences are that (i) the V-term for the variance appears in a symmetrised form with z_j and $\overline{z_j}$, (ii) the error term (3.9) deteriorates as $\Im z_i \approx 0$, and (iii) the expectation (3.11) has an additional subleading term which is even present in case $\kappa_4 = 0$ (second term in (3.12)).

Finally, in order to show that $I_{\eta_0}^{\eta_c}$ in (3.7) is negligible, we prove that $\langle G^{z_1}(\mathrm{i}\eta_1)\rangle$ and $\langle G^{z_2}(\mathrm{i}\eta_2)\rangle$ are asymptotically independent if z_1, z_2 and z_1, $\overline{z}_2$ are far enough from each other, they are far away from the real axis, they are well inside $\mathbf{D}$, and $\eta_0 \leq \eta_1, \eta_2 \leq \eta_c$. These regimes of the parameters z_1, z_2 represent the overwhelming part of the $\mathrm{d}^2 z_1\, \mathrm{d}^2 z_2$ integration in the calculation of $\mathbf{E}|I_{\eta_0}^{\eta_c}|^2$. The following proposition is the direct analogue of [22, Prop. 3.5].

Proposition 3.4 (Independence of resolvents with small imaginary part). *Fix $p \in \mathbf{N}$. For any sufficiently small $\omega_h, \omega_d > 0$ there exist $\omega_*, \delta_0, \delta_1$ with $\omega_h \ll \delta_m \ll \omega_* \ll 1$, for $m = 0, 1$, such that for any choice of $z_1, \dots, z_p$ with*

$$|z_l| \leq 1 - n^{-\omega_h},\ |z_l - z_m| \geq n^{-\omega_d},\ |z_l - \overline{z}_m| \geq n^{-\omega_d},\ |z_l - \overline{z}_l| \geq n^{-\omega_d},$$

with $l, m \in [p]$, $l \neq m$, it follows that

$$\mathbf{E}\prod_{l=1}^p \langle G^{z_l}(\mathrm{i}\eta_l)\rangle = \prod_{l=1}^p \mathbf{E}\langle G^{z_l}(\mathrm{i}\eta_l)\rangle + \mathcal{O}\left(\frac{n^{p(\omega_h+\delta_0)+\delta_1}}{n^{\omega_*}}\right), \tag{3.13}$$

for any $\eta_1, \dots, \eta_p \in [n^{-1-\delta_0}, n^{-1+\delta_1}]$.

As in the complex case [22], one key ingredient for both Propositions 3.3 and 3.4 is a local law for products of resolvents G_1, G_2 for $G_i = G^{z_i}(w_i)$. We remark that local laws for products of resolvents have also been derived for (generalized) Wigner matrices [30, 46] and for sample covariance matrices [20], as well as for the addition of random matrices [8].

Note that the deterministic approximation to G_1G_2 is not given simply by M_1M_2 where $M_i := M^{z_i}(w_i)$ from (3.4). To describe the correct approximation, as in [22, Section 5], we define the *stability operator*

$$\widehat{\mathcal{B}} = \widehat{\mathcal{B}}_{12} = \widehat{\mathcal{B}}(z_1, z_2, w_1, w_2) := 1 - M_1\mathcal{S}[\cdot]M_2, \tag{3.14}$$

acting on the space of $2n \times 2n$ matrices. Here the linear *covariance* or *self-energy operator* $\mathcal{S}\colon \mathbf{C}^{2n\times 2n} \to \mathbf{C}^{2n\times 2n}$ is defined as

$$\mathcal{S}\left[\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right] := \widetilde{\mathbf{E}}\widetilde{W}\begin{pmatrix} A & B \\ C & D \end{pmatrix}\widetilde{W} = \begin{pmatrix} \langle D\rangle & 0 \\ 0 & \langle A\rangle \end{pmatrix}, \quad \widetilde{W} = \begin{pmatrix} 0 & \widetilde{X} \\ \widetilde{X}^* & 0 \end{pmatrix}, \quad \widetilde{X} \sim \mathrm{Gin}_{\mathbf{C}}, \tag{3.15}$$

i.e. it averages the diagonal blocks and swaps them. Here $\widetilde{\mathbf{E}}$ denotes the expectation with respect to $\widetilde{X}$, $\langle A\rangle = n^{-1}\mathrm{Tr}A$ and $\mathrm{Gin}_{\mathbf{C}}$ stands for the standard complex Ginibre ensemble. The ultimate equality in (3.15) follows directly from $\mathbf{E}\,\widetilde{x}_{ab}^2 = 0$, $\mathbf{E}|\widetilde{x}_{ab}|^2 = n^{-1}$. Note that as a matter of choice we define the stability operator (3.14) with the covariance operator $\mathcal{S}$ corresponding to the complex rather than the real Ginibre ensemble. However, to leading order there is no difference between the two and the present choice is more consistent with the companion paper [22]. The effect of this discrepancy will be estimated in a new error term (see (6.4) later).

For any deterministic matrix B we define

$$M_B^{z_1,z_2}(w_1, w_2) := \widehat{\mathcal{B}}_{12}^{-1}[M^{z_1}(w_1)BM^{z_2}(w_2)], \tag{3.16}$$

which turns out to be the deterministic approximation to G_1BG_2. Indeed, from the local law for G_1, G_2, Theorem 3.1, and [22, Thm. 5.2] we immediately conclude the following theorem.

Theorem 3.5 (Local law for $G^{z_1}BG^{z_2}$)**.** *Fix $z_1, z_2 \in \mathbf{C}$ with $|1 - |z_i|| \ge \epsilon$, for some $\epsilon > 0$ and $w_1, w_2 \in \mathbf{C}$ with $|\eta_i| := |\Im w_i| \ge n^{-1}$ such that*

$$\eta_* := \min\{|\eta_1|, |\eta_2|\} \ge n^{-1+\epsilon_*}|\widehat{\beta}_*|^{-1},$$

for some small $\epsilon_* > 0$, where $\widehat{\beta}_*$ is the, in absolute value, smallest eigenvalue of $\widehat{\mathcal{B}}_{12}$ defined in (3.14)*. Then, for any bounded deterministic matrix B, $\|B\| \lesssim 1$, the product of resolvents $G^{z_1}BG^{z_2} = G^{z_1}(w_1)BG^{z_2}(w_2)$ is well approximated by $M_B^{z_1,z_2} = M_B^{z_1,z_2}(w_1, w_2)$ defined in* (3.16) *in the sense that*

$$\begin{aligned} |\langle A(G^{z_1}BG^{z_2} - M_B^{z_1,z_2})\rangle| &\le \frac{C_\epsilon\|A\|n^\xi}{n\eta_*|\eta_1\eta_2|^{1/2}|\widehat{\beta}_*|}\Big(\eta_*^{1/12} + \frac{\eta_*^{1/4}}{|\widehat{\beta}_*|} + \frac{1}{\sqrt{n\eta_*}} + \frac{1}{(|\widehat{\beta}_*|n\eta_*)^{1/4}}\Big), \\ |\langle \boldsymbol{x}, (G^{z_1}BG^{z_2} - M_B^{z_1,z_2})\boldsymbol{y}\rangle| &\le \frac{C_\epsilon\|\boldsymbol{x}\|\|\boldsymbol{y}\|n^\xi}{(n\eta_*)^{1/2}|\eta_1\eta_2|^{1/2}|\widehat{\beta}_*|} \end{aligned} \tag{3.17}$$

for some C_ϵ with very high probability for any deterministic $A, \boldsymbol{x}, \boldsymbol{y}$ and $\xi > 0$. If $w_1, w_2 \in \mathrm{i}\mathbf{R}$ we may choose $C_\epsilon = 1$, otherwise we can choose $C_\epsilon \le \epsilon^{-100}$.

An effective lower bound on $\Re\widehat{\beta}_*$, hence on $|\widehat{\beta}_*|$, will be given in Lemma 6.1 later.

The paper is organised as follows: In Section 4 we prove Theorem 2.2 by combining Propositions 3.3 and 3.4. In Section 5 we prove the local law for G away from the imaginary axis, Theorem 3.1. In Section 6 we prove Proposition 3.3, the Central Limit Theorem for resolvents using Theorem 3.5. In Section 7 we prove Proposition 3.4 again using Theorem 3.5, and conclude Theorem 2.8.

Note that Theorem 3.5, the local law for $G^{z_1}BG^{z_2}$, is used in two different contexts. Traces of $AG^{z_1}BG^{z_2}$, for some deterministic matrices $A, B \in \mathbf{C}^{2n\times 2n}$, naturally arise along the cumulant expansion for $\prod_i \langle G_i - \mathbf{E}\, G_i\rangle$ in Proposition 3.3. The proof of Proposition 3.4 is an analysis of weakly correlated DBMs, where the correlations are given by eigenvector overlaps (1.6), whose estimate is reduced to an upper bound on $\langle \Im G^{z_1} \Im G^{z_2}\rangle$.

4 Central limit theorem for linear statistics: proof of Theorem 2.2

From Propositions 3.3 and 3.4 we conclude Theorem 2.2 analogously to [22, Section 4], we only describe the few minor modifications.

Proof of Theorem 2.2. We explain the three modifications compared with the proof of [22, Theorem 2.2]. First, there are two additional terms in in the variance (3.10) and expectation (3.12) of the resolvent CLT, compared to [22, Eqs. (3.14)–(3.15)]. These additional terms result in additional explicit terms in (2.8) and (2.7). For the expectation in (2.8) we have

$$
\begin{aligned}
&-\frac{1}{2\pi \mathrm{i}}\int_{\mathbf{C}} \Delta f(z) \frac{\mathrm{i}}{4n}\int_0^\infty \partial_\eta \log\Big(1-u^2+2u^3|z|^2-u^2(z^2+\overline{z}^2)\Big)\,\mathrm{d}\eta\,\mathrm{d}^2 z \qquad (4.1)\\
&=\frac{1}{4\pi}\int_{\mathbf{D}} \frac{f(\Re z)-f(z)}{(\Im z)^2}\,\mathrm{d}^2 z - \frac{1}{2\pi}\int_0^{2\pi} f(e^{\mathrm{i}\theta})\,\mathrm{d}\theta + \frac{1}{2\pi}\int_{-1}^1 \frac{f(x)}{\sqrt{1-x^2}}\,\mathrm{d}x + \frac{f(1)+f(-1)}{4}
\end{aligned}
$$

and for the variance in (2.7) we have

$$
\begin{aligned}
&-\frac{1}{8\pi^2}\int_{\mathbf{C}} \mathrm{d}^2 z_1 \int_{\mathbf{C}} \mathrm{d}^2 z_2 \Delta f(z_1)\overline{\Delta g(z_2)}\int_0^\infty \mathrm{d}\eta_1 \int_0^\infty \mathrm{d}\eta_2 V(z_1,\overline{z_2},\eta_1,\eta_2)\\
&\qquad = \frac{1}{4\pi}\langle \nabla g(\bar{\cdot}), \nabla f\rangle_{L^2(\mathbf{D})} + \frac{1}{2}\langle g(\bar{\cdot}), f\rangle_{\dot H^{1/2}(\partial\mathbf{D})},
\end{aligned} \qquad (4.2)
$$

so that together with contribution from $V(z_1, z_2, \eta_1, \eta_2)$ in (3.10) we have

$$
\begin{aligned}
&\frac{1}{4\pi}\langle \nabla g + \nabla g(\bar{\cdot}), \nabla f\rangle_{L^2(\mathbf{D})} + \frac{1}{2}\langle g+g(\bar{\cdot}), f\rangle_{\dot H^{1/2}(\partial\mathbf{D})}\\
&\qquad = \frac{1}{2\pi}\langle \nabla P_{\mathrm{sym}} g, \nabla P_{\mathrm{sym}} f\rangle_{L^2(\mathbf{D})} + \langle P_{\mathrm{sym}} g, P_{\mathrm{sym}} f\rangle_{\dot H^{1/2}(\partial\mathbf{D})}.
\end{aligned}
$$

The identities (4.1)–(4.2) will be proven separately below. The other two modifications concern the error terms in (3.9) and (3.11). Namely, there is an additional factor including $(\Im z_l)^{-2}$ (cf. [22, Eqs. (3.13), (3.15)]), and, finally, (3.13) holds under the additional assumption that $|z_l - \overline{z}_m| \ge n^{-\omega_d}$, and $|z_l - \overline{z}_l| \ge n^{-\omega_d}$ (cf. [22, Prop. 3.5]). Both these issues can be handled in the same way as the constraints on $|z_l - z_m|$ have been treated in [22, Section 4] (see e.g. [22, Eq. (4.11)]). This means that we additionally exclude the regimes of negligible volume $|z_l - \overline{z}_m| < n^{-\omega_d}$ or $|z_l - \overline{z}_l| < n^{-\omega_d}$ from the $\mathrm{d}z_1 \ldots \mathrm{d}z_p$-integral in [22, Eqs. (4.10), (4.22)] using the almost optimal *a priori* bound from [22, Lemma 4.3]. □

Proof of (4.1). With the short-hand notation $z = x + \mathrm{i}y$, we compute

$$
\begin{aligned}
&\int_0^\infty \frac{\mathrm{i}}{4n}\partial_\eta \log\Big(1-u^2+2u^3|z|^2-u^2(z^2+\overline{z}^2)\Big)\,\mathrm{d}\eta\\
&\qquad = -\frac{\mathrm{i}}{4n}\begin{cases} \log 4 + 2\log|y|, & |z|\le 1,\\ \log\big|(x^2+y^2)^2+1-2(x^2-y^2)\big| - \log\big|(x^2+y^2)^2\big|, & |z|>1, \end{cases}
\end{aligned} \qquad (4.3)
$$

using that $u = 1 + \mathcal{O}(\eta)$ for $|z| \leq 1$ and $u = |z|^{-2} + \mathcal{O}(\eta)$ for $|z| > 1$, so that for (4.1) we need to compute

$$\frac{1}{4}\int_{\mathbf{C}} \Delta f(z)\big[(\log 4 + 2\log|y|)\mathbf{1}(|z| \leq 1) + (\log|z-1|^2 + \log|z+1|^2 - 2\log|z|^2)\mathbf{1}(|z| \geq 1)\big]\,\mathrm{d}^2 z. \tag{4.4}$$

We may assume that f is symmetric with respect to the real axis, i.e. $f = P_{\mathrm{sym}} f$ with P_{sym} as in (2.6) since $L_n(f - P_{\mathrm{sym}} f) = 0$ by symmetry of the spectrum and therefore $L_n(f) = L_n(P_{\mathrm{sym}} f)$. Since the functions in (4.4) are singular we introduce an ϵ-regularisation which enables us to perform integration by parts. In particular, the integral in (4.4) is equal to the $\epsilon \to 0$ limit of

$$\begin{aligned}\int_{\mathbf{C}} \partial_z \partial_{\overline{z}} f(z)\big[&(\log 4 + 2\log|y|)\mathbf{1}(|z| \leq 1, |y| \geq \epsilon)\\ &+ (\log|z-1|^2 + \log|z+1|^2 - 2\log|z|^2)\mathbf{1}(|z| \geq 1, |z \pm 1| \geq \epsilon)\big]\,\mathrm{d}^2 z,\end{aligned} \tag{4.5}$$

where $|z \pm 1| \geq \epsilon$ denotes that $|z-1| \geq \epsilon$ and $|z+1| \geq \epsilon$, and we used that the contribution from the regimes $|y| \leq \epsilon$ and $|z \pm 1| \leq \epsilon$ are negligible as $\epsilon \to 0$. In the following equalities should be understood in the $\epsilon \to 0$ limit.

Since

$$\log|z-1|^2 + \log|z+1|^2 - 2\log|z|^2 = \log 4 + 2\log|y|$$

for $|z| = 1$, when integrating by parts in (4.5), the terms where either $\mathbf{1}(|z| \leq 1)$ or $\mathbf{1}(|z| > 1)$ are differentiated are equal to zero, using that

$$\partial_z \mathbf{1}(|z| \geq 1)\,\mathrm{d}^2 z = \frac{\mathrm{i}}{2}\mathbf{1}(|z| = 1)\,\mathrm{d}\overline{z}. \tag{4.6}$$

We remark that (4.6) is understood in the sense of distributions, i.e. the equality holds when tested against compactly supported test functions f:

$$-\int_{\mathbf{C}} \partial_z f(z)\mathbf{1}(|z| \geq 1)\,\mathrm{d}^2 z = \frac{\mathrm{i}}{2}\int_{|z|=1} f(z)\,\mathrm{d}\overline{z}.$$

Moreover, with a slightly abuse of notation in (4.6) by $\mathbf{1}(|z| = 1)\,\mathrm{d}\overline{z}$ we denote the clockwise contour integral over the unit circle. This notation is used in the remainder of this section.

Then, performing integration by parts with respect to $\partial_{\overline{z}}$, we conclude that (4.5) is equal to

$$-\int_{\mathbf{C}} \partial_z f(z)\left[\frac{\mathrm{i}}{y}\mathbf{1}(|z| \leq 1, |y| \geq \epsilon) + \left(\frac{1}{\overline{z}-1} + \frac{1}{\overline{z}+1} - \frac{2}{\overline{z}}\right)\mathbf{1}(|z| \geq 1, |z \pm 1| \geq \epsilon)\right]\mathrm{d}^2 z. \tag{4.7}$$

In order to get (4.7) we used that

$$|\partial_z f(x + \mathrm{i}\epsilon) - \partial_z f(x - \mathrm{i}\epsilon)| \cdot |\log \epsilon| \lesssim \epsilon^{\delta'},$$

for some small fixed $\delta' > 0$, by $f \in H^{2+\delta}$, and similarly all the other ϵ-boundary terms tend to zero. This implies that when the $\partial_{\overline{z}}$ derivative hits the ϵ-boundary terms then these give a negligible contribution as $\epsilon \to 0$. We now consider the two terms in (4.7) separately.

Since the integral of y^{-1} over $\mathbf{D}$ is zero we can rewrite the first term in (4.7) as

$$-\int_{\mathbf{C}} \partial_z (f(z) - f(x))\frac{\mathrm{i}}{y}\mathbf{1}(|z| \leq 1, |y| \geq \epsilon)\,\mathrm{d}^2 z.$$

Then performing integration by parts we conclude that the first term in (4.7) is equal to

$$-\frac{1}{2}\int_{\mathbf{D}}\frac{f(x+\mathrm{i}y)-f(x)}{y^2}\,\mathrm{d}x\,\mathrm{d}y-\frac{\mathrm{i}}{2}\int_0^{2\pi}\frac{f(e^{\mathrm{i}\theta})-f(\cos\theta)}{\sin\theta}e^{-\mathrm{i}\theta}\,\mathrm{d}\theta \tag{4.8}$$

where we used that

$$\left|\frac{f(x,\epsilon)-2f(x,0)+f(x,-\epsilon)}{\epsilon}\right|\lesssim\epsilon^{\delta'},$$

to show that the terms when the ∂_z derivative hits the ϵ-boundary terms go to zero as $\epsilon\to 0$. Note that the integrals in (4.8) are absolutely convergent since f is symmetric with respect to the real axis. For the second term in (4.8) we further compute

$$\begin{aligned}\int_0^{2\pi}\frac{f(e^{\mathrm{i}\theta})-f(\cos\theta)}{\sin\theta}e^{-\mathrm{i}\theta}\,\mathrm{d}\theta&=\int_0^{2\pi}\frac{f(e^{\mathrm{i}\theta})-f(\cos\theta)}{\sin\theta}(\cos\theta-\mathrm{i}\sin\theta)\,\mathrm{d}\theta\\&=-\mathrm{i}\int_0^{2\pi}\Big(f(e^{\mathrm{i}\theta})-f(\cos\theta)\Big)\,\mathrm{d}\theta\end{aligned} \tag{4.9}$$

where we used that the term with $\cos\theta/\sin\theta$ is zero by symmetry.

With defining the domain

$$\Omega_\epsilon:=\{|z|\geq 1\}\cap\{|z\pm 1|\geq\epsilon\},$$

the second term in (4.7) is equal to

$$-\int_{\Omega_\epsilon}\partial_z f(z)\left(\frac{1}{\overline{z}-1}+\frac{1}{\overline{z}+1}-\frac{2}{\overline{z}}\right)\mathrm{d}^2z. \tag{4.10}$$

Since

$$\frac{1}{\overline{z}-1}+\frac{1}{\overline{z}+1}-\frac{2}{\overline{z}}$$

is anti-holomorphic on Ω_ϵ, performing integration by parts with respect to ∂_z in (4.10), we obtain

$$-\int_{\Omega_\epsilon}\partial_z f(z)\left(\frac{1}{\overline{z}-1}+\frac{1}{\overline{z}+1}-\frac{2}{\overline{z}}\right)\mathrm{d}^2z=\frac{\mathrm{i}}{2}\int_{\partial\Omega_\epsilon}f(z)\left(\frac{1}{\overline{z}-1}+\frac{1}{\overline{z}+1}-\frac{2}{\overline{z}}\right)\mathrm{d}\overline{z}. \tag{4.11}$$

Taking the limit $\epsilon\to 0$ in the r.h.s. of (4.11) we conclude

$$\begin{aligned}\lim_{\epsilon\to 0}\frac{\mathrm{i}}{2}\int_{\partial\Omega_\epsilon}f(z)\left(\frac{1}{\overline{z}-1}+\frac{1}{\overline{z}+1}-\frac{2}{\overline{z}}\right)\mathrm{d}\overline{z}&=\frac{\pi}{2}\big[f(1)+f(-1)\big]-\int_0^{2\pi}f(e^{\mathrm{i}\theta})\,\mathrm{d}\theta\\&\quad+\lim_{\epsilon\to 0}\left(\int_\epsilon^{\pi-\epsilon}+\int_{\pi+\epsilon}^{2\pi-\epsilon}\right)f(e^{\mathrm{i}\theta})\frac{e^{-2\mathrm{i}\theta}}{e^{-2\mathrm{i}\theta}-1}\,\mathrm{d}\theta.\end{aligned} \tag{4.12}$$

The last term in (4.12) simplifies to

$$\begin{aligned}\lim_{\epsilon\to 0}\left(\int_\epsilon^{\pi-\epsilon}+\int_{\pi+\epsilon}^{2\pi-\epsilon}\right)f(e^{\mathrm{i}\theta})\frac{e^{-2\mathrm{i}\theta}}{e^{-2\mathrm{i}\theta}-1}\,\mathrm{d}\theta&=\lim_{\epsilon\to 0}\left(\int_\epsilon^{\pi-\epsilon}+\int_{\pi+\epsilon}^{2\pi-\epsilon}\right)f(e^{\mathrm{i}\theta})\Big[\frac{\mathrm{i}}{2}\frac{\cos\theta}{\sin\theta}+\frac{1}{2}\Big]\,\mathrm{d}\theta\\&=\frac{1}{2}\int_0^{2\pi}f(e^{\mathrm{i}\theta})\,\mathrm{d}\theta,\end{aligned} \tag{4.13}$$

by symmetry. By combining (4.8)–(4.13) we conclude (4.1). □

Proof of (4.2). By change of variables $z_2\to\overline{z_2}$ we can then write

$$\begin{aligned}&\int_{\mathbf{C}}\mathrm{d}^2z_1\int_{\mathbf{C}}\mathrm{d}^2z_2\int_0^\infty\mathrm{d}\eta_1\int_0^\infty\mathrm{d}\eta_2\Delta f(z_1)\Delta\overline{g(z_2)}V(z_1,\overline{z_2},\eta_1,\eta_2)\\&=\int_{\mathbf{C}}\mathrm{d}^2z_1\int_{\mathbf{C}}\mathrm{d}^2z_2\int_0^\infty\mathrm{d}\eta_1\int_0^\infty\mathrm{d}\eta_2\Delta f(z_1)\Delta\overline{g(\overline{z_2})}V(z_1,z_2,\eta_1,\eta_2)\end{aligned} \tag{4.14}$$

such that [22, Lemma 4.8] is applicable and (4.2) follows. □

5 Local law away from the imaginary axis: proof of Theorem 3.1

The goal of this section is to prove a local law for $G = G^z(w)$ for z in the bulk, as stated in Theorem 3.1. We do not follow the precise ϵ-dependence in the proof explicitly but it can be checked from the arguments below that $C_\epsilon = \epsilon^{-100}$ clearly suffices. We denote the unique solution to the deterministic matrix equation (see e.g. [1])

$$-1 = \mathcal{S}[M]M + ZM + wM, \quad Z := \begin{pmatrix} 0 & z \\ \overline{z} & 0 \end{pmatrix}, \quad \Im M > 0, \quad \Im w > 0 \tag{5.1}$$

by $M = M^z(w)$, where we recall the definition of $\mathcal{S}$ from (3.15). The solution to (5.1) is given by (3.4). To keep notations compact, we first introduce a commonly used (see, e.g. [28]) notion of high-probability bound.

Definition 5.1 (Stochastic Domination)**.** *If*

$$X = \left(X^{(n)}(u) \,\middle|\, n \in \mathbf{N}, u \in U^{(n)} \right) \quad \text{and} \quad Y = \left(Y^{(n)}(u) \,\middle|\, n \in \mathbf{N}, u \in U^{(n)} \right)$$

are families of non-negative random variables indexed by n, and possibly some parameter u in a set $U^{(n)}$, then we say that X is stochastically dominated by Y, if for all $\epsilon, D > 0$ we have

$$\sup_{u \in U^{(n)}} \mathbf{P}\left[X^{(n)}(u) > n^\epsilon Y^{(n)}(u) \right] \le n^{-D}$$

for large enough $n \ge n_0(\epsilon, D)$. In this case we use the notation $X \prec Y$. Moreover, if we have $|X| \prec Y$ for families of random variables X, Y, we also write $X = \mathcal{O}_\prec(Y)$.

Let us assume that some a-priori bounds

$$|\langle \boldsymbol{x}, (G - M)\boldsymbol{y} \rangle| \prec \Lambda, \qquad |\langle A(G - M) \rangle| \prec \xi \tag{5.2}$$

for some deterministic control functions Λ and ξ depending on w, z have already been established, uniformly in $\boldsymbol{x}, \boldsymbol{y}, A$ under the constraint $\|\boldsymbol{x}\|, \|\boldsymbol{y}\|, \|A\| \le 1$. From the resolvent equation $1 = (W - Z - w)G$ we obtain

$$-1 = -WG + ZG + wG = \mathcal{S}[G]G + ZG + wG - \underline{WG}, \tag{5.3}$$

where we introduced the *self-renormalisation*, denoted by underlining, of a random variable of the form $Wf(W)$ for some regular function f as

$$\underline{Wf(W)} := Wf(W) - \widetilde{\mathbf{E}}\widetilde{W}(\partial_{\widetilde{W}} f)(W), \qquad \widetilde{W} = \begin{pmatrix} 0 & \widetilde{X} \\ \widetilde{X}^* & 0 \end{pmatrix}, \quad \widetilde{X} \sim \mathrm{Gin}_{\mathbf{C}}, \tag{5.4}$$

with $\widetilde{X}$ independent of X. The choice of defining the self-renormalisation in terms of the complex rather than real Ginibre ensemble has the consequence that an additional error term needs to be estimated. For real Ginibre we have

$$\mathbf{E}\, WG = -\mathbf{E}\, \mathcal{S}[G]G - \mathbf{E}\, \mathcal{T}[G]G, \quad \mathcal{T}\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right] = \frac{1}{n} \begin{pmatrix} 0 & c^t \\ b^t & 0 \end{pmatrix},$$

but the renormalisation comprises only the $\mathcal{S}[G]$ term, i.e.

$$\underline{WG} = WG + \mathbf{E}\, \mathcal{S}[G]G,$$

thus the $\mathcal{T}$-term needs to be estimated. By the Ward identity $GG^* = G^*G = \eta^{-1}\Im G$ it follows that

$$|\langle \boldsymbol{x}, \mathcal{T}[G]G\boldsymbol{y} \rangle| \le \frac{1}{n}\sqrt{\langle \boldsymbol{x}, GG^*\boldsymbol{x} \rangle}\sqrt{\langle \boldsymbol{y}, G^*G\boldsymbol{y} \rangle} = \frac{1}{n\eta}\sqrt{\langle \boldsymbol{x}, \Im G\boldsymbol{x} \rangle}\sqrt{\langle \boldsymbol{y}, \Im G\boldsymbol{y} \rangle} \prec \frac{\Lambda + \rho}{n\eta}, \tag{5.5}$$

where $\rho := \pi^{-1}\Im m$ from (3.3). By [29, Theorem 4.1] it follows that

$$|\langle \boldsymbol{x}, (WG + \mathcal{S}[G]G + \mathcal{T}[G]G)\boldsymbol{y}\rangle| \prec \sqrt{\frac{\rho+\Lambda}{n\eta}}, \quad |\langle A(WG + \mathcal{S}[G]G + \mathcal{T}[G]G)\rangle| \prec \frac{\rho+\Lambda}{n\eta}$$

and therefore, together with the bound (5.5) on the $\mathcal{T}$-term we obtain

$$|\langle \boldsymbol{x}, \underline{WG}\boldsymbol{y}\rangle| \prec \sqrt{\frac{\rho+\Lambda}{n\eta}}, \quad |\langle A\underline{WG}\rangle| \prec \frac{\rho+\Lambda}{n\eta}. \tag{5.6}$$

We now consider the *stability operator* $\mathcal{B} := 1 - M\mathcal{S}[\cdot]M$ which expresses the stability of (5.1) against small perturbations. Since $\mathcal{S}$ only depends on the four block traces of the input matrix, and M is a multiple of the identity matrix in each block, the operator $\mathcal{B}$ can be understood as an operator acting on 2×2 matrices after taking a *partial trace*. Henceforth for all practical purposes we may identify $\mathcal{B}$ with this four dimensional operator. Written as a 4×4 matrix, it is given by

$$\mathcal{B} = \begin{pmatrix} B_1 & 0 \\ B_2 & 1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1-u^2|z|^2 & -m^2 \\ -m^2 & 1-u^2|z|^2 \end{pmatrix}, \quad B_2 = \begin{pmatrix} muz & muz \\ mu\overline{z} & mu\overline{z} \end{pmatrix}, \tag{5.7}$$

with m, u defined in (3.2)–(3.4). Here the rows and columns of $\mathcal{B}$ are ordered in such a way that 2×2 matrices are mapped to vectors as in

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \begin{pmatrix} a \\ d \\ b \\ c \end{pmatrix}.$$

We first record some spectral properties of $\mathcal{B}$ in the following lemma, the proof of which we defer to the end of the section. Note that $\mathcal{B}^*$ refers to the adjoint of $\mathcal{B}$ with respect to the scalar product $\langle A, B\rangle = (2n)^{-1}\mathrm{Tr}A^*B$, for any deterministic matrices $A, B \in \mathbf{C}^{2n\times 2n}$.

