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A_1 Fefferman–Stein inequality for maximal functions of martingales in uniformly smooth spaces

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Abstract

Let f be a martingale with values in a uniformly p-smooth Banach space and w any positive weight. We show that $\mathbb{E}(f^* \cdot w) \lesssim \mathbb{E}(S_p f \cdot w^*)$, where $\cdot^*$ is the martingale maximal operator and S_p is the ℓ^p sum of martingale increments.

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1 Introduction

A Banach space $(X, |\cdot|)$ is called (p, C_{sm})-*smooth* (with $p \in [1, 2]$ and $C_{\mathrm{sm}} \in \mathbb{R}_{>0}$) if, for every $x, y \in X$, we have

$$\frac{1}{2}\Big(|x+y|^p + |x-y|^p\Big) \le |x|^p + C_{\mathrm{sm}}^p |y|^p. \tag{1.1}$$

The most basic examples are that, for any $r \in (1, 2]$, any L^r space is $(r, 1)$-smooth, see [14, (10.33)] (this is also a consequence of Clarkson's inequality), and, for any $r \in [2, \infty)$, any L^r space is $(2, r-1)$-smooth, this follows from [14, (10.37)] and Jensen's inequality. In general, unless X is zero-dimensional, we must have $C_{\mathrm{sm}} \ge 1$, as can be seen by taking $x = 0$ in (1.1).

Our main result is the following.

Theorem 1.1. *Let $p \in (1, 2]$. Let $(f_n)_{n\in\mathbb{N}}$ be a martingale on a filtered probability space $(\Omega, (\mathcal{F}_n)_n)$ with values in a (p, C_{sm})-smooth Banach space X and $w : \Omega \to \mathbb{R}_{\ge 0}$ a measurable function (called a* weight*). Then,*

$$\mathbb{E}(f^* w) \le 84 p' C_{\mathrm{sm}} \mathbb{E}(S_p f \cdot w^*), \tag{1.2}$$

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where p' denotes the Hölder conjugate $1/p' + 1/p = 1$, and

$$S_p f = \big(|f_0|^p + \sum_{n=1}^{\infty} |f_n - f_{n-1}|^p\big)^{1/p}, \quad f^* = \sup_{n\in\mathbb{N}} |f_n|, \quad w^* = \sup_{n\in\mathbb{N}} \mathbb{E}(w|\mathcal{F}_n).$$

In order to put Theorem 1.1 into context, we list the previously known cases (in each of which the inequality (1.2) is in fact known with a smaller constant).

1. The unweighted ($w = 1$) scalar ($X = \mathbb{R}$) case is one of the Burkholder–Davis–Gundy inequalities [4].
2. The scalar ($X = \mathbb{R}$) case, which served as the main inspiration for this work, was proved in [12].
3. The unweighted ($w = 1$) case is one of the implications in the characterization of martingale type, see [14, Theorem 10.60].

We follow [12] in calling the inequality (1.2) a Fefferman–Stein inequality, in reference to [7, §3], where the first inequality involving the pair of weights w, w^* appeared (see [9, Theorem 3.2.3] for a martingale version). In order to distinguish this result from many others going back to Fefferman and Stein, we prepend the designation "A_1", which in the one-weight theory stands for the condition $w^* \le [w]_{A_1} w$. The pair w, w^* can be seen as satisfying a two-weight version of the A_1 condition.

For dyadic martingales, assuming $w \in A_\infty$, an inequality similar to (1.2) with w^* replaced by w is known [8, Theorem 2]. The recent result [2, Theorem 1.3] (which applies to martingale transforms in place of the square function) suggests that no such inequality is possible for general martingales.

The advantage of weighted estimates such as (1.2) is that they can be easily extrapolated to estimates for other moments, see Appendix A. We illustrate the extrapolation idea with a basic argument, which shows that the linear dependence on C_{sm} in (1.2) is optimal. Assume that the inequality

$$\mathbb{E}(f^* w) \le K \mathbb{E}(S_p f \cdot w^*)$$

holds for all weights w. By Hölder's inequality and Doob's maximal inequality, see e.g. [9, Theorem 3.2.2], for any $r \in (1, \infty)$, we obtain

$$\mathbb{E}(f^* w) \le K \|S_p f\|_{L^r} \|w^*\|_{L^{r'}} \le K r \|S_p f\|_{L^r} \|w\|_{L^{r'}}.$$

Since $L^{r'}$ is the dual space of L^r, this implies

$$\|f^*\|_{L^r} \le K r \|S_p f\|_{L^r}. \tag{1.3}$$

Incidentally, the linear growth in r of the constant in the inequality (1.3) is optimal in the scalar case $X = \mathbb{R}$, $p = 2$, see [3, Theorem 3.2].

Let now X be a Banach space such that the inequality (1.3) holds with $p = r \in (1, 2]$ for all martingales f with values in X. By Pisier's renorming theorem [14, Theorem 10.22], the space X admits an equivalent norm that is (p, CK)-smooth for some C depending only on p. In this sense, the linear dependence of (1.2) on C_{sm} is optimal.

The dependence of the bound (1.2) on p' does not seem natural, since it does not appear in the corresponding non-maximal bound (3.1). Also, the $p = 1$ bound clearly holds with constant 1. Therefore, we find it reasonable to conjecture that $84p'$ in (1.2) can be replaced by a constant that does not depend on p.

1.1 Non-martingale version

The proof of Theorem 1.1 in fact yields a more general statement, involving processes with a structure that was introduced in [15, Theorem 3.1]. Let $(\Omega, (\mathcal{F}_n)_{n\in\mathbb{N}})$ be a filtered probability space, $(g_n)_{n\in\mathbb{N}}$ be a martingale, and $(f_n)_{n\in\mathbb{N}}$, $(\tilde{f}_n)_{n\in\mathbb{N}}$ be adapted processes with values in a (p, C_{sm})-smooth Banach space X. Assume that $f_0 = \tilde{f}_0 = 0$, and for every $n \in \mathbb{N}_{>0}$ we have

$$f_n = \tilde{f}_{n-1} + (g_n - g_{n-1}), \quad |\tilde{f}_n| \leq |f_n|.$$

Then,

$$\mathbb{E}(f^* w) \leq 84 p' C_{\mathrm{sm}} \mathbb{E}(S_p g \cdot w^*). \tag{1.4}$$

As in (1.3), for $r \in [1, \infty)$, this implies

$$\|f^*\|_{L^r} \leq 84 p' C_{\mathrm{sm}} r \|S_p g\|_{L^r}. \tag{1.5}$$

The Rosenthal-type inequality in [15, Theorem 3.1] states that, if X is a $(2, C_{\mathrm{sm}})$-space and $r \in [2, \infty)$, then

$$\|f^*\|_{L^r} \leq 30 r \|\sup_n |g_n - g_{n-1}|\|_{L^r} + 40 C_{\mathrm{sm}} r^{1/2} \|sg\|_{L^r}, \tag{1.6}$$

where sg is the conditional square function:

$$sg = \Big(\sum_n \mathbb{E}(|g_n - g_{n-1}|^2 | \mathcal{F}_{n-1}) \Big)^{1/2}.$$

For $r \geq 2$, (1.6) implies (1.5), since

$$\|sg\|_{L^r} \leq (r/2)^{1/2} \|S_2 g\|_{L^r}, \quad r \in [2, \infty),$$

by Doob's maximal inequality and duality. On the other hand, the version of (1.6) for $r < 2$ in [15, Corollary 3.6] is not obviously related to (1.5).

1.2 Outline of the article

The proof of Theorem 1.1 is based on the Bellman function technique; we refer to the books [11, 17] for other instances of this technique.

In Section 2, we review the characterization of uniform smoothness that will be used in the proofs of our main results.

In Section 3, we prove the inequality (3.1), which is a non-maximal version of Theorem 1.1. The proof of that inequality uses a Bellman function that is adapted from [12]. Although that inequality will not be used in the proof of Theorem 1.1, the Bellman function estimate in Proposition 3.1 will be used again there.

In Section 4, we prove the full Theorem 1.1. This is accomplished using a Bellman function that combines features present in the articles [1] and [12].

In Appendix A, we give a sample application of the weighted bound (1.2).

