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On limit theory for functionals of stationary increments Lévy driven moving averages

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Abstract

In this paper we present new limit theorems for variational functionals of stationary increments Lévy driven moving averages in the high frequency setting. More specifically, we will show the "law of large numbers" and a "central limit theorem", which heavily rely on the kernel, the driving Lévy process and the properties of the functional under consideration. The first order limit theory consists of three different cases. For one of the appearing limits, which we refer to as the ergodic type limit, we prove the associated weak limit theory, which again consists of three different cases. Our work is related to [10, 7], who considered power variation functionals of stationary increments Lévy driven moving averages. However, the asymptotic theory of the present paper is more complex. In particular, the weak limit theorems are derived for an arbitrary Appell rank of the involved functional.

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1 Introduction

The last two decades have witnessed a great progress in limit theory for high frequency functionals of continuous time stochastic processes. The interest in infill asymptotics has been motivated by the increasing availability of high frequency data in natural and social sciences such as finance, physics, biology or medicine. Limit theorems in the high frequency framework are an important probabilistic tool for the analysis of small scale fluctuations of the underlying stochastic process and have numerous applications in mathematical statistics e.g. in the field of parametric estimation and testing. Such limit theory has been investigated in various model classes including Itô semimartingales

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(see e.g. [6, 22, 23]), (multi)fractional Brownian motion and related processes (see e.g. [2, 3, 4, 5, 19, 25]), and many others.

In this paper we investigate the asymptotic theory for high frequency functionals of stationary increments Lévy driven moving averages. More specifically, we focus on an infinitely divisible process with stationary increments $(X_t)_{t\geq 0}$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, given as

$$X_t = \int_{-\infty}^{t} \big\{ g(t-s) - g_0(-s) \big\}\, dL_s, \tag{1.1}$$

where $L = (L_t)_{t\in\mathbb{R}}$ is a two-sided Lévy process with no Gaussian component and $L_0 = 0$, and $g, g_0 : \mathbb{R} \to \mathbb{R}$ are continuous functions vanishing on $(-\infty, 0)$. In particular, this class of stochastic processes contains the linear fractional stable motion, which has the form (1.1) with $g(s) = g_0(s) = s_+^{\alpha}$ and the driving Lévy process L is symmetric stable. The linear fractional stable motion is the most common heavy-tailed self-similar process, and hence exhibit both the Joseph and Noah effects of Mandelbrot, cf. [32, Chapter 7]. Fractional Lévy processes are other examples of processes of the form (1.1), see e.g. [28, Chapter 2.6.8]. Recent papers address various topics on linear fractional stable motions including analysis of semimartingale property [8], fine scale behavior [11, 18], simulation techniques [16] and statistical inference [1, 17, 26, 29]. We consider the class of variational functionals of the type

$$V(f;k)^n := a_n \sum_{i=k}^{n} f(b_n \Delta_{i,k}^n X), \tag{1.2}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a measurable function, $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ are suitable normalising sequences, and the operator $\Delta_{i,k}^n X$ denotes the kth order increments of X defined as

$$\Delta_{i,k}^n X := \sum_{j=0}^{k} (-1)^j \binom{k}{j} X_{(i-j)/n}, \qquad i \geq k. \tag{1.3}$$

The usual first and second order increments take the forms $\Delta_{i,1}^n X = X_{i/n} - X_{(i-1)/n}$ and $\Delta_{i,2}^n X = X_{i/n} - 2X_{(i-1)/n} + X_{(i-2)/n}$. The reason for considering general kth order increments lies in statistical applications. Indeed, using higher order increments, with $k \geq 2$, is often desirable since this gives rise to better convergence rates for various estimators (cf. [26]). This fact is also seen in our asymptotic results Theorems 2.1, 2.5 and 2.6. The choice of the normalising sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ depends on the interplay between the form of the kernel g, the infinitesimal properties of the driving Lévy process L and the growth/smoothness of the function f.

The asymptotic behaviour of statistics of the form (1.2) in the context of power variation, i.e. $f(x) = |x|^p$ for some $p > 0$, has been characterized in the work [10, 7]. Further papers on related topics include [27] that investigate asymptotic normality for functionals of the type (1.2) in the low frequency setting and for *bounded* functions f (the article [29] extends the results of [27] to certain unbounded functions). Much more is known about weak limit theory for statistics of *discrete* moving averages driven by heavy tailed i.i.d. noise; we refer to [21, 35, 36] among others. However, the asymptotic theory is investigated mostly for bounded functions f and under assumptions on the kernel and the noise process, which are not comparable to ours. We will conclude the discussion of related literature by mentioning the two papers [11, Section 5] and [18], which show "law of large numbers" results of the ergodic type in the context of fractional Lévy processes.

The aim of this work is to investigate the limit theorems for general functionals $V(f;k)^n$. We will start with first order asymptotic results, which consist of three different limits depending on the interplay between f, g and L. More specifically, the "laws of large numbers" include stable convergence towards a certain random variable, ergodic type convergence to a constant when the driving motion L is assumed to be symmetric β-stable and convergence in probability to an integral of some stochastic process. In the second step we will also prove three weak limit theorems associated with the ergodic type convergence, consisting of a central limit theorem and two convergence results towards stable distributions. Motivated by statistical applications, such as parametric estimation of linear fractional stable motion (cf. [1, 17, 26, 29]), we will apply our theory to functions f of the form

$$\begin{aligned} &f_1(x) = |x|^p, \quad p>0 && \text{(power variation)} \\ &f_2(x) = |x|^{-p}\mathbb{1}_{\{x\neq 0\}}, \quad p\in(0,1) && \text{(negative power variation)} \\ &f_3(x) = \cos(ux) \text{ or } \sin(ux) && \text{(empirical characteristic function)} \\ &f_4(x) = \mathbb{1}_{(-\infty,u]}(x) && \text{(empirical distribution function)} \\ &f_5(x) = \log(|x|)\mathbb{1}_{\{x\neq 0\}} && \text{(log-variation)} \end{aligned} \tag{1.4}$$

among others. One of the major difficulties when showing weak limit theorems lies in the fact that the ideas suggested in e.g. [21, 27, 35, 36] in the setting of bounded functions f do not directly extend to a more general class of functions (also the proofs in [10] for the power variation case use the specific form of the function $f(x) = |x|^p$). As it has been noticed in earlier papers on discrete moving averages (see e.g. [35, 36] and references therein) the *Appell rank* of the function f often plays an important role for the weak limit theory. It is defined as $m_\rho^\star = \min\{m\in\mathbb{N}:\ \Phi_\rho^{(m)}(0)\neq 0\}$ with

$$\Phi_\rho(x) := \mathbb{E}[f(x+\rho S)] - \mathbb{E}[f(\rho S)],$$

where S is a symmetric β-stable random variable with scale parameter 1, $\rho>0$ and $\Phi_\rho^{(m)}$ denotes the mth derivative of $x\mapsto\Phi_\rho(x)$. In this paper we will obtain weak limit theorems for an arbitrary Appell rank without assuming boundedness or a specific form of the function f. This is an important improvement over the existing results on limit theory for heavy tailed moving averages, which have never been investigated in this general setting. Our key observation is that it is much more convenient to impose assumptions on the function Φ, which are easy to check for all practical examples, rather than on the function f itself.

The paper is structured as follows. Section 2 presents the required assumptions, the main results and some remarks and examples. We present some preliminaries in Section 3. The proofs of the first order asymptotic results are collected in Section 4. Section 5 is devoted to the proofs of weak limit theorems, with a few more technical results postponed to Section 6.

2 The setting and main results

We start by introducing various definitions, notations and assumptions that will be important for the presentation of the main results. We recall that the *Blumenthal–Getoor index* of L is defined as

$$\beta := \inf\left\{r\geq 0: \int_{-1}^{1}|x|^r\,\nu(dx)<\infty\right\}\in[0,2],$$

where ν denotes the Lévy measure of L. Furthermore, $\Delta L_s := L_s - L_{s-}$ with $L_{s-} := \lim_{u\uparrow s,\,u<s} L_u$ stands for the jump size of L at point s. If L is stable with index of stability $\beta\in(0,2)$, the index of stability and the Blumenthal-Getoor index coincide, and both will be denoted by β. Let $\mathbb{F} = (\mathcal{F}_t)_{t\in\mathbb{R}}$ be the filtration generated by the Lévy process

L and $(T_m)_{m\geq 1}$ be a sequence of $\mathbb{F}$-stopping times that exhausts the jumps of $(L_t)_{t\geq 0}$. That is, $\{T_m(\omega): m \geq 1\} = \{t \geq 0 : \Delta L_t(\omega) \neq 0\}$ and $T_m(\omega) \neq T_n(\omega)$ for all $m \neq n$ with $T_m(\omega) < \infty$.

Our first set of conditions, which has been originally introduced in [10], concerns the behaviour of the Lévy measure ν at infinity and the functional form of the kernel g:

Assumption (A): *The function $g\colon \mathbb{R} \to \mathbb{R}$ satisfies*

$$g(t) \sim t^{\alpha} \qquad \text{as } t \downarrow 0 \quad \text{for some } \alpha > 0, \tag{2.1}$$

where $g(t) \sim w(t)$ as $t \downarrow 0$ means that $\lim_{t\downarrow 0} g(t)/w(t) = 1$. For some $\theta \in (0,2]$ it holds that $\limsup_{t\to\infty} \nu(x\colon |x| \geq t)t^{\theta} < \infty$ and $g - g_0$ is a bounded function in $L^{\theta}(\mathbb{R}_+)$. Furthermore, $g \in C^k((0,\infty))$ where k is as in (1.3), i.e. denoting the order of increments under consideration. Assume moreover there exists a $\delta > 0$ such that $|g^{(k)}(t)| \leq Ct^{\alpha-k}$ for all $t \in (0,\delta)$, and such that both $|g'|$ and $|g^{(k)}|$ are in $L^{\theta}((\delta,\infty))$, and are decreasing on (δ,∞).

Assumption (A) ensures in particular that the process X, introduced in (1.1), is well-defined in the sense of [30], see [10, Section 2.4]. When L is a β-stable Lévy process, we may and do choose $\theta = \beta$. By adjusting the Lévy measure ν, we may also include the case where (2.1) is replaced by $g(t) \sim c_0 t^{\alpha}$ as $t \downarrow 0$ for some $c_0 \neq 0$.

The limiting behaviour of $V(f;k)^n$ depends on the interplay of the order k of the increments introduced in (1.3), the Blumenthal-Getoor index β of the driving Lévy process, and the power α in (2.1) characterizing the behaviour of g at 0. Throughout this paper we reserve the symbols k, α, and β for these quantities and never use them in a different context.

For Theorem 2.1(i) below, we need to slightly strengthen Assumption (A) if $\theta = 1$:

Assumption (A-log): *In addition to (A) suppose that*

$$\int_{\delta}^{\infty} |g^{(k)}(s)|^{\theta} \log(1/|g^{(k)}(s)|)\, ds < \infty,$$

with δ and θ as in (A).

In order to formulate our main results, we require some more notation. For $p > 0$ we denote by $C^p(\mathbb{R})$ the space of $r := [p]$-times continuous differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that $f^{(r)}$ is locally $(p-r)$-Hölder continuous if $p \notin \mathbb{N}$. We introduce the function $h_k\colon \mathbb{R} \to \mathbb{R}$ by

$$h_k(x) := \sum_{j=0}^{k} (-1)^j \binom{k}{j} (x-j)_+^{\alpha}, \qquad x \in \mathbb{R}, \tag{2.2}$$

where $y_+ := \max\{y, 0\}$ for all $y \in \mathbb{R}$. We recall that a sequence $(Z^n)_{n\in\mathbb{N}}$ of random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in a Polish space $(E, \mathcal{E})$ converges stably in law to Z, which is defined on an extension $(\Omega', \mathcal{F}', \mathbb{P}')$ of the original probability space, if for all bounded continuous $g : E \to \mathbb{R}$ and for all bounded $\mathcal{F}$-measurable random variables Y it holds that

$$\mathbb{E}[g(Z^n)Y] \to \mathbb{E}'[g(Z)Y],$$

where $\mathbb{E}'$ denotes the expectation on the extended space. We denote the stable convergence in law by $Z^n \xrightarrow{\mathcal{L}-s} Z$, and refer to [20, 31] for more details. Note, in particular, that stable convergence in law is a stronger property than convergence in law, but a weaker property than convergence in probability. In the framework of stochastic processes we write $Z^n \xrightarrow{\text{u.c.p.}} Z$ for uniform convergence in probability, i.e. when $\sup_{t\in[0,T]} |Z^n_t - Z_t| \xrightarrow{\mathbb{P}} 0$ holds for all $T > 0$. Furthermore, we denote by $Z^n \xrightarrow{f.i.d.i.} Z$ the stable convergence of finite dimensional distributions.

2.1 Law of large numbers

Our first theorem presents the "law of large numbers" for the statistic $V(f;k)^n$ defined at (1.2). The sequence $(U_m)_{m\geq 1}$ below is i.i.d. $\mathcal{U}(0,1)$-distributed, defined on an extension $(\Omega',\mathcal{F}',\mathbb{P}')$ and independent of $\mathbb{F}$. Here and throughout the paper we denote by $S\beta S(\rho)$ the symmetric β-stable distribution with scale parameter $\rho>0$, that is $Y\sim S\beta S(\rho)$ if $\mathbb{E}[\exp(i\theta Y)]=\exp(-|\rho\theta|^\beta)$ for all $\theta\in\mathbb{R}$.