Lemma 5.2. *Let $w \in \mathbf{H}$, $z \in \mathbf{C}$ be bounded spectral parameters, $|w| + |z| \lesssim 1$. Then the operator $\mathcal{B}$ has the trivial eigenvalues 1 with multiplicity 2, and furthermore has two non-trivial eigenvalues, and left and right eigenvectors*

$$\mathcal{B}[E_-] = (1 + m^2 - u^2|z|^2)E_- \quad \mathcal{B}^*[E_-] = \overline{(1 + m^2 - u^2|z|^2)}E_-,$$
$$\mathcal{B}[V_r] = (1 - m^2 - u^2|z|^2)V_r, \quad \mathcal{B}^*[V_l] = \overline{(1 - m^2 - u^2|z|^2)}V_l,$$

where $E_- := (E_1 - E_2)/\sqrt{2}$ and

$$E_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_2 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad V_r := \begin{pmatrix} m^2 + u^2|z|^2 & -2muz \\ -2mu\overline{z} & m^2 + u^2|z|^2 \end{pmatrix}, \quad V_l := \frac{1}{\overline{\langle V_r\rangle}}. \tag{5.8}$$

Moreover, for the second non-trivial eigenvalue we have the lower bound

$$|1 - m^2 - u^2|z|^2| \gtrsim \begin{cases} \Im m, & |1-|z|| \ge \epsilon, \\ (\Im m)^2, & |1-|z|| < \epsilon. \end{cases} \tag{5.9}$$

Corresponding to the two non-trivial eigenvalues of $\mathcal{B}$ we define the *spectral projections*

$$\mathcal{P}_* := \langle E_-, \cdot\rangle E_-, \quad \mathcal{P} := \langle V_l, \cdot\rangle V_r, \quad \mathcal{Q}_* := 1 - \mathcal{P}_*, \quad \mathcal{Q} := 1 - \mathcal{P}_* - \mathcal{P}.$$

From (5.1) and (5.3) it follows that

$$\mathcal{B}[G - M] = M\mathcal{S}[G-M](G-M) - M\underline{WG}. \tag{5.10}$$

We now distinguish the two cases $\rho \sim 1$ and $\rho \ll 1$. In the former we obtain

$$\|\mathcal{Q}_*\mathcal{B}^{-1}\|_{\|\cdot\|\to\|\cdot\|} \lesssim \frac{1}{|1-m^2-u^2|z|^2|} \lesssim 1 \tag{5.11}$$

by (5.9). Since $\langle E_-, G\rangle = \langle E_-, M\rangle = 0$ by block symmetry, it follows that

$$G - M = \mathcal{Q}_*[G-M] = \mathcal{Q}_*\mathcal{B}^{-1}\mathcal{B}[G-M]$$

and thus

$$\begin{aligned}\langle \boldsymbol{x}, (G-M)\boldsymbol{y}\rangle &= \operatorname{Tr}\left[(\mathcal{Q}_*\mathcal{B}^{-1})^*[\boldsymbol{x}\boldsymbol{y}^*]\right]^* \mathcal{B}[G-M] \\ &= \sum_{i=1}^4 \langle \boldsymbol{x}_i, (M\mathcal{S}[G-M](G-M) - M\underline{WG})\boldsymbol{y}_i\rangle \\ &= \mathcal{O}_\prec\left(\xi\Lambda + \sqrt{\frac{\rho+\Lambda}{n\eta}}\right),\end{aligned} \tag{5.12a}$$

where we used that the image of $\boldsymbol{x}\boldsymbol{y}^*$ under $(\mathcal{Q}_*\mathcal{B}^{-1})^*$ is of rank at most 4, hence it can be written as $\sum_{i=1}^4 \boldsymbol{x}_i\boldsymbol{y}_i^*$ with vectors of bounded norm. Similarly, for general matrices A we find

$$\begin{aligned}\langle A(G-M)\rangle &= \langle\left[(\mathcal{Q}\mathcal{B}^{-1})^*[A^*]\right]^* \mathcal{B}[G-M]\rangle \\ &= \langle\left[(\mathcal{Q}_*\mathcal{B}^{-1})^*[A^*]\right]^* (M\mathcal{S}[G-M](G-M) - M\underline{WG})\rangle \\ &= \mathcal{O}_\prec\left(\frac{\rho+\Lambda}{n\eta} + \xi^2\right).\end{aligned} \tag{5.12b}$$

In the complementary case $\rho \ll 1$ we similarly decompose

$$G - M = \mathcal{P}[G-M] + \mathcal{P}_*[G-M] + \mathcal{Q}[G-M] = \theta V_r + \mathcal{Q}[G-M], \quad \theta := \langle V_l, G-M\rangle. \tag{5.13}$$

Now we apply $\mathcal{B}$ to both sides of (5.13) and take the inner product with V_l to obtain

$$\langle V_l, \mathcal{B}[G-M]\rangle = (1-m^2-u^2|z|^2)\theta + \langle V_l, \mathcal{B}\mathcal{Q}[G-M]\rangle \tag{5.14}$$

from (5.10). For the spectral projection $\mathcal{Q}$ we find

$$\mathcal{B}^{-1}\mathcal{Q} = \mathcal{Q}\mathcal{B}^{-1} = \begin{pmatrix} 0 & 0 \\ B_3 & 1\end{pmatrix}, \quad B_3 = \frac{mu}{m^2+u^2|z|^2}\begin{pmatrix} z & z \\ \overline{z} & \overline{z}\end{pmatrix}. \tag{5.15}$$

Thus it follows that

$$\|\mathcal{B}^{-1}\mathcal{Q}\|_{\|\cdot\|\to\|\cdot\|} \lesssim \frac{|muz|}{|m^2+u^2|z|^2|} \lesssim 1 \tag{5.16}$$

since in the regime $\rho \ll 1$ we have $|1-m^2-u^2|z|^2| \ll 1$ due to $|\Im u^2| \ll 1$ which follows by a simple calculation.

By using (5.10) in (5.14) it follows that

$$|\theta| \prec \frac{1}{\rho}\left(\frac{\rho+\Lambda}{n\eta} + \xi^2\right) \tag{5.17}$$

from (5.2), (5.6) since, due to $||z|-1| \gtrsim \epsilon$, we have $|1-m^2-u^2|z|^2| \geq \rho$ according to (5.9). For general vectors $\boldsymbol{x}, \boldsymbol{y}$ it follows from (5.13), (5.17) and inserting $1 = \mathcal{B}^{-1}\mathcal{B}$ similarly to (5.12) that

$$\begin{aligned}\langle \boldsymbol{x}, (G-M)\boldsymbol{y}\rangle &= \mathcal{O}_\prec\left(\frac{\rho+\Lambda}{\rho n\eta} + \frac{\xi^2}{\rho}\right) + \left\langle \left[(\mathcal{Q}\mathcal{B}^{-1})^*[\boldsymbol{x}\boldsymbol{y}^*]\right]^* \mathcal{B}[G-M]\right\rangle \\ &= \mathcal{O}_\prec\left(\frac{\rho+\Lambda}{\rho n\eta} + \frac{\xi^2}{\rho}\right) + \sum_{i=1}^4 \langle \boldsymbol{x}_i, (M\mathcal{S}[G-M](G-M) - M\underline{WG})\boldsymbol{y}_i\rangle \\ &= \mathcal{O}_\prec\left(\frac{\rho+\Lambda}{\rho n\eta} + \frac{\xi^2}{\rho} + \xi\Lambda + \sqrt{\frac{\rho+\Lambda}{n\eta}}\right),\end{aligned} \tag{5.18a}$$

and

$$\begin{aligned}\langle A(G-M)\rangle &= \mathcal{O}_\prec\left(\frac{\rho+\Lambda}{\rho n\eta} + \frac{\xi^2}{\rho}\right) + \left\langle \left[(\mathcal{Q}\mathcal{B}^{-1})^*[A^*]\right]^* \mathcal{B}[G-M]\right\rangle \\ &= \mathcal{O}_\prec\left(\frac{\rho+\Lambda}{\rho n\eta} + \frac{\xi^2}{\rho}\right) + \left\langle \left[(\mathcal{Q}\mathcal{B}^{-1})^*[A^*]\right]^* (M\mathcal{S}[G-M](G-M) - M\underline{WG})\right\rangle \\ &= \mathcal{O}_\prec\left(\frac{\rho+\Lambda}{\rho n\eta} + \frac{\xi^2}{\rho}\right).\end{aligned} \tag{5.18b}$$

By using the bounds in (5.12) and (5.18) in the two complementary regimes we improve the input bound in (5.2). We can iterate this procedure and obtain

$$|\langle \boldsymbol{x}, (G-M)\boldsymbol{y}\rangle| \prec \frac{1}{n\eta} + \sqrt{\frac{\rho}{n\eta}}, \qquad |\langle A(G-M)\rangle| \prec \frac{1}{n\eta}. \tag{5.19}$$

In order to make sure the iteration yields an improvement one needs an priori bound on ξ of the form $\xi \ll 1$ since otherwise ξ^2 is difficult to control. For large η such an a priori bound is trivially available which can then be iteratively bootstrapped by monotonicity down to the optimal $\eta \gg n^{-1}$. For details on this standard argument the reader is referred to e.g. [4, Section 3.3]. Then the local law for any $\eta > 0$ readily follows by exactly the same argument as in [23, Appendix A]. This completes the proof of Theorem 3.1. □

Proof of Lemma 5.2. The fact that $\mathcal{B}$ has the eigenvalue 1 with multiplicity 2, and the claimed form of the remaining two eigenvalues and corresponding eigenvectors can be checked by direct computations. Taking the imaginary part of (3.2) we have

$$(1 - |m|^2 - |u|^2|z|^2)\Im m = (|m|^2 + |u|^2|z|^2)\Im w, \tag{5.20}$$

which implies

$$|m|^2 + |u|^2|z|^2 < 1, \qquad \lim_{\Im w\to 0}(|m|^2 + |u|^2|z|^2) = 1, \quad \Re w \in \overline{\operatorname{supp}\rho} \tag{5.21}$$

as $\Im m$ and $\Im w$ have the same sign. Here $\operatorname{supp}\rho$ should be understood as the support of the self-consistent density of states, as defined in (3.3), restricted to the real axis. The second bound in (5.9) then follows from (5.21) and

$$|1 - m^2 - u^2|z|^2| \ge \Re(1 - m^2 - u^2|z|^2) = 1 - (\Re m)^2 + (\Im m)^2 - \Re(u^2)|z|^2 \gtrsim (\Im m)^2. \tag{5.22}$$

The bound (5.22) can be improved in the case $\rho \ll 1$ if w is near a *regular edge* of ρ, i.e. where ρ locally vanishes as a square-root. According to [24, Eq. (15b)] the density

ρ has two regular edges $\pm\sqrt{\mathfrak{e}_+}$ if $|z| \le 1-\epsilon$, and four regular edges in $\pm\sqrt{\mathfrak{e}_+}, \pm\sqrt{\mathfrak{e}_-}$ for $|z| \ge 1+\epsilon$, where

$$\mathfrak{e}_\pm := \frac{8(1-|z|^2)^2 \pm (1+8|z|^2)^{3/2} - 36(1-|z|^2) + 27}{8|z|^2} \gtrsim 1.$$

By the explicit form of $\mathfrak{e}_\pm$ it follows that $\mathfrak{e}_\pm \gtrsim 1$ whenever $|1-|z|| \ge \epsilon$. In contrast, if $|z| = 1$, then ρ has a *cusp* singularity in 0 where it locally vanishes like a cubic root. Near a regular edge we have $\Im m \lesssim \sqrt{\Im w}$, and therefore from (5.20)

$$(1-|m|^2 - |u|^2|z|^2) \gtrsim \sqrt{\Im w} \gtrsim \Im m$$

and it follows that

$$|1-m^2-u^2|z|^2| \gtrsim \Im m,$$

proving also the first inequality in (5.9). □

6 CLT for resolvents: proof of Proposition 3.3

The goal of this section is to prove the CLT for resolvents, as stated in Proposition 3.3. The proof is very similar to [22, Section 6] and we focus on the differences specific to the real case. Within this section we consider resolvents $G_1, \dots, G_p$ with $G_i = G^{z_i}(\mathrm{i}\eta_i)$ and $\eta_i \ge n^{-1}$. As a first step we recall the leading-order approximation of $G = G_i$

$$\langle G - M \rangle = -\langle \underline{WG}A \rangle + \mathcal{O}_\prec\left(\frac{1}{|\beta|(n\eta)^2}\right), \quad A := (\mathcal{B}^*)^{-1}[1]^* M \tag{6.1}$$

from [22, Eq. (6.9)], where the stability operator $\mathcal{B}$ has been defined in (5.7). Here β is the eigenvalue of $\mathcal{B}$ with eigenvector $(1,1,0,0)$ and is bounded by (see [22, Eq. (6.8b)])

$$|\beta| \gtrsim |1-|z|| + \eta^{2/3}. \tag{6.2}$$

One important input for the proof of Proposition 3.3 is a lower bound on the eigenvalues of the stability operator $\widehat{\mathcal{B}}$, defined in (3.14), the proof of which we defer to the end of the section. Note that the two-body stability operator $\widehat{\mathcal{B}}$ and its eigenvalues $\widehat{\beta}, \widehat{\beta}_*$ are consistently decorated by hats $\widehat{(\cdot)}$ to distinguish them from their one-body analogues $\mathcal{B}, \beta$. We will consistently equip $\mathcal{B}, \widehat{\mathcal{B}}$ and their eigenvalues, $\beta, \widehat{\beta}, \widehat{\beta}_*$ with indices when instead of M they are defined with the help of $M_i = M^{z_i}(w_i)$; e.g. $\widehat{\beta}_*^{1i}$ is the lowest eigenvalue of $\widehat{\mathcal{B}}_{1i} = \widehat{\mathcal{B}}(z_1, z_i, w_1, w_i)$ defined analogously to (3.14).

Lemma 6.1. *For $z_1, z_2 \in \mathbf{C}$, $w_1, w_2 \in \mathbf{C} \setminus \mathbf{R}$ such that $|z_i|, |w_i| \lesssim 1$ the two non-trivial eigenvalues $\widehat{\beta}, \widehat{\beta}_*$ of $\widehat{\mathcal{B}}$ satisfy*

$$\min\{\Re\widehat{\beta}, \Re\widehat{\beta}_*\} \gtrsim |z_1 - z_2|^2 + \min\{|w_1 + \overline{w_2}|, |w_1 - \overline{w_2}|\}^2 + |\Im w_1| + |\Im w_2| \tag{6.3}$$

Proof of Proposition 3.3. The proof of Proposition 3.3 goes in two steps. First, we use (6.1) and a cumulant expansion in order to prove the asymptotic representation of the expectation in (3.11). In the second step we then turn to the computation of higher moments and establish an asymptotic Wick theorem in the form of (3.8).

We use the notation Δ^{ab} for the matrix $(\Delta^{ab})_{cd} = \delta_{ac}\delta_{bd}$ and decompose $W = \sum_{ab} w_{ab}\Delta^{ab}$. For each a, b we then perform a cumulant expansion and obtain

$$\mathbf{E}\langle \underline{WG}A \rangle = -\frac{1}{n}\sum_{ab}{}' \mathbf{E}\langle \Delta^{ab} G \Delta^{ab} GA \rangle + \sum_{k\ge 2}\sum_{ab}\sum_{\boldsymbol{\alpha}\in\{ab,ba\}^k} \frac{\kappa(ab,\boldsymbol{\alpha})}{k!}\mathbf{E}\,\partial_{\boldsymbol{\alpha}}\langle \Delta^{ab} GA \rangle, \tag{6.4}$$

which has an additional term compared to the complex case [22, Eq. (6.11)] since the self-renormalisation (5.4) was chosen such that it only takes the $\kappa(ab, ba) = 1$ and not the $\kappa(ab,ab)=1$ cumulant into account. Here $\kappa(ab,cd,ef,\dots)$ denotes the joint cumulant of the random variables $w_{ab},w_{cd},w_{ef},\dots$, and we denote partial derivatives by $\partial_{\boldsymbol\alpha}:=\partial_{w_{\alpha_1}}\cdots\partial_{w_{\alpha_k}}$ for tuples $\boldsymbol\alpha=(\alpha_1,\dots,\alpha_k)$, with $\alpha_i\in[n]\times[n]$. In (6.4) we introduced the notation

$$\sum_{ab}{}' := \sum_{a\le n}\sum_{b>n}+\sum_{a>n}\sum_{b\le n}.$$

We note that by Assumption 2.1 the cumulants $\kappa(\alpha_1,\dots,\alpha_k)$ satisfy the scaling

$$|\kappa(\alpha_1,\dots,\alpha_k)|\lesssim n^{-k/2}. \tag{6.5}$$

For the second term in (6.4) we find exactly as in [22, Eqs. (6.13)–(6.10)] that

$$\sum_{k\ge2}\sum_{ab}\sum_{\boldsymbol\alpha\in\{ab,ba\}^k}\frac{\kappa(ab,\boldsymbol\alpha)}{k!}\partial_{\boldsymbol\alpha}\langle\Delta^{ab}GA\rangle=\frac{\mathrm{i}\kappa_4}{4n}\partial_\eta(m^4)+\mathcal{O}_\prec\left(\frac{1}{|\beta|}\Big(\frac{1}{n^{3/2}(1+\eta)}+\frac{1}{(n\eta)^2}\Big)\right). \tag{6.6}$$

For the first term in (6.4), which is new compared to [22, Eq. (6.11)], we rewrite

$$\frac1n\sum_{ab}{}'\langle\Delta^{ab}G\Delta^{ab}GA\rangle=\frac1n\langle GAEG^tE'\rangle=\frac1n\langle G^zAEG^{\overline z}E'\rangle,$$

where we used that $(G^z)^t=G^{\overline z}$, and the convention that formulas containing (E,E') are understood so that the matrices E,E' are summed over the assignments $(E,E')=(E_1,E_2)$ and $(E,E')=(E_2,E_1)$ with

$$E_1:=\begin{pmatrix}1&0\\0&0\end{pmatrix}\qquad E_2:=\begin{pmatrix}0&0\\0&1\end{pmatrix}.$$

From the local law [22, Theorem 5.2] for products of resolvents and the bound on $|\widehat\beta_*|$ from Lemma 6.1 we can thus conclude

$$\begin{aligned}\frac1n\sum_{ab}{}'\langle\Delta^{ab}G\Delta^{ab}GA\rangle&=\frac1n\langle M_{AE}^{z,\overline z}E'\rangle+\mathcal{O}_\prec\left(\frac{1}{|z-\overline z|^2}\frac{1}{(n\eta)^2}\right)\\&=\frac mn\frac{m^4+m^2u^2|z|^2-2u^4|z|^4+2u^2(x^2-y^2)}{(1-m^2-u^2|z|^2)(1+u^4|z|^4-m^4-2u^2(x^2-y^2))}\\&\quad+\mathcal{O}_\prec\left(\frac{1}{|z-\overline z|^2}\frac{1}{(n\eta)^2}\right),\end{aligned} \tag{6.7}$$

where $z=x+\mathrm{i}y$, and the second step follows by explicitly computing the inverse

$$M_{AE}^{z,\overline z}=(1-M^z\mathcal{S}[\cdot]M^{\overline z})^{-1}[M^zAEM^{\overline z}]$$

in terms of the entries of M, noting that $m^z=m^{\overline z}$ and $u^z=u^{\overline z}$. Then, using the definition $v:=-\mathrm{i}m>0$ and that

$$|z|^2u^2+v^2=u,\qquad u'=-\frac{2uv}{1+u-|z|^2u^2},\qquad v^2=u(1-|z|^2u)$$

we obtain

$$\begin{aligned}&m\frac{m^4+m^2u^2|z|^2-2u^4|z|^4+2u^2(x^2-y^2)}{(1-m^2-u^2|z|^2)(1+u^4|z|^4-m^4-2u^2(x^2-y^2))}\\&\quad=-\frac{\mathrm{i}u'}{2}\frac{u-3|z|^2u^2+2u(x^2-y^2)}{1-u^2+2u^3|z|^2-2u^2(x^2-y^2)}.\end{aligned} \tag{6.8}$$

Now (3.11) follows from combining (6.1) and (6.4)–(6.8).

We now turn to the computation of higher moments for which we recall from (6.1) and (3.11) that

$$\begin{aligned}\prod_{i\in[p]}\langle G_i-\mathbf{E}\,G_i\rangle&=\prod_{i\in[p]}\langle G_i-M_i-\mathcal{E}_i\rangle+\mathcal{O}_\prec\Big(\frac{\psi}{n\eta}\Big)\\&=\prod_{i\in[p]}\langle -\underline{WG_i}A_i-\mathcal{E}_i\rangle+\mathcal{O}_\prec\Big(\frac{\psi}{n\eta}\Big)\end{aligned}\tag{6.9}$$

with A_i as in (6.1) and $\mathcal{E}_i$ as in (3.12), and

$$\psi:=\prod_i\Big(\frac{1}{|\beta_i|}+\frac{1}{(\Im z_i)^2}\Big)\frac{1}{n\eta_i}\le\prod_i\Big(\frac{1}{|1-|z_i||}+\frac{1}{(\Im z_i)^2}\Big)\frac{1}{n\eta_i}\tag{6.10}$$

with the bound on β_i from (6.2). We begin with the cumulant expansion of $\underline{WG_1}$ to obtain

$$\begin{aligned}&\mathbf{E}\prod_{i\in[p]}\langle -\underline{WG_i}A_i-\mathcal{E}_i\rangle\\&=\mathbf{E}\Big(\frac{1}{n}\sum_{ab}{}'\langle\Delta^{ab}G_1\Delta^{ab}G_1A_1\rangle-\langle\mathcal{E}_1\rangle\Big)\prod_{i\neq1}\langle-\underline{WG_i}A_i-\mathcal{E}_i\rangle\\&\quad+\sum_{i\neq1}\mathbf{E}\,\widehat{\mathbf{E}}\langle\widehat{W}G_1A_1\rangle\langle\widehat{W}G_iA_i-\underline{WG_i\widehat{W}G_i}A_i\rangle\prod_{j\neq1,i}\langle-\underline{WG_j}A_j-\mathcal{E}_j\rangle\\&\quad+\sum_{k\ge2}\sum_{ab}\sum_{\boldsymbol{\alpha}\in\{ab,ba\}^k}\frac{\kappa(ba,\boldsymbol{\alpha})}{k!}\mathbf{E}\,\partial_{\boldsymbol{\alpha}}\Big[\langle-\Delta^{ba}G_1A_1\rangle\prod_{i\neq1}\langle-\underline{WG_i}A_i-\mathcal{E}_i\rangle\Big],\end{aligned}\tag{6.11}$$

where, compared to [22, Eq. (6.17)], the first line on the rhs. has an additional term specific to the real case, and $\widehat{W}$, as opposed to $\widetilde{W}$ in (5.4), is the Hermitisation of an independent real Ginibre matrix $\widehat{X}$ with expectation $\widehat{\mathbf{E}}$. The expansion of the third line on the rhs. of (6.11) is completely analogous to [22] since for cumulants of degree at least three nothing specific to the complex case was used. Therefore we obtain, from combining[2] [22, Eqs. (6.26), (6.29)], that

$$\begin{aligned}&\sum_{k\ge2}\sum_{ab}\sum_{\boldsymbol{\alpha}\in\{ab,ba\}^k}\frac{\kappa(ba,\boldsymbol{\alpha})}{k!}\mathbf{E}\,\partial_{\boldsymbol{\alpha}}\Big[\langle-\Delta^{ba}G_1A_1\rangle\prod_{i\neq1}\langle-\underline{WG_i}A_i-\mathcal{E}_i\rangle\Big]\\&=-\frac{\mathrm{i}\kappa_4}{4n}\partial_{\eta_1}(m_1^4)\,\mathbf{E}\prod_{i\neq1}\langle-\underline{WG_i}A_i-\mathcal{E}_i\rangle+\sum_{i\neq1}\frac{\kappa_4U_1U_i}{2n^2}\,\mathbf{E}\prod_{j\neq1,i}\langle-\underline{WG_j}A_j-\mathcal{E}_j\rangle+\mathcal{O}\Big(\frac{n^\xi\psi}{\sqrt{n\eta_*}}\Big),\end{aligned}\tag{6.12}$$

where

$$U_i:=-\sqrt{2}\langle M_i\rangle\langle M_iA_i\rangle=\frac{\mathrm{i}}{\sqrt{2}}\partial_{\eta_i}m_i^2.$$

Recall the definition of $\mathcal{E}_i$ in (3.12), then using (6.7)–(6.8) and (6.12) in (6.11) we thus have

$$\begin{aligned}&\mathbf{E}\prod_{i\in[p]}\langle-\underline{WG_i}A_i-\mathcal{E}_i\rangle\\&=\sum_{i\neq1}\mathbf{E}\Big(\frac{\kappa_4U_1U_i}{2n^2}+\widehat{\mathbf{E}}\langle\widehat{W}G_1A_1\rangle\langle\widehat{W}G_iA_i-\underline{WG_i\widehat{W}G_i}A_i\rangle\Big)\prod_{j\neq1,i}\langle-\underline{WG_j}A_j-\mathcal{E}_j\rangle\\&\qquad+\mathcal{O}\Big(\frac{n^\xi\psi}{\sqrt{n\eta_*}}\Big).\end{aligned}\tag{6.13}$$

[2] Note that the definition of $\mathcal{E}$ in [22, Eq. (6.8c)] differs from (3.12) in the present paper.

It remains to consider the variance term in (6.13) for which we use the identity

$$\widehat{\mathbf{E}}\langle\widehat{W}A\rangle\langle\widehat{W}B\rangle=\frac{1}{2n^2}\langle AE(B+B^t)E'\rangle=\frac{\langle AE_1(B+B^t)E_2\rangle+\langle AE_2(B+B^t)E_1\rangle}{2n^2}\tag{6.14}$$

in order to compute

$$\begin{aligned}&\widehat{\mathbf{E}}\langle\widehat{W}G_1A_1\rangle\langle\widehat{W}G_iA_i-\underline{WG_i\widehat{W}G_i}A_i\rangle\\&\quad=\frac{1}{2n^2}\langle G_1A_1E(G_iA_i+A_i^tG_i^t)E'-G_1A_1E(\underline{G_iA_iWG_i}+\underline{G_i^tWA_i^tG_i^t})E'\rangle,\end{aligned}\tag{6.15}$$

where, compared to [22, Eqs. (6.18)–(6.19)], there is an additional term with transposition. Here the self-renormalisation e.g. in $\underline{G_iA_iWG_i}$ is defined analogously to (5.4) with the derivative acting on both G_i's. For the second term in (6.15) we identify the leading order contribution using the fact that $G^z(w)^t=G^{\overline{z}}(w)$ and denoting $G_{\overline{i}}=G^{\overline{z_i}}(\mathrm{i}\eta_i)$ as

$$\begin{aligned}\langle G_1A_1E(\underline{G_iA_iWG_i}+\underline{G_i^tWA_i^tG_{\overline{i}}^t})E'\rangle&=-\langle G_1\mathcal{S}[G_1A_1EG_iA_i]G_iE'+G_1\mathcal{S}[G_1A_1EG_{\overline{i}}]A_i^tG_{\overline{i}}E'\rangle\\&\quad+\langle\underline{G_1A_1EG_iA_iWG_iE'}+\underline{G_1A_1EG_{\overline{i}}WA_i^tG_{\overline{i}}E'}\rangle\end{aligned}\tag{6.16}$$

for which we use the local law from Theorem 3.5 to conclude that the main terms in (6.15) are

$$\begin{aligned}&\langle G_1A_1E(G_iA_i+A_i^tG_{\overline{i}})E'+G_1\mathcal{S}[G_1A_1EG_iA_i]G_iE'+G_1\mathcal{S}[G_1A_1EG_{\overline{i}}]A_i^tG_{\overline{i}}E'\rangle\\&\quad=\widehat{V}_{1,i}+\mathcal{O}_\prec\left(\frac{1}{n|\widehat{\beta}_*^{1i}|^2\eta_*^{1i}|\eta_1\eta_i|^{1/2}}+\frac{1}{n^2|\widehat{\beta}_*^{1i}|^2(\eta_*^{1i})^2|\eta_1\eta_i|}\right)\\&\widehat{V}_{1,i}:=\langle M_{A_1E}^{z_1,z_i}A_iE'+M_{A_1EA_i^t}^{z_1,\overline{z_i}}E'+\mathcal{S}[M_{A_1E}^{z_1,z_i}A_i]M_{E'}^{z_i,z_1}+\mathcal{S}[M_{A_1E}^{z_1,\overline{z_i}}]A_i^tM_{E'}^{\overline{z_i},z_1}\rangle,\end{aligned}\tag{6.17}$$

where $|\widehat{\beta}_*^{1i}|\gtrsim|z_1-z_i|^2$ from Lemma 6.1, and $\eta_*^{1i}:=\min\{\eta_1,\eta_i\}$. By an explicit computation similarly to [22, Eq. (6.23)] it follows that

$$\widehat{V}_{1,i}=V(z_1,z_i,\eta_1,\eta_i)+V(z_1,\overline{z_i},\eta_1,\eta_i)\tag{6.18}$$

with V being exactly as in the complex case, i.e. as in (3.10). For the error term in (6.16) we claim that

$$\mathbf{E}|\langle\underline{G_1A_1EG_iA_iWG_iE'}\rangle|^2+\mathbf{E}|\langle\underline{G_1A_1EG_{\overline{i}}WA_i^tG_{\overline{i}}E'}\rangle|^2\lesssim\Big(\frac{1}{n\eta_1\eta_i\eta_*^{1i}}\Big)^2.\tag{6.19}$$

The CLT for resolvents, as stated in (3.8) follows from inserting (6.15)–(6.19) into (6.13), and iteration of (6.13) for the remaining product.