2 General facts about uniformly smooth spaces

We will use the regularity properties of the norm on a uniformly smooth Banach space that can be found e.g. in [16, Lemma 2.1]. We take the opportunity to streamline the deduction of these properties from (1.1). The following lemma is a minor variant of [5, Lemma I.1.3] (there, the case $\phi(x) = |x|$ is considered).

Lemma 2.1. *Let $(X, |\cdot|)$ be a Banach space, $\phi : X \to \mathbb{R}$ a convex function, and $x \in X$ such that*

$$L := \limsup_{y \to 0} \frac{|\phi(x+y) - \phi(x)|}{|y|} < \infty, \quad \textit{and} \tag{2.1}$$

$$\lim_{y\to 0} \frac{\phi(x+y)+\phi(x-y)-2\phi(x)}{|y|} = 0. \tag{2.2}$$

Then ϕ is Fréchet differentiable at x, and its derivative satisfies $|\phi'(x)|_{X'} \leq L$.

Proof. Convexity implies that, for any $y \in X$, the function $t \mapsto \frac{\phi(x+ty)-\phi(x)}{t}$ is monotonically increasing in $t \in (0,\infty)$. Therefore, there exist one-sided directional derivatives

$$A(y) := \lim_{t\to 0+} \frac{\phi(x+ty)-\phi(x)}{t},$$

and $|A(y)| \leq L|y|$ by (2.1). We will show that A is the Fréchet derivative of ϕ at x. From (2.2), it follows that $A(y)+A(-y)=0$ for all $y \in X$. Hence, again by (2.2), we obtain

$$\begin{aligned}
&\sup_{|y|\leq 1} \frac{\phi(x+ty)-\phi(x)}{t} - A(y) \\
&\quad \leq \sup_{|y|\leq 1} \frac{\phi(x+ty)-\phi(x)}{t} - A(y) + \frac{\phi(x-ty)-\phi(x)}{t} - A(-y) \\
&\quad = \sup_{|y|\leq 1} \frac{\phi(x+ty)+\phi(x-ty)-2\phi(x)}{t} \xrightarrow{t\to 0} 0.
\end{aligned}$$

This shows that the difference quotients of ϕ converge to A locally uniformly. It remains to show that A is linear. To this end, we first observe that A is convex, since it is the limit of the convex functions $y \mapsto (\phi(x+ty)-\phi(x))/t$. Then also $y \mapsto -A(y) = A(-y)$ is convex, so that A is concave. It follows that A is affine. Finally, $A(0)=0$. $\square$

Let $(X, |\cdot|)$ be a (p, C_{sm})-smooth Banach space with $p \in (1,2]$ and

$$\phi(x) := |x|^p, \quad x \in X.$$

The hypothesis (2.2) of Lemma 2.1 follows directly from the definition (1.1). It is also easy to see that, for any $x \in X$, the hypothesis (2.1) holds with $L = L(x) = p|x|^{p-1}$. Therefore, Lemma 2.1 implies that the function ϕ is Fréchet differentiable, and $|\phi'(x)|_{X'} \leq p|x|^{p-1}$.

Let $C_{\mathrm{H}} \in [0,\infty]$ be the smallest constant such that, for any $x,y \in X$, we have

$$|\phi'(x)-\phi'(y)|_{X'} \leq C_{\mathrm{H}}^p |x-y|^{p-1}. \tag{2.3}$$

The proof of [5, Lemma V.3.5] (with $\alpha = p-1$) shows that

$$C_{\mathrm{H}}^p \leq 2^p C_{\mathrm{sm}}^p. \tag{2.4}$$

Conversely, for any $x,y \in X$, we have

$$\begin{aligned}
\frac{1}{2}\Big(|x+y|^p + |x-y|^p\Big) &= \frac{1}{2}\Big(\phi(x) + \int_0^1 \phi'(x+ty)y \,\mathrm{d}t + \phi(x) + \int_0^1 \phi'(x-ty)(-y)\,\mathrm{d}t\Big) \\
&\leq \phi(x) + \frac{1}{2}\int_0^1 |\phi'(x+ty)-\phi'(x-ty)|_{X'} |y| \,\mathrm{d}t \\
&\leq \phi(x) + \frac{1}{2}\int_0^1 C_{\mathrm{H}}^p |2ty|^{p-1}|y| \,\mathrm{d}t = |x|^p + \frac{2^{p-2}C_{\mathrm{H}}^p}{p}|y|^p.
\end{aligned}$$

Therefore, $C_{\mathrm{sm}}^p \leq 2^{p-2}C_{\mathrm{H}}^p/p$, so the conditions (2.3) and (1.1) are equivalent. However, we find the condition (2.3) more convenient to use, so all subsequent results will be formulated in terms of C_{H}. We note that $C_{\mathrm{H}}^p \geq p$, as can be seen by considering a one-dimensional subspace of X.

3 Bellman function for the martingale

In this section, we adapt the Bellman function from [12] to our setting. This will allow us to prove the inequality

$$\mathbb{E}(|f|w) \leq 9C_{\mathrm{H}}\mathbb{E}(S_p f \cdot w^*). \tag{3.1}$$

Note that, unlike in (1.2), the constant on the right-hand side of (3.1) does not explicitly depend on p.

For $x \in X$, $q \geq 0$, and $0 \leq u \leq v$, let

$$U(x, q, u, v) := u(|x|^p / C_{\mathrm{H}}^p + q)^{1/p} - Cvq^{1/p} + \tilde{C}vq^{1/p}\ln(1 + u/v). \tag{3.2}$$

We denote the x- and the u-derivatives of U by U_x and U_u, respectively. Note that U is indeed Fréchet differentiable in x, and the derivative is given by

$$U_x(x, q, u, v)h = \frac{\phi'(x)h}{pC_{\mathrm{H}}^p(|x|^p/C_{\mathrm{H}}^p + q)^{1-1/p}}.$$

The main feature of the function (3.2) is the following concavity property.,

Proposition 3.1. *Suppose that $C = 9$ and $\tilde{C} = 4\sqrt{2}$. Then, for any $x, d \in X$, $q, u, v \in \mathbb{R}_{\geq 0}$, and $e \in \mathbb{R}$ with $u \leq v$ and $0 \leq u + e$, we have*

$$U(x + d, q + |d|^p, u + e, (u + e) \vee v) \leq U(x, q, u, v) + U_x(x, q, u, v)d + U_u(x, q, u, v)e. \tag{3.3}$$

Before turning to the verification of 3.3, let us quickly show why it is useful.

Proof of (3.1) assuming Proposition 3.1. Let $w_n := \mathbb{E}(w|\mathcal{F}_n)$ and $w_n^* := \max_{n' \leq n} w_{n'}$. For each n, we apply Proposition 3.1 with

$$x = f_n, \quad q = q_n = |f_0|^p + \sum_{m=1}^{n} |f_n - f_{n-1}|^p,$$
$$u = w_n, \quad v = w_n^*, \quad d = f_{n+1} - f_n, \quad e = w_{n+1} - w_n. \tag{3.4}$$

Taking the conditional expectation on both sides of the resulting inequality, we obtain

$$\mathbb{E}U(f_{n+1}, q_{n+1}, w_{n+1}, w_{n+1}^*) \leq \mathbb{E}U(f_n, q_n, w_n, w_n^*).$$

Iterating this inequality, we obtain

$$\mathbb{E}(w_N|f_N|/C_{\mathrm{H}} - Cw_N^* q_N^{1/p}) \leq \mathbb{E}U(f_N, q_N, w_N, w_N^*) \leq \mathbb{E}U(f_0, q_0, w_0, w_0^*) \leq 0.$$

This implies (3.1). □

Unlike in the scalar case in [12], it does not seem possible to directly use the Bellman function (3.2) to deduce the maximal estimate (1.2). However, Proposition 3.1 will be used in the proof of Proposition 4.1, which will in turn imply the maximal estimate.

We did not attempt to optimize the numerical values of $C, \tilde{C}$ in Proposition 3.1. Also the conditions (3.16) and (3.17), according to which these values are chosen, can be improved by a more careful choice of numerical constants at various places in the proof. However, we should like to point out that the main loss compared to [12] is due to the use of the estimate (3.9) in several denominators.

Proof of Proposition 3.1. The inequality (3.3) is quite delicate for small values of d and e, and quite sloppy for large values. This can be seen by looking at the asymptotic behavior of (3.3) for $d \to \infty$ or $e \to \infty$, which is dominated by the term $-C((u + e) \vee v)(q + |d|^p)^{1/p}$. Accordingly, we distinguish the following cases.