Theorem 2.1. *Suppose Assumption (A) holds and assume that the Blumenthal–Getoor index satisfies $\beta<2$. Let θ,α and k be as in Assumption (A). The following hold.*

(i) Let $k>\alpha$ and suppose that (A-log) holds if $\theta=1$. If f is such that $f\in C^p(\mathbb{R})$ and $f^{(j)}(0)=0$ for $j=0,\dots,[p]$, for some $p>\beta\vee\frac{1}{k-\alpha}$. Then, taking $a_n=1$ and $b_n=n^\alpha$ we have the stable convergence

$$V(f;k)^n \xrightarrow{\mathcal{L}-s} \sum_{m:\,T_m\in[0,1]}\sum_{l=0}^{\infty} f\big(\Delta L_{T_m}h_k(l+U_m)\big).$$

(ii) Suppose that L is a symmetric β-stable Lévy process with scale parameter $\rho_L>0$. Moreover, assume that $\mathbb{E}[|f(L_1)|]<\infty$, and $H:=\alpha+1/\beta<k$. Then, setting $a_n=1/n$ and $b_n=n^H$, we obtain

$$V(f;k)^n \xrightarrow{\mathbb{P}} \mathbb{E}[f(\rho_0 S)], \tag{2.3}$$

where $S\sim S\beta S(1)$ and $\rho_0=\rho_L\|h_k\|_{L^\beta(\mathbb{R})}$.

(iii) Suppose that $(1\vee\beta)(k-\alpha)<1$ and that f is continuous and satisfies $|f(x)|\leq C(1\vee|x|^q)$ for all $x\in\mathbb{R}$, for some $q,C>0$ with $q(k-\alpha)<1$. With the normalising sequences $a_n=1/n$ and $b_n=n^k$ it holds that

$$V(f;k)^n \xrightarrow{\mathbb{P}} \int_0^1 f(F_u)\,du$$

where $(F_u)_{u\in\mathbb{R}}$ is defined by

$$F_u=\int_{-\infty}^u g^{(k)}(u-s)\,dL_s \quad \text{a.s. for all } u\in\mathbb{R}. \tag{2.4}$$

Theorem 2.1 may be viewed as a generalization of [10, Theorem 1.1] from power variation to general functionals. The limiting random variable in Theorem 2.1(i) is indeed well-defined, as we show in Lemma 4.1 below. We remark that one of the conditions of Theorem 2.1(i) is the restriction $\alpha<k-1/p$. This restriction on the parameter α gets weaker when p gets larger, but on the other hand the condition $f\in C^p(\mathbb{R})$ is stronger for a larger p. Thus, there is a trade-off between these two conditions.

The three cases of the theorem are closely related to the three limits for the power variation derived in [10, Theorem 1.1]. Let us briefly explain the main intuition behind Theorems 2.1(ii) and (iii). We use the symbol $\approx$ to denote that the difference of left and right hand side term converge to 0, in probability.

The crucial step in the proof of Theorem 2.1(ii) is the approximation

$$\Delta^n_{i,k}X\approx\Delta^n_{i,k}Y \qquad \text{in probability}$$

where $(Y_t)_{t\in[0,\infty)}$ is the linear fractional stable motion defined via

$$Y_t:=\int_{\mathbb{R}}\{(t-s)_+^\alpha-(-s)_+^\alpha\}\,dL_s.$$

It is well known that the process Y is H-self-similar and its increment process is ergodic (see e.g. [15]). Hence, under assumptions of Theorem 2.1(ii), we may conclude by Birkhoff's ergodic theorem for e.g. $k=1$:

$$V(f;1)^n \approx \frac{1}{n}\sum_{i=1}^{n} f\left(n^H(Y_{i/n}-Y_{(i-1)/n})\right) \stackrel{d}{=} \frac{1}{n}\sum_{i=1}^{n} f\left(Y_i - Y_{i-1}\right) \xrightarrow{\text{a.s.}} \mathbb{E}[f(Y_1-Y_0)].$$

This is exactly the statement of (2.3) for the case $k=1$.

Under the assumptions of Theorem 2.1(iii) it turns out that the stochastic process F defined at (2.4) is a version of the kth derivative of X. Hence, we conclude by Taylor expansion:

$$V(f;k)^n = \frac{1}{n}\sum_{i=k}^{n} f(n^k\Delta_{i,k}^n X) \approx \frac{1}{n}\sum_{i=k}^{n} f\left(F_{(i-1)/n}\right) \xrightarrow{\mathbb{P}} \int_0^1 f(F_u)\,du, \quad \text{as } n\to\infty.$$

This explains the statement of Theorem 2.1(iii).

Remark 2.2. In contrast to the power variation case investigated in [10], the assumptions of Theorems 2.1(i) and (ii), and of Theorems 2.1(i) and (iii), are not mutually exclusive, and hence two limit theorems can hold at the same time. This phenomenon appears already in the simpler setting of Lévy processes. Assume for example that L is a symmetric β-stable Lévy process and consider the function $f(x)=\sin^2(x)$. If $k=1$ and we choose $a_n=b_n=1$ we deduce the convergence

$$\sum_{i=1}^{n} \sin^2(\Delta_{i,1}^n L) \xrightarrow{\text{a.s.}} \sum_{m:\, T_m\in[0,1]} \sin^2(\Delta L_{T_m}) < \infty,$$

using, in particular, $|f(x)|\le Cx^2$. On the other hand when we choose the normalising sequences $a_n=n^{-1}$ and $b_n=n^{1/\beta}$ we readily deduce by the strong law of large numbers that

$$\frac{1}{n}\sum_{i=1}^{n} \sin^2(n^{1/\beta}\Delta_{i,1}^n L) \stackrel{\text{d}}{=} \frac{1}{n}\sum_{i=1}^{n} \sin^2(L_i-L_{i-1}) \xrightarrow{\text{a.s.}} \mathbb{E}[\sin^2(L_1)].$$

This example shows that we can obtain two different limits for two different scalings. □

In the next step we present a functional version of Theorem 2.1. For this purpose we introduce the sequence of processes

$$V(f;k)_t^n := a_n \sum_{i=k}^{[nt]} f(b_n\Delta_{i,k}^n X).$$

In the proposition below we will use the Skorokhod M_1-topology, which was introduced in [34]. For a detailed exposition we refer to [39].

Proposition 2.3. *Suppose Assumption (A) holds and assume that the Blumenthal–Getoor index satisfies $\beta<2$. We have the following three cases:*

(i) Under the conditions of Theorem 2.1(i) we have the stable convergence

$$V(f;k)_t^n \xrightarrow{f.i.d.i.} V(f;k)_t := \sum_{m:T_m\in[0,t]} \sum_{l=0}^{\infty} f\big(\Delta L_{T_m} h_k(l+U_m)\big).$$

Moreover, the stable convergence also holds with respect to Skorokhod M_1-topology if additionally the following assumption is satisfied:

(FC) Each of the two functions $x\mapsto f(x)\mathbb{1}_{\{x\ge 0\}}$ and $x\mapsto f(x)\mathbb{1}_{\{x<0\}}$ is either non-negative or non-positive.

(ii) Under the conditions of Theorem 2.1(ii) we have

$$V(f;k)_t^n \xrightarrow{u.c.p.} t\mathbb{E}[f(\rho_0 S)],$$

where S and ρ_0 have been introduced in (2.3).

(iii) Under the conditions of Theorem 2.1(iii) we have

$$V(f;k)_t^n \xrightarrow{u.c.p.} \int_0^t f(F_u)\,du$$

where $(F_u)_{u\in\mathbb{R}}$ has been defined at (2.4).

We remark that the uniform convergence results of Proposition 2.3(ii) and (iii) are easily obtained from Theorem 2.1(ii) and (iii) by the following argument. Observe the decomposition $f = f^+ - f^-$, where f^+ (resp. f^-) denotes the positive (resp. negative) part of f. Then f^+, f^- satisfy the same assumptions as f in the setting of Theorem 2.1(ii) and (iii). Furthermore, since $f^+, f^- \geq 0$, the statistics $V(f^+;k)_t^n$ and $V(f^-;k)_t^n$ are increasing in t and the corresponding limits in Proposition 2.3(ii) and (iii) are continuous in t. Consequently, the uniform convergence is obtained from the pointwise convergence by Dini's theorem.

2.2 Weak limit theorems

In this section we present weak limit theorems associated to the ergodic type limit from Theorem 2.1(ii). Throughout this section we assume that $\mathbb{E}[|f(S)|] < \infty$, where $S \sim S\beta S(1)$. As mentioned in the introduction, the crucial quantity in this context is the function Φ_ρ defined via

$$\Phi_\rho(x) = \mathbb{E}[f(x+\rho S)] - \mathbb{E}[f(\rho S)], \qquad x \in \mathbb{R},\ \rho > 0.$$

Similarly to limit theory for discrete moving averages, see e.g. [21, 35, 36], the Appell rank of the function f often plays a key role for the asymptotic behaviour of the statistic $V(f;k)^n - \mathbb{E}[f(\rho_0 S)]$. In our setting, the Appell rank $m_\rho^\star$ is defined as

$$m_\rho^\star := \min\{r \in \mathbb{N} \,:\, \Phi_\rho^{(r)}(0) \neq 0\},$$

where $\Phi_\rho^{(r)}(x) := \frac{\partial^r}{\partial x^r}\Phi_\rho(x)$ for $r = 1, 2, \ldots$. Note that we have Appell rank one if and only if $\Phi'_\rho(0) \neq 0$, and Appell rank greater or equal two if and only if $\Phi'_\rho(0) = 0$. The Appell rank is an analogue of the Hermite rank used in the context of Gaussian processes. However, the non-Gaussian case is usually much more complicated due to the lack of orthogonal series expansions. While the Appell rank $m_\rho^\star$ usually depends on the parameter ρ, we always have that $m_\rho^\star = 1$ for all $\rho > 0$ in the framework of the imaginary part of the characteristic function $f_3(x) = \sin(ux)$ and the empirical distribution function f_4 (cf. Remark 6.7). Moreover, $m_\rho^\star > 1$ for all $\rho > 0$ when f is an even function, in fact, in this case we have that $0 = \frac{\partial}{\partial x}\Phi_\rho(0) = \frac{\partial^2}{\partial x \partial \rho}\Phi_\rho(0)$ (cf. Remark 6.7). Indeed, $m_\rho^\star > 1$ for all $\rho > 0$ therefore holds in the setting of power variations f_1 and f_2, real part of the characteristic function $f_3(x) = \cos(ux)$ and the log-variation f_5.

For our weak limit theorems we will need the following smoothness assumptions on Φ_ρ:

Assumption (B): *The function $(\rho, x) \mapsto \Phi_\rho(x)$ is $C^{1,2}((0,\infty)\times\mathbb{R})$, and for all $\varepsilon \in (0,1)$ there are $p \in [0,1]$ and $C > 0$ such that, for all $\rho \in [\varepsilon, \varepsilon^{-1}]$ and $x, y \in \mathbb{R}$*

$$\Big|\Phi_\rho(x) - \Phi_\rho(y)\Big| \leq C|x-y|^p, \tag{2.5}$$

$$\Big|\frac{\partial^{j+r}}{\partial x^j \partial \rho^r}\Phi_\rho(x)\Big| \leq C \qquad \text{for all } j = 0,1,2 \text{ and } r = 0,1 \text{ with } r + j > 0. \tag{2.6}$$

Note that (2.6) implies Lipschitz continuity of Φ_ρ, and therefore the p-Hölder assumption (2.5) may be viewed as a growth condition on Φ_ρ. In particular, (2.5) implies that $|\Phi_\rho(x)| \leq C|x|^p$. Note also that (2.6) implies (2.5) with $p = 1$, however, in several cases we need $p < 1$. Before presenting our main weak limit theorems, we remark that Assumption (B) is satisfied for our key examples, its proof is postponed to the end of Section 6.

Remark 2.4. The following two classes of functions satisfy the assumption (B).

(i) (*Bounded functions*). Any bounded measurable function f satisfies (B) for any $p \in [0,1]$. This covers, in particular, the empirical distribution function $f_4(x) = \mathbb{1}_{(-\infty,u]}(x)$, and the empirical characteristic functions $f_3(x) = \sin(ux)$ or $f_3 = \cos(ux)$ from (1.4), where $u \in \mathbb{R}$ is a fixed real number.

(ii) (*A class of unbounded functions*). Suppose that $f \in L^1_{\text{loc}}(\mathbb{R})$ and there exists $K > 0$ and $q \leq 1$ such that $f \in C^3([-K,K]^c)$ and $|f'(x)|, |f''(x)|, |f'''(x)| \leq C$ and $|f'(x)| \leq C|x|^{q-1}$ for $|x| > K$. Then f satisfies (B) with $p = q$ when $q > 0$, and $p = 0$ when $q < 0$. This covers, in particular, the power functions $f(x) = |x|^q \mathbb{1}_{\{x \neq 0\}}$ where $q \in (-1,0) \cup (0,1]$, that is, f_1 and f_2 from (1.4). Furthermore, the logarithmic function $f_5(x) = \log(|x|)\mathbb{1}_{\{x \neq 0\}}$ from (1.4) is also covered by the above condition and hence satisfies (B). In this case we may choose any $p \in (0,1]$. □

In the following we will need to strengthen Assumption (A).

