Electronic Journal of Probability

Electron. J. Probab. **25** (2020), article no. 150, 1–30.
ISSN: 1083-6489 https://doi.org/10.1214/20-EJP551

Approximation of Hilbert-Valued Gaussians on Dirichlet structures*

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Abstract

We introduce a framework to derive quantitative central limit theorems in the context of non-linear approximation of Gaussian random variables taking values in a separable Hilbert space. In particular, our method provides an alternative to the usual (non-quantitative) finite dimensional distribution convergence and tightness argument for proving functional convergence of stochastic processes. We also derive four moments bounds for Hilbert-valued random variables with possibly infinite chaos expansion, which include, as special cases, all finite-dimensional four moments results for Gaussian approximation in a diffusive context proved earlier by various authors. Our main ingredient is a combination of an infinite-dimensional version of Stein's method as developed by Shih and the so-called Gamma calculus. As an application, rates of convergence for the functional Breuer-Major theorem are established.

Keywords: Gaussian measures on Hilbert spaces; Dirichlet structures; Stein's method on Banach spaces; Gaussian approximation; probabilistic metrics; functional limit theorems; fourth moment conditions.
MSC2020 subject classifications: 46G12; 46N30; 60B12; 60F17.
Submitted to EJP on February 18, 2020, final version accepted on November 11, 2020.
Supersedes arXiv:1905.05127.

1 Introduction

Random variables taking values in Hilbert spaces play an important role in many fields of mathematics and statistics, both at a theoretical and applied level. For example, they arise naturally in statistics, in particular in the field of functional data analysis or machine learning (for example in the context of Reproducing Kernel Hilbert Spaces). An important and classical topic is the asymptotic analysis of sequences of such random variables.

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In the linear case, i.e., when looking at normalized sums of i.i.d. random variables, the asymptotic behaviour is very well understood, with central limit theorems including error bounds being available in Banach or more general infinite-dimensional spaces (see [1]). Here, (separable) Hilbert spaces have the distinguished property of being the only infinite-dimensional Banach spaces for which convergence of such sums is equivalent to finite variances (square integrability of the norms) of the components.

In the non-linear case, where the sum is replaced by a general transformation, much less is known, except when the dimension of the Hilbert space is finite. In this case, Nourdin and Peccati ([30]) have introduced the very powerful combination of Stein's method and Malliavin calculus, which yields quantitative central limit theorems for a very wide class of square integrable real-valued transformations of arbitrary Gaussian processes. Since its inception, this approach, which is now known as the *Malliavin-Stein method*, has had a very substantial impact with numerous generalizations and applications. We refer to the monograph [31] for an overview.

In this paper, we lift the theory to infinite-dimension, thus obtaining quantitative central limit theorems for square-integrable and Hilbert-valued random variables. The setting we will be working in is that of a diffusive Markov generator L, acting on $L^2(\Omega; K)$, where K is a real separable Hilbert space. Our main result (see Section 3 for unexplained definitions and Theorem 3.2 for a precise statement) then states that for random variables F in the domain of the associated carré du champ operator Γ and centered, non-degenerate Gaussians Z on K with covariance operator S, one has

$$d(F, Z) \leq \frac{1}{2}\sqrt{\mathrm{E}\left(\|\Gamma(F, -L^{-1}F) - S\|_{\mathrm{HS}}^2\right)}. \tag{1.1}$$

Here, $\|\cdot\|_{\mathrm{HS}}$ denotes the Hilbert-Schmidt norm, L^{-1} the pseudo-inverse of the generator L and d is a probability metric generating a topology which is stronger than convergence in distribution.

Some examples of random variables F fitting our framework are homogeneous sums of i.i.d. Gaussians with Hilbert-valued coefficients (or more generally a polynomial chaos with distributions coming from a diffusion generator), stochastic integrals of the form $F_t = \int_0^\infty u_{t,s}\,\mathrm{d}B_s$, where B is Brownian motion and the kernel u is such that the trajectories of F are Hölder-continuous of order less than one half, or multiple Wiener-Itô integrals.

Proceeding from the general bound (1.1), we generalize and refine the two most important results of the finite-dimensional Malliavin-Stein framework:

The first results are quantifications of so-called *Fourth Moment Theorems* (first discovered in [39] and substantially generalized in [24, 2, 7]), which state that for a sequence of eigenfunctions of the carré du champ operator satisfying a chaotic property, convergence in distribution to a Gaussian is equivalent to convergence of the second and fourth moment. We prove that such quantitative Fourth Moment Theorems continue to hold in infinite-dimension, i.e., that if F is a chaotic eigenfunction of the carré du champ operator and Z is a Gaussian having the same covariance operator as F, then one has (see Section 3.2 for precise statements)

$$d(F, Z) \leq \frac{1}{2}\sqrt{\mathrm{E}\left(\|F\|^4\right)\left(\mathrm{E}\left(\|F\|^4\right) - \mathrm{E}\left(\|Z\|^4\right)\right)}. \tag{1.2}$$

The fact that the moment difference on the right-hand side is non-negative will follow from our analysis. In fact, we prove a more general version of (1.2) for K-valued random variables F which have a possibly infinite chaos expansion and whose covariance operator not necessarily coincides with the one of Z. Even in finite dimensions, such bounds are new.

The second type of results we obtain are so called *contraction bounds* in the special case where L is the Ornstein-Uhlenbeck generator. Here, the chaotic eigenfunctions of the generator become multiple Hilbert-valued Wiener-Itô integrals, so that, using a Hilbert-valued version of Malliavin calculus, their moments can be expressed in terms of contraction norms of their kernels (see (4.3) for a definition). Such contractions are important in applications due to their relatively straightforward computability. We provide bounds on $d(F, Z)$ (F and Z as in (1.1)) in terms of such contraction norms, again for random variables with possibly infinite chaos expansions. As an application, we explain how these bounds can be used to obtain rates of convergence in the functional Breuer-Major Theorem.

From a theoretical point of view, our results described above contain and extend, in a unified way, all previously established fourth moment theorems as well as carré du champ, fourth moment and contraction bounds for Gaussian approximation in a diffusive, finite-dimensional context (see [24, 2, 10, 7, 30, 33, 27, 36, 39]).

Furthermore, in the context of weak convergence of stochastic processes, our approach is an alternative to the usual method of proving finite-dimensional distribution convergence and tightness, with the advantage of yielding rates of convergence.

The existing literature on quantitative limit theorems in a non-linear and infinite-dimensional context is rather scarce. Barbour extended Stein's method to a functional setting in [4] for diffusion approximations by a Brownian motion. This has recently been applied and extended by Kasprzak in [18, 19, 20]. Coutin and Decreusefond ([12]) combined Stein's method with integration by parts techniques in a separable Hilbert space setting. While the general theme of this latter reference is similar to ours, the results are very different: the bounds in [12] are stated in terms of partial traces and require explicit evaluations of isometries as all calculations are done in $\ell^2(\mathbb{N})$; furthermore, no carré du champ, moment, or contraction bounds are provided.

The rest of the paper is organized as follows. In Section 2, after recalling basic notions of probability theory on Hilbert spaces (Subsection 2.1), we provide an outline of Stein's method on abstract Wiener spaces (Subsection 2.2) and introduce the Dirichlet structure framework we will be working in (Subsection 2.3). The main results are contained in Section 3: we start by proving the aforementioned general carré du champ bound (1.1) in Section 3.1 and then provide quantitative fourth moment theorems in Section 3.2. Section 4 is devoted to the special case of the Ornstein-Uhlenbeck generator. First, we present the associated Malliavin calculus in a Hilbert space setting (Subsection 4.1), which then leads to quantitative refined Fourth Moment and contraction bounds (Subsection 4.2). Finally, in Section 5, rates of convergence for the functional Breuer-Major theorem are established.

2 Preliminaries

2.1 Probability on Hilbert spaces

Let K be a separable real Hilbert space, $\mathcal{B}(K)$ the family of Borel sets of K and $(\Omega, \mathcal{F}, P)$ a complete probability space. A K-valued random variable X is a measurable map from $(\Omega, \mathcal{F})$ to $(K, \mathcal{B}(K))$. Such random variables are characterized by the property that for any continuous linear functional $\varphi \in K^*$, the function $\varphi(X) \colon \Omega \to \mathbb{R}$ is a (real-valued) random variable. As usual, the distribution or law of a random variable X is defined to be the push-forward probability measure $P \circ X^{-1}$ on $(K, \mathcal{B}(K))$. The set of all K-valued random variables is a vector space over the field of real numbers. If the Lebesgue integral $\mathrm{E}\left(\|X\|_K\right) = \int_\Omega \|X\|_K \,\mathrm{d}P$ exists and is finite, then the Bochner integral $\int_\Omega X \,\mathrm{d}P$ exists in K and is called the expectation of X. Slightly abusing notation, we denote this integral by $\mathrm{E}\left(X\right)$ as well. It will always be clear from the context if $\mathrm{E}\left(\cdot\right)$

denotes Lebesgue or Bochner integration with respect to P. For $p \geq 1$, we denote by $L^p(\Omega; K)$ the Banach space of all equivalence classes (under almost sure equality) of K-valued random variables X with finite p-th moment, i.e., such that

$$\|X\|_{L^p(\Omega,K)} = \left(\mathrm{E}\left(\|X\|_K^p\right)\right)^{1/p} < \infty.$$

Note that for all $X \in L^p(\Omega; K)$, the Bochner integral $\mathrm{E}\,(X) \in B$ exists. If $X \in L^2(\Omega; K)$, the covariance operator $S\colon K \to K$ of X is defined by

$$Su = \mathrm{E}\left(\langle X, u\rangle X\right). \tag{2.1}$$

It is a positive, self-adjoint trace-class operator and verifies the identity

$$\operatorname{tr} S = \mathrm{E}\left(\|X\|^2\right). \tag{2.2}$$

We denote by $\mathcal{S}_1(K)$ the Banach space of all trace class operators on K with norm $\|T\|_{\mathcal{S}_1(K)} = \operatorname{tr}|T|$, where $|T| = \sqrt{TT^*}$. The subspace of Hilbert-Schmidt operators will be denoted by $\mathrm{HS}(K)$, its inner product and associated norm by $\langle\cdot,\cdot\rangle_{\mathrm{HS}(K)}$ and $\|\cdot\|_{\mathrm{HS}(K)}$, respectively. Recall that

$$\|\cdot\|_{\mathrm{op}} \leq \|\cdot\|_{\mathrm{HS}(K)} \leq \|\cdot\|_{\mathcal{S}_1},$$

where $\|\cdot\|_{\mathrm{op}}$ denotes the operator norm.

When there is no ambiguity about what Hilbert space K underlies $\langle\cdot,\cdot\rangle_K$, $\|\cdot\|_K$, $\mathcal{S}_1(K)$ or $\mathrm{HS}(K)$, we will drop the K dependency and just write $\langle\cdot,\cdot\rangle$, $\|\cdot\|$, $\mathcal{S}_1$, HS, and so on.

2.2 Gaussian measures and Stein's method on abstract Wiener spaces

In this section, we introduce Gaussian measures, the associated abstract Wiener spaces and Stein's method. We present the theory in a Banach space setting as specializing to Hilbert spaces brings no significant advantages at this point. Standard references for Gaussian measures and abstract Wiener spaces are the books [5, 23], Stein's method on abstract Wiener space has been introduced by Shih in [41].

2.2.1 Abstract Wiener spaces

Let H be a real separable Hilbert space with inner product $\langle\cdot,\cdot\rangle_H$ and define a norm $\|\cdot\|$ on H (not necessarily induced by another inner product) that is weaker than $\|\cdot\|_H$. Denote by B the Banach space obtained as the completion of H with respect to the norm $\|\cdot\|$ (note that if the $\|\cdot\|$ norm happens to be induced by an inner product, then B is actually a Hilbert space), and define i to be the canonical embedding of H into B. Then, the triple (i, H, B) is called an abstract Wiener space. We identify B^* as a dense subspace of H^* under the adjoint operator i^* of i, so that we have the continuous embeddings $B^* \subset H \subset B$, where, as usual, H is identified with its dual. The abstract Wiener measure p on B is characterized as the Borel measure on B satisfying

$$\int_B e^{\mathrm{i}\langle x,\eta\rangle_{B,B^*}} p(dx) = e^{-\frac{\|\eta\|_H^2}{2}},$$

for any $\eta \in B^*$, where $\langle\cdot,\cdot\rangle_{B,B^*}$ denotes the dual pairing in B.

2.2.2 Gaussian measures on Banach and Hilbert spaces

For a Banach space B, we denote by $\mathcal{B}(B)$ its family of Borel sets.

Definition 2.1. *Let B be a real separable Banach space. A Gaussian measure ν is a probability measure on $(B, \mathcal{B}(B))$, such that every linear functional $x \in B^*$, considered as a random variable on $(B, \mathcal{B}(B), \nu)$, has a Gaussian distribution (on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$). The Gaussian measure ν is called* centered *(or* non-degenerate*), if these properties hold for the distributions of every $x \in B^*$.*

We see from the definition that every abstract Wiener measure is a Gaussian measure and, conversely, for any Gaussian measure ν on a separable Banach space B, there exists a Hilbert space H such that the triple (i, H, B) is an abstract Wiener space with associated abstract Wiener measure ν (see [22, Lemma 2.1]). The space H is called the Cameron-Martin space.

2.2.3 Stein characterization of abstract Wiener measures

Let B be real separable Banach space with norm $\|\cdot\|$ and let Z be a B-valued random variable on some probability space $(\Omega, \mathcal{F}, P)$ such that the distribution μ_Z of Z is a non-degenerate Gaussian measure on B with zero mean. Let (i, H, B) be the abstract Wiener space associated to the Wiener measure μ_Z, as described in the previous subsection. Let $\{P_t \colon t \geq 0\}$ denote the Ornstein-Uhlenbeck semigroup associated with μ_Z and defined, for any $\mathcal{B}(B)$-measurable function f and $x \in B$, by

$$P_t f(x) = \int_B f\left(e^{-t}x + \sqrt{1 - e^{-2t}}y\right) \mu_Z(dy), \quad t \geq 0,$$

provided such an integral exists. We have the following Stein lemma for abstract Wiener measures (see [41, Theorem 3.1]).