In order to conclude the proof of Proposition 3.3 it remains to prove (6.19). Introduce the shorthand notation G_{i1i} for generic finite sums of products of G_i,G_1,G_i (or $G_{\overline{i}}$ in place of G_i) with arbitrary bounded deterministic matrices, e.g. $G_iE'G_1A_1EG_iA_i$ appearing in the first term in (6.19). We will prove the more general claim

$$\mathbf{E}|\langle\underline{WG_{i1i}}\rangle|^2\lesssim\Big(\frac{1}{n\eta_1\eta_i\eta_*^{1i}}\Big)^2.\tag{6.20}$$

The proof is similar to [22, Eq. (6.32)]. Therefore we focus on the differences. In the cumulant expansion of (6.20) there is an additional term compared to [22, Eq. (6.33)] given by

$$\begin{aligned}&\frac{1}{n}\sum_{ab}{}'\,\mathbf{E}\langle\Delta^{ab}G_i\Delta^{ab}G_{i1i}+\Delta^{ab}G_{i1}\Delta^{ab}G_{1i}+\Delta^{ab}G_{i1i}\Delta^{ab}G_i\rangle\langle\underline{WG_{i1i}}\rangle\\&\quad=\frac{1}{n}\mathbf{E}\langle G_{1iii}+G_{i1i1}\rangle\langle\underline{WG_{i1i}}\rangle,\end{aligned}\tag{6.21}$$

where we combined two terms of type G_{1iii} into one since in our convention G_{1iii} is a short-hand notation for generic sums of products. We now perform another cumulant expansion of (6.21) to obtain

$$\begin{aligned} &\frac{1}{n}\mathbf{E}\langle G_{1iii}+G_{i1i1}\rangle\langle \underline{WG_{i1i}}\rangle \\ &=\frac{1}{n^2}\mathbf{E}\langle G_{1iii}+G_{i1i1}\rangle^2 \\ &\quad+\frac{1}{n}\mathbf{E}\,\widetilde{\mathbf{E}}\langle\widetilde{W}(G_{1iii1}+G_{iii1i}+G_{ii1ii}+G_{i1iii}+G_{1i1i1}+G_{i1i1i})\rangle\langle\widetilde{W}G_{i1i}\rangle \\ &\quad+\sum_{k\ge 2}\mathcal{O}\Big(\frac{1}{n^{(k+3)/2}}\Big)\sum_{ab}{}'\sum_{\boldsymbol{\alpha}\in\{ab,ba\}^k}\mathbf{E}\,\partial_{\boldsymbol{\alpha}}\Big[\langle G_{1iii}+G_{i1i1}\rangle\langle\Delta^{ab}G_{i1i}\rangle\Big], \end{aligned} \tag{6.22}$$

where the first line on the rhs. corresponds to the term where the remaining W acts on G_{i1i} within its own trace as in (6.21), and in the last line we used the scaling bound (6.5) for κ. In order to estimate (6.21) we recall [22, Lemma 5.8].

Lemma 6.2. *Let $w_1, w_2, \dots, z_1, z_2, \dots,$ denote arbitrary spectral parameters with $\eta_i = \Im w_i > 0$. Let $G_j = G^{z_j}(w_j)$, then with $G_{j_1\dots j_k}$ we denote generic products of resolvents $G_{j_1}, \dots, G_{j_k}$, or their adjoints/transpositions (in that order, each G_{j_i} appears exactly once) with bounded deterministic matrices in between, e.g. $G_{1i1} = A_1G_1A_2G_iA_3G_1A_4$.*

(i) For $j_1, \dots j_k$ we have the isotropic bound

$$|\langle \boldsymbol{x}, G_{j_1\dots j_k}\boldsymbol{y}\rangle| \prec \|\boldsymbol{x}\|\|\boldsymbol{y}\|\sqrt{\eta_{j_1}\eta_{j_k}}\Big(\prod_{n=1}^k \eta_{j_n}\Big)^{-1}. \tag{6.23a}$$

(ii) For $j_1, \dots, j_k$ and any $1 \le s < t \le k$ we have the averaged bound

$$|\langle G_{j_1\dots j_k}\rangle| \prec \sqrt{\eta_{j_s}\eta_{j_t}}\Big(\prod_{n=1}^k \eta_{j_n}\Big)^{-1}. \tag{6.23b}$$

Since only η_1, η_i play a role within the proof of (6.19), we drop the indices from η_*^{1i} and use the notation $\eta_* = \eta_*^{1i}$. For the first term in (6.22) we use (6.23b) to obtain

$$\frac{1}{n^2}|\langle G_{1iii}+G_{i1i1}\rangle|^2 \prec \frac{1}{n^2\eta_1^2\eta_i^2\eta_*^2}. \tag{6.24}$$

Similarly for the second term we use (6.14) and again (6.23b) to bound it by

$$\begin{aligned} &\frac{1}{n}|\widetilde{\mathbf{E}}\langle\widetilde{W}(G_{1iii1}+G_{iii1i}+G_{ii1ii}+G_{i1iii}+G_{1i1i1}+G_{i1i1i})\rangle\langle\widetilde{W}G_{i1i}\rangle| \\ &\qquad\prec \frac{1}{n^3\eta_1^2\eta_i^2\eta_*^3} \le \frac{1}{n^2\eta_1^2\eta_i^2\eta_*^2} \end{aligned} \tag{6.25}$$

since $\eta_* \ge 1/n$. Finally, for the last term of (6.22) we estimate

$$\bigg|\mathcal{O}\Big(\frac{1}{n^{(k+7)/2}}\Big)\sum_{ab}{}'\sum_{c}\sum_{\boldsymbol{\alpha}}\partial_{\boldsymbol{\alpha}}\Big[(G_{1iii}+G_{i1i1})_{cc}(G_{i1i})_{ba}\Big]\bigg| \prec \frac{1}{n^2\eta_1^2\eta_i^2\eta_*^2} \tag{6.26}$$

for any $k \ge 2$. Indeed, for $k \ge 3$ the claim (6.26) follows trivially from (6.23a) and the observation that the bound (6.23a) remains invariant under the action of derivatives. Indeed, differentiating a term like $(G_{i1i})_{ab}$ gives rise to the terms $(G_i)_{aa}(G_{i1i})_{bb}$, $(G_{i1})_{ab}(G_{1i})_{ab}, \dots$ for all of which (6.23a) gives the same estimate as for $(G_{i1i})_{ab}$ since the presence of an additional factor of G_1 or G_i is compensated by the fact that the same type of G appears two additional times as the first or last factor in some product. For the $k=2$ case we observe that by parity at least one factor will be off-diagonal in the sense that it has two distinct summation indices from $\{a,b,c\}$ giving rise to an additional factor of $(n\eta_*)^{-1/2}$ by summing up one of the indices with the Ward identity. For example, for the term with $(G_{1iii})_{cc}(G_{i1})_{bb}(G_{1i})_{aa}(G_i)_{ba}$ we estimate

$$\begin{aligned} n^{-9/2}\Big|\sum_{ab}{}' \sum_c (G_{1iii})_{cc}(G_{i1})_{bb}(G_{1i})_{aa}(G_i)_{ba}\Big| &\prec n^{-9/2}\frac{n}{\eta_1^{3/2}\eta_i^{7/2}}\sum_{ab}{}'|(G_i)_{ba}| \\ &\le n^{-3}\frac{1}{\eta_1^{3/2}\eta_i^{7/2}}\sum_b \sqrt{\sum_a |(G_i)_{ba}|^2} \\ &= n^{-3}\frac{1}{\eta_1^{3/2}\eta_i^{4}}\sum_b \sqrt{(\Im G_i)_{bb}} \prec \frac{1}{n^2\eta_1^{3/2}\eta_i^4}. \end{aligned}$$

Thus, in general we obtain a bound of

$$\frac{1}{n^{3/2}}\Big(\frac{1}{\eta_1^{3/2}\eta_i^{7/2}}+\frac{1}{\eta_1^{5/2}\eta_i^{5/2}}\Big)\frac{1}{\sqrt{n\eta_*}} \lesssim \frac{1}{n^2\eta_1^2\eta_i^2\eta_*^2}.$$

By combining (6.24)–(6.26) we obtain a bound of $(n\eta_1\eta_i\eta_*)^{-2}$ on the additional term (6.21). The remaining terms can be estimated as in [22, Eq. (6.32)] and we conclude the proof of (6.20) and thereby Proposition 3.3. □

Proof of Lemma 6.1. The claim (6.3) is equivalent to the claim

$$\max\{\Re\tau,\Re\tau_*\} \le 1 - c\big[|z_1-z_2|^2 + \min\{|w_1+\overline{w_2}|^2, |w_1-\overline{w_2}|^2\} + |\Im w_1| + |\Im w_2|\big], \quad c>0, \tag{6.27}$$

where τ,τ_* are the eigenvalues of the matrix

$$R := \begin{pmatrix} z_1\overline{z_2}u_1u_2 & m_1m_2 \\ m_1m_2 & \overline{z_1}z_2u_1u_2 \end{pmatrix}, \tag{6.28}$$

thus $\widehat{\beta} = 1-\tau$, $\widehat{\beta}_* = 1-\tau_*$. We first check that (6.27) holds true ineffectively, i.e. with $c=0$. We claim that

$$\max \Re \operatorname{Spec}(A) \le \lambda_{\max}\Big(\frac{A+A^*}{2}\Big) := \max \operatorname{Spec}\Big(\frac{A+A^*}{2}\Big) \tag{6.29}$$

holds for any square matrix A. Indeed, suppose that $A\boldsymbol{x} = \lambda\boldsymbol{x}$, $\|\boldsymbol{x}\|=1$ and $(A+A^*)/2 \le M$ in the sense of quadratic forms. We then compute

$$0 \ge \Big\langle \boldsymbol{x}, \Big(\frac{A+A^*}{2} - M\Big)\boldsymbol{x}\Big\rangle = \frac{\langle \boldsymbol{x}, A\boldsymbol{x}\rangle + \langle A\boldsymbol{x},\boldsymbol{x}\rangle}{2} - M = \Re\lambda - M,$$

from which (6.29) follows by choosing M to be the largest eigenvalue of $(A+A^*)/2$.

Since R is such that its entrywise real part is given by $\Re R = (R+R^*)/2$, from (6.29) we conclude the chain of inequalities

$$\max\{\Re\tau,\Re\tau_*\} \le \lambda_{\max}\begin{pmatrix} \Re(z_1\overline{z_2}u_1u_2) & \Re(m_1m_2) \\ \Re(m_1m_2) & \Re(\overline{z_1}z_2u_1u_2)\end{pmatrix} \tag{6.30a}$$

$$= (\Re u_1u_2)(\Re z_1\overline{z_2}) + \sqrt{\big(|\Im u_1u_2||\Im z_1\overline{z_2}| + |\Re m_1m_2|\big)^2 - 2|\Im u_1u_2||\Im z_1\overline{z_2}||\Re m_1m_2|} \tag{6.30b}$$

$$\le (\Re u_1u_2)(\Re z_1\overline{z_2}) + |\Im u_1u_2||\Im z_1\overline{z_2}| + |\Re m_1m_2| \tag{6.30c}$$

$$\le \Big|(\Re u_1u_2)(\Re z_1\overline{z_2}) + |\Im u_1u_2||\Im z_1\overline{z_2}|\Big| + |\Re m_1m_2| \tag{6.30d}$$

$$= \sqrt{|z_1z_2u_1u_2|^2 - \big(\Re u_1u_2|\Im z_1\overline{z_2}| - \Re z_1\overline{z_2}|\Im u_1u_2|\big)^2} + \sqrt{|m_1m_2|^2 - [\Im m_1m_2]^2} \tag{6.30e}$$

$$\le |z_1z_2u_1u_2| + |m_1m_2| \tag{6.30f}$$

$$= \sqrt{(|u_1z_1|^2 + |m_1|^2)(|u_2z_2|^2 + |m_2|^2) - (|u_1z_1m_2| - |u_2z_2m_1|)^2} \tag{6.30g}$$

$$\le \sqrt{(|m_1|^2 + |z_1u_1|^2)(|m_2|^2 + |z_2u_2|^2)} \tag{6.30h}$$

$$\le 1, \tag{6.30i}$$

where in the last step we used (5.21).

We now assume that for some $0 \le \epsilon \ll 1$ we have

$$\max\{\Re\tau,\Re\tau_*\} \ge 1 - \epsilon^2, \tag{6.31}$$

i.e. that all inequalities in (6.30a)–(6.30i) are in fact equalities up to an ϵ^2 error. The assertion (6.27) is then equivalent to

$$|z_1 - z_2| + \min\{|w_1 + \overline{w_2}|, |w_1 - \overline{w_2}|\} + \sqrt{|\Im w_1|} + \sqrt{|\Im w_2|} \lesssim \epsilon, \tag{6.32}$$

the proof of which we present now.

The fact that (6.30h)–(6.30i) is ϵ^2-saturated implies the saturation

$$|m_i|^2 + |z_iu_i|^2 = 1 + \mathcal{O}(\epsilon^2), \tag{6.33}$$

and, consequently,

$$|u_i| \sim 1. \tag{6.34}$$

Indeed, suppose that $|u_i| \ll 1$, then on the one hand since $u_i = u_i^2|z_i|^2 - m_i^2$, it follows that $|m_i| \ll 1$, while on the other hand $|1 - |m_i|^2| \ll 1$ from (6.33) which would be a contradiction. From (5.20) it follows that

$$|m_i|^2 + |u_i|^2|z_i|^2 \le 1 - c\,\Im w_i,$$

from which we conclude $|\Im w_1| + |\Im w_2| \lesssim \epsilon^2$, i.e. the bound on the last two terms in (6.32). The ϵ^2-saturation of (6.30g)–(6.30h) implies that

$$\begin{aligned}\mathcal{O}(\epsilon) = |u_1z_1m_2| - |u_2z_2m_1| &= \sqrt{1 - |m_1|^2}|m_2| - \sqrt{1 - |m_2|^2}|m_1| + \mathcal{O}(\epsilon^2) \\ &= \sqrt{1 - |u_1z_1|^2}|u_2z_2| - \sqrt{1 - |u_2z_2|^2}|u_1z_1| + \mathcal{O}(\epsilon^2).\end{aligned}$$

Thus it follows that

$$|m_1| = |m_2| + \mathcal{O}(\epsilon), \qquad |z_1u_1| = |z_2u_2| + \mathcal{O}(\epsilon). \tag{6.35}$$

In the remainder of the proof we distinguish the cases

(C1) $\epsilon \ll |z_1|$ and $|m_1| \sim 1$, (C2) $|z_1| \lesssim \epsilon$,

(C3) $|m_1| \lesssim \sqrt{\epsilon}$ and $|z_1| \sim 1$,

(C4) $\sqrt{\epsilon} \ll |m_1| \ll 1$ and $|z_1| \sim 1$,

where we note that this list is exhaustive since $|z_1| \ll 1$ implies $|m_1| \sim 1$ from (6.33).

In case (C1) we have $|z_2| \sim |z_1|$ and $|m_1| \sim |m_2| \sim 1$ from (6.34)–(6.35). By the near-saturation of (6.30e)–(6.30f) it follows that $\Im m_1 m_2 = \mathcal{O}(\epsilon)$ and therefore with (6.35) that

$$m_1 = \pm \overline{m_2} + \mathcal{O}(\epsilon), \tag{6.36}$$

hence $|\Re m_1 m_2| \sim 1$. From the ϵ^2-saturation of (6.30b)–(6.30c) and (6.30e)–(6.30f) it then follows that

$$|\Im u_1 u_2| \left| \Im \frac{z_1 \overline{z_2}}{|z_1 z_2|} \right| = \mathcal{O}\left(\frac{\epsilon^2}{|z_1|^2} \right), \quad (\Re u_1 u_2) \left| \Im \frac{z_1 \overline{z_2}}{|z_1 z_2|} \right| = \left(\Re \frac{z_1 \overline{z_2}}{|z_1 z_2|} \right) |\Im u_1 u_2| + \mathcal{O}\left(\frac{\epsilon}{|z_1|} \right), \tag{6.37}$$

and (6.37) implies

$$|\Im u_1 u_2| + \left| \Im \frac{z_1 \overline{z_2}}{|z_1 z_2|} \right| \lesssim \frac{\epsilon}{|z_1|}. \tag{6.38}$$

Indeed, the first equality in (6.37) implies that at least one of the two factors is at most of size $\epsilon/|z_1| \ll 1$ in which case the second equality implies that the other factor satisfies the same bound since $|u_1 u_2| \sim 1$. Thus there exists some $c \in \mathbf{R}$, $|c| \sim 1$ such that $z_2 = c z_1 + \mathcal{O}(\epsilon)$ and $u_2 = \pm |c|^{-1} \overline{u_1} + \mathcal{O}(\epsilon/|z_1|)$ since the two proportionality constants c and $\pm |c|^{-1}$ are related by (6.35). On the other hand, from the MDE (3.2) we have that

$$u_2 = u_2^2 |z_2|^2 - m_2^2 = \overline{u_1}^2 |z_1|^2 - \overline{m_1}^2 + \mathcal{O}(\epsilon) = \overline{u_1} + \mathcal{O}(\epsilon) \tag{6.39}$$

and thus $|c| = 1 + \mathcal{O}(\epsilon/|z_1|)$. Finally, since (6.30c)–(6.30d) is assumed to be saturated up to an ϵ^2-error, $\Re u_1 u_2$ and $\Re z_1 \overline{z_2}$ have the same sign which, together with (6.39), fixes $c > 0$, and we conclude $z_2 = z_1 + \mathcal{O}(\epsilon)$. Finally, with

$$w_2 = \frac{m_2}{u_2} - m_2 = \pm \left(\frac{\overline{m_1}}{\overline{u_1}} - \overline{m_1} \right) + \mathcal{O}(\epsilon) = \pm \overline{w_1} + \mathcal{O}(\epsilon) \tag{6.40}$$

the claim (6.32) follows.

In case (C2) the conclusion $z_2 = z_1 + \mathcal{O}(\epsilon)$ follows trivially from (6.35) and (6.34). Next, just as in case (C1), we conclude (6.36) and therefore from (3.2) that

$$u_2 = u_2^2 |z_2|^2 - m_2^2 = -\overline{m_1}^2 + \mathcal{O}(\epsilon) = \overline{u_1} + \mathcal{O}(\epsilon),$$

and thus (6.32) follows just as in (6.40).

Finally, we consider the case $|m_i| \ll 1$, i.e. (C3) and (C4). If $|m_i| \ll 1$, then from (6.33), $|1 - |z_i u_i|^2| \ll 1$, and therefore from (3.2), $|1 - |u_i|| \ll 1$ and consequently $|1 - u_i |z_i|^2| = |m_i^2 / u_i| \ll 1$ and $|1 - u_i| + |1 - |z_i|^2| \ll 1$. If $|m_1| \lesssim \sqrt{\epsilon}$, then it follows from (6.35) that also $|m_2| \lesssim \sqrt{\epsilon}$. From solving the equation (3.2) for u_i we find

$$u_i = \frac{1 + \sqrt{1 + 4|z_i|^2 m_i^2}}{2|z_i|^2} = \frac{1}{|z_i|^2} + \mathcal{O}(|m_i|^2), \tag{6.41}$$

where the sign choice is fixed due to $|1 - u_i| \ll 1$.

In case (C3) from $|m_i| \lesssim \sqrt{\epsilon}$ it follows that $u_i = |z_i|^{-2} + \mathcal{O}(\epsilon)$, and thus with (6.30e), (6.30f) and $\Re u_1 u_2 \sim 1$ we can conclude

$$|\Im z_1 \overline{z_2}| = \frac{\Re z_1 \overline{z_2}}{\Re u_1 u_2} |\Im u_1 u_2| + \mathcal{O}(\epsilon) = \mathcal{O}(\epsilon), \quad |\Im u_1 u_2| = \mathcal{O}(\epsilon). \tag{6.42}$$

Together with (6.35) and the saturation of (6.30c)–(6.30d), we obtain $z_1 = z_2 + \mathcal{O}(\epsilon)$ and $u_1 = \overline{u_2} + \mathcal{O}(\epsilon)$ by the same argument as after (6.38). Equation (3.2) implies that $m_2 = \pm\overline{m_1} + \mathcal{O}(\epsilon)$ and we are able to conclude (6.32) just as in (6.40).

In case (C4) from (6.35) we have $|m_2| \sim |m_1|$. By saturation of (6.30e)–(6.30f) it follows that

$$\Im \frac{m_1 m_2}{|m_1 m_2|} = \mathcal{O}\left(\frac{\epsilon}{|m_1|}\right)$$

and therefore, together with (6.35) we conclude that (6.36) also holds in this case. Now we use the saturation of (6.30b)–(6.30c) to conclude

$$|\Im u_1 u_2||\Im z_1 \overline{z_2}||\Re m_1 m_2| \lesssim \epsilon^2 \Big(|\Re m_1 m_2| + |\Im u_1 u_2||\Im z_1 \overline{z_2}|\Big).$$

Together with the fact that $|\Im u_1 u_2||\Im z_1 \overline{z_2}| \lesssim |m_i|^2 \sim |\Re m_1 m_2|$ from (6.36), (6.41), this implies $|\Im u_1 u_2||\Im z_1 \overline{z_2}| \lesssim \epsilon^2$. Finally, the ϵ^2-saturation of (6.30e)–(6.30f) shows that (6.37) (with $|z_1| \sim |z_2| \sim 1$) also holds in case (C4) and we are able to conclude (6.32) just like in case (C1). □

7 Asymptotic independence of resolvents: proof of Proposition 3.4

For any fixed $z \in \mathbf{C}$ let H^z be defined in (3.1). Recall that we denote the eigenvalues of H^z by $\{\lambda^z_{\pm i}\}_{i\in[n]}$, with $\lambda^z_{-i} = -\lambda^z_i$, and by $\{\boldsymbol{w}^z_{\pm i}\}_{i\in[n]}$ their corresponding orthonormal eigenvectors. As a consequence of the symmetry of the spectrum of H^z with respect to zero, its eigenvectors are of the form $\boldsymbol{w}^z_{\pm i} = (\boldsymbol{u}^z_i, \pm \boldsymbol{v}^z_i)$, for any $i \in [n]$. The eigenvectors of H^z are not well defined if H^z has multiple eigenvalues. This minor inconvenience can be easily solved by a tiny Gaussian regularization (see (7.17) and Remark 7.5 later).

Convention 7.1. We omitted the index $i = 0$ in the definition of the eigenvalues of H^z. In the remainder of this section we always assume that all the indices are not zero, e.g we use the notation

$$\sum_{i=-n}^{n} := \sum_{i=-n}^{-1} + \sum_{i=1}^{n},$$

and we use $|i| \leq A$, for some $A > 0$, to denote $0 < |i| \leq A$, etc.

The main result of this section is the proof of Proposition 3.4 which follows by Proposition 7.2 and the local law in Theorem 3.1.

Proposition 7.2 (Asymptotic independence of small eigenvalues of H^{z_l})**.** *Fix $p \in \mathbf{N}$, and let $\{\lambda^{z_l}_{\pm i}\}_{i=1}^n$ be the eigenvalues of H^{z_l}, with $l \in [p]$. For any $\omega_d, \omega_h, \omega_f > 0$ sufficiently small constants such that $\omega_h \ll \omega_f \ll \omega_d \ll 1$, there exist constants $\omega, \widehat{\omega}, \delta_0, \delta_1 > 0$, with $\omega_h \ll \delta_m \ll \widehat{\omega} \ll \omega \ll \omega_f$, for $m = 0, 1$, such that for any fixed $z_1, \dots, z_p \in \mathbf{C}$ so that $|z_l| \leq 1 - n^{-\omega_h}$, $|z_l - z_m|, |z_l - \overline{z}_m|, |z_l - \overline{z}_l| \geq n^{-\omega_d}$, with $l, m \in [p]$, $l \neq m$, it follows that*

$$\begin{aligned}\mathbf{E} \prod_{l=1}^{p} \frac{1}{n} \sum_{|i_l| \leq n^{\widehat{\omega}}} \frac{\eta_l}{(\lambda^{z_l}_{i_l})^2 + \eta_l^2} &= \prod_{l=1}^{p} \mathbf{E} \frac{1}{n} \sum_{|i_l| \leq n^{\widehat{\omega}}} \frac{\eta_l}{(\lambda^{z_l}_{i_l})^2 + \eta_l^2} \\ &\quad + \mathcal{O}\left(\frac{n^{\widehat{\omega}}}{n^{1+\omega}} \sum_{l=1}^{p} \frac{1}{\eta_l} \times \prod_{m=1}^{p} \left(1 + \frac{n^\xi}{n\eta_m}\right) + \frac{n^{p\xi + 2\delta_0} n^{\omega_f}}{n^{3/2}} \sum_{l=1}^{p} \frac{1}{\eta_l} + \frac{n^{p\delta_0 + \delta_1}}{n^{\widehat{\omega}}} \right),\end{aligned} \tag{7.1}$$

for any $\xi > 0$, where $\eta_1, \dots, \eta_p \in [n^{-1-\delta_0}, n^{-1+\delta_1}]$ and the implicit constant in $\mathcal{O}(\cdot)$ may depend on p.

Proof of Proposition 3.4. Let ρ^{z_l} be the self consistent density of states of H^{z_l}, and define its quantiles $\gamma_i^{z_l}$ by

$$\frac{i}{n} = \int_0^{\gamma_i^{z_l}} \rho^{z_l}(x)\,\mathrm{d}x, \qquad i \in [n],$$

and $\gamma_{-i}^{z_l} = -\gamma_i^{z_l}$ for $i \in [n]$. Then, using the local law in Theorem 3.1, by standard application of Helffer-Sjöstrand formula (see e.g. [28, Lemma 7.1, Theorem 7.6] or [32, Section 5] for a detailed derivation), we conclude the following rigidity bound

$$|\lambda_i^{z_l} - \gamma_i^{z_l}| \le \frac{n^{100\omega_h}}{n}, \qquad |i| \le n^{1-10\omega_h}, \tag{7.2}$$

with very high probability, uniformly in $|z_l| \le 1 - n^{-\omega_h}$. Then Proposition 3.4 follows by Proposition 7.2 and (7.2) exactly as in [22, Section 7.1]. We remark that in the current case we additionally require that $|z_l - \overline{z}_m|, |z_l - \overline{z}_l| \gtrsim n^{-\omega_d}$ compared to [22, Proposition 7.2], but this does not cause any change in the proof in [22, Section 7.1]. □

Section 7 is divided as follows: in Section 7.1 we state the main technical results needed to prove Proposition 7.2 and conclude its proof. In Section 7.2 we prove Theorem 2.8, which will follow by the results stated in Section 7.1. In Section 7.3 we estimate the overlaps of eigenvectors, corresponding to small indices, of H^{z_l}, H^{z_m} for $l \neq m$; this is the main input to prove the asymptotic independence in Proposition 7.2. In Section 7.4 and Section 7.6 we prove several technical results stated in Section 7.1. In Section 7.5 we present Proposition 7.14 which is a modification of the path-wise coupling of DBMs close to zero from [22, Proposition 7.14] to the case when the driving martingales in the DBM have a small correlation. This is needed to deal with the (small) correlation of $\boldsymbol{\lambda}^{z_l}$, the eigenvalues of H^{z_l}, for different l's.

7.1 Overview of the proof of Proposition 7.2

The main result of this section is the proof Proposition 7.2, which is essentially about the asymptotic independence of the eigenvalues $\lambda_i^{z_l}$, $\lambda_j^{z_m}$, for $l \neq m$ and small indices i and j. We do not prove this feature directly, instead we will compare $\lambda_i^{z_l}$, $\lambda_j^{z_m}$ with similar eigenvalues $\mu_i^{(l)}$, $\mu_j^{(m)}$ coming from independent Ginibre matrices, for which independence is straightforward by construction. The comparison is done by exploiting the strong local equilibration of the Dyson Brownian motion (DBM) in several steps. For convenience, we record the sequence of approximations in Figure 1. We remark that $z_1, \dots, z_p$ are fixed as in Proposition 7.2 throughout this section.