Assumption (A2): *Suppose that Assumption (A) holds for α and k. In addition, assume that $|g^{(k)}(t)| \leq Ct^{\alpha-k}$ for all $t > 0$, and for the function $\zeta : (0,\infty) \to \mathbb{R}$ defined as $\zeta(t) = g(t)t^{-\alpha}$ the limit $\lim_{t \downarrow 0} \zeta^{(j)}$ exists in $\mathbb{R}$ for all $j = 0, ..., k$.*

In the following two theorems we present weak limit results associated with Theorem 2.1(ii) in the case of "short memory" (small α) or "long memory" (large α). The long memory case depends heavily on the Appell rank of the function f, whereas the short memory case does not depend on the Appell rank. In the theorems below we follow the notation of Theorem 2.1, i.e. L is a symmetric β-stable Lévy process with scale parameter ρ_L, (X_t) is given by (1.1), $H = \alpha + 1/\beta$, $\rho_0 = \rho_L \|h_k\|_{L^\beta(\mathbb{R})}$, $S \sim S\beta S(1)$, $a_n = 1/n$ and $b_n = n^H$.

Theorem 2.5 ("Short memory"). *Assume that (A), (A2) and (B) hold, that p in (B) is such that $p < \beta/2$, and $\mathbb{E}[f(L_1)^2] < \infty$. Assume furthermore that $\alpha < k - 2/\beta$. We then have*

$$\sqrt{n}\Big(V(f;k)^n - \mathbb{E}[f(\rho_0 S)]\Big) \xrightarrow{\mathcal{L}} \mathcal{N}(0,\eta^2), \tag{2.7}$$

where the variance is given as $\eta^2 := \lim_{m\to\infty} \eta_m^2$ with η_m defined in (5.16).

Theorem 2.6 ("Long memory"). *Assume that (A), (A2) and (B) hold*
(i) (Appell rank=1). If $m^\star_{\rho_0} = 1$, $p = 1$, $\beta \in (1,2)$ and $\alpha + 1/\beta < k < \alpha + 1$, then we have the convergence in law

$$n^{k-\alpha-1/\beta}\Big(V(f;k)^n - \mathbb{E}[f(\rho_0 S)]\Big) \xrightarrow{\mathcal{L}} S\beta S(\sigma), \tag{2.8}$$

where the scale parameter σ is given by (5.22).
(ii) (Appell rank>1). If $p < \beta/2$, $\frac{\partial}{\partial x}\Phi_\rho(0) = 0 = \frac{\partial^2}{\partial x \partial \rho}\Phi_\rho(0)$, for all $\rho \in (0,\infty)$, and $\alpha + 1/\beta < k < \alpha + 2/\beta$, then we have the convergence in law

$$n^{1-\frac{1}{(k-\alpha)\beta}}\Big(V(f;k)^n - \mathbb{E}[f(\rho_0 S)]\Big) \xrightarrow{\mathcal{L}} \mathcal{S}((k-\alpha)\beta, 0, \rho_1, \eta_1), \tag{2.9}$$

where the right hand side denotes the $(k-\alpha)\beta$-stable distribution with location parameter 0, scale parameter ρ_1 and skewness parameter η_1, which are specified in (5.46).

Remark 2.7. (i) We note that the limiting distribution in Theorem 2.6(i) is only non-degenerate in the Appell rank one case, or more precisely when $\frac{\partial}{\partial x}\Phi_{\rho_0}(0) \neq 0$, which follows from (5.40).

(ii) We also remark that the condition $m_\rho^\star \geq 2$ in Theorem 2.6(ii) is required to hold for *all* $\rho > 0$, which is in strong contrast to the discrete framework of e.g. [36] where only assumptions on $m_{\rho_0}^\star$ are made. The reason for our stronger condition on the Appell rank is the fact that the scaled increments $n^H \Delta_{i,k}^n X$ are only asymptotically SβS(ρ_0)-distributed.

(iii) Theorems 2.5 and 2.6(ii) give a rather complete picture of possible limits when the Appell rank is strictly large than one. Indeed, we cover all cases $\alpha \in (0, k-1/\beta)$ except the critical value of $\alpha = k - 2/\beta$. This is not the case for the setting of Appell rank one. Not only we need to assume that $\beta \in (1,2)$, but we also have that $k - 2/\beta < k-1$. Hence, the limit theory in the framework of $\beta \in (0,1]$, and also $\beta \in (1,2)$ with $\alpha \in [k-2/\beta, k-1]$, is still an open problem.

(iv) Notice that Theorem 2.5, which has the fastest rate of convergence, never holds for $k=1$ since $\beta \in (0,2)$. Hence, for the purpose of statistical estimation, it makes sense to use higher values of k to end up in the setting of Theorem 2.5. We refer to [26] for more details on statistical applications using higher order increments. □

Similarly to Proposition 2.3 one might be able to prove the functional versions of Theorems 2.5 and 2.6. However, we dispense with the precise exposition of these results in this paper.

2.3 Outline of the proofs of Theorems 2.5 and 2.6

The strategy of the three proofs Theorems 2.5, 2.6(i) and 2.6(ii) are quite different, and are briefly outlined in the following.

- For the proof of Theorem 2.5 we approximate $V(f;k)^n$ by

$$V_{n,m} = \sum_{i=k}^{n} \big(f(n^H \Delta_{i,k}^n X^m) - \mathbb{E}[f(n^H \Delta_{i,k}^n X^m)] \big), \quad \text{where}$$
$$X_t^m = \int_{t-m/n}^{t} \big\{ g(t-s) - g_0(-s) \big\}\, dL_s.$$

  More precisely, the main part of the proof is to show

$$\lim_{m\to\infty} \limsup_{n\to\infty} \mathbb{E}[n^{-1}(V(f;k)^n - V_{n,m})^2] = 0.$$

  It is then sufficient to establish asymptotic normality of $(V_{n,m})_{n\in\mathbb{N}}$ for each $m \geq 1$, which follows by the central limit theorem for m-dependent sequences of random variables. This general approach to deriving central limit theorems is popular in the literature, see [27] for an example.

- The main idea of the proof of Theorem 2.6(i) is to approximate $V(f;k)^n$, in a suitable sense, by a linear functional V_n of $(n^H \Delta_{i,k}^n X)_{i=k}^n$ given by

$$V_n = c_n \sum_{i=k}^{n} n^H \Delta_{i,k}^n X, \qquad n \in \mathbb{N},$$

  where c_n are certain chosen constants. With such an approximation in hand, the proof boils down to showing that the $S\beta S$-stable random variables V_n converge in distribution.

- For the proof of Theorem 2.6(ii) we decompose $V(f;k)^n$ as

$$V(f;k)^n = \sum_{r=k}^{n} K_r + \sum_{r=k}^{n} Z_r \tag{2.10}$$

where $\{Z_r\}_{k\geq n}$ is suitable defined i.i.d. sequence of random variables to be defined in (5.43) below. We argue that the first sum, on the right-hand side of (2.10), is asymptotically negligible and that the random variables Z_r are in the domain of attraction of a $(k-\alpha)\beta$-stable random variable with location parameter 0, scale parameter ρ_1 and skewness parameter η_1 as defined in (5.46) in the proof. Similar decompositions have been applied to derive stable limit theorems for discrete time moving averages, see for example [21].

3 Preliminaries

Throughout all our proofs we denote by C a generic positive constant that does not depend on n or ω, but may change from line to line. For a random variable Y and $q>0$ we denote $\|Y\|_q = \mathbb{E}[|Y|^q]^{1/q}$. Throughout this paper we will repeatedly use the fact that if L is a symmetric β-stable Lévy process with scale parameter ρ_L, then for each measurable function ψ with $\int_{-\infty}^{\infty} |\psi(s)|^\beta \, ds < \infty$ the integral $\int_{\mathbb{R}} \psi(s)\, dL_s$ is a symmetric β-stable random variable with scale parameter

$$\rho_L \left(\int_{\mathbb{R}} |\psi(s)|^\beta \, ds \right)^{1/\beta} = \rho_L \|\psi\|_{L^\beta(\mathbb{R})}, \tag{3.1}$$

see [32, Proposition 3.4.1]. We will also frequently use the notation

$$g_{i,k}^n(s) := \sum_{j=0}^{k} (-1)^j \binom{k}{j} g((i-j)/n - s), \tag{3.2}$$

which leads to the expression

$$\Delta_{i,k}^n X = \int_{-\infty}^{i/n} g_{i,k}^n(s)\; dL_s \tag{3.3}$$

for the the kth order increments of X. For the functions $g_{i,k}^n$ we have the following simple estimates from [10].

Lemma 3.1 (Lemma 3.1 in [10])**.** *Suppose that Assumption (A) is satisfied. It holds that*

$$\begin{aligned}
&|g_{i,k}^n(s)| \leq C(i/n - s)^\alpha && \text{for } s \in [(i-k-1)/n, i/n],\\
&|g_{i,k}^n(s)| \leq Cn^{-k}((i-k)/n - s)^{\alpha-k} && \text{for } s \in (i/n-\delta, (i-k-1)/n), \text{ and}\\
&|g_{i,k}^n(s)| \leq Cn^{-k}\big(\mathbb{1}_{[(i-k)/n-\delta, i/n-\delta]}(s) + g^{(k)}((i-k)/n - s)\mathbb{1}_{(-\infty,(i-k)/n-\delta)}(s)\big),\\
& && \text{for } s \in (-\infty, i/n-\delta].
\end{aligned}$$

We briefly recall the definition and some properties of the Skorokhod M_1-topology, as it is not as widely used as the J_1-topology. It was originally introduced by Skorokhod [34] by defining a metric on the completed graphs of càdlàg functions, where the completed graph of ϕ is defined as

$$\Gamma_\phi = \{(x,t) \in \mathbb{R}\times\mathbb{R}_+ \,:\, x = \alpha\phi(t-) + (1-\alpha)\phi(t), \text{ for some } \alpha \in [0,1]\}.$$

The M_1-topology is weaker than the J_1-topology but still strong enough to make many important functionals, such as supremum and infimum, continuous. It can be shown that the stable convergence in Theorem 2.1(i) does not hold with respect to the J_1-topology (cf. [9]). Since the M_1-topology is metrizable, it is completely characterized through convergence of sequences, which we describe in the following. A sequence ϕ_n of functions in $\mathbb{D}(\mathbb{R}_+,\mathbb{R})$ converges to $\phi \in \mathbb{D}(\mathbb{R}_+,\mathbb{R})$ with respect to the Skorokhod M_1-topology if and only if $\phi_n(t) \to \phi(t)$ for all t in a dense subset of $[0,\infty)$, and for all $t_\infty \in [0,\infty)$ it holds that

$$\lim_{\delta\downarrow 0} \limsup_{n\to\infty} \sup_{0\le t\le t_\infty} w(\phi_n, t, \delta) = 0.$$

Here, the oscillation function w is defined as

$$w(\phi,t,\delta) = \sup_{0\vee(t-\delta)\le t_1<t_2<t_3\le(t+\delta)\wedge t_\infty} \{|\phi(t_2) - [\phi(t_1),\phi(t_3)]|\},$$

where for $b<a$ the interval $[a,b]$ is defined to be $[b,a]$, and $|a-[b,c]| := \inf_{d\in[b,c]} |a-d|$. We refer to [39] for more details on the M_1-topology.

4 Proof of Theorem 2.1

4.1 Proofs of Theorem 2.1(i) and Proposition 2.3(i)

We concentrate on the proof of Proposition 2.3(i), since it is a stronger statement than Theorem 2.1(i). The proof is divided into three parts. First, we assume that L is a compound Poisson process and show the finite dimensional stable convergence for the statistic $V(f;k)^n_t$. Thereafter we argue that the convergence holds in the functional sense with respect to the M_1-topology, when f satisfies condition (FC). Finally, the results are extended to general Lévy processes by truncation. For this step, an isometry for Lévy integrals, which is due to [30], plays a key role.

Since $C^q(\mathbb{R}) \subset C^p(\mathbb{R})$ for $p<q$ we may and do assume that $p \notin \mathbb{N}$. Note that, if $f \in C^p(\mathbb{R})$ and $f^{(j)}(0) = 0$ for all $j = 0,\dots,[p]$, then for any $N>0$ there exists a constant C_N such that

$$|f^{(j)}(x)| \le C_N |x|^{p-j}, \quad \text{for all } x\in[-N,N], \text{ and } j = 0,\dots[p]. \tag{4.1}$$

By the assumption $p > \frac{1}{k-\alpha}$, this implies the following estimate to be used in the proof below. For all $N>0$ there is a constant C_N such that

$$|f^{(j)}(x)| \le C_N |x|^{\gamma_j}, \quad \text{for all } x\in[-N,N], \text{ and } j = 0,\dots,[p], \tag{4.2}$$

where $\gamma_j = \frac{p-j}{p(k-\alpha)}$. The following lemma ensures in particular that the limit in Theorem 2.1(i) exists.