Theorem 2.2. *Let X be a B-valued random variable with distribution μ_X.*

(i) If B is finite-dimensional, then $\mu_X = \mu_Z$ if and only if

$$\mathrm{E}\left[\langle X, \nabla f(X)\rangle_{B,B^*} - \Delta_G f(X)\right] = 0 \tag{2.3}$$

for any twice differentiable function f on B such that $\mathrm{E}\left[\left\|\nabla^2 f(Z)\right\|_{\mathcal{S}_1(H)}\right] < \infty$.

(ii) If B is infinite-dimensional, then $\mu_X = \mu_Z$ if and only if (2.3) *holds for any twice H-differentiable function f on B such that $\nabla f(x) \in B^*$ for any $x \in B$, $\mathrm{E}\left[\left\|\nabla^2 f(Z)\right\|_{\mathcal{S}_1(H)}\right] < \infty$ and $\mathrm{E}\left[\|\nabla f(Z)\|^2_{B^*}\right] < \infty$.*

The notion of an H-derivative appearing in Theorem 2.2 was introduced by Gross in [16] and is defined as follows. A function $f \colon U \to W$ from an open set U of B into a Banach space W is said to be H-differentiable at a point $x \in U$ if the map $\phi(h) = f(x+h)$, $h \in H$, regarded as a function defined in a neighborhood of the origin of H is Fréchet differentiable at 0. The Fréchet derivative $\phi'(0)$ at $0 \in H$ is called the H-derivative of f at $x \in B$. The H-derivative of f at x in the direction $h \in H$ is denoted by $\langle \nabla f(x), h\rangle_H$. The k-th order H-derivatives of f at x can be defined inductively and are denoted by $\nabla^k f(x)$ for $k \geq 2$, provided they exist. If f is scalar-valued, $\nabla f(x) \in H^* \approx H$ and $\nabla^2 f(x)$ is regarded as a bounded linear operator from H into H^* for any $x \in U$, and the notation $\left\langle \nabla^2 f(x)h, k\right\rangle_H$ stands for the action of the linear form $\nabla^2 f(x)(h, \cdot)$, $h \in H$, on $k \in H$, denoted by $\nabla^2 f(x)(h, k)$. Furthermore, if $\nabla^2 f(x)$ is a trace-class operator on H, we can define the so-called Gross Laplacian $\Delta_G f(x)$ of f at x appearing in (2.3) by $\Delta_G f(x) = \mathrm{tr}_H(\nabla^2 f(x))$.

Remark 2.3 (On the relation between Fréchet and H-derivatives)**.** An H-derivative $\nabla f(x)$ at $x \in B$ determines an element in B^* if there is a constant $C > 0$ such that

$|\langle \nabla f(x), h \rangle_H| \leq C \|h\|_H$ for any $h \in H$. Then, $\nabla f(x)$ defines an element of B^* by continuity and we denote this element by $\nabla f(x)$ as well. Now, if f is also twice Fréchet differentiable on B, then $\nabla f(x)$ coincides with the first-order Fréchet derivative $f'(x)$ at $x \in B$ and is automatically in B^*. Furthermore, $\nabla^2 f(x)$ coincides with the restriction of the second-order Fréchet derivative $f''(x)$ to $H \times H$ at $x \in B$. In this framework, since for any $x \in B$, $f''(x)$ is a bounded linear operator from B into B^*, Goodman's theorem (see [23, Chapter 1, Theorem 4.6]) implies that $\nabla^2 f(x)$ is a trace-class operator on H and that, consequently, the Gross Laplacian $\Delta_G f(x)$ is well-defined. Twice Fréchet differentiability hence constitutes a sufficient condition for the existence of the Gross Laplacian.

2.2.4 Stein's equation and its solutions for abstract Wiener measures

In view of the above Stein lemma (Theorem 2.2), the associated Stein equation is given by

$$\Delta_G f(x) - \langle x, \nabla f(x) \rangle_{B,B^*} = h(x) - \mathrm{E}[h(Z)], \qquad x \in B, \tag{2.4}$$

where h is given in some class of test functionals. Shih showed in [41] that

$$f_h(x) = -\int_0^\infty (P_u h(x) - \mathrm{E}[h(Z)])\, du, \qquad x \in B \tag{2.5}$$

solves the Stein equation (2.4) whenever h is an element of $\mathrm{ULip}\text{-}1(B)$, the Banach space of scalar-valued uniformly 1-Lipschitz functions h on B with the norm $\|h\| = \|h\|_{\mathrm{ULip}} + |h(0)|$, where

$$\|h\|_{\mathrm{ULip}} = \sup_{x \neq y \in B} \frac{|h(x) - h(y)|}{\|x - y\|} < \infty.$$

In what follows, we will consider test functions from the space $C_b^k(K)$ of real-valued, k-times Fréchet differentiable functions on a separable Hilbert space K with bounded derivatives up to order k. A function h thus belongs to $C_b^k(K)$ whenever

$$\|h\|_{C_b^k(K)} = \sup_{j=1,\dots,k} \sup_{x \in K} \left\| \nabla^j h(x) \right\|_{K^{\otimes j}} < \infty.$$

The following lemma collects some properties of the Stein solution f_h for a given function $h \in C_b^k(K)$.

Lemma 2.4. *Let K be a separable Hilbert space, $k \geq 1$ and $h \in C_b^k(K)$. Then the Stein solution f_h defined in* (2.5) *also belongs to $C_b^k(K)$ and furthermore one has that*

$$\sup_{u \in K} \left\| \nabla^j f_h(u) \right\|_{K^{\otimes j}} \leq \frac{1}{j} \|h\|_{C_b^j(K)}, \qquad j \in \mathbb{N}, \qquad j \leq k. \tag{2.6}$$

Proof. As for any $x \in K$, $f_h(x) = -\int_0^\infty (P_u h(x) - \mathrm{E}[h(Z)])\, du$, we have, for any $j = 1, \dots, k$,

$$\nabla^j f_h(x) = -\int_0^\infty \nabla^j P_u h(x) du,$$

so that

$$\begin{aligned} \|f_h\|_{C_b^k(K)} &= \sup_{j=1,\dots,k} \sup_{x \in K} \left\| -\int_0^\infty \nabla^j P_u h(x) du \right\|_{K^{\otimes j}} \\ &\leq \sup_{j=1,\dots,k} \sup_{x \in K} \int_0^\infty \left\| \nabla^j P_u h(x) \right\|_{K^{\otimes j}} du. \end{aligned}$$

Using the property of the semigroup P that $\nabla^j P_u h(x) = e^{-ju} P_u \nabla^j h(x)$, and the fact that P is contractive yields

$$\begin{aligned}
\|f_h\|_{C_b^k(K)} &\leq \sup_{j=1,\dots,k} \sup_{x\in K} \int_0^\infty e^{-ju} \left\| P_u \nabla^j h(x) \right\|_{K^{\otimes j}} du \\
&\leq \sup_{j=1,\dots,k} \sup_{x\in K} \int_0^\infty e^{-ju} \left\| \nabla^j h(x) \right\|_{K^{\otimes j}} du \\
&= \sup_{j=1,\dots,k} \sup_{x\in K} \frac{1}{j} \left\| \nabla^j h(x) \right\|_{K^{\otimes j}} \\
&\leq \|h\|_{C_b^k(K)} < \infty,
\end{aligned}$$

proving that $f_h \in C_b^k(K)$. The bound (2.6) can be derived similarly. □

2.3 Dirichlet structures

In this section, a Dirichlet structure for Hilbert-valued random variables is introduced, which will be the framework we work in. We start by recalling the well-known definition in the case of real-valued random variables (full details can for example be found in [6, 14, 26, 3], where the latter reference emphasizes the equivalent notion of a Markov triple). Given a probability space $(\Omega, \mathcal{F}, P)$, a Dirichlet structure $(\mathbb{D}, \mathcal{E})$ on $L^2(\Omega; \mathbb{R})$ with associated carré du champ operator Γ consists of a Dirichlet domain $\mathbb{D}$, which is a dense subset of $L^2(\Omega; \mathbb{R})$ and a carré du champ operator $\Gamma \colon \mathbb{D} \times \mathbb{D} \to L^1(\Omega; \mathbb{R})$ characterized by the following properties.

– Γ is bilinear, symmetric ($\Gamma(f, g) = \Gamma(g, f)$) and positive $\Gamma(f, f) \geq 0$,
– for all $m, n \in \mathbb{N}$, all Lipschitz and continuously differentiable functions $\varphi \colon \mathbb{R}^m \to \mathbb{R}$ and $\psi \colon \mathbb{R}^n \to \mathbb{R}$ and all $f = (f_1, \dots, f_m) \in \mathbb{D}^m$, $g = (g_1, \dots, g_n) \in \mathbb{D}^n$, it holds that

$$\Gamma(\varphi(f), \psi(g)) = \sum_{i=1}^m \sum_{j=1}^n \partial_i \varphi(f) \partial_j \psi(g) \Gamma(f_i, g_j), \tag{2.7}$$

– the induced positive linear form $f \mapsto \mathcal{E}(f, f)$, where

$$\mathcal{E}(f, g) = \frac{1}{2} \operatorname{E}(\Gamma(f, g))$$

is closed in $L^2(\Omega; \mathbb{R})$, i.e., $\mathbb{D}$ is complete when equipped with the norm

$$\|\cdot\|_{\mathbb{D}}^2 = \|\cdot\|_{L^2(\Omega;\mathbb{R})}^2 + \mathcal{E}(\cdot).$$

Here and in the following, $\operatorname{E}(\cdot)$ denotes expectation on $(\Omega, \mathcal{F})$ with respect to P. The form $f \to \mathcal{E}(f, f)$ is called a Dirichlet form, and, as is customary, we will write $\mathcal{E}(f)$ for $\mathcal{E}(f, f)$. Every Dirichlet form gives rise to a strongly continuous semigroup $\{P_t \colon t \geq 0\}$ on $L^2(\Omega; \mathbb{R})$ and an associated symmetric Markov generator $-L$, defined on a dense subset $\operatorname{dom}(-L) \subseteq \mathbb{D}$. We will often switch between $-L$ and L, as these two operators only differ by sign. There are two important relations between Γ and L. The first one is the integration by parts formula

$$\operatorname{E}(\Gamma(f, g)) = -\operatorname{E}(fLg) = -\operatorname{E}(gLf), \tag{2.8}$$

valid whenever $f, g \in \mathbb{D}$, the second one is the relation

$$\Gamma(f, g) = \frac{1}{2}\left(L(fg) - gLf - fLg\right),$$

which holds for all $f, g \in \operatorname{dom}(L)$ such that $fg \in \operatorname{dom}(L)$. If $-L$ is diagonalizable with spectrum $\{0 = \lambda_0 < \lambda_1 < \dots\}$, a pseudoinverse $-L^{-1}$ can be introduced via spectral calculus as follows: if $f = \sum_{i=0}^{\infty} f_i$ with $f_i \in \ker(L + \lambda_i \operatorname{Id})$, then

$$-L^{-1} f = \sum_{i=1}^{\infty} \frac{1}{\lambda_i} f_i.$$

It follows that

$$LL^{-1} f = f - \mathrm{E}(f).$$

Consider now such a Dirichlet structure on $L^2(\Omega; \mathbb{R})$ with diagonalizable generator as given and denote the Dirichlet domain, Dirichlet form, carré du champ operator, its associated infinitesimal generator and pseudo-inverse by $\widetilde{\mathbb{D}}$, $\widetilde{\mathcal{E}}$, $\widetilde{\Gamma}$, $\widetilde{L}$ and $\widetilde{L}^{-1}$, respectively, in order to distinguish these objects from their extensions to the Hilbert-valued setting to be introduced below.

Given a separable Hilbert space K, one has that $L^2(\Omega; K)$ is isomorphic to $L^2(\Omega; \mathbb{R}) \otimes K$. The Dirichlet structure on $L^2(\Omega; \mathbb{R})$ can therefore be extended to $L^2(\Omega; K)$ via a tensorization procedure as follows.

Let $\{0 = \lambda_0 < \lambda_1 < \dots\}$ be the spectrum of $-\widetilde{L}$ and $\mathcal{A}$ the set of all functions F of the form

$$F = \sum_{(i,j) \in I} f_{p_i} \otimes k_j, \tag{2.9}$$

where $I \subseteq \mathbb{N} \times \mathbb{N}$ is finite, the f_{p_i} are eigenfunctions of $-\widetilde{L}$ with eigenvalue $\lambda_{p_i} \geq 0$ and the k_j form an orthonormal basis in K. Then $\mathcal{A}$ is dense in $L^2(\Omega; K)$. For $F \in \mathcal{A}$ of the form (2.9) and analogously $G = \sum_{(i',j') \in I'} f_{p_{i'}} \times k_{j'} \in \mathcal{A}$, define linear operators L, L^{-1} by

$$\begin{aligned} LF &= \sum_{(i,j) \in I} (\widetilde{L} f_{p_i}) \otimes k_j = -\sum_{(i,j) \in I} \lambda_{p_i} f_{p_i} \otimes k_j, \\ L^{-1}(F) &= \sum_{(i,j) \in I} (\widetilde{L}^{-1} f_{p_i}) \otimes k_j = -\sum_{\substack{(i,j) \in I \\ p_i \neq 0}} \frac{1}{\lambda_{p_i}} f_{p_i} \otimes k_j, \end{aligned}$$

a bilinear and positive operator Γ by

$$\Gamma(F, G) = \frac{1}{2} \sum_{(i,j) \in I} \sum_{(i',j') \in I'} \widetilde{\Gamma}(f_{p_i}, f_{p_{i'}}) \otimes (k_j \otimes k_{j'} + k_{j'} \otimes k_j)$$

and a bilinear, positive and symmetric form $\mathcal{E}$ by

$$\mathcal{E}(F, G) = \mathrm{E}(\operatorname{tr} \Gamma(F, G)),$$

where in the definition of $\mathcal{E}$ we identify $\Gamma(F, G) \in L^2(\Omega; \mathbb{R}) \otimes K \otimes K \simeq L^2(\Omega; \mathcal{L}(K, K))$ with a random operator on K, whose action is given by

$$\Gamma(F, G) u = \frac{1}{2} \sum_{(i,j) \in I} \sum_{(i',j') \in I'} \widetilde{\Gamma}(f_{p_i}, f_{p_{i'}}) (\langle k_j, u \rangle \otimes k_{j'} + \langle k_{j'}, u \rangle \otimes k_j), \quad u \in K.$$

For all $F, G \in \mathcal{A}$, the operator $\Gamma(F, G)$ is then of trace class and an element of $L^1(\Omega; \mathcal{S}_1)$. It is standard to verify that the definitions of L, L^{-1} and Γ do not depend on the choice of the basis of K. Furthermore, from the well-known results for $\widetilde{L}$, $\widetilde{\Gamma}$ and $\widetilde{\mathcal{E}}$, we can extend them as follows.