First, via a standard Green's function comparison argument (GFT) in Lemma 7.3 we prove that we may replace X by an i.i.d. matrix with a small Gaussian component. In the next step we make use of this Gaussian component and interpret the eigenvalues $\boldsymbol{\lambda}^z$ of H^z as the short-time evolution $\boldsymbol{\lambda}^z(t)$ of the eigenvalues of an auxiliary matrix H_t^z according to the Dyson Brownian motion. Proposition 7.2 is thus reduced to proving asymptotic independence of the flows $\boldsymbol{\lambda}^{z_l}(t)$ for different $l \in [p]$ after a short time $t = t_f$, a bit bigger than n^{-1}. The corresponding DBM describing the eigenvalues of H_t^z (see (7.14) later) differs from the standard DBM in two related aspects: (i) the driving martingales are weakly correlated, (ii) the interaction term has a coefficient slightly deviating from one. Note that the stochastic driving terms b_i in (7.14) are martingales but not Brownian motions (see Appendix B for more details). Both effects come from the small but non-trivial overlap of the eigenvectors $\boldsymbol{w}_i^{z_l}$ with $\overline{\boldsymbol{w}}_j^{z_l}$. They also influence the well-posedness of the DBM, so an extra care is necessary. We therefore define two comparison processes. First we regularise the DBM by (i) setting the coefficient of the interaction equal to one, (ii) slightly reducing the diffusion term, and (iii) cutting off the possible large values of the correlation. The resulting process, denoted by $\mathring{\boldsymbol{\lambda}}(t)$

(see (7.22) later), will be called the *regularised DBM*. Second, we artificially remove the correlation in the driving martingales for large indices. This *partially correlated DBM*, defined in (7.27) below, will be denoted by $\widetilde{\boldsymbol{\lambda}}(t)$. We will show that in both steps the error is much smaller than the relevant scale $1/n$. After these preparations, we can directly compare the *partially correlated DBM* $\widetilde{\boldsymbol{\lambda}}(t)$ with its Ginibre counterpart $\widetilde{\boldsymbol{\mu}}(t)$ (see (7.29) later) since their distribution is the same. Finally, we remove the partial correlation in the process $\widetilde{\boldsymbol{\mu}}(t)$ by comparing it with a purely independent Ginibre DBM $\boldsymbol{\mu}(t)$, defined in (7.24) below.

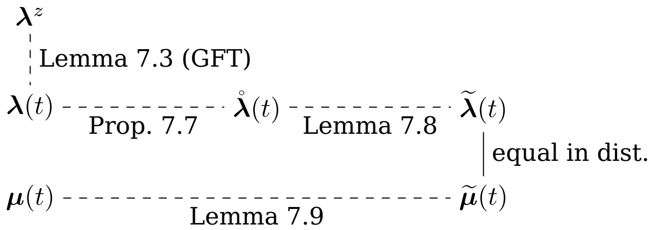

Figure 1: Proof overview for Proposition 7.2: The collections of eigenvalues $\boldsymbol{\lambda}^{z_l}$ of H^{z_l} for different l's are approximated by several stochastic processes. The processes $\boldsymbol{\mu} = \boldsymbol{\mu}^{(l)}$ are independent for different l's by definition.

Now we define these processes precisely. From now on we assume that $p = 2$ in Proposition 7.2 to make our presentation clearer. The case $p \geq 3$ is completely analogous. Consider the Ornstein-Uhlenbeck (OU) flow

$$d\widehat{X}_t = -\frac{1}{2}\widehat{X}_t \, \mathrm{d}t + \frac{\mathrm{d}\widehat{B}_t}{\sqrt{n}}, \qquad \widehat{X}_0 = X, \tag{7.3}$$

for a time

$$t_f := \frac{n^{\omega_f}}{n}, \tag{7.4}$$

with some small exponent $\omega_f > 0$ given as in Proposition 7.2, in order to add a small Gaussian component to X. Here $\widehat{B}_t$ in (7.3) is a standard matrix valued real Brownian motion, i.e. $\widehat{B}_{ab}$, $a, b \in [n]$ are i.i.d. standard real Brownian motions, independent of $\widehat{X}_0$. Then we can construct an i.i.d. matrix $\check{X}_{t_f}$ such that

$$\widehat{X}_{t_f} \stackrel{\mathrm{d}}{=} \check{X}_{t_f} + \sqrt{ct_f}U, \tag{7.5}$$

for some explicit constant $c > 0$ very close to 1, and U is a real Ginibre matrix independent of $\check{X}_{t_f}$. Using a simple Green's function comparison argument (GFT), by [22, Lemma 7.5], we conclude the following lemma.

Lemma 7.3. *The eigenvalues of H^{z_l} and the eigenvalues of $\widehat{H}^{z_l}_{t_f}$, with $t_f = n^{-1+\omega_f}$ obtained from replacing X by $\widehat{X}_{t_f}$, are close in the sense that for any sufficiently small $\omega_f, \delta_0, \delta_1 > 0$ it holds*

$$\mathbf{E}\prod_{l=1}^{2}\frac{1}{n}\sum_{|i_l|\leq n}\frac{\eta_l}{(\lambda_{i_l}(H^{z_l}))^2+\eta_l^2} = \mathbf{E}\prod_{l=1}^{2}\frac{1}{n}\sum_{|i_l|\leq n}\frac{\eta_l}{(\lambda_{i_l}(\widehat{H}^{z_l}_{t_f}))^2+\eta_l^2} + \mathcal{O}\left(\frac{n^{2\xi+2\delta_0}t_f}{n^{1/2}}\sum_{l=1}^{2}\frac{1}{\eta_l}\right), \tag{7.6}$$

where $\eta_l \in [n^{-1-\delta_0}, n^{-1+\delta_1}]$.

Next, we consider the matrix flow

$$\mathrm{d}X_t = \frac{\mathrm{d}B_t}{\sqrt{n}}, \qquad X_0 = \check{X}_{t_f}, \tag{7.7}$$

and denote by H_t^z the Hermitisation of $X_t - z$. Here B_t is a real standard matrix valued Brownian motion independent of X_0 and $\widehat{B}_t$. Note that by construction X_{ct_f} is such that

$$X_{ct_f} \stackrel{\mathrm{d}}{=} \widehat{X}_{t_f}. \tag{7.8}$$

Denote the eigenvalues and eigenvectors of H_t^z by

$$\boldsymbol{\lambda}^z(t) = \{\lambda_{\pm i}^z(t) \mid i \in [n]\}, \quad \{\boldsymbol{w}_{\pm i}^z(t) \mid i \in [n]\} = \{(\boldsymbol{u}_i^z(t), \pm \boldsymbol{v}_i^z(t)) \mid i \in [n]\},$$

and the resolvent by $G_t^z(w) := (H_t^z - w)^{-1}$ for $w \in \mathbf{H}$. For any $\boldsymbol{w} = (\boldsymbol{u}, \boldsymbol{v})$, with $\boldsymbol{u}, \boldsymbol{v} \in \mathbf{C}^n$ define the projections $P_1, P_2 \colon \mathbf{C}^{2n} \to \mathbf{C}^n$ by

$$P_1 \boldsymbol{w} = \boldsymbol{u}, \qquad P_2 \boldsymbol{w} = \boldsymbol{v}, \tag{7.9}$$

and, for any $z, z' \in \mathbf{C}$, define the *eigenvector overlaps* by

$$\Theta_{ij}^{z,z'} = \Theta_{ij}^{z,z'}(t) := 4\Re[\langle P_1 \boldsymbol{w}_j^{z'}(t), P_1 \boldsymbol{w}_i^z(t)\rangle \langle P_2 \boldsymbol{w}_i^z(t), P_2 \boldsymbol{w}_j^{z'}(t)\rangle], \quad |i|, |j| \le n. \tag{7.10}$$

Note that by the spectral symmetry of H_t^z it holds

$$\Theta_{ij}^{z,z} = \delta_{i,j} - \delta_{i,-j}, \quad \Theta_{ij}^{z,z'} = \Theta_{ji}^{z',z}, \quad |\Theta_{ij}^{z,z'}| \le 1, \tag{7.11}$$

for any $|i|, |j| \le n$. The coefficients $\Theta_{ij}^{z,z'}(t)$ are small with high probability due to the following lemma whose proof is postponed to Section 7.3.

Lemma 7.4 (Eigenvectors overlaps are small)**.** *For any sufficiently small constants $\omega_h, \omega_d > 0$, there exists $\omega_E > 0$ so that for any $z, z' \in \mathbf{C}$ such that $|z|, |z'| \le 1 - n^{-\omega_h}$, $|z - z'| \ge n^{-\omega_d}$, we have*

$$\sup_{0 \le t \le T} \sup_{|i|,|j| \le n} \left| \Theta_{ij}^{z,z'}(t) \right| \le n^{-\omega_E}, \tag{7.12}$$

with very high probability for any fixed $T \ge 0$.

Most of the DBM analysis is performed for a fixed $z \in \{z_1, z_2\}$, with z_1, z_2 as in Proposition 7.2, for this purpose we introduce the notation

$$\Lambda_{ij}^z(t) := \Theta_{ij}^{z,\overline{z}}(t), \tag{7.13}$$

for any $|i|, |j| \le n$. In particular, note $\Theta_{ij}^{z,\overline{z}} = \Theta_{ij}^{\overline{z},z}$ and so that by (7.11) it follows that $\Lambda_{ij}^z(t) = \Lambda_{ji}^z(t)$.

By the derivation of the DBM in Appendix B, using the fact that $\overline{\boldsymbol{w}}^z = \boldsymbol{w}^{\overline{z}}$, for $z = z_l$ with $l \in [2]$, it follows that (7.7) induces the flow

$$\mathrm{d}\lambda_i^z(t) = \frac{\mathrm{d}b_i^z}{\sqrt{n}} + \frac{1}{2n} \sum_{j \ne i} \frac{1 + \Lambda_{ij}^z(t)}{\lambda_i^z(t) - \lambda_j^z(t)} \,\mathrm{d}t, \quad \lambda_i^z(0) = \lambda_i^z, \qquad |i| \le n, \tag{7.14}$$

on the eigenvalues $\{\lambda_i^z(t)\}_{|i| \le n}$ of H_t^z. Here $\{\lambda_i^z\}_{|i| \le n}$ are the eigenvalues of the initial matrix H^z. The martingales $\{b_i^z\}_{i \in [n]}$, with $b_i^z(0) = 0$, and $\Lambda_{ij}^z(t)$, the overlap of eigenvectors in (7.13), (7.10), are defined on a probability space Ω_b equipped with the filtration

$$(\mathcal{F}_{b,t})_{0 \le t \le T} := \big(\sigma(X_0, (B_s)_{0 \le s \le t})\big)_{0 \le t \le T}, \tag{7.15}$$

where B_s is defined in (7.7). The martingale differentials in (7.14) are such that (see (B.16)–(B.17))

$$\begin{aligned} \mathrm{d}b_i^z &:= \mathrm{d}B_{ii}^z + \mathrm{d}\overline{B_{ii}^z}, \quad \text{with} \quad \mathrm{d}B_{ij}^z := \langle \boldsymbol{u}_i^z, (\mathrm{d}B)\boldsymbol{v}_i^z\rangle, \quad i, j \in [n], \\ \mathbf{E}\big[\mathrm{d}b_i^z \,\mathrm{d}b_j^z \bigm| \mathcal{F}_{b,t}\big] &= \frac{\delta_{ij} + \Lambda_{ij}^z(t)}{2} \,\mathrm{d}t, \quad i, j \in [n], \end{aligned} \tag{7.16}$$

and $\mathrm{d}b_{-i}^z = -\,\mathrm{d}b_i^z$ for $i \in [n]$. Here we used the notation Ω_b for the probability space to emphasize that is the space where the martingales $\boldsymbol{b}^z$ are defined, since in Section 7.1.2 we will introduce another probability space which we will denote by Ω_β.

In the remainder of this section we will apply Lemma 7.4 for $z = z_1, z' = z_2$ and $z = z_1, z' = \overline{z}_2$ and $z = z_l, z' = \overline{z}_l$, for $l \in [2]$, with z_1, z_2 fixed as in Proposition 7.2. We recall that throughout this section we assumed that $p = 2$ in Proposition 7.2. Note that $\Lambda_{ij}^{z_1}$, $\Lambda_{ij}^{z_2}$, $\Theta_{ij}^{z_1,z_2}$, $\Theta_{ij}^{z_1,\overline{z}_2}$ with $|i|,|j| \le n$, are not well-defined if $H_t^{z_1}$, $H_t^{z_2}$ have multiple eigenvalues. This minor inconvenience can easily be resolved by a tiny regularization as in [19, Lemma 6.2] (which is the singular values counterpart of [17, Proposition 2.3]). Using this result, we may, without loss of generality, assume that the eigenvalues of $H_t^{z_l}$ are almost surely distinct for any fixed time $t \ge 0$. Indeed, if this were not the case then we replace $H_0^{z_l}$ by

$$H_{0,\mathrm{reg}}^{z_l} := \begin{pmatrix} 0 & X - z_l + e^{-n}Q \\ X^* - \overline{z}_l + e^{-n}Q^* & 0 \end{pmatrix}, \tag{7.17}$$

with Q being a complex $n \times n$ Ginibre matrix independent of X, i.e. we may regularize X by adding an exponentially small Gaussian component. Then, by [19, Lemma 6.2], $H_{t,\mathrm{reg}}^{z_l}$, the evolution of $H_{0,\mathrm{reg}}^{z_l}$ along the flow (7.7), does not have multiple eigenvalues almost surely; additionally, the eigenvalues of $H_{0,\mathrm{reg}}^{z_l}$ and the ones of $H_0^{z_l}$ are exponentially close. Hence, by Fubini's theorem, $\{\Lambda_{ij}^{z_l}(t)\}_{|i|,|j|\le n}$, with $l \in [2]$, and $\{\Theta_{ij}^{z_1,z_2}(t)\}_{|i|,|j|\le n}$, $\{\Theta_{ij}^{z_1,\overline{z}_2}(t)\}_{|i|,|j|\le n}$ are well-defined for almost all $t \ge 0$; we set them equal to zero whenever they are not well defined.

Remark 7.5. The perturbation of X in (7.17) is exponentially small, hence does not change anything in the proof of the local laws in Theorem 3.1 and Theorem 3.5 or in the Green's function comparison (GFT) argument in Lemma 7.3, since these proofs deal with scales much bigger than e^{-n}. This implies that any local law or GFT result which holds for $H_t^{z_l}$ then holds true for $H_{t,\mathrm{reg}}^{z_l}$ as well. Hence, in the remainder of this section we assume that [19, Lemma 6.2] holds true for $H_t^{z_l}$ (the unperturbed matrix).

The process (7.14) is well-defined in the sense of Proposition 7.6, whose proof is postponed to Section 7.6.

Proposition 7.6 (The DBM in (7.14) is well-posed)**.** *Fix $z \in \{z_1, z_2\}$, and let H_t^z be defined by the flow* (7.7)*. Then the eigenvalues $\boldsymbol{\lambda}(t)$ of H_t^z are the unique strong solution to* (7.14) *on $[0,T]$, for any $T > 0$, such that $\boldsymbol{\lambda}(t)$ is adapted to the filtration $(\mathcal{F}_{b,t})_{0\le t\le T}$, $\boldsymbol{\lambda}(t)$ is γ-Hölder continuous for any $\gamma \in (0,1/2)$, and*

$$\mathbf{P}\Big(\lambda_{-n}(t) < \cdots < \lambda_{-1}(t) < 0 < \lambda_1(t) < \cdots < \lambda_n(t),\ \textit{for almost all } t \in [0,T]\Big) = 1.$$

In order to prove that the term Λ_{ij}^z in (7.14) is irrelevant, we will couple the driving martingales in (7.14) with the ones of a DBM that does not have the additional term Λ_{ij}^z (see (7.22) below). For this purpose we have to consider the correlation of $\{b_i^{z_1}\}_{|i|\le n}$, $\{b_i^{z_2}\}_{|i|\le n}$ for two different $z_1, z_2 \in \mathbf{C}$ as in Proposition 7.2. In the following we will focus only on the driving martingales with positive indices, since the ones with negative indices are defined by symmetry. The martingales $\boldsymbol{b}^{z_l} = \{b_i^{z_l}\}_{i\in[n]}$, with $l = 1, 2$, are defined on a common probability space equipped with the filtration $(\mathcal{F}_{b,t})_{0\le t\le T}$ from (7.15).

We consider $\boldsymbol{b}^{z_1}$, $\boldsymbol{b}^{z_2}$ jointly as a $2n$-dimensional martingale $(\boldsymbol{b}^{z_1}, \boldsymbol{b}^{z_2})$. Define the naturally reordered indices

$$\mathfrak{i} = (l-1)n + i, \qquad \mathfrak{j} = (m-1)n + j,$$

with $l, m \in [2]$, $i, j \in [n]$, and $\mathfrak{i}, \mathfrak{j} \in [2n]$. Then the correlation between $\boldsymbol{b}^{z_1}$, $\boldsymbol{b}^{z_2}$ is given by

$$C_{\mathfrak{i}\mathfrak{j}}(t)\,\mathrm{d}t := \mathbf{E}\big[\mathrm{d}b_i^{z_l}\,\mathrm{d}b_j^{z_m}\ \big|\ \mathcal{F}_{b,t}\big] = \frac{\Theta_{ij}^{z_l,z_m}(t) + \Theta_{ij}^{z_l,\overline{z}_m}(t)}{2}\,\mathrm{d}t \qquad \mathfrak{i}, \mathfrak{j} \in [2n]. \tag{7.18}$$

Note that $C(t)$ is a positive semi-definite matrix. In particular, taking also negative indices into account, for a fixed $z \in \{z_1, z_2\}$, the family of martingales $\boldsymbol{b}^z = \{b_i^z\}_{|i| \le n}$ is such that

$$\mathbf{E}\big[\mathrm{d}b_i^z\,\mathrm{d}b_j^z \mid \mathcal{F}_{b,t}\big] = \frac{\delta_{i,j} - \delta_{i,-j} + \Lambda_{ij}^z(t)}{2}\,\mathrm{d}t, \qquad |i|,|j| \le n. \tag{7.19}$$

7.1.1 Comparison of $\boldsymbol{\lambda}$ with the regularised process $\mathring{\boldsymbol{\lambda}}$

By Lemma 7.4 the overlaps $\Theta_{ij}^{z,z'}$ are typically small for any $z, z' \in \mathbf{C}$ such that $|z|, |z'| \le 1 - n^{-\omega_h}$ and $|z - z'| \ge n^{-\omega_d}$. We now define their cut-off versions (see (7.21) below). We only consider positive indices, since negative indices are defined by symmetry. Throughout this section we use the convention that regularised objects will be denoted by circles. Let z_l, with $l \in [2]$ be fixed throughout Section 7 as in Proposition 7.2. Define the $2n \times 2n$ matrix $\mathring{C}(t)$ by

$$\mathring{C}_{\mathfrak{i}\mathfrak{j}}(t) := \frac{\mathring{\Theta}_{ij}^{z_l,z_m}(t) + \mathring{\Theta}_{ij}^{z_l,\overline{z}_m}(t)}{2} \qquad i,j \in [n], \quad \mathfrak{i},\mathfrak{j} \in [2n], \tag{7.20}$$

where $\mathring{\Theta}_{ij}^{z_l,z_l} = \delta_{ij}$ for $i,j \in [n]$, and

$$\begin{aligned}\mathring{\Theta}_{ij}^{z_1,z_2}(t) &:= \Theta_{ij}^{z_1,z_2}(t) \cdot \mathbf{1}\left(\mathcal{A}(t) \le n^{-\omega_E}\right), \quad \mathring{\Theta}_{ij}^{z_l,\overline{z}_m}(t) := \Theta_{ij}^{z_l,\overline{z}_m}(t) \cdot \mathbf{1}\left(\mathcal{A}(t) \le n^{-\omega_E}\right), \\ \mathcal{A}(t) = \mathcal{A}^{z_1,z_2}(t) &:= \max_{|i|,|j| \le n} |\Lambda_{ij}^{z_1}(t)| + |\Lambda_{ij}^{z_2}(t)| + |\Theta_{ij}^{z_1,\overline{z}_2}(t)| + |\Theta_{ij}^{z_1,z_2}(t)|\end{aligned} \tag{7.21}$$

for any $l, m \in [2]$, recalling that $\Lambda_{ij}^{z_l} = \Theta_{ij}^{z_l,\overline{z}_l}$. Note that by Lemma 7.4 it follows that $\mathring{C}(t) = C(t)$ on a set of very high probability, and $\mathring{C}(t) = \frac{1}{2}I$, with I the $2n \times 2n$ identity matrix, on the complement of this set, for any $t \in [0,T]$. In particular, $\mathring{C}(t)$ is positive semi-definite for any $t \in [0,T]$, since $C(t)$, defined as a covariance in (7.18), is positive semi-definite. The purpose of the cut-off in (7.20) it is to ensure the well-posedness of the process (7.22) below.

We compare the processes $\boldsymbol{\lambda}^{z_l}(t)$ in (7.14) with the *regularised processes* $\mathring{\boldsymbol{\lambda}}^{z_l}(t)$ defined, for $z = z_l$, by

$$\mathrm{d}\mathring{\lambda}_i^z = \frac{\mathrm{d}\mathring{b}_i^z}{\sqrt{n(1+n^{-\omega_r})}} + \frac{1}{2n}\sum_{j \ne i} \frac{1}{\mathring{\lambda}_i^z - \mathring{\lambda}_j^z}\,\mathrm{d}t, \qquad \mathring{\lambda}_i^z(0) = \lambda_i^z(0), \qquad |i| \le n, \tag{7.22}$$

with $\omega_r > 0$ such that $\omega_f \ll \omega_r \ll \omega_E$. We organise the martingales $\boldsymbol{b}^{z_1}, \boldsymbol{b}^{z_2}$ with positive indices into a single $2n$-dimensional vector $\boldsymbol{b} = (\boldsymbol{b}^{z_1}, \boldsymbol{b}^{z_2})$ with a correlation structure given by (7.18). Then by Doob's martingale representation theorem [41, Theorem 18.12] there exists a standard Brownian motion $\mathfrak{w} = (\mathfrak{w}^{(1)}, \mathfrak{w}^{(2)}) \in \mathbf{R}^{2n}$ realized on an extension $(\widetilde{\Omega}_b, \widetilde{\mathcal{F}}_{b,t})$ of the original probability space $(\Omega_b, \mathcal{F}_{b,t})$ such that $\mathrm{d}\boldsymbol{b} = \sqrt{C}\,\mathrm{d}\mathfrak{w}$, with $\sqrt{C} = \sqrt{C(t)}$ the matrix square root of $C(t)$. Moreover, $\mathfrak{w}(t)$ and $C(t)$ are adapted to the filtration $\widetilde{\mathcal{F}}_{b,t}$. Then the martingales $\mathring{\boldsymbol{b}}^{z_l} = \{\mathring{b}_i^{z_l}\}_{i \in [n]}$, with $l \in [2]$, are defined by $\mathring{\boldsymbol{b}}^{z_l}(0) = 0$ and

$$\begin{pmatrix} \mathrm{d}\mathring{\boldsymbol{b}}^{z_1}(t) \\ \mathrm{d}\mathring{\boldsymbol{b}}^{z_2}(t) \end{pmatrix} := \sqrt{\mathring{C}(t)} \begin{pmatrix} \mathrm{d}\mathfrak{w}^{(1)}(t) \\ \mathrm{d}\mathfrak{w}^{(2)}(t) \end{pmatrix}, \tag{7.23}$$

where $\sqrt{\mathring{C}(t)}$ denotes the matrix square root of the positive semi-definite matrix $\mathring{C}(t)$. For negative indices we define $\mathring{b}_{-i} = -\mathring{b}_i$, with $i \in [n]$. The purpose of the additional factor $1 + n^{-\omega_r}$ in (7.22) is to ensure the well-posedness of the process, since $\mathring{\boldsymbol{b}}^z$ is a small deformation of a family of i.i.d. Brownian motions with variance $1/2$, and the well-posedness of (7.22) is already critical for those Brownian motions (it corresponds to the GOE case, i.e. $\beta = 1$). The well-posedness of the process (7.22) is proven in Appendix A. The main result of this section is the following proposition, whose proof is deferred to Section 7.4.

Proposition 7.7 (The *regularised process* $\mathring{\boldsymbol{\lambda}}$ is close to $\boldsymbol{\lambda}$)**.** *For any sufficiently small $\omega_d, \omega_h, \omega_f > 0$ such that $\omega_h \ll \omega_f \ll 1$ there exist small constants $\widehat{\omega}, \omega > 0$ such that $\omega_h \ll \widehat{\omega} \ll \omega \ll \omega_f$, and that for $|z_l - \overline{z}_l|, |z_l - \overline{z}_m|, |z_l - z_m| \geq n^{-\omega_d}$, $|z_l| \leq 1 - n^{-\omega_h}$, with $l \neq m$, it holds*

$$|\lambda_i^{z_l}(ct_f) - \mathring{\lambda}_i^{z_l}(ct_f)| \leq n^{-1-\omega}, \qquad |i| \leq n^{\widehat{\omega}},$$

with very high probability, where $t_f = n^{-1+\omega_f}$ and $c > 0$ is defined in (7.5)*.*

7.1.2 Definition of the partially correlated processes $\widetilde{\boldsymbol{\lambda}}$, $\widetilde{\boldsymbol{\mu}}$

The construction of the *partially correlated processes* for $\mathring{\boldsymbol{\lambda}}^{z_l}(t)$ is exactly the same as in the complex case [22, Section 7.2]; we present it here as well for completeness. We want to compare the correlated processes $\mathring{\boldsymbol{\lambda}}^{z_l}(t)$, with $l = 1, 2$, defined on a probability space $\widetilde{\Omega}_b$ equipped with a filtration $\widetilde{\mathcal{F}}_{b,t}$ with carefully constructed independent processes $\boldsymbol{\mu}^{(l)}(t)$, $l = 1, 2$ on a different probability space Ω_β equipped with a filtration $\mathcal{F}_{\beta,t}$, which is defined in (7.25) below. We choose $\boldsymbol{\mu}^{(l)}(t)$ to be a *complex Ginibre DBM*, i.e. it is given as the solution of

$$\mathrm{d}\mu_i^{(l)}(t) = \frac{\mathrm{d}\beta_i^{(l)}}{\sqrt{2n}} + \frac{1}{2n}\sum_{j\neq i}\frac{1}{\mu_i^{(l)}(t) - \mu_j^{(l)}(t)}\,\mathrm{d}t, \quad \mu_i^{(l)}(0) = \mu_i^{(l)}, \qquad |i| \leq n, \tag{7.24}$$

with $\mu_i^{(l)}$ the singular values, taken with positive and negative sign, of independent complex Ginibre matrices $X^{(l)}$, and $\boldsymbol{\beta}^{(l)} = \{\beta_i^{(l)}\}_{i\in[n]}$ being independent vectors of i.i.d. standard real Brownian motions, and $\beta_{-i}^{(l)} = -\beta_i^{(l)}$ for $i \in [n]$. The filtration $\mathcal{F}_{\beta,t}$ is defined by

$$\left(\mathcal{F}_{\beta,t}\right)_{0\leq t\leq T} := \left(\sigma(X^{(l)}, (\boldsymbol{\beta}_s^{(l)})_{0\leq s\leq t}, (\widetilde{\boldsymbol{\zeta}}_s^{(l)})_{0\leq s\leq t}; l \in [2])\right)_{0\leq t\leq T}, \tag{7.25}$$

with $\widetilde{\boldsymbol{\zeta}}^{(l)}$ standard real i.i.d. Brownian motions, independent of $\boldsymbol{\beta}^{(l)}$, which will be used later in the definition of the processes in (7.29).

The comparison of $\mathring{\boldsymbol{\lambda}}^{z_l}(t)$ and $\boldsymbol{\mu}^{(l)}(t)$ is done via two intermediate *partially correlated processes* $\widetilde{\boldsymbol{\lambda}}^{(l)}(t)$, $\widetilde{\boldsymbol{\mu}}^{(l)}(t)$ so that for a time $t \geq 0$ large enough $\widetilde{\lambda}_i^{(l)}(t)$, $\widetilde{\mu}_i^{(l)}(t)$ for small indices i will be close to $\mathring{\lambda}_i^{z_l}(t)$ and $\mu_i^{(l)}(t)$, respectively, with very high probability. Additionally, the processes $\widetilde{\boldsymbol{\lambda}}^{(l)}$, $\widetilde{\boldsymbol{\mu}}^{(l)}$ will be constructed such that they have the same joint distribution:

$$\left(\widetilde{\boldsymbol{\lambda}}^{(1)}(t), \widetilde{\boldsymbol{\lambda}}^{(2)}(t)\right)_{0\leq t\leq T} \stackrel{\mathrm{d}}{=} \left(\widetilde{\boldsymbol{\mu}}^{(1)}(t), \widetilde{\boldsymbol{\mu}}^{(2)}(t)\right)_{0\leq t\leq T}, \tag{7.26}$$

for any $T > 0$.

Fix $\omega_A > 0$ such that $\omega_h \ll \omega_A \ll \omega_f$, and for $l \in [2]$ define the process $\widetilde{\boldsymbol{\lambda}}^{(l)}(t)$ to be the solution of

$$\mathrm{d}\widetilde{\lambda}_i^{(l)}(t) = \frac{1}{2n}\sum_{j\neq i}\frac{1}{\widetilde{\lambda}_i^{(l)}(t) - \widetilde{\lambda}_j^{(l)}(t)}\,\mathrm{d}t + \begin{cases} \left(n(1+n^{-\omega_r})\right)^{-1/2}\mathrm{d}\mathring{b}_i^{z_l}, & |i| \leq n^{\omega_A} \\ (2n)^{-1/2}\,\mathrm{d}\widetilde{b}_i^{(l)}, & n^{\omega_A} < |i| \leq n, \end{cases} \tag{7.27}$$

with initial data $\widetilde{\boldsymbol{\lambda}}^{(l)}(0)$ being the singular values, taken with positive and negative sign, of independent complex Ginibre matrices $\widetilde{Y}^{(l)}$ independent of $\boldsymbol{\lambda}^{z_l}(0)$. Here $\mathrm{d}\mathring{b}_i^{z_l}$ is the martingale differential from (7.22) which is used for small indices in (7.27). For large indices we define the driving martingales to be an independent collection $\{\{\widetilde{b}_i^{(l)}\}_{i=n^{\omega_A}+1}^{n} \,|$

$l \in [2]\}$ of two vector-valued i.i.d. standard real Brownian motions which are also independent of $\{\{\mathring{b}^{z_l}_{\pm i}\}_{i=1}^n \mid l \in [2]\}$, and that $\widetilde{b}^{(l)}_{-i} = -\widetilde{b}^{(l)}_i$ for $i \in [n]$. The martingales $\mathring{\boldsymbol{b}}^{z_l}$, with $l \in [2]$, and $\{\{\widetilde{b}^{(l)}_i\}_{i=n^{\omega_A}+1}^n \mid l \in [2]\}$ are defined on a common probability space that we continue to denote by $\widetilde{\Omega}_b$ with the common filtration $\widetilde{\mathcal{F}}_{b,t}$, given by

$$\big(\widetilde{\mathcal{F}}_{b,t}\big)_{0\le t\le T} := \big(\sigma(X_0, \widetilde{Y}^{(l)}, (B_s)_{0\le s\le t}, (\widetilde{\boldsymbol{b}}^{(l)})_{0\le s\le t}; l \in [2])\big)_{0\le t\le T}.$$

The well-posedness of (7.27), and of (7.29) below, readily follows by exactly the same arguments as in Appendix A.