1. $|d|^p \le q/2$ and $u+e \le v$,
2. $|d|^p \ge q/2$, $u+e \le v$,
3. $u+e \ge v$.

Throughout the proof, let

$$\psi(t) := |x+td|^p/C_{\mathrm{H}}^p, \quad a := \psi(0), \quad b := \psi(0). \tag{3.5}$$

As a consequence of Lemma 2.1 and (2.3), we have

$$|\psi'(t)| \le p|x+td|^{p-1}|d|/C_{\mathrm{H}}^p = p(\psi(t))^{1-1/p}|d|/C_{\mathrm{H}} \le p(\psi(t))^{1-1/p}|d|, \tag{3.6}$$

$$|\psi'(t) - \psi'(\tilde{t})| \le |t-\tilde{t}|^{p-1}|d|^p. \tag{3.7}$$

Case 1. Suppose that $u+e \le v$ and $|d|^p \le q/2$.
We have

$$|\psi(t) - \psi(0) - t\psi'(0)| \le \int_0^t |\psi'(\tilde{t}) - \psi'(0)| \,\mathrm{d}\tilde{t} \le \int_0^t \tilde{t}^{p-1}|d|^p \,\mathrm{d}\tilde{t} = \frac{1}{p}|td|^p. \tag{3.8}$$

By the AMGM inequality, we have

$$\begin{aligned}
|\psi'(0)| &\le p\psi(0)^{1-1/p}|d| \le \frac{1}{2}\psi(0) + p^{p-1}|d|^p \le \frac{1}{2}\psi(0) + q,\\
|\psi'(0)| &\le p\psi(0)^{1-1/p}|d| \le \psi(0) + |d|^p \le \psi(0) + q/2.
\end{aligned}$$

This implies in particular that, for any $t \in [0,1]$, we have

$$\psi(0) + \psi'(0)t + q \ge \max(\psi(0)/2, q/2). \tag{3.9}$$

Let

$$\begin{aligned}
G(t) &:= U(x+td, q+|td|^p, u+te, v)\\
&= (u+te)(\psi(t)+q+|td|^p)^{1/p} - v(q+|td|^p)^{1/p}(C - \tilde{C}\ln(1+(u+te)/v)).
\end{aligned}$$

The claim is then equivalent to $G(1) \le G(0) + G'(0)$. Let also

$$H(t) := (u+te)(\psi(0)+\psi'(0)t+q)^{1/p} - C_5 v(q+|td|^p)^{1/p} - vq^{1/p}(C_6 - \tilde{C}\ln(1+(u+te)/v)),$$

where the splitting $C = C_5 + C_6$ will be chosen later. Then $H(0) = G(0)$ and $H'(0) = G'(0)$. In view of (3.8), we have

$$\psi(1) + q + |d|^p \le \psi(0) + \psi'(0) + \frac{1}{p}|d|^p + q + |d|^p, \tag{3.10}$$

and it follows that

$$\begin{aligned}
G(1) - H(1) &= (u+e)(\psi(1)+q+|d|^p)^{1/p} - v(q+|d|^p)^{1/p}(C - \tilde{C}\ln(1+(u+e)/v))\\
&\quad - \Big((u+e)(\psi(0)+\psi'(0)+q)^{1/p} - C_5 v(q+|d|^p)^{1/p} - vq^{1/p}(C_6 - \tilde{C}\ln(1+(u+e)/v))\Big)\\
&\le (u+e)(\psi(0)+\psi'(0)+q+(1+1/p)|d|^p)^{1/p} - v(q+|d|^p)^{1/p}(C_6 - \tilde{C}\ln(1+(u+e)/v))\\
&\quad - \Big((u+e)(\psi(0)+\psi'(0)+q)^{1/p} - vq^{1/p}(C_6 - \tilde{C}\ln(1+(u+e)/v))\Big)\\
&= K(|d|^p) - K(0),
\end{aligned}$$

where

$$K(s) = (u+e)(\psi(0)+\psi'(0)+q+(1+1/p)s)^{1/p} - v(q+s)^{1/p}(C_6 - \tilde{C}\ln(1+(u+e)/v)).$$

We have

$$\begin{aligned} K'(s) &= \frac{(u+e)(1+1/p)}{p(\psi(0)+\psi'(0)+q+(1+1/p)s)^{1-1/p}} \\ &\quad - \frac{v}{p(q+s)^{1-1/p}}(C_6 - \tilde{C}\ln(1+(u+e)/v)) \\ &\leq \frac{v(1+1/p)}{p(q/2+s/2)^{1-1/p}} - \frac{v}{p(q+s)^{1-1/p}}(C_6 - \tilde{C}\ln(2)) \leq 0 \end{aligned}$$

provided that

$$C_6 \geq 2^{1-1/p}(1+1/p) + \tilde{C}\ln(2). \tag{3.11}$$

Next, to show that $H(1) \leq H(0) + H'(0)$, we show that $H'(t) \leq H'(0)$ for $t \in [0,1]$. We compute

$$\begin{aligned} H'(t) = e(a+bt+q)^{1/p} &+ \frac{(u+te)b}{p(a+bt+q)^{1-1/p}} \\ &- \frac{C_5 v t^{p-1}|d|^p}{(q+|td|^p)^{1-1/p}} + eq^{1/p}\tilde{C}/(1+(u+te)/v), \end{aligned}$$

$$\begin{aligned} H''(t) &= \frac{2eb}{p(a+bt+q)^{1-1/p}} - \frac{(1-1/p)(u+te)b^2}{p(a+bt+q)^{2-1/p}} \\ &\quad - \frac{(p-1)C_5 v t^{p-2}|d|^p}{(q+|td|^p)^{1-1/p}} + \frac{p(1-1/p)C_5 v t^{2p-2}|d|^{2p}}{(q+|td|^p)^{2-1/p}} - e^2 q^{1/p}\tilde{C}/(1+(u+te)/v)^2/v \\ &\leq \frac{2|e|a^{1-1/p}|d|}{(a+bt+q)^{1-1/p}} - \frac{(p-1)C_5 v t^{p-2}|d|^p q}{(q+|td|^p)^{2-1/p}} - e^2 q^{1/p}\tilde{C}/(4v) \\ &\leq 2^{2-1/p}|e||d| - \frac{(p-1)C_5 v t^{p-2}|d|^p}{(3/2)^{2-1/p}q^{1-1/p}} - e^2 q^{1/p}\tilde{C}/(4v). \end{aligned}$$

Integrating this inequality, we obtain

$$H'(t) - H'(0) = \int_0^t H''(\tilde{t})\,\mathrm{d}\tilde{t} \leq 2^{2-1/p}|e||d|t - \frac{C_5 v t^{p-1}|d|^p}{(3/2)^{2-1/p}q^{1-1/p}} - e^2 q^{1/p}\tilde{C}/(4v)t.$$

By the AMGM inequality,

$$\begin{aligned} |e||d| &= \Big(\frac{v|d|^p}{q^{1-1/p}}\Big)^{1/p}\Big(\frac{|e|^{p'}q^{1/p}}{v^{p'/p}}\Big)^{1-1/p} \\ &\leq \Big(\frac{v|d|^p}{q^{1-1/p}}\Big)^{1/p}\Big(\frac{|e|^2 q^{1/p}}{v}\Big)^{1-1/p} \\ &\leq \frac{1}{p}\frac{v|d|^p}{q^{1-1/p}} + \frac{1}{p'}\frac{|e|^2 q^{1/p}}{v} \end{aligned} \tag{3.12}$$

Hence, $H'(t) \leq H'(0)$ provided that

$$C_5 \geq 3^{2-1/p}/p, \quad \tilde{C} \geq 2^{4-1/p}/p'. \tag{3.13}$$

Case 2. Suppose now $|d|^p \geq q/2$, $u+e \leq v$. Let

$$I(t) := U(x+td, q+|td|^p, u+e, v) - U(x,q,u,v) - U_x(x,q,u,v)td - U_u(x,q,u,v)e.$$