Proposition 2.5. *The operators L, L^{-1}, $\mathcal{E}$ and Γ introduced above can be extended to* $\operatorname{dom}(L)$, $\operatorname{dom}(L^{-1})$ *and* $\operatorname{dom}(\Gamma) = \operatorname{dom}(\mathcal{E}) = \mathbb{D} \times \mathbb{D}$*, given by*

$$\operatorname{dom}(L) = \left\{ F \in L^2(\Omega; K) \colon \sum_{p=1}^{\infty} \lambda_p^2 \, \widetilde{\pi}_p \left(\|F\|^2 \right) < \infty \right\}$$
$$\operatorname{dom}(L^{-1}) = L^2(\Omega; K)$$

and

$$\mathbb{D} = \left\{ F \in L^2(\Omega; K) \colon \sum_{p=1}^{\infty} \lambda_p \, \widetilde{\pi}_p \left(\|F\|^2 \right) < \infty \right\},$$

respectively, where $\widetilde{\pi}_p$ denotes the orthogonal projection onto

$$\ker(\widetilde{L} + \lambda_p \operatorname{Id}) \subseteq L^2(\Omega; \mathbb{R}).$$

In particular, one has

$$\mathcal{A} \subseteq \operatorname{dom}(L) \subseteq \mathbb{D} \subseteq \operatorname{dom}(L^{-1}) = L^2(\Omega; K),$$

where all inclusions are dense.

Throughout this article, the extensions of L, L^{-1} and Γ to their maximal domains will still be denoted by the same symbols. The operators just defined yield a Dirichlet structure $(\Gamma, \mathbb{D})$ on $L^2(\Omega; K)$, which is a natural counterpart to the given structure $(\widetilde{\Gamma}, \widetilde{\mathbb{D}})$ on $L^2(\Omega; \mathbb{R})$. The following theorem summarizes its main features.

Theorem 2.6. *For a Dirichlet structure $(\mathbb{D}, \Gamma)$ on $L^2(\Omega; K)$, consisting of a dense subspace $\mathbb{D}$ of $L^2(\Omega; K)$ and a carré du champ operator $\Gamma \colon \mathbb{D} \times \mathbb{D} \to L^1(\Omega; \mathcal{S}_1)$ as introduced above, the following is true.*

(i) *Γ is bilinear, almost surely positive (i.e., $\Gamma(F, F) \geq 0$ as an operator on K), symmetric in its arguments and self-adjoint ($\langle \Gamma(F, G) u, v \rangle = \langle u, \Gamma(F, G) v \rangle$ for all $u, v \in K$).*

(ii) *The Dirichlet domain $\mathbb{D}$, endowed with the norm*

$$\|F\|_{\mathbb{D}}^2 = \|F\|_{L^2(\Omega; K)}^2 + \|\Gamma(F, F)\|_{L^1(\Omega; \mathcal{S}_1)}$$

is complete, so that Γ is closed.

(iii) *For all Lipschitz and Fréchet differentiable operators φ, ψ on K and $F, G \in \mathbb{D}$, one has that $\varphi(F), \psi(G) \in \mathbb{D}$ and the diffusion identity*

$$\Gamma(\varphi(F), \psi(G)) = \frac{1}{2} \big(\nabla \varphi(F)^* \Gamma(F, G) \nabla \psi(G) + \nabla \psi(G)^* \Gamma(F, G) \nabla \varphi(F) \big) \tag{2.10}$$

holds, where $\nabla \varphi(F)$ and $\nabla \psi(G)$ denote the Fréchet derivatives of φ and ψ at F and G, respectively, and $\nabla \varphi(F)^*$, $\nabla \psi(G)^*$ are their adjoints in K.

(iv) *The associated generator $-L$ acting on $L^2(\Omega; K)$ is positive, symmetric, densely defined and has the same spectrum as $-\widetilde{L}$.*

(v) *There exists a compact pseudo-inverse L^{-1} of L such that*

$$L L^{-1} F = F - \operatorname{E}(F)$$

for all $F \in L^2(\Omega; K)$, where the expectation on the right is a Bochner integral (well defined as $F \in L^2(\Omega; K)$).

(vi) The integration by parts formula

$$\mathrm{E}\left(\operatorname{tr}\Gamma(F,G)\right) = -\,\mathrm{E}\left(\langle LF, G\rangle\right) = -\,\mathrm{E}\left(\langle F, LG\rangle\right) \tag{2.11}$$

is satisfied for all $F, G \in \operatorname{dom}(-L)$.

(vii) The carré du champ Γ *and the generators* L *and* $\widetilde{L}$ *are connected through the identity*

$$\operatorname{tr}\Gamma(F,G) = \frac{1}{2}\left(\widetilde{L}\,\langle F, G\rangle - \langle LF, G\rangle - \langle F, LG\rangle\right),$$

valid for $F, G \in \operatorname{dom}(L)$.

(viii) The fundamental identity

$$\langle \Gamma(F,G)u, v\rangle = \frac{1}{2}\left(\widetilde{\Gamma}\left(\langle F, u\rangle, \langle G, v\rangle\right) + \widetilde{\Gamma}\left(\langle G, u\rangle, \langle F, v\rangle\right)\right), \tag{2.12}$$

connecting Γ *and its one-dimensional counterpart* $\widetilde{\Gamma}$ *is valid for all* $F, G \in \mathbb{D}$ *and all* $u, v \in K$.

Proof. Parts $(i) - (ii)$ and $(iv) - (viii)$ are straightforward to verify. In order to prove (iii), write

$$F = \sum_{p=0}^{\infty}\sum_{i=1}^{\infty} f_p \otimes k_i \qquad \text{and} \qquad G = \sum_{p=0}^{\infty}\sum_{i=1}^{\infty} g_p \otimes k_i,$$

where the f_p and g_p are eigenfunctions of $\widetilde{L}$ with eigenvalue $-\lambda_p$, and $\{k_i \colon i \in \mathbb{N}\}$ is an orthonormal basis of K. Let $K_n = \operatorname{span}\{k_i \colon 1 \leq i \leq n\}$ and ρ_n be the orthogonal projection onto $L^2(\Omega; K_n)$, so that

$$\rho_n(F) = \sum_{p=0}^{\infty}\sum_{i=1}^{n} f_p \otimes k_i \qquad \text{and} \qquad \rho_n(G) = \sum_{p=0}^{\infty}\sum_{i=1}^{n} g_p \otimes k_i.$$

Denote by $\mathfrak{i}_n \colon K_n \to \mathbb{R}^n$ the canonical isometric isomorphism mapping K_n to $\mathbb{R}^n$ so that $\xi_n = \mathfrak{i}_n \circ \rho_n(F) \in \mathbb{R}^n$ and $\upsilon_n = \mathfrak{i}_n \circ \rho_n(G) \in \mathbb{R}^n$.

Let $\widetilde{\varphi}_n = \varphi \circ \mathfrak{i}_n^{-1}$ and $\widetilde{\psi}_n = \psi \circ \mathfrak{i}_n^{-1}$. Then $\widetilde{\varphi}_n \colon \mathbb{R}^n \to K$ is Lipschitz and Fréchet differentiable, with Fréchet derivative given by

$$\nabla\widetilde{\varphi}_n(x)(y) = \nabla\varphi(\mathfrak{i}_n^{-1}(x))(\mathfrak{i}_n^{-1}(y))$$

for all $x, y \in \mathbb{R}^n$ and an analogous result is true for $\nabla\widetilde{\psi}_n$. Therefore, via

$$\Gamma\left(\varphi(\rho_n(F)), \psi(\rho_n(G))\right) = \Gamma\left(\widetilde{\varphi}_n(\xi_n), \widetilde{\psi}_n(\upsilon_n)\right)$$

and identity (2.12), the assertion can be transformed into an equivalent assertion for $\widetilde{\Gamma}$, which can then be verified by tedious but straightforward calculations, using the diffusion property (2.7) for $\widetilde{\Gamma}$ and then letting $n \to \infty$. □

The most important example in our context is the Dirichlet structure given by the Ornstein-Uhlenbeck generator of a Hilbert-valued Ornstein-Uhlenbeck semigroup. Here, $-L = \delta D$, where D and δ denote the Malliavin derivative and divergence operator, and the carré du champ operator is given by $\Gamma(X,Y) = \langle DX, DY\rangle_{\mathfrak{H}}$, where $\mathfrak{H}$ is the Hilbert space associated to the underlying isonormal Gaussian process (see Section 4 for full details). The corresponding eigenspaces are known as Wiener chaos and spanned by the infinite-dimensional Hermite polynomials. In the same way, one can obtain Jacobi, Laguerre or other polynomial chaoses (see for example [2] for the real-valued case). We refer to the monographs quoted at the beginning of this section for further numerous examples.

3 Approximation of Hilbert-valued Gaussians

In this section, we combine Stein's method introduced in Section 2.2 with the Dirichlet structure defined in Section 2.3 in order to derive bounds on a probabilistic metric between the laws of square integrable random variables and a Gaussian, both taking values in some separable Hilbert space.

Throughout the whole section, this separable Hilbert space will be denoted by K, and we furthermore assume as given a Dirichlet structure on $L^2(\Omega; K)$ as introduced in the previous section, with Dirichlet domain $\mathbb{D}$, carré du champ operator Γ and associated generator L.

The probabilistic distance we use is the well-known d_2-metric, given by

$$d_2(F, G) = \sup_{\substack{h \in C_b^2(K) \\ \|h\|_{C_b^2(K)} \leq 1}} \left| \mathrm{E}\left(h(F)\right) - \mathrm{E}\left(h(G)\right) \right|, \tag{3.1}$$

where $C_b^2(K)$ denotes the twice Fréchet differentiable, real-valued functions on K with uniformly bounded first and second derivatives (see Section 2.2.4). In an infinite-dimensional context, this distance has already been used in [12] and, in a weakened form, also in [4]. As already observed in [12], it metrizes convergence in distribution:

Lemma 3.1. *If $\{F_n : n \in \mathbb{N}_0\}$ is a sequence of K-valued random variables such that*

$$d_2(F_n, F_0) \to 0$$

as $n \to \infty$, then the law of F_n converges in distribution to the law of F_0, i.e.,

$$\mathrm{E}\left(h(F_n)\right) \to \mathrm{E}\left(h(F_0)\right)$$

as $n \to \infty$, for all bounded, real-valued and continuous functions h on K.

Proof. The proof given in [12, Lemma 4.1] for $K = \ell^2(\mathbb{N})$ continues to work without any modification. □

3.1 An abstract carré du champ bound

The following general bound between the laws of a square integrable K-valued random variable in the Dirichlet domain $\mathbb{D}$ and an arbitrary Gaussian random variable holds.

Theorem 3.2. *Let Z be a centered, non-degenerate Gaussian random variable on K with covariance operator S and let $F \in \mathbb{D}$. Then*

$$d_2(F, Z) \leq \frac{1}{2} \left\| \Gamma(F, -L^{-1}F) - S \right\|_{L^2(\Omega; \mathrm{HS})}. \tag{3.2}$$

If K has dimension $d < \infty$, then

$$d_W(F, Z) \leq C_{S,d} \left\| \Gamma(F, -L^{-1}F) - S \right\|_{L^2(\Omega; \mathrm{HS})}, \tag{3.3}$$

where d_W denotes the Wasserstein distance, and

$$C_{S,d} = \sqrt{d \left\| S \right\|_{\mathrm{op}}} \left\| S^{-1} \right\|_{\mathrm{op}}.$$

Proof. To prove (3.2), it suffices to show that for $h \in C_b^2(K)$ one has

$$\left| \mathrm{E}\left(\operatorname{tr}_H \nabla^2 f_h(F) - \left(F, \nabla f_h(F)\right)_{K,K^*} \right) \right| \\ \leq \frac{1}{2} \left\| \nabla^2 h \right\|_{C_b^2(K)} \left\| \Gamma(F, -L^{-1}F) - S \right\|_{L^2(\Omega; \mathrm{HS})}, \tag{3.4}$$

where f_h is the Stein solution given by (2.5). Indeed, using Stein's equation (see (2.4)), the left hand side of (3.4) is equal to $|\mathrm{E}(h(X)) - \mathrm{E}(h(Z))|$, so that the assertion follows after taking the supremum over h.