Lemma 4.1. *Let $t>0$ be fixed. Under conditions of Theorem 2.1(i) there exists a finite random variable $K>0$ such that*

$$\sum_{m:\,T_m\in[0,t]} \sum_{l=0}^{\infty} \big|f\big(\Delta L_{T_m} h_k(l+U_m)\big)\big| \le K, \quad \textit{and} \tag{4.3}$$

$$\sum_{m:\,T_m\in[0,t]} \sum_{l=0}^{n-1} \big|f\big(\Delta L_{T_m} n^\alpha g_{i_m+l,n}(T_m)\big)\big| \le K, \quad \textit{for all } n, \tag{4.4}$$

where i_m denotes the random index such that $T_m \in \big(\frac{i_m-1}{n}, \frac{i_m}{n}\big]$.

Proof. Throughout the proof, K denotes a positive random variable that does not depend on n, but may change from line to line. For the first inequality note that $|h_k(l+U_m)| \le C(l-k)^{\alpha-k}$ for all $l>k$ and $|h_k(l+U_m)| \le C$ for $l\in\{0,...,k\}$. This implies in particular

$$|\Delta L_{T_m}(\omega) h_k(l+U_m)| \le \begin{cases} C(l-k)^{\alpha-k}\sup_{s\in[0,t]}|\Delta L_s|, & \text{for } l>k \\ C\sup_{s\in[0,t]}|\Delta L_s|, & \text{for } l\in\{0,...,k\}. \end{cases}$$

Therefore, we find by (4.1) a random variable K such that

$$\big|f\big(\Delta L_{T_m} h_k(l+U_m)\big)\big| \le K\big|\Delta L_{T_m} h_k(l+U_m)\big|^p$$

for all $l \ge 0$ and all m. Consequently, the left-hand side of (4.3) is dominated by

$$K\Bigg(\sum_{m:\,T_m\in[0,t]} |\Delta L_{T_m}|^p + \sum_{m:\,T_m\in[0,t]} |\Delta L_{T_m}|^p \sum_{l=k+1}^{\infty} (l-k)^{(\alpha-k)p}\Bigg) \le \widetilde{K},$$

for some random variable $\widetilde{K}$, where we used that $(\alpha-k)p < -1$, and that

$$\sum_{m:\,T_m\in[0,t]} |\Delta L_{T_m}|^p < \infty, \quad \text{since } p > \beta.$$

The inequality (4.4) follows by the same arguments since Lemma 3.1 implies the existence of a constant $C > 0$ such that for all $n \in \mathbb{N}$

$$\begin{aligned} &n^\alpha g_{i_m+l,n}(T_m) \le C && \text{for } l \in \{0,...,k\}, \text{ and} \\ &n^\alpha g_{i_m+l,n}(T_m) \le C(l-k)^{\alpha-k}, && \text{for } l \in \{k+1,...,n-1\}. \end{aligned}$$

□

4.1.1 Compound Poisson process as driving process

In this subsection, we show the finite dimensional stable convergence of $V(f;k)^n_t$ under the assumption that L is a compound Poisson process. The extension to functional convergence when condition (FC) is satisfied follows in the next subsection, the extension to general L thereafter.

Let $0 \le T_1 < T_2 < ...$ denote the jump times of $(L_t)_{t\ge0}$. For $\varepsilon > 0$ we define

$$\begin{aligned}\Omega_\varepsilon = \big\{\omega\in\Omega : &\text{ for all } m \text{ with } T_m(\omega)\in[0,t] \text{ we have } |T_m(\omega)-T_{m-1}(\omega)| > \varepsilon \\ &\text{and } \Delta L_s(\omega) = 0 \text{ for all } s\in[-\varepsilon,0] \text{ and } |\Delta L_s(\omega)| \le \varepsilon^{-1} \text{ for all } s\in[0,t]\big\}.\end{aligned}$$

We note that $\Omega_\varepsilon \uparrow \Omega$, as $\varepsilon \downarrow 0$. Letting

$$M_{i,n,\varepsilon} := \int_{i/n-\varepsilon}^{i/n} g^n_{i,k}(s)\, dL_s, \quad \text{and} \quad R_{i,n,\varepsilon} := \int_{\infty}^{i/n-\varepsilon} g^n_{i,k}(s)\, dL_s,$$

we have the decomposition $\Delta^n_{i,k}X = M_{i,n,\varepsilon} + R_{i,n,\varepsilon}$. It turns out that $M_{i,n,\varepsilon}$ is the asymptotically dominating term, whereas $R_{i,n,\varepsilon}$ is negligible as $n\to\infty$. We show that, on Ω_ε,

$$\sum_{i=k}^{[nt]} f(n^\alpha M_{i,n,\varepsilon}) \xrightarrow{f.i.d.i.} Z_t, \quad \text{where} \quad Z_t := \sum_{m:\,T_m\in[0,t]} \sum_{l=0}^{\infty} f(\Delta L_{T_m} h_k(l+U_m)), \tag{4.5}$$

as $n\to\infty$. Here $(U_m)_{m\ge1}$ are independent identically $\mathcal{U}([0,1])$-distributed random variables, defined on an extension $(\Omega', \mathcal{F}', \mathbb{P}')$ of the original probability space, that are independent of $\mathcal{F}$. For this step, the following expression for the left hand side is instrumental. On Ω_ε it holds that

$$\sum_{i=k}^{[nt]} f(n^\alpha M_{i,n,\varepsilon}) = V^{n,\varepsilon}_t,$$

where

$$V_t^{n,\varepsilon} := \sum_{m:T_m\in(0,[nt]/n]} \sum_{l=0}^{v_t^m} f(n^\alpha \Delta L_{T_m} g_{i_m+l,n}(T_m)). \tag{4.6}$$

Here, i_m denotes the random index such that $T_m \in ((i_m-1)/n, i_m/n]$, and v_t^m is defined as

$$v_t^m = v_t^m(\varepsilon, n) := \begin{cases} [\varepsilon n] \wedge ([nt]-i_m) & \text{if } T_m - ([\varepsilon n]+i_m)/n > -\varepsilon, \\ [\varepsilon n] - 1 \wedge ([nt]-i_m) & \text{if } T_m - ([\varepsilon n]+i_m)/n \leq -\varepsilon. \end{cases} \tag{4.7}$$

Additionally, we set $v_t^m = \infty$ if $T_m > [nt]/n$. The following lemma proves (4.5).

Lemma 4.2. *On Ω_ε, for $r \geq 1$, and $0 \leq t_1 < \cdots < t_r \leq t$, we have the stable convergence*

$$(V_{t_1}^{n,\varepsilon}, \dots, V_{t_r}^{n,\varepsilon}) \xrightarrow{\mathcal{L}-s} (Z_{t_1}, \dots, Z_{t_r}), \quad \text{as } n \to \infty.$$

Proof. By arguing as in [10, Section 5.1], we deduce for any $d \geq 1$ the stable convergence in law

$$\{n^\alpha g_{i_m+l,n}(T_m)\}_{l,m\leq d} \xrightarrow{\mathcal{L}-s} \{h_k(l+U_m)\}_{l,m\leq d}$$

as $n \to \infty$. Defining

$$V_t^{n,d} := \sum_{m\leq d:\, T_m\in(0,[nt]/n]} \sum_{l=0}^{d} f(n^\alpha \Delta L_{T_m} g_{i_m+l,n}(T_m)) \quad \text{and}$$

$$Z_t^d := \sum_{m\leq d:\, T_m\in(0,t]} \sum_{l=0}^{d} f(\Delta L_{T_m} h_k(l+U_m)),$$

we obtain by the continuous mapping theorem for stable convergence in law

$$(V_{t_1}^{n,d}, \dots, V_{t_r}^{n,d}) \xrightarrow{\mathcal{L}-s} (Z_{t_1}^d, \dots, Z_{t_r}^d), \quad \text{as } n \to \infty, \tag{4.8}$$

for all $d \geq 1$. Therefore, by a standard approximation argument (cf. [13, Theorem 3.2]), it is sufficient to show that

$$\limsup_{n\to\infty} \left\{ \max_{t\in\{t_1,\dots,t_r\}} |V_t^{n,\varepsilon} - V_t^{n,d}| \right\} \xrightarrow{\text{a.s.}} 0, \qquad \text{as } d \to \infty, \quad \text{and} \tag{4.9}$$

$$\sup_{s\in[0,t]} |Z_s^d - Z_s| \xrightarrow{\text{a.s.}} 0, \qquad \text{as } d \to \infty. \tag{4.10}$$

For all $s \in [0,t]$ and sufficiently large n we have

$$\begin{aligned} |V_s^{n,d} - V_s^{n,\varepsilon}| &\leq \sum_{m\leq d:\, T_m\in(0,[ns]/n]} \sum_{l=d\wedge v_t^m}^{d\vee v_t^m} |f(\Delta L_{T_m} n^\alpha g_{i_m+l,n}(T_m))| \\ &\quad + \sum_{m>d:\, T_m\in(0,[ns]/n]} \sum_{l=0}^{v_t^m} |f(\Delta L_{T_m} n^\alpha g_{i_m+l,n}(T_m))| \\ &\leq \sum_{m:\, T_m\in(0,t]} \sum_{l=d\wedge v_t^m}^{n-1} |f(\Delta L_{T_m} n^\alpha g_{i_m+l,n}(T_m))| \\ &\quad + \sum_{m>d:\, T_m\in(0,[nt]/n]} \sum_{l=0}^{n-1} |f(\Delta L_{T_m} n^\alpha g_{i_m+l,n}(T_m))|. \end{aligned}$$

Therefore, (4.9) follows from Lemma 4.1 by the dominated convergence theorem since the random index $v_t^m = v_t^m(n,\omega)$ satisfies $\liminf_{n\to\infty} v_t^m(n,\omega) = \infty$, almost surely. Lemma 4.1 also implies (4.10), since

$$\sup_{s\in[0,t]} |Z_s^d - Z_s| \leq$$

$$\sum_{m\leq d:\, T_m\in(0,t]} \sum_{l=d+1}^{\infty} |f(\Delta L_{T_m} h_k(l+U_m))| + \sum_{m>d:\, T_m\in(0,t]} \sum_{l=0}^{\infty} |f(\Delta L_{T_m} h_k(l+U_m))|.$$

The lemma now follows from (4.8), (4.9) and (4.10). □

Recalling the decomposition (4.5) and applying the triangle inequality, the proof can be completed by showing that

$$J_n := \sum_{i=k}^{[nt]} |f(n^\alpha \Delta_{i,k}^n X) - f(n^\alpha M_{i,n,\varepsilon})| \xrightarrow{\text{a.s.}} 0, \tag{4.11}$$

as $n\to\infty$. We first argue that, on Ω_ε, $\{n^\alpha M_{i,n,\varepsilon}, n^\alpha \Delta_{i,k}^n X\}_{n\in\mathbb{N}, i\in\{k,\dots,[nt]\}}$ are uniformly bounded by a constant on Ω_ε, which will allow us to apply the estimate (4.1). The random variables $M_{i,n,\varepsilon}$ satisfy by construction either $|n^\alpha M_{i,n,\varepsilon}| = 0$ or $|n^\alpha M_{i,n,\varepsilon}| = |n^\alpha g_{i,k}^n(T_m)\Delta L_{T_m}|$ for some m, where we recall that on Ω_ε it holds that $T_m - T_{m-1} > \varepsilon$. Consequently, they are uniformly bounded by Lemma 3.1, where we used that $k>\alpha$ and that the jumps of L are bounded on Ω_ε. The uniform boundedness of $n^\alpha \Delta_{i,k}^n X = n^\alpha(M_{i,n,\varepsilon} + R_{i,n,\varepsilon})$ follows by [10, Eqs. (4.8), (4.12)] which implies that for any $\eta>0$

$$\sup_{n\in\mathbb{N},\ i\in\{k,\dots,[nt]\}} \left\{ n^{k-\eta} |R_{i,n,\varepsilon}| \right\} < \infty, \quad \text{almost surely.} \tag{4.12}$$

In order to show (4.11) we apply Taylor expansion for f at $n^\alpha M_{i,n,\varepsilon}$, and bound the terms in the Taylor expansion using (4.1) and the following lemma.