For $|td|^p \le q/2$, we showed $I(t) \le 0$ in the previous step. Hence, it suffices to show $I'(t) \le 0$ for all $t \in [0,1]$ such that $|td|^p \ge q/2$. We have

$$\begin{aligned}
I'(t) &= \frac{(u+e)(\psi'(t) + pt^{p-1}|d|^p)}{p(\psi(t)+q+|td|^p)^{1-1/p}} \\
&\qquad - \frac{vt^{p-1}|d|^p}{(q+|td|^p)^{1-1/p}}(C - \tilde{C}\ln(1+(u+e)/v)) - \frac{u\psi'(0)}{p(\psi(0)+q)^{1-1/p}} \\
&\le \frac{v|d|(\psi(t)^{1-1/p} + |td|^{p-1})}{(\psi(t)+q+|td|^p)^{1-1/p}} - \frac{vt^{p-1}|d|^p}{(q+|td|^p)^{1-1/p}}(C - \tilde{C}\ln(2)) + \frac{v\psi(0)^{1-1/p}|d|}{(\psi(0)+q)^{1-1/p}} \\
&\qquad \le 2^{1/p} v|d| - \frac{v|d|}{3^{1-1/p}}(C - \tilde{C}\ln 2) + v|d| \le 0
\end{aligned}$$

provided that

$$C \ge 3^{1-1/p}(1+2^{1/p}) + \tilde{C}\ln 2. \tag{3.14}$$

Case 3. Suppose now that $u+e \ge v$. We want to show

$$J(e) := U(x+d, q+|d|^p, u+e, u+e) - U(x,q,u,v) - U_x(x,q,u,v)d - U_u(x,q,u,v)e \le 0.$$

For $u+e=v$, we have shown that $J(e) \le 0$ in the previous steps. Hence, it suffices to show $J'(e) \le 0$ for $e \ge v-u$. We have

$$\begin{aligned}
J'(e) &= (|x+d|^p/C_{\mathrm{H}}^p + q + |d|^p)^{1/p} - (q+|d|^p)^{1/p}(C - \tilde{C}\ln 2) \\
&\qquad - (|x|^p/C_{\mathrm{H}}^p + q)^{1/p} - q^{1/p}\tilde{C}/(1+u/v) \\
&\le |x+d|/C_{\mathrm{H}} + q^{1/p} + |d| - (q+|d|^p)^{1/p}(C - \tilde{C}\ln 2) - |x|/C_{\mathrm{H}} - q^{1/p}\tilde{C}/2 \\
&\qquad \le 2|d| - |d|(C - \tilde{C}\ln 2) - q^{1/p}(\tilde{C}/2 - 1) \le 0
\end{aligned}$$

provided that

$$\tilde{C} \ge 2, \quad C \ge 2 + \tilde{C}\ln 2. \tag{3.15}$$

The inequalities (3.11), (3.13), (3.14), (3.15) can be summarized as

$$\tilde{C} \ge \max(2, 2^{4-1/p}/p'), \tag{3.16}$$

$$C \ge \max(2, 3^{1-1/p}(1+2^{1/p}), 3^{2-1/p}/p + 2^{1-1/p}(1+1/p)) + \tilde{C}\ln 2. \tag{3.17}$$

Plotting these functions, we see that the inequalities are satisfied with the claimed values of $C, \tilde{C}$. □

4 Bellman function for the maximal function

In this section, we combine the Bellman functions from [12] and [1]. For $x \in X$, $|x| \le m$, $q \ge 0$, and $0 \le u \le v$, let

$$\begin{aligned}
&U(x,m,q,u,v) \\
&\quad := u(m^p/C_{\mathrm{H}}^p + q)^{1/p} - \frac{u}{p}\frac{m^p/C_{\mathrm{H}}^p - |x|^p/C_{\mathrm{H}}^p}{(m^p/C_{\mathrm{H}}^p + q)^{1-1/p}} - Cvq^{1/p} + \tilde{C}vq^{1/p}\ln(1+u/v) \\
&\quad = \frac{u}{p'}(m^p/C_{\mathrm{H}}^p + q)^{1/p} + \frac{u}{p}\frac{q + |x|^p/C_{\mathrm{H}}^p}{(m^p/C_{\mathrm{H}}^p + q)^{1-1/p}} - Cvq^{1/p} + \tilde{C}vq^{1/p}\ln(1+u/v)
\end{aligned} \tag{4.1}$$

Evidently, the function (4.1) is a modification of (3.2). The most obvious such modification would be to replace $|x|$ by m; the more sophisticated modification in (4.1) is chosen in such a way that the left-hand side of (4.2) becomes differentiable in d. The following concavity property is the main feature of the function (4.1).

Proposition 4.1. *Let $p \in (1,2]$, $C = 21$, and $\tilde{C} = 4\sqrt{2}$. Then, for any $x, d \in X$, $m, q, u, v \in \mathbb{R}_{\geq 0}$, and $e \in \mathbb{R}$ with $|x| \leq m$, $u \leq v$, and $u + e \geq 0$, we have*

$$\begin{aligned} U\big(x+d, m \vee |x+d|, q+|d|^p, u+e, (u+e) \vee v\big) \\ \leq U(x,m,q,u,v) + U_x(x,m,q,u,v)d + U_u(x,m,q,u,v)e. \end{aligned} \tag{4.2}$$

The numerical value of C, which comes out of the condition (4.12), is again probably far from optimal.

Proof of Theorem 1.1 assuming Proposition 4.1. We apply Proposition 4.1 with the same parameters as in (3.4), and additionally

$$m = f_n^* := \max_{n' \leq n} |f_{n'}|.$$

Taking the conditional expectation on both sides of the resulting inequality, we obtain

$$\mathbb{E}U(f_{n+1}, f_{n+1}^*, q_{n+1}, w_{n+1}, w_{n+1}^*) \leq \mathbb{E}U(f_n, f_n^*, q_n, w_n, w_n^*).$$

Iterating this inequality, we obtain

$$\mathbb{E}\Big(\frac{w_N f_N^*}{p' C_{\mathrm{H}}} - C w_N^* q_N^{1/p}\Big) \leq \mathbb{E}U(f_N, f_N^*, q_N, w_N, w_N^*) \leq \mathbb{E}U(f_0, f_0^*, q_0, w_0, w_0^*) \leq 0.$$

This implies

$$\mathbb{E}(f^* w) \leq 21 p' C_{\mathrm{H}} \mathbb{E}(S_p f \cdot w^*), \tag{4.3}$$

which in turn implies (1.2) in view of (2.4).

A similar argument also shows (1.4). □

Remark 4.2. Proposition 4.1 can also be used to recover a non-maximal bound similar to (3.1) (but with a larger absolute constant). This is because, by the AMGM inequality,

$$\frac{w_N |f_N|}{C_{\mathrm{H}}} \leq w_N ((|f_N|/C_{\mathrm{H}})^p + q_N)^{1/p} \leq \frac{w_N}{p'} ((f_N^*/C_{\mathrm{H}})^p + q_N)^{1/p} + \frac{w_N}{p} \frac{q_N + |f_N|^p / C_{\mathrm{H}}^p}{((f_N^*/C_{\mathrm{H}})^p + q_N)^{1-1/p}}.$$

Proof of Proposition 4.1. Due to an additional maximum in (4.2), we have to distinguish a few more cases than in Section 3. The main distinction is according to the ordering of $|x+d|$ and m, since this ordering substantially affects the shape of the function (4.1). The cases are as follows.

1. $|x+d| \leq m$
   (a) $|d|^p \leq q/2$, $u + e \leq v$,
   (b) $|d|^p \leq q/2$, $u + e \geq v$,
   (c) $|d|^p \geq q/2$.
2. $|x+d| \geq m$
   (a) $|d|^p \leq q/2$,
   (b) $|d|^p \geq q/2$.

Similarly as in Proposition 3.1, only the cases 1a and 2a are delicate.

We continue to use the notation (3.5) and the estimates (3.6), (3.7).

Case 1. First, we consider the case $|x+d| \leq m$.

Case 1a. We consider the subcase $u + e \leq v$, $|d|^p \leq q/2$.

