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Consistent particle systems and duality

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Abstract

We consider consistent particle systems, which include independent random walkers, the symmetric exclusion and inclusion processes, as well as the dual of the Kipnis-Marchioro-Presutti model. Consistent systems are such that the distribution obtained by first evolving n particles and then removing a particle at random is the same as the one given by a random removal of a particle at the initial time followed by evolution of the remaining $n-1$ particles.

In this paper we discuss two main results. Firstly, we show that, for reversible systems, the property of consistency is equivalent to self-duality, thus obtaining a novel probabilistic interpretation of the self-duality property. Secondly, we show that consistent particle systems satisfy a set of recursive equations. This recursions implies that factorial moments of a system with n particles are linked to those of a system with $n-1$ particles, thus providing substantial information to study the dynamics. In particular, for a consistent system with absorption, the particle absorption probabilities satisfy universal recurrence relations.

Since particle systems with absorption are often dual to boundary-driven non-equilibrium systems, the consistency property implies recurrence relations for expectations of correlations in non-equilibrium steady states. We illustrate these relations with several examples.

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1 Introduction

Non-equilibrium systems, i.e. systems carrying a current, such as systems coupled to reservoirs or bulk-driven systems, are of great interest in non-equilibrium statistical

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mechanics. In particular one would like to understand the properties of their stationary distribution, that is often called in the mathematical physics literature the "*non-equilibrium steady state*", where "non-equilibrium" refers to the absence of reversibility, often manifested through the presence of currents. However, detailed information such as explicit closed form expression for correlation functions are rarely available. In stochastic systems, there are a few integrable systems, such as the symmetric and asymmetric exclusion process in dimension one, where one can obtain such closed form formulas for all correlation functions in the non-equilibrium steady states via matrix product ansatz solution [4]. Another powerful tool in the analysis of non-equilibrium systems is duality, a technique which allows to connect correlation functions of order n in the non-equilibrium steady state to a system of n dual particles. Duality is strictly weaker than integrability, i.e., there are many systems which have useful dualities but are not integrable. Duality also allows to study time-dependent expectations and time-dependent correlation functions. For a correlation function of order n, one needs the law of n dual particles.

In particle systems with duality [12, 1, 2, 15], whenever we couple the system to appropriately chosen reservoirs, they are dual to particle systems where the driving reservoirs are replaced by absorbing boundaries. As a consequence, the computation of correlation functions of the non-equilibrium steady state can be reduced to the computation of absorption probabilities of dual particles. These absorption probabilities can usually be obtained explicitly (in closed form) for a single particle, and in some exceptional cases, such as the symmetric simple exclusion process on a chain [5, 4], also for many particles.

In this paper, we focus on the property of consistency of particle systems. Consistency is a property of particle systems which generalizes duality and which has a simple probabilistic interpretation. Intuitively, consistent particle systems are those where the operation of randomly removing a particle commutes with the time-evolution. This property implies that there are intertwining relations between the dynamics with n and $n-1$ particles, and as a consequence simplifying recursion relations for moments and absorption probabilities.

More in detail, consistent particle systems are defined as a family of permutation invariant particle systems, indexed by the total number of particles, that additionally satisfy the following property: if we marginalize on the first $n-1$ coordinates of the process $\{X^{(n)}(t),\, t \geq 0\}$ describing the positions of n particles, we obtain (in distribution) the process $\{X^{(n-1)}(t) :\geq 0\}$ describing the positions of $n-1$ particles. This property holds trivially for independent particles, but here we are particularly interested in proving and using this property for systems of interacting particles. The consistency property has already been observed and used in the context of the Kipnis-Marchioro-Presutti (KMP) model (see [12]), where local equilibrium is proved via consistency.

Another motivation to study consistency is to provide a more probabilistic interpretation of (self-) duality. Indeed, in many systems duality is understood from a probabilistic construction such as the graphical construction for the symmetric exclusion process. However, there is no general theorem that provides a probabilistic interpretation of duality. In this paper we show that consistency implies self-duality, which provides a more probabilistic understanding of self-duality. More precisely we show that a particle system is consistent if and only if the process generator commutes with the so-called annihilation operator. This commutation relation formalizes the property that for consistent particle systems the two operations of "removing a particle at random" and "dynamical evolution" yield the same result (in distribution) in whatever order they are performed. The characterization of consistency as a symmetry property, i.e., an operator commuting with the generator, allows us to establish a direct link between consistency and duality

[8].

A second line of research of this paper is concerned with exploiting the consistency property to obtain detailed information about the dynamics, especially in the context of non-equilibrium systems, i.e., in the presence of boundary reservoirs where particles can enter and leave the system. Indeed, such boundary driven systems are dual to absorbing systems, where the reservoirs are replaced by absorbing boundaries. We show that the addition of absorbing sites does not change the consistency property, and as a consequence we obtain a set of recursive equations from the consistency property of the absorbing dual. These recursions imply that factorial moments of a system with n particles are linked to those of a system with $n-1$ particles. Although the set of recursive equations is not enough to fully solve the dynamics, it yields substantial simplifications. We apply these recursion relations to show universal properties of non-equilibrium steady states of boundary-driven systems, such as (generalized) exclusion and inclusion processes coupled to boundary reservoirs.

To conclude this introduction, we summarize the main results of the paper.

a) We characterize particle systems which have the consistency property. We show the relation between consistency and duality, thus providing a probabilistic interpretation of duality.

b) We show that consistency is conserved upon adding absorbing sites, which provides consistency for duals of systems with boundary reservoirs.

c) We show that consistency of absorbing systems leads to universal recurrence relations for the factorial moments of the particle occupation numbers. Via duality this translates into universal recurrence relations for the factorial moments in the non-equilibrium steady state.

The rest of our paper is organized as follows. In section 2 we give the basic definitions yielding the distinction between the coordinate process, which specifies the positions of all the particles, and the configuration process, which instead provides the number of particles at each site. We also recall the definition of the annihilation operator and its interpretation as a particle removal operator. In section 3 we introduce the notion of consistent permutation-invariant particle systems and show that this property is equivalent to a commutation property between the generator of the configuration process and the annihilation operator. We also write the set of recursion relation implied by consistency in the general setting. In section 4, we show the relation between consistency and (self-)duality. In section 5 we show that consistency is preserved by adding sites where particles can be absorbed (independently for different particles) and specify the recursion relations in terms of the absorption probabilities. Sections 6 and 7 are dedicated to a special class of consistent particle systems, including the case of an integrable systems (the open symmetric exclusion process that is solved by matrix product ansatz) and a non-integrable case (the symmetric inclusion process). We show in both cases the form taken by consistency equations.

2 Preliminary definitions

2.1 Coordinate process and configuration process

We consider a system of n particles moving on a countable set of vertices V, with cardinality $|V|$. Their *positions* are denoted by $(x_1, \ldots, x_n) \in V^n$. The set of functions $g : V^n \to \mathbb{R}$ is denoted by $\mathscr{C}_n$. A *configuration* of n particles is the n-tuple of their positions modulo permutations of labels. In the following configurations will be denoted by $\eta = (\eta_x)_{x\in V}$, where η_x will be an element of $\Lambda \subseteq \mathbb{N}$ to be interpreted as the number of particles at vertex x. For $\mathbf{x} = (x_1, \ldots, x_n) \in V^n$ the associated configuration is denoted by

$$\varphi(\mathbf{x}) := \sum_{i=1}^{n} \delta_{x_i} \tag{2.1}$$

where δ_z is the configuration having only one particle located at $z \in V$, i.e. for $y \in V$,

$$(\delta_z)_y = \begin{cases} 1 & \text{if } y = z, \\ 0 & \text{otherwise.} \end{cases}$$

We view φ as a map from n-tuples with arbitrary n to configurations, i.e., $\varphi : \cup_{n=1}^{\infty} V^n \to \mathbb{N}^V$. We then define Ω_n as the set of configurations of n particles, i.e.

$$\Omega_n = \Big\{\eta \in \Lambda^V : |\eta| = n\Big\}, \qquad \text{with} \qquad |\eta| := \sum_{x \in V} \eta_x$$

where $\Lambda \subseteq \mathbb{N}$ is the single-site state space, namely the set of possible occupation numbers for each site. We denote by $\mathscr{E}_n$ the set of functions $f : \Omega_n \to \mathbb{R}$ and define $\Omega = \{\eta \in \Lambda^V : |\eta| < \infty\}$ the set of finite particle configurations, namely $\Omega = \cup_{n \in \mathbb{N}} \Omega_n$.

In this paper, we shall consider both examples with a finite state space, such as the partial exclusion processes [11, 14] (with a restriction on the number of particles per site, i.e. $\Lambda = \{0, \ldots, \alpha\}$ where $\alpha \in \mathbb{N}$ denotes the maximal number of particles per site), as well as examples with $\Lambda = \mathbb{N}$, such as the inclusion process [8] or the independent random walk process (see Section 6 for a treatment of these processes).

In some cases not all elements of V^n give rise to allowed configurations of Ω. For instance, for the partial exclusion process, it is not guaranteed that $\mathbf{x} \in V^n$ does not contain more than α particles at any site, i.e. $\varphi(\mathbf{x}) \in \{0, \ldots, \alpha\}^V$. For this reason we define the set V_n of n-tuples $\mathbf{x} \in V^n$ such that the associated configuration $\varphi(\mathbf{x})$ is an element of Ω_n. For the examples of independent random walkers and inclusion process $V_n = V^n$, whereas, in the case of the partial exclusion process, $V_n \neq V^n$.

With these preliminaries we next specify the distinction between a coordinate process and a configuration process.

Definition 2.1 (Coordinate process)**.** *We shall call a* coordinate process with n particles*, denoted by $\{X^{(n)}(t),\, t \geq 0\}$, the stochastic process taking values in V_n that describes the positions of particles in the course of time. Namely, for $i = 1, \ldots, n$, the random variable $X_i^{(n)}(t)$ denotes the position of the i^{th} particle at time $t \geq 0$. We denote by $\{X(t), t \geq 0\}$ a family of coordinate-processes $\big(\{X^{(n)}(t),\, t \geq 0\}, n \in \mathbb{N}\big)$, labeled by the number of particles $n \in \mathbb{N}$.*

Throughout this paper we shall restrict to coordinate processes that are Markov processes.

Definition 2.2 (Configuration process)**.** *We shall call a* configuration process*, denoted by $\{\eta(t),\, t \geq 0\}$, the stochastic process, taking values in Ω that describes the joint occupancy numbers of all the sites in the course of time. More precisely, for $i \in V$, the random variable $\eta_i(t)$ denotes the number of particles at site i at time $t \geq 0$.*

Throughout this paper we shall restrict to configuration processes that conserve the number of particles, i.e. if the process $\{\eta(t),\, t \geq 0\}$ is started from $\eta \in \Omega_n$ then $\eta(t) \in \Omega_n$ for all later times $t > 0$. A configuration process is naturally induced by a coordinate process using the map φ defined in (2.1). There can be several coordinate processes whose image under the map φ yields the same configuration process. This leads us to the following definition.

Definition 2.3 (Compatibility)**.** *A family of coordinate processes $\{X(t),\, t \geq 0\}$ and a configuration process $\{\eta(t),\, t \geq 0\}$ are compatible if for all $n \in \mathbb{N}$ the following holds:*

whenever $\varphi(X_1(0),\dots,X_n(0))=\eta(0)$ *then*

$$\{\varphi(X^{(n)}(t)),\, t\ge 0\}=\{\eta(t),\, t\ge 0\}$$

where the equality is in distribution.

Of course it is not guaranteed that, starting from a Markov coordinate process, the mapping φ defined in (2.1) induces a compatible configuration process that is also Markov. To further discuss this point we need to introduce the notion of permutation invariance. We denote by Σ_n the set of permutations of n elements. Moreover we define the operator U_φ mapping functions $f:\Omega_n\to\mathbb{R}$ to functions $U_\varphi f\in\mathscr{C}_n$ via $U_\varphi f=f\circ\varphi$.

Definition 2.4 (Permutation invariance)**.**

a) *A family of coordinate Markov processes* $\{X(t),\, t\ge 0\}$ *is said to be* permutation-invariant *if, for every* $n\in\mathbb{N}$ *and permutation* $\sigma\in\Sigma_n$*, the processes:* $\{(X_1^{(n)}(t),\dots,X_n^{(n)}(t)),\, t\ge 0\}$ *and* $\{(X_{\sigma(1)}^{(n)}(t),\dots,X_{\sigma(n)}^{(n)}(t)),\, t\ge 0\}$ *are equal in distribution.*

b) *A function* $g\in\mathscr{C}_n$ *is said to be* permutation-invariant *if*

$$g(x_1,\dots,x_n)=g(x_{\sigma(1)},\dots,x_{\sigma(n)})\qquad \textit{for all}\quad \sigma\in\Sigma_n.$$

Equivalently, a function $g\in\mathscr{C}_n$ *is permutation-invariant if there exists a function* $f:\Omega_n\to\mathbb{R}$ *such that* $g=U_\varphi f$*.*

c) *A probability measure* μ_n *on* V_n *is called* permutation-invariant *if, for all* $n\in\mathbb{N}$ *and* $\sigma\in\Sigma_n$*, under* μ_n*, the random vectors* $(X_1,\dots,X_n)$ *and* $(X_{\sigma(1)},\dots,X_{\sigma(n)})$ *have the same distribution.*

We denote by L_n the infinitesimal generator of the n-particle coordinate process $\{X^{(n)}(t),\, t\ge 0\}$, and by $S_n(t)$ the related semigroup, i.e., for $g\in\mathscr{C}_n$

$$S_n(t)g(\mathbf{x}):=\mathbb{E}_{\mathbf{x}}[g(X^{(n)}(t))]$$

where $\mathbb{E}_{\mathbf{x}}$ denotes expectation when the coordinate process is started from $\mathbf{x}\in V_n$.

The following lemma shows that to permutation-invariant coordinate processes one can naturally associate a compatible configuration process enjoying the Markov property.