Lemma 4.3. *Let $\psi:\mathbb{R}\to\mathbb{R}$ be continuous and such that $|\psi(x)|\leq C|x|^\gamma$ for all $x\in[-1,1]$ for some $\gamma\in(0,1/(k-\alpha))$. It holds on Ω_ε that*

$$\limsup_{n\to\infty} \left\{ n^{(k-\alpha)\gamma-1} \sum_{i=k}^{[nt]} |\psi(n^\alpha M_{i,n,\varepsilon})| \right\} \leq C, \quad \textit{a.s.}$$

Proof. We have on Ω_ε

$$\sum_{i=k}^{[nt]} |\psi(n^\alpha M_{i,n,\varepsilon})| = W_t^{n,\varepsilon},$$

where

$$W_t^{n,\varepsilon} := \sum_{m:T_m\in(0,[nt]/n]} \sum_{l=0}^{v_{t_\infty}^m} |\psi(n^\alpha \Delta L_{T_m} g_{i_m+l,n}(T_m))|,$$

and $v_{t_\infty}^m$ is the random index defined in (4.7). By Lemma 3.1 the random variables $n^\alpha g_{i_m+l,n}(T_m)$ are bounded for $l=0,\dots,k$. For $l\in\{k+1,\dots,n-1\}$, Lemma 3.1 implies that $n^\alpha g_{i_m+l,n}(T_m)\leq C(l-k)^{\alpha-k}$. Since the random index $v_{t_\infty}^m$ satisfies $v_{t_\infty}^m < n$ for all m, we obtain on Ω_ε

$$\sum_{i=k}^{[nt]} |\psi(n^\alpha M_{i,n,\varepsilon})| \leq C \sum_{m:T_m\in(0,t]} \left(\sum_{l=0}^{k} |n^\alpha g_{i_m+l,n}(T_m)|^\gamma + \sum_{l=k+1}^{n} |(l-k)^{\alpha-k}|^\gamma \right).$$

It follows by comparison with the integral $\int_{k+1}^n (s-k)^{(\alpha-k)\gamma}\, ds$ that the right hand side multiplied with $n^{(k-\alpha)\gamma-1}$ is convergent, where we used that $(\alpha-k)\gamma\in(-1,0)$ and that the number of jumps of $L(\omega)$ in $[0,t]$ is uniformly bounded for $\omega\in\Omega_\varepsilon$. □

Considering the sum J_n in (4.11), Taylor expansion up to order $r=[p]$ shows that

$$J_n \le \sum_{i=k}^{[nt]} \big|n^\alpha R_{i,n,\varepsilon} f'(n^\alpha M_{i,n,\varepsilon})\big| + \cdots + \frac{1}{r!}\sum_{i=k}^{[nt]} \big|(n^\alpha R_{i,n,\varepsilon})^r f^{(r)}(n^\alpha M_{i,n,\varepsilon})\big| + TR_r$$
$$:= S_1 + \cdots + S_r + TR_r, \tag{4.13}$$

where TR_r denotes the Taylor rest term. Recalling the estimate (4.2), we can now estimate the jth Taylor monomial S_j for $j=0,\dots,[p]$ by applying Lemma 4.3 on $\psi = f^{(j)}$, where we remark that $\gamma_j = \frac{p-j}{p(k-\alpha)} \in (0, 1/(k-\alpha))$. Using (4.12) and recalling that $p > k-\alpha$, we obtain that for sufficiently small $\eta > 0$

$$\frac{1}{j!}\sum_{i=k}^{[nt]} \big|(n^\alpha R_{i,n,\varepsilon})^j f^{(j)}(n^\alpha M_{i,n,\varepsilon})\big| \le Cn^{-j/p-\eta} \sum_{i=k}^{[nt]} \big|f^{(j)}(n^\alpha M_{i,n,\varepsilon})\big|$$
$$\le Cn^{-\eta}, \tag{4.14}$$

where the second inequality follows from Lemma 4.3 since $(k-\alpha)\gamma_j - 1 = -j/p$. For the Taylor rest term TR_r we obtain by the mean value theorem:

$$TR_r = \frac{1}{r!}\sum_{i=k}^{[nt]} \big|(n^\alpha R_{i,n,\varepsilon})^r \big(f^{(r)}(\xi_{i,n}) - f^{(r)}(n^\alpha M_{i,n,\varepsilon})\big)\big|,$$

with $\xi_{i,n} \in (n^\alpha |M_{i,n,\varepsilon}|, n^\alpha |X_{i,n,\varepsilon}|)$. Since $n^\alpha |M_{i,n,\varepsilon}|$ and $n^\alpha |X_{i,n,\varepsilon}|$ are bounded and $f^{(r)}$ is locally $(p-r)$-Hölder continuous, it follows that

$$TR_r \le Cn \sup_{n\in\mathbb{N},\ i\in\{k,\dots,[nt]\}} |n^\alpha R_{i,n,\varepsilon}|^p.$$

From (4.12) it follows that $TR_r \to 0$ as $n\to\infty$, where we recall that $(\alpha-k)p < -1$. Together with (4.13) and (4.14) this implies $J_n \xrightarrow{\text{a.s.}} 0$, and it follows that

$$\sup_{s\in[0,t]} \left\{ \left| V(f;k)_s^n - \sum_{i=k}^{[ns]} f(n^\alpha M_{i,n,\varepsilon}) \right| \right\} \xrightarrow{\text{a.s.}} 0$$

on Ω_ε. Now, the proposition follows from Lemma 4.2 by letting $\varepsilon \to 0$.

4.1.2 Functional convergence

In this subsection we show that if f satisfies (FC) and under the assumption that L is a compound Poisson process, the convergence in Proposition 2.3(i) holds in the functional sense with respect to the Skorokhod M_1-topology. To this end, we denote by $\xrightarrow{\mathcal{L}_{M_1}-s}$ the stable convergence of càdlàg processes on $\mathbb{D}([0,t];\mathbb{R})$ equipped with the Skorokhod M_1-topology. We first replace (FC) by the following stronger auxiliary assumption.

(FC') It holds that f is either non-negative or non-positive.

This assumption puts us into the comfortable situation that our limiting process is monotonic. Recall the definition of the processes $V^{n,\varepsilon}$ and Z introduced in (4.5) and (4.6), respectively. In Lemma 4.2 the stable convergence of the finite dimensional distributions of $V^{n,\varepsilon}$ to Z was shown. The functional convergence $V^{n,\varepsilon} \xrightarrow{\mathcal{L}_{M_1}-s} Z$ on Ω_ε follows from the following lemma.

Lemma 4.4. *The sequence of $\mathbb{D}([0,t])$-valued random variables $(V^{n,\varepsilon}\mathbb{1}_{\Omega_\varepsilon})_{n\ge1}$ is tight with respect to the Skorokhod M_1-topology.*

Proof. It is sufficient to show that the conditions of [39, Theorem 12.12.3] are satisfied. Condition (i) is satisfied, since the family of real valued random variables $(V_t^{n,\varepsilon})_{n\geq 1}$ is tight by Lemma 4.2. Condition (ii) is satisfied, since the oscillating function w_s introduced in [39, Chapter 12, (5.1)] satisfies $w_s(V^{n,\varepsilon},\theta)=0$ for all $\theta>0$ and all n, since $V^{n,\varepsilon}$ is monotonic by assumption (FC'). □

Lemma 4.5. *Let $(X_n)_{n\in\mathbb{N}}$ be a tight sequence of stochastic processes in $(\mathbb{D}([0,t]),M_1)$, and let X be a stochastic process in $(\mathbb{D}([0,t]),M_1)$ such that $X_n \stackrel{f.i.d.i.}{\longrightarrow} X$. Then $X_n \stackrel{\mathcal{L}-s}{\longrightarrow} X$ in $(\mathbb{D}([0,t]),M_1)$.*

Proof. In the following we equip $\mathbb{D}([0,t])$ with the the M_1-metric, and recall that $\mathbb{D}([0,t])$ is a Polish space, see [39, Section 12.8]. For any subsequence $(n_k)_{k\in\mathbb{N}}$, $(X_{n_k})_{k\in\mathbb{N}}$ is tight in $\mathbb{D}([0,t])$ and hence there exists a subsequence $(k_l)_{l\in\mathbb{N}}$ such that $(X_{n_{k_l}})_{l\in\mathbb{N}}$ converges stably in law in $\mathbb{D}([0,t])$, cf. Proposition 3.4(1) in [20]. Since $X_n \stackrel{f.i.d.i.}{\longrightarrow} X$ it follows that $X_{n_{k_l}} \stackrel{\mathcal{L}-s}{\longrightarrow} X$ in $\mathbb{D}([0,t])$, which implies $X_n \stackrel{\mathcal{L}-s}{\longrightarrow} X$ in $\mathbb{D}([0,t])$ since $(n_k)_{k\in\mathbb{N}}$ was arbitrarily chosen. □

The functional convergence in Proposition 2.3(i) follows when f satisfies (FC') by Lemmas 4.2, 4.4 and 4.5. Now, for general f satisfying condition (FC) we decompose $f=f_++f_-$ with $f_+(x)=f(x)\mathbb{1}_{\{x>0\}}$ and $f_-(x)=f(x)\mathbb{1}_{\{x<0\}}$. Both functions f_+ and f_- satisfy (FC'), and the functional convergence of $V(f_+;k)^n$ and $V(f_-;k)^n$ follows, with the corresponding limits denoted by Z^+ and Z^-. Note that Z^+ jumps exactly at those times, where the Lévy process L jumps upward, and Z^- at those, where it jumps downward. In particular, Z^+ and Z^- do not jump at the same time, which implies that summation is continuous at (Z^+,Z^-) with respect to the M_1-topology (cf. [39, Theorem 12.7.3]). Thus, an application of the continuous mapping theorem yields the convergence of $V(f;k)^n=V(f_+;k)^n+V(f_-;k)^n$ towards $Z=Z^++Z^-$. Let us stress that indeed the sole reason why the extra condition (FC) is required for functional convergence is that summation is not continuous on the Skorokhod space in general, and the convergence of $V(f_+;k)^n$ and $V(f_-;k)^n$ does not necessarily imply the convergence of $V(f;k)^n$.

4.1.3 Extension to infinite activity Lévy processes

In this section we extend the results of Proposition 2.3(i) to moving averages driven by a general Lévy process L, by approximating L by a sequence of compound Poisson processes $(\hat{L}(j))_{j\geq 1}$. To this end we introduce the following notation. Let N be the jump measure of L, that is $N(A):=\#\{t:(t,\Delta L_t)\in A\}$ for measurable $A\subset\mathbb{R}\times(\mathbb{R}\setminus\{0\})$, and define for $j\in\mathbb{N}$

$$X_t(j):=\int_{(-\infty,t]\times[-\frac{1}{j},\frac{1}{j}]}\{(g(t-s)-g_0(-s))x\}N(ds,dx).$$

Denote $\hat{X}_t(j):=X_t-X_t(j)$. The results of the last section show that Proposition 2.3(i) holds for $\hat{X}(j)$, since it is a moving average driven by a compound Poisson process. By letting $j\to\infty$ we will show that the theorem remains valid for X by deriving the following approximation result

Lemma 4.6. *Suppose that f satisfies the conditions of Proposition 2.3(i). It holds that*

$$\lim_{j\to\infty}\limsup_{n\to\infty}\mathbb{P}\left(\sup_{s\in[0,t]}|V(X,f;k)^n_s-V(\hat{X}(j),f;k)^n_s|>\varepsilon\right)=0,\quad \textit{for all } \varepsilon>0. \tag{4.15}$$

Proof. In the following we say that a family $\{Y_{n,j}\}_{n,j\in\mathbb{N}}$ of random variables is *asymptotically tight* if for any $\varepsilon>0$ there is an $N>0$ such that

$$\limsup_{n\to\infty}\mathbb{P}(|Y_{n,j}|>N)<\varepsilon,\quad \text{for all } j\in\mathbb{N}.$$

We deduce first for $p>\beta\vee\frac{1}{k-\alpha}$ the asymptotic tightness of the two families

$$\left\{\sum_{i=k}^{[nt]}|n^\alpha\Delta_{i,k}^n\hat{X}(j)|^p\right\}_{n,j\in\mathbb{N}}\quad\text{and}\quad\left\{\max_{i=k,\dots,[nt]}|n^\alpha\Delta_{i,k}^n\hat{X}(j)|\right\}_{n,j\in\mathbb{N}},\tag{4.16}$$

and tightness of

$$\left\{\sum_{i=k}^{[nt]}|n^\alpha\Delta_{i,k}^nX|^p\right\}_{n\in\mathbb{N}}\quad\text{and}\quad\left\{\max_{i=k,\dots,[nt]}|n^\alpha\Delta_{i,k}^nX|\right\}_{n\in\mathbb{N}}.\tag{4.17}$$

The authors of [10] showed the stable convergences in law

$$\sum_{i=k}^{[nt]}|n^\alpha\Delta_{i,k}^n\hat{X}(j)|^p\xrightarrow{\mathcal{L}-s}Z_j,\qquad\text{and}\qquad\sum_{i=k}^{[nt]}|n^\alpha\Delta_{i,k}^nX|^p\xrightarrow{\mathcal{L}-s}Z,\tag{4.18}$$

where Z_j and Z are defined as in [10, Eq. (4.34)]. The asymptotic tightness of the first family of random variables in (4.16) follows from the tightness of the family $\{Z_j\}_{j\in\mathbb{N}}$, see [10, Eq. (4.35)]. The asymptotic tightness of the second family of random variables from (4.16) follows from the tightness of the first family by the estimate $\max_{i=1,\dots,n}|a_i|\le\left(\sum_{i=1}^n|a_i|^p\right)^{1/p}$ for $a_1,\dots,a_n\in\mathbb{R}$. The second statement of (4.18) implies (4.17) by similar arguments. The (asymptotic) tightness of the two families on the right-hands side of (4.16) and (4.17) allows us, for the proof of (4.15), to assume that $|n^\alpha\Delta_{i,k}^n\hat{X}(j)|$ and $|n^\alpha\Delta_{i,k}^nX|$ are uniformly bounded by some $N_0>0$.