Let

$$\begin{aligned} G(t) &:= U(x+td, m, q+|td|^p, u+te, v) \\ &= \frac{u+te}{p'}(m^p/C_{\mathrm{H}}^p + q + |td|^p)^{1/p} + \frac{u+te}{p}\frac{q+|td|^p+|x+td|^p/C_{\mathrm{H}}^p}{(m^p/C_{\mathrm{H}}^p+q+|td|^p)^{1-1/p}} \\ &\quad - v(q+|td|^p)^{1/p}(C - \tilde{C}\ln(1+(u+te)/v)). \end{aligned}$$

With this notation, the claim (4.2) turns into $G(1) \le G(0) + G'(0)$.

With a splitting $C = C_1 + C_2$ to be chosen later, let

$$\begin{aligned} H(t) := \frac{u+te}{p'}(m^p/C_{\mathrm{H}}^p + q)^{1/p} + \frac{u+te}{p}\frac{q+\psi(0)+\psi'(0)t}{(m^p/C_{\mathrm{H}}^p+q)^{1-1/p}} \\ - C_1 v(q+|td|^p)^{1/p} - vq^{1/p}(C_2 - \tilde{C}\ln(1+(u+te)/v)). \end{aligned}$$

Then $G(0) = H(0)$ and $G'(0) = H'(0)$. By (3.10), we have

$$\begin{aligned} G(1) - H(1) &= \frac{u+e}{p'}(m^p/C_{\mathrm{H}}^p + q + |d|^p)^{1/p} + \frac{u+e}{p}\frac{q+|d|^p+|x+d|^p/C_{\mathrm{H}}^p}{(m^p/C_{\mathrm{H}}^p+q+|d|^p)^{1-1/p}} \\ &- v(q+|d|^p)^{1/p}(C - \tilde{C}\ln(1+(u+e)/v)) - \Big(\frac{u+e}{p'}(m^p/C_{\mathrm{H}}^p+q)^{1/p} + \frac{u+e}{p}\frac{q+\psi(0)+\psi'(0)}{(m^p/C_{\mathrm{H}}^p+q)^{1-1/p}} \\ &\qquad - C_1 v(q+|d|^p)^{1/p} - vq^{1/p}(C_2 - \tilde{C}\ln(1+(u+e)/v))\Big) \\ &\le \frac{u+e}{p'}(m^p/C_{\mathrm{H}}^p + q + |d|^p)^{1/p} + \frac{u+e}{p}\frac{q+|d|^p+\psi(0)+\psi'(0)+|d|^p/p}{(m^p/C_{\mathrm{H}}^p+q+|d|^p)^{1-1/p}} \\ &- v(q+|d|^p)^{1/p}(C_2 - \tilde{C}\ln(1+(u+e)/v)) - \Big(\frac{u+e}{p'}(m^p/C_{\mathrm{H}}^p+q)^{1/p} + \frac{u+e}{p}\frac{q+\psi(0)+\psi'(0)}{(m^p/C_{\mathrm{H}}^p+q)^{1-1/p}} \\ &\qquad - vq^{1/p}(C_2 - \tilde{C}\ln(1+(u+e)/v))\Big) = K(|d|^p) - K(0), \end{aligned}$$

where

$$\begin{aligned} K(s) = \frac{u+e}{p'}(m^p/C_{\mathrm{H}}^p + q + s)^{1/p} + \frac{u+e}{p}\frac{q+s+a+b+s/p}{(m^p/C_{\mathrm{H}}^p+q+s)^{1-1/p}} \\ - v(q+s)^{1/p}(C_2 - \tilde{C}\ln(1+(u+e)/v)). \end{aligned}$$

In order to show that $G(1) \le H(1)$, we compute

$$\begin{aligned} K'(s) &= \frac{u+e}{p'}\frac{1}{p(m^p/C_{\mathrm{H}}^p+q+s)^{1-1/p}} \\ &\quad + \frac{u+e}{p}\Big(\frac{1+1/p}{(m^p/C_{\mathrm{H}}^p+q+s)^{1-1/p}} - \frac{(1-1/p)(q+s+a+b+s/p)}{(m^p/C_{\mathrm{H}}^p+q+s)^{2-1/p}}\Big) \\ &\quad - \frac{v}{p(q+s)^{1/p}}(C_2 - \tilde{C}\ln(1+(u+e)/v)) \\ &\le (u+e)\frac{2/p}{(m^p/C_{\mathrm{H}}^p+q+s)^{1-1/p}} - \frac{v}{p(q+s)^{1/p}}(C_2 - \tilde{C}\ln(2)). \\ &\le \frac{2v}{p}\frac{1}{(q+s)^{1/p}} - \frac{v}{p(q+s)^{1/p}}(C_2 - \tilde{C}\ln(2)) \le 0 \end{aligned}$$

provided that

$$C_2 \ge 2 + \tilde{C}\ln(2). \tag{4.4}$$

Now, it suffices to show $H(1) \le H(0) + H'(0)$. This will follow from $H'(t) \le H'(0)$ for $t \in [0,1]$. Compute

$$\begin{aligned} H'(t) &= \frac{e}{p'}(m^p/C_{\mathrm{H}}^p + q)^{1/p} + \frac{e}{p}\frac{q+a+bt}{(m^p/C_{\mathrm{H}}^p+q)^{1-1/p}} + \frac{u+te}{p}\frac{b}{(m^p/C_{\mathrm{H}}^p+q)^{1-1/p}} \\ &\quad - \frac{t^{p-1}|d|^p C_1 v}{(q+|td|^p)^{1/p}} + eq^{1/p}\tilde{C}/(1+(u+te)/v), \end{aligned}$$

and

$$\begin{aligned} H''(t) &= \frac{2e}{p}\frac{b}{(m^p/C_{\mathrm{H}}^p+q)^{1-1/p}} \\ &\quad - \frac{(p-1)t^{p-2}|d|^p C_1 v}{(q+|td|^p)^{1-1/p}} + \frac{(1-1/p)pt^{p-1}|d|^p t^{p-1}|d|^p C_1 v}{(q+|td|^p)^{2-1/p}} \\ &\quad - \frac{e^2}{v} q^{1/p}\tilde{C}/(1+(u+te)/v)^2 \\ &\le 2|e|\frac{a^{1/p'}|d|}{(m^p/C_{\mathrm{H}}^p+q)^{1-1/p}} \\ &\quad - \frac{(p-1)t^{p-2}|d|^p C_1 v}{(q+|td|^p)^{1-1/p}}\Big(1 - \frac{|td|^p}{q+|td|^p}\Big) - \frac{e^2}{v}q^{1/p}\tilde{C}/4 \\ &\le 2|e||d| - \frac{(p-1)t^{p-2}|d|^p C_1 v}{(3/2)^{2-1/p}q^{1-1/p}} - \frac{e^2}{v}q^{1/p}\tilde{C}/4. \end{aligned}$$

Integrating this inequality, we obtain

$$H'(t) - H'(0) \le 2|e||d|t - \frac{t^{p-1}|d|^p C_1 v}{(3/2)^{2-1/p}q^{1-1/p}} - \frac{e^2 q^{1/p}\tilde{C}}{4v}t.$$

Recalling (3.12), we see that $H'(t) \le H'(0)$ provided that

$$C_1 \ge 2\cdot(3/2)^{2-1/p}/p, \quad \tilde{C} \ge 2^3/p'. \tag{4.5}$$

Case 1b. We keep the assumptions $|x+d| \le m$ and $|d|^p \le q/2$. Now, we consider the case $u+e \ge v$. In particular, $e \ge v-u \ge 0$. Let

$$\begin{aligned} J(e) &:= U\big(x+d, m, q+|d|^p, u+e, u+e\big) \\ &\qquad - U(x,m,q,u,v) - U_x(x,m,q,u,v)d - U_u(x,m,q,u,v)e. \end{aligned}$$

For $e = v-u$, we showed that $J(e) \le 0$ in the previous case. Hence, it suffices to show that $J'(e) \le 0$ for $e \ge v-u$. We have

$$\begin{aligned} J'(e) &= \frac{1}{p'}(m^p/C_{\mathrm{H}}^p + q + |d|^p)^{1/p} + \frac{1}{p}\frac{q+|d|^p+|x+d|^p/C_{\mathrm{H}}^p}{(m^p/C_{\mathrm{H}}^p+q+|d|^p)^{1-1/p}} - (q+|d|^p)^{1/p}(C - \tilde{C}\ln 2) \\ &\quad - \frac{1}{p'}(m^p/C_{\mathrm{H}}^p+q)^{1/p} - \frac{1}{p}\frac{q+|x|^p/C_{\mathrm{H}}^p}{(m^p/C_{\mathrm{H}}^p+q)^{1-1/p}} - \tilde{C}q^{1/p}/(1+u/v) \\ &\le \frac{|d|}{p'} + \frac{1}{p}\frac{|d|^p+|x+d|^p/C_{\mathrm{H}}^p - |x|^p/C_{\mathrm{H}}^p}{(m^p/C_{\mathrm{H}}^p+q)^{1-1/p}} - (q+|d|^p)^{1/p}(C-\tilde{C}\ln 2). \end{aligned}$$