Identifying K^* with K, using the integration by parts formula (2.11) and the diffusion property (2.10) for the carré du champ, we can write

$$\begin{aligned}\mathrm{E}\left((F, \nabla f_h(F))_{K,K^*}\right) &= \mathrm{E}\left(\langle F, \nabla f_h(F)\rangle_K\right) \\ &= \mathrm{E}\left(\left\langle LL^{-1}F, \nabla f_h(F)\right\rangle_K\right) \\ &= \mathrm{E}\left(\mathrm{tr}_K\, \Gamma\left(\nabla f_h(F), -L^{-1}F\right)\right) \\ &= \mathrm{E}\left(\mathrm{tr}_K\left(\nabla^2 f_h(F)\Gamma(F, -L^{-1}F)\right)\right).\end{aligned}$$

Now let H be the Cameron-Martin space associated to Z as introduced in Section 2.2. As the covariance operator S of Z is compact and one-to-one, it holds that $S = \sum_{i\in\mathbb{N}} \lambda_i \langle \cdot, e_k\rangle_K e_i$ for some $\lambda_i > 0$ and an orthonormal basis $\{e_i : i \in \mathbb{N}\}$ of H consisting of eigenvectors. Then $\{k_i : i \in \mathbb{N}\}$, where $k_i = \frac{1}{\sqrt{\lambda_i}} e_i$, is an orthonormal basis of K, as $H = \sqrt{S}(K)$. It thus follows that

$$\mathrm{tr}_H\, \nabla^2 f_h(F) = \sum_{i\in\mathbb{N}} \nabla^2 f_h(F)(e_i, e_i) = \sum_{i\in\mathbb{N}} \nabla^2 f_h(F)\,(S\,k_i, k_i) = \mathrm{tr}_K\left(\nabla^2 f_h(F) S\right).$$

Combining the last two calculations yields that

$$\mathrm{E}\left(\langle F, \nabla f_h(F)\rangle_{K,K^*} - \mathrm{tr}_H\, \nabla^2 f_h(F)\right) = \mathrm{E}\left(\mathrm{tr}_K\left(\nabla^2 f_h(F)\left(\Gamma(F, -L^{-1}F) - S\right)\right)\right),$$

and, taking absolute values and applying Hölder's inequality for the Schatten norms, we get

$$\begin{aligned}&\left|\mathrm{E}\left(\langle F, \nabla f_h(F)\rangle_{K,K^*} - \mathrm{tr}_H\, \nabla^2 f_h(F)\right)\right| \\ &\qquad = \left|\mathrm{E}\left(\mathrm{tr}_K\left(\nabla^2 f_h(F)\left(\Gamma(F, -L^{-1}F) - S\right)\right)\right)\right| \\ &\qquad \le \mathrm{E}\left(\mathrm{tr}_K\left|\nabla^2 f_h(F)\left(\Gamma(F, -L^{-1}F) - S\right)\right|\right) \\ &\qquad \le \mathrm{E}\left(\left\|\nabla^2 f_h(F)\right\|_{\mathrm{HS}(K)} \left\|\Gamma(F, -L^{-1}F) - S\right\|_{\mathrm{HS}(K)}\right) \\ &\qquad \le \left\|\nabla^2 f_h(F)\right\|_{L^2(\Omega;\mathrm{HS}(K))} \left\|\Gamma(F, -L^{-1}F) - S\right\|_{L^2(\Omega;\mathrm{HS}(K))}.\end{aligned} \tag{3.5}$$

An application of Lemma 2.4 now yields (3.4), finishing the proof of (3.2). If K has finite-dimension d, one can proceed as in [31, Proposition 4.3.2] to obtain that

$$\left\|\nabla^2 f_h(F)\right\|_{L^2(\Omega;\mathrm{HS}(K))} \le C_{S,d}\, \|h\|_{\mathrm{Lip}},$$

where $\|\cdot\|_{\mathrm{Lip}}$ denotes the Lipschitz norm. The Wasserstein distance is then obtained by approximating Lipschitz functions in $C_b^2(K)$ (for example by convoluting a Gaussian kernel). □

If Z is a K-valued Gaussian random variable with covariance operator S, then, taking L to be the Ornstein-Uhlenbeck generator (see the forthcoming Section 4), one has that $\Gamma(Z, -L^{-1}Z) = S$. Therefore, taking F to be Gaussian in Theorem 3.2 yields a bound on the distance between two Gaussians Z_1, Z_2 in terms of the Hilbert-Schmidt norm of their covariance operators S_1, S_2. We state this as a corollary.

Corollary 3.3. *Let Z_1, Z_2 be two centered, non-degenerate Gaussian random variables on K with covariance operators S_1, S_2, respectively. Then, it holds that*

$$d_2(Z_1, Z_2) \le \frac{1}{2}\, \|S_1 - S_2\|_{\mathrm{HS}}.$$

We continue with some remarks on Theorem 3.2.

Remark 3.4.

(i) Note that the proof of Theorem 3.2 does not use diagonalizability of L, so that this assumption can be replaced by weaker conditions guaranteeing that a pseudo-inverse can still be defined (in a finite-dimensional context, this has been done in [7]). However, we will not need this level of generality.

(ii) While $\left\|\Gamma(F,-L^{-1}F)-S\right\|_{\mathrm{HS}}$ is almost surely finite for any $F \in \mathbb{D}$, it might be that $\left\|\Gamma(F,-L^{-1}F)-S\right\|_{L^2(\Omega;\mathrm{HS})}$ is infinite. A simple sufficient condition for finiteness of the latter norm is that F has finite chaos decomposition (see Section 3.2). In the case of an infinite decomposition, some control on the tail is needed.

(iii) In principle, Theorem 3.2 can also be used to prove weak convergence in a Banach space setting. Starting from a Gaussian random variable on a separable Banach space B, it is always possible (see [22, Lemma 2.1]) to densely embed B in a separable Hilbert space K such that the Borel sets of B are generated by the inner product of K. Then, by applying our methods, one obtains weak convergence in K, which in turn implies weak convergence in B.

3.2 Fourth Moment bounds via chaos expansions

In this section, we show how the carré du champ bounds obtained in Theorem 3.2 can be further estimated by the first four moments of the approximating random variable or sequence. For this, we need to assume that the generator satisfies the following, generalized version of an abstract polynomial chaos property first stated in [2] for the finite-dimensional case.

Definition 3.5. *Denote by $\widetilde{L}$ the one-dimensional counterpart of L as introduced in Section 2.3 and recall that L and $\widetilde{L}$ have the same spectrum. Let λ, η be two of their common eigenvalues. Two eigenvectors $F \in \ker(L+\lambda\,\mathrm{Id})$ and $G \in \ker(L+\eta\,\mathrm{Id})$ are called* jointly chaotic*, if*

$$\langle F,G\rangle_K \in \bigoplus_{\substack{\alpha\in\Lambda\\ \alpha\leq\lambda+\eta}} \ker\left(\widetilde{L}+\alpha\,\mathrm{Id}\right),$$

where Λ denotes the spectrum of L. An eigenvector $F \in \ker(L+\lambda\,\mathrm{Id})$ is called chaotic *if it is jointly chaotic with itself, i.e., if*

$$\|F\|_K^2 \in \bigoplus_{\substack{\alpha\in\Lambda\\ \alpha\leq 2\lambda}} \ker\left(\widetilde{L}+\alpha\,\mathrm{Id}\right).$$

The generator L is called chaotic, if any two of its eigenfunctions are jointly chaotic.

Prime examples of chaotic generators are those whose eigenspaces consist of (closures of) multivariate polynomials, such as the Hilbert-valued Ornstein-Uhlenbeck generator, Laguerre or Jacobi generators, in finite or infinite dimension. The Ornstein-Uhlenbeck case will be covered in depth in Section 4, precise definitions for the other two generators can for example be found in [2].

We will also make use of the following covariance condition.

Definition 3.6. *A random variable $F \in L^2(\Omega;K)$ is said to satisfy the* covariance condition *if it holds that*

$$2\,\mathrm{Cov}\left(\langle F,u\rangle,\langle F,v\rangle\right)^2 \leq \mathrm{Cov}\left(\langle F,u\rangle^2,\langle F,v\rangle^2\right) \tag{3.6}$$

for any two orthonormal vectors $u,v \in K$.

It will be proved later that both the covariance condition and the chaotic property is satisfied whenever F is an eigenfunction of the Ornstein-Uhlenbeck generator.

Now we can state the main result of this section.

Theorem 3.7. *Let $F \in \mathbb{D}$ with chaos expansion $F = \sum_{p=1}^{\infty} F_p$, where $LF_p = -\lambda_p F_p$ and assume that L is chaotic and its eigenfunctions verify the covariance assumption* (3.6)*. Denote the covariance operators of F_p by S_p, so that F has covariance operator $S = \sum_{p=1}^{\infty} S_p$. Then*

$$\left\| \Gamma(F, -L^{-1}F) - S \right\|_{L^2(\Omega;\mathrm{HS}(K))} \leq \sqrt{M(F) + C(F)}, \tag{3.7}$$

where

$$M(F) = \frac{1}{\sqrt{3}} \sum_{p,q=1}^{\infty} c_{p,q} \sqrt{\mathrm{E}\left(\|F_p\|^4\right)\left(\mathrm{E}\left(\|F_q\|^4\right) - \mathrm{E}\left(\|F_q\|^2\right)^2 - 2\left\|S_q\right\|_{\mathrm{HS}}^2\right)}, \tag{3.8}$$

$$C(F) = \sum_{\substack{p,q \in \mathbb{N} \\ p \neq q}} a_{p,q} \operatorname{Cov}\left(\|F_p\|^2, \|F_q\|^2\right), \tag{3.9}$$

and the constants $a_{p,q}$ and $c_{p,q}$ are given by $a_{p,q} = (\lambda_p + \lambda_q)/2\lambda_q$ and

$$c_{p,q} = \begin{cases} 1 + \sqrt{3} & \text{if } p = q, \\ a_{p,q} & \text{if } p \neq q, \end{cases}$$

respectively.

Before proving Theorem 3.7, let us give the following restatement of M in terms of fourth moments only.

Proposition 3.8. *In the setting of Theorem 3.7, it holds that*

$$M(F) = \frac{1}{\sqrt{3}} \sum_{p,q=1}^{\infty} c_{p,q} \sqrt{\mathrm{E}\left(\|F_p\|^4\right)\left(\mathrm{E}\left(\|F_q\|^4\right) - \mathrm{E}\left(\|Z_q\|^4\right)\right)}, \tag{3.10}$$

where the Z_p are centered Gaussian random variables with the same covariance operators as the F_p.

Proof. Using similar arguments as in [35, Proof of Theorem 4.2] combined with [17, Theorem 2.1], a straightforward calculation yields

$$\mathrm{E}\left(\|Z_p\|^4\right) = \mathrm{E}\left(\|F_p\|^2\right)^2 + 2\left\|S_p\right\|_{\mathrm{HS}}^2. \tag{3.11}$$

□

Proof of Theorem 3.7. The idea of the proof is to transfer the Dirichlet structure from $L^2(\Omega; K)$ to $L^2(\Omega; \mathbb{R})$ by expanding in an orthonormal basis and working on the coefficients, afterwards reassembling everything again. To this end, let $\{e_i : i \in \mathbb{N}\}$ be an orthonormal basis of K and denote $F_i = \langle F, e_i \rangle$, as well as $F_{p,i} = \langle F_p, e_i \rangle$ for $i \in \mathbb{N}$. Note that

$$\left\langle \Gamma(F, -L^{-1}F)e_i, e_j \right\rangle = \widetilde{\Gamma}(F_i, -\widetilde{L}^{-1}F_j),$$

where $\widetilde{\Gamma}$ and $\widetilde{L}$ are the real-valued counterparts of Γ and L (see Section 2.3). To improve readability, we will not make any notational distinction between the real-valued and Hilbert-valued case and therefore denote $\widetilde{\Gamma}$ and $\widetilde{L}$ by the symbols Γ and L as well throughout the proof. The meaning can always unambigously be inferred from the context, depending on whether the arguments are K- or $\mathbb{R}$-valued.

Define the cross-covariance operators $C_{p,q} : K \to K$ via the identity

$$\mathrm{E}\left(\langle F_p, k \rangle \langle F_q, l \rangle\right) = \langle C_{p,q} k, l \rangle, \qquad k, l \in K.$$

Then, $C_{p,p} = S_p$ and, by orthogonality, $C_{p,q} = 0$ if $p \neq q$. Therefore,

$$S = \sum_{p=1}^{\infty} S_p = \sum_{p,q=1}^{\infty} C_{p,q},$$

and consequently

$$\begin{aligned}\left\|\Gamma(F,-L^{-1}F)-S\right\|_{L^2(\Omega;\mathrm{HS}(K))} &\leq \sum_{p,q=1}^{\infty}\left\|\Gamma(F_p,-L^{-1}F_q)-C_{p,q}\right\|_{L^2(\Omega;\mathrm{HS}(K))} \\ &= \sum_{p,q=1}^{\infty}\sqrt{\sum_{i,j=1}^{\infty}\mathrm{E}\left(\left(\Gamma(F_{p,i},-L^{-1}F_{q,j})-\mathrm{E}\left(F_{p,i}F_{q,j}\right)\right)^2\right)} \\ &= \sum_{p,q=1}^{\infty}\sqrt{\sum_{i,j=1}^{\infty}\mathrm{Var}\left(\Gamma(F_{p,i},-L^{-1}F_{q,j})\right)}, \qquad (3.12)\end{aligned}$$

Note that all carré du champ operators appearing in the double sum (3.12) are acting on real valued random variables, so that known results from the finite-dimensional theory can be applied.