Notice that $\mathring{\boldsymbol{\lambda}}(t)$ and $\widetilde{\boldsymbol{\lambda}}(t)$ differ in two aspects: the driving martingales with large indices for $\widetilde{\boldsymbol{\lambda}}(t)$ are set to be independent, and the initial conditions are different. Lemma 7.8 below states that these differences are negligible for our purposes (i.e. after time ct_1 the two processes at small indices are closer than the rigidity scale $1/n$). Its proof is postponed to Section 7.5.1. Let $\rho_{\mathrm{sc}}(E) = \frac{1}{2\pi}\sqrt{4-E^2}$ denote the semicircle density.

Lemma 7.8 (The partially correlated process $\widetilde{\boldsymbol{\lambda}}$ is close to $\mathring{\boldsymbol{\lambda}}$)**.** *Let $\mathring{\boldsymbol{\lambda}}^{z_l}(t)$, $\widetilde{\boldsymbol{\lambda}}^{(l)}(t)$, with $l \in [2]$, be the processes defined in (7.22) and (7.27), respectively. For any sufficiently small $\omega_h, \omega_f > 0$ such that $\omega_h \ll \omega_f \ll 1$ there exist constants $\omega, \widehat{\omega} > 0$ such that $\omega_h \ll \widehat{\omega} \ll \omega \ll \omega_f$, and that for $|z_l| \le 1 - n^{-\omega_h}$ it holds*

$$|\rho^{z_l}(0)\mathring{\lambda}^{z_l}_i(ct_f) - \rho_{\mathrm{sc}}(0)\widetilde{\lambda}^{(l)}_i(ct_f)| \le n^{-1-\omega}, \qquad |i| \le n^{\widehat{\omega}}, \tag{7.28}$$

with very high probability, where $t_f := n^{-1+\omega_f}$ and $c > 0$ is defined in (7.5).

Finally, $\widetilde{\boldsymbol{\mu}}^{(l)}(t)$, the comparison process of $\boldsymbol{\mu}^{(l)}(t)$, is given as the solution of the following DBM

$$\mathrm{d}\widetilde{\mu}^{(l)}_i(t) = \frac{1}{2n}\sum_{j\ne i} \frac{1}{\widetilde{\mu}^{(l)}_i(t) - \widetilde{\mu}^{(l)}_j(t)}\,\mathrm{d}t + \begin{cases} \big(n(1+n^{-\omega_r})\big)^{-1/2}\,\mathrm{d}\mathring{\zeta}^{z_l}_i, & |i| \le n^{\omega_A}, \\ (2n)^{-1/2}\,\mathrm{d}\widetilde{\zeta}^{(l)}_i, & n^{\omega_A} < |i| \le n, \end{cases} \tag{7.29}$$

with initial data $\widetilde{\boldsymbol{\mu}}^{(l)}(0) = \boldsymbol{\mu}^{(l)}$. We now explain how to construct the driving martingales in (7.29) so that (7.26) is satisfied. For this purpose we closely follow [22, Eqs. (7.22)–(7.29)]. We only consider positive indices, since the negative indices are defined by symmetry. Define the $2n^{\omega_A}$-dimensional martingale $\underline{\mathring{\boldsymbol{b}}} := \{\{\mathring{b}^{z_l}_i\}_{i\in[n^{\omega_A}]} \mid l \in [2]\}$. Throughout this section underlined vectors or matrices denote their restriction to the first $i \in [n^{\omega_A}]$ indices within each l-group, i.e.

$$\boldsymbol{v} \in \mathbf{C}^{2n} \Longrightarrow \underline{\boldsymbol{v}} \in \mathbf{C}^{2n^{\omega_A}}, \quad \text{with} \quad \underline{v}_i := \begin{cases} v_i & \text{if} \quad i \in [n^{\omega_A}] \\ v_{i+n^{\omega_A}} & \text{if} \quad i \in n + [n^{\omega_A}]. \end{cases}$$

Then we define $\underline{\mathring{C}}(t)$ as the $2n^{\omega_A} \times 2n^{\omega_A}$ positive semi-definite matrix which consists of the four blocks corresponding to index pairs $\{(i,j) \in [n^{\omega_A}]^2\}$ of the matrix $\mathring{C}(t)$ defined in (7.20). Similarly to (7.23), by Doob's martingale representation theorem, we obtain $\mathrm{d}\underline{\mathring{\boldsymbol{b}}} = (\underline{\mathring{C}})^{1/2}\,\mathrm{d}\boldsymbol{\theta}$ with $\boldsymbol{\theta}(t) := \{\{\theta^{(l)}_i(t)\}_{i\in[n^{\omega_A}]} \mid l \in [2]\}$ a family of i.i.d. standard real Brownian motions. We define an independent copy $\underline{\mathring{C}}^{\#}(s)$ of $\underline{\mathring{C}}(s)$ and $\underline{\boldsymbol{\beta}} := \{\{\beta^{(l)}_i\}_{i\in[n^{\omega_A}]} \mid l \in [2]\}$ such that $(\underline{\mathring{C}}^{\#}(t), \underline{\boldsymbol{\beta}}(t))$ has the same joint distribution as $(\underline{\mathring{C}}(t), \boldsymbol{\theta}(t))$. We then define the families $\underline{\mathring{\boldsymbol{\zeta}}} := \{\{\mathring{\zeta}^{z_l}_i\}_{i\in[n^{\omega_A}]} \mid l \in [2]\}$ by $\underline{\mathring{\boldsymbol{\zeta}}}(0) = 0$ and

$$\mathrm{d}\underline{\mathring{\boldsymbol{\zeta}}}(t) := \Big(\underline{\mathring{C}}^{\#}(t)\Big)^{1/2}\,\mathrm{d}\underline{\boldsymbol{\beta}}(t), \tag{7.30}$$

and extend this to negative indices by $\zeta^{z_l}_{-i} = -\zeta^{z_l}_i$ for $i \in [n^{\omega_A}]$. For indices $n^{\omega_A} < |i| \le n$, instead, we choose $\{\widetilde{\zeta}^{(l)}_{\pm i}\}_{i=n^{\omega_A}+1}^n$ to be independent families (independent of each other for different l's, and also independent of $\boldsymbol{\beta}$) of i.i.d. Brownian motions defined on the same probability space Ω_β. Note that (7.26) follows by the construction in (7.30).

Similarly to Lemma 7.8 we also have that $\boldsymbol{\mu}(t)$ and $\widetilde{\boldsymbol{\mu}}(t)$ are close thanks to the carefully designed relation between their driving Brownian motions. The proof of this lemma is postponed to Section 7.5.1.

Lemma 7.9 (The partially correlated process $\widetilde{\boldsymbol{\mu}}$ is close to $\boldsymbol{\mu}$)**.** *For any sufficiently small $\omega_d, \omega_h, \omega_f > 0$, there exist constants $\omega, \widehat{\omega} > 0$ such that $\omega_h \ll \widehat{\omega} \ll \omega \ll \omega_f$, and that for $|z_l - z_m|, |z_l - \overline{z}_m|, |z_l - \overline{z}_l| \geq n^{-\omega_d}$, $|z_l| \leq 1 - n^{-\omega_h}$, with $l, m \in [2]$, $l \neq m$, it holds*

$$\left|\mu_i^{(l)}(ct_f) - \widetilde{\mu}_i^{(l)}(ct_f)\right| \leq n^{-1-\omega}, \quad |i| \leq n^{\widehat{\omega}}, \quad l \in [2], \tag{7.31}$$

with very high probability, where $t_f = n^{-1+\omega_f}$ and $c > 0$ is defined in (7.5)*.*

7.1.3 Proof of Proposition 7.2

In this section we conclude the proof of Proposition 7.2 using the comparison processes defined in Section 7.1.1 and Section 7.1.2. We recall that $p = 2$ for simplicity. More precisely, we use that the processes $\boldsymbol{\lambda}^{z_l}(t)$, $\mathring{\boldsymbol{\lambda}}^{z_l}(t)$ and $\mathring{\boldsymbol{\lambda}}^{z_l}(t)$, $\widetilde{\boldsymbol{\lambda}}^{(l)}(t)$ and $\widetilde{\boldsymbol{\mu}}^{(l)}(t)$, $\boldsymbol{\mu}^{(l)}(t)$ are close path-wise at time t_1, as stated in Proposition 7.7, Lemma 7.8, and Lemma 7.9, respectively, choosing $\omega, \widehat{\omega}$ as the minimum of the ones in the statements of this three results. In particular, by these results and Lemma 7.3 we readily conclude the following lemma, whose proof is postponed to the end of this section.

Lemma 7.10. *Let $\boldsymbol{\lambda}^{z_l}$ be the eigenvalues of H^{z_l}, and let $\boldsymbol{\mu}^{(l)}(t)$ be the solution of* (7.24)*. Let $\omega, \widehat{\omega}, \omega_h > 0$ given as above, and define $\nu_{z_l} := \rho_{sc}(0)/\rho^{z_l}(0)$, then for any small $\omega_f > 0$ such that $\omega_h \ll \omega_f$ there exists δ_0, δ_1 such that $\omega_h \ll \delta_m \ll \widehat{\omega}$, for $m = 0, 1$, and that*

$$\mathbf{E} \prod_{l=1}^{2} \frac{1}{n} \sum_{|i_l| \leq n^{\widehat{\omega}}} \frac{\eta_l}{(\lambda_{i_l}^{z_l})^2 + \eta_l^2} = \mathbf{E} \prod_{l=1}^{2} \frac{1}{n} \sum_{|i_l| \leq n^{\widehat{\omega}}} \frac{\eta_l}{(\mu_{i_l}^{(l)}(ct_f)\nu_{z_l})^2 + \eta_l^2} + \mathcal{O}(\Psi), \tag{7.32}$$

where $t_f = n^{-1+\omega_f}$, $\eta_l \in [n^{-1-\delta_0}, n^{-1+\delta_1}]$, and the error term is given by

$$\Psi := \frac{n^{\widehat{\omega}}}{n^{1+\omega}} \left(\sum_{l=1}^{2} \frac{1}{\eta_l} \right) \cdot \prod_{l=1}^{2} \left(1 + \frac{n^{\xi}}{n\eta_l} \right) + \frac{n^{2\xi + 2\delta_0} t_f}{n^{1/2}} \sum_{l=1}^{2} \frac{1}{\eta_l} + \frac{n^{2(\delta_1 + \delta_0)}}{n^{\widehat{\omega}}}. \tag{7.33}$$

We remark that Ψ in (7.33) denotes a different error term compared with the error terms in (3.9) and (6.10).

By the definition of the processes $\boldsymbol{\mu}^{(l)}(t)$ in (7.24) it follows that $\boldsymbol{\mu}^{(l)}(t)$, $\boldsymbol{\mu}^{(m)}(t)$ are independent for $l \neq m$ and so that

$$\mathbf{E} \prod_{l=1}^{2} \frac{1}{n} \sum_{|i_l| \leq n^{\widehat{\omega}}} \frac{\eta_l}{(\mu_{i_l}^{(l)}(ct_f)\nu_{z_l})^2 + \eta_l^2} = \prod_{l=1}^{2} \mathbf{E} \frac{1}{n} \sum_{|i_l| \leq n^{\widehat{\omega}}} \frac{\eta_l}{(\mu_{i_l}^{(l)}(ct_f)\nu_{z_l})^2 + \eta_l^2}. \tag{7.34}$$

Then, similarly to Lemma 7.10, we conclude that

$$\prod_{l=1}^{2} \mathbf{E} \frac{1}{n} \sum_{|i_l| \leq n^{\widehat{\omega}}} \frac{\eta_l}{(\lambda_{i_l}^{z_l})^2 + \eta_l^2} = \prod_{l=1}^{2} \mathbf{E} \frac{1}{n} \sum_{|i_l| \leq n^{\widehat{\omega}}} \frac{\eta_l}{(\mu_{i_l}^{(l)}(ct_f)\nu_{z_l})^2 + \eta_l^2} + \mathcal{O}(\Psi). \tag{7.35}$$

Finally, combining (7.32)–(7.35) we conclude the proof of Proposition 7.2. □

We remark that in order to prove (7.35) it would not be necessary to introduce the additional comparison processes $\widetilde{\boldsymbol{\lambda}}^{(l)}$ and $\widetilde{\boldsymbol{\mu}}^{(l)}$ of Section 7.1.2, since in (7.35) the product is outside the expectation, so one can compare the expectations one by one;

the correlation between these processes for different l's plays no role. Hence, already the usual coupling (see e.g. [15, 18, 45]) between the processes $\boldsymbol{\lambda}^{z_l}(t)$, $\boldsymbol{\mu}^{(l)}(t)$ defined in (7.14) and (7.24), respectively, would be sufficient to prove (7.35). On the other hand, the comparison processes $\mathring{\boldsymbol{\lambda}}^{z_l}(t)$ are anyway needed in order to remove the coefficients Λ_{ij} (which are small with very high probability) from the interaction term in (7.14).

We conclude this section with the proof of Lemma 7.10.

Proof of Lemma 7.10. In the following, to simplify notations, we assume that the scaling factors ν_{z_l} are equal to one. First of all, we notice that the summation over the indices $n^{\widehat{\omega}} < |i| \le n$ in (7.6) can be removed, using the eigenvalue rigidity (7.2) similarly to [22, Eq. (7.6)–(7.7)], at a price of an additional error term $n^{2(\delta_1+\delta_0)-\widehat{\omega}}$:

$$\mathbf{E}\prod_{l=1}^2 \frac{1}{n}\sum_{|i_l|\le n^{\widehat{\omega}}} \frac{\eta_l}{(\lambda_{i_l}(H^{z_l}))^2+\eta_l^2} = \mathbf{E}\prod_{l=1}^2 \frac{1}{n}\sum_{|i_l|\le n} \frac{\eta_l}{(\lambda_{i_l}(H^{z_l}))^2+\eta_l^2} + \mathcal{O}\left(\frac{n^{2(\delta_1+\delta_0)}}{n^{\widehat{\omega}}}\right). \tag{7.36}$$

The error term is negligible by choosing δ_0, δ_1 to be such that $\omega_h \ll \delta_m \ll \widehat{\omega}$, for $m = 0, 1$. Then, from the GFT Lemma 7.3, and (7.8), using (7.36) again, this time for $\lambda_{i_l}^{z_l}(ct_f)$, we have that

$$\begin{aligned}\mathbf{E}\prod_{l=1}^2 \frac{1}{n}\sum_{|i_l|\le n^{\widehat{\omega}}} \frac{\eta_l}{(\lambda_{i_l}(H^{z_l}))^2+\eta_l^2} &= \mathbf{E}\prod_{l=1}^2 \frac{1}{n}\sum_{|i_l|\le n^{\widehat{\omega}}} \frac{\eta_l}{(\lambda_{i_l}^{z_l}(ct_f))^2+\eta_l^2} \\ &\quad + \mathcal{O}\left(\frac{n^{2\xi+2\delta_0}t_f}{n^{1/2}}\sum_{l=1}^2\frac{1}{\eta_l} + \frac{n^{2(\delta_1+\delta_0)}}{n^{\widehat{\omega}}}\right).\end{aligned} \tag{7.37}$$

We remark that the rigidity for $\lambda_{i_l}^{z_l}(ct_f)$ is obtained by Theorem 3.1 exactly as in (7.2). Next, by the same computations as in [22, Lemma 7.8] by writing the difference of l.h.s. and r.h.s. of (7.38) as a telescopic sum and then using the very high probability bound from Proposition 7.7 we get

$$\mathbf{E}\prod_{l=1}^2 \frac{1}{n}\sum_{|i_l|\le n^{\widehat{\omega}}} \frac{\eta_l}{(\lambda_{i_l}^{z_l}(ct_f))^2+\eta_l^2} = \mathbf{E}\prod_{l=1}^2 \frac{1}{n}\sum_{|i_l|\le n^{\widehat{\omega}}} \frac{\eta_l}{(\mathring{\lambda}_{i_l}^{(l)}(ct_f))^2+\eta_l^2} + \mathcal{O}(\Psi). \tag{7.38}$$

Similarly to (7.38), by Lemma 7.8 it also follows that

$$\mathbf{E}\prod_{l=1}^2 \frac{1}{n}\sum_{|i_l|\le n^{\widehat{\omega}}} \frac{\eta_l}{(\mathring{\lambda}_{i_l}^{z_l}(ct_f))^2+\eta_l^2} = \mathbf{E}\prod_{l=1}^2 \frac{1}{n}\sum_{|i_l|\le n^{\widehat{\omega}}} \frac{\eta_l}{(\widetilde{\lambda}_{i_l}^{(l)}(ct_f))^2+\eta_l^2} + \mathcal{O}(\Psi). \tag{7.39}$$

By (7.26) it readily follows that

$$\mathbf{E}\prod_{l=1}^2 \frac{1}{n}\sum_{|i_l|\le n^{\widehat{\omega}}} \frac{\eta_l}{(\widetilde{\lambda}_{i_l}^{(l)}(ct_f))^2+\eta_l^2} = \mathbf{E}\prod_{l=1}^2 \frac{1}{n}\sum_{|i_l|\le n^{\widehat{\omega}}} \frac{\eta_l}{(\widetilde{\mu}_{i_l}^{(l)}(ct_f))^2+\eta_l^2}. \tag{7.40}$$

Moreover, by (7.31), similarly to (7.38), we conclude

$$\mathbf{E}\prod_{l=1}^2 \frac{1}{n}\sum_{|i_l|\le n^{\widehat{\omega}}} \frac{\eta_l}{(\widetilde{\mu}_{i_l}^{(l)}(ct_f))^2+\eta_l^2} = \mathbf{E}\prod_{l=1}^2 \frac{1}{n}\sum_{|i_l|\le n^{\widehat{\omega}}} \frac{\eta_l}{(\mu_{i_l}^{(l)}(ct_f))^2+\eta_l^2} + \mathcal{O}(\Psi). \tag{7.41}$$

Combining (7.37)–(7.41), we conclude the proof of (7.32). □

Finally, we conclude Section 7.1 by listing the scales needed in the entire Section 7 and explain the dependences among them.

7.1.4 Relations among the scales in the proof of Proposition 7.2

Throughout Section 7 various scales are characterized by exponents of n, denoted by ω's, that we will also refer to scales for simplicity.

All the scales in the proof of Proposition 7.2 depend on the exponents $\omega_d, \omega_h, \omega_f \ll 1$. We recall that ω_d, ω_h are the exponents such that Lemma 7.4 on eigenvector overlaps holds under the assumption $|z_l - z_m|, |z_l - \overline{z}_m|, |z_l - \overline{z}_l| \geq n^{-\omega_d}$, and $|z_l| \leq 1 - n^{-\omega_h}$. The exponent ω_f determines the time $t_f = n^{-1+\omega_f}$ to run the DBM so that it reaches its local equilibrium and thus to prove the asymptotic independence of $\lambda_i^{z_l}(ct_f)$ and $\lambda_j^{z_m}(ct_f)$, with $c > 0$ defined in (7.5), for small indices i, j and $l \neq m$.

The most important scales in the proof of Proposition 7.2 are $\omega, \widehat{\omega}, \delta_0, \delta_1, \omega_E$. The scale ω_E is determined in Lemma 7.4 and it controls the correlations among the driving martingales originating from the eigenvector overlaps in (7.11)–(7.13). The scale ω gives the $n^{-1-\omega}$ precision of the coupling between various processes while $\widehat{\omega}$ determines the range of indices $|i| \leq n^{\widehat{\omega}}$ for which this coupling is effective. These scales are chosen much bigger than ω_h and they are determined in Proposition 7.7, Lemma 7.8 and Lemma 7.9, that describe these couplings. Each of these results gives an upper bound on the scales $\omega, \widehat{\omega}$, at the end we will choose the smallest of them. Finally, δ_0, δ_1 describe the scale of the range of the η's in Proposition 7.2. These two scales are determined in Lemma 7.10, given $\omega, \widehat{\omega}$ from the previous step. Putting all these steps together, we constructed $\omega, \widehat{\omega}, \delta_0, \delta_1$ claimed in Proposition 7.2 and hence also in Proposition 3.4. These scales are related as

$$\omega_h \ll \delta_m \ll \widehat{\omega} \ll \omega \ll \omega_f \ll \omega_E \ll 1, \qquad \omega_E = 4\omega_d, \tag{7.42}$$

for $m = 0, 1$.

Along the proof of Proposition 7.2 four auxiliary scales, $\omega_L, \omega_A, \omega_r, \omega_c$, are also introduced. The scale ω_L describes the range of interaction in the short range approximation processes $\widehat{\boldsymbol{x}}^{z_l}(t, \alpha)$ (see (7.60) later), while ω_A is the scale for which we can (partially) couple the driving martingales of the regularized processes $\mathring{\boldsymbol{\lambda}}^{z_l}(t)$ with the driving Brownian motions of Ginibre processes $\boldsymbol{\mu}^{(l)}(t)$. The scale ω_c is a cut-off in the energy estimate in Lemma 7.13, see (7.68). Finally, ω_r reduces the variance of the driving martingales by a factor $(1 + n^{-\omega_r})^{-1}$ to ensure the well-posedness of the processes $\mathring{\boldsymbol{\lambda}}^{z_l}(t)$, $\widetilde{\boldsymbol{\lambda}}^{(l)}(t)$, $\widetilde{\boldsymbol{\mu}}^{(l)}$, $\boldsymbol{x}^{z_l}(t, \alpha)$ defined in (7.22), (7.27), (7.29), and (7.48), respectively. These scales are inserted in the chain (7.42) as follows

$$\omega_h \ll \omega_A \ll \omega_f \ll \omega_L \ll \omega_c \ll \omega_r \ll \omega_E. \tag{7.43}$$

Note that there are no relations required among ω_A and $\omega, \widehat{\omega}, \delta_m$.

7.2 Universality and independence of the singular values of $X - z_1, X - z_2$ close to zero: proof of Theorems 2.8 and 2.10

In the following we present only the proof of Theorem 2.10, since the proof of Theorem 2.8 proceeds exactly in the same way. Universality of the joint distribution of the singular values of $X - z_1$ and $X - z_2$ follows by universality for the joint distribution of the eigenvalues of H^{z_1} and H^{z_2}, which is defined in (1.2), since the eigenvalues of H^{z_l} are exactly the singular values of $X - z_l$ taken with positive and negative sign. From now on we only consider the eigenvalues of H^{z_l}, with $z_l \in \mathbf{C}$ such that $|\Im z_l| \sim 1$, $|z_1 - z_2|, |z_1 - \overline{z}_2| \sim 1$, and $|z_l| \leq 1 - \epsilon$ for some small fixed $\epsilon > 0$.

For $l \in [2]$, denote by $\{\lambda_i^{z_l}\}_{|i| \leq n}$ the eigenvalues of H^{z_l} and by $\{\lambda_i^{z_l}(t)\}_{|i| \leq n}$ their evolution under the DBM flow (7.14). Define $\{\mu_i^{(l)}(t)\}_{|i| \leq n}$, for $l \in [2]$, to be the solution of (7.24) with initial data $\{\mu_i^{(l)}\}_{|i| \leq n}$, which are the eigenvalues of independent complex

Ginibre matrices $\widetilde{X}^{(1)}, \widetilde{X}^{(2)}$. Then, defining the comparison processes $\mathring{\boldsymbol{\lambda}}^{z_l}(t)$, $\widetilde{\boldsymbol{\lambda}}^{(l)}(t)$, $\widetilde{\boldsymbol{\mu}}^{(l)}(t)$ as in Sections 7.1.1–7.1.2, and combining Proposition 7.7, Lemma 7.8, and Lemma 7.9, we conclude that for any sufficiently small $\omega_f > 0$ there exist $\omega, \widehat{\omega} > 0$ such that $\widehat{\omega} \ll \omega \ll \omega_f$, and that

$$|\rho^{z_l}(0)\lambda_i^{z_l}(ct_f) - \rho_{\mathrm{sc}}(0)\mu_i^{(l)}(ct_f)| \le n^{-1-\omega}, \qquad |i| \le n^{\widehat{\omega}}, \tag{7.44}$$

with very high probability, with $c > 0$ defined in (7.5).

Then, by a simple Green's function comparison argument (GFT) as in Lemma 7.3, using (7.44), by exactly the same computations as in the proof of [21, Proposition 3.1 in Section 7] adapted to the bulk scaling, i.e. changing $\mathfrak{b}_{r,t_1} \to 0$ and $N^{3/4} \to 2n$, using the notation therein, we conclude Theorem 2.10.

7.3 Bound on the eigenvector overlaps

In this section we prove the bound on the eigenvector overlaps, as stated in Lemma 7.4. For any $T > 0$, and any $t \in [0, T]$, denote by ρ_t^z the self consistent density of states (scDOS) of the Hermitised matrix H_t^z, and define its quantiles by

$$\frac{i}{n} = \int_0^{\gamma_i^z(t)} \rho_t^z(x)\,\mathrm{d}x, \qquad i \in [n], \tag{7.45}$$

and $\gamma_{-i}^z(t) = -\gamma_i^z(t)$ for $i \in [n]$. Similarly to (7.2), as a consequence of Theorem 3.1 and the fact that the eigenvalues of $H_t^{z_l}$ are γ-Hölder continuous in time for any $\gamma \in (0, 1/2)$ by Weyl's inequality, by standard application of Helffer-Sjöstrand formula, we conclude the following rigidity bound

$$\sup_{0 \le t \le T} |\lambda_i^{z_l}(t) - \gamma_i^{z_l}(t)| \le \frac{n^{100\omega_h}}{n^{2/3}(n+1-i)^{1/3}}, \qquad i \in [n], \tag{7.46}$$

with very high probability, uniformly in $|z_l| \le 1 - n^{-\omega_h}$. A bound similar to (7.46) holds for negative indices as well. We remark that the Hölder continuity of the eigenvalues of $H_t^{z_l}$ is used to prove (7.46) uniformly in time, using a standard grid argument.

The main input to prove Lemma 7.4 is Theorem 3.5 combined with Lemma 6.1.

Proof of Lemma 7.4. Recall that $P_1 \boldsymbol{w}_i^z = \boldsymbol{u}_i^z$ and $P_2 \boldsymbol{w}_i^z = \mathrm{sign}(i)\boldsymbol{v}_i^z$, for $|i| \le n$, by (7.9). In the following we consider $z, z' \in \mathbf{C}$ such that $|z|, |z'| \le 1 - n^{-\omega_h}$, $|z - z'| \ge n^{-\omega_d}$, for some sufficiently small $\omega_h, \omega_d > 0$.

Eigenvector overlaps can be estimated by traces of products of resolvents. More precisely, for any $\eta \ge n^{-2/3+\epsilon_*}$, for some small fixed $\epsilon_* > 0$, and any $|i_0|, |j_0| \le n$, using the rigidity bound (7.46), similarly to [22, Eq. (7.43)], we have that

$$\begin{aligned} |\langle \boldsymbol{u}_{i_0}^z(t), \boldsymbol{u}_{j_0}^{z'}(t)\rangle|^2 &\lesssim \eta^2 \operatorname{Tr}\big(\Im G^z(\gamma_{i_0}^z(t) + \mathrm{i}\eta)\big)E_1\big(\Im G^{z'}(\gamma_{j_0}^{z'}(t) + \mathrm{i}\eta)\big)E_1, \\ |\langle \boldsymbol{v}_{i_0}^z(t), \boldsymbol{v}_{j_0}^{z'}(t)\rangle|^2 &\lesssim \eta^2 \operatorname{Tr}\big(\Im G^z(\gamma_{i_0}^z(t) + \mathrm{i}\eta)\big)E_2\big(\Im G^{z'}(\gamma_{j_0}^{z'}(t) + \mathrm{i}\eta)\big)E_2, \end{aligned} \tag{7.47}$$

with E_1, E_2 defined in (5.8). By Theorem 3.5, combined with Lemma 6.1, choosing $\eta = n^{-12/23}$, say, the error term in the r.h.s. of (3.17) is bounded by $n^{-1/23}n^{2\omega_d + 100\omega_h}$, hence we conclude the bound in (7.12) for any fixed time $t \in [0, T]$, choosing $\omega_E = -(2\omega_d + 100\omega_h - 1/23)$, for any $\omega_h \ll \omega_d \le 1/100$.

Moreover, the bound (7.12) holds uniformly in time by a union bound, using a standard grid argument and Hölder continuity in the form

$$\|\Im G_t^z \Im G_t^{z'} - \Im G_s^z \Im G_s^{z'}\| \lesssim n^3 \left(\|H_t^z - H_s^z\| + \|H_t^{z'} - H_s^{z'}\|\right) \lesssim n^{7/2}|t - s|^{1/2}$$

for any $s, t \in [0, T]$, where the spectral parameters in the resolvents have imaginary parts at least $\eta > 1/n$. This concludes the proof of Lemma 7.4. $\square$

7.4 Proof of Proposition 7.7

Throughout this section we use the notation $z = z_l$, with $l \in [2]$, with z_1, z_2 fixed as in Proposition Proposition 7.7.