Lemma 2.5. *Let* $\{X(t),\, t\ge 0\}$ *be a family of permutation-invariant coordinate Markov processes with generators* L_n*,* $n\in\mathbb{N}$*. Define the operator* $\mathscr{L}$ *acting on functions* $f:\Omega\to\mathbb{R}$ *as*

$$\mathscr{L}f(\eta):=L_n(U_\varphi f)(\mathbf{x})\qquad \textit{for all}\quad \eta\in\Omega_n \ \textit{and}\ \mathbf{x}\in V_n\ :\ \varphi(\mathbf{x})=\eta \tag{2.2}$$

or, equivalently,

$$L_nU_\varphi=U_\varphi\mathscr{L}\qquad \textit{on}\quad \mathscr{E}_n. \tag{2.3}$$

Then $\mathscr{L}$ *is the infinitesimal generator of a Markov process* $\{\eta(t),t\ge 0\}$ *that is a configuration process compatible with* $\{X(t),\, t\ge 0\}$*.*

Proof. For a permutation $\sigma\in\Sigma_n$ and a function $g\in\mathscr{C}_n$ we define the operator

$$T_\sigma g(x_1,\dots,x_n)=g(x_{\sigma(1)},\dots,x_{\sigma(n)}). \tag{2.4}$$

From the permutation invariance of the family of coordinate Markov processes $\{X(t),t\ge 0\}$ it follows that

$$[L_n,T_\sigma]=0\qquad \text{for all}\quad n\in\mathbb{N}\quad\text{and}\quad \sigma\in\Sigma_n \tag{2.5}$$

where $[\cdot,\cdot]$ denotes the commutator. Let $f:\Omega\to\mathbb{R}$ then, by definition, $U_\varphi f$ is a permutation-invariant function. Hence, from (2.5) it follows

$$T_\sigma L_n U_\varphi f = L_n T_\sigma U_\varphi f = L_n U_\varphi f \qquad \text{for all} \quad f\in\mathscr{E}_n. \tag{2.6}$$

This means that $L_n U_\varphi f$ is permutation-invariant, hence there exists a function $\tilde f:\Omega\to\mathbb{R}$ such that

$$L_n(U_\varphi f)(\mathbf{x}) = \tilde f(\varphi(\mathbf{x})) = U_\varphi \tilde f(\mathbf{x})$$

namely $L_n U_\varphi f = U_\varphi \tilde f$. Then it is possible to define the operator $\mathscr{L}$ acting on functions $f:\Omega\to\mathbb{R}$ such that $\mathscr{L}f=\tilde f$, and then (2.3) is satisfied. From (2.6) we have that

$$T_\sigma S_n(t) U_\varphi f = S_n(t) U_\varphi f \qquad \text{for all} \quad f\in\mathscr{E}_n \tag{2.7}$$

and then also $S_n(t)U_\varphi f$ is a permutation-invariant function at all times. Now, if we denote by $\mathscr{S}(t)$ the semigroup associated to $\mathscr{L}$, it follows that

$$\mathscr{S}(t)f(\eta) = S_n(t)U_\varphi f(\mathbf{x}) \qquad \text{for all} \quad \eta\in\Omega_n \text{ and } \mathbf{x}\in V_n \ : \ \varphi(\mathbf{x})=\eta \tag{2.8}$$

namely

$$U_\varphi \mathscr{S}(t) = S_n(t) U_\varphi \qquad \text{on} \quad \mathscr{E}_n \tag{2.9}$$

From this it follows that $\mathscr{L}$ is the generator of a Markov process that is a configuration process compatible with $\{X(t),\, t\ge 0\}$. □

2.2 Annihilation operator

We continue by introducing the operators that remove particles either in the coordinate process (see (2.10) below) or in the configuration process (see (2.13)).

Definition 2.6 (Particle removal operators)**.** *For $n\in\mathbb{N}$, $1\le i\le n$ we denote by $\pi_i^{(n)}:\mathscr{C}_{n-1}\to\mathscr{C}_n$ the removal operator of the i^{th} labeled particle, acting on functions $g\in\mathscr{C}_{n-1}$ as follows:*

$$(\pi_i^{(n)}g)(x_1,\dots,x_n) = g(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n) \qquad \textit{for all} \quad x_i\in V \tag{2.10}$$

and we denote by $\Pi^{(n)}:\mathscr{C}_{n-1}\to\mathscr{C}_n$ the operator acting on $g\in\mathscr{C}_{n-1}$ via

$$\Pi^{(n)}g = \sum_{i=1}^n \pi_i^{(n)} g. \tag{2.11}$$

We define the "single-site annihilation operator" a acting on functions $f:\mathbb{N}_0\to\mathbb{R}$ as

$$af(n) = \begin{cases} nf(n-1) & \textit{if } n\ge 1 \\ 0 & \textit{if } n=0 \end{cases} \tag{2.12}$$

and the "annihilation operator" working on functions $f:\Omega\to\mathbb{R}$ as

$$\mathscr{A}f(\eta) = \sum_{x\in V} a_x f(\eta) \quad \textit{with} \quad a_x f(\eta) = \begin{cases} \eta_x f(\eta-\delta_x) & \textit{if } \eta_x\ge 1 \\ 0 & \textit{if } \eta_x = 0 \end{cases} \tag{2.13}$$

i.e. a_x denotes the operator a working on the variable η_x.

We remind the reader here that we are restricting to configurations $\eta \in \Omega$, i.e. with a finite number of particles and therefore the sum in (2.13) is a finite sum.

The annihilation operator is crucially important in the explanation of self-dualities for several particle systems [6, 7]. In particular, as it will be shown later, the fact that the generator $\mathscr{L}$ of the configuration process and the annihilation operator $\mathscr{A}$ commute, i.e. $[\mathscr{L}, \mathscr{A}] = 0$, is enough to obtain a self-duality when the process has a reversible measure. Such a commutation relation for the configuration process is equivalent to an intertwining relation between the coordinate process with n particles and the coordinate process with $n-1$ particles. This equivalence is the object of the next theorem.

Theorem 2.7. *Let $\{X(t), t \geq 0\}$ be a family of coordinate Markov processes with generators L_n, $n \in \mathbb{N}$ and $\{\eta(t), t \geq 0\}$ a compatible Markov configuration process with generator $\mathscr{L}$ defined in* (2.2)*. Then the following statements are equivalent:*

a) The generators of the coordinate process with n and $n-1$ particles restricted to permutation invariant functions are intertwined via $\Pi^{(n)}$, i.e., for every $n \in \mathbb{N}$, and for all $g \in \mathscr{C}_{n-1}$ permutation-invariant

$$(L_n \Pi^{(n)})(g) = (\Pi^{(n)} L_{n-1})(g) \tag{2.14}$$

b) The generator of the configuration process commutes with the total annihilation operator, i.e.

$$[\mathscr{L}, \mathscr{A}] = 0 \tag{2.15}$$

Proof. We first show that

$$\Pi^{(n)} U_\varphi = U_\varphi \mathscr{A} \qquad \text{on} \quad \mathscr{E}_n. \tag{2.16}$$

This means that

$$(\Pi^{(n)}(f \circ \varphi))(\mathbf{x}) = \mathscr{A} f(\eta) \qquad \text{for all} \quad \eta \in \Omega_n \ \text{ and } \ \mathbf{x} \in V_n \ : \ \varphi(\mathbf{x}) = \eta \tag{2.17}$$

Let $\mathbf{x} = (x_1, \ldots, x_n)$ and $\eta := \varphi(\mathbf{x}) = \sum_{i=1}^n \delta_{x_i}$, then we have

$$\varphi(x_1, \ldots, x_{l-1}, x_{l+1}, \ldots, x_n) = \left(\sum_{i=1}^n \delta_{x_i}\right) - \delta_{x_l} = \eta - \delta_{x_l}$$

As a consequence:

$$\begin{aligned} (\Pi^{(n)}(f \circ \varphi))(\mathbf{x}) &= \sum_{l=1}^n f(\eta - \delta_{x_l}) \\ &= \sum_{x \in V} \eta_x f(\eta - \delta_x) \end{aligned} \tag{2.18}$$

where the last step follows because every $x \in V$ is counted exactly η_x times in the sum $\sum_{l=1}^n f(\eta - \delta_{x_l})$. This proves (2.16). Suppose now that $[\mathscr{L}, \mathscr{A}] = 0$, then on $\mathscr{E}_n$ we have

$$L_{n-1}\Pi^{(n)} U_\varphi = L_{n-1} U_\varphi \mathscr{A} = U_\varphi \mathscr{L} \mathscr{A} = U_\varphi \mathscr{A} \mathscr{L} = \Pi^{(n)} U_\varphi \mathscr{L} = \Pi^{(n)} L_n U_\varphi$$

where the equalities follow from (2.16), (2.3) and the commutation relation. Then (2.14) follows since, for all $g \in \mathscr{C}_n$ permutation-invariant, $g = U_\varphi f$ for some $f \in \mathscr{E}_n$. The reverse implication is proved analogously. □

Remark 2.8. The probabilistic interpretation of (2.14) is as follows: if we remove a randomly chosen particle, evolve the process, evaluate a permutation-invariant function at time $t > 0$ and finally take expectation, then we can as well first evolve the process, remove a randomly chosen particle at time $t > 0$, evaluate the same permutation-invariant function and take expectation. In other words, the operations "removing a randomly chosen particle" and "time evolution in the process followed by expectation" commute as long as we restrict to permutation-invariant functions. See also Proposition 3.6.

3 Consistency

In this section we first define the consistency property and give some characterization of consistent particle systems. We then discuss the implication of consistency in the form a set of recursive equations.

3.1 Definition of consistency and relation to annihilation operator

Based on Theorem 2.7 we define consistent particle systems as follows.

Definition 3.1 (Consistency)**.**

a) A family of coordinate Markov processes $\{X(t),\, t \geq 0\}$ *is said to be* consistent *if the processes* $\{(X_1^{(n)}(t), \ldots, X_{n-1}^{(n)}(t)),\, t \geq 0\}$ *and* $\{(X_1^{(n-1)}(t), \ldots, X_{n-1}^{(n-1)}(t)),\, t \geq 0\}$ *are equal in distribution for every* $n \in \mathbb{N}$*.*

b) A configuration process $\{\eta(t),\ t \geq 0\}$ *is said to be* consistent *if its generator* $\mathscr{L}$ *commutes with the annihilation operator, i.e.* $[\mathscr{L}, \mathscr{A}] = 0$*.*

c) A family of probability measures μ_n *on* V_n*, indexed by* $n \in \mathbb{N}$*, is called* consistent *if for all* $n \geq 2$*, under* μ_n*, the distribution of* $(x_1, \ldots, x_{n-1})$ *equals* μ_{n-1}*.*

Let $\{\mu_n, n \in \mathbb{N}\}$ be a consistent collection of probability measures on $V_n, n \in \mathbb{N}$. If additionally $\{\mu_n, n \in \mathbb{N}\}$ is permutation-invariant, then *every* m*-dimensional marginal* of μ_n, $m \leq n$, coincides with μ_m. A simple example of such a consistent permutation-invariant family is

$$\mu_n = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \delta_{(x_{\sigma(1)}, \ldots, x_{\sigma(n)})} \qquad \text{for some } \mathbf{x} \in V_n$$

Analogously, if $\{X(t),\, t \geq 0\}$ is a family of permutation-invariant coordinate processes, consistency implies that, for all $n \in \mathbb{N}$, any m-dimensional marginal of $\{X^{(n)}(t),\, t \geq 0\}$, $m \leq n$, is equal in distribution to $\{X^{(m)}(t),\, t \geq 0\}$, i.e., for all $1 \leq i_1 < \ldots < i_m \leq n$,

$$\{X_{i_1}^{(n)}(t), \ldots, X_{i_m}^{(n)}(t),\, t \geq 0\} = \{X_1^{(m)}(t), \ldots, X_m^{(m)}(t),\, t \geq 0\} \qquad \text{in distribution} \tag{3.1}$$

In the following theorem we show that, if a configuration process is consistent then a compatible coordinate process is also consistent provided that its initial distribution at time $t = 0$ is consistent and permutation invariant.

Theorem 3.2. *Let* $\{X(t),\, t \geq 0\}$ *be a family of coordinate Markov processes with generators* L_n*,* $n \in \mathbb{N}$*, and let* $\{\eta(t),\, t \geq 0\}$ *be a compatible configuration process. Assume that:*

i) $\{\eta(t),\ t \geq 0\}$ *is consistent;*

ii) the probability measures $\{\mu_n, n \in \mathbb{N}\}$ *on* V_n*,* $n \in \mathbb{N}$*, form a consistent family which is also permutation-invariant.*

Then we have consistency of the family of coordinate processes starting from $\{\mu_n, n \in \mathbb{N}\}$, i.e., for all $n \in \mathbb{N}$, $g \in \mathscr{C}_{n-1}$ permutation-invariant,

$$\mathbb{E}^{(n)}_{\mu_n}\Big[g(X^{(n)}_1(t),\dots,X^{(n)}_{n-1}(t))\Big] = \mathbb{E}^{(n-1)}_{\mu_{n-1}}\Big[g(X^{(n)}_1(t),\dots,X^{(n)}_{n-1}(t))\Big] \tag{3.2}$$

where $\mathbb{E}^n_{\mu_n}$ denotes expectation w.r.t. the Markov process $\{X^{(n)}(t),\, t \geq 0\}$, started initially with distribution μ_n.

Proof. Let $g \in \mathscr{C}_{n-1}$ be a permutation-invariant function, then we have $\pi^{(n)}_l g = \pi^{(n)}_k g$ for $k, l \in \{1,\dots,n\}$. Then, by consistency and permutation invariance of μ_n, we have

$$\int \pi_l g(x_1,\dots,x_n)\mu_n(dx_1\dots dx_n) = \int g(x_1,\dots,x_{n-1})\mu_{n-1}(dx_1\dots dx_{n-1})$$

for all $n \in \mathbb{N}$ and all $l \in \{1,\dots,n\}$. Therefore, using (2.14), we have

$$\begin{aligned}
\mathbb{E}^{(n)}_{\mu_n}\Big[g(X^{(n)}_1(t),\dots,X^{(n)}_{n-1}(t))\Big] &= \mathbb{E}^{(n)}_{\mu_n}\Big[\pi^{(n)}_n(g(X^{(n)}_1(t),\dots,X^{(n)}_{n-1}(t),X^{(n)}_n(t))\Big] \\
&= \int S_n(t)(\pi^{(n)}_n g)(x_1,\dots,x_n)\mu_n(dx_1\dots dx_n) \\
&= \frac{1}{n}\int \left(S_n(t)\left(\sum_{k=1}^n \pi^{(n)}_k g\right)\right)(x_1,\dots,x_n)\mu_n(dx_1\dots dx_n) \\
&= \frac{1}{n}\int S_n(t)(\Pi^{(n)} g)(x_1,\dots,x_n)\mu_n(dx_1\dots dx_n) \\
&= \frac{1}{n}\int (\Pi^{(n)} S_{n-1}(t) g)(x_1,\dots,x_n)\mu_n(dx_1\dots dx_n) \\
&= \int (S_{n-1}(t) g)(x_1,\dots,x_{n-1})\mu_{n-1}(dx_1\dots dx_n) \\
&= \mathbb{E}^{(n-1)}_{\mu_{n-1}}\Big[g(X^{(n)}_1(t),\dots,X^{(n)}_{n-1}(t))\Big].
\end{aligned} \tag{3.3}$$

This concludes the proof. □

3.2 Characterization of consistent configuration processes

In this section we provide a class of configuration processes exhibiting the consistency property. We restrict ourselves to models where only one particle jumps at a time. We further assume the jump rate to depend only on the number of particles hosted by the departure and arrival sites. We consider here generators of the form

$$\mathscr{L} := \sum_{\{i,j\}\in E} \mathscr{L}_{i,j} \tag{3.4}$$

where the summation is over the set E of (non-oriented) edges $\{i, j\}$ of the complete graph with vertices V and

$$\mathscr{L}_{i,j} f(\eta) = c_{i,j}(\eta_i,\eta_j)[f(\eta^{i,j}) - f(\eta)] + c_{j,i}(\eta_j,\eta_i)[f(\eta^{j,i}) - f(\eta)]. \tag{3.5}$$

with $\eta^{i,j} := \eta - \delta_i + \delta_j$ and $c_{i,j} : \Lambda \times \Lambda \to \mathbb{R}^+$ being the hopping rate for a particle to jump from site i to site j.