Using (3.8), we obtain

$$\begin{aligned} J'(e) &\le \frac{|d|}{p'} + \frac{1}{p}\frac{|d|^p + |b| + |d|^p/p}{(m^p/C_{\mathrm{H}}^p+q)^{1-1/p}} - (q+|d|^p)^{1/p}(C-\tilde{C}\ln 2) \\ &\le (1/p)|d| + \frac{1}{p}\frac{|d|^p(1+1/p) + pa^{p/p'}|d|}{(m^p/C_{\mathrm{H}}^p+q)^{1-1/p}} - (q+|d|^p)^{1/p}(C-\tilde{C}\ln 2) \\ &\le (1/p + (1+(1/p+1/p^2)^p)^{1/p})|d| - (q+|d|^p)^{1/p}(C-\tilde{C}\ln 2). \end{aligned}$$

This is ≤ 0 provided that

$$C \ge 1/p + (1 + (1/p + 1/p^2)^p)^{1/p} + \tilde{C} \ln 2. \tag{4.6}$$

Case 1c. Suppose now that still $|x + d| \le m$, but now $|d|^p \ge q/2$. Let $\tilde{v} := (u + e) \vee v$. Let

$$\begin{aligned} I(t) := U(x + td, m, q + |td|^p, u + e, \tilde{v}) \\ - U(x, m, q, u, v) - U_x(x, m, q, u, v)td - U_u(x, m, q, u, v)e. \end{aligned}$$

For $|td|^p \le q/2$, we showed $I(t) \le 0$ in the previous steps. Hence, it suffices to show $I'(t) \le 0$ for $t \in [0, 1]$ such that $|td|^p \ge q/2$. We have

$$\begin{aligned} I'(t) &= \frac{u+e}{p'} \frac{t^{p-1}|d|^p}{(m^p/C_{\mathrm{H}}^p + q + |td|^p)^{1-1/p}} \\ &+ \frac{u+e}{p}\Big(\frac{pt^{p-1}|d|^p + \psi'(t)}{(m^p/C_{\mathrm{H}}^p + q + |td|^p)^{1-1/p}} - \frac{(q + |td|^p + |x+td|^p/C_{\mathrm{H}}^p) \cdot (1-1/p)pt^{p-1}|d|^p}{(m^p/C_{\mathrm{H}}^p + q + |td|^p)^{2-1/p}}\Big) \\ &\qquad - \frac{\tilde{v}t^{p-1}|d|^p}{(q + |td|^p)^{1-1/p}}(C - \tilde{C}\ln(1 + (u+e)/\tilde{v})) - \frac{u\psi'(0)}{p(m^p/C_{\mathrm{H}}^p + q)^{1-1/p}}. \\ &\le \tilde{v}\Big(\frac{(1+1/p')t^{p-1}|d|^p + \psi(t)^{1/p'}|d|}{(m^p/C_{\mathrm{H}}^p + q + |td|^p)^{1-1/p}}\Big) - \frac{\tilde{v}t^{p-1}|d|^p}{(q+|td|^p)^{1-1/p}}(C - \tilde{C}\ln 2) + \frac{\tilde{v}a^{1/p'}|d|}{(m^p/C_{\mathrm{H}}^p + q)^{1-1/p}} \\ &\le \tilde{v}|d|((1 + 1/p')^p + 1)^{1/p} - \frac{\tilde{v}t}{3^{1-1/p}}(C - \tilde{C}\ln 2) + \tilde{v}|d| \le 0 \end{aligned}$$

provided that

$$C \ge 3^{1-1/p}(1 + (1 + (1/p')^p)^{1/p}) + \tilde{C} \ln 2. \tag{4.7}$$

Case 2. Suppose now that $|x + d| > m$. Let $\tilde{v} := v \vee (u + e)$. In this case, the claim (4.2) becomes

$$\begin{aligned} (u+e)(|x+d|^p/C_{\mathrm{H}}^p + q + |d|^p)^{1/p} - C\tilde{v}(q + |d|^p)^{1/p} + \tilde{C}\tilde{v}(q + |d|^p)^{1/p}\ln(1 + (u+e)/\tilde{v}) \\ \le \frac{u}{p'}(m^p/C_{\mathrm{H}}^p + q)^{1/p} + \frac{u}{p}\frac{q + |x|^p/C_{\mathrm{H}}^p}{(m^p/C_{\mathrm{H}}^p + q)^{1-1/p}} - Cvq^{1/p} + \tilde{C}vq^{1/p}\ln(1 + u/v) \\ + \frac{u\psi'(0)}{p(m^p/C_{\mathrm{H}}^p + q)^{1-1/p}} + \Big(\frac{1}{p'}(m^p/C_{\mathrm{H}}^p + q)^{1/p} + \frac{1}{p}\frac{q + |x|^p/C_{\mathrm{H}}^p}{(m^p/C_{\mathrm{H}}^p + q)^{1-1/p}} + \tilde{C}q^{1/p}/(1 + u/v)\Big)e. \end{aligned} \tag{4.8}$$

Case 2a. Suppose first $|d|^p \le q/2$. Let a splitting $C = C_3 + C_4$ be chosen later. By Proposition 3.1, we have

$$\begin{aligned} (u+e)(|x+d|^p/C_{\mathrm{H}}^p + q + |d|^p)^{1/p} - C_3\tilde{v}(q + |d|^p)^{1/p} + \tilde{C}\tilde{v}(q + |d|^p)^{1/p}\ln(1 + (u+e)/\tilde{v}) \\ \le u(|x|^p/C_{\mathrm{H}}^p + q)^{1/p} - C_3vq^{1/p} + \tilde{C}vq^{1/p}\ln(1 + u/v) \\ + \frac{u\psi'(0)}{p(|x|^p/C_{\mathrm{H}}^p + q)^{1/p}} + \Big((|x|^p/C_{\mathrm{H}}^p + q)^{1/p} + \tilde{C}q^{1/p}/(1 + u/v)\Big)e \end{aligned}$$

with

$$C_3 = 9, \quad \tilde{C} = 4\sqrt{2}. \tag{4.9}$$

Note that this value of $\tilde{C}$ is compatible with (4.5).

By the AMGM inequality, we have

$$(|x|^p/C_{\mathrm{H}}^p + q)^{1/p} \le \frac{1}{p'}(m^p + q)^{1/p} + \frac{1}{p}\frac{q + |x|^p/C_{\mathrm{H}}^p}{(m^p + q)^{1-1/p}}.$$

Multiplying this inequality by $u+e$ and inserting it on the right-hand side of (4.8), we see that it suffices to show

$$C_4 v q^{1/p} - C_4 \tilde{v}(q+|d|^p)^{1/p} \leq \frac{u\psi'(0)}{p(m^p/C_{\mathrm{H}}^p+q)^{1-1/p}} - \frac{u\psi'(0)}{p(|x|^p/C_{\mathrm{H}}^p+q)^{1-1/p}}.$$

If $\psi'(0) \leq 0$, then the right-hand side is positive, so there is nothing to show. Let us assume $\psi'(0) > 0$. The claim is then equivalent to

$$\frac{u\psi'(0)}{p}\Big(\frac{1}{(|x|^p/C_{\mathrm{H}}^p+q)^{1-1/p}} - \frac{1}{(m^p/C_{\mathrm{H}}^p+q)^{1-1/p}}\Big) \leq C_4\tilde{v}(q+|d|^p)^{1/p} - C_4 v q^{1/p}.$$

Since $0 \leq u \leq v \leq \tilde{v}$ and $|\psi'(0)| \leq p a^{1/p'}|d|$, it suffices to show

$$\frac{|x|^{p/p'}|d|}{C_{\mathrm{H}}^p}\Big(\frac{1}{(|x|^p/C_{\mathrm{H}}^p+q)^{1-1/p}} - \frac{1}{(m^p/C_{\mathrm{H}}^p+q)^{1-1/p}}\Big) \leq C_4(q+|d|^p)^{1/p} - C_4 q^{1/p}.$$