Remark 7.11. In the remainder of this section we assume that $|z| \le 1 - \epsilon$, with some positive $\epsilon > 0$ instead of $n^{-\omega_h}$, in order to make our presentation clearer. One may follow the ϵ-dependence throughout the proofs and find that all the estimates deteriorate with some fixed ϵ^{-1} power, say ϵ^{-100}. Thus, when $|z| \le 1 - n^{-\omega_h}$ is assumed, we get an additional factor $n^{100\omega_h}$ but this does not play any role since ω_h is the smallest exponent (e.g. see Proposition 7.7) in the analysis of the processes (7.14), (7.22).

The proof of Proposition 7.7 consists of several parts that we first sketch. The process $\mathring{\boldsymbol{\lambda}}^z(t)$ differs from $\boldsymbol{\lambda}^z(t)$ in three aspects: (i) the coefficients $\Lambda^z_{ij}(t)$ in the SDE (7.14) for $\boldsymbol{\lambda}^z(t)$ are removed; (ii) large values of the correlation of the driving martingales is cut off, and (iii) the martingale term is slightly reduced by a factor $(1 + n^{\omega_r})^{-1/2}$. We deal with these differences in two steps. The substantial step is the first one, from Section 7.4.1 to Section 7.4.4, where we handle (i) by interpolation, using short range approximation and energy method. This is followed by a more technical second step in Section 7.4.5, where we handle (ii) and (iii) using a stopping time controlled by a well chosen Lyapunov function to show that the correlation typically remains below the cut-off level.

A similar analysis has been done in [17, Section 4] (which has been used in the singular value setup in [19, Eq. (3.13)]) but our more complicated setting requires major modifications. In particular, (7.14) has to be compared to [17, Eq. (4.1)] with $\mathrm{d}M_i = 0$, $Z_i = 0$, and identifying Λ^z_{ij} with γ_{ij}, using the notations therein. One major difference is that we now have a much weaker estimate $|\Lambda^z_{ij}| \le n^{-\omega_E}$ than the bound $|\gamma_{ij}| \le n^{-1+a}$, for some small fixed $a > 0$, used in [17]. We therefore need to introduce an additional cut-off function $\boldsymbol{\chi}$ in the energy estimate in Section 7.4.4.

7.4.1 Interpolation process

In order to compare the processes $\boldsymbol{\lambda}^z$ and $\mathring{\boldsymbol{\lambda}}^z$ from (7.14) and (7.22) we start with defining an interpolation process, for any $\alpha \in [0, 1]$, as

$$\mathrm{d}x^z_i(t,\alpha) = \frac{\mathrm{d}\mathring{b}^z_i}{\sqrt{n(1+n^{-\omega_r})}} + \frac{1}{2n}\sum_{j\neq i}\frac{1+\alpha\mathring{\Lambda}^z_{ij}(t)}{x^z_i(t,\alpha)-x^z_j(t,\alpha)}\,\mathrm{d}t, \quad x^z_i(0,\alpha) = \lambda^z_i(0), \quad |i| \le n. \tag{7.48}$$

We recall that $\omega_f \ll \omega_r \ll \omega_E$. We use the notation $x^z_i(t,\alpha)$ instead of $z_i(t,\alpha)$ as in [17, Eq. (4.12)] to stress the dependence of $x^z_i(t,\alpha)$ on $z \in \mathbf{C}$. The well-posedness of the process (7.48) is proven in Appendix A for any fixed $\alpha \in [0, 1]$. In particular, the particles keep their order $x^z_i(t,\alpha) < x^z_{i+1}(t,\alpha)$. Additionally, by Lemma A.2 it follows that the differentiation with respect to α of the process $\boldsymbol{x}^z(t,\alpha)$ is well-defined.

Note that the process $\boldsymbol{x}^z(t,\alpha)$ does not fully interpolate between $\mathring{\boldsymbol{\lambda}}^z(t)$ and $\boldsymbol{\lambda}^z(t)$; it handles only the removal of the $\mathring{\Lambda}_{ij}$ term. Indeed, it holds $\boldsymbol{x}^z(t,0) = \mathring{\boldsymbol{\lambda}}^z(t)$ for any $t \in [0,T]$, but $\boldsymbol{x}^z(t,1)$ is not equal to $\boldsymbol{\lambda}^z(t)$. Thus we will proceed in two steps as already explained:

Step 1 The process $\boldsymbol{x}^z(t,\alpha)$ does not change much in $\alpha \in [0,1]$ for particles close to zero (by Lemma 7.13 below), i.e. $x^z_i(t,1) - x^z_i(t,0)$ is much smaller than the rigidity scale $1/n$ for small indices;

Step 2 The process $\boldsymbol{x}^z(t,1)$ is very close to $\boldsymbol{\lambda}^z(t)$ for all indices (see Lemma 7.14 below).

We start with the analysis of the interpolation process $\boldsymbol{x}^z(t,\alpha)$, then in Section 7.4.5 we state and prove Lemma 7.14.

7.4.2 Local law for the interpolation process

In order to analyse the interpolation process $\boldsymbol{x}^z(t,\alpha)$, we first need to establish a local law for the Stieltjes transform of the empirical particle density. This will be used for a rigidity estimate to identify the location of $x_i(t,\alpha)$ with a precision $n^{-1+\epsilon}$, for some small $\epsilon>0$, that is above the final target precision but it is needed as an a priori bound. Note that, unlike for $\boldsymbol{\lambda}^z(t)$, for $\boldsymbol{x}^z(t,\alpha)$ there is no obvious matrix ensemble behind this process, so local law and rigidity have to be proven directly from its defining equation (7.48).

Define the Stieltjes transform of the empirical particle density by

$$m_n(w,t,\alpha)=m_n^z(w,t,\alpha):=\frac{1}{2n}\sum_{|i|\le n}\frac{1}{x_i^z(t,\alpha)-w}, \tag{7.49}$$

and denote the Stieltjes transform of ρ^z, the *self-consistent density of states (scDOS)* of H^z, by $m^z(w)$. Moreover, we denote the Stieltjes transform of ρ_t^z, the free convolution of ρ^z with the semicircular flow up to time t, by $m_t^z(w)$. Using the definition of the quantiles $\gamma_i^z(t)$ in (7.45), by Theorem 3.1 we have that

$$\begin{aligned}\sup_{|\Re w|\le 10c_1}\ \sup_{n^{-1+\gamma}\le\Im w\le 10}\ \sup_{\alpha\in[0,1]}|m_n(w,0,\alpha)-m^z(w)|&\le\frac{n^\xi C_\epsilon}{n\Im w},\\ \sup_{|i|\le 10c_2 n}\ \sup_{\alpha\in[0,1]}|x_i^z(0,\alpha)-\gamma_i^z(0)|&\le\frac{C_\epsilon n^\xi}{n},\end{aligned} \tag{7.50}$$

with very high probability for any $\xi>0$, uniformly in $|z|\le 1-\epsilon$, for some small fixed $c_1,c_2,\gamma>0$. We recall that $C_\epsilon\le\epsilon^{-100}$. The rigidity bound in the second line of (7.50) follows by a standard application of Helffer-Sjöstrand formula.

In Lemma 7.12 we prove that (7.50) holds true uniformly in $0\le t\le t_f$. For its proof, similarly to [17, Section 4.5], we follow the analysis of [40, Section 3.2] using (7.50) as an input.

Lemma 7.12 (Local law and rigidity)**.** *Fix $|z|\le 1-\epsilon$, and assume that* (7.50) *holds with some $\gamma,c_1,c_2,C_\epsilon>0$, then*

$$\begin{aligned}\sup_{|\Re w|\le 10c_1}\ \sup_{n^{-1+\gamma}\le\Im w\le 10}\ \sup_{\alpha\in[0,1]}\ \sup_{0\le t\le t_f}|m_n^z(w,t,\alpha)-m_t(w)|&\le\frac{C_\epsilon n^\xi}{n\Im w},\\ \sup_{|i|\le 10c_2 n}\ \sup_{\alpha\in[0,1]}\ \sup_{0\le t\le t_f}|x_i^z(t,\alpha)-\gamma_i^z(t)|&\le\frac{C_\epsilon n^\xi}{n},\end{aligned} \tag{7.51}$$

with very high probability for any $\xi>0$, with $\gamma_i^z(t)\sim i/n$ for $|i|\le 10c_2n$ and $t\in[0,t_f]$.

Proof. Differentiating (7.49), by (7.48) and Itô's formula, we get

$$\begin{aligned}\mathrm{d}m_n=m_n(\partial_w m_n)\,\mathrm{d}t-\frac{1}{2n^{3/2}\sqrt{1+n^{-\omega_r}}}\sum_{|i|\le n}\frac{\mathrm{d}\mathring{b}_i}{(x_i-w)^2}+\frac{\alpha}{4n^2}\sum_{|i|,|j|\le n}\frac{\mathring{\Lambda}_{ij}}{(x_i-w)^2(x_j-w)}\,\mathrm{d}t\\ +\frac{1}{4n^2}\sum_{|i|\le n}\frac{\big[1-\alpha-n^{-\omega_r}(1+n^{-\omega_r})^{-1}\big]\mathring{\Lambda}_{ii}}{(x_i-w)^3}\,\mathrm{d}t.\end{aligned} \tag{7.52}$$

Note that by (7.20)–(7.21) it follows that

$$\mathring{\Lambda}_{ij}(t)=\Lambda_{ij}(t),\qquad \big(\mathring{b}_i(s)\big)_{0\le s\le t}=\big(b_i(s)\big)_{0\le s\le t}, \tag{7.53}$$

with very high probability uniformly in $0\le t\le t_f$, where Λ_{ij} and $(b_i(s))_{0\le s\le t}$ are defined in (7.10)–(7.13) and (7.15)–(7.16), respectively.

The equation (7.52) is the analogue of [40, Eq. (3.20)] with some differences. First, the last two terms are new and need to be estimated, although the penultimate term in (7.52) already appeared in [17, Eq. (4.62)] replacing $\mathring{\Lambda}_{ij}$ by $\widehat{\gamma}_{ij}$, using the notation therein. Second, the martingales in the second term in the r.h.s. of (7.52) are correlated. Hence, in order to apply the results in [40, Section 3.2] we prove that these additional terms are bounded as in [17, Eq. (4.64)]. Note that in [17, Eq. (4.64)] the corresponding term to the penultimate term in the r.h.s. of (7.52) is estimated using that $\widehat{\gamma}_{ij} \le n^{-1+a}$, for some small $a > 0$. In our case, however, the bound on $|\mathring{\Lambda}|$ is much weaker and a crude estimate by absolute value is not affordable. We will use (7.53) and then the explicit form of Λ_{ij} in (7.10)–(7.13), that enables us to perform the two summations and write this term as the trace of the product of two operators (see (7.57) later).

Since $|\mathring{\Lambda}_{ii}| \le n^{-\omega_E}$ by its definition below (7.21), the last term in (7.52) is easily bounded by

$$\left| \frac{1}{4n^2} \sum_{|i|\le n} \frac{(1-\alpha-n^{-\omega_r}(1+n^{-\omega_r})^{-1})\mathring{\Lambda}_{ii}}{(x_i-w)^3} \right| \le \frac{\Im m_n(w)}{n^{1+\omega_E}(\Im w)^2}. \tag{7.54}$$

Next, we proceed with the estimate of the penultimate term in (7.52). Define the operators

$$T(t,\alpha) := \sum_{|i|\le n} f(x_i(t,\alpha)) \boldsymbol{w}_i(t)[\boldsymbol{w}_i(t)]^*, \qquad S(t,\alpha) := \sum_{|i|\le n} g(x_i(t,\alpha)) \overline{\boldsymbol{w}}_i(t)[\overline{\boldsymbol{w}}_i(t)]^*, \tag{7.55}$$

where $\{\boldsymbol{w}_i(t)\}_{|i|\le n}$ are the orthonormal eigenvectors in the definition of $\Lambda_{ij}(t)$ in (7.10), and for any fixed $w \in \mathbf{H}$ the functions $f, g \colon \mathbf{R} \to \mathbf{C}$ are defined as

$$f(x) := \frac{1}{(x-w)^2}, \qquad g(x) := \frac{1}{x-w}. \tag{7.56}$$

Then, using the definitions (7.55)–(7.56) and (7.53), we bound the last term in the first line of (7.52) as

$$\begin{aligned} \left| \frac{\alpha}{4n^2} \sum_{|i|,|j|\le n} \frac{\mathring{\Lambda}_{ij}}{(x_i-w)^2(x_j-w)} \, \mathrm{d}t \right| &= \left| \frac{\alpha}{2n^2} \left[\mathrm{Tr}\big(P_1 T P_2 P_2 S P_1\big) + \overline{\mathrm{Tr}\big(P_1 T P_2 P_2 S P_1\big)} \right] \right| \\ &\lesssim \frac{1}{n^2} \left[\Im w \, \mathrm{Tr}\big[P_1 T P_2 (P_1 T P_2)^*\big] + \frac{\mathrm{Tr}\big[P_1 S P_2 (P_1 S P_2)^*\big]}{\Im w} \right] \\ &\lesssim \frac{1}{n^2} \left[\Im w \sum_{|i|\le n} |f(x_i)|^2 + \frac{1}{\Im w} \sum_{|i|\le n} |g(x_i)|^2 \right] \lesssim \frac{\Im m_n(w)}{n(\Im w)^2}, \end{aligned} \tag{7.57}$$

with very high probability uniformly in $0 \le t \le t_f$. Note that in the first equality of (7.57) we used that $\mathring{\Lambda}_{ij}(t) = \Lambda_{ij}(t)$ for any $0 \le t \le t_f$ with very high probability by (7.53).

Finally, in order to conclude the proof, we estimate the martingale term in (7.52). For this purpose, using that $\mathbf{E}[\mathrm{d}\mathring{b}_i \, \mathrm{d}\mathring{b}_j \,|\, \mathcal{F}_{b,t}] = (\delta_{i,j} - \delta_{i,-j} + \mathring{\Lambda}_{ij})/2 \, \mathrm{d}t$ and proceeding similarly to (7.57), we estimate its quadratic variation by

$$\begin{aligned}\frac{1}{4n^3(1+n^{-\omega_r})}\sum_{|i|,|j|\le n}\frac{\mathbf{E}[\mathrm{d}\mathring{b}_i\,\mathrm{d}\mathring{b}_j\mid\mathcal{F}_{b,t}]}{(x_i-w)^2(x_j-\overline{w})^2}&=\frac{1}{8n^3(1+n^{-\omega_r})}\sum_{|i|\le n}\frac{1}{|x_i-w|^4}\,\mathrm{d}t\\&\quad+\frac{1}{8n^3(1+n^{-\omega_r})}\sum_{|i|\le n}\frac{1}{(x_i+w)^2(x_i-\overline{w})^2}\,\mathrm{d}t\\&\quad+\frac{1}{8n^3(1+n^{-\omega_r})}\sum_{|i|,|j|\le n}\frac{\mathring{\Lambda}_{ij}}{(x_i-w)^2(x_j-\overline{w})^2}\,\mathrm{d}t\\&\lesssim\frac{\Im m_n(w)}{n^2(\Im w)^3}+\frac{1}{n^3}\operatorname{Tr}\big[P_1TP_2(P_1TP_2)^*\big]\,\mathrm{d}t\\&\lesssim\frac{\Im m_n(w)}{n^2(\Im w)^3},\end{aligned}\tag{7.58}$$

where the operator T is defined in (7.55), and in the penultimate inequality we used that $\mathring{\Lambda}_{ij}(t)=\Lambda_{ij}(t)$ for any $0\le t\le t_f$ with very high probability.

Combining (7.54), (7.57), and (7.58) we immediately conclude the proof of the first bound in (7.51) using the arguments of [40, Section 3.2]. The rigidity bound in the second line of (7.51) follows by a standard application of Helffer-Sjöstrand (see also below (7.50)). □

7.4.3 Short range approximation

Since the main contribution to the dynamics of $x_i^z(t,\alpha)$ comes from the nearby particles, in this section we introduce a *short range approximation* process $\widehat{\boldsymbol{x}}^z(t,\alpha)$, which will very well approximate the original process $\boldsymbol{x}^z(t,\alpha)$ (see (7.63) below). The actual interpolation analysis comparing $\alpha=0$ and $\alpha=1$ will then be performed on the short range process $\widehat{\boldsymbol{x}}^z(t,\alpha)$ in Section 7.4.4.

Fix $\omega_L>0$ so that $\omega_f\ll\omega_L\ll\omega_E$, and define the index set

$$\mathcal{A}:=\{(i,j)\mid|i-j|\le n^{\omega_L}\}\cup\{(i,j)\mid|i|,|j|>5c_2n\},\tag{7.59}$$

with $c_2>0$ defined in (7.51). We remark that in [17, Eq. (4.69)] the notation ω_l is used instead of ω_L; we decided to change this notation in order to not create confusion with ω_l defined in [22, Eq. (7.67)]. Then we define the short range approximation $\widehat{\boldsymbol{x}}^z(t,\alpha)$ of the process $\boldsymbol{x}^z(t,\alpha)$ by

$$\begin{aligned}\mathrm{d}\widehat{x}_i^z(t,\alpha)&=\frac{\mathrm{d}\mathring{b}_i^z}{\sqrt{n}}+\frac{1}{2n}\sum_{\substack{j:(i,j)\in\mathcal{A},\\ j\ne i}}\frac{1+\alpha\mathring{\Lambda}_{ij}(t)}{\widehat{x}_i^z(t,\alpha)-\widehat{x}_j^z(t,\alpha)}\,\mathrm{d}t+\frac{1}{2n}\sum_{\substack{j:(i,j)\notin\mathcal{A},\\ j\ne i}}\frac{1}{x_i^z(t,0)-x_j^z(t,0)}\,\mathrm{d}t,\\ \widehat{x}_i^z(0,\alpha)&=x_i^z(0,\alpha),\qquad |i|\le n.\end{aligned}\tag{7.60}$$

The well-posedness of the process (7.60) follows by nearly identical computations as in the proof of Proposition A.1.

In order to check that the *short range approximation* $\widehat{\boldsymbol{x}}^z(t,\alpha)$ is close to the process $\boldsymbol{x}^z(t,\alpha)$, defined in (7.48), we start with a trivial bound on $|x_i^z(t,\alpha)-x_i^z(t,0)|$ (see (7.61) below) to estimate the difference of particles far away from zero in (7.62), for which we do not have the rigidity bound in (7.51). Notice that by differentiating (7.48) in α and estimating $|\mathring{\Lambda}_{ij}|$ trivially by $n^{-\omega_E}$, it follows that

$$\sup_{0\le t\le t_f}\sup_{|i|\le n}\sup_{\alpha\in[0,1]}|x_i^z(t,\alpha)-x_i^z(t,0)|\lesssim n^{-\omega_E/2},\tag{7.61}$$

similarly to [17, Lemma 4.3].

By the rigidity estimate (7.51), the weak global estimate (7.61) to estimate the contribution of the far away particles for which we do not know rigidity, and the bound $|\mathring{\Lambda}_{ij}| \le n^{-\omega_E}$ from (7.21) it follows that

$$\left| \frac{1}{2n} \sum_{\substack{j:(i,j)\notin\mathcal{A}, \\ j\neq i}} \frac{1}{x_i^z(t,0)-x_j^z(t,0)} - \frac{1}{2n} \sum_{\substack{j:(i,j)\notin\mathcal{A}, \\ j\neq i}} \frac{1+\alpha\mathring{\Lambda}_{ij}(t)}{x_i^z(t,\alpha)-x_j^z(t,\alpha)} \right| \lesssim n^{-\omega_E/2} + n^{-\omega_L+\xi}, \tag{7.62}$$

for any $\xi > 0$ with very high probability uniformly in $0 \le t \le t_f$. Hence, by exactly the same computations as in [45, Lemma 3.8], it follows that

$$\sup_{\alpha\in[0,1]} \sup_{|i|\le n} \sup_{0\le t\le t_f} |x_i^z(t,\alpha) - \widehat{x}_i^z(t,\alpha)| \le \frac{n^{2\omega_f}}{n} \left(\frac{1}{n^{\omega_E/2}} + \frac{1}{n^{\omega_L}} \right). \tag{7.63}$$

Note that (7.63) implies that the second estimate in (7.51) holds with x_i^z replaced by $\widehat{x}_i^z$. In order to conclude the proof of Proposition 7.7 in the next section we differentiate in the process $\widehat{\boldsymbol{x}}^z$ in α and study the deterministic (discrete) PDE we obtain from (7.60) after the α-derivation. Note that the α-derivative of $\widehat{\boldsymbol{x}}^z$ is well defined by Lemma A.2.

7.4.4 Energy estimate

Define $v_i = v_i^z(t,\alpha) := \partial_\alpha \widehat{x}_i^z(t,\alpha)$, for any $|i| \le n$. In the remainder of this section we may omit the z-dependence since the analysis is performed for a fixed $z \in \mathbf{C}$ such that $|z| \le 1-\epsilon$, for some small fixed $\epsilon > 0$. By (7.60) it follows that $\boldsymbol{v}$ is the solution of the equation

$$\partial_t v_i = -(B\boldsymbol{v})_i + \xi_i, \quad v_i(0) = 0, \qquad |i| \le n, \tag{7.64}$$

where

$$(B\boldsymbol{v})_i := \sum_{j:(i,j)\in\mathcal{A}} B_{ij}(v_j - v_i), \quad B_{ij} = B_{ij}(t,\alpha) := \frac{1+\alpha\mathring{\Lambda}_{ij}(t)}{2n(\widehat{x}_i(t,\alpha)-\widehat{x}_j(t,\alpha))^2} \mathbf{1}((i,j)\in\mathcal{A}), \tag{7.65}$$

and

$$\xi_i = \xi_i(t,\alpha) := \frac{1}{2n} \sum_{j:(i,j)\in\mathcal{A}} \frac{\mathring{\Lambda}_{ij}(t)}{\widehat{x}_i(t,\alpha)-\widehat{x}_j(t,\alpha)}.$$

Before proceeding with the optimal estimate of the ℓ^∞-norm of $\boldsymbol{v}$ in (7.67), we give the following crude bound

$$\sup_{|i|\le n} \sup_{0\le t\le t_f} \sup_{\alpha\in[0,1]} |v_i(t,\alpha)| \lesssim 1, \tag{7.66}$$

that will be needed as an a priori estimate for the more precise result later. The bound (7.66) immediately follows by exactly the same computations as in [17, Lemma 4.7] using that $|\mathring{\Lambda}_{ij}| \le n^{-\omega_E}$.

The main technical result to prove Step 1 towards Proposition 7.7 is the following lemma. In particular, after integration in α, Lemma 7.13 proves that the processes $\boldsymbol{x}^z(t,1)$ and $\boldsymbol{x}^z(t,0)$ are closer than the rigidity scale $1/n$.

Lemma 7.13. *For any small $\omega_f > 0$ there exist small constants $\omega, \widehat{\omega} > 0$ such that $\widehat{\omega} \ll \omega \ll \omega_f$ and*

$$\sup_{\alpha\in[0,1]} \sup_{|i|\le n^{\widehat{\omega}}} \sup_{0\le t\le t_f} |v_i(t)| \le n^{-1-\omega}, \tag{7.67}$$

with very high probability.

This lemma is based upon the finite speed of propagation mechanism for the dynamics (7.64) [31, Lemma 9.6]. Our proof follows [13, Lemma 6.2] that introduced a carefully chosen special cut-off function.

Proof. In order to bound $|v_i(t)|$ for small indices we will bound $\|\boldsymbol{v}\boldsymbol{\chi}\|_\infty$ for an appropriate cut-off vector $\boldsymbol{\chi}$ supported at a few coordinates around zero. More precisely, we will use an energy estimate to control $\|\boldsymbol{v}\boldsymbol{\chi}\|_2$ and then we use the trivial bound $\|\boldsymbol{v}\boldsymbol{\chi}\|_\infty \le \|\boldsymbol{v}\boldsymbol{\chi}\|_2$. This bound would be too crude without the cut-off.

Let $\varphi(x)$ be a smooth cut-off function which is equal to zero for $|x| \ge 1$, it is equal to one if $|x| \le 1/2$. Fix a small constant $\omega_c > 0$ such that $\omega_f \ll \omega_L \ll \omega_c \ll \omega_E$, and define

$$\chi(x) := e^{-2xn^{1-\omega_c}}\varphi((2c_2)^{-1}x), \tag{7.68}$$

for any $x > 0$, with the constant $c_2 > 0$ defined in (7.51). It is trivial to see that χ is Lipschitz, i.e.

$$|\chi(x) - \chi(y)| \lesssim e^{-(x\wedge y)n^{1-\omega_c}}|x-y|n^{1-\omega_c}, \tag{7.69}$$

for any $x, y \ge 0$, and that

$$|\chi(x) - \chi(y)| \lesssim e^{-(x+y)n^{1-\omega_c}}|x-y|n^{1-\omega_c}, \tag{7.70}$$

if additionally $|x-y| \le n^{\omega_c}/(2n)$. Finally we define the vector $\boldsymbol{\chi}$ by

$$\chi_i = \chi(\widehat{x}_i) := e^{-2|\widehat{x}_i|n^{1-\omega_c}}\varphi((2c_1)^{-1}\widehat{x}_i). \tag{7.71}$$

Note that χ_i is exponentially small if $n^{3\omega_c/2} \le |i| \le n$ by rigidity (7.51) and the fact that $\gamma_i^z \sim i/n$, for $n^{3\omega_c/2} \le |i| \le 10c_2n$. We remark that the lower bound $n^{3\omega_c/2}$ on $|i|$ is arbitrary, since χ_i is exponentially small for any $|i|$ much bigger than n^{ω_c}. Moreover, as a consequence of (7.51) we have that

$$\widehat{x}_i \sim \frac{i}{n} \quad \text{for} \quad n^\xi \le |i| \le 10c_2n, \tag{7.72}$$

with very high probability for any $\xi > 0$.

By (7.64) it follows that

$$\begin{aligned}
\partial_t\|\boldsymbol{v}\boldsymbol{\chi}\|_2^2 = \partial_t \sum_{|i|\le n} v_i^2\chi_i^2 &= -2\sum_i \chi_i^2 v_i(Bv)_i + \frac{1}{n}\sum_{(i,j)\in\mathcal{A}} \frac{\chi_i^2 v_i \mathring{\Lambda}_{ij}}{\widehat{x}_i - \widehat{x}_j} \\
&= -\sum_{(i,j)\in\mathcal{A}} B_{ij}(v_i\chi_i - v_j\chi_j)^2 + \frac{1}{2n}\sum_{(i,j)\in\mathcal{A}} \frac{(v_i\chi_i - v_j\chi_j)\mathring{\Lambda}_{ij}}{\widehat{x}_i - \widehat{x}_j}\chi_i \\
&\quad + \sum_{(i,j)\in\mathcal{A}} B_{ij}v_iv_j(\chi_i-\chi_j)^2 + \frac{1}{2n}\sum_{(i,j)\in\mathcal{A}} \frac{(\chi_i-\chi_j)\mathring{\Lambda}_{ij}}{\widehat{x}_i - \widehat{x}_j} v_j\chi_j,
\end{aligned} \tag{7.73}$$

where, in order to symmetrize the sums, we used that the operator B and the set $\mathcal{A}$ are symmetric, i.e. $B_{ij} = B_{ji}$ (see (7.65)) and $(i,j) \in \mathcal{A} \Leftrightarrow (j,i) \in \mathcal{A}$, and that $\mathring{\Lambda}_{ij} = \mathring{\Lambda}_{ji}$.

We start estimating the terms in the second line of the r.h.s. of (7.73). The most critical term is the first one because of the $(\widehat{x}_i - \widehat{x}_j)^{-2}$ singularity of B_{ij}. We write this term as

$$\sum_{(i,j)\in\mathcal{A}} B_{ij}v_iv_j(\chi_i-\chi_j)^2 = \left(\sum_{\substack{(i,j)\in\mathcal{A},\\ |i-j|\le n^{\omega_L}}} + \sum_{\substack{(i,j)\in\mathcal{A},\\ |i-j|> n^{\omega_L}}} \right) B_{ij}v_iv_j(\chi_i-\chi_j)^2. \tag{7.74}$$

Then, using (7.70), $\|\boldsymbol{v}\|_\infty \lesssim 1$ by (7.66), $|\mathring{\Lambda}_{ij}| \le n^{-\omega_E}$ by (7.21), the rigidity (7.72), and that $\omega_L \ll \omega_c$, we bound the first sum by

$$\begin{aligned}
&\left|\sum_{\substack{(i,j)\in\mathcal{A},\\ |i-j|\le n^{\omega_L}}} B_{ij} v_i v_j (\chi_i-\chi_j)^2\right| \\
&\quad\lesssim \frac{1}{n}\sum_{\substack{(i,j)\in\mathcal{A},\\ |i-j|\le n^{\omega_L}}} \frac{1+|\mathring{\Lambda}_{ij}|}{(\widehat{x}_i-\widehat{x}_j)^2}|v_i v_j| \frac{n^2|\widehat{x}_i-\widehat{x}_j|^2}{n^{2\omega_c}} e^{-2(|\widehat{x}_i|+|\widehat{x}_j|)n^{1-\omega_c}} \\
&\quad\lesssim n^{1-2\omega_c}\left(\sum_{|i|,|j|\le n^{3\omega_c/2}} + \sum_{\substack{|i|\le n^{3\omega_c/2},|j|\ge n^{3\omega_c/2},\\ |i-j|\le n^{\omega_L}}}\right)|v_i||v_j| e^{-2(|\widehat{x}_i|+|\widehat{x}_j|)n^{1-\omega_c}} \\
&\quad\lesssim n^{1-\omega_c/2}\|\boldsymbol{v}\boldsymbol{\chi}\|_2^2 + e^{-\frac{1}{2}n^{\omega_c/2}},
\end{aligned} \tag{7.75}$$

with very high probability. In the last inequality we trivially inserted φ to reproduce $\boldsymbol{\chi}$, using that $\varphi((2c_2)^{-1}|\widehat{x}_i|) = \varphi((2c_2)^{-1}|\widehat{x}_j|) = 1$ with very high probability uniformly in $0 \le t \le t_f$ if $|i|, |j| \le c_2 n$ by the rigidity estimate in (7.72).