We then call $\mathscr{L}_{i,j}$ the single-edge generator corresponding to the (non-oriented) edge $\{i, j\}$. We will characterize the processes such that the annihiliation operator commutes with each single edge generator. These processes are then automatically consistent.

Theorem 3.3. *Let* $\{\eta(t),\ t \geq 0\}$ *be a configuration process with generator* (3.4)-(3.5)*. We have that* $[\mathscr{L}_{i,j}, \mathscr{A}] = 0$ *for all* $\{i,j\} \in E$ *if and only if the rates are of the form:*

$$c_{i,j}(\kappa, m) = \kappa \cdot \Big(\theta(\{i,j\}) \cdot m + \alpha(i,j)\Big) \tag{3.6}$$

for some $\theta : E \to \mathbb{R}$ *and* $\alpha : V^2 \to \mathbb{R}^+$ *such that* $\theta(\{i,j\}) \cdot m + \alpha(i,j) \geq 0$ *for all possible values of* $(\kappa, m) \in \Lambda \times \Lambda$*. As a consequence the corresponding configuration process with rates* (3.6) *is consistent.*

Remark 3.4. By requiring that the r.h.s. of (3.6) are rates we have implicitly assumed they are non-negative. This implies both restrictions on the functions θ and α and on the set Λ, i.e the allowed values of the occupations $(\eta_i)_{i\in V}$.

Proof. We have have to characterize the generators $\mathscr{L}$ such that for all $i, j \in V$

$$[\mathscr{L}_{i,j}, \mathscr{A}] = 0. \tag{3.7}$$

Recalling that $\mathscr{A} = \sum_{x\in V} a_x$, using the assumption that the rates only depend on the number of particles in the departure and arrival sites (3.5), we have that

$$[\mathscr{L}_{i,j}, a_x] = 0 \qquad \text{for all} \quad x \neq i, j \tag{3.8}$$

Therefore we have that (3.7) is satified if and only if for all i, j

$$[\mathscr{L}_{i,j}, a_i + a_j] = 0 \tag{3.9}$$

We thus impose

$$\mathscr{L}_{i,j}(a_i + a_j)f = (a_i + a_j)\mathscr{L}_{i,j}f \tag{3.10}$$

for all functions $f : \Lambda \times \Lambda \to \mathbb{R}$ of two variables. One one hand we have that

$$\begin{aligned} \mathscr{L}_{i,j}(a_i + a_j)f(\kappa, m) &= c_{i,j}(\kappa, m)\left[(\kappa-1)f(\kappa-2, m+1) - \kappa f(\kappa-1, m)\right] \\ &+ c_{j,i}(m, \kappa)\left[(\kappa+1)f(\kappa, m-1) - \kappa f(\kappa-1, m)\right] \\ &+ c_{i,j}(\kappa, m)\left[(m+1)f(\kappa-1, m) - m f(\kappa, m-1)\right] \\ &+ c_{j,i}(m, \kappa)\left[(m-1)f(\kappa+1, m-2) - m f(\kappa, m-1)\right]; \end{aligned} \tag{3.11}$$

on the other hand

$$\begin{aligned} (a_i + a_j)\mathscr{L}_{i,j}f(\kappa, m) &= \kappa c_{i,j}(\kappa-1, m)\left[f(\kappa-2, m+1) - f(\kappa-1, m)\right] \\ &+ \kappa c_{j,i}(m, \kappa-1)\left[f(\kappa, m-1) - f(\kappa-1, m)\right] \\ &+ m c_{i,j}(\kappa, m-1)\left[f(\kappa-1, m) - f(\kappa, m-1)\right] \\ &+ m c_{j,i}(m-1, \kappa)\left[f(\kappa+1, m-2) - f(\kappa, m-1)\right]. \end{aligned} \tag{3.12}$$

Imposing (3.10) for all f is equivalent to imposing the following conditions for all $m, \kappa \in \Lambda$:

$$c_{i,j}(\kappa, m)(\kappa-1) = c_{i,j}(\kappa-1, m)\kappa \tag{3.13}$$

$$c_{j,i}(m, \kappa)(m-1) = c_{j,i}(m-1, \kappa)m \tag{3.14}$$

and

$$\begin{aligned} &(m+1)c_{i,j}(\kappa, m) - \kappa c_{j,i}(m, \kappa) - \kappa c_{i,j}(\kappa, m) = \\ &\qquad = m c_{i,j}(\kappa, m-1) - \kappa c_{j,i}(m, \kappa-1) - \kappa c_{i,j}(\kappa-1, m) \\ &(\kappa+1)c_{j,i}(m, \kappa) - m c_{i,j}(\kappa, m) - m c_{j,i}(m, \kappa) = \\ &\qquad = \kappa c_{j,i}(m, \kappa-1) - m c_{i,j}(\kappa, m-1) - m c_{j,i}(m-1, \kappa) \end{aligned} \tag{3.15}$$

Iterating the conditions (3.13)-(3.14) produces

$$c_{i,j}(\kappa,m)=\kappa\cdot b_{i,j}(m),\qquad b_{i,j}(m):=c_{i,j}(1,m) \tag{3.16}$$
$$c_{j,i}(m,\kappa)=m\cdot b_{j,i}(\kappa),\qquad b_{j,i}(\kappa):=c_{j,i}(1,\kappa) \tag{3.17}$$

for some functions $b_{i,j},b_{j,i}:\Lambda\to\mathbb{R}$. Inserting (3.16) and (3.17) in the conditions (3.15) we get

$$b_{i,j}(m)-b_{i,j}(m-1)=b_{j,i}(\kappa)-b_{j,i}(\kappa-1)\qquad \text{for all}\quad \kappa,m\in\Lambda \tag{3.18}$$

Since the conditions (3.18) must be satisfied for all values of κ and m, the only possibility is that there exists a constant $\theta=\theta(\{i,j\})$ such that

$$b_{i,j}(m)-b_{i,j}(m-1)=\theta(\{i,j\})=b_{j,i}(\kappa)-b_{j,i}(\kappa-1)\qquad \text{for all}\quad \kappa,m\in\Lambda. \tag{3.19}$$

Hence, iterating both for $b_{i,j}$ and for $b_{j,i}$ we deduce that there exist two constants, $\alpha_{i,j}$ and $\alpha_{j,i}$ such that

$$b_{i,j}(m)=\theta(\{i,j\})m+\alpha(i,j),\qquad b_{j,i}(\kappa)=\theta(\{i,j\})\kappa+\alpha(j,i) \tag{3.20}$$

Then, substituting (3.20) in (3.16)-(3.17) we obtain

$$c_{i,j}(\kappa,m)=\kappa\cdot\Big(\theta(\{i,j\})\cdot m+\alpha(i,j)\Big) \tag{3.21}$$
$$c_{j,i}(m,\kappa)=m\cdot\Big(\theta(\{i,j\})\cdot \kappa+\alpha(j,i)\Big) \tag{3.22}$$

that concludes the proof. □

Remark 3.5. The class of models with rate of the form (3.6) contains some classical models such as symmetric partial exclusion process, independent random walkers and symmetric inclusion process, see section 6 and 7 below. Compared to the class of models with factorized self-duality, described in [13], the class of consistent processes introduced in Theorem 3.3 is larger and in particular contains models which do not have product invariant measures (namely when $\alpha(i,j)\neq\alpha(j,i)$).

3.3 Recursion relations

In this section we analyze the consequences of consistency. More precisely we prove recursive relations that give the transition probabilities of a system with m particles in terms of the transition probabilities for the system with $m-1$ particles. As a consequence we obtain recursive relations for time dependent factorial moments of consistent configuration processes. Specifically, in Theorem 3.8 below, we show that time dependent factorial moments of order m in a system with $n>m$ particles can be expressed in terms of m-particle transition probabilities. These recursion relations will be particularly useful when we deal with systems having absorbing sites (cf. sections 5.1 and 5.2). In the reversible setting they are equivalent with self-duality (cf. section 4).

To prove Theorem 3.8, we state two preparatory propositions.

Proposition 3.6. *Let $\{\eta(t),\, t\geq 0\}$ a consistent configuration process on a finite lattice V, then, for all $f:\Omega\to\mathbb{R}$ we have*

$$\sum_{i\in V}\mathbb{E}_\eta[\eta_i(t)f(\eta(t)-\delta_i)]=\sum_{i\in V}\eta_i\mathbb{E}_{\eta-\delta_i}[f(\eta(t))] \tag{3.23}$$

Proof. From the commutation of the generator of the configuration process $\mathscr{L}$ with the annihilation operator $\mathscr{A}$, we obtain the commutation of the semigroup of the configuration process $\mathscr{S}(t) = e^{t\mathscr{L}}$ with $\mathscr{A}$. We then remark that (3.23) can be written as

$$[\mathscr{S}(t)(\mathscr{A}f)](\eta) = [\mathscr{A}(\mathscr{S}(t)f)](\eta)$$

this concludes the proof. □

We define now the function

$$F(\xi,\eta) := \prod_{j\in V} \binom{\eta_j}{\xi_j}, \qquad \xi,\eta\in\Omega \tag{3.24}$$

and prove a recursive relation for its expectation.

Proposition 3.7. *Let $\{\eta(t),\, t\geq 0\}$ a consistent configuration process on a finite lattice V then, for all $\eta,\xi\in\Omega$ such that $1\leq|\xi|<|\eta|$, we have*

$$\mathbb{E}_\eta\left[F(\xi,\eta(t))\right] = \frac{1}{(|\eta|-|\xi|)}\cdot\sum_{i\in V}\eta_i\mathbb{E}_{\eta-\delta_i}\left[F(\xi,\eta(t))\right], \qquad \forall\, t\geq 0\,. \tag{3.25}$$

Proof. We fix $\eta,\xi\in\Omega$ such that $|\xi|\in\{1,\dots,|\eta|-1\}$ and use (3.23) for the function $f=F(\xi,\cdot)$. We get

$$\sum_{i\in V}\eta_i\mathbb{E}_{\eta-\delta_i}\left[F(\xi,\eta(t))\right] = \sum_{i\in V}\mathbb{E}_\eta\left[\eta_i(t)\binom{\eta_i(t)-1}{\xi_i}\prod_{\substack{j\in V\\ j\neq i}}\binom{\eta_j(t)}{\xi_j}\right] \tag{3.26}$$

The right-hand side of (3.26) is equal to

$$\sum_{i\in V}\mathbb{E}_\eta\left[(\eta_i(t)-\xi_i)\cdot\prod_{j\in V}\binom{\eta_j(t)}{\xi_j}\right] = (|\eta|-|\xi|)\cdot\mathbb{E}_\eta\left[F(\xi,\eta(t))\right] \tag{3.27}$$

Combining together (3.26) and (3.27) we obtain (3.25). □

The function $F(\xi,\eta)$ defined in (3.24) has a precise combinatorial meaning that is useful to understand with the help of the coordinate notation. Let us first give some preliminary notations. We denote by $[n]$ the set of the first n natural numbers: $[n] := \{1,\dots,n\}$, for $n\in\mathbb{N}$. Then, for $m\leq n$ we define the set $C_{m,n}$ of combinations of m elements chosen in $[n]$:

$$C_{m,n} := \{(i_1,\dots,i_m):\ i_j\in[n],\ \forall\, j\in[m]\ \text{s.t.}\ \ i_1<i_2<\ldots<i_m\}\subset[n]^m. \tag{3.28}$$

Let now $I:=(i_1,\dots,i_m)$ be an element of $[n]^m$, and let $\mathbf{x}\in V_n$ the vector of the positions of n particles in V, then we denote by $\mathbf{x}_I$ the m-uple:

$$\mathbf{x}_I := (x_{i_1},\dots,x_{i_m})\in V_m \tag{3.29}$$

i.e. the vector of the positions of the particles with labels in I. We define, moreover the following ordering between elements of the coordinates state spaces. For $\mathbf{x},\mathbf{y}\in\cup_{n\in\mathbb{N}}V_n$, we say that

$$\mathbf{y}\leq\mathbf{x} \quad \text{ if and only if } \quad |\mathbf{y}|\leq|\mathbf{x}| \ \text{ and } \ \exists\, I\in C_{|\mathbf{y}|,|\mathbf{x}|} \ \text{ s.t. } \ \mathbf{y}=\mathbf{x}_I. \tag{3.30}$$

For what concerns the combinatoric interpretation of F, we have that the value $F(\xi,\eta)$ is equal to the number of ways to choose, for each site $i\in V$, ξ_i particles out of η_i. Then, for any fixed particles labelling of the configuration η, i.e. for any $\mathbf{x} \in V_{|\eta|}$ such that $\varphi(\mathbf{x}) = \eta$, we can write

$$F(\xi,\eta) = \left|\{I \in C_{|\xi|,|\eta|} :\ \varphi(\mathbf{x}_I) = \xi\}\right| \cdot \mathbf{1}_{\xi\le\eta}. \tag{3.31}$$

In other words, for any fixed labelling $\mathbf{x}$ of particles in the configuration η, it is the number of ways to select $|\xi|$ particles out of $|\eta|$ in such a way that the corresponding configuration in Ω is ξ. In view of the ordering (3.30), we can also rewrite (3.31) as follows:

$$F(\xi,\eta) \quad = \quad \left|\{\mathbf{y} \in V_{|\xi|} : \mathbf{y} \le \mathbf{x},\ \varphi(\mathbf{y}) = \xi\}\right| \cdot \mathbf{1}_{\xi\le\eta}, \qquad \forall\, \mathbf{x} :\ \varphi(\mathbf{x}) = \eta \tag{3.32}$$