For $p = q$, Theorem 3.2 in [2] yields

$$\mathrm{Var}\left(\Gamma(F_{q,j},-L^{-1}F_{q,j})\right) \leq \frac{1}{3}\left(\mathrm{E}\left(F_{q,j}^4\right)-3\,\mathrm{E}\left(F_{q,j}^2\right)^2\right), \qquad (3.13)$$

which, together with the covariance condition (3.6), implies that

$$\begin{aligned}\sum_{j=1}^{\infty}\mathrm{Var}\left(\Gamma(F_{q,j},-L^{-1}F_{q,j})\right) &\leq \frac{1}{3}\sum_{j=1}^{\infty}\left(\mathrm{E}\left(F_{q,j}^4\right)-3\,\mathrm{E}\left(F_{q,j}^2\right)^2\right) \\ &\leq \frac{1}{3}\sum_{i,j=1}^{\infty}\left(\mathrm{E}\left(F_{q,i}^2F_{q,j}^2\right)-\mathrm{E}\left(F_{q,i}^2\right)\mathrm{E}\left(F_{q,j}^2\right)-2\,\mathrm{E}\left(F_{q,i}F_{q,j}\right)^2\right) \\ &= \frac{1}{3}\left(\mathrm{E}\left(\|F_q\|^4\right)-\mathrm{E}\left(\|F_q\|^2\right)^2-2\left\|S_q\right\|_{\mathrm{HS}}^2\right). \qquad (3.14)\end{aligned}$$

For $p \neq q$, similar calculations as in [10, Proof of Theorem 1.2] (which in turn relied on the main ideas of [2]) lead to

$$\begin{aligned}\mathrm{Var}\left(\Gamma(F_{p,i},-L^{-1}F_{q,j})\right) \leq a_{p,q}\Big(&\mathrm{E}\left(F_{p,i}^2F_{q,j}^2\right)-\mathrm{E}\left(F_{p,i}^2\right)\mathrm{E}\left(F_{q,j}^2\right)-2\,\mathrm{E}\left(F_{p,i}F_{q,j}\right)^2 \\ &-\mathrm{E}\left(F_{p,i}^2\left(\Gamma(F_{q,j},-L^{-1}F_{q,j})-\mathrm{E}\left(F_{q,j}^2\right)\right)\right)\Big),\end{aligned}$$

so that

$$\begin{aligned}&\sum_{i,j=1}^{\infty}\mathrm{Var}\left(\Gamma(F_{p,i},-L^{-1}F_{q,j})\right) \\ &\leq a_{p,q}\left(\mathrm{E}\left(\|F_p\|^2\|F_q\|^2\right)-\mathrm{E}\left(\|F_p\|^2\right)\mathrm{E}\left(\|F_q\|^2\right)-2\sum_{i,j=1}^{\infty}\mathrm{E}\left(F_{p,i}F_{q,j}\right)^2\right. \\ &\qquad\left.-\sum_{j=1}^{\infty}\mathrm{E}\left(\|F_p\|^2\left(\Gamma(F_{q,j},-L^{-1}F_{q,j})-\mathrm{E}\left(F_{q,j}^2\right)\right)\right)\right) \\ &\leq a_{p,q}\left(\mathrm{E}\left(\|F_p\|^2\|F_q\|^2\right)-\mathrm{E}\left(\|F_p\|^2\right)\mathrm{E}\left(\|F_q\|^2\right)-2\sum_{i,j=1}^{\infty}\mathrm{E}\left(F_{p,i}F_{q,j}\right)^2\right. \\ &\qquad\left.+\sqrt{\mathrm{E}\left(\|F_p\|^4\right)}\sqrt{\sum_{j=1}^{\infty}\mathrm{Var}\left(\Gamma(F_{q,j},-L^{-1}F_{q,j})\right)}\right).\end{aligned}$$

Together with (3.14), we thus get for $p = q$ that

$$\begin{aligned}
&\sum_{i,j=1}^{\infty} \operatorname{Var}\left(\Gamma(F_{p,i}, -L^{-1}F_{p,j})\right) \\
&\qquad \leq \mathrm{E}\left(\|F_p\|^4\right) - \mathrm{E}\left(\|F_p\|^2\right)^2 - 2\|S_p\|_{\mathrm{HS}}^2 \\
&\qquad\qquad + \sqrt{\frac{1}{3}\mathrm{E}\left(\|F_p\|^4\right)}\sqrt{\mathrm{E}\left(\|F_p\|^4\right) - \mathrm{E}\left(\|F_p\|^2\right)^2 - 2\|S_p\|_{\mathrm{HS}}^2} \\
&\qquad \leq \frac{1+\sqrt{3}}{\sqrt{3}}\sqrt{\mathrm{E}\left(\|F_p\|^4\right)}\sqrt{\mathrm{E}\left(\|F_p\|^4\right) - \mathrm{E}\left(\|F_p\|^2\right)^2 - 2\|S_p\|_{\mathrm{HS}}^2}
\end{aligned}$$

and for $p \neq q$ that

$$\begin{aligned}
&\sum_{i,j=1}^{\infty} \operatorname{Var}\left(\Gamma(F_{p,i}, -L^{-1}F_{q,j})\right) \\
&\qquad \leq a_{p,q}\left(\mathrm{E}\left(\|F_p\|^2\|F_q\|^2\right) - \mathrm{E}\left(\|F_p\|^2\right)\mathrm{E}\left(\|F_q\|^2\right)\right) \\
&\qquad\qquad + \frac{a_{p,q}}{\sqrt{3}}\sqrt{\mathrm{E}\left(\|F_p\|^4\right)}\sqrt{\mathrm{E}\left(\|F_q\|^4\right) - \mathrm{E}\left(\|F_q\|^2\right)^2 - 2\|S_q\|_{\mathrm{HS}}^2},
\end{aligned}$$

from which the asserted bound follows. □

Inspecting the proof of Theorem 3.7, it becomes apparent that for the case where $F = F_p$ is a chaotic eigenfunction, we can remove one square root. In other words, the following holds.

Corollary 3.9. *If F is a chaotic eigenfunction of L and Z a centered, non-degenerate Gaussian on K, both having covariance operator S, then*

$$\begin{aligned}
&\left\|\Gamma(F, -L^{-1}F) - S\right\|_{L^2(\Omega;\mathrm{HS}(K))} \\
&\qquad \leq \frac{1+\sqrt{3}}{\sqrt{3}}\sqrt{\mathrm{E}\left(\|F\|^4\right)\left(\mathrm{E}\left(\|F\|^4\right) - \mathrm{E}\left(\|F\|^2\right)^2 - 2\|S\|_{\mathrm{HS}}^2\right)} \\
&\qquad = \frac{1+\sqrt{3}}{\sqrt{3}}\sqrt{\mathrm{E}\left(\|F\|^4\right)\left(\mathrm{E}\left(\|F\|^4\right) - \mathrm{E}\left(\|Z\|^4\right)\right)}. \qquad (3.15)
\end{aligned}$$

Combining Theorems 3.7 and 3.2, the following moment bound is obtained.

Theorem 3.10. *Let Z be a centered Gaussian, non-degenerate random variable on K, assume that L is chaotic and let $F \in L^2(\Omega; K)$ with chaos expansion $F = \sum_{p=1}^{\infty} F_p$, where $LF_p = -\lambda_p F_p$. Denote the covariance operators of Z, F and F_p by S and T and S_p, respectively. Then the following two statements are true.*

(i) If F_p satisfies the covariance condition (3.6) for all $p \in \mathbb{N}$, then

$$d_2(F, Z) \leq \frac{1}{2}\left(\sqrt{M(F) + C(F)} + \|S - T\|_{\mathrm{HS}}\right) \qquad (3.16)$$

where the quantities $M(F)$ and $C(F)$ are given by (3.8) *(or equivalently* (3.10)*) and* (3.9)*, respectively.*

(ii) If $F = F_p$ for some eigenfunction $F_p \in \ker(L + \lambda_p \operatorname{Id})$, then

$$d_2(F, Z) \leq \frac{1+\sqrt{3}}{2\sqrt{3}}\sqrt{\mathrm{E}\left(\|F\|^4\right)\left(\mathrm{E}\left(\|F\|^4\right) - \mathrm{E}\left(\|F\|^2\right)^2 - 2\|S_p\|_{\mathrm{HS}}^2\right)} + \frac{1}{2}\|S - S_p\|_{\mathrm{HS}}.$$

Remark 3.11. In applications, given $F = \sum_{p=1}^{\infty} F_p$, it might sometimes be favorable to apply Theorem 3.10 to the truncated series $G_N = \sum_{p=1}^{N} F_p$ via the simple estimate

$$d_2(F,Z) \leq d_2(G_N,Z) + d_2(G_N,F),$$

so that the expressions $M(G_N)$ and $C(G_N)$ are no longer infinite series but finite sums. To handle the additional term $d_2(G_N,F)$, one then needs control on the tails $\mathrm{E}\left(\|F - G_N\|\right)$, for example via

$$\mathrm{E}\left(\|F-G_N\|\right)^2 \leq \mathrm{E}\left(\|F-G_N\|^2\right) = \sum_{p=N+1}^{\infty} \operatorname{tr} S_p.$$

Of course, in the setting of Theorem 3.10, if K is assumed to have finite dimension d, then the right hand side of (3.16) also bounds the Wasserstein distance $d_W(F,Z)$ (with constant $1/2$ replaced by $C_{s,d}$ of Theorem 3.2). Let us now state two central limit theorems which are direct consequences of Theorem 3.10. The first one is an abstract Fourth Moment Theorem.

Theorem 3.12 (Abstract fourth moment theorem)**.** *Let Z be a centered, non-degenerate Gaussian random variable on K and $\{F_n \colon n \in \mathbb{N}\}$ be a sequence of K-valued chaotic eigenfunctions such that $\mathrm{E}\left(\|F_n\|^2\right) \to \mathrm{E}\left(\|Z\|^2\right)$. Consider the following two asymptotic relations, as $n \to \infty$:*

(i) *F_n converges in distribution to Z;*

(ii) $\mathrm{E}\left(\|F_n\|^4\right) \to \mathrm{E}\left(\|Z\|^4\right)$.

Then, (ii) implies (i), and the converse implication holds whenever the moment sequence $\left\{\|F_n\|^4 : n \geq 1\right\}$ is uniformly integrable.

Proof. Denote the covariance operators of Z and the F_n by S and S_n, respectively. Then by assumption $\operatorname{tr}(S_n - S) \to 0$. The fact that (ii) implies (i) is a direct consequence of Theorem 3.10. The converse implication follows immediately if the additional uniform integrability condition is assumed to hold. □

Remark 3.13. (i) As is well known, a sufficient condition for uniform integrability of the sequence $\left\{\|F_n\|^4 : n \geq 1\right\}$ is given by $\sup_{n\geq 1} \mathrm{E}\left[\|F_n\|^{4+\varepsilon}\right] < \infty$ for some $\varepsilon > 0$.

(ii) Theorem 3.12 is a Hilbert-valued generalization of the Gaussian Fourth Moment Theorems derived in [2] ($K = \mathbb{R}$) and [10] ($K = \mathbb{R}^d$ with Euclidean inner product). As further special cases, taking L to be the Ornstein-Uhlenbeck generator on $L^2(\Omega,\mathbb{R})$, the classical Fourth Moment Theorem of [39] ($K = \mathbb{R}$) and Theorem 4.2 of [35] ($K = \mathbb{R}^d$ with Euclidean inner product) are included. Further details on these latter two cases will be provided in Section 4.2.

For functionals with infinite chaos expansions, the corresponding limit theorem reads as follows. Again, the proof is a straightforward application of Theorem 3.10.

Theorem 3.14. *Let Z be a centered, non-degenerate Gaussian random variable on K with covariance operator S and let $\{F_n \colon n \in \mathbb{N}\}$ be a sequence of square integrable, K-valued random variables with chaos decomposition*

$$F_n = \sum_{p=1}^{\infty} F_{p,n},$$

where, for each $n,p \geq 1$, $F_{p,n}$ is a chaotic eigenfunction associated to the eigenvalue $-\lambda_p$ (of the operator $-L$) and verifying the covariance condition (3.6)*. For $n,p \in \mathbb{N}$, let S_n and $S_{p,n}$ be the covariance operators of F_n and $F_{p,n}$, respectively. Suppose that:*

(i) There exists a sequence $\{S_p \colon p \in \mathbb{N}\}$ *of covariance operators such that* $S = \sum_{p=1}^{\infty} S_p$ *is of trace class,* $\operatorname{tr}(S_{p,n} - S_p) \to 0$ *as* $n \to \infty$ *and*

$$\lim_{N\to\infty} \sup_{n\in\mathbb{N}} \sum_{p=N}^{\infty} \operatorname{tr}(S_{p,n}) = 0 \tag{3.17}$$

(ii) For all $p, q \in \mathbb{N}$*, it holds that*

$$\mathrm{E}\left(\|F_{p,n}\|^4\right) - \mathrm{E}\left(\|F_{p,n}\|^2\right)^2 - 2\,\|S_{p,n}\|_{\mathrm{HS}}^2 \to 0$$

and, if $p \neq q$*,*

$$\mathrm{E}\left(\|F_{p,n}\|^2\,\|F_{q,n}\|^2\right) - \mathrm{E}\left(\|F_{p,n}\|^2\right)\mathrm{E}\left(\|F_{q,n}\|^2\right) \to 0$$

as $n \to \infty$*.*

Then F_n *converges in distribution to* Z *as* $n \to \infty$*.*

Proof. For $N \in \mathbb{N}$, define $F_{n,N} = \sum_{p=1}^{N} F_{p,n}$, $R_{n,N} = F_n - F_{n,N} = \sum_{p=N+1}^{\infty} F_{p,n}$ and let Z_N be a centered Gaussian random variable on K with covariance operator $\sum_{p=1}^{N} S_p$. Now let $\varepsilon > 0$ and note that

$$d_2(F_n, Z) \leq d_2(F_n, F_{n,N}) + d_2(F_{n,N}, Z_N) + d_2(Z_N, Z). \tag{3.18}$$

For $h \in C_b^2(K)$, we get by Lipschitz-continuity that

$$d_2(F_n, F_{n,N}) \leq \mathrm{E}\left(\|R_{n,N}\|\right) \leq \sqrt{\mathrm{E}\left((\|R_{n,N}\|)^2\right)} \leq \sqrt{\sum_{p=N+1}^{\infty} \operatorname{tr}(S_{p,n})}.$$

Similarly,

$$d_2(Z_N, Z) \leq \sqrt{\sum_{p=N+1}^{\infty} \operatorname{tr}(S_p)}$$

The above two calculations, together with assumption *(i)*, yield the existence of $N \in \mathbb{N}$, not dependent of n, such that

$$d_2(F_n, F_{n,N}) + d_2(Z_N, Z) < \varepsilon. \tag{3.19}$$

By assumption *(ii)* and Theorem 3.10, we also have that

$$d_2(F_{n,N}, Z_N) \to 0$$

as $n \to \infty$, so that in view of (3.18),

$$0 \leq \lim_{n\to\infty} d(F_n, Z) < \varepsilon.$$

The assertion follows as ε was arbitrary. $\square$

Although we stated Theorems 3.12 and 3.14 in a qualitative way, it should be clear that the convergences in both results are actually quantified by Theorem 3.10.