Define the set

$$\mathcal{A}_1 := \{(i,j) \mid |i|,|j| \ge 5c_2 n\} \cap \{(i,j) \mid |i-j| > n^{\omega_L}\} = \mathcal{A} \cap \{(i,j) \mid |i-j| > n^{\omega_l}\},$$

which is symmetric. The second sum in (7.74), using (7.69), (7.66), and rigidity from (7.72), is bounded by

$$\left|\sum_{(i,j)\in\mathcal{A}_1} B_{ij} v_i v_j (\chi_i-\chi_j)^2\right| \lesssim n^{1-2\omega_c}\sum_{(i,j)\in\mathcal{A}_1} e^{-2(|\widehat{x}_i|\wedge|\widehat{x}_j|)n^{1-\omega_c}} \le e^{-n/2}, \tag{7.76}$$

with very high probability.

Next, we consider the second term in the second line of the r.h.s. of (7.73). Using (7.70), and that $|\mathring{\Lambda}_{ij}| \le n^{-\omega_E}$, proceeding similarly to (7.75)–(7.76), we bound this term as

$$\begin{aligned}
\left|\frac{1}{n}\sum_{(i,j)\in\mathcal{A}} \frac{(\chi_i-\chi_j)\mathring{\Lambda}_{ij}}{\widehat{x}_i-\widehat{x}_j} v_j\chi_j\right| &\lesssim \left|\sum_{\substack{(i,j)\in\mathcal{A}\\ |i-j|\le n^{\omega_L}}} \frac{(\chi_i-\chi_j)\mathring{\Lambda}_{ij}}{n(\widehat{x}_i-\widehat{x}_j)} v_j\chi_j\right| + \left|\sum_{(i,j)\in\mathcal{A}_1} \frac{(\chi_i-\chi_j)\mathring{\Lambda}_{ij}}{n(\widehat{x}_i-\widehat{x}_j)} v_j\chi_j\right| \\
&\lesssim \sum_{\substack{(i,j)\in\mathcal{A}\\ |i-j|\le n^{\omega_L}}} \frac{|\mathring{\Lambda}_{ij}|}{|\widehat{x}_i-\widehat{x}_j|}\frac{|\widehat{x}_i-\widehat{x}_j|}{n^{\omega_c}}|v_j|\chi_j e^{-(|\widehat{x}_i|+|\widehat{x}_j|)n^{1-\omega_c}} + e^{-n/2} \\
&\lesssim \frac{1}{n^{\omega_c+\omega_E}}\sum_{|i|,|j|\le n^{3\omega_c/2}} |v_j|\chi_j + e^{-\frac{1}{2}n^{\omega_c/2}} \\
&\lesssim \frac{1}{n^{\omega_c/4+\omega_E}}\|\boldsymbol{v}\boldsymbol{\chi}\|_2 + e^{-\frac{1}{2}n^{\omega_c/2}},
\end{aligned} \tag{7.77}$$

with very high probability uniformly in $0 \le t \le t_f$.

Finally, we consider the first line in the r.h.s. of (7.73). Since $1+\alpha\mathring{\Lambda}_{ij} \ge 1/2$, we conclude that

$$\begin{aligned}\left|\frac{1}{n}\sum_{(i,j)\in\mathcal{A}}\frac{(v_i\chi_i-v_j\chi_j)\mathring{\Lambda}_{ij}}{\widehat{x}_i-\widehat{x}_j}\chi_i\right| \\ &\le \frac{1}{C}\sum_{(i,j)\in\mathcal{A}}B_{ij}(v_i\chi_i-v_j\chi_j)^2+\frac{C}{n}\sum_{(i,j)\in\mathcal{A}}|\mathring{\Lambda}_{ij}|^2\chi_i^2 \\ &\le \frac{1}{C}\sum_{(i,j)\in\mathcal{A}}B_{ij}(v_i\chi_i-v_j\chi_j)^2+\frac{C}{n}\sum_{|i|,|j|\le n^{3\omega_c/2}}|\mathring{\Lambda}_{ij}|^2\chi_i^2+e^{-\frac{1}{2}n^{\omega_c/2}} \\ &\le \frac{1}{C}\sum_{(i,j)\in\mathcal{A}}B_{ij}(v_i\chi_i-v_j\chi_j)^2+\frac{n^{3\omega_c}}{n^{1+2\omega_E}},\end{aligned} \tag{7.78}$$

for some large $C>0$. The error term in the r.h.s. of (7.78) is affordable since $\omega_c\ll\omega_E$.

Hence, combining (7.73)–(7.78), we conclude that

$$\partial_t\|\boldsymbol{v}\boldsymbol{\chi}\|_2^2\lesssim -\frac{1}{2}\sum_{(i,j)\in\mathcal{A}}B_{ij}(v_i\chi_i-v_j\chi_j)^2+n^{1-\omega_c/2}\|\boldsymbol{v}\boldsymbol{\chi}\|_2^2+n^{-\omega_c/4-\omega_E}\|\boldsymbol{v}\boldsymbol{\chi}\|_2+\frac{n^{3\omega_c}}{n^{1+2\omega_E}}, \tag{7.79}$$

with very high probability uniformly in $0\le t\le t_f$. Then, ignoring the negative first term, integrating (7.79) from 0 to $t_f=n^{-1+\omega_f}$, and using that $n^{1-\omega_c/2}t_f=n^{\omega_f-\omega_c/2}$ with $\omega_f\ll\omega_c\ll\omega_E$, we get

$$\sup_{0\le t\le t_f}\|\boldsymbol{v}\boldsymbol{\chi}\|_2^2\le\frac{n^{3\omega_c}t_f}{n^{1+2\omega_E}}.$$

Hence, using the bound

$$\sup_{0\le t\le t_f}\sup_{|i|\le n^{\widehat{\omega}}}|v_i(t)|\le\sup_{0\le t\le t_f}\|\boldsymbol{v}\boldsymbol{\chi}\|_2\le\sqrt{\frac{n^{3\omega_c}t_f}{n^{1+2\omega_E}}},$$

we conclude (7.67) for some $\omega,\widehat{\omega}>0$ such that $\widehat{\omega}\ll\omega\ll\omega_f\ll\omega_L\ll\omega_c\ll\omega_E$. □

With this proof we completed the main Step 1 in the proof of Proposition 7.7, the analysis of the interpolation process $\boldsymbol{x}^z(t,\alpha)$.

7.4.5 The processes $\boldsymbol{\lambda}(t)$ and $\boldsymbol{x}^z(t,1)$ are close

In Step 2 towards the proof of Proposition 7.7, we now prove that the processes $\boldsymbol{\lambda}(t)$ and $\boldsymbol{x}^z(t,1)$ are very close for any $t\in[0,t_f]$:

Lemma 7.14. *Let $\boldsymbol{\lambda}^z(t)$, $\boldsymbol{x}^z(t,1)$ be defined in* (7.14) *and* (7.48)*, respectively, and let $t_f=n^{-1+\omega_f}$, then*

$$\sup_{|i|\le n}\sup_{0\le t\le t_f}|x_i^z(t,1)-\lambda_i^z(t)|\lesssim\frac{n^{\omega_f}}{n^{1+\omega_r}}. \tag{7.80}$$

with very high probability.

Proof of Proposition 7.7. Proposition 7.7 follows by exactly the same computations as in [17, Section (4.10)], combining (7.80), (7.63), (7.66)–(7.67). □

Proof of Lemma 7.14. The proof of this lemma closely follows [17, Lemma 4.2]. We remark that in our case $\mathrm{d}M_i=Z_i=0$ compared to [17, Lemma 4.2], using the notation therein. Recall the definitions of $C(t),\Lambda_{ij}^{z_l}(t),\Theta_{ij}^{z_1,z_2}(t),\Theta_{ij}^{z_1,\overline{z}_2}(t)$ and $\mathring{C}(t)$, $\mathring{\Lambda}_{ij}^{z_l}(t)$, $\mathring{\Theta}_{ij}^{z_1,z_2}(t)$,

$\mathring{\Theta}_{ij}^{z_1,\overline{z}_2}(t)$ in (7.18), (7.10),(7.13) and (7.20)–(7.21), respectively. In the following we may omit the z-dependence. Introduce the stopping times

$$\tau_1 := \inf\Big\{ t \geq 0 \,\Big|\, \exists |i|, |j| \leq n; l \in [2] \text{ s.t. } |\Lambda_{ij}^{z_l}(t)| + |\Theta_{ij}^{z_1,z_2}(t)| + |\Theta_{ij}^{z_1,\overline{z}_2}(t)| > n^{-\omega_E} \Big\}, \tag{7.81}$$

$$\tau_2 := \inf\{ t \geq 0 \,|\, \exists |i| \leq n \text{ s.t. } |x_i(t,1)| + |\lambda_i(t)| > 2R \}, \tag{7.82}$$

for some large $R > 0$, and

$$\tau := \tau_1 \wedge \tau_2 \wedge t_f. \tag{7.83}$$

Note that $|\lambda_i(t)| \leq R$ with very high probability, since $\boldsymbol{\lambda}(t)$ are the eigenvalues of H_t^z, whose norm is typically bounded. Furthermore, by (7.61) and the fact that the process $\boldsymbol{x}(t,0)$ stays bounded by [40, Section 3] it follows that $|x_i(t,\alpha)| \leq R$ for any $t \in [0,t_f]$ and $\alpha \in [0,1]$. We remark that the analysis in [40, Section 3] is done for a process of the form (7.48), with $\alpha = 0$, when it has i.i.d. driving Brownian motions, but the same results apply for our case as well since the correlation in (7.20) does not play any role (see (7.58)). This, together with Lemma 7.13 applied for $z = z_1, z' = z_2$ and $z = z_1, z' = \overline{z}_2$ and $z = z_l, z' = \overline{z}_l$, implies that

$$\tau = t_f$$

with very high probability. In particular, $\mathring{\Theta}_{ij}(t) = \Theta_{ij}(t)$ for any $t \leq \tau$, hence

$$C(t) = \mathring{C}(t) \tag{7.84}$$

for any $t \leq \tau$.

In the remainder of the proof, omitting the time- and z-dependence, we use the notation $\boldsymbol{x} = \boldsymbol{x}^z(t,1)$, $\boldsymbol{\lambda} = \boldsymbol{\lambda}(t)$. Define

$$u_i := \lambda_i - x_i, \qquad |i| \leq n,$$

then, as a consequence of (7.84), subtracting (7.14) and (7.48), it follows that

$$\mathrm{d}u_i = \sum_{j \neq i} B_{ij}(u_j - u_i)\,\mathrm{d}t + \frac{A_n}{\sqrt{n}}\,\mathrm{d}b_i, \tag{7.85}$$

for any $0 \leq t \leq \tau$, where

$$B_{ij} = \frac{1 + \Lambda_{ij}}{2n(\lambda_i - \lambda_j)(x_i - x_j)} > 0, \tag{7.86}$$

since $|\Lambda_{ij}(t)| = |\mathring{\Lambda}_{ij}(t)| \leq n^{-\omega_E}$, and

$$A_n = \frac{1}{\sqrt{1 + n^{-\omega_r}}} - 1 = \mathcal{O}(n^{-\omega_r}). \tag{7.87}$$

Let $\nu := n^{1+\omega_r}$, and define the Lyapunov function

$$F(t) := \frac{1}{\nu} \log \left(\sum_{|i| \leq n} e^{\nu u_i(t)} \right). \tag{7.88}$$

By Itô's lemma, for any $0 \leq t \leq \tau$, we have that

$$\begin{aligned} \mathrm{d}F = {} & \frac{1}{\sum_{|i|\leq n} e^{\nu u_i}} \sum_{|i|\leq n} e^{\nu u_i} \sum_{j \neq i} B_{ij}(u_j - u_i)\,\mathrm{d}t + \frac{n^{-1/2} A_n}{\sum_{|i|\leq n} e^{\nu u_i}} \sum_{|i|\leq n} e^{\nu u_i}\,\mathrm{d}b_i \\ & + \frac{n^{-1}\nu A_n^2}{4\sum_{|i|\leq n} e^{\nu u_i}} \sum_{|i|\leq n} e^{\nu u_i}(1 + \Lambda_{ii})\,\mathrm{d}t - \frac{4n^{-1}\nu A_n^2}{\left(\sum_{|i|\leq n} e^{\nu u_i}\right)^2} \sum_{|i|,|j|\leq n} e^{\nu u_i} e^{\nu u_j}\, \mathbf{E}\Big[\mathrm{d}b_i\,\mathrm{d}b_j \,\Big|\, \widetilde{\mathcal{F}}_{b,t}\Big]. \end{aligned} \tag{7.89}$$

Note that the first term in the r.h.s. of (7.89) is negative since the map $x \mapsto e^{\nu x}$ is increasing. The second and third term in the r.h.s. of (7.89), using that $1+\Lambda_{ii}\le 2$, are bounded exactly as in [17, Eqs. (4.37)–(4.38)] by

$$\frac{n^\xi t_f^{1/2}}{n^{1/2+\omega_r}}+\frac{t_f\nu}{n^{1+2\omega_r}},$$

with very high probability for any $\xi>0$.

Note that

$$\sum_{|i|,|j|\le n} e^{\nu u_i}e^{\nu u_j}\,\mathbf{E}\Big[\mathrm{d}b_i\,\mathrm{d}b_j\,\Big|\,\widetilde{\mathcal{F}}_{b,t}\Big]\ge 0,$$

hence, the last term in the r.h.s. of (7.89) is always non positive. This implies that

$$\sup_{0\le t\le t_f} F(t)\le F(0)+\frac{t_f\nu A_n^2}{n}+\frac{n^\xi t_f^{1/2}A_n}{n^{1/2}},$$

for any $\xi>0$. Then, since

$$F(0)=\frac{\log(2n)}{n^{1+\omega_r}},\qquad F(t)\ge\sup_{|i|\le n}u_i(t),$$

we conclude the upper bound in (7.80). Then noticing that $u_{-i}=-u_i$ for $i\in[n]$, we conclude the lower bound as well. □

7.5 Path-wise coupling close to zero: proof of Lemmata 7.8–7.9

This section is the main technical result used in the proof of Lemmata 7.8–7.9. In Proposition 7.17 we will show that the points with small indices in the two processes become very close to each other on a certain time scale $t_f=n^{-1+\omega_f}$, for any small $\omega_f>0$.

The main result of this section (Proposition 7.17) is stated for general deterministic initial data $\boldsymbol{s}(0)$ satisfying a certain regularity condition (see Definition 7.16 later) even if for its applications in the proof of Proposition 7.2 we only consider initial data which are eigenvalues of i.i.d. random matrices. The initial data $\boldsymbol{r}(0)$, without loss of generality, are assumed to be the singular values of a Ginibre matrix (see also below (7.91) for a more detailed explanation). For notational convenience we formulate the result for two general processes $\boldsymbol{s}$ and $\boldsymbol{r}$ and later we specialize them to our application.

Fix a small constant $0<\omega_r\ll 1$, and define the processes $s_i(t)$, $r_i(t)$ to be the solution of

$$\mathrm{d}s_i(t)=\sqrt{\frac{1}{2n(1+n^{-\omega_r})}}\,\mathrm{d}\mathfrak{b}_i^s(t)+\frac{1}{2n}\sum_{j\ne i}\frac{1}{s_i(t)-s_j(t)}\,\mathrm{d}t,\qquad 1\le|i|\le n,\tag{7.90}$$

and

$$\mathrm{d}r_i(t)=\sqrt{\frac{1}{2n(1+n^{-\omega_r})}}\,\mathrm{d}\mathfrak{b}_i^r(t)+\frac{1}{2n}\sum_{j\ne i}\frac{1}{r_i(t)-r_j(t)}\,\mathrm{d}t,\qquad 1\le|i|\le n,\tag{7.91}$$

with initial data $s_i(0)=s_i$, $r_i(0)=r_i$, where $\boldsymbol{s}=\{s_{\pm i}\}_{i\in[n]}$ and $\boldsymbol{r}=\{r_{\pm i}\}_{i\in[n]}$ are two independent sets of particles such that $s_{-i}=-s_i$ and $r_{-i}=-r_i$ for $i\in[n]$. The driving martingales $\{\mathfrak{b}_i^s\}_{i\in[n]}$, $\{\mathfrak{b}_i^r\}_{i\in[n]}$ in (7.90)–(7.91) are two families satisfying Assumption 7.15 below, and they are such that $\mathfrak{b}_{-i}^s=-\mathfrak{b}_i^s$, $\mathfrak{b}_{-i}^r=-\mathfrak{b}_i^r$ for $i\in[n]$. The coefficient $(1+n^{-\omega_r})^{-1/2}$ ensures the well-posedness of the processes (7.90)–(7.91) (see Appendix A), but it does not play any role in the proof of Proposition 7.17 below.

For convenience we also assume that $\{r_{\pm i}\}_{i=1}^n$ are the singular values of $\widetilde{X}$, with $\widetilde{X}$ a Ginibre matrix. This is not a restriction; indeed, once a process with general initial data $\boldsymbol{s}$ is shown to be close to the reference process with Ginibre initial data, then processes with any two initial data will be close.

On the correlation structure between the two families of i.i.d. Brownian motions $\{\mathfrak{b}_i^s\}_{i=1}^n$, $\{\mathfrak{b}_i^r\}_{i=1}^n$ and the initial data $\{s_{\pm i}\}_{i\in[n]}$ we make the following assumptions.

Assumption 7.15. *Fix $\omega_K, \omega_Q > 0$ such that $\omega_K \ll \omega_r \ll \omega_Q \ll 1$, with ω_r defined in* (7.90)–(7.91)*, and define the n-dependent parameter $K = K_n = n^{\omega_K}$. Suppose that the families $\{\mathfrak{b}_{\pm i}^s\}_{i=1}^n$, $\{\mathfrak{b}_{\pm i}^r\}_{i=1}^n$ in* (7.90)–(7.91) *are realised on a common probability space with a common filtration $\mathcal{F}_t$. Let*

$$L_{ij}(t)\,\mathrm{d}t := \mathbf{E}\big[\big(\mathrm{d}\mathfrak{b}_i^s(t) - \mathrm{d}\mathfrak{b}_i^r(t)\big)\big(\mathrm{d}\mathfrak{b}_j^s(t) - \mathrm{d}\mathfrak{b}_j^r(t)\big) \,\big|\, \mathcal{F}_t\big] \tag{7.92}$$

denote the covariance of the increments conditioned on $\mathcal{F}_t$. The processes satisfy the following assumptions:

(a) The two families of martingales $\{\mathfrak{b}_i^s\}_{i=1}^n$, $\{\mathfrak{b}_i^r\}_{i=1}^n$ are such that

$$\mathbf{E}\big[\mathrm{d}\mathfrak{b}_i^{q_1}(t)\,\mathrm{d}\mathfrak{b}_j^{q_2}(t) \,\big|\, \mathcal{F}_t\big] = \big[\delta_{ij}\delta_{q_1 q_2} + \Xi_{ij}^{q_1,q_2}(t)\big]\,\mathrm{d}t, \quad |\Xi_{ij}^{q_1,q_2}(t)| \le n^{-\omega_Q}, \tag{7.93}$$

for any $i,j \in [n]$, $q_1, q_2 \in \{s,r\}$. The quantities in (7.93) *for negative i,j-indices are defined by symmetry.*

(b) The subfamilies $\{\mathfrak{b}_{\pm i}^s\}_{i=1}^K$, $\{\mathfrak{b}_{\pm i}^r\}_{i=1}^K$ are very strongly dependent in the sense that for any $|i|,|j| \le K$ it holds

$$|L_{ij}(t)| \le n^{-\omega_Q} \tag{7.94}$$

with very high probability for any fixed $t \ge 0$.

Definition 7.16 ((g,G)-regular points [22, Definition 7.12])**.** *Fix a very small $\nu > 0$, and choose g, G such that*

$$n^{-1+\nu} \le g \le n^{-2\nu}, \qquad G \le n^{-\nu}.$$

A set of $2n$-points $\boldsymbol{s} = \{s_i\}_{|i|\le n}$ on $\mathbf{R}$ is called (g,G)-regular *if there exist constants $c_\nu, C_\nu > 0$ such that*

$$c_\nu \le \frac{1}{2n}\Im \sum_{i=-n}^{n} \frac{1}{s_i - (E + \mathrm{i}\eta)} \le C_\nu, \tag{7.95}$$

for any $|E| \le G$, $\eta \in [g, 10]$, and if there is a constant C_s large enough such that $\|\boldsymbol{s}\|_\infty \le n^{C_s}$. Moreover, $c_\nu, C_\nu \sim 1$ if $\eta \in [g, n^{-2\nu}]$ and $c_\nu \ge n^{-100\nu}$, $C_\nu \le n^{100\nu}$ if $\eta \in (n^{-2\nu}, 10]$.

Let $\rho_{\mathrm{fc},t}(E)$ be the scDOS of the particles $\{s_{\pm i}(t)\}_{i\in[n]}$ that is given by the semicircular flow acting on the scDOS of the initial data $\{s_{\pm i}(0)\}_{i\in[n]}$, see [45, Eqs. (2.5)–(2.6)].

Proposition 7.17 (Path-wise coupling close to zero)**.** *Let the processes $\boldsymbol{s}(t) = \{s_{\pm i}(t)\}_{i\in[n]}$, $\boldsymbol{r}(t) = \{r_{\pm i}(t)\}_{i\in[n]}$ be the solutions of* (7.90) *and* (7.91)*, respectively, and assume that the driving martingales in* (7.90)–(7.91) *satisfy Assumption 7.15 for some $\omega_K, \omega_Q > 0$. Additionally, assume that $\boldsymbol{s}(0)$ is (g,G)-regular in the sense of Definition 7.16 and that $\boldsymbol{r}(0)$ are the singular values of a Ginibre matrix. Then for any small $\omega_f, \nu > 0$ such that $\nu \ll \omega_K \ll \omega_f \ll \omega_Q$ and that $gn^\nu \le t_f \le n^{-\nu}G^2$, there exist constants $\omega, \widehat{\omega} > 0$ such that $\nu \ll \widehat{\omega} \ll \omega \ll \omega_f$, and*

$$|\rho_{\mathrm{fc},t_1}(0)s_i(t_f) - \rho_{\mathrm{sc}}(0)r_i(t_f)| \le n^{-1-\omega}, \qquad |i| \le n^{\widehat{\omega}}, \tag{7.96}$$

with very high probability, where $t_f := n^{-1+\omega_f}$.

Proof. The proof of Proposition 7.17 is nearly identical to the proof of [22, Proposition 7.14], which itself follows the proof of fixed energy universality in [15, 45], adapted to the block structure (3.1) in [18] (see also [14] for a different technique to prove universality, adapted to the block structure in [62]). We will not repeat the whole proof, just explain the modification. The only difference of Proposition 7.17 compared to [22, Proposition 7.14] is that here we allow the driving martingales in (7.90)–(7.91) to have a (small) correlation (compare Assumption 7.15 with a non zero $\Xi_{ij}^{q_1,q_2}$ to [22, Assumption 7.11]). The additional pre-factor $(1+n^{-\omega_r})^{-1/2}$ does not play any role.

The correlation of the driving martingales in (7.90)–(7.91) causes a difference in the estimate of [22, Eq. (7.83)]. In particular, the bound on

$$\mathrm{d}M_t = \frac{1}{2n}\sum_{|i|\le n}(w_i-f_i)f_i'\,\mathrm{d}C_i(t,\alpha), \qquad \mathrm{d}C_i(t,\alpha) := \frac{\alpha\,\mathrm{d}\mathfrak{b}^s + (1-\alpha)\,\mathrm{d}\mathfrak{b}^r}{\sqrt{2n(1+n^{-\omega_r})}}, \tag{7.97}$$

using the notation in [22, Eq. (7.83)], will be slightly different. In the remainder of the proof we present how [22, Eqs. (7.83)–(7.87)] changes in the current setup. Using that by [45, Eqs. (3.119)–(3.120)] we have

$$|f_i| + |f_i'| + |w_i| \le n^{-D}, \qquad n^{\omega_A} < |i| \le n, \tag{7.98}$$

for $\omega_A = \omega_K$ (with ω_K defined in Assumption 7.15), and for any $D>0$ with very high probability, we bound the quadratic variation of (7.97) by

$$\mathrm{d}\langle M\rangle_t = \frac{1}{4n^2}\sum_{1\le |i|,|j|\le n^{\omega_A}}(w_i-f_i)(w_j-f_j)f_i'f_j'\,\mathbf{E}[\mathrm{d}C_i(\alpha,t)\,\mathrm{d}C_j(\alpha,t)\mid\mathcal{F}_t] + \mathcal{O}\left(n^{-100}\right). \tag{7.99}$$

Here we estimated the regime when $|i|$ or $|j|$ are larger than n^{ω_A} differently compared to [22, Eq. (7.84)], since, unlike in [22, Eq. (7.84)], $\mathbf{E}[\mathrm{d}C_i(t,\alpha)\,\mathrm{d}C_j(t,\alpha)\mid\mathcal{F}_t]\ne\delta_{ij}$, hence here we anyway need to estimate the double sum using (7.98).

Then, by (a)–(b) of Assumption 7.15, for $|i|,|j|\le n^{\omega_A}$ we have

$$\begin{aligned}\mathbf{E}[\mathrm{d}C_i(t,\alpha)\,\mathrm{d}C_j(t,\alpha)\mid\mathcal{F}_t] &= \frac{\delta_{ij}+\alpha^2\Xi_{ij}^{s,s}(t)+(1-\alpha)^2\Xi_{ij}^{r,r}(t)}{2n(1+n^{-\omega_r})}\,\mathrm{d}t \\ &\quad + \frac{\alpha(1-\alpha)}{2n(1+n^{-\omega_r})}\,\mathbf{E}\big[\big(\mathrm{d}\mathfrak{b}_i^s\,\mathrm{d}\mathfrak{b}_j^r + \mathrm{d}\mathfrak{b}_i^r\,\mathrm{d}\mathfrak{b}_j^s\big)\,\big|\,\mathcal{F}_t\big],\end{aligned} \tag{7.100}$$

and that

$$\begin{aligned}\big|\mathbf{E}\big[\mathrm{d}\mathfrak{b}_i^s\,\mathrm{d}\mathfrak{b}_j^r\,\big|\,\mathcal{F}_t\big]\big| &= \big|\mathbf{E}\big[(\mathrm{d}\mathfrak{b}_i^s-\mathrm{d}\mathfrak{b}_i^r)\,\mathrm{d}\mathfrak{b}_j^r\,\big|\,\mathcal{F}_t\big] + (\delta_{ij}+\Xi_{ij}^{r,r}(t))\,\mathrm{d}t\big| \\ &\lesssim (|L_{ii}(t)|^{1/2} + |\Xi_{ij}^{r,r}(t)| + \delta_{ij})\,\mathrm{d}t,\end{aligned} \tag{7.101}$$

where in the last step we used Kunita-Watanabe inequality for the quadratic variation $(\mathrm{d}\mathfrak{b}_i^s-\mathrm{d}\mathfrak{b}_i^r)\,\mathrm{d}\mathfrak{b}_j^r$.

Combining (7.99)–(7.101), and adding back the sum over $n^{\omega_A}<|i|\le n$ of $(w_i-f_i)^2(f_i')^2$ at the price of an additional error $\mathcal{O}(n^{-100})$, omitting the t-dependence, we finally conclude that

$$\begin{aligned}\mathrm{d}\langle M\rangle_t &\lesssim \frac{1}{n^3}\sum_{1\le|i|\le n}(w_i-f_i)^2(f_i')^2\,\mathrm{d}t \\ &\quad + \frac{1}{n^3}\sum_{|i|,|j|\le n^{\omega_A}}\left(|L_{ii}|^{1/2}+|\Xi_{ij}^{s,s}|+|\Xi_{ij}^{r,r}|\right)\big|(w_i-f_i)(w_j-f_j)f_i'f_j'\big|\,\mathrm{d}t + \mathcal{O}\left(n^{-100}\right).\end{aligned} \tag{7.102}$$

Since $|L_{ii}| + |\Xi_{ij}^{q_1,q_2}| \le n^{-\omega_Q}$, for any $|i|,|j| \le n$, $q_1, q_2 \in \{s,r\}$, and $\omega_A = \omega_K \ll \omega_Q$ by (7.93)–(7.94), using Cauchy-Schwarz in (7.102), we conclude that

$$\mathrm{d}\langle M \rangle_t \lesssim \frac{1}{n^3} \sum_{1\le |i| \le n} (w_i - f_i)^2 (f_i')^2 \,\mathrm{d}t + \mathcal{O}\left(n^{-100}\right), \tag{7.103}$$

which is exactly the same bound as in [22, Eq. (7.88)] (except for the tiny error $\mathcal{O}(n^{-100})$ that is negligible). Proceeding exactly as in [22], we conclude the proof of Proposition 7.17. □

7.5.1 Proof of Lemma 7.8 and Lemma 7.9

The fact that the processes $\mathring{\boldsymbol{\lambda}}(t)$, $\widetilde{\boldsymbol{\lambda}}(t)$ and $\widetilde{\boldsymbol{\mu}}(t)$, $\boldsymbol{\mu}(t)$ satisfy the hypotheses of Proposition 7.17 for the choices $\nu = \omega_h$, $\omega_K = \omega_A$, $\omega_Q = \omega_E$, and $\Xi_{ij}^{q_1,q_2} = \Theta_{ij}^{z_1,\overline{z}_2}$ follows by Lemma 7.4 applied for $z = z_1, z' = z_2$ and $z = z_1, z' = \overline{z}_2$ and $z = z_l, z' = \overline{z}_l$, and exactly the same computations as in [22, Section 7.5]. We remark that the processes $\boldsymbol{\mu}^{(l)}(t)$ do not have the additional coefficient $(1 + n^{-\omega_r})$ in the driving Brownian motions, but this does not play any role in the application of Proposition 7.17 since it causes an error term $n^{-1-\omega_r}$ that is much smaller then the bound $n^{-1-\omega}$ in (7.31). Then, by Proposition 7.17, the results in Lemma 7.8 and Lemma 7.9 immediately follow. □

7.6 Proof of Proposition 7.6

First of all we notice that $\boldsymbol{\lambda}(t)$ is γ-Hölder continuous for any $\gamma \in (0, 1/2)$ by Weyl's inequality. Then the proof of Proposition 7.6 consists of two main steps, (i) proving that the eigenvalues $\boldsymbol{\lambda}(t)$ are a strong solution of (7.14) as long as there are no collisions, and (ii) proving that there are no collisions for almost all $t \in [0, T]$.