This suggests that the term F disappears when switching from the configurations to coordinate variables in the summations of the following type:

$$\sum_{\xi\in\Omega_m} F(\xi,\eta) f(\xi) \tag{3.33}$$

indeed, for any $\mathbf{x} \in V_{|\eta|}$ such that $\varphi(\mathbf{x}) = \eta$ we have that (3.33) is equal to

$$\begin{aligned}\sum_{\xi\in\Omega_m} F(\xi,\varphi(\mathbf{x})) f(\xi) \quad &= \quad \sum_{\substack{\xi\in\Omega_m\\ \xi\le\varphi(\mathbf{x})}} \left|\{I \in C_{m,n} :\ \varphi(\mathbf{x}_I) = \xi\}\right| \cdot f(\xi)\\ = \sum_{\substack{\xi\in\Omega_m\\ \xi\le\varphi(\mathbf{x})}} \sum_{\substack{I\in C_{m,n}:\\ \varphi(\mathbf{x}_I)=\xi}} f(\varphi(\mathbf{x}_I)) \quad &= \quad \sum_{I\in C_{m,n}} f(\varphi(\mathbf{x}_I)) = \sum_{\substack{\mathbf{y}\in V_m\\ \mathbf{y}\le\mathbf{x}}} f(\varphi(\mathbf{y})).\end{aligned} \tag{3.34}$$

Theorem 3.8. *Let $\{\eta(t),\ t \ge 0\}$ be a consistent configuration process on a finite lattice V. Let $\mathbf{x} = (x_1, \ldots, x_n) \in V_n$, then, for all $\xi \in \Omega_m$ with $m \in \{1, \ldots, n-1\}$,*

$$\mathbb{E}_{\varphi(\mathbf{x})}\left[F(\xi,\eta(t))\right] = \sum_{I\in C_{m,n}} \mathbb{P}_{\varphi(\mathbf{x}_I)}\left(\eta(t) = \xi\right). \tag{3.35}$$

Proof. Let $\eta := \varphi(\mathbf{x}) \in \Omega_n$ and ξ as in the hypothesis and define $\kappa := n - |\xi| = n - m$. From Lemma 3.7 we have that, if $\kappa \in \{1, \ldots, n-1\}$,

$$\begin{aligned}\mathbb{E}_\eta\left[F(\xi,\eta(t))\right] \quad &= \quad \frac{1}{n-m} \sum_{i\in V} \eta_i\, \mathbb{E}_{\eta-\delta_i}\left[F(\xi,\eta(t))\right]\\ &= \quad \frac{1}{\kappa} \sum_{j_1=1}^{n} \mathbb{E}_{\eta-\varphi(x_{j_1})}\left[F(\xi,\eta(t))\right]\end{aligned} \tag{3.36}$$

Now, if $n - 1 = |\eta - \varphi(x_{j_1})| > m = n - \kappa$, namely, if $\kappa \ge 2$ this can be iterated since

$$\mathbb{E}_{\eta-\varphi(x_{j_1})}\left[F(\xi,\eta(t))\right] = \frac{1}{\kappa-1} \sum_{\substack{j_2=1\\ j_2\ne j_1}}^{n} \mathbb{E}_{\eta-\varphi(x_{j_1},x_{j_2})}\left[F(\xi,\eta(t))\right] \tag{3.37}$$

We denote by $J = (j_1, \ldots, j_\kappa)$ an element of the set $C_{\kappa,n}$ defined in (3.28), hence, iterating the argument used in (3.36) κ times we get:

$$\begin{aligned}\mathbb{E}_\eta\left[F(\xi,\eta(t))\right] \quad &= \quad \frac{1}{\kappa!} \sum_{j_1=1}^{n} \sum_{\substack{j_2=1\\ j_2\ne j_1}}^{n} \cdots \sum_{\substack{j_\kappa=1\\ j_\kappa\notin\{j_1,\ldots,j_{\kappa-1}\}}}^{n} \mathbb{E}_{\eta-\varphi(x_{j_1},\ldots,x_{j_\kappa})}\left[F(\xi,\eta(t))\right]\\ &= \quad \sum_{J\in C_{\kappa,n}} \mathbb{E}_{\eta-\varphi(x_{j_1},\ldots,x_{j_\kappa})}\left[F(\xi,\eta(t))\right]\\ &= \quad \sum_{J\in C_{n-\kappa,n}} \mathbb{E}_{\varphi(x_{j_1},\ldots,x_{j_{n-\kappa}})}\left[F(\xi,\eta(t))\right]\\ &= \quad \sum_{J\in C_{m,n}} \mathbb{E}_{\varphi(\mathbf{x}_J)}\left[F(\xi,\eta(t))\right]\end{aligned} \tag{3.38}$$

The theorem follows from the fact that now, for $I \in C_{m,n}$, $|\varphi(\mathbf{x}_I)| = m = |\xi|$ and from the observation that

$$F(\xi, \eta) = \mathbf{1}_{\eta=\xi}, \qquad \text{for} \quad |\eta| = |\xi| \tag{3.39}$$

so that $\mathbb{E}_{\varphi(\mathbf{y})}\left[F(\xi, \eta(t))\right] = \mathbb{P}_{\varphi(\mathbf{y})}\left(\eta(t) = \xi\right)$. This concludes the proof. □

Remark 3.9. The message of Theorem 3.8 is the following. Suppose we initialize the configuration-process $\{\eta(t),\, t \geq 0\}$ from a configuration η with $|\eta| = n$ particles. Now fix $1 \leq m \leq n-1$. Then (3.35) allows to compute all the factorial moments (of the occupation numbers) of order m in terms of the transition probabilities of the process initialized with m particles. In other words, it is possible to gain information about the system with n particles in terms of the dynamics of $m < n$ particles. Unfortunately the information provided by Theorem 3.8 is not complete, in the sense that (3.35) does not give the full distribution of the system with n particles. This is due to the fact that the factorial-moments of order n are still missing.

We close this section with a specialization of Theorem 3.8 that will be useful later.

Definition 3.10. *For a configuration process $\{\eta(t),\ t \geq 0\}$ we define an associated random walk process $\{X^{\mathrm{rw}}(t),\, t \geq 0\}$ which is such that when $\eta(0) = \delta_u$ then $\eta(t) = \delta_{X^{\mathrm{rw}}(t)}$, with $X^{\mathrm{rw}}(0) = u$.*

Remark 3.11. Notice that the random walk X^{rw} is only associated to the configuration process started with a single particle at time 0. Later on, in section 5 below, we will then consider independent copies of these walks and a configuration process $\{\eta^{\mathrm{irw}}(t); t \geq 0\}$ associated to them via the map φ.

Corollary 3.12. *Let $\{\eta(t),\, t \geq 0\}$ a consistent configuration process on a finite lattice V and let $\mathbf{x} = (x_1, \ldots, x_n) \in V_n$, then, for all $u \in V$ we have*

$$\mathbb{E}_{\varphi(\mathbf{x})}\left[\eta_v(t)\right] = \sum_{j=1}^{n} \mathbf{P}_{x_j}(X^{rw}(t) = v) \tag{3.40}$$

where $\mathbf{P}_u$, is the path space measure of the random walk $\{X^{\mathrm{rw}}(t),\, t \geq 0\}$ on V starting from $u \in V$ associated to the configuration process $\{\eta(t),\ t \geq 0\}$ as defined above in Definition 3.10.

Proof. The result immediately follows by applying Theorem 3.8 to the case $\xi = \delta_v$. □

4 Consistency and self-duality

In this section we show that consistency implies a form of self-duality whenever a process admits a strictly-positive reversible measure. We start by recalling the definition of *duality* (we refer to [11] for more background on duality, see also [1, 10]).

Definition 4.1. *Let $\{Y_t\}_{t\geq 0}$, $\{\widehat{Y}_t\}_{t\geq 0}$ be two Markov processes with state spaces Ω and $\widehat{\Omega}$ and $D : \Omega \times \widehat{\Omega} \to \mathbb{R}$ a bounded measurable function. The processes $\{Y_t\}_{t\geq 0}$, $\{\widehat{Y}_t\}_{t\geq 0}$ are said to be dual with respect to D if*

$$\mathbb{E}_y\big[D(Y_t, \widehat{y})\big] = \widehat{\mathbb{E}}_{\widehat{y}}\big[D(y, \widehat{Y}_t)\big] \tag{4.1}$$

for all $y \in \Omega, \widehat{y} \in \widehat{\Omega}$ and $t > 0$. In (4.1) $\mathbb{E}_y$ is the expectation with respect to the law of the process $\{Y_t\}_{t\geq 0}$ started at y, while $\widehat{\mathbb{E}}_{\widehat{y}}$ denotes expectation with respect to the law of the process $\{\widehat{Y}_t\}_{t\geq 0}$ initialized at $\widehat{y}$.

We say that a process is self-dual when the dual process coincides with the original process.

It is useful to express the duality property in terms of generators of the two processes. If L denotes the generator of $\{Y_t\}_{t\geq 0}$ and $\widehat{L}$ denotes the generator of $\{\widehat{Y}_t\}_{t\geq 0}$, then (assuming that the duality functions are in the domain of the generators), the above definition is equivalent to $LD(\cdot,\widehat{y})(y) = \widehat{L}D(y,\cdot)(\widehat{y})$ where L acts on the first variable and $\widehat{L}$ acts on the second variable. The conditions to have the equality between semigroup duality and generator duality are further discussed in [10].

In order to prove our theorem on the relation between consistency and self-duality we recall two general results on self-duality from [7].

a) *Trivial duality function from a reversible measure.*

If a Markov process $\{Y_t : t \geq 0\}$ with countable state-space Ω has a strictly-positive reversible measure ν, then the function $D : \Omega \times \Omega \to \mathbb{R}$ given by

$$D(y,\widehat{y}) = \frac{\delta_{y,\widehat{y}}}{\nu(y)} \tag{4.2}$$

is a self-duality function.

b) *New duality functions via symmetries.*

If $D : \Omega \times \Omega \to \mathbb{R}$ is a self-duality function and S is a symmetry of $\mathscr{L}$ (namely an operator acting on functions $f : \Omega \to \mathbb{R}$ commuting with $\mathscr{L}$, $[S,\mathscr{L}] = 0$) then SD is a self-duality function, where $SD(y,\widetilde{y}) := SD(y,\cdot)(\widetilde{y})$.

Lemma 4.2. *Let $\mathscr{A}$ denote the annihilation operator defined in* (2.13)*. For $\xi,\eta \in \Omega$ denote by $h_\xi(\eta) = \delta_{\xi,\eta}$ the Kronecker delta. Then we have*

$$(e^{\mathscr{A}} h_\xi)(\eta) = F(\xi,\eta) \tag{4.3}$$

with F as in (3.24)*.*

Proof. Recalling the definition of $\mathscr{A} = \sum_{x\in V} a_x$ where a_x denotes the operator a working on the variable η_x (see (2.12)-(2.13)), in order to prove (4.3) it is sufficient to show that for all $n,k \in \mathbb{N}_0$

$$e^a h_k(n) = \binom{n}{k} \tag{4.4}$$

with $h_k(n) = \delta_{k,n}$. For a function $f : \mathbb{N}_0 \to \mathbb{R}$ we have

$$e^a f(n) = \sum_{r=0}^{n} \binom{n}{r} f(n-r).$$

Inserting $f = h_k$ gives (4.4). □

We then obtain the following result.

Theorem 4.3. *Let $\{\eta(t),\, t \geq 0\}$ be a reversible configuration process with reversible measure ν that is strictly-positive. Then the process is consistent if and only if it is self-dual with self-duality function*

$$D(\xi,\eta) = \frac{1}{\nu(\xi)} \cdot F(\xi,\eta) \tag{4.5}$$

with $F(\cdot,\cdot)$ as in (3.24)*.*

Proof. We first prove that reversibility and consistency implies that (4.5) is a self-duality function. From reversibility we know that the function

$$D^{\mathrm{rev}}(\xi,\eta) = \frac{\delta_{\xi,\eta}}{\nu(\xi)}$$

is a self-duality function. If we now act with $e^{\mathscr{A}}$ on D^{rev} in the η-variable we produce (4.5) via Lemma 4.2. This produces a new self-duality function because, by assumption, $\mathscr{A}$ is a symmetry of $\mathscr{L}$.

Now we prove that reversibility and self-duality with duality function (4.5) implies consistency. Fix $\eta \in \Omega_n$ and let $\xi = \eta - \delta_x$ for some $x \in V$, then $\xi \in \Omega_{n-1}$. Then,

$$\nu(\xi) \cdot D(\xi, \eta) = F(\xi, \eta) = \eta_x. \tag{4.6}$$

Let L_n be the operators implicitly defined by (2.3). Because D is a self-duality function and ν is a strictly-positive function on Ω, satisfying detailed balance w.r.t. $\mathscr{L}$, the operator $K : \mathscr{E}_{n-1} \to \mathscr{E}_n$

$$Kf(\eta) := \sum_{\xi \in \Omega_{n-1}} \nu(\xi) D(\xi, \eta) f(\xi) = \sum_{\xi \in \Omega_{n-1}} F(\xi, \eta) f(\xi)$$

intertwines between L_n and L_{n-1}, i.e., $K(L_{n-1} f) = L_n K f$ for $f \in \mathscr{E}_{n-1}$ (see [9] for the connection between duality functions and corresponding intertwining kernel operators). Now, via (4.6) we obtain

$$Kf(\eta) = \sum_x \eta_x f(\eta - \delta_x) = \mathscr{A} f(\eta)$$

and conclude that $\mathscr{A}(L_{n-1} f) = L_n \mathscr{A} f$ for $f \in \mathscr{E}_{n-1}$, which is exactly the commutation property $[\mathscr{L}, \mathscr{A}] = 0$. The process is thus consistent. □

5 Consistency for systems with absorbing sites

In this section we study consistency for systems of interacting particles with absorbing sites. These systems are important as they emerge as dual processes of boundary-driven non-equilibrium systems connected to reservoirs. In Section 7 we will analyze the consequences of consistency in the context of non-equilibrium systems.

Let $\{\eta(t),\, t \geq 0\}$ denote a configuration process with generator $\mathscr{L}$. Let $V^* \supset V$ be a countable set of sites containing V, and call $V^{\mathrm{abs}} := V^* \setminus V$. We will define a configuration process on $\mathbb{N}^{V^*}$ as follows: inside V particles move according to the generator $\mathscr{L}$, and additionally every particle at site i moves with rate $r(i,j)$ to a site $j \in V^{\mathrm{abs}}$, independently from each other. Particles arriving at sites $j \in V^{\mathrm{abs}}$ are absorbed and, after absorption, do not move anymore. We thus obtain what we call the process with absorbing sites V^{abs} and absorption rates $r(i,j)$. The generator of this process is given by

$$\mathscr{L}^{\mathrm{abs}} f(\eta) = \mathscr{L} f(\eta) + \mathscr{H} f(\eta) \qquad \text{for} \quad f : \mathbb{N}^{V^*} \to \mathbb{R} \tag{5.1}$$

where $\mathscr{L}$ only works on the variables $\{\eta_i, i \in V\}$ and where $\mathscr{H}$ denotes the absorption part of the generator, i.e.,

$$\mathscr{H} f(\eta) = \sum_{i \in V, j \in V^{\mathrm{abs}}} r(i,j) \eta_i \left[f(\eta^{i,j}) - f(\eta) \right] \qquad \text{with} \quad \eta^{i,j} := \eta - \delta_i + \delta_j \tag{5.2}$$

We can rewrite $\mathscr{H}$ as follows

$$\mathscr{H} = \sum_{i \in V, j \in V^{\mathrm{abs}}} r(i,j) [a_i a_j^\dagger - a_i a_i^\dagger] \tag{5.3}$$

where a_i is the annihilation operator at site i defined in (2.13) and $a_i^\dagger$ is the creation operator defined via

$$a_i^\dagger f(\eta) \quad = \quad f(\eta + \delta_i).$$

A process with generator (5.1) is called an absorbing extension of the generator $\mathscr{L}$.