Consider first the case $p<1$. By local Hölder-continuity of f of order p we have that

$$\sup_{s\in[0,t]}|V(f,X;k)_s^n-V(f,\hat{X}(j);k)_s^n|\le C_{N_0}\sum_{i=k}^{[nt]}|n^\alpha\Delta_{i,k}^nX(j)|^p,$$

and (4.15) follows from [10, Lemma 4.2], where we used that $p>\beta\vee\frac{1}{(k-\alpha)}$. Let now $p>1$. We can find $\xi_{i,n,j}\in[n^\alpha\Delta_{i,k}^n\hat{X}(j),n^\alpha\Delta_{i,k}^nX]$ such that $|f(n^\alpha\Delta_{i,k}^n\hat{X}(j))-f(n^\alpha\Delta_{i,k}^nX)|=|n^\alpha\Delta_{i,k}^nX(j)f'(\xi_{i,n,j})|$, and with $\gamma=\frac{p-1}{p}\left(\beta\vee\frac{1}{k-\alpha}\right)$ we obtain by (4.1) that

$$\begin{aligned}|f(n^\alpha\Delta_{i,k}^n\hat{X}(j))-f(n^\alpha\Delta_{i,k}^nX)|&\le C_{N_0}|n^\alpha\Delta_{i,k}^nX(j)||\xi_{i,n,j}|^{p-1}\\&\le C_{N_0}|n^\alpha\Delta_{i,k}^nX(j)||\xi_{i,n,j}|^\gamma\le C|n^\alpha\Delta_{i,k}^nX(j)|^{\gamma+1}+C|n^\alpha\Delta_{i,k}^nX(j)||n^\alpha\Delta_{i,k}^nX|^\gamma,\end{aligned}$$

where in the second inequality we used that $\gamma<p-1$ by assumption and that $\xi_{i,n,j}\in[-N_0,N_0]$, and the last inequality follows from the triangle inequality. Thus, in order to complete the proof of (4.15), it is sufficient to show that for all $\varepsilon>0$ we obtain

$$\lim_{j\to\infty}\limsup_{n\to\infty}\mathbb{P}\left(\sum_{i=k}^{[nt]}|n^\alpha\Delta_{i,k}^nX(j)|^{\gamma+1}>\varepsilon\right)=0,\quad\text{and}\tag{4.19}$$

$$\lim_{j\to\infty}\limsup_{n\to\infty}\mathbb{P}\left(\sum_{i=k}^{[nt]}|n^\alpha\Delta_{i,k}^nX(j)||n^\alpha\Delta_{i,k}^nX|^\gamma>\varepsilon\right)=0.\tag{4.20}$$

By definition it holds that $\gamma+1>\beta\vee\frac{1}{k-\alpha}$, and (4.19) follows from [10, Lemma 4.2]. For (4.20) we choose Hölder conjugates θ_1 and $\theta_2=\theta_1/(\theta_1-1)$ with $\theta_1\in\left(\beta\vee\frac{1}{k-\alpha},p\right)$, where we used that $p > 1$. The Hölder inequality and the estimate $\mathbb{P}(|XY| > \varepsilon) \leq \mathbb{P}(|X| > \varepsilon/N) + P(|Y| > N)$ for any $N > 0$ leads to the decomposition

$$\begin{aligned}
&\mathbb{P}\Big(\sum_{i=k}^{[nt]} |n^{\alpha}\Delta_{i,k}^n X(j)||n^{\alpha}\Delta_{i,k}^n X|^{\gamma} > \varepsilon\Big) \\
&\qquad \leq \mathbb{P}\Big(\sum_{i=k}^{[nt]} |n^{\alpha}\Delta_{i,k}^n X(j)|^{\theta_1} > \Big(\frac{\varepsilon}{N}\Big)^{\theta_1}\Big) + \mathbb{P}\Big(\sum_{i=k}^{[nt]} |n^{\alpha}\Delta_{i,k}^n X|^{\gamma\theta_2} > N^{\theta_2}\Big) \\
&\qquad =: J_{n,j,N}^1 + J_{n,j,N}^2.
\end{aligned}$$

Since $\theta_1 > \beta \vee \frac{1}{k-\alpha}$, yet another application of [10, Lemma 4.2] yields that

$$\lim_{j\to\infty} \limsup_{n\to\infty} J_{n,j,N}^1 = 0 \quad \text{for all } N > 0\text{, and all } \varepsilon > 0.$$

Moreover, $\theta_1 < p$ implies $\gamma\theta_2 > \beta \vee \frac{1}{k-\alpha}$. Therefore, it follows from the asymptotic tightness of the first family of random variables from (4.17) that

$$\limsup_{j\to\infty} \limsup_{n\to\infty} J_{n,j,N}^2 \to 0, \quad \text{as } N \to \infty,$$

which completes the proof of the lemma. □

Finally, the proof of Proposition 2.3(i) can be completed by letting $j \to \infty$. More precisely, we introduce for $j \in \mathbb{N}$ the stopping times

$$T_{m,j} := \begin{cases} T_m & \text{if } |\Delta L_{T_m}| > 1/j, \\ \infty & \text{else.} \end{cases}$$

The results of the last two subsections show that

$$V(\hat{X}(j), f; k)_t^n \xrightarrow{f.i.d.i.} Z_t^j := \sum_{m:\, T_{m,j}\in[0,t]} \sum_{l=0}^{\infty} f(\Delta L_{T_{m,j}} h_k(l - U_m)),$$

and that the convergence holds in the functional sense with respect to the M_1-topology if f satisfies (FC). From Lemma 4.1 and an application of the dominated convergence theorem it follows that

$$\sup_{s\in[0,t]} |Z_s - Z_s^j| \xrightarrow{\text{a.s.}} 0, \quad \text{as } j \to \infty.$$

Proposition 2.3(i) follows therefore from Lemma 4.6 and a standard approximation argument (cf. [13, Theorem 3.2]). □

4.2 Proof of Theorem 2.1(ii)

As mentioned earlier the proof relies upon replacing the increments of X by the increments of its tangent process, which is the linear fractional stable motion. To make this approximation precise we will use a scaling argument to transfer the Theorem 2.1(ii) into to a low-frequency result. For all $n \geq 1$ let $g_n(x) = n^{\alpha} g(x/n)$, $x \in \mathbb{R}$, and set

$$\phi_k^n(x) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} g_n(x-j), \qquad \text{and} \qquad Y_i^n = \int_{\mathbb{R}} \phi_n(i-s)\, dL_s.$$

The self-similarity of L of index $1/\beta$ implies that for all $n \in \mathbb{N}$ that

$$\{n^H \Delta_{i,k}^n X \colon i = k, \dots, n\} \stackrel{d}{=} \{Y_i^n \colon i = k, \dots, n\}. \tag{4.21}$$

In fact, to show (4.21) it is enough to note that for all $\theta_k,\dots,\theta_n\in\mathbb{R}$ we have that

$$\begin{aligned}\mathbb{E}\Big[\exp\Big(i\sum_{j=k}^n\theta_j n^H\Delta_{j,k}^n X\Big)\Big]&=\mathbb{E}\Big[\exp\Big(i\int_{\mathbb{R}}\Big(\sum_{j=k}^n\theta_j n^H g_{j,k}^n(s)\Big)\,dL_s\Big)\Big]\\&=\exp\Big(-\rho_L^\beta\int_{\mathbb{R}}\Big|\sum_{j=k}^n\theta_j n^H g_{j,k}^n(s)\Big|^\beta ds\Big)=\exp\Big(-\rho_L^\beta\int_{\mathbb{R}}\Big|\sum_{j=k}^n\theta_j\phi_k^n(j-u)\Big|^\beta du\Big)\\&=\mathbb{E}\Big[\exp\Big(i\sum_{j=k}^n\theta_j Y_j^n\Big)\Big],\end{aligned}\tag{4.22}$$

where the first equality follows by (3.3), the second equality follows by (3.1), the third equality follows by the substitution $u=ns$ and the definitions of ϕ_n and g_n, and the fourth equality follows also follows by (3.1). From (4.21) we obtain for all $n\in\mathbb{N}$ that

$$V(f;X)^n\overset{d}{=}\frac{1}{n}\sum_{i=k}^n f(Y_i^n).\tag{4.23}$$

For fixed $n\in\mathbb{N}$, $\{(Y_i^\infty,Y_i^n)\}_{i=k,\dots}$ is a two-dimensional stationary sequence. Indeed, this follows by a substitution argument similar to the one used in (4.22). Hence, by the triangle inequality we have that

$$\begin{aligned}\mathbb{E}\Big[\Big|\frac{1}{n}\sum_{i=k}^n f(Y_i^n)-\frac{1}{n}\sum_{i=k}^n f(Y_i^\infty)\Big|\Big]&\le\frac{1}{n}\sum_{i=k}^n\mathbb{E}[|f(Y_i^n)-f(Y_i^\infty)|]\\&\le\mathbb{E}[|f(Y_k^n)-f(Y_k^\infty)|].\end{aligned}\tag{4.24}$$

From [10, Eq. (4.44)] we deduce that $\mathbb{E}[|Y_k^n-Y_k^\infty|^p]\to0$ as $n\to\infty$ for all $p<\beta$, which by Lemma 6.5 used on $p=1$ implies that $\mathbb{E}[|f(Y_k^n)-f(Y_k^\infty)|]$ as $n\to\infty$, and hence the right-hand side of (4.24) converges to zero as $n\to\infty$.

Furthermore, set

$$Y_i^\infty=\int_{\mathbb{R}}h_k(i-s)\,dL_s,\qquad i\in\mathbb{N},$$

(recall the definition of h_k in (2.2)). We note that for $H=\alpha+1/\beta<1$, Y^∞ is the k-order increments of the linear fractional stable motion. When $H=\alpha+1/\beta\ge1$, the linear fractional stable motion is not well-defined, but Y^∞ remains well-defined when $H<k$ since h_k is locally bounded and satisfied $|h_k(x)|\le Kx^{\alpha-k}$ for $x\ge k+1$, and therefore $h_k\in L^\beta(\mathbb{R})$. Process $(Y_t^\infty)_{t\in\mathbb{R}}$ is mixing since it is a symmetric stable moving average, see e.g. [15]. This implies, in particular, that the discrete time stationary sequence $\{Y_j\}_{j\in\mathbb{Z}}$ is mixing and hence ergodic. According to Birkhoff's ergodic theorem (cf. [24, Theorem 10.6])

$$\frac{1}{n}\sum_{i=k}^n f(Y_i^\infty)\to\mathbb{E}[f(Y_k^\infty)]\qquad\text{almost surely and in }L^1\tag{4.25}$$

as $n\to\infty$. We now conclude that $V(f;X)^n\xrightarrow{\mathbb{P}}\mathbb{E}[f(Y_k^\infty)]$ as $n\to\infty$ by (4.23), (4.24) and (4.25), which completes the proof of Theorem 2.1(ii).

4.3 Proof of Theorem 2.1(iii)

Let us first remark that the growth condition $|f(x)|\le C(1\vee|x|^q)$ for some q with $q(k-\alpha)<1$ is weaker for larger q and can therefore be thought of as

$$|f(x)|\le C|x|^{\frac{1}{k-\alpha}-\varepsilon}\quad\text{for }|x|\to\infty,$$

if $k > \alpha$. Whereas for $k \leq \alpha$ we require only that f is of polynomial growth. Since by the assumptions of the theorem we have $k - \alpha < 1$, we may and do assume that $q > 1$. We recall that a function $\xi : \mathbb{R} \to \mathbb{R}$ is absolutely continuous if there exists a locally integrable function ξ' such that

$$\xi(t) - \xi(s) = \int_s^t \xi'(u)\, du, \quad \text{for all } s < t.$$

This implies that ξ is differentiable almost everywhere and the derivative coincides with ξ' almost everywhere. If ξ' can be chosen absolutely continuous we say that ξ is two times absolutely continuous, and similarly we define k-times absolute continuity.

By an application of [14, Theorem 5.1] it has been shown in [10, Lemma 4.3] that under the condition $(k - \alpha)(1 \vee \beta) > 1$ the process X admits a k-times absolutely continuous version and the k-th derivative is a version of the process $(F_u)_{u\in\mathbb{R}}$ defined in (2.4). Moreover, [10, Lemma 4.3] shows that for every $q \geq 1$, $q \neq \theta$ with $q(k - \alpha) < 1$ the process F admits a version with sample paths in $L^q([0,1])$, almost surely, which implies $\int_0^1 |f(F_u)|\, du \leq \int_0^1 C(1 \wedge |F_u|^q) du \leq C + C\int_0^1 |F_u|^q du < \infty$. With these prerequisites at hand, Theorem 2.1(iii) is a consequence of the following Lemma, which despite its intuitive statement requires some work. We denote by $W^{k,q}$ the space of k-times absolutely continuous functions ξ on $[0,1]$ satisfying $\xi^{(k)} \in L^q([0,1])$.