4 Hilbert-valued Wiener structures

In this section, we apply our general results to the special Dirichlet structure induced by the Ornstein-Uhlenbeck generator. This leads to Hilbert-valued Wiener chaos and a carré du champ operator given by a gradient of Hilbert-valued Malliavin derivatives. The eigenfunctions are multiple Wiener-Itô integrals with Hilbert-valued deterministic kernels. This additional structure allows to express the moment bounds of the previous sections in terms of kernel contractions, which in the finite-dimensional case have already proved themselves to be very useful in applications, due to their comparatively easy computability when compared to moments.

4.1 Malliavin calculus

We present some basic elements of the Malliavin calculus on Hilbert spaces, as introduced in [25] (based on the earlier work [15]). Expository references are [21, Chapter 4], [11, Chapter 5]. The authoritative reference for Malliavin calculus on $L^2(\Omega;\mathbb{R})$ is [38], a further excellent exposition can be found in [31, Chapters 1 and 2].

4.1.1 The Malliavin derivative and divergence operators

Let $\{W(h)\colon h\in\mathfrak{H}\}$ be an isonormal Gaussian process with underlying separable Hilbert space $\mathfrak{H}$, that is $\{W(h)\colon h\in\mathfrak{H}\}$ is a centered family of Gaussian random variables, defined on a complete probability space $(\Omega,\mathcal{F},P)$, satisfying

$$\mathrm{E}[W(h_1)W(h_2)] = \langle h_1,h_2\rangle_{\mathfrak{H}}\,, \qquad h_1,h_2\in\mathfrak{H}.$$

We assume that the σ-algebra $\mathcal{F}$ is generated by W. Let K be another separable Hilbert space and denote by $\mathcal{S}\otimes K$ the class of smooth K-valued random variables $F\colon\Omega\to K$ of the form $F=f\left(W(h_1),\dots,W(h_n)\right)\otimes v$, where $f\in C_b^\infty\left(\mathbb{R}^n\right)$, $h_1,\dots,h_n\in\mathfrak{H}$, $v\in K$, and linear combinations thereof. $\mathcal{S}\otimes K$ is dense in $L^2(\Omega;K)$ and for $F\in\mathcal{S}\otimes K$, define the Malliavin derivative DF of F as the $\mathfrak{H}\otimes K$-valued random variable given by

$$DF=\sum_{i=1}^n \partial_i f\left(W(h_1),\dots,W(h_n)\right) h_i\otimes v. \tag{4.1}$$

It can be shown that D is a closable operator from $L^2\left(\Omega;K\right)$ into $L^2\left(\Omega;\mathfrak{H}\otimes K\right)$, and from now we continue to use the symbol D to denote the closure. The domain of D, denoted by $\mathbb{D}^{1,2}(K)$, is the closure of $\mathcal{S}\otimes K$ with respect to the Sobolev norm $\|F\|^2_{\mathbb{D}^{1,2}(K)} = \|F\|^2_{L^2(\Omega;K)}+\|DF\|^2_{L^2(\Omega;\mathfrak{H}\otimes K)}$. Similarly, for $k\geq 2$, let $\mathbb{D}^{k,2}(K)$ denote the closure of $\mathcal{S}\otimes K$ with respect to the Sobolev norm $\|F\|^2_{\mathbb{D}^{k,2}(K)} = \|F\|^2_{L^2(\Omega;K)}+\sum_{i=1}^k \left\|D^iF\right\|^2_{L^2(\Omega;\mathfrak{H}^{\otimes i}\otimes K)}$. For any $k\geq 2$, the operator D^k can be interpreted as the iteration of the Malliavin derivative operator defined in (4.1). As D is a closed linear operator from $\mathbb{D}^{1,2}(K)$ to $L^2(\Omega;\mathfrak{H}\otimes K)$, it has an adjoint operator, denoted by δ, which maps a subspace of $L^2(\Omega;\mathfrak{H}\otimes K)$ into $L^2(\Omega;K)$ through the duality relation

$$\mathrm{E}\big[\langle DF,\eta\rangle_{\mathfrak{H}\otimes K}\big] = \mathrm{E}[\langle F,\delta(\eta)\rangle_K]\,,$$

for any $F\in\mathbb{D}^{1,2}(K)$ and $\eta\in\mathrm{dom}(\delta)$. The domain of δ, denoted by $\mathrm{dom}(\delta)$, is the subset of random variables $\eta\in L^2(\Omega;\mathfrak{H}\otimes K)$ such that $\left|\mathrm{E}\big[\langle DF,\eta\rangle_{\mathfrak{H}\otimes K}\big]\right|\leq C_\eta\,\|F\|_{L^2(\Omega;K)}$, for all $F\in\mathbb{D}^{1,2}(K)$, where C_η is a positive constant depending only on η. Since D is a form of gradient, its adjoint δ should be interpreted as a divergence, so that it is referred to as the divergence operator. Similarly, for any $k\geq 2$, we denote by δ^k the adjoint of D^k as an operator from $L^2(\Omega;\mathfrak{H}^{\otimes k}\otimes K)$ to $L^2(\Omega;K)$ with domain $\mathrm{dom}(\delta^k)$.

4.1.2 Multiple integrals and chaos decomposition

Any K-valued random variable $F \in L^2(\Omega; K)$ can be decomposed as

$$F = \sum_{n=0}^{\infty} \delta^n(f_n), \tag{4.2}$$

where the kernel $f_n \in \mathfrak{H}^{\odot n} \otimes K$ are uniquely determined by F, where $\mathfrak{H}^{\odot n}$ denotes the n-fold symmetrized tensor product of $\mathfrak{H}$. The representation (4.2) is called the chaos decomposition of F, and for each $n \geq 0$, $\delta^n(f_n)$ is an element of the closure of $\mathfrak{H}_n \otimes K$ with respect to the norm on $L^2(\Omega; K)$, where the so-called n-th Wiener chaos $\mathfrak{H}_n$ is defined to be closed linear subspace of $L^2(\Omega)$ generated by the random variables $\left\{H_n(W(h))\colon h \in \mathfrak{H},\ \|h\|_{\mathfrak{H}} = 1\right\}$, where H_n is the n-th Hermite polynomial given by $H_n(x) = (-1)^n \mathrm{e}^{x^2/2} \left(\frac{d}{dx}\right)^n \mathrm{e}^{-x^2/2}$ (recall that $\mathfrak{H}_0$ is identified with $\mathbb{R}$). For any $n \geq 0$, the K-valued random variable $\delta^n(f_n)$ is usually denoted by $I_n(f_n)$ and called the (K-valued) multiple Wiener integral of order n of f_n. In the particular case where $K = \mathbb{R}$, these integrals coincide with the ones defined in [38]. Denote by J_n the linear operator on $L^2(\Omega)$ given by the orthogonal projection onto $\mathfrak{H}_n$, and by J_n^K the extension of $J_n \otimes \mathrm{Id}_K$ to $L^2(\Omega; K)$. Then, it holds that $J_n^K F = I_p(f_n)$. Let $\{e_k \colon k \geq 0\}$ be an orthonormal basis of $\mathfrak{H}$. Given $f \in \mathfrak{H}^{\odot n}$ and $g \in \mathfrak{H}^{\odot m}$, for every $r = 0, \dots, n \wedge m$, the r-th contraction of f and g is the element of $\mathfrak{H}^{\otimes(n+m-2r)}$ defined as

$$f \otimes_r g = \sum_{i_1, \dots, i_r = 0}^{\infty} \langle f, e_{i_1} \otimes \cdots \otimes e_{i_r} \rangle_{\mathfrak{H}^{\otimes r}} \otimes \langle g, e_{i_1} \otimes \cdots \otimes e_{i_r} \rangle_{\mathfrak{H}^{\otimes r}}. \tag{4.3}$$

We denote by $f \widetilde{\otimes}_r g$ the symmetrization (average over all permutations of the arguments) of $f \otimes_r g$. Given an orthonormal basis $\{v_k \colon k \geq 0\}$ of K, the following multiplication formula is satisfied by K-valued multiple Wiener integrals: for two arbitrary basis elements v_i, v_j of K and for $f \in \mathfrak{H}^{\odot n} \otimes K$ and $g \in \mathfrak{H}^{\odot m} \otimes K$, define $f_i = \langle f, v_i \rangle_K$ and $g_j = \langle g, v_j \rangle_K$. Then

$$I_n(f_i) I_m(g_j) = \sum_{r=0}^{n \wedge m} r! \binom{n}{r} \binom{m}{r} I_{n+m-2r}(f_i \widetilde{\otimes}_r g_j). \tag{4.4}$$

Finally, the action of the Malliavin derivative operator on a K-valued multiple Wiener integral of the form $I_n(f) \in L^2(\Omega; K)$, where $f \in \mathfrak{H}^{\odot n} \otimes K$, is given by $DI_n(f) = nI_{n-1}(f(\cdot)) \in L^2(\Omega; \mathfrak{H} \otimes K)$.

4.2 Fourth moment and contraction bounds

In this section, we are going to apply our abstract results to the Dirichlet structure given by the Ornstein-Uhlenbeck generator, acting on $L^2(\Omega; K)$, where K is a real, separable Hilbert space and the σ-algebra of the underlying probability space is generated by an isonormal Gaussian process W, indexed by a real, separable Hilbert space $\mathfrak{H}$. The Ornstein-Uhlenbeck generator, commonly denoted by $-L$ in this context, is then defined as $-L = \delta D$. Its spectrum is given by the non-negative integers and the eigenspace asociated to the eigenvalue $p \in \mathbb{N}_0$ consists of K-valued multiple Wiener-Itô integrals of order p. The product formula (4.4) furthermore shows that each of these eigenfunctions is chaotic in the sense of Definition 3.5. The carré du champ operator is given by $\Gamma(F, G) = \langle DF, DG \rangle_{\mathfrak{H}}$, where $\mathfrak{H}$ denotes the underlying Hilbert space on which the isonormal Gaussian process is defined. The operator $\Gamma(F, G)$ thus acts on K via

$$\Gamma(F, G)u = \langle \langle DF, u \rangle_K, DG \rangle_{\mathfrak{H}}.$$

Using this concrete structure, our bounds can be expressed in terms of kernel contractions. In applications, such contractions have proven to be very useful, as they are typically easier to evaluate than moments (see, among many others, [30, 32] for the context of Breuer-Major theorems, for instance).

Throughout the rest of this section, we assume a Dirichlet structure as introduced in the above paragraph as given.

We start by proving that the covariance condition (3.6) always holds in the present context.

Lemma 4.1. *For $p \in \mathbb{N}$ and $f \in \mathfrak{H}^{\odot p} \otimes K$, let $F = I_p(f)$ be a multiple integral with values in K. Then F satisfies the covariance condition* (3.6).

Proof. Let $u, v \in K$. For better legibility, we will write $I_p(f_u) = I_p(\langle f, u\rangle)$ and $I_p(f_v) = I_p(\langle f, v\rangle)$. By the product formula for multiple integrals, we get that

$$\begin{aligned} \mathrm{E}\left(I_p(f_u)^2 I_p(f_v)^2\right) &= \sum_{r=0}^{p} a_{p,r}^2 (2p-2r)! \left\langle f_u \widetilde{\otimes}_r f_v, f_u \widetilde{\otimes}_r f_v \right\rangle_{\mathfrak{H}^{\otimes(2p-2r)}} \\ &= (2p)! \left\| f_u \widetilde{\otimes} f_v \right\|^2_{\mathfrak{H}^{\otimes 2p}} + (p!)^2 \left\langle f_u, f_v \right\rangle^2_{\mathfrak{H}^{\otimes p}} \\ &\qquad + \sum_{r=1}^{p-1} a_r^2 (2p-2r)! \left\| f_u \widetilde{\otimes}_r f_v \right\|^2_{\mathfrak{H}^{\otimes(2p-2r)}}, \end{aligned}$$

where $a_{p,r} = r! \binom{p}{r}^2$. A straightforward modification of the calculations given in [31, Pages 97-98] yields

$$\begin{aligned} &(2p)! \left\| f_u \widetilde{\otimes} f_v \right\|^2_{\mathfrak{H}^{\otimes 2p}} \\ &\quad = p!^2 \left\| f_u \right\|^2_{\mathfrak{H}^{\otimes p}} \left\| f_v \right\|^2_{\mathfrak{H}^{\otimes p}} + p!^2 \left\langle f_u, f_v \right\rangle^2_{\mathfrak{H}^{\otimes p}} + p!^2 \sum_{r=1}^{p-1} \binom{p}{r}^2 \left\| f_u \otimes_r f_v \right\|^2_{\mathfrak{H}^{\otimes(2p-2r)}} \\ &\quad = \mathrm{E}\left(I_p(f_u)^2\right) \mathrm{E}\left(I_p(f_v)^2\right) + \mathrm{E}\left(I_p(f_u) I_p(f_v)\right)^2 + p!^2 \sum_{r=1}^{p-1} \binom{p}{r}^2 \left\| f_u \otimes_r f_v \right\|^2_{\mathfrak{H}^{\otimes(2p-2r)}}. \end{aligned}$$

Combining the last two calculations and rearranging terms gives

$$\begin{aligned} &\mathrm{E}\left(I_p(f_u)^2 I_p(f_v)^2\right) - \mathrm{E}\left(I_p(f_u)^2\right) \mathrm{E}\left(I_p(f_v)^2\right) - 2\,\mathrm{E}\left(I_p(f_u) I_p(f_v)\right)^2 \\ &\quad = \sum_{r=1}^{p-1} \left(a_r^2 (2p-2r)! \left\| f_u \widetilde{\otimes}_r f_v \right\|^2_{\mathfrak{H}^{\otimes(2p-2r)}} + p!^2 \binom{p}{r}^2 \left\| f_u \otimes_r f_v \right\|^2_{\mathfrak{H}^{\otimes(2p-2r)}} \right) > 0, \end{aligned} \tag{4.5}$$

which is the desired inequality. ☐

The contraction bound is as follows.