By concavity of $\cdot^{1/p}$ and convexity of $\cdot^{1/p-1}$, this will follow from

$$\frac{|x|^{p/p'}|d|}{C_{\mathrm{H}}^p}(|x|^p/C_{\mathrm{H}}^p - m^p/C_{\mathrm{H}}^p)(1/p-1)(|x|^p/C_{\mathrm{H}}^p+q)^{1/p-2} \leq C_4|d|^p(1/p)(q+|d|^p)^{1/p-1}.$$

Since $|d|^p \leq q/2$, this will follow from

$$\frac{|x|^{p/p'}|d|}{C_{\mathrm{H}}^{2p}}(p-1)(m^p-|x|^p)(|x|^p/C_{\mathrm{H}}^p+q)^{1/p-2} \leq C_4|d|^p(3q/2)^{1/p-1}.$$

This will follow from

$$\frac{|d|}{C_{\mathrm{H}}^p}(p-1)(m^p-|x|^p)(|x|^p/C_{\mathrm{H}}^p+q)^{-1} \leq C_4|d|^p(3q/2)^{1/p-1}.$$

Since $|x| \leq m \leq |x+d| \leq |x|+|d|$ and using the elementary inequality $(a+b)^p \leq a+pa^{p-1}b+b^p$ (which holds for any $a,b \geq 0$ and $p \in [1,2]$, as can be seen by differentiating both sides in b), we have

$$\begin{aligned} m^p/C_{\mathrm{H}}^p+q &\leq (|x|+|d|)^p/C_{\mathrm{H}}^p+q \leq (|x|^p+p|x|^{p-1}|d|+|d|^p)/C_{\mathrm{H}}^p+q \\ &\leq |x|^p/C_{\mathrm{H}}^p + p(|x|/C_{\mathrm{H}})^{p-1}|d| + 2q \leq |x|^p/C_{\mathrm{H}}^p + p((|x|/C_{\mathrm{H}})^p/p' + |d|^p/p) + 2q \\ &\leq 3(|x|^p/C_{\mathrm{H}}^p+q), \end{aligned}$$

so it suffices to show

$$3(p-1)\frac{|d|}{C_{\mathrm{H}}^p}(m^p-|x|^p) \leq C_4|d|^p(3q/2)^{1/p-1}(m^p/C_{\mathrm{H}}^p+q).$$

$$\Longleftarrow\ 3(p-1)\frac{|d|}{C_{\mathrm{H}}^p}p(m-|x|)m^{p-1} \leq C_4|d|^p(3q/2)^{1/p-1}(m^p/C_{\mathrm{H}}^p+q).$$

$$\Longleftarrow\ 3(p-1)\frac{|d|}{C_{\mathrm{H}}^p}p(m-|x|)m^{p-1} \leq C_4|d|^p(3/2)^{1/p-1}(m^p/C_{\mathrm{H}}^p+q)^{1/p}.$$

$$\Longleftarrow\ 3(p-1)|d|p(m-|x|)m^{p-1} \leq C_4|d|^p(3/2)^{1/p-1}m.$$

Since $m-|x| \leq \min(|d|,m)$, this holds provided that

$$C_4 \geq 3(p-1)p(3/2)^{1-1/p}. \tag{4.10}$$

Case 2b. Suppose now $|d|^p \geq q/2$. Let

$$\begin{aligned} I(t) := (u+e)(|x+td|^p/C_{\mathrm{H}}^p+q+|td|^p)^{1/p} - C\tilde{v}(q+|td|^p)^{1/p} + \tilde{C}\tilde{v}(q+|td|^p)^{1/p}\ln(1+(u+e)/\tilde{v}) \\ - \frac{u}{p'}(m^p/C_{\mathrm{H}}^p+q)^{1/p} - \frac{u}{p}\frac{q+|x|^p/C_{\mathrm{H}}^p}{(m^p/C_{\mathrm{H}}^p+q)^{1-1/p}} + Cvq^{1/p} - \tilde{C}vq^{1/p}\ln(1+u/v) \\ - \frac{u\psi'(0)t}{p(m^p/C_{\mathrm{H}}^p+q)^{1-1/p}} - \Big(\frac{1}{p'}(m^p/C_{\mathrm{H}}^p+q)^{1/p} + \frac{1}{p}\frac{q+|x|^p/C_{\mathrm{H}}^p}{(m^p/C_{\mathrm{H}}^p+q)^{1-1/p}} + \tilde{C}q^{1/p}/(1+u/v)\Big)e. \end{aligned}$$

The claim (4.8) is equivalent to $I(1) \leq 0$. From the previous cases, we know that $I(t) \leq 0$ if t is so small that either $|x+td| = m$ or $|td|^p \leq q/2$. Hence, it suffices to show $I'(t) \leq 0$ for all $t \in [0,1]$ such that $|x+td| \geq m$ and $|td|^p \geq q/2$. We compute

$$\begin{aligned} I'(t) &= (u+e)\frac{\psi'(t)+pt^{p-1}|d|^p}{p(|x+td|^p/C_{\mathrm{H}}^p+q+|td|^p)^{1-1/p}} \\ &\quad - \frac{\tilde{v}t^{p-1}|d|^p}{(q+|td|^p)^{1-1/p}}(C-\tilde{C}\ln(1+(u+e)/\tilde{v})) - \frac{u\psi'(0)}{p(m^p/C_{\mathrm{H}}^p+q)^{1-1/p}} \\ &\leq (u+e)|d|\frac{|x+td|^{p/p'}/C_{\mathrm{H}}^p+|td|^{p-1}}{(|x+td|^p/C_{\mathrm{H}}^p+q+|td|^p)^{1-1/p}} \\ &\quad - \frac{\tilde{v}t^{p-1}|d|^p}{(q+|td|^p)^{1-1/p}}(C-\tilde{C}\ln(2)) + \frac{u|x|^{p/p'}|d|/C_{\mathrm{H}}^p}{(m^p/C_{\mathrm{H}}^p+q)^{1-1/p}} \\ &\leq (u+e)|d|2^{1/p} - \frac{\tilde{v}|d|}{3^{1-1/p}}(C-\tilde{C}\ln(2)) + u|d|. \end{aligned}$$

This is negative provided that

$$C \geq 3^{1-1/p}(1+2^{1/p}) + \tilde{C}\ln(2). \tag{4.11}$$

Conclusion of the proof. The conditions (4.4), (4.5), (4.6), (4.7), (4.9), (4.10), (4.11) amount to $\tilde{C} = 4\sqrt{2}$ and

$$\begin{aligned} C \geq \max(2\cdot(3/2)^{2-1/p}/p+2, 1/p+(1+(1/p+1/p^2)^p)^{1/p}, \\ 3^{1-1/p}(1+(1+(1/p')^p)^{1/p}), 9+3(p-1)p(3/2)^{1-1/p}, \\ 3^{1-1/p}(1+2^{1/p})) + \tilde{C}\ln(2). \end{aligned} \tag{4.12}$$

The latter condition holds for $C = 21$. □

A Extrapolation

Here, we show how weighted estimates such as (1.2) can be used to obtain a different kind of vector-valued estimates, where the vector space is a UMD Banach function space. We recall that a Banach space X is UMD if and only if every martingale transform with bounded coefficients on a filtered probability space Ω defines a bounded operator on every $L^r(\Omega, X)$ with $r \in (1,\infty)$; we refer to [9, Section 4] for many equivalent characterizations (including via the eponymous unconditional summability of martingale differences) and examples of UMD spaces. We start with a very simple extrapolation result, which uses only the duality argument introduced in [7].