Lemma 4.7. *Let $\xi \in W^{k,q}$, and suppose that f is continuous and $|f(x)| \leq C(1 \vee |x|^q)$ for some $q \geq 1$. As $n \to \infty$ it holds that*

$$V(\xi; f, k)^n := n^{-1} \sum_{i=k}^n f(n^k \Delta_{i,k}^n \xi) \to \int_0^1 f(\xi_s^{(k)})\, ds. \tag{4.26}$$

Proof. Assume first $\xi \in \mathcal{C}^{k+1}([0,t])$. Taylor approximation shows that

$$n^k \Delta_{i,k}^n \xi = \xi^{(k)}_{\frac{i-k}{n}} + a_{i,n},$$

where $|a_{i,n}| \leq C/n$ for all $n \geq 1$, $k \leq i \leq n$. We can therefore assume without loss of generality that f has compact support and admits a concave modulus of continuity ω_f, i.e. a continuous increasing function $\omega_f : [0,\infty) \to [0,\infty)$ with $\omega_f(0) = 0$ such that $|f(x) - f(y)| \leq \omega_f(|x-y|)$ for all x, y. We have by Jensen's inequality that

$$\limsup_{n\to\infty} \left| V(\xi, f, k)^n - \frac{1}{n}\sum_{i=k}^n f\big(\xi^{(k)}_{\frac{i-k}{n}}\big) \right| \leq \limsup_{n\to\infty} \left\{ \omega_f\left(\frac{1}{n}\sum_{i=k}^n |a_{i,n}| \right) \right\} = 0.$$

The result follows by the convergence of Riemann sums

$$\frac{1}{n}\sum_{i=k}^n f\big(\xi^{(k)}_{\frac{i-k}{n}}\big) \to \int_0^1 f(\xi_s^{(k)})\, ds.$$

In the following we extend the result to general $\xi \in W^{k,q}$ by approximating ξ with a sequence $(\xi^m)_{m\geq 1}$ of functions in $C^{k+1}([0,1])$. To this end, choose ξ^m such that

$$\int_0^1 |\xi_s^{(k)} - \xi_s^{m,(k)}|^q\, ds \leq 1/m, \quad \text{for all } m. \tag{4.27}$$

Indeed, the existence of such a sequence follows since continuous functions are dense in $L^q([0,1])$. Note that (4.27) and Jensen's inequality imply that $\int_0^1 |\xi_s^{(k)} - \xi_s^{m,(k)}|\, ds \leq C/m^{1/q}$, since we assumed $q \geq 1$. Since $\xi^{m,(k)}$ converges in $L^q([0,1])$, the family $(|\xi^{m,(k)}|^q)_{m\geq 1}$ is uniformly integrable. Hence, by the assumption $|f(x)| \leq C(1 \vee |x|^q)$

for $x \in \mathbb{R}$, we obtain uniform integrability of $\{f(\xi^{m,(k)})_{m\geq 1}\}$. By continuity of f, we have that $f(\xi^{m,(k)}) \to f(\xi^{(k)})$ in measure (with respect to the Lebesgue measure on $[0,1]$), and thus by uniform integrability also in $L^1([0,1])$:

$$\limsup_{m\to\infty} \int_0^1 |f(\xi_s^{(k)}) - f(\xi_s^{m,(k)})|\, ds = 0.$$

Hence, (4.26) follows if we show

$$\limsup_{m\to\infty} \sup_{n\in\mathbb{N}} |V(\xi;f,k)^n - V(\xi^m;f,k)^n| = 0. \tag{4.28}$$

In order to show (4.28) we split the sum

$$|V(\xi;f,k)^n - V(\xi^m;f,k)^n| \leq \frac{1}{n}\sum_{i=k}^{n} \left| f(n^k \Delta_{i,k}^n \xi) - f(n^k\Delta_{i,k}^n \xi^m)\right|$$

into sums over the following sets of indices, where N and M are positive constants:

$$\begin{aligned} A_n^N &= \{i \in \{k,...,n\} \,:\, n^k|\Delta_{i,k}^n \xi| > N\} \\ B_{m,n}^{N,M} &= \{i \in \{k,...,n\} \,:\, n^k|\Delta_{i,k}^n \xi| \leq N,\ n^k|\Delta_{i,k}^n \xi^m| > M\} \\ C_{m,n}^{N,M} &= \{i \in \{k,...,n\} \,:\, n^k|\Delta_{i,k}^n \xi| \leq N,\ n^k|\Delta_{i,k}^n \xi^m| \leq M\}, \end{aligned}$$

and estimate the corresponding sums separately. The following relationship between $\Delta_{i,k}^n\xi$ and $\xi^{(k)}$ will be essential. For all $\xi \in W^{k,q}$ we have

$$\Delta_{i,k}^n \xi = \int_{\frac{i-1}{n}}^{i/n} \int_{s_1-1/n}^{s_1} \cdots \int_{s_{k-1}-1/n}^{s_{k-1}} \xi_{s_k}^{(k)} ds_k \dots ds_1.$$

In particular, it follows that

$$|n^k\Delta_{i,k}^n \xi| \leq \int_{[0,1]^k} n^k |\xi_{s_k}^{(k)}| \mathbb{1}_{\{(s_1,\dots,s_k)\in[(i-k)/n,i/n]^k\}}\, ds_k \dots\, ds_1 = k^{k-1}\int_{\frac{i-k}{n}}^{i/n} n|\xi_s^{(k)}|\, ds. \tag{4.29}$$

The A_n^N term: We show that for given $\varepsilon > 0$ we can find sufficiently large N such that, for a suitable constant C,

$$\begin{aligned} &\limsup_{m\to\infty}\sup_{n\in\mathbb{N}} \left\{ n^{-1}\sum_{i\in A_n^N} \left|f(n^k\Delta_{i,k}^n\xi) - f(n^k\Delta_{i,k}^n\xi^m)\right| \right\} \\ &\quad\leq \limsup_{m\to\infty}\sup_{n\in\mathbb{N}} \left\{ n^{-1}C\sum_{i\in A_n^N} |n^k\Delta_{i,k}^n\xi|^q + n^{-1}C\sum_{i\in A_n^N} |n^k\Delta_{i,k}^n\xi^m|^q \mathbb{1}_{\{|n^k\Delta_{i,k}^n\xi^m|>1\}} \right. \\ &\qquad\qquad \left. + n^{-1}C\sum_{i\in A_n^N} |f(n^k\Delta_{i,k}^n\xi^m)|\mathbb{1}_{\{|n^k\Delta_{i,k}^n\xi^m|\leq 1\}} \right\} \\ &\quad =: \limsup_{m\to\infty}\sup_{n\in\mathbb{N}} \left\{ CI_{1,n,N} + CI_{2,n,m,N} + CI_{3,n,m,N} \right\} \leq \varepsilon, \end{aligned} \tag{4.30}$$

where the first inequality follows from $|f(x)| \leq C(1\vee |x|^q)$. First we consider $I_{1,n,N}$. By (4.29) we have for all $i \in A_n^N$

$$N < k^{k-1}\int_{\frac{i-k}{n}}^{i/n} |\xi_s^{(k)}| n\, ds \leq k^{k-1}\int_{\frac{i-k}{n}}^{i/n} n|\xi_s^{(k)}|\mathbb{1}_{\{|\xi_s^{(k)}|>C_{0,k}\}}\, ds + \frac{N}{2},$$

where $C_{0,k} := N(2k^k)^{-1}$. Therefore, again by (4.29), it follows that

$$|n^k \Delta_{i,k}^n \xi| \le k^{k-1} \int_{\frac{i-k}{n}}^{i/n} |\xi_s^{(k)}| n \, ds \le 2k^{k-1} \int_{\frac{i-k}{n}}^{i/n} |\xi_s^{(k)}| n \, ds - N$$
$$\le 2k^{k-1} \int_{\frac{i-k}{n}}^{i/n} |\xi_s^{(k)}| \mathbb{1}_{\{|\xi_s^{(k)}| > C_{0,k}\}} n \, ds. \tag{4.31}$$

Consequently, recalling that $q \ge 1$, we have by Jensen's inequality

$$n^{-1} \sum_{i \in A_n^N} |n^k \Delta_{i,k}^n \xi|^q \le (2k^{k-1})^q k^{q-1} n^{-1} \sum_{i \in A_n^N} \int_{\frac{i-k}{n}}^{i/n} |\xi_s^{(k)}|^q \mathbb{1}_{\{|\xi_s^{(k)}| > C_{0,k}\}} n \, ds$$
$$\le (2k^k)^q \int_0^1 |\xi_s^{(k)}|^q \mathbb{1}_{\{|\xi_s^{(k)}| > C_{0,k}\}} \, ds. \tag{4.32}$$

It follows for sufficiently large $N > 0$ that

$$\limsup_{m \to \infty} \sup_{n \in \mathbb{N}} \{I_{1,n,N}\} \le \varepsilon. \tag{4.33}$$

Next, we argue that the same holds for the $I_{2,n,m,N}$ term. By (4.27) and Minkowski's inequality it follows for any $A \in \mathcal{B}([0,1])$ that $\int_A |\xi_s^{m,(k)}|^q \, ds \le 2^{q-1} \int_A |\xi_s^{(k)}|^q \, ds + C/m$. Consequently, it holds that

$$n^{-1} \sum_{i \in A_n^N} |n^k \Delta_{i,k}^n \xi^m|^q \mathbb{1}_{\{|n^k \Delta_{i,k}^n \xi^m| > 1\}} \le C n^{-1} \sum_{i \in A_n^N} \int_{\frac{i-k}{n}}^{i/n} |\xi_s^{m,(k)}|^q n \, ds$$
$$\le C \sum_{i \in A_n^N} \int_{\frac{i-k}{n}}^{i/n} |\xi_s^{(k)}|^q \, ds + \frac{C}{m} \le C \sum_{i \in A_n^N} \int_{\frac{i-k}{n}}^{i/n} |\xi_s^{(k)}|^q \mathbb{1}_{\{|\xi_s^{(k)}| > C_{0,k}\}} \, ds + \frac{C}{m}$$
$$\le C \int_0^1 |\xi_s^{(k)}|^q \mathbb{1}_{\{|\xi_s^{(k)}| > C_{0,k}\}} \, ds + \frac{C}{m},$$

where the first inequality follows from (4.29) and the third from (4.31). This shows that for sufficiently large N it holds that

$$\limsup_{m \to \infty} \sup_{n \in \mathbb{N}} \{I_{2,n,m,N}\} \le \varepsilon. \tag{4.34}$$

Next, we estimate the term $I_{3,n,m,N}$. Introducing the notation

$$D_{m,n} = \{i \in \{k, ..., n\} \, : \, n^k |\Delta_{i,k}^n \xi^{(m)}| \le 1\}$$

we have

$$I_{3,n,m,N} = n^{-1} \sum_{i \in A_n^N \cap D_{m,n}} |f(n^k \Delta_{i,k}^n \xi^{(m)})| \le n^{-1} |A_n^N \cap D_{m,n}| \sup_{x \in (-1,1)} |f(x)| \tag{4.35}$$

where $|A_n^N \cap D_{m,n}|$ denotes the number of elements of $A_n^N \cap D_{m,n}$. Using (4.29) we have for all $i \in A_n^N \cap D_{m,n}$

$$N - 1 \le n^k |\Delta_{i,k}^n (\xi^{(k)} - \xi^{m,(k)})| \le k^{k-1} \int_{\frac{i-k}{n}}^{i/n} |\xi_s^{(k)} - \xi_s^{m,(k)}| n \, ds,$$

and it follows that

$$|A_n^N \cap D_{m,n}| \; = \; \sum_{i \in A_n^N \cap D_{m,n}} 1 \; \le \; \sum_{i \in A_n^N \cap D_{m,n}} \frac{k^{k-1}}{N-1} \int_{\frac{i-k}{n}}^{i/n} |\xi_s^{(k)} - \xi_s^{m,(k)}| n \, ds$$
$$\le \frac{nk^k}{N-1} \int_0^1 |\xi_s^{(k)} - \xi_s^{m,(k)}| \, ds \; \le \; \frac{nk^k}{(N-1)m^{1/q}}, \tag{4.36}$$

where in the last inequality we used (4.27) and Jensen's inequality. With (4.35) it follows that for all $N > 1$ we have

$$\limsup_{m\to\infty} \sup_{n\in\mathbb{N}} \{I_{3,n,m,N}\} = 0. \tag{4.37}$$

Combining (4.33), (4.34) and (4.37) we conclude that (4.30) holds for sufficiently large N.