Theorem 4.2. *For $F \in \mathbb{D}^{1,4}$ with covariance operator S and chaos decomposition $F = \sum_{p=1}^{\infty} F_p$, where $F_p = I_p(f)$ and $f \in \mathfrak{H}^{\odot p} \otimes K$, one has*

$$\left\| \left\langle DF, -DL^{-1}F \right\rangle_{\mathfrak{H}} - S \right\|_{L^2(\Omega; \mathrm{HS}(K))} \leq \widetilde{M}(F) + \widetilde{C}(F), \tag{4.6}$$

where

$$\widetilde{M}(F) = \sum_{p=1}^{\infty} \sqrt{\sum_{r=1}^{p-1} c_{p,p}(r)^2 \left\| f_p \otimes_r f_p \right\|^2_{\mathfrak{H}^{\otimes(2p-2r)} \otimes K^{\otimes 2}}},$$

$$\widetilde{C}(F) = \sum_{\substack{1 \leq p,q \leq \infty \\ p \neq q}} \sqrt{\sum_{r=1}^{p \wedge q} c_{p,q}(r)^2 \left\| f_p \otimes_r f_q \right\|^2_{\mathfrak{H}^{\otimes(p+q-2r)} \otimes K^{\otimes 2}}}$$

and the positive constants $c_{p,q}(r)$ are given by

$$c_{p,q}(r) = p^2(r-1)!\binom{p-1}{r-1}\binom{q-1}{r-1}(p+q-2r)!. \tag{4.7}$$

Proof. Let $\{e_i : i \in \mathbb{N}\}$ be an orthonormal basis of K and abbreviate the inner products $\langle F_p, e_i\rangle$ and $\langle f_p, e_i\rangle$ by $F_{p,i}$ and $f_{p,i}$, respectively. As in the proof of Theorem 3.7, it follows that

$$\left\|\left\langle DF, -DL^{-1}F\right\rangle_{\mathfrak{H}} - S\right\|_{L^2(\Omega;\mathrm{HS}(K))} \leq \sum_{p,q=1}^{\infty}\sqrt{\sum_{i,j=1}^{\infty}\mathrm{Var}\left(\langle DF_{p,i}, -DL^{-1}F_{q,j}\rangle_{\mathfrak{H}}\right)},$$

so that we can apply the finite-dimensional result [31, Lemma 6.2.1] and obtain

$$\begin{aligned}
&\mathrm{Var}\left(\left\langle DF_{p,i}, -DL^{-1}F_{q,j}\right\rangle_{\mathfrak{H}}\right) \\
&\qquad = \begin{cases} \sum_{r=1}^{p\wedge q} c_{p,q}(r)^2 \left\|f_{p,i}\widetilde{\otimes}_r f_{q,j}\right\|^2_{\mathfrak{H}^{\otimes(p+q-2r)}} & \text{if } p \neq q, \\ \sum_{r=1}^{p-1} c_{p,p}(r)^2 \left\|f_{p,i}\widetilde{\otimes}_r f_{p,j}\right\|^2_{\mathfrak{H}^{\otimes(2p-2r)}} & \text{if } p = q, \end{cases} \qquad (4.8)\\
&\qquad \leq p^2 \begin{cases} \sum_{r=1}^{p\wedge q} c_{p,q}(r)^2 \left\|f_{p,i}\otimes_r f_{q,j}\right\|^2_{\mathfrak{H}^{\otimes(p+q-2r)}} & \text{if } p \neq q, \\ \sum_{r=1}^{p-1} c_{p,p}(r)^2 \left\|f_{p,i}\otimes_r f_{p,j}\right\|^2_{\mathfrak{H}^{\otimes(2p-2r)}} & \text{if } p = q. \end{cases}
\end{aligned}$$

The assertion follows after summing over i and j. □

Combined with Theorem 3.2, the contraction bound just obtained yields the following result.

Theorem 4.3. *Let Z be a centered Gaussian random variable on K with covariance operator S and $F \in \mathbb{D}^{1,4}$ with covariance operator T and chaos decomposition $F = \sum_{p=1}^{\infty} I_p(f_p)$, where, for each $p \geq 1$, $f_p \in \mathfrak{H}^{\odot p} \otimes K$. Then*

$$d_2(F, Z) \leq \frac{1}{2}\left(\widetilde{M}(F) + \widetilde{C}(F) + \|S - T\|_{\mathrm{HS}}\right),$$

where the quantities $\widetilde{M}(F)$ and $\widetilde{C}(F)$ are defined as in Theorem 4.2.

As special cases for $K = \mathbb{R}$, Theorem 4.3 includes the main results of [13] and [33] (as usual in finite dimension, d_2 can be replaced by the Wasserstein distance – see the proof of Theorem 3.2).

Let us now show how the results proved in Section 3.2 can be refined in the Wiener chaos setting. We start with the Fourth Moment Theorem.

Theorem 4.4 (Infinite-dimensional Fourth Moment Theorem)**.** *Let Z be a centered Gaussian random variable on K with covariance operator S, and, for $p \geq 1$, let $\{F_n : n \in \mathbb{N}\} = \{I_p(f_n) : n \in \mathbb{N}\}$ be a sequence of K-valued multiple integrals such that $\mathrm{tr}(S_n - S) \to 0$ as $n \to \infty$. Then, as $n \to \infty$, the following assertions are equivalent.*

(i) *F_n converges in distribution to Z,*

(ii) $\mathrm{E}\left(\|F_n\|^4\right) \to \mathrm{E}\left(\|Z\|^4\right)$,

(iii) $\|f_n \otimes_r f_n\|_{\mathfrak{H}^{(2p-2r)}\otimes K^{\otimes 2}} \to 0$ *for all* $r = 1, \ldots, p-1$,

(iv) $\left\|f_n \widetilde{\otimes}_r f_n\right\|_{\mathfrak{H}^{(2p-2r)}\otimes K^{\otimes 2}} \to 0$ *for all* $r = 1, \ldots, p-1$,

(v) $\left\|\langle DF_n, DF_n\rangle_{\mathfrak{H}} - pS_n\right\|_{L^2(\Omega;\mathrm{HS}(K))} \to 0$.

Proof. As $\mathrm{tr}(S_n - S) \to 0$ as $n \to \infty$, hypercontractivity of Wiener chaos implies that for any $r \geq 2$, $\sup_n \mathrm{E}[\|F_n\|^r] < \infty$, which yields that *(i)* implies *(ii)* by uniform integrability. Summing (4.5) over i and j and using (3.11) yields the implication *(ii)⇒(iii)* (and also *(ii)⇒(iv)*). The fact that $\left\|f_n \widetilde{\otimes}_r f_n\right\|_{\mathfrak{H}^{(2p-2r)} \otimes K^{\otimes 2}} \leq \|f_n \otimes_r f_n\|_{\mathfrak{H}^{(2p-2r)} \otimes K^{\otimes 2}}$ gives *(iii)⇒(iv)* and the implication *(iv)⇒(v)* follows by summing (4.8) over i and j. Finally, *(v)⇒(i)* is a consequence of Theorem 4.2. □

The corresponding Fourth Moment Theorems for random variables with infinite chaos expansion (Theorem 3.14 in Section 3.2) can be expressed using contractions as follows:

Theorem 4.5. *Let $\{F_n \colon n \in \mathbb{N}\}$ be a sequence of square integrable K-valued random variables with chaos decomposition*

$$F_n = \sum_{p=1}^{\infty} I_p(f_{p,n}), \tag{4.9}$$

where, for each $n, p \geq 1$, $f_{p,n} \in \mathfrak{H}^{\odot p} \otimes K$. Suppose that:

(i) for every $p \in \mathbb{N}$ there exists $f_p \in \mathfrak{H}^{\odot p} \otimes K$ such that $\|f_{n,p} - f_p\|_{\mathfrak{H}^{\otimes p} \otimes K} \to 0$,

$$\sum_{p=1}^{\infty} p! \, \|f_p\|^2_{\mathfrak{H}^{\otimes p} \otimes K} < \infty$$

and

$$\lim_{N \to \infty} \sup_{n \geq 1} \sum_{p=N+1}^{\infty} p! \, \|f_{p,n}\|^2_{\mathfrak{H}^{\otimes p} \otimes K} = 0.$$

(ii) for all $p \in \mathbb{N}$ and $r = 1, \ldots, p-1$, it holds that

$$\|f_{p,n} \otimes_r f_{p,n}\|_{\mathfrak{H}^{\otimes 2(p-r)} \otimes K^{\otimes 2}} \to 0.$$

Then F_n converges in distribution to a centered Gaussian Z with covariance operator S given by

$$S = \sum_{p=1}^{\infty} \mathrm{E}\left(\|f_p\|^2_{\mathfrak{H}^{\otimes p}}\right),$$

where, with some slight abuse of notation, $\mathrm{E}\left(\|f_p\|^2_{\mathfrak{H}^{\otimes p}}\right) \in K \otimes K \simeq L(K, K)$ denotes the covariance operator of $I_p(f_p)$.

Proof. For $p, n \in \mathbb{N}$, let S_p and $S_{p,n}$ be the covariance operators of $I_p(f_p)$ and $I_p(f_{p,n})$, respectively. Then

$$\begin{aligned}
|\mathrm{tr}(S_{p,n} - S_p)| &= \left|\mathrm{E}\left(\|I_p(f_{p,n})\|^2_K - \|I_p(f_p)\|^2_K\right)\right| \\
&\leq \sqrt{\mathrm{E}\left(\|I_p(f_{p,n} - I_p(f_p)\|^2_K\right) \mathrm{E}\left(\|I_p(f_{p,n}) + I_p(f_p)\|^2_K\right)} \\
&= p! \, \|f_{p,n} - f_p\|_{\mathfrak{H}^{\otimes p} \otimes K} \, \|f_{p,n} + f_p\|_{\mathfrak{H}^{\otimes p} \otimes K},
\end{aligned}$$

which tends to zero as $n \to \infty$ by assumption (i). As $\mathrm{tr}(S_{p,n}) = \mathrm{E}\left(\|I_{p,n}\|^2_K\right) = p! \, \|f_{p,n}\|^2_{\mathfrak{H}^{\otimes p} \otimes K}$, the same assumption also implies that

$$\lim_{N \to \infty} \sup_{n \geq 1} \sum_{p=N+1}^{\infty} \mathrm{tr}(S_{p,n}) = 0.$$

The rest of the proof can now be done as in Theorem 3.14, using the bound provided by Theorem 4.2. □

5 Quantifying the functional Breuer-Major Theorem

In this section, we will give rates of convergence for a functional version of the seminal Breuer-Major Theorem. To introduce the setting, let $X = \{X_t \colon t \geq 0\}$ be a centered, stationary Gaussian process and define $\rho(k) = \mathrm{E}\left(X_0 X_k\right)$ such that $\mathrm{E}\left(X_s X_t\right) = \rho(t-s) = \rho(s-t)$. Assume $\rho(0) = 1$, denote the standard Gaussian measure on $\mathbb{R}$ by γ and let $\varphi \in L^2(\mathbb{R}, \gamma)$ be of Hermite rank $d \geq 1$, so that φ can be expanded in the form

$$\varphi(x) = \sum_{i=d}^{\infty} c_i H_i(x), \qquad c_d \neq 0, \tag{5.1}$$

where $H_i(x) = (-1)^i \mathrm{e}^{x^2/2} \left(\frac{d}{dx}\right)^i \mathrm{e}^{-x^2/2}$ is the ith Hermite polynomial. The Breuer-Major Theorem then states that under the condition

$$\sum_{k \in \mathbb{Z}} \rho(k)^d < \infty,$$

the finite-dimensional distributions of the stochastic process $\{U_n(t) \colon t \in [0,1]\}$ given by

$$U_n(t) = \frac{1}{\sqrt{n}} \sum_{i=0}^{\lfloor nt \rfloor} \varphi(X_i) \tag{5.2}$$

converge in law to those of a scaled Brownian motion σW, where $W = \{W_t \colon t \in [0,1]\}$ is a standard Brownian motion and the scaling is given by

$$\sigma^2 = \sum_{p=d}^{\infty} p! c_p^2 \sum_{k \in \mathbb{Z}} \rho(k)^p. \tag{5.3}$$

After its discovery by Breuer and Major (see [8]), it took more than twenty years until progress was made towards quantifying this result. Taking X to be the normalized increment process of a fractional Brownian motion, Nourdin and Peccati ([30]), as an illustration of the Malliavin-Stein method introduced in the same reference, were able to associate rates to the normal convergence of the *chaotic projections* of the coordinate sequences of U_n, i.e., to the random sequence

$$\frac{1}{\sqrt{n}} \sum_{k=1}^{n} H_p\left(n^H \left(B^H_{\frac{k+1}{n}} - B^H_{\frac{k}{n}}\right)\right), \tag{5.4}$$

where H_p denotes the pth Hermite polynomial and B^H is a fractional Brownian motion with Hurst index H. Note that the random variables defined in (5.4) can be represented as multiple integrals of order p and therefore are elements of the pth Wiener chaos. Recently, the Breuer-Major Theorem has been intensively studied, and very strong results have been obtained concerning the coordinate sequence, providing rates of convergence in total variation distance for general functions φ under rather weak assumptions (see [37, 29, 34]). Turning to infinite-dimension, it also has been proved recently in [9] and [28] that the process U_n converges in distribution towards a scaled Brownian motion in the Skorohod space or in the space of continuous functions (replacing the Gauss brackets in the sum by a linear interpolation).