Definition 5.1. *Let $\{X(t),\, t \geq 0\}$ be a family of coordinate Markov processes on the lattice V. Then we define its absorbing extension $\{X^{\text{abs}}(t),\, t \geq 0\}$ to the lattice V^* as the family of coordinate Markov processes $\{X^{\text{abs},(\text{n})}(t),\, t \geq 0\}$, $n \in \mathbb{N}$, on $(V^*)^n$ defined by adding to the jumps of $\{X^{(n)}(t),\, t \geq 0\}$, $n \in \mathbb{N}$, the additional jumps from $i \in V$ to $j \in V^{\text{abs}}$ at rate $r(i,j)$, that particles perform independently from each other.*

Define the set Ω^{abs} of configurations on absorbing sites:

$$\Omega^{\text{abs}} := \{\zeta \in \mathbb{N}_0^{V^{\text{abs}}} : \|\zeta\| < \infty\}. \tag{5.4}$$

We have the following lemma.

Lemma 5.2. *Let $\{X(t),\, t \geq 0\}$ be a family of coordinate Markov processes on the lattice V compatible with the configuration process on Ω having generator $\mathscr{L}$. Then its absorbing extension $\{X^{\text{abs}}(t),\, t \geq 0\}$ to the lattice V^* is compatible with the configuration process on $\Omega^* := \Omega \times \Omega^{\text{abs}}$ with generator $\mathscr{L}^{\text{abs}}$ in (5.1).*

Proof. It follows immediately from the definition of absorbing extensions. □

We then have the following results.

Lemma 5.3. *Let $\{\eta(t),\ t \geq 0\}$ be a consistent configuration process on a lattice V, then every absorbing extension to a lattice $V^* \supset V$ is a consistent process.*

Proof. Let $\mathscr{A}$ denote the annihilation operator (2.13). We want to prove that

$$[\mathscr{L}^{\text{abs}}, \mathscr{A}^{\text{abs}}] = 0 \qquad \text{for} \quad \mathscr{A}^{\text{abs}} f(\eta) := \sum_{i \in V^*} a_i \tag{5.5}$$

with a_i as defined in (2.13). Since $[\mathscr{L}, \mathscr{A}] = 0$ by assumption, and as a consequence $[\mathscr{L}, \mathscr{A}^{\text{abs}}] = 0$, we only have to prove that

$$[\mathscr{H}, \mathscr{A}^{\text{abs}}] = 0.$$

Using (5.3) and the fact that operators working on variables at different sites commute, we have to show that for all $i, j \in V^*$

$$[a_i a_j^\dagger - a_i a_i^\dagger, a_i + a_j] = 0$$

This in turn follows from the commutation relations $[a_i, a_j] = 0, [a_i, a_j^\dagger] = \delta_{i,j}$. □

Theorem 5.4. *Let $\{\eta(t),\ t \geq 0\}$ be a consistent configuration process on the lattice V, and let $\{X(t),\, t \geq 0\}$ be a family of coordinate Markov processes compatible with it. Then its absorbing extension $\{X^{\text{abs}}(t),\, t \geq 0\}$ to $V^* \supset V$ is consistent if started from a consistent family $\{\mu_n, n \in \mathbb{N}\}$ of probability measures on $(V^*)^n$, $n \in \mathbb{N}$, which is also permutation-invariant. Namely for all $n \in \mathbb{N}$, $g : (V^*)^n \to \mathbb{R}$ permutation-invariant,*

$$\mathbb{E}^{(n)}_{\mu_n}\left[g(X_1^{\text{abs},(\text{n})}(t), \ldots, X_{n-1}^{\text{abs},(\text{n})}(t))\right] = \mathbb{E}^{(n-1)}_{\mu_{n-1}}\left[g(X_1^{\text{abs},(\text{n})}(t), \ldots, X_{n-1}^{\text{abs},(\text{n})}(t))\right] \tag{5.6}$$

where $\mathbb{E}^n_{\mu_n}$ denotes expectation w.r.t. the Markov process $\{X^{\text{abs},(\text{n})}(t),\, t \geq 0\}$, started initially with distribution μ_n.

Proof. It follows from Lemma 5.2, Lemma 5.3 and Theorem 3.2. □

5.1 Recursion relations for absorption probabilities

In what follows we denote by $\mathbb{P}_\eta(\eta(\infty) = \zeta)$ the probability that eventually $\eta(t)$ settles in the absorbing configuration $\zeta \in \Omega^{\text{abs}}$ starting from the initial configuration $\eta \in \Omega^*$, i.e.,

$$\mathbb{P}_\eta(\eta(\infty) = \zeta) := \lim_{t \to \infty} \mathbb{P}_\eta(\eta(t) = \zeta). \tag{5.7}$$

Similarly,

$$\mathbb{E}_\eta[f(\eta(\infty))] = \lim_{t\to\infty} \mathbb{E}_\eta[f(\eta(t))]. \tag{5.8}$$

Then, as a consequence of Lemma 5.3 and Theorem 3.8 we have the following recursion relations for the absorption probabilities.

Theorem 5.5. *Let $\{\eta(t),\, t \geq 0\}$ be a consistent configuration process on a finite lattice V^* with generator $\mathscr{L}^{\rm abs} = \mathscr{L} + \mathscr{H}$. Let $\mathbf{x} = (x_1, \ldots, x_n) \in (V^*)^n$, then, for all $\zeta \in \Omega_m^{\rm abs}$ with $m \in \{1, \ldots, n-1\}$, we have*

$$\mathbb{E}_{\varphi(\mathbf{x})}\left[F(\zeta, \eta(\infty))\right] = \sum_{I\in C_{m,n}} \mathbb{P}_{\varphi(\mathbf{x}_I)}\left(\eta(\infty) = \zeta\right). \tag{5.9}$$

The relations (5.9) express combinations of absorption probabilities from the initial configuration $\varphi(\mathbf{x})$ in terms of combinations of absorption probabilities from an initial configuration η' with less particles. Although these equations are not sufficient to determine the absorption probabilities in closed form, they are still considerably simplifying the problem of computing them as they imply severe restrictions.

The following Corollary immediately follows by specializing (5.9) to the case $\zeta = \delta_v$ for some $v \in V^{\rm abs}$.

Corollary 5.6. *Let $\{\eta(t),\, t \geq 0\}$ be a consistent configuration process on a finite lattice V^* with generator $\mathscr{L}^{\rm abs} = \mathscr{L} + \mathscr{H}$. Let $\mathbf{x} = (x_1, \ldots, x_n) \in (V^*)^n$, then, for all $v \in V^{\rm abs}$ we have*

$$\mathbb{E}_{\varphi(\mathbf{x})}\left[\eta_v(\infty)\right] = \sum_{j=1}^n \mathbf{P}_{x_j}(X^{\rm rw}(\infty) = v) \tag{5.10}$$

where $\mathbf{P}_u$ is the path-space measure of the random walk $\{X^{\rm rw}(t),\, t \geq 0\}$ on V^* starting from $u \in V^*$ associated to the configuration process $\{\eta(t),\ t \geq 0\}$ as in Definition 3.10.

5.2 Systems with two absorbing states

We now consider the situation in which the system contains only two absorbing states, $|V^{\rm abs}| = 2$, say $V = \{1, \ldots, N\}$ and $V^{\rm abs} = \{0, N+1\}$. In this case, due to the conservation of particle number, it is sufficient to know the absorption probability in one of the two states, say state 0. The following proposition is an immediate consequence of Theorem 5.5.

Proposition 5.7. *Let $\{\eta(t),\, t \geq 0\}$ be a consistent configuration process on the lattice $V^* = V \cup V^{\rm abs}$, $V = \{1, \ldots, N\}$, $V^{\rm abs} = \{0, N+1\}$, with generator $\mathscr{L}^{\rm abs} = \mathscr{L} + \mathscr{H}$. Let $\mathbf{x} = (x_1, \ldots, x_n) \in (V^*)^n$, then, for all $m \in \{1, \ldots, n-1\}$, we have*

$$\mathbb{E}_{\varphi(\mathbf{x})}\left[\binom{\eta_0(\infty)}{m}\right] = \sum_{I\in C_{m,n}} \mathbb{P}_{\varphi(\mathbf{x}_I)}\left(\eta_0(\infty) = m\right). \tag{5.11}$$

Proof. This follows by applying Theorem 5.5 to the case $\zeta = m\delta_0$, for $1 \leq m \leq n-1$. □

Remark 5.8. Notice that (5.11) can be viewed as a linear system of $n-1$ independent equations in the $n+1$ variables $\{\mathbb{P}_\eta\left(\eta_0(\infty) = m\right),\ m = 0, \ldots, n\}$. Indeed we consider inductively the r.h.s. of (5.11) as a "known quantity" because it concerns an absorption probability of less than n particles. For instance if $n = 2$ and $m = 1$ the r.h.s. of (5.11) contains only the absorption probability of one particle. Complementing these with the normalization condition $\sum_{m=0}^n \mathbb{P}_\eta\left(\eta_0(\infty) = m\right) = 1$ we obtain n independent equations. This is still not sufficient to get a closed form expression for the absorption probability (which represent $n+1$ unknowns), since one independent equation is still missing.

Generating function method

Let $\{\eta(t),\, t \geq 0\}$ be a consistent process on the lattice $V^* = V \cup V^{\mathrm{abs}}$, $V = \{1, \dots, N\}$, $V^{\mathrm{abs}} = \{0, N+1\}$, with generator $\mathscr{L}^{\mathrm{abs}} = \mathscr{L} + \mathscr{H}$. We define the function

$$G(\eta, z) := \mathbb{E}_\eta \left[z^{\eta_0(\infty)} \right], \qquad \eta \in \Omega^*,\ z \geq 0 \tag{5.12}$$

The function $G(\eta, \cdot)$ is the probability generating function of the number of absorbed particles at 0 starting from the configuration η. Here we define as usual $G(\eta, 0)$ by continuous extension, i.e.,

$$G(\eta, 0) := \lim_{z \to 0} G(\eta, z) = \mathbb{P}_\eta(\eta_0(\infty) = 0)$$

which is equal to the probability that all the particles in η are eventually absorbed at $N+1$. We then have the following recursion relation.

Proposition 5.9. *Let $\{\eta(t),\, t \geq 0\}$ be a consistent configuration process on the lattice $V^* = V \cup V^{\mathrm{abs}}$, $V = \{1, \dots, N\}$, $V^{\mathrm{abs}} = \{0, N+1\}$, with generator $\mathscr{L}^{\mathrm{abs}} = \mathscr{L} + \mathscr{H}$. For all $\eta \in \Omega_n^*$ we have*

$$(1-z) G'(\eta, z) + n G(\eta, z) = \sum_{i \in V^*} \eta_i G(\eta - \delta_i, z) \tag{5.13}$$

and, as a consequence,

$$G(\eta, z) = (1-z)^n G(\eta, 0) + (1-z)^n \sum_{i \in V^*} \eta_i \int_0^z \frac{1}{(1-u)^{n+1}}\, G(\eta - \delta_i, u) du \tag{5.14}$$

Proof. Because of commutation of the generator of the absorbing system with the annihilation operator (2.13) we can write

$$\lim_{t \to \infty} \mathbb{E}_\eta[(\mathscr{A} z^{\eta_0})(t)] = \sum_{i \in V^*} \eta_i G(\eta - \delta_i, z)$$

which leads to

$$\mathbb{E}_\eta[\eta_0(\infty) z^{\eta_0(\infty)-1}] + \mathbb{E}_\eta \left[(n - \eta_0(\infty)) z^{\eta_0(\infty)} \right] = \sum_{i \in V^*} \eta_i G(\eta - \delta_i, z)$$

which gives (5.13). We can now "integrate" the recursion (5.13) as follows. Putting $G(\eta, z) = (1-z)^n H(\eta, z)$ and substituting in (5.13) we find,

$$(1-z)^{n+1} H'(\eta, z) = \sum_{i \in V^*} \eta_i G(\eta - \delta_i, z).$$

Noticing that $H(\eta, 0) = G(\eta, 0)$, by integrating we obtain (5.14). □

The recursion (5.14) can be iterated until we are left with one particle configurations for which, in ideal situations, the generating function can be computed, as we will see in the last section. It is crucial then to obtain a formula for the function

$$\mathscr{G}(\eta, z) := G(\eta, z) - G^{\mathrm{irw}}(\eta, z) \tag{5.15}$$

that is the difference between the generating function of our process $\{\eta(t),\, t \geq 0\}$ and the generating function $G^{\mathrm{irw}}(\eta, z)$ of the auxiliary process $\{\eta^{\mathrm{irw}}(t),\, t \geq 0\}$ of independent random walkers defined as follows. Let $\{X^{\mathrm{rw}}(t),\, t \geq 0\}$ be the random walker on V^* associated to the configuration process $\{\eta(t),\, t \geq 0\}$ as in Definition 3.10. Now let

$\{X^{\mathrm{irw}}(t),\, t \geq 0\}$ be the family of coordinate processes $\{X^{\mathrm{irw},(\mathrm{n})}(t),\, t \geq 0\}$, $n \geq 1$ on $(V^*)^n$ whose coordinates are n independent copies of $X^{\mathrm{rw}}(t)$:

$$X^{\mathrm{irw},(\mathrm{n})}(t) = (X_1^{\mathrm{rw}}(t), \dots, X_n^{\mathrm{rw}}(t)), \qquad t \geq 0.$$

Then we define $\{\eta^{\mathrm{irw}}(t),\, t \geq 0\}$ as the configuration process compatible with $\{X^{\mathrm{irw}}(t),\, t \geq 0\}$ and we denote by $\mathbb{P}_\eta^{\mathrm{irw}}$ the related path-space measure conditioned to $\eta^{\mathrm{irw}}(0) = \eta$.

It is clear that for $|\eta| = 1$, $\mathscr{G}(\eta, z) = 0$. In the next theorem we obtain a formula for the difference function $\mathscr{G}(\eta, \cdot)$ when $|\eta| \geq 2$.