Proposition A.1 (Banach function space valued extrapolation)**.** *For every $r \in (1,\infty)$ and every UMD Banach function space X over a σ-finite measure space (S,Σ,μ), there exists a constant $C_{r,X} < \infty$ such that the following holds.*

Let $(\Omega, (\mathcal{F}_n)_{n\in\mathbb{N}})$ be a filtered probability space and let $f, g : \Omega \times S \to \mathbb{R}_{\geq 0}$ be measurable functions such that, for some $A < \infty$, μ-almost every $s \in S$, and every weight

w on Ω, we have

$$\int_\Omega f(\cdot,s)w \le A\int_\Omega g(\cdot,s)w^*. \tag{A.1}$$

Then, we have

$$\|f\|_{L^r(\Omega,X)} \le C_{r,X}A\|g\|_{L^r(\Omega,X)}. \tag{A.2}$$

Proposition A.1 is not new, in the sense that it also follows from the usual Rubio de Francia extrapolation argument and (a martingale version of) [10, Theorem 2.4.1]. However, the direct proof is substantially easier.

Proof. Truncating f, we may assume that the left-hand side of (A.2) is finite.

Recall that the Banach function space X has an *associate Banach function space* X', which is again a Banach function space on (S,Σ,μ) with the property that, for every $h\in X'$, we have

$$\|h\|_{X'} = \sup_{f\in X,\|f\|_X\le 1}\int_S fh\,\mathrm{d}\mu,$$

and all these integrals converge absolutely. In particular, X' is isomorphic to a closed subspace of the dual space of X. The associate space is *norming* in the sense that, for every $f\in X$, we have

$$\|f\|_X = \sup_{h\in X',\|h\|_{X'}\le 1}\int_S fh\,\mathrm{d}\mu,$$

see e.g. [20, §71] for the proof of this fact.

Next, we need a measurable selection of $h(f)$ that almost extremize the supremum on the right-hand side. In concrete spaces X, it is frequently possible to explicitly find such a selection. For an abstract Banach function space X, it seems necessary to reduce to a finite-dimensional subspace first.

Since X is UMD, it is reflexive [9, Theorem 4.3.3]. It follows that the norm of X is absolutely continuous [20, §73, Theorem 2]. Therefore, a version of the dominated convergence theorem holds in X [20, §72, Theorem 2]. Hence, we can approximate f by simple functions in $L^r(\Omega,X)$, that is, finite linear combinations of characteristic functions of product subsets of $\Omega\times S$.

Let $\epsilon>0$, and let $\mathfrak{f}$ be such a simple function with

$$\|f-\mathfrak{f}\|_{L^r(\Omega,X)} < \epsilon.$$

Then $\mathfrak{f}(\omega,\cdot)$ takes values in a finite-dimensional subspace of X as $\omega\in\Omega$ varies. On this finite-dimensional subspace, we may choose a measurable map $f\mapsto h(f)$ such that

$$\|h(f)\|_{X'} \le \|f\|_X^{r-1}, \quad \text{and} \tag{A.3}$$

$$(1+\epsilon)\int_S fh(f)\,\mathrm{d}\mu \ge \|f\|_X^r. \tag{A.4}$$

We do this by first choosing such a map on the unit sphere, using compactness of the unit sphere, and then extend it by homogeneity.

By the hypothesis (A.1) with the weights

$$w(\omega,s) := h(\mathfrak{f}(\omega,\cdot))(s),$$

we obtain

$$\int_\Omega\int_S f(\omega,s)w(\omega,s)\,\mathrm{d}\mu(s)\,\mathrm{d}\omega \le A\int_S\int_\Omega g(\omega,s)w^*(\omega,s)\,\mathrm{d}\mu(s)\,\mathrm{d}\omega, \tag{A.5}$$

where $w^*(\omega,s) := \sup_{n\geq 0} \mathbb{E}(w(\cdot,s)|\mathcal{F}_n)(\omega)$. By duality,

$$(\text{A.5}) \leq A \int_\Omega \|g(\omega,\cdot)\|_X \|w^*(\omega,\cdot)\|_{X'} \, \mathrm{d}\omega \leq A\|g\|_{L^r(\Omega,X)} \|w^*\|_{L^{r'}(\Omega,X')}.$$

Since X' is a closed subspace of the Banach space dual of X, it is again UMD [9, Proposition 4.2.17]. By the UMD-valued maximal inequality [18, Theorem 3.2], we have

$$\|w^*\|_{L^{r'}(\Omega,X')} \leq C_{r',X'} \|w\|_{L^{r'}(\Omega,X')}.$$

By (A.3), we have

$$\|w\|_{L^{r'}(\Omega,X')} \leq \|\mathfrak{f}\|^{r-1}_{L^r(\Omega,X)}. \tag{A.6}$$

It follows that

$$(\text{A.5}) \leq AC_{r',X'} \|g\|_{L^r(\Omega,X)} \|\mathfrak{f}\|^{r-1}_{L^r(\Omega,X)}.$$

Finally, by duality and (A.6), we have

$$\int_\Omega \int_S (f-\mathfrak{f})(\omega,s) w(\omega,s) \, \mathrm{d}\mu(s) \, \mathrm{d}\omega \leq \|f-\mathfrak{f}\|_{L^r(\Omega,X)} \|\mathfrak{f}\|^{r-1}_{L^r(\Omega,X)}.$$

Combining the last two inequalities, we obtain

$$\begin{aligned}\|\mathfrak{f}\|^r_{L^r(\Omega,X)} &\leq (1+\epsilon) \int_\Omega \int_S \mathfrak{f}(\omega,s) h(\mathfrak{f}(\omega,\cdot))(s) \, \mathrm{d}\mu(s) \, \mathrm{d}\omega \\ &\leq (1+\epsilon)(AC_{r',X'} \|g\|_{L^r(\Omega,X)} + \|f-\mathfrak{f}\|_{L^r(\Omega,X)}) \|\mathfrak{f}\|^{r-1}_{L^r(\Omega,X)}.\end{aligned}$$

Since ϵ can be chosen arbitrarily small, this implies the claim (A.2). □

Remark A.2. The space $L^r(\Omega)$ in Proposition A.1 can be replaced by another Banach function space Y, provided that $Y'(X')$ is a norming subspace of the dual space of $Y(X)$, and, most importantly, that the martingale maximal operator is bounded on $Y'(X')$. One example is when Y is a weighted L^r space and $X = \mathbb{R}$; the appropriate maximal bounds in this case have been proved in [6].

As a direct consequence of the weighted BDG inequality (1.2) (or rather just the scalar case from [12]) and Proposition A.1, we recover the following BDG-type inequality.

Corollary A.3 ([18, Theorem 1.1])**.** *Let $r, S, X, C_{r,X}$ be as in Proposition A.1. Let $(\Omega, (\mathcal{F}_n)_{n\in\mathbb{N}})$ be a filtered probability space. Let $f : \mathbb{N} \times \Omega \times S \to \mathbb{R}$ be a function with $f_0(\omega,s) = 0$ such that*

1. *for every n, $f_n(\cdot,\cdot)$ is $\mathcal{F}_n \times \Sigma$-measurable, and*
2. *for almost every $s \in S$, $(f_n(\cdot,s))_{n\in\mathbb{N}}$ is a martingale with respect to $(\mathcal{F}_n)_n$.*

Then,

$$\|\sup_{n\in\mathbb{N}} |f_n(\cdot,\cdot)|\|_{L^r(\Omega,X)} \leq 16(\sqrt{2}+1) C_{r,X} \Big\| \Big(\sum_{n\geq 1} |f_n(\cdot,\cdot) - f_{n-1}(\cdot,\cdot)|^2 \Big)^{1/2} \Big\|_{L^r(\Omega,X)}. \tag{A.7}$$

Proof. By the monotone convergence theorem, we may consider a finite sequence of times $n \leq N$, so that the left-hand side of (A.7) is finite if the right-hand side is. Let

$$f(\omega,s) := \max_{n\leq N} |f_n(\omega,s)|.$$

This is an $\mathcal{F}_N \times \Sigma$-measurable function, and for a.e. ω we have $f(\omega,\cdot) \in X$. By [12, Theorem 1.1], the hypothesis (A.1) of Proposition A.1 holds for the above function f with $A = 16(\sqrt{2}+1)$ and

$$g(\omega,s) = \Big(\sum_{n=1}^N |f_n(\omega,s) - f_{n-1}(\omega,s)|^2 \Big)^{1/2}.$$

□

Remark A.4. In [18, Theorem 1.1], also a converse inequality to (A.7) has been proved. That converse inequality does not follow from the main result of [13], due to the restriction to weights that are almost surely continuous in time in that result. In [19], we extend the main result of [13] in such a way that it recovers the converse to (A.7).

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