The proof that the eigenvalues $\boldsymbol{\lambda}(t)$ are a solution of (7.14) is deferred to Appendix B. The fact that there are no collisions for almost all $t \in [0, T]$ is ensured by [19, Lemma 6.2] following nearly the same computations as in [17, Theorem 5.2] (see also [19, Theorem 6.3] for its adaptation to the 2×2 block structure). The only difference in our case compared to the proof of [17, Theorem 5.2] is that the martingales $\mathrm{d}M_i(t)$ (cf. [17, Eq. (5.4)]) are defined as

$$\mathrm{d}M_i(t) := \frac{\mathrm{d}b_i^z(t)}{\sqrt{n}}, \qquad |i| \le n, \tag{7.104}$$

with $\{b_i^z\}_{i\in[n]}$ having non trivial covariance (7.16). This fact does not play any role in that proof, since the only information about $\mathrm{d}\boldsymbol{M} = \{\mathrm{d}M_i\}_{|i|\le n}$ used in [17, Theorem 5.2] is that it has bounded quadratic variation and that $\boldsymbol{M}(t)$ is γ-Hölder continuous for any $\gamma \in (0, 1/2)$, which is clearly the case for $\mathrm{d}\boldsymbol{M}$ defined in (7.104). □

A The interpolation process is well defined

We recall that the eigenvectors of H^z are of the form $\boldsymbol{w}_{\pm i}^z = (\boldsymbol{u}_i^z, \pm \boldsymbol{v}_i^z)$ for any $i \in [n]$, as a consequence of the symmetry of the spectrum of H^z with respect to zero. Consider the matrix flow

$$\mathrm{d}X_t = \frac{\mathrm{d}B_t}{\sqrt{n}}, \qquad X_0 = X, \tag{A.1}$$

with B_t being a standard real matrix valued Brownian motion. Let H_t^z denote the Hermitisation of $X_t - z$, and $\{\boldsymbol{w}_i^z(t)\}_{|i|\le n}$ its eigenvectors. We recall that the eigenvectors $\{\boldsymbol{w}_i^z(t)\}_{|i|\le n}$ are almost surely well defined, since H_t^z does not have multiple eigenvalues almost surely by (7.17). We set the eigenvectors equal to zero where they are not well defined. Recall the definitions of the coefficients $\Lambda_{ij}^z(t)$, $\mathring{\Lambda}_{ij}^z(t)$ from (7.10), (7.13) and (7.21), respectively. Set

$$\Delta_n := \{(x_i)_{|i|\le n} \in \mathbf{R}^{2n} \mid 0 < x_1 < \cdots < x_n,\, x_{-i} = -x_i,\, \forall i \in [n]\},$$

and let $C(\mathbf{R}_+, \Delta_n)$ be the space of continuous functions $f : \mathbf{R}_+ \to \Delta_n$. Let $\omega_E > 0$ be the exponent in (7.21), and let $\omega_r > 0$ be such that $\omega_r \ll \omega_E$. In this appendix we prove that for any $\alpha \in [0,1]$ the system of SDEs

$$\mathrm{d}x_i^z(t,\alpha) = \frac{\mathrm{d}\mathring{b}_i^z(t)}{\sqrt{n(1+n^{-\omega_r})}} + \frac{1}{2n}\sum_{j\ne i} \frac{1+\alpha\mathring{\Lambda}_{ij}^z(t)}{x_i^z(t,\alpha)-x_j^z(t,\alpha)}\,\mathrm{d}t, \quad x_i^z(0,\alpha) = x_i(0), \quad |i| \le n, \tag{A.2}$$

with $\boldsymbol{x}(0) \in \Delta_n$, admits a strong solution for any $t \ge 0$. For $T > 0$, by (7.20), the martingales $\{\mathring{b}_i^z\}_{|i|\le[n]}$, defined on a filtration $(\widetilde{\mathcal{F}}_{b,t})_{0\le t\le T}$, are such that $\mathring{b}_{-i}^z = -\mathring{b}_i^z$ for $i \in [n]$, and that

$$\mathbf{E}\Big[\mathrm{d}\mathring{b}_i^z\,\mathrm{d}\mathring{b}_j^z \;\Big|\; \widetilde{\mathcal{F}}_{b,t}\Big] = \frac{\delta_{i,j} - \delta_{i,-j} + \mathring{\Lambda}_{ij}^z(t)}{2}\,\mathrm{d}t, \qquad |i|,|j| \le n. \tag{A.3}$$

The main result of this section is Proposition A.1 below. Its proof follows closely [17, Proposition 5.4], which is inspired by the proof of [5, Lemma 4.3.3]. We nevertheless present the proof of Proposition A.1 for completeness, explaining the differences compared with [17, Proposition 5.4] as a consequence of the correlation in (A.3).

Proposition A.1. *Fix any $z \in \mathbf{C}$, and let $\boldsymbol{x}(0) \in \Delta_n$. Then for any fixed $\alpha \in [0,1]$ there exists a unique strong solution $\boldsymbol{x}(t,\alpha) = \boldsymbol{x}^z(t,\alpha) \in C(\mathbf{R}_+, \Delta_n)$ to the system of SDE* (A.2) *with initial condition $\boldsymbol{x}(0)$.*

We will mostly omit the z-dependence since the analysis of (A.2) is done for any fixed $z \in \mathbf{C}$; in particular, we will use the notation $\mathring{\Lambda}_{ij} = \mathring{\Lambda}_{ij}^z$. By (7.10), (7.13) and (7.21) it follows that $\mathring{\Lambda}_{ij}(t) = \mathring{\Lambda}_{ji}(t)$, and that $|\mathring{\Lambda}_{ij}(t)| \le n^{-\omega_E}$, for any $t \ge 0$.

Proof. We follow the notations used in the proof of [17, Proposition 5.4] to make the comparison clearer. Moreover, we do not keep track of the n-dependence of the constants, since throughout the proof n is fixed. By a simple time rescaling, we rewrite the process (A.2) as

$$\mathrm{d}x_i(t,\alpha) = \mathrm{d}\mathring{b}_i(t) + \frac{1}{2}\sum_{j\ne i}\frac{1+\theta_{ij}(t)}{x_i(t,\alpha)-x_j(t,\alpha)}\,\mathrm{d}t, \qquad |i| \le n, \tag{A.4}$$

where $\theta_{ij}(t) := \alpha\mathring{\Lambda}_{ij}(1+n^{-\omega_r}) + n^{-\omega_r}$ is such that $\theta_{ij}(t) = \theta_{ji}(t)$. Note that $c_1 \le \theta_{ij}(t) \le c_2$ for any $t \ge 0$ and $\alpha \in [0,1]$, with $c_1 = n^{-\omega_r}/2$, $c_2 = 1$. For any $\epsilon > 0$ define the bounded Lipschitz function $\phi_\epsilon \colon \mathbf{R} \to \mathbf{R}$ as

$$\phi_\epsilon(x) := \begin{cases} x^{-1}, & |x| \ge \epsilon, \\ \epsilon^{-2}x, & |x| < \epsilon, \end{cases}$$

that cuts off the singularity of x^{-1} at zero.

Introduce the system of cut-off SDEs

$$\mathrm{d}x_i^\epsilon(t,\alpha) = \mathrm{d}\mathring{b}_i(t) + \frac{1}{2}\sum_{j\ne i}(1+\theta_{ij}(t))\phi_\epsilon(x_i^\epsilon(t,\alpha) - x_j^\epsilon(t,\alpha))\,\mathrm{d}t, \quad |i| \le n, \tag{A.5}$$

which admits a unique strong solution (see e.g. [43, Theorem 2.9 of Section 5]) as a consequence of ϕ_ϵ being Lipschitz and the fact that $\mathrm{d}\mathring{\boldsymbol{b}} = (\mathring{C})^{1/2}\,\mathrm{d}\mathfrak{w}$ (see (7.23)). Define the stopping times

$$\tau_\epsilon = \tau_\epsilon(\alpha) := \inf\left\{ t \,\middle|\, \min_{|i|,|j|\le n} \left| x_i^\epsilon(t,\alpha) - x_j^\epsilon(t,\alpha)\right| \le \epsilon \quad \text{or} \quad \|\boldsymbol{x}^\epsilon(t,\alpha)\|_\infty \ge \epsilon^{-1} \right\}. \tag{A.6}$$

By strong uniqueness we have that $\boldsymbol{x}^{\epsilon_2}(t,\alpha) = \boldsymbol{x}^{\epsilon_1}(t,\alpha)$ for any $t \in [0,\tau_{\epsilon_2}]$ if $0<\epsilon_1<\epsilon_2$. Note that $\tau_{\epsilon_2} \le \tau_{\epsilon_1}$ for $\epsilon_1 < \epsilon_2$, thus the limit $\tau = \tau(\alpha) := \lim_{\epsilon\to 0}\tau_\epsilon(\alpha)$ exists, and $\boldsymbol{x}(t,\alpha) := \lim_{\epsilon\to 0}\boldsymbol{x}^\epsilon(t,\alpha)$ defines a strong solution to (A.4) on $[0,\tau)$. Moreover, by continuity in time, $\boldsymbol{x}(t,\alpha)$ remains ordered as $0 < x_1(t,\alpha) < \dots < x_n(t,\alpha)$ and $x_{-i}(t,\alpha) = -x_i(t,\alpha)$ for $i \in [n]$. Additionally, for the square of the ℓ^2-norm $\|\boldsymbol{x}\|_2^2 = \sum_i x_i^2$ a simple calculation shows that

$$\mathrm{d}\|\boldsymbol{x}(t,\alpha)\|_2^2 = \frac{1}{2}\left(\sum_{j\ne i}(1+\theta_{ij}) + \sum_{|i|,|j|\le n} \mathring{\Lambda}_{ij} \right)\mathrm{d}t + \mathrm{d}M_1, \tag{A.7}$$

with $\mathrm{d}M_1$ being a martingale term. This implies that $\mathbf{E}\|\boldsymbol{x}(t\wedge s)\|_2^2 \le c(1+t)$ for any stopping time $s<\tau$ and for any $t\ge 0$, where c depends on n.

Let $a>0$ be a large constant that we will choose later in the proof, and define a_k recursively by $a_0 := a$, $a_{k+1} := a_k^5$ for $k \ge 0$. Consider the Lyapunov function

$$f(\boldsymbol{x}) := -2\sum_{k\ne l} a_{|k-l|}\log|x_k - x_l|. \tag{A.8}$$

Then by Itô's formula we get

$$\mathrm{d}f(\boldsymbol{x}) = A(\boldsymbol{x}(t,\alpha))\,\mathrm{d}t + \mathrm{d}M_2(t), \tag{A.9}$$

with

$$\begin{aligned} A(\boldsymbol{x}(t,\alpha)) :=& -2\sum_{l\ne i, j\ne i} \frac{(1+\theta_{ij})a_{|i-l|}}{(x_i(t,\alpha)-x_l(t,\alpha))(x_i(t,\alpha)-x_j(t,\alpha))} + \sum_{|i|\le n}\frac{a_{|2i|}}{(2x_i(t,\alpha))^2} \\ &+ \sum_{j\ne i}\frac{a_{|i-j|}(1+\mathring{\Lambda}_{ii}(t)-\mathring{\Lambda}_{ij}(t))}{(x_i(t,\alpha)-x_j(t,\alpha))^2}, \end{aligned} \tag{A.10}$$

where $\mathrm{d}M_2$ is a martingale given by

$$\mathrm{d}M_2(t) = -2\sum_{j\ne i}\frac{a_{|i-j|}\,\mathrm{d}\mathring{b}_i(t)}{x_i(t,\alpha)-x_j(t,\alpha)}.$$

In the following we will often omit the time dependence. Note that the term in (A.10) containing $\mathring{\Lambda}_{ii} - \mathring{\Lambda}_{ij}$ is new compared to [17, Eq. (5.39)], since it comes from the correlation of the martingales $\{\mathring{b}_i\}_{|i|\le n}$, whilst in [17, Eq. (5.39)] i.i.d. Brownian motions have been considered. In the remainder of the proof we show that the term $\mathring{\Lambda}_{ii} - \mathring{\Lambda}_{ij}$ is negligible using the fact that $|\mathring{\Lambda}_{ij}| \le n^{-\omega_E}$, and so that this term can be absorbed in the negative term coming from the first sum in the r.h.s. of (A.10) for $l=j$.

We now prove that $A(\boldsymbol{x}(t,\alpha)) \le 0$ if $a>0$ is sufficiently large. Firstly, we write $A(\boldsymbol{x}(t,\alpha))$ as

$$\begin{aligned} A(\boldsymbol{x}(t,\alpha)) =& -2\sum_{\substack{l\ne i, j\ne i\\ j\ne l}} \frac{(1+\theta_{ij})a_{|i-l|}}{(x_i-x_l)(x_i-x_j)} - \sum_{j\ne \pm i}\frac{a_{|i-j|}(1+2\theta_{ij}-\mathring{\Lambda}_{ii}+\mathring{\Lambda}_{ij})}{(x_i-x_j)^2} \\ &-2\sum_{|i|\le n}\frac{a_{|2i|}(\theta_{-i,i}-\mathring{\Lambda}_{ii})}{(2x_i)^2}. \end{aligned} \tag{A.11}$$

Then, using that the first sum in (A.11) is non-positive for $(i-l)(i-j) > 0$, and that $c_1 \le \theta_{ij} \le c_2$, with $c_1 = n^{-\omega_r}$, we bound $A(\boldsymbol{x}(t,\alpha))$ as follows

$$\begin{aligned} A(\boldsymbol{x}(t,\alpha)) \le -2(1+c_2) \sum_{|i|\le n} \sum_{(i-l)(i-j)<0} \frac{a_{|i-l|}}{(x_i-x_l)(x_i-x_j)} \\ - c_1 \sum_{j\ne i} \frac{a_{|i-j|}}{(x_i-x_j)^2} - \sum_{j\ne \pm i} \frac{a_{|i-j|}}{(x_i-x_j)^2}. \end{aligned} \tag{A.12}$$

In (A.12) we used that

$$\theta_{ij} - \mathring{\Lambda}_{ii} + \mathring{\Lambda}_{ij} \ge \frac{c_1}{2}, \qquad \theta_{-i,i} - \mathring{\Lambda}_{ii} \ge \frac{c_1}{2},$$

since $\theta_{ij} \ge c_1 = n^{-\omega_r}$ and $|\mathring{\Lambda}_{ij}| \le n^{-\omega_E}$, where $\omega_r \ll \omega_E$. This shows that the correlations of the martingales $\{\mathring{b}_i\}_{|i|\le n}$ is negligible. Note that the r.h.s. of (A.12) has exactly the same form as [17, Eq. (5.42)], since the third term in (A.12) is non-positive. Hence, following exactly the same computations as in [17, Eqs. (5.43)–(5.46)], choosing $a > n^{10}$, we conclude that

$$A(\boldsymbol{x}(t,\alpha)) \le \left[\frac{2(1+c_2)}{a} - c_1\right] \sum_{j\ne i} \frac{a_{|i-j|}}{(x_i-x_j)^2}, \tag{A.13}$$

which is negative for a sufficiently large.

Fix $a > 0$ large enough so that $A(\boldsymbol{x}(t,\alpha)) \le 0$, then for any stopping time $s < \tau$, and any $t \ge 0$ we have

$$\mathbf{E}[f(\boldsymbol{x}(t\wedge s,\alpha))] \le \mathbf{E}[f(\boldsymbol{x}(0,\alpha))]. \tag{A.14}$$

Hence, by [17, Eqs. (5.48)–(5.49)], using that $\mathbf{E}\|\boldsymbol{x}(t\wedge\tau_\epsilon)\|_2^2 \le c(1+t)$, it follows that

$$\log(\epsilon^{-1})\,\mathbf{P}(\tau_\epsilon < t) \le c,$$

and so that $\mathbf{P}(\tau < t) = 0$, letting $\epsilon \to 0$. Since $t \ge 0$ is arbitrary, this implies that $\mathbf{P}(\tau < +\infty) = 0$, i.e. (A.4) has a unique strong solution on $(0,\infty)$ such that $\boldsymbol{x}(t,\alpha) \in \Delta_n$ for any $t \ge 0$ and $\alpha \in [0,1]$. □

Additionally, by a similar argument as in [17, Proposition 5.5], we conclude the following lemma.

Lemma A.2. *Let $\boldsymbol{x}(t,\alpha)$ be the unique strong solution of* (A.2) *with initial data $\boldsymbol{x}(0,\alpha) \in \Delta_n$, for any $\alpha \in [0,1]$, and assume that there exists $L > 0$ such that $\|\boldsymbol{x}(0,\alpha_1) - \boldsymbol{x}(0,\alpha_2)\|_2 \le L|\alpha_1 - \alpha_2|$, for any $\alpha_1, \alpha_2 \in [0,1]$. Then $\boldsymbol{x}(t,\alpha)$ is Lipschitz in $\alpha \in [0,1]$ for any $t \ge 0$ on an event Ω such that $\mathbf{P}(\Omega) = 1$, and its derivative satisfies*

$$\begin{aligned} \partial_\alpha x_i(t,\alpha) = \partial_\alpha x_i(0,\alpha) + \frac{1}{2n}\int_0^t \sum_{j\ne i} \frac{[1+\alpha\mathring{\Lambda}_{ij}(s)][\partial_\alpha x_j(s,\alpha) - \partial_\alpha x_i(s,\alpha)]}{(x_i(s,\alpha)-x_j(s,\alpha))^2}\,\mathrm{d}s \\ + \frac{1}{2n}\int_0^t \sum_{j\ne i} \frac{\mathring{\Lambda}_{ij}(s)}{x_i(s,\alpha)-x_j(s,\alpha)}\,\mathrm{d}s. \end{aligned} \tag{A.15}$$

B Derivation of the DBM for singular values in the real case

Let X be an $n\times n$ real random matrix, and define $Y^z := X - z$. Consider the matrix flow (A.1) defined on a probability space Ω equipped with a filtration $(\mathcal{F}_t)_{0\le t\le T}$, and denote by H_t^z the Hermitisation of $X_t - z$. We now derive (7.14), under the assumption that the eigenvalues are all distinct. This derivation is easily made complete by the argument in the proof Proposition 7.6 in Section 7.6.

Let $\{\lambda_i^z(t), -\lambda_i^z(t)\}_{i\in[n]}$ be the eigenvalues of H_t^z, and denote by $\{\boldsymbol{w}_i^z(t), \boldsymbol{w}_{-i}^z(t)\}_{i\in[n]}$ their corresponding orthonormal eigenvectors, i.e. for any $i,j\in[n]$, omitting the t-dependence, we have that

$$H^z\boldsymbol{w}_{\pm i}^z = \pm\lambda_i^z\boldsymbol{w}_{\pm i}^z, \qquad (\boldsymbol{w}_i^z)^*\boldsymbol{w}_j^z = \delta_{ij}, \qquad (\boldsymbol{w}_i^z)^*\boldsymbol{w}_{-j}^z = 0. \tag{B.1}$$

In particular, for any $i\in[n]$, by the block structure of H^z it follows that

$$\boldsymbol{w}_{\pm i}^z = (\boldsymbol{u}_i^z, \pm\boldsymbol{v}_i^z), \qquad Y^z\boldsymbol{v}_i^z = \lambda_i^z\boldsymbol{u}_i^z, \qquad (Y^z)^*\boldsymbol{u}_i^z = \lambda_i^z\boldsymbol{v}_i^z. \tag{B.2}$$

Moreover, since $\{\boldsymbol{w}_{\pm i}^z\}_{i=1}^n$ is an orthonormal basis, we conclude that

$$(\boldsymbol{u}_i^z)^*\boldsymbol{u}_i^z = (\boldsymbol{v}_i^z)^*\boldsymbol{v}_i^z = \frac{1}{2}. \tag{B.3}$$

In the following, for any fixed entry x_{ab} of X, we denote the derivative in the x_{ab} direction by

$$\dot{f} := \frac{\partial f}{\partial x_{ab}}, \tag{B.4}$$

where $f = f(X)$ is a function of the matrix X. From now on we only consider positive indices $1\le i\le n$. We may also drop the z and t dependence to make our notation lighter. For any $i,j\in[n]$, differentiating (B.1) we obtain

$$\dot{H}\boldsymbol{w}_i + H\dot{\boldsymbol{w}}_i = \dot{\lambda}_i\boldsymbol{w}_i + \lambda_i\dot{\boldsymbol{w}}_i, \tag{B.5}$$

$$\dot{\boldsymbol{w}}_i^*\boldsymbol{w}_j + \boldsymbol{w}_i^*\dot{\boldsymbol{w}}_j = 0, \tag{B.6}$$

$$\boldsymbol{w}_i^*\dot{\boldsymbol{w}}_i + \dot{\boldsymbol{w}}_i^*\boldsymbol{w}_i = 0. \tag{B.7}$$

Note that (B.7) implies that $\Re[\boldsymbol{w}_i^*\dot{\boldsymbol{w}}_i] = 0$. Moreover, since the eigenvectors are defined modulo a phase, we can choose eigenvectors such that $\Im[\boldsymbol{w}_i^*\dot{\boldsymbol{w}}_i] = 0$ for any $t\ge 0$ hence $\boldsymbol{w}_i^*\dot{\boldsymbol{w}}_i = 0$. Then, multiplying (B.5) by $\boldsymbol{w}_i^*$ we conclude that

$$\dot{\lambda}_i = \boldsymbol{u}_i^*\dot{Y}\boldsymbol{v}_i + \boldsymbol{v}_i^*\dot{Y}^*\boldsymbol{u}_i. \tag{B.8}$$

Moreover, multiplying (B.5) by $\boldsymbol{w}_j^*$, with $j\neq i$, and by $\boldsymbol{w}_{-j}^*$, we get

$$(\lambda_i - \lambda_j)\boldsymbol{w}_j^*\dot{\boldsymbol{w}}_i = \boldsymbol{w}_j^*\dot{H}\boldsymbol{w}_i, \qquad (\lambda_i + \lambda_j)\boldsymbol{w}_{-j}^*\dot{\boldsymbol{w}}_i = \boldsymbol{w}_{-j}^*\dot{H}\boldsymbol{w}_i, \tag{B.9}$$

respectively. By (B.7) and $\boldsymbol{w}_i^*\dot{\boldsymbol{w}}_i = 0$ it follows that

$$\dot{\boldsymbol{w}}_i = \sum_{\substack{j\in[n],\\ j\neq i}} (\boldsymbol{w}_j^*\dot{\boldsymbol{w}}_i)\boldsymbol{w}_j + \sum_{j\in[n]} (\boldsymbol{w}_{-j}^*\dot{\boldsymbol{w}}_i)\boldsymbol{w}_{-j}, \tag{B.10}$$

hence, by (B.9), we conclude

$$\dot{\boldsymbol{w}}_i = \sum_{j\neq i} \frac{\boldsymbol{v}_j^*\dot{Y}^*\boldsymbol{u}_i + \boldsymbol{u}_j^*\dot{Y}\boldsymbol{v}_i}{\lambda_i - \lambda_j}\boldsymbol{w}_j + \sum_j \frac{\boldsymbol{u}_j^*\dot{Y}\boldsymbol{v}_i - \boldsymbol{v}_j^*\dot{Y}^*\boldsymbol{u}_i}{\lambda_i + \lambda_j}\boldsymbol{w}_{-j}. \tag{B.11}$$

Throughout this appendix we use the convention that for any vectors $\boldsymbol{v}\in\mathbf{C}^n$ we denote its entries by $v(a)$, with $a\in[n]$. By (B.8)–(B.11) it follows that

$$\frac{\partial\lambda_i}{\partial x_{ab}} = 2\Re[u_i^*(a)v_i(b)], \tag{B.12}$$

and that

$$\begin{aligned}\frac{\partial w_i}{\partial x_{ab}}(k) = \sum_{j\neq i}&\left[\frac{u_j^*(a)v_i(b) + v_j^*(b)u_i(a)}{\lambda_i - \lambda_j}w_j(k) + \frac{u_j^*(a)v_i(b) - v_j^*(b)u_i(a)}{\lambda_i + \lambda_j}w_{-j}(k)\right]\\ &+ \frac{u_i^*(a)v_i(b) - v_i^*(b)u_i(a)}{2\lambda_i}w_{-i}(k).\end{aligned}$$

By Ito's formula we have that

$$\mathrm{d}\lambda_i = \sum_{ab} \frac{\partial \lambda_i}{\partial x_{ab}}\,\mathrm{d}x_{ab} + \frac{1}{2}\sum_{ab}\sum_{kl} \frac{\partial^2 \lambda_i}{\partial x_{ab}\partial x_{kl}}\,\mathrm{d}x_{ab}\,\mathrm{d}x_{kl}. \tag{B.13}$$

Then we compute

$$\begin{aligned}
&\frac{\partial^2 \lambda_i}{\partial x_{ab}\partial x_{kl}} \\
&\quad= 2\Re\left[\frac{\partial v_i^*}{\partial x_{ab}}(l)u_i(k) + v_i^*(l)\frac{\partial u_i}{\partial x_{ab}}(k)\right] \\
&\quad= 2\Re\Bigg[\sum_{j\neq i}\left[\frac{u_j(a)v_i^*(b)+v_j(b)u_i^*(a)}{\lambda_i-\lambda_j}v_j^*(l)u_i(k) - \frac{u_j(a)v_i^*(b)-v_j(b)u_i^*(a)}{\lambda_i+\lambda_j}v_j^*(l)u_i(k)\right] \\
&\qquad - \frac{u_i(a)v_i^*(b)-v_i(b)u_i^*(a)}{2\lambda_i}v_i^*(l)u_i(k) + \frac{u_i^*(a)v_i(b)-v_i^*(b)u_i(a)}{2\lambda_i}u_i(k)v_i^*(l) \\
&\qquad + \sum_{j\neq i}\left[\frac{u_j^*(a)v_i(b)+v_j^*(b)u_i(a)}{\lambda_i-\lambda_j}u_j(k)v_i^*(l) + \frac{u_j^*(a)v_i(b)-v_j^*(b)u_i(a)}{\lambda_i+\lambda_j}u_j(k)v_i^*(l)\right]\Bigg].
\end{aligned} \tag{B.14}$$

Hence, combining (B.12)–(B.14), we finally conclude that

$$\begin{aligned}
\mathrm{d}\lambda_i^z = \frac{\mathrm{d}b_i^z}{\sqrt{n}} + \frac{1}{2n}\sum_{j\neq i}\left[\frac{1+4\Re[\langle \overline{u_j^z}, u_i^z\rangle\langle v_i^z, \overline{v_j^z}\rangle]}{\lambda_i^z-\lambda_j^z} + \frac{1+4\Re[\langle \overline{u_j^z}, u_i^z\rangle\langle v_i^z, -\overline{v_j^z}\rangle]}{\lambda_i^z+\lambda_j^z}\right]\mathrm{d}t \\
+ \frac{1+4\Re[\langle \overline{u_i^z}, u_i^z\rangle\langle v_i^z, -\overline{v_i^z}\rangle]}{4n\lambda_i^z}\,\mathrm{d}t.
\end{aligned} \tag{B.15}$$

In (B.15) we used the convention that for any vector $\boldsymbol{v} \in \mathbf{C}^n$ by $\overline{\boldsymbol{v}}$ we denote the vector with entries $\overline{v}(a) = \overline{v(a)}$, for any $a \in [n]$. The driving martingales in (B.15) are defined as

$$\mathrm{d}b_i^z := \mathrm{d}B_{ii}^z + \mathrm{d}\overline{B_{ii}^z}, \quad \text{with} \quad \mathrm{d}B_{ij}^z := \sum_{ab} (u_i^z)^*(a)\,\mathrm{d}B_{ab} v_j^z(b), \tag{B.16}$$

with $B = B_t$ the matrix valued Brownian motion in (A.1), and their covariance given by

$$\mathbf{E}\big[\mathrm{d}b_i^z\,\mathrm{d}b_j^z \,\big|\, \mathcal{F}_t\big] = \frac{\delta_{ij} + 4\Re\left[\langle \overline{u_j^z}, u_i^z\rangle\langle v_i^z, \overline{v_j^z}\rangle\right]}{2}\,\mathrm{d}t. \tag{B.17}$$

Note that $\{b_i^z\}_{i\in[n]}$ defined in (B.16) are not Brownian motions, as a consequence of the non deterministic quadratic variation (B.17).

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