Theorem 5.10. *Let $\{\eta(t),\, t \geq 0\}$ be a consistent configuration process on the lattice $V^* = V \cup V^{\mathrm{abs}}$, $V = \{1, \dots, N\}$, $V^{\mathrm{abs}} = \{0, N+1\}$, with generator $\mathscr{L}^{\mathrm{abs}} = \mathscr{L} + \mathscr{H}$. For all $\mathbf{x} \in (V^*)^n$ we have*

$$\mathscr{G}(\varphi(\mathbf{x}), z) = \sum_{\kappa=2}^{n} z^{n-\kappa}(1-z)^\kappa \sum_{I \in C_{\kappa,n}} \mathscr{G}(\varphi(\mathbf{x}_I), 0), \tag{5.16}$$

with

$$\mathscr{G}(\eta, 0) = \mathbb{P}_\eta(\eta_0(\infty) = 0) - \mathbb{P}_\eta^{\mathrm{irw}}(\eta_0(\infty) = 0). \tag{5.17}$$

Proof. We proceed by induction on n. We first consider the case $n = 2$. In this case

$$\sum_{i \in V^*} \eta_i G(\eta - \delta_i, z) = \sum_{i \in V^*} \eta_i G^{\mathrm{irw}}(\eta - \delta_i, z) \tag{5.18}$$

because $G(\zeta, z)$ and $G^{\mathrm{irw}}(\zeta, z)$ coincide on configuration ζ with one particle. Therefore,

$$\sum_{i \in V^*} \eta_i \mathscr{G}(\eta - \delta_i, z) = 0 \tag{5.19}$$

and we obtain from Proposition 5.9,

$$\mathscr{G}(\varphi(\mathbf{x}), z) = (1-z)^2 \mathscr{G}(\varphi(\mathbf{x}), 0) \tag{5.20}$$

which is (5.16) for $n = 2$. Now we assume that (5.16) is true for $n-1$ and prove the induction step. First of all we notice that we can rewrite

$$\sum_{i \in V^*} (\varphi(\mathbf{x}))_i G(\varphi(\mathbf{x}) - \delta_i, z) = \sum_{j=1}^{n} G(\varphi(\mathbf{x}) - \delta_{x_j}, z). \tag{5.21}$$

where we used that for $\eta = \varphi(x)$ we have $\eta_i = \sum_{j=1}^n \delta_{x_j,i}$ and we then exchanged the sums. Using (5.14) and the induction hypothesis we have that

$$\begin{aligned}
\frac{1}{(1-z)^n} \cdot \mathscr{G}(\varphi(\mathbf{x}), z) - \mathscr{G}(\varphi(\mathbf{x}), 0) \qquad &= \sum_{j=1}^{n} \int_0^z \frac{1}{(1-u)^{n+1}} \mathscr{G}(\varphi(\mathbf{x}) - \delta_{x_j}, u)\, du \\
&= \sum_{j=1}^{n} \sum_{\kappa=2}^{n-1} \sum_{\substack{J \in C_{\kappa,n}: \\ j \notin J}} \mathscr{G}(\varphi(\mathbf{x}_J), 0) \int_0^z \frac{u^{n-\kappa-1}}{(1-u)^{n-\kappa+1}}\, du
\end{aligned}$$

Calling $m = n - \kappa - 1$, we have

$$\begin{aligned}
\int_0^z \frac{u^m}{(1-u)^{m+2}}\, du \quad &= \quad \int_0^z \frac{d}{du}\Big[\frac{1}{m+1}\Big(\frac{u}{1-u}\Big)^{m+1}\Big]\, du \\
&= \quad \frac{1}{m+1}\left(\frac{z}{1-z}\right)^{m+1}
\end{aligned} \tag{5.22}$$

Hence

$$\frac{1}{(1-z)^n}\cdot \mathscr{G}(\varphi(\mathbf{x}),z)-\mathscr{G}(\varphi(\mathbf{x}),0)$$
$$=\sum_{\kappa=2}^{n-1}\left(\frac{z}{1-z}\right)^{n-\kappa}\frac{1}{n-\kappa}\sum_{j=1}^{n}\sum_{\substack{J\in C_{\kappa,n}:\\ j\notin J}}\mathscr{G}(\varphi(\mathbf{x}_J),0) \tag{5.23}$$

Consider now

$$\sum_{j_0=1}^{n}\sum_{\substack{J\in C_{\kappa,n}:\\ j_0\notin J}}\mathscr{G}(\varphi(\mathbf{x}_J),0)=\frac{1}{\kappa!}\sum_{j_0=1}^{n}\sum_{\substack{j_1=1\\ j_1\neq j_0}}^{n}\dots\sum_{\substack{j_\kappa=1\\ j_\kappa\neq j_0,\dots j_{\kappa-1}}}^{n}\mathscr{G}(\varphi(x_{j_1},\dots,x_{j_\kappa}),0)$$
$$=\frac{1}{\kappa!}\sum_{j_1=1}^{n}\sum_{\substack{j_2=1\\ j_2\neq j_1}}^{n}\dots\sum_{\substack{j_\kappa=1\\ j_\kappa\neq j_1,\dots j_{\kappa-1}}}^{n}\sum_{\substack{j_0=1\\ j_0\neq j_1,\dots j_\kappa}}^{n}\mathscr{G}(\varphi(x_{j_1},\dots,x_{j_\kappa}),0)$$
$$=\frac{n-\kappa}{\kappa!}\sum_{j_1=1}^{n}\sum_{\substack{j_2=1\\ j_2\neq j_1}}^{n}\dots\sum_{\substack{j_\kappa=1\\ j_\kappa\neq j_1,\dots j_{\kappa-1}}}^{n}\mathscr{G}(\varphi(x_{j_1},\dots,x_{j_\kappa}),0)$$
$$=(n-\kappa)\sum_{J\in C_{\kappa,n}}\mathscr{G}(\varphi(\mathbf{x}_J),0)$$

Thus, from (5.23) we get

$$\frac{1}{(1-z)^n}\cdot \mathscr{G}(\varphi(\mathbf{x}),z)-\mathscr{G}(\varphi(\mathbf{x}),0)=\sum_{\kappa=2}^{n-1}\left(\frac{z}{1-z}\right)^{n-\kappa}\sum_{J\in C_{\kappa,n}}\mathscr{G}(\varphi(\mathbf{x}_J),0) \tag{5.24}$$

that concludes the proof. □

6 A class of consistent processes

In this section we consider a natural class of consistent configuration processes $\{\eta(t),\, t\geq 0\}$. These can be obtained, with a certain choice of the parameters, as particular cases of the more general class of processes produced in the characterization Theorem 3.3. These processes do not constitute the entire class of processes exibithing the consistency property. Nevertheless they are paradigmatic examples within this class, as they are well known in the literature. The processes we consider are of three types: partial exclusion processes, inclusion processes and independent random walkers. For the sake of synthesis we formally define a unique generator and let this be parametrized by a constant $\theta\in\mathbb{R}$ tuning the attractive or repulsive nature of particle-interaction. The generator is given by

$$\mathscr{L}_\theta f(\eta):=\sum_{i,j\in V}p(i,j)\,\eta_i(1+\theta\eta_j)[f(\eta^{i,j})-f(\eta)] \tag{6.1}$$

acting on functions $f:\Omega_\theta\to\mathbb{R}$, with Ω_θ to be defined. Here V is a finite set, $p:V\times V\to\mathbb{R}$ is a symmetric function: $p(i,j)=p(j,i)$ and $\eta^{i,j}:=\eta-\delta_i+\delta_j$.

Remark that if the configuration contains only one particle then this particle moves according to a continuous-time random walk jumping from i to j at rate $p(i,j)$, which is not depending on the interaction parameter θ. The parameter $\theta\in\mathbb{R}$ can also be negative, under the condition that $\frac{1}{|\theta|}$ is integer: $\theta\in\mathbb{R}^+\cup\{\alpha<0:\,-1/\alpha\in\mathbb{N}\}$ and, according to its sign we recover one of three cases: the partial symmetric exclusion-process SEP$(1/|\theta|)$, the symmetric inclusion process SIP$(1/\theta)$, and the independent-random-walkers process IRW:

$$\mathscr{L}_\theta = \begin{cases} \mathscr{L}^{\mathrm{irw}} & \text{for } \theta = 0 \\ \theta \mathscr{L}^{\mathrm{SIP}(1/\theta)} & \text{for } \theta > 0 \\ |\theta| \mathscr{L}^{\mathrm{SEP}(1/|\theta|)} & \text{for } \theta < 0,\, \frac{1}{|\theta|} \in \mathbb{N} \end{cases} \tag{6.2}$$

Also the state space Ω_θ where configurations η live changes according to the choice of θ, we have:

$$\Omega_\theta = \Lambda_\theta^V, \qquad \text{with} \quad \Lambda_\theta = \begin{cases} \mathbb{N} & \text{for } \theta \geq 0 \\ \{1, 2, \ldots, \frac{1}{|\theta|}\} & \text{for } \theta < 0,\, \frac{1}{|\theta|} \in \mathbb{N} \end{cases} \tag{6.3}$$

These processes have been introduced in [7] and broadly studied due to their algebraic properties. In [7] it has been proved that $[\mathscr{L}, \mathscr{A}] = 0$ where $\mathscr{A}$ is the annihilation operator defined in (2.13). Notice that it is possible to rewrite the generator as

$$\mathscr{L}_\theta f(\eta) := \sum_{\{i,j\} \in E} p(\{i,j\})\, \mathscr{L}_{\theta,i,j} f(\eta) \tag{6.4}$$

where now the summation is over the edges $\{i,j\} \in E$ of the complete graph with vertices V and

$$\mathscr{L}_{\theta,i,j} f(\eta) = \eta_i(1 + \theta\eta_j)[f(\eta^{i,j}) - f(\eta)] + \eta_j(1 + \theta\eta_i)[f(\eta^{j,i}) - f(\eta)]. \tag{6.5}$$

It is possible to verify that the commutation relation with the annihilation operator holds true at the level of each bond, namely $[\mathscr{L}_{\theta,i,j}, \mathscr{A}] = 0$ for all $i, j \in V$.

In [7, 1] the reversible measures of these processes have been identified, together with their duality properties. The processes admit an infinite family of reversible homogeneous product measures $\nu_{\rho,\theta}$ on Ω_θ labelled by the density parameter $\rho := \langle \eta_i \rangle_{\nu_{\rho,\theta}} > 0$, $i \in V$, with marginals

$$\nu_{\rho,\theta}(\eta_i = n) = \frac{\rho^n}{n!} \cdot \begin{cases} e^{-\rho} & \text{for } \theta = 0 \\ \\ (1 + \theta\rho)^{-n-\frac{1}{\theta}} \cdot \theta^n \cdot (1/\theta)^{(n)} & \text{for } \theta > 0 \\ \\ (1 + \theta\rho)^{-n-\frac{1}{\theta}} \cdot |\theta|^n \cdot (1/|\theta|)_n & \text{for } \theta < 0 \text{ and } -1/\theta \in \mathbb{N} \end{cases} \tag{6.6}$$

where $(a)^{(n)}$ and $(a)_n$ are the Pochhammer symbols for rising and falling factorials:

$$(a)^{(n)} := \frac{\Gamma(a+n)}{\Gamma(a)} \qquad \text{and} \qquad (a)_n := \frac{\Gamma(a+1)}{\Gamma(a+1-n)} \tag{6.7}$$

In other words, the reversible homogeneous measures is a product of Poisson distributions if $\theta = 0$, a product of Negative-Binomial distributions if $\theta > 0$ and a product of Binomial distributions if $\theta < 0$ and $-1/\theta \in \mathbb{N}$. Hence, from Theorem 4.3, the processes are self-dual with duality functions of the form $D(\xi, \eta) = F(\xi, \eta)/\nu_\rho(\xi)$ with F as in (3.24) (modulo a factor that depends on the total number of dual particles $|\xi|$ which is a conserved quantity for the dual dynamics). More precisely, the self-duality functions are given by [7]

$$D_\theta(\xi, \eta) = \prod_{i \in V} d_\theta(\xi_i, \eta_i), \quad \text{with} \quad d_\theta(k, n) = \frac{n!}{(n-k)!} \cdot \begin{cases} 1 & \text{for } \theta = 0 \\ \\ \frac{1}{(1/\theta)^{(k)}} & \text{for } \theta > 0 \\ \\ \frac{1}{(1/|\theta|)_k} & \text{for } \theta < 0,\, \frac{1}{|\theta|} \in \mathbb{N} \end{cases} \tag{6.8}$$

From consistency we know that for the processes with generator (6.1) Theorem 3.8 holds true.

6.1 Adding absorbing sites

Let now $V^* = V \cup V^{\text{abs}} \subseteq \mathbb{Z}^d$ with V^{abs} a finite set of absorbing sites. Define $\mathscr{L}_\theta^{\text{abs}} := \mathscr{L}_\theta + \mathscr{H}$ with $\mathscr{L}_\theta$ as in (6.1) and $\mathscr{H}$ as in (5.2):

$$\mathscr{L}_\theta^{\text{abs}} f(\eta) := \sum_{i,j\in V} p(i,j)\,\eta_i(1+\theta\eta_j)[f(\eta^{i,j}) - f(\eta)] + \sum_{i\in V, j\in V^{\text{abs}}} r(i,j)\,\eta_i\left[f(\eta^{i,j}) - f(\eta)\right]. \tag{6.9}$$

From Lemma 5.3 we know that $[\mathscr{L}_\theta^{\text{abs}}, \mathscr{A}^{\text{abs}}] = 0$, and as a consequence for these processes we can apply Theorem 5.5 when studying their absorption probabilities.

6.2 Adding reservoirs

We now add to the bulk generator (6.1) additional terms describing the action of external reservoirs, each acting on one of the sites of a subset $V^{\text{ext}} \subseteq V$. The generator has the following form:

$$\begin{aligned}\mathscr{L}_\theta^{\text{res}} f(\eta) &= \sum_{i,j\in V} p(i,j)\,\eta_i(1+\theta\eta_j)[f(\eta^{i,j}) - f(\eta)] \qquad (6.10)\\ &+ \sum_{i\in V^{\text{ext}}} c(i)\left\{\rho_i(1+\theta\eta_i)[f(\eta+\delta_i) - f(\eta)] + (1+\theta\rho_i)\eta_i[f(\eta-\delta_i) - f(\eta)]\right\}.\end{aligned}$$

for some $\rho, c : V^{\text{ext}} \to [0, +\infty)$. Particles are injected and removed from the system through the sites $i \in V^{\text{ext}}$ (which we call the sites coupled to reservoirs) via additional birth-and-death processes. Here $c(i)$ is the global rate at which the external reservoirs acts on the site i and ρ_i is the density imposed by the reservoir on that site. This means that the birth-and-death process at site $i \in V^{\text{ext}}$ has $\nu_{\rho_i,\theta}$ as stationary measure, where we recall that the single-site measure $\nu_{\rho,\theta}$ was defined in (6.6). In particular if the densities of all reservoirs are equal, i.e. $\rho_i = \rho$ for all $i \in V^{\text{ext}}$, then the process with generator (6.10) is reversible with respect to the product measure $\otimes_{i\in V}\nu_{\rho,\theta}$, which is called the "equilibrium state" in statistical physics. In all other cases, we have that the system has a stationary state which is called the "non-equilibrium steady state", with non-trivial correlations as soon as $\theta \neq 0$. In order to avoid uninteresting degenerate cases, we will always choose $p(i,j)$ and $c(i)$ in such a way that there exists a unique stationary measure, which we denote by μ_θ^{st}.