In this section, it will be shown how, using our bounds, one can associate rates to the aforementioned functional convergences, taking place in a suitable Hilbert space K containing $D([0,1])$ and $C_0([0,1])$, respectively. The rates are obtained through the contraction bounds obtained in the previous section, which allow a natural and straightforward lifting of the one-dimensional results. We illustrate this method on [32, Example

2.5], where $\varphi = H_p$ and $\rho(k) = k^\alpha l(k)$ for some $\alpha < 0$ and a slowly varying function l. This latter assumption on ρ for example includes the case where X is the increment process of a fractional Brownian motion. Also, for simplicity, we set $K = L^2([0,1])$. Our results also allow the analysis of more general functions φ and smaller Hilbert spaces K with finer topologies, such as the Besov-Liouville (see [40] for definitions and [12] for proofs of related functional limit theorems in this space) or other reproducing kernel Hilbert spaces, but as the calculations are more involved and also quite lenghty and technical, we decided to focus on the general picture in this article and will provide full details on this topic in a dedicated followup work.

The statement is as follows.

Theorem 5.1. *Let $\{U_n(t) \colon t \in [0,1]\}$ be the stochastic process defined in* (5.2)*, considered as a sequence of random variables taking values in $L^2([0,1])$, assume that $\varphi = H_p$ for some $p \in \mathbb{N}$ and that the covariance function ρ of the underlying centered, stationary Gaussian process is of the form $\rho(k) = |k|^\alpha l(|k|)$, where $\alpha < -1/p$ and l is a slowly varying function. Then there exists a constant $C > 0$, such that*

$$d_2(U_n, \sigma W) \leq C r_\alpha(n) \tag{5.5}$$

where σ is defined in (5.3)*, W denotes a standard Brownian motion on $L^2([0,1])$ and the rate function is given by*

$$r_\alpha(n) = \begin{cases} n^{-1/2} & \text{if } \alpha < -1, \\ n^{\alpha/2} l(n) & \text{if } \alpha \in \left(-1, -\frac{1}{p-1}\right), \\ n^{(\alpha q+1)/2} l^2(n) & \text{if } \alpha \in \left(-\frac{1}{p-1}, \frac{-1}{p}\right). \end{cases}$$

Remark 5.2. Theorem 5.1 applies to the case where $X_i = B^H_{i+1} - B^H_i$ is the increment process of a fractional Brownian motion with Hurst index $H \leq \frac{2p-1}{2p}$. In this case, the corresponding bound reads

$$d_2(U_n, \sigma W) \leq C \begin{cases} n^{-1/2} & \text{if } H \in \left(0, \frac{1}{2}\right), \\ n^{H-1} & \text{if } H \in \left[\frac{1}{2}, \frac{2p-3}{2p-2}\right] \\ n^{(2pH-2p+1)/2} & \text{if } H \in \left(\frac{2p-3}{2p-2}, \frac{2p-1}{2p}\right). \end{cases}$$

See [32, Example 2.6] for further details. See also [31, Exercise 7.5.1] for a particular (and simpler) case where the function φ is taken to be the p-th Hermite polynomial H_p.

Proof of Theorem 5.1. Throughout this proof, C denotes a positive constant which might change from line to line. Let $\mathfrak{H}$ be the Hilbert space obtained by the closure of the set of all finite linear combinations of indicator functions $1_{[0,t]}$, $t \geq 0$ with respect to the inner product

$$\left\langle 1_{[0,s]}, 1_{[0,t]} \right\rangle_{\mathfrak{H}} = \rho(t-s)$$

and let $\mathcal{X}$ be an isonormal Gaussian process on $\mathfrak{H}$ (for details on this construction, see [31, Example 2.1.5]). Then

$$\mathrm{E}\left(\mathcal{X}(1_{[0,i]})\mathcal{X}(1_{[0,j]})\right) = \left\langle 1_{[0,i]}, 1_{[0,j]} \right\rangle_{\mathfrak{H}} = \rho(j-i) = \mathrm{E}\left(X_i X_j\right),$$

where expectations are taken over the respective probability spaces of $\mathcal{X}$ and X. Furthermore, note that U_n has the same law as $I_p(f_{n,t})$, where

$$f_{n,t}(x) = \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor - 1} g_p(i,x) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n-1} 1_{[\frac{i+1}{n},1]}(t) g(i,x) \tag{5.6}$$

and $g(i,x) = \prod_{j=1}^{p} 1_{[0,i]}(x_j) \geq 0$. Let us denote by T_n the covariance operator of U_n and define

$$\sigma_n^2 = p! \sum_{k \in \mathbb{Z}} \rho(k)^p \left(1 - \frac{|k|}{n}\right) 1_{\{|k|<n\}}.$$

Now

$$d_2\left(U_n, \sigma W\right) \leq d_2\left(U_n, \sigma_n W\right) + d_2\left(\sigma_n W, \sigma W\right), \tag{5.7}$$

and applying Corollary 3.3 together with the identity (2.2), we obtain

$$\begin{aligned} d_2(\sigma_n W, \sigma W) &\leq \frac{1}{2} \left\| \sigma_n^2 S - \sigma^2 S \right\|_{L^1(\Omega; \mathcal{S}_1(L^2([0,1])))} \\ &= \frac{1}{2} \left| \sigma_n^2 - \sigma^2 \right| \operatorname{tr} S = \frac{1}{2} \left| \sigma_n^2 - \sigma^2 \right| \leq C \left(n^{-1} + n^{\alpha p + 1} l(n) \right), \end{aligned} \tag{5.8}$$

where the last inequality follows after a straightforward calculation, using the same estimate as in [32, Example 2.5]. Furthermore, by Theorem 4.3,

$$d_2\left(U_n, \sigma_n W\right) \leq \frac{1}{2} \sum_{r=1}^{p-1} p \left\| f_{n,\cdot} \otimes_r f_{n,\cdot} \right\|_{\mathfrak{H}^{\otimes(2p-2r)} \otimes L^2([0,1])^{\otimes 2}} + \frac{1}{2} \left\| T_n - \sigma_n^2 S \right\|_{\mathrm{HS}(L^2([0,1]))}. \tag{5.9}$$

Lemma 5.3 yields that

$$\left\| T_n - \sigma_n^2 S \right\|_{HS} \leq C \left(n^{-1} + n^{\alpha p + 1} l(n) \right). \tag{5.10}$$

Plugging (5.10) into (5.9), then together with (5.8) into (5.7) and noting that

$$\frac{n^{-1 \vee (1 - \alpha p)}}{r_\alpha(n)} \to 0$$

as $n \to \infty$, it remains to show that

$$\sum_{r=1}^{p-1} \left\| f_{n,\cdot} \otimes_r f_{n,\cdot} \right\|_{\mathfrak{H}^{\otimes(2p-2r)} \otimes L^2([0,1])^{\otimes 2}} \leq C\, r_\alpha(n). \tag{5.11}$$

Now, as for any $s_1, s_2 \in [0,1]$, it holds that

$$\left\langle 1_{[\frac{s_1}{n},1]}(\cdot), 1_{[\frac{s_2}{n},1]}(\cdot) \right\rangle_{L^2([0,1])} \leq 1,$$

we have that for $r = 1, \dots, p \wedge q$,

$$\left\| f_{n,t} \otimes_r f_{n,t} \right\|_{\mathfrak{H}^{\otimes(p+q-2r)} \otimes L^2([0,1])^{\otimes 2}} \leq \left\| f_{n,1} \otimes_r f_{n,1} \right\|_{\mathfrak{H}^{\otimes(p+q-2r)}}.$$

In other words, the contraction norms of the kernels of the stochastic process $\{U_n(t) \colon t \in [0,1]\}$ are bounded by those of the random variable $U_n(1)$, so that (5.11) follows from the one-dimensional calculations in [32, Example 2.5]. □

Lemma 5.3. *In the setting of Theorem 5.1, it holds that*

$$\left\| T_n - \sigma_n^2 S \right\|_{\mathrm{HS}(L^2([0,1]))} \leq C \left(n^{-1} + n^{\alpha p + 1} l(n) \right). \tag{5.12}$$

Proof. The operator $K_n = T_n - \sigma_n^2 S$ is a Hilbert-Schmidt integral operator of the form $K_n f(t) = \int_0^1 k_n(s,t) f(s)\, \mathrm{d}s$, with kernel k_n given by

$$k_n(s,t) = \mathrm{E}\left(U_n(s) U_n(t)\right) - (s \wedge t) \sigma_n^2.$$

Note that by orthogonality

$$\mathrm{E}\left(U_n(s)U_n(t)\right) = p!\left\langle f_{n,s}, f_{n,t}\right\rangle_{\mathfrak{H}^{\otimes p}}, \tag{5.13}$$

where the kernels $f_{n,\cdot}$ are given by (5.6). Now

$$\begin{aligned}
\left\langle f_{p,n,s}, f_{p,n,t}\right\rangle_{\mathfrak{H}^{\otimes p}} &= \frac{1}{n}\sum_{i,j=1}^{n} 1_{[\frac{i}{n},1]}(s) 1_{[\frac{j}{n},1]}(t)\rho\left(|i-j|\right)^p \\
&= \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1-i}^{n-i} 1_{[\frac{i}{n},1]}(s) 1_{[\frac{j+i}{n},1]}(t)\rho\left(|j|\right)^p \\
&= \frac{1}{n}\sum_{j=-(n-1)}^{n-1}\sum_{i=1}^{n} 1_{[1-j,n-j]}(i) 1_{[\frac{i}{n},1]}(s) 1_{[\frac{j+i}{n},1]}(t)\rho(|j|)^p \\
&= \frac{1}{n}\sum_{j=-(n-1)}^{n-1}\sum_{i=1}^{n} 1_{[1-j,n-j]}(i) 1_{[\frac{i}{n},1]}(s) 1_{[\frac{j+i}{n},1]}(t)\rho(|j|)^p \\
&= A_n + B_n + C_n,
\end{aligned} \tag{5.14}$$

where the terms A_n, B_n and C_n are obtained by decomposing the sum over j according to

$$\sum_{j=-(n-1)}^{n-1}\beta(j) = \sum_{j=-(n-1)}^{-1}\beta(j) + \beta(0) + \sum_{j=1}^{n-1}\beta(j),$$

where

$$\beta(j) = \frac{1}{n}\sum_{i=1}^{n} 1_{[1-j,n-j]}(i) 1_{[\frac{i}{n},1]}(s) 1_{[\frac{j+i}{n},1]}(t)\rho(|j|)^p.$$

Now, we have

$$\begin{aligned}
A_n &= \frac{1}{n}\sum_{j=-(n-1)}^{-1} \#\left\{1 \le i \le n \colon 1-j \le i,\, i \le ns,\, i \le nt-j\right\}\rho\left(|j|\right)^p \\
&= \frac{1}{n}\sum_{j=-(n-1)}^{-1}\left(\lfloor ns \wedge (nt-j)\rfloor + j\right)\rho\left(|j|\right)^p \\
&= \sum_{j=-(n-1)}^{-1}\rho\left(|j|\right)^p \times \begin{cases} \frac{\lfloor ns\rfloor}{n} + \frac{j}{n} & \text{if } t-s > \frac{j}{n} \\ \frac{\lfloor nt\rfloor}{n} & \text{if } t-s \le \frac{j}{n} \end{cases}, \\
B_n &= \frac{1}{n}\#\left\{1 \le i \le n \colon i \le n(s\wedge t)\right\} = \frac{\lfloor n(s\wedge t)\rfloor}{n}
\end{aligned}$$

and

$$\begin{aligned}
C_n &= \frac{1}{n}\sum_{j=1}^{n-1}\#\left\{1 \le i \le n \colon i \le n-j,\, i \le ns,\, i \le nt-j\right\}\rho\left(|j|\right)^p \\
&= \frac{1}{n}\sum_{j=1}^{n-1}\left((n-j)\wedge\lfloor ns \wedge (nt-j)\rfloor\right)\rho\left(|j|\right)^p \\
&= \sum_{j=1}^{n-1}\rho\left(|j|\right)^p \times \begin{cases} \frac{\lfloor ns\rfloor}{n} & \text{if } t-s > \frac{j}{n} \\ \frac{\lfloor nt\rfloor}{n} - \frac{j}{n} & \text{if } t-s \le \frac{j}{n} \end{cases}.
\end{aligned}$$

Plugging (5.14) into (5.13) and using formula (2.1) for σ_n, this yields

$$\begin{aligned}&\mathrm{E}\left(U_n(s)U_n(t)\right) - (s\wedge t)\sigma_n \\ &\qquad = p!\left(A_n + B_n + C_n - (s\wedge t)\sum_{j=-(n-1)}^{n-1}\rho\left(|j|\right)^p\left(1-\frac{|j|}{n}\right)\right)\end{aligned}$$

and after a tedious but straightforward calculation (similarly as in [32, Proof of Theorem 2.2]), one arrives at

$$\begin{aligned}|k_n(s,t)| = \left|\mathrm{E}\left(U_n(s)U_n(t)\right) - (s\wedge t)\sigma_n\right| &\lesssim \frac{1}{n}\left(1 + p!\sum_{j=1}^{n-1} j\rho(|j|)^p\right) \\ &\lesssim n^{-1} + n^{\alpha p+1}l(n),\end{aligned}$$

where we have used Karamata's theorem to obtain the last estimate (see [32, Example 2.5] for details). Consequently,

$$\left\|T_n - \sigma_n^2 S\right\|_{\mathrm{HS}(L^2([0,1]))} = \|k_n\|_{L^2([0,1]^2)} \leq \sup_{s,t\in[0,1]}|k_n(s,t)| \leq C\,\left(n^{-1} + n^{\alpha p+1}l(n)\right)$$

as asserted. □

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Acknowledgments. The authors are grateful to two anonymous reviewers for their very careful reading of an earlier version of this manuscript, as well as numerous insightful comments and remarks.