The rates of the birth-and-death processes modeling the reservoirs preserve the duality property. Indeed, in [1] we proved that the process with generator (6.10) is dual to a system with absorbing sites, i.e. generated by (6.9). Here the set of absorbing site V^{abs} is a "copy" of V^{ext}, i.e., such that $|V^{\text{abs}}| = |V^{\text{ext}}|$ and the rates $r(\cdot,\cdot)$ of the absorbing part of the generator (6.9) are given by:

$$r(i,j) = \mathbf{1}_{i\in V^{\text{ext}}, j=i^*}\cdot c(i)$$

where $*$ is a bijection $* : V^{\text{ext}} \to V^{\text{abs}}$ that assigns to each site $i \in V^{\text{ext}}$ a corresponding absorbing site $i^* \in V^{\text{abs}}$. From now on we will denote by $\{\eta(t),\, t \geq 0\}$ the process with reservoirs with generator (6.10) and state space $\Omega_\theta = \Lambda_\theta^V$ (as in (6.3)) and by $\{\xi(t),\, t \geq 0\}$ the dual process with absorbing sites and with generator:

$$\mathscr{L}_\theta^{\text{Dual}} f(\xi) := \sum_{i,j\in V} p(i,j)\,\xi_i(1+\theta\xi_j)[f(\xi^{i,j}) - f(\xi)] + \sum_{i\in V^{\text{ext}}} c(i)\,\xi_i\left[f(\xi^{i,i^*}) - f(\xi)\right] \tag{6.11}$$

and state space $\Omega_\theta^* = \Lambda_\theta^{V^*}$, $V^* = V \cup V^{\text{abs}}$. The duality function $\widehat{D}_\theta : \Omega_\theta^* \times \Omega_\theta \to \mathbb{R}$ between the two processes is the following:

$$\widehat{D}_\theta(\xi,\eta) = \prod_{i\in V} d_\theta(\xi_i,\eta_i)\cdot \prod_{i\in V^{\text{abs}}} \rho_i^{\xi_i} = D_\theta(\xi,\eta)\cdot \prod_{i\in V^{\text{abs}}} \rho_i^{\xi_i} \tag{6.12}$$

with d as in (6.8) and where we extend the definition of density function ρ to the absorbing sites, $\rho : V^{\mathrm{ext}} \cup V^{\mathrm{abs}}$, by identifying: $\rho_i = \rho_{i^*}$. See [1] for a proof of the duality relation.

An important consequence of the duality property is the information that it gives about the non-equilibrium stationary measure μ_θ^{st} of the process generated by (6.10). Indeed, the expectations of duality functions under μ_θ^{st} can be expressed in terms of the absorption probabilities of the dual process. The key relation is

$$\int \widehat{D}_\theta(\xi,\eta)\mu_\theta^{\mathrm{st}}(d\eta) = \mathbb{E}_\xi\Big[\prod_{i\in V^{\mathrm{abs}}} \rho_i^{\xi_i(\infty)}\Big] \tag{6.13}$$

which is obtained from the duality relation by taking the limit as time goes to infinity, and using that asymptotically the dual process voids all the sites $i \in V$ implying that $D_\theta(\xi(\infty),\eta) = 1$. Notice that now, thanks to the consistency property of the dual process $\{\xi(t),\, t \geq 0\}$, we can apply Theorem 5.5 to get information about the moments in (6.13). In the next section we specialize to the case of two reservoirs (and then two dual absorbing sites) for which we can use the generating-function method developed in Section 5.2.

We remark that independently of the value of the parameter θ, if the dual configuration $\xi = \sum_{i=1}^n \delta_{x_i}$ where all x_i are mutually different elements of V, then

$$\widehat{D}_\theta(\xi,\eta) = \prod_{i=1}^n \eta_{x_i} \tag{6.14}$$

In particular, when the dual process is initialized with only one particle, then this particle will move as a symmetric random walk. As a consequence we have for all $x \in V$

$$\mathbb{E}_{\mu_\theta^{\mathrm{st}}}(\eta_x) = \int \widehat{D}_\theta(\delta_x,\eta)\mu_\theta^{\mathrm{st}}(d\eta) \quad = \sum_{y\in V^{\mathrm{abs}}} \rho_y q(x,y) \tag{6.15}$$

where $q(x,y)$ is the probability for the walker starting at x to be absorbed at y eventually:

$$q(x,y) := \mathbf{P}_x(X^{\mathrm{rw}}(\infty) = y) \tag{6.16}$$

In the case $\theta = 0$ the process with generator (6.10) is made of independent walkers with reservoirs, and the dual process is made of independent walkers absorbed at the absorbing sites. In such a case one can show by duality that the non-equilibrium steady state is an inhomogeneous product of Poisson measures with parameters given by the local density (6.15). See [1] for a proof in the case with two reservoirs.

6.3 Instantaneous thermalization models

Another class of models sharing the consistency property is the class of processes obtained as the "instantaneous thermalization limit" of the processes defined in (6.1) which we now briefly recall. An important example in this class is the dual KMP model, see [1, 12]. An instantaneous thermalization process gives rise, for each couple of nearest neighbouring sites, to an instantaneous redistribution of the total number of particles. For each bond, the total number of particles in that bond $\eta_i + \eta_j$ is redistributed according to the stationary measure of the original process at equilibrium on that bond, conditioned to the conservation of $\eta_i + \eta_j$. The generators of the instantaneous thermalization processes can thus be defined by

$$\mathscr{L}_\theta^{\mathrm{th}} f(\eta) := \sum_{\{i,j\}\in E} p(i,j)\mathscr{L}_{\theta,i,j}^{\mathrm{th}} f(\eta) \qquad \text{with} \qquad \mathscr{L}_{\theta,i,j}^{\mathrm{th}} := \lim_{t\to\infty}\left(e^{t\mathscr{L}_{\theta,i,j}} - \mathbf{1}\right) f \tag{6.17}$$

with $\mathscr{L}_{\theta,i,j}$ as in (6.5). Since we have that the commutation $[\mathscr{L}_{\theta,i,j}, \mathscr{A}] = 0$ for all $i, j \in V$, it follows that also $\mathscr{L}^{\text{th}}_{\theta,i,j}$ commutes with $\mathscr{A}$ and then the thermalized models (6.17) are also consistent, namely $[\mathscr{L}^{\text{th}}_{\theta}, \mathscr{A}] = 0$. A more explicit expression of the generator is given by:

$$\mathscr{L}^{\text{th}}_{\theta,i,j} := \sum_{m=0}^{\eta_i+\eta_j} [f(\eta^{i,j,m}) - f(\eta)] \ \cdot \bar{\nu}_\theta(m \mid \eta_i + \eta_j) \tag{6.18}$$

where

$$\eta_k^{i,j,m} := \begin{cases} \eta_k & \text{for } k \neq i, j \\ m & \text{for } k = i \\ \eta_i + \eta_j - m & \text{for } k = j \end{cases} \tag{6.19}$$

and $\bar{\nu}_\theta(m \mid M) := \nu_{\theta,\rho}(\eta_i = m \mid \eta_i + \eta_j = M)$ with $\nu_{\theta,\rho}$ the reversible measure defined in (6.6). The process (6.18) is self-dual with duality function (6.8) (see Section 5 of [1]).

Also for the instantaneous thermalization models it is possible to add absorbing boundaries in such a way to preserve the consistency property, and also in this case there is duality with the system with absorbing boundaries and a system with reservoirs (if the action of reservoirs is properly chosen, see [1]).

In the next section we use consistency to obtain an expression for the n-points stationary correlation function for the system with reservoirs (6.10) in a specific setting. This result can be easily extended to the thermalized models with reservoirs, since, as we have seen here, they share the same commutation property and as a consequence consistency property.

7 Correlation functions in non-equilibrium steady states

Here we consider processes in the class of models introduced in the previous section, and, as in Section 5.2, we restrict to the case where $V = \{1, \ldots, N\}$, with $V^{\text{ext}} = \{1, N\}$ and $V^{\text{abs}} = \{0, N+1\}$. In the spirit of the previous section we assign to each site coupled to a reservoir an absorbing site, namely we say that $1^* = 0$ and $N^* = N + 1$. Then we denote by $\{\eta(t), t \geq 0\}$ on Ω_θ the process with generator:

$$\begin{aligned} \mathscr{L}^{\text{res}}_\theta f(\eta) \quad &= \quad \sum_{i,j\in V} p(i,j)\, \eta_i(1+\theta\eta_j)[f(\eta^{i,j}) - f(\eta)] \\ &+ \quad c_\ell \left\{\rho_\ell(1+\theta\eta_1)[f(\eta+\delta_1) - f(\eta)] + (1+\theta\rho_\ell)\eta_1[f(\eta-\delta_1) - f(\eta)]\right\}. \\ &+ \quad c_r \left\{\rho_r(1+\theta\eta_N)[f(\eta+\delta_N) - f(\eta)] + (1+\theta\rho_r)\eta_N[f(\eta-\delta_N) - f(\eta)]\right\}. \end{aligned} \tag{7.1}$$

for some $c_\ell, c_r, \rho_\ell, \rho_r \geq 0$, and by $\{\xi(t),\, t \geq 0\}$ its dual process with state space Ω^*_θ and generator:

$$\begin{aligned} \mathscr{L}^{\text{Dual}}_\theta f(\xi) \quad &:= \quad \sum_{i,j\in V} p(i,j)\, \xi_i(1+\theta\xi_j)[f(\xi^{i,j}) - f(\xi)] \\ &+ \quad c_\ell\, \xi_1 \left[f(\xi^{1,0}) - f(\xi)\right] + c_r\, \xi_N \left[f(\xi^{N,N+1}) - f(\xi)\right] \end{aligned} \tag{7.2}$$

with duality function given by

$$\widehat{D}_\theta(\xi,\eta) = \rho_\ell^{\xi_0} \cdot D_\theta(\xi,\eta) \cdot \rho_r^{\xi_{N+1}}. \tag{7.3}$$

Hence, the formula for the non-equilibrium stationary state (6.13) becomes:

$$\int \widehat{D}_\theta(\xi,\eta)\mu^{\text{st}}_\theta(d\eta) = \rho_r^{|\xi|} \cdot \mathbb{E}_\xi\left[\left(\frac{\rho_\ell}{\rho_r}\right)^{\xi_0(\infty)}\right] = \rho_r^{|\xi|} \cdot G\left(\xi, \frac{\rho_\ell}{\rho_r}\right) \tag{7.4}$$

where $G(\xi,\cdot)$ is the generating-function defined in Section 5.2, equation (5.12). The following theorem then follows by combining (7.4) with Theorem 5.10 and expresses the difference between the expectations of the duality functions in the non-equilibrium stationary measure and their non-interacting counterparts in terms of the probabilities of κ dual particles being all absorbed at one end.

Theorem 7.1. *Let $\mathbf{x}\in V_n$, with $n\geq 2$, then*

$$\int \widehat{D}_\theta(\varphi(\mathbf{x}),\eta)\mu_\theta^{\rm st}(d\eta) - \int \widehat{D}_0(\varphi(\mathbf{x}),\eta)\mu_0^{\rm st}(d\eta) = \sum_{\kappa=2}^{n} \gamma_\kappa(\mathbf{x})\cdot(\rho_r-\rho_\ell)^\kappa\,\rho_\ell^{n-\kappa} \tag{7.5}$$

with

$$\begin{aligned} \gamma_\kappa(\mathbf{x}) &:= \sum_{\mathbf{y}\in C_\kappa(\mathbf{x})} \mathscr{G}(\varphi(\mathbf{y}),0) \\ \mathscr{G}(\xi,0) &= \mathbb{P}_\xi(\xi_0(\infty)=0) - \mathbb{P}_\xi^{\rm irw}(\xi_0(\infty)=0) \end{aligned} \tag{7.6}$$

and

$$C_\kappa(\mathbf{x}) = \{(x_{i_1},\dots,x_{i_\kappa}) :\ (i_1,\dots,i_\kappa)\in C_{\kappa,n}\} \tag{7.7}$$

The following corollary of Theorem 7.1 specializes to expectations of products of occupation numbers at different sites in the non-equilibrium steady state.

Corollary 7.2. *For every $\mathbf{x}=(x_1,\dots,x_n)\in V_n$ such that $x_i\neq x_j$ for all $i,j\in\{1,\dots,n\}$, $i\neq j$, we have*

$$\mathbb{E}_{\mu_\theta^{\rm st}}\left[\prod_{i=1}^n \eta_{x_i}\right] - \prod_{i=1}^n \mathbb{E}_{\mu_\theta^{\rm st}}[\eta_{x_i}] = \sum_{\kappa=2}^n \gamma_\kappa(\mathbf{x})\cdot(\rho_r-\rho_\ell)^\kappa\,\rho_\ell^{n-\kappa} \tag{7.8}$$

where $\mathbb{E}_{\mu_\theta^{\rm st}}[\eta_x]=\sum_{y\in}\rho_y\, q(x,y)$ with $q(\cdot,\cdot)$ as in (6.16) *and $\gamma_\kappa(\mathbf{x})$ as in* (7.6).

Proof. Using (6.15), (6.14) and Theorem 7.1, we have for all θ

$$\int \widehat{D}_\theta(\varphi(\mathbf{x}),\eta)\mu_\theta^{\rm st}(d\eta) = \mathbb{E}_{\mu_\theta^{\rm st}}\left[\prod_{i=1}^n \eta_{x_i}\right] \tag{7.9}$$

and then, in particular, for $\theta=0$, from the product-nature of $\mu_0^{\rm st}$ we have

$$\int \widehat{D}_0(\varphi(\mathbf{x}),\eta)\mu_0^{\rm st}(d\eta) = \prod_{i=1}^n \mathbb{E}_{\mu_0^{\rm st}}[\eta_{x_i}] \tag{7.10}$$

This concludes the proof. □

Remark 7.3. Specializing the corollary to $n=2$ we obtain information on the covariance. We see that the explicit dependence on the boundary densities ρ_ℓ and ρ_r that turns out to be a quadratic function of their difference: for $x\neq y$,

$$\begin{aligned} \mathrm{cov}_{\mu_\theta^{\rm st}}(\eta_x,\eta_y) &:= \mathbb{E}_{\mu_\theta^{\rm st}}[\eta_x\eta_y] - \mathbb{E}_{\mu_\theta^{\rm st}}[\eta_x]\cdot\mathbb{E}_{\mu_\theta^{\rm st}}[\eta_y] \\ &= (\rho_r-\rho_\ell)^2\cdot\mathscr{G}(\delta_x+\delta_y,0) \\ &= (\rho_r-\rho_\ell)^2\cdot\Big(\mathbb{P}_{\delta_x+\delta_y}(\xi_0(\infty)=0) - \mathbb{P}_{\delta_x+\delta_y}^{\rm irw}(\xi_0(\infty)=0)\Big). \end{aligned} \tag{7.11}$$

The covariance is *exactly* quadratic in $(\rho_\ell-\rho_r)$ with a multiplying factor (i.e. the difference of the two absorption probabilities above) not depending on ρ_ℓ and ρ_r. This multiplying factor is non-positive for exclusion particles, $\theta<0$ (by Liggett's inequality [11], chapter 8) and non-negative for inclusion particles, $\theta>0$ (by the analogue of Ligget's inequality from [8]).

7.1 Examples

In this section we consider three concrete examples where the non-equilibrium steady state expectations are known in closed form, and we explicitly verify the recursion (5.14).

We restrict to the situation where $V = \{1, \ldots, N\}$ is a a one-dimensional chain and particles jump only to nearest neigbors, and interact (in a symmetric way) only if sitting in neighboring sites. More precisely we choose the function $p(\cdot, \cdot)$ in (7.1) and (7.2) as

$$p(i,j) = \mathbf{1}_{j=i\pm 1} \tag{7.12}$$

Moreover we choose the reservoirs clocks to have rates 1:

$$c_\ell = c_r = 1 \tag{7.13}$$

Independent walkers (case $\theta = 0$)

In this case the absorption probabilities of the dual-process:

$$\mathscr{L}_0^{\text{Dual}} f(\xi) \quad := \quad \sum_{i=1}^{N} \xi_i [f(\xi^{i,i+1}) - 2f(\xi) + f(\xi^{i,i-1})]$$

can be explicitly computed and are determined by the single-walker absorption probabilities which are given by:

$$q_x^+ := q(x, N+1) = \mathbf{P}_x(X^{\text{rw}}(\infty) = N+1) = \frac{x}{N+1}, \qquad x \in \{0, 1, \ldots, N+1\}. \tag{7.14}$$

Here $\mathbf{P}_x$ is the probability law of the random walker $\{X^{\text{rw}}(t),\, t \geq 0\}$ on $V^* = \{0, 1, \ldots, N+1\}$, starting from $x \in V^*$, with rate-one nearest-neighbour jumps and with $\{0, N+1\}$ absorbing states.

If we now consider the corresponding system with reservoirs:

$$\begin{aligned} \mathscr{L}_0^{\text{res}} f(\eta) \quad &= \quad \sum_{i=1}^{N-1} \eta_i [f(\eta^{i,i+1}) - f(\eta)] + \sum_{i=2}^{N} \eta_i [f(\eta^{i,i-1}) - f(\eta)] \\ &+ \quad \{\rho_\ell [f(\eta + \delta_1) - f(\eta)] + \eta_1 [f(\eta - \delta_1) - f(\eta)]\} \\ &+ \quad \{\rho_r [f(\eta + \delta_N) - f(\eta)] + \eta_N [f(\eta - \delta_N) - f(\eta)]\} \end{aligned} \tag{7.15}$$

we have that the stationary measure μ_0^{st} is a non-homogeneous product measure with Poisson-distributed marginals:

$$\mu_0^{\text{st}} \sim \otimes_{x=1}^{N} \text{Pois}\left(\rho_\ell + (\rho_r - \rho_\ell) q_x^+\right) \tag{7.16}$$

with q_x^+ as in (7.14).

Notice that in this case, since we can easily compute the generating function $G^{\text{irw}}(\cdot, \cdot)$, it is possible to directly verify the recursion relation (5.13). For $\xi \in \Omega^*$ we have

$$G^{\text{irw}}(\xi, z) = \prod_{i=0}^{N+1} \left(z \left(1 - q_i^+\right) + q_i^+ \right)^{\xi_i}.$$

As a consequence,

$$\begin{aligned}
&(1-z)\,\frac{d}{dz}G^{\mathrm{irw}}(\xi,z) = \\
&\quad = (1-z)\sum_{j=0}^{N+1}\left\{\xi_j(1-q_j^+)\big(z(1-q_j^+)+q_j^+\big)^{\xi_j-1}\prod_{\substack{i=0\\ i\neq j}}^{N+1}\big(z(1-q_i^+)+q_i^+\big)^{\xi_i}\right\} \\
&\quad = \sum_{j=1}^{N}\left\{\xi_j\left(-z\left(1-q_j^+\right)-q_j^++1\right)\left(z\left(1-q_j^+\right)+q_j^+\right)^{\xi_j-1}\prod_{\substack{i=1\\ i\neq j}}^{N}\big(z(1-q_i^+)+q_i^+\big)^{\xi_i})\right\} \\
&\quad = -|\xi|\,G^{\mathrm{irw}}(\xi,z)+\sum_{j=0}^{N+1}\xi_j G^{\mathrm{irw}}(\xi-\delta_j,z)
\end{aligned}$$

and then (5.13) is satisfied. Since we explicitly know both the generating function $G^{\mathrm{irw}}(\cdot,\cdot)$ and the stationary measure (7.16), it is possible to verify a posteriory the duality relation (7.4). It is also possible to verify that, by iterating the recursion relation in its integrated form (5.14) one can recover the generating function $G^{\mathrm{irw}}(\xi,z)$ starting from the knowledge of $G^{\mathrm{irw}}(\xi,0)$.

Remark 7.4. As a further application of the recursion (5.13) in the same spirit, one can easily show by induction that if the probabilities for all the particles to be absorbed at zero factorize, i.e., if $G(\xi,0)=\prod_i G(\delta_i,0)^{\xi_i}$ for all ξ, then the generating function factorizes, i.e., $G(\xi,z)=\prod_i G(\delta_i,z)^{\xi_i}$ and as a consequence the system has a product invariant non-equilibrium stationary measure μ_0^{st}.

Interacting walkers (case $\theta \neq 0$)

For the interacting case, we consider two special cases. The first example is the simple exclusion process. This is the only interacting model in the class for which there is a full knowledge of the n-points correlation functions. The second example is a special case of the inclusion process for which an exact formula is known for the two-point correlations [6].

For the interacting case, specialized to nearest neighbor jumps and rate 1 reservoirs, the process has generator

$$\begin{aligned}
\mathscr{L}_\theta^{\mathrm{res}} f(\eta) &= \sum_{i=1}^{N-1}\eta_i(1+\theta\eta_{i+1})[f(\eta^{i,i+1})-f(\eta)]+\sum_{i=2}^{N}\eta_i(1+\theta\eta_{i-1})[f(\eta^{i,i-1})-f(\eta)] \\
&+ \{\rho_\ell(1+\theta\eta_1)[f(\eta+\delta_1)-f(\eta)]+(1+\theta\rho_\ell)\eta_1[f(\eta-\delta_1)-f(\eta)]\} \\
&+ \{\rho_r(1+\theta\eta_N)[f(\eta+\delta_N)-f(\eta)]+(1+\theta\rho_r)\eta_N[f(\eta-\delta_N)-f(\eta)]\}
\end{aligned}$$

Simple exclusion process (case $\theta = -1$)

For exclusion process the matrix product ansatz gives an algebraic procedure to calculate all correlation functions. This provides a recursion relation for the correlation functions (formula (A.7) in [3]) that reads

$$\begin{aligned}
\mathbb{E}^{\mathrm{N}}_{\mu^{\mathrm{st}}}[\eta_{x_1}\eta_{x_2}\ldots\eta_{x_m}] &= (\rho_\ell-\rho_r)\Big(1-\frac{x_m}{N+1}\Big)\,\mathbb{E}^{\mathrm{N-1}}_{\mu^{\mathrm{st}}}[\eta_{x_1}\eta_{x_2}\ldots\eta_{x_{m-1}}] \\
&+ \rho_r\,\mathbb{E}^{\mathrm{N}}_{\mu^{\mathrm{st}}}[\eta_{x_1}\eta_{x_2}\ldots\eta_{x_{m-1}}]
\end{aligned} \tag{7.17}$$

where $\mathbb{E}^{\mathrm{N}}_{\mu^{\mathrm{st}}}$ denotes expectation in the non-equilibrium steady states of a system of size N and $\mu^{\mathrm{st}}=\mu^{\mathrm{st}}_{-1}$. In this section we fix $\xi\in\Omega_m$, with $\xi=\varphi(\mathbf{x})$, and $\mathbf{x}=(x_1,\ldots,x_m)$ with

$1 \le x_1 < x_2 < \ldots < x_m \le N$ and denote by $q_\xi^{(N)}(k)$ the absorption probabilities:

$$q_\xi^{(N)}(k) := \mathbb{P}_\xi(\xi_0(\infty) = k), \qquad k \in \{0, 1, \ldots, m\} \tag{7.18}$$

in a system of size N. Duality yields

$$\mathbb{E}_{\mu^{st}}^{N}[\eta_{x_1}\eta_{x_2}\ldots\eta_{x_m}] \;=\; \rho_r^m \sum_{k=0}^{m} \left(\frac{\rho_\ell}{\rho_r}\right)^k q_\xi^{(N)}(k) \tag{7.19}$$

Inserting (7.19) in (7.17) the principle of identity of polynomials turns the recurrence relation for the correlation functions into a recurrence relation for the absorption probabilities:

$$q_\xi^{(N)}(k) = \mathbf{1}_{k\neq 0} \cdot q_{\xi-\delta_{x_m}}^{(N-1)}(k-1) \cdot q_{\delta_{x_m}}^{(N)}(1) + \mathbf{1}_{k\neq m} \cdot \left[q_{\xi-\delta_{x_m}}^{(N)}(k) - q_{\xi-\delta_{x_m}}^{(N-1)}(k) \cdot q_{\delta_{x_m}}^{(N)}(1)\right] \tag{7.20}$$

where $k \in \{0, 1, 2, \ldots, m\}$. Introducing the generating function

$$G^{(N)}(\xi, z) = \sum_{k=0}^{m} z^k \, q_\xi^{(N)}(k) \tag{7.21}$$

the recursion relation of the absorption probabilities (7.20) implies the recursion relation for the generating function

$$G^{(N)}(\xi, z) = (z-1)\, q_{\delta_{x_m}}^{(N)}(1) \cdot G^{(N-1)}(\xi - \varphi(x_m), z) + G^{(N)}(\xi - \varphi(x_m), z) \tag{7.22}$$

Clearly, if $m = 1$, namely $\xi = \delta_x = \varphi(x)$ for some $x \in V$ we have

$$G^{(N)}(\varphi(x), z) = \frac{x}{N+1} + \Big(1 - \frac{x}{N+1}\Big) z \tag{7.23}$$

Thus for the exclusion process the probability generating function for the number of particles absorbed at zero can be computed by iterating (7.22).

For $m = 2$ and $x < y$ the recurrence (7.22) gives

$$G^{(N)}(\varphi(x, y), z) = (z-1)\, q_{\delta_y}^{(N)}(1) \cdot G^{(N-1)}(\varphi(x), z) + G^{(N)}(\varphi(x), z) \tag{7.24}$$

Using (7.23) we get

$$G^{(N)}(\varphi(x, y), z) = (z-1)\Big(1 - \frac{y}{N+1}\Big)\Big[\frac{x}{N} + \Big(1 - \frac{x}{N}\Big)z\Big] + \frac{x}{N+1} + \Big(1 - \frac{x}{N+1}\Big)z \tag{7.25}$$

and one can check that this expression satisfies (5.13) with $m = 2$, i.e.

$$(1-z)\frac{d}{dz}G^{(N)}(\varphi(x, y), z) + 2G^{(N)}(\varphi(x, y), z) = G^{(N)}(\varphi(x), z) + G^{(N)}(\varphi(y), z) \tag{7.26}$$

For $m = 3$ and $x < y < u$ the recurrence (7.22) gives

$$G^{(N)}(\varphi(x, y, u), z) = (z-1)\, q_{\delta_u}^{(N)}(1) \cdot G^{(N-1)}(\varphi(x, y), z) + G^{(N)}(\varphi(x, y), z)$$

Using (7.25) we get

$$\begin{aligned} G^{(N)}(\varphi(x, y, u), z) \;&=\; (z-1)^2\Big(1 - \frac{u}{N+1}\Big)\Big(1 - \frac{y}{N}\Big)\Big[\frac{x}{N-1} + \Big(1 - \frac{x}{N-1}\Big)z\Big] \\ &+\; (z-1)\Big(2 - \frac{u+y}{N+1}\Big)\Big[\frac{x}{N} + \Big(1 - \frac{x}{N}\Big)z\Big] \\ &+\; \frac{x}{N+1} + \Big(1 - \frac{x}{N+1}\Big)z \end{aligned} \tag{7.27}$$

One can check that this expression satisfies (5.13) with $m = 3$, i.e.

$$\begin{aligned} &(1-z)\,\frac{d}{dz}G^{(N)}(\varphi(x, y, u), z) + 3G^{(N)}(\varphi(x, y, u), z) \\ &\qquad = G^{(N)}(\varphi(x, y), z) + G^{(N)}(\varphi(y, u), z) + G^{(N)}(\varphi(u, y), z) \end{aligned} \tag{7.28}$$

Inclusion process with $\theta = 2$

For the inclusion process SIP(2) we can verify (5.13) for the case $m = 2$ using the results in Section 5 of [6]. Writing out

$$G(\varphi(x,y),z) = q_{\varphi(x,y)}(0) + zq_{\varphi(x,y)}(1) + z^2 q_{\varphi(x,y)}(2) \tag{7.29}$$

we see that (5.13) is equivalent to

$$2q_{\varphi(x,y)}(0) + q_{\varphi(x,y)}(1) = \frac{x+y}{N+1} \tag{7.30}$$

From Eq. (5.6) in [6] giving the two-point correlation function for the Brownian Energy process with reservoirs, and using the fact that the inclusion process with absorbing boundaries (see eq. (3.2) in [6]) is dual to it, we can read off the absorption probabilities as:

$$q_{\varphi(x,y)}(0) = \frac{x(2+y)}{(N+1)(N+3)} \tag{7.31}$$

and

$$q_{\varphi(x,y)}(1) = 1 - \Big(1 - \frac{x}{N+3}\Big)\Big(1 - \frac{y}{N+1}\Big) - \frac{x(2+y)}{(N+1)(N+3)} \tag{7.32}$$

Thus equation (7.30) is verified.

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