The $B^{N,M}_{m,n}$ term: We show that for any $\varepsilon > 0$ and any $N > 0$ we can find a sufficiently large M such that

$$\begin{aligned}
&\limsup_{m\to\infty} \sup_{n\in\mathbb{N}} \Big\{ n^{-1} \sum_{i\in B^{N,M}_{m,n}} \big| f(n^k \Delta^n_{i,k}\xi) - f(n^k\Delta^n_{i,k}\xi^m)\big| \Big\} \\
&\quad \le \limsup_{m\to\infty} \sup_{n\in\mathbb{N}} \Big\{ n^{-1} \sum_{i\in B^{N,M}_{m,n}} \big| f(n^k \Delta^n_{i,k}\xi)| + n^{-1} \sum_{i\in B^{N,M}_{m,n}} |n^k\Delta^n_{i,k}\xi^m|^q \Big\} \\
&\quad =: \limsup_{m\to\infty} \sup_{n\in\mathbb{N}} \{ J^1_{n,m,N,M} + J^2_{n,m,N,M}\} < \varepsilon.
\end{aligned} \tag{4.38}$$

The argument for $J^1_{n,m,N,M}$ is similar to the one used for $I_{3,m,n,N}$ above. We assume that $M > N$. For $i \in B^{N,M}_{m,n}$ it holds by (4.29) that

$$M - N < n^k |\Delta^n_{i,k}(\xi - \xi^m)| \le k^{k-1} n \int_{\frac{i-k}{n}}^{i/n} |\xi_s - \xi^m_s|\, ds.$$

Consequently, arguing as in (4.36), we obtain for all $m \in \mathbb{N}$

$$|B^{N,M}_{m,n}| \le \frac{k^k n}{M-N} \int_0^1 |\xi_s - \xi^m_s|\, ds \le \frac{k^k n}{(M-N)m^{1/q}},$$

where $|B^{N,M}_{m,n}|$ denotes the number of elements in $B^{N,M}_{m,n}$. Then, it follows that for all $M > N$

$$\begin{aligned}
\limsup_{m\to\infty} \sup_{n\in\mathbb{N}} \{J^1_{n,m,N,M}\} &\le \limsup_{m\to\infty} \sup_{n\in\mathbb{N}} \{ n^{-1} |B^{N,M}_{m,n}| \sup_{s\in[-N,N]} |f(s)| \} \\
&\le \limsup_{m\to\infty} \sup_{n\in N} \Big\{ \frac{k^k}{(M-N)m^{1/q}} \sup_{s\in[-N,N]} |f(s)| \Big\} = 0.
\end{aligned} \tag{4.39}$$

For $J^2_{n,m,N,M}$ we obtain by arguing as in (4.32) with $\xi^{(k)}$ replaced by $\xi^{m,(k)}$ and N replaced by M that

$$J^2_{n,m,N,M} \le (2k^k)^q \int_0^1 |\xi^{m,(k)}_s|^q \mathbb{1}_{\{|\xi^{m,(k)}_s| > M/2k^k\}}\, ds,$$

for all m, n, N. Since $(|\xi^{m,(k)}|^q)_{m\ge 1}$ is uniformly integrable we can for $\varepsilon > 0$ find sufficiently large M such that

$$\limsup_{m\to\infty} \sup_{n\in\mathbb{N}} \{ J^2_{n,m,N,M}\} \le \varepsilon. \tag{4.40}$$

Now, (4.38) follows from (4.39) and (4.40).

The $C^{N,M}_{m,n}$ term: We show that for all $N, M > 0$ we have that

$$\limsup_{m\to\infty} \sup_{n\in\mathbb{N}} \Big\{ n^{-1} \sum_{i\in C^{N,M}_{m,n}} \big| f(n^k\Delta^n_{i,k}\xi) - f(n^k\Delta^n_{i,k}\xi^m)\big| \Big\} = 0. \tag{4.41}$$

Since $|n^k\Delta^n_{i,k}\xi| \leq N$ and $|n^k\Delta^n_{i,k}\xi^m| \leq M$ for all $i \in C^{N,M}_{m,n}$, we can replace f by a continuous function $\widetilde{\Phi}_{N,M}$ with compact support, such that $f(x) = \widetilde{\Phi}_{N,M}(x)$ for all $x \in [-(N\vee M), N\vee M]$. Denote by $\widetilde{\omega}_{N,M}$ the concave modulus of continuity for $\widetilde{\Phi}_{N,M}$. It holds that

$$\begin{aligned}
&\sup_{n\in\mathbb{N}}\Big\{n^{-1}\sum_{i\in C^{N,M}_{m,n}}\big|f(n^k\Delta^n_{i,k}\xi)-f(n^k\Delta^n_{i,k}\xi^m)\big|\Big\}\\
&\quad=\sup_{n\in\mathbb{N}}\Big\{n^{-1}\sum_{i\in C^{N,M}_{m,n}}\big|\widetilde{\Phi}_{N,M}(n^k\Delta^n_{i,k}\xi)-\widetilde{\Phi}_{N,M}(n^k\Delta^n_{i,k}\xi^m)\big|\Big\}\\
&\quad\leq\sup_{n\in\mathbb{N}}\Big\{\widetilde{\omega}_{N,M}\Big(n^{-1}\sum_{j=k}^{n}n^k|\Delta^n_{i,k}\xi-\Delta^n_{i,k}\xi^m|\Big)\Big\}\leq\widetilde{\omega}_{N,M}\Big(k^k\int_0^1|\xi^{(k)}_s-\xi^{m,(k)}_s|\,ds\Big),
\end{aligned}$$

where the first inequality follows by Jensen's inequality from concavity of $\widetilde{\omega}_{N,M}$, and we used (4.29) in the last inequality. Now, (4.41) follows by (4.27).

Finally, by (4.30), (4.38) and (4.41), for any $\varepsilon > 0$ we can find sufficiently large N, M such that

$$\limsup_{m\to\infty}\sup_{n\to\infty}\Big(n^{-1}\sum_{i=k}^{n}\big|f(n^k\Delta^n_{i,k}\xi)-f(n^k\Delta^n_{i,k}\xi^m)\big|\Big)<\varepsilon.$$

By letting $\varepsilon \to 0$ we obtain (4.28) and the proof of the lemma is complete. □

5 Proofs of Theorems 2.5 and 2.6

Before carrying out the proofs we will introduce some notation and estimates to be used in the following.

Definitions and notation: For any function ψ on the real line we denote

$$D^k\psi(s) := \sum_{j=0}^{k}(-1)^j\binom{k}{j}\psi(s-j).$$

Furthermore, set

$$g_n(s) := n^\alpha g(s/n), \qquad \phi^n_t(s) := D^k g_n(t-s), \quad \text{and} \quad Y^n_t := \int_{-\infty}^{t}\phi^n_t(s)\,dL_s, \tag{5.1}$$

for $n \in \mathbb{N}$. By our assumptions on the function g it holds that $g_n(s) \to s^\alpha_+$, and consequently $\phi^n_t(s) \to h_k(t-s)$ as $n \to \infty$, where h_k was defined in (2.2). Therefore, we complement (5.1) by defining

$$\phi^\infty_t(s) := h_k(t-s), \quad \text{and} \quad Y^\infty_t := \int_{-\infty}^{t}h_k(t-s)\,dL_s.$$

We recall that $(\mathcal{F}_t)_{t\in\mathbb{R}}$ denotes the filtration generated by L and introduce additionally the σ-algebras

$$\mathcal{F}^1_s := \sigma(L_r - L_u \,|\, s \leq r, u \leq s+1),$$

remarking that $(\mathcal{F}^1_s)_{s\in\mathbb{R}}$ is not a filtration. We denote

$$U^n_{j,r} := \int_r^{r+1}\phi^n_j(s)\,dL_s, \quad \text{where } n \in \mathbb{N}\cup\{\infty\} \text{ and } j \geq k,$$

and introduce the notation

$$\rho^n_j := \rho_L\|\phi^n_j\|_{L^\beta(\mathbb{R}\setminus[0,1])}, \quad \text{and} \quad \rho^n := \rho_L\|\phi^n_1\|_{L^\beta(\mathbb{R})}. \tag{5.2}$$

Note that $Y_r^n \sim S\beta S(\rho^n)$ for all $r \geq k$ and $n \in \mathbb{N}$, which follows by (3.1).
Preliminary estimates: For $\xi < \beta$ and $\gamma > 0$ there is a $C > 0$ such that for all $\rho \in (0,1]$ and $S \sim S\beta S(1)$ we have

$$\mathbb{E}[|\rho S|^\xi \wedge |\rho S|^\gamma] \leq \begin{cases} C\rho^\beta & \text{for } \gamma > \beta, \\ C\rho^\gamma & \text{for } \gamma < \beta, \end{cases} \tag{5.3}$$

where the first case follows by [10, Lemma 5.5], and the second case is a standard estimate. The function ϕ_j^n introduced above satisfies the estimate

$$\|\phi_j^n\|_{L^\beta([0,1])} \leq C j^{\alpha-k}, \tag{5.4}$$

for all $j \in \mathbb{N}$ and all $n \in \mathbb{N} \cup \{\infty\}$, which follows from Taylor expansion and the condition (A2) in Section 2. Moreover, ϕ_j^n satisfies the following estimate that has been derived in [10, Eq. (5.92)]. There exists a $C > 0$ such that for all $n \in \mathbb{N}$ and $j \in \mathbb{N}$

$$\|\phi_j^n - \phi_j^\infty\|_{L^\beta([0,1])} \leq C n^{-1} j^{\alpha-k+1}. \tag{5.5}$$

Remark 5.1. In the proofs of Theorems 2.5 and 2.6 we may and do replace $\mathbb{E}[f(\rho_0 S)]$ by $\mathbb{E}[f(n^H \Delta_{i,k}^n X)]$ in (2.7), (2.8) and (2.9). Indeed, to show this claim we first show that the function $\rho \mapsto G(\rho) := \mathbb{E}[f(\rho S)]$ is continuously differentiable on $(0,\infty)$. Let g_β denote the density of a SβS random variable. By substitution we have that

$$G(\rho) = \int_{\mathbb{R}} f(u) g_\beta(u/\rho)\, du. \tag{5.6}$$

Since $\mathbb{E}[|f(S)|] < \infty$ it follows that $\int |f(u)|(1 \wedge |u|^{-1-\beta})\, du < \infty$, cf. [38, Theorem 1.2]. We have that $g_\beta \in C^\infty(\mathbb{R})$, according to [33, Remark 28.2], and for all $r \geq 1$, the rth derivative of g_β satisfies

$$|g_\beta^{(r)}(x)| \leq C(1 \wedge |x|^{-1-\beta-r}), \qquad x \in \mathbb{R}. \tag{5.7}$$

Indeed, to show the estimate (5.7) we use the dual representation for stable densities given in [40, (2.5.5)], which implies that

$$g_\beta(x) = x^{-1-\beta} \tilde{g}(x^{-\beta}), \qquad x > 0, \tag{5.8}$$

where $\tilde{g}$ is the density of a $1/\beta$-distribution. By r-times differentiation of (5.8), the estimate (5.7) follows. Hence, from the estimate (5.7) used on $r = 1$ and (5.6), it follows that $G \in C^1((0,\infty))$. By [10, Lemma 5.3] we have that

$$\Big| n^H \rho_L \|g_{i,k}^n\|_{L^\beta(\mathbb{R})} - \rho_0 \Big| \leq \begin{cases} C n^{-1} & \text{for } \alpha \in (0, k - 2/\beta) \\ C n^{(\alpha-k)\beta+1} & \text{for } \alpha \in (k-2/\beta, k-1/\beta). \end{cases} \tag{5.9}$$

Hence, for large enough n, we obtain the estimate

$$\Big| \mathbb{E}[f(n^H \Delta_{i,k}^n X)] - \mathbb{E}[f(\rho_0 S)] \Big| \leq \Big(\max_{x \in [\rho_0-\varepsilon, \rho_0+\varepsilon]} |G'(x)| \Big) \Big| n^H \rho_L \|g_{i,k}^n\|_{L^\beta(\mathbb{R})} - \rho_0 \Big|, \tag{5.10}$$

and by (5.10) and (5.9) it follows that

$$a_n \Big| \mathbb{E}[f(n^H \Delta_{i,k}^n X)] - \mathbb{E}[f(\rho_0 S)] \Big| \to 0 \qquad \text{as } n \to \infty, \tag{5.11}$$

where $a_n = \sqrt{n}$ for Theorem 2.5, $a_n = n^{k-\alpha-1/\beta}$ for Theorem 2.6(i), and $a_n = n^{1-\frac{1}{(k-\alpha)\beta}}$ for Theorem 2.6(ii). Eq. (5.11) proves the above claim that we may replace $\mathbb{E}[f(\rho_0 S)]$ by $\mathbb{E}[f(n^H \Delta_{i,k}^n X)]$ in Theorems 2.5 and 2.6. □

We have that $\{n^H \Delta^n_{r,k} X\}_{r=k,\dots,n} \overset{\mathrm{d}}{=} \{Y^n_r\}_{r=k,\dots,n}$, cf. (4.21), and to deduce Theorems 2.5 and 2.6 we show, cf. Remark 5.1, convergence in distribution for the properly normalised version of

$$S_n := \sum_{r=k}^{n} \big(f(Y^n_r) - \mathbb{E}[f(Y^n_r)]\big) = \sum_{r=k}^{n} V^n_r, \tag{5.12}$$

where we denoted $V^n_r := f(Y^n_r) - \mathbb{E}[f(Y^n_r)]$ for brevity.

5.1 Proof of Theorem 2.5

We recall the definition of Y^n_r and S_n from (5.1) and (5.12), and define additionally, for $a < b$, $a, b \in [0, \infty]$ and $m \geq 0$,

$$Y^{n,[a,b]}_r = \int_{r-b}^{r-a} \phi^n_r(s)\, dL_s, \qquad Y^{n,m}_r = Y^{n,[0,m]}_r,$$
$$S_{n,m} = \sum_{r=k}^{n} \big(f(Y^{n,m}_r) - \mathbb{E}[f(Y^{n,m}_r)]\big).$$

By [13, Theorem 3.2], the statement of the theorem follows if we show the following three results

$$\lim_{m\to\infty} \limsup_{n\to\infty} \mathbb{E}[n^{-1}(S_n - S_{m,n})^2] = 0, \tag{5.13}$$

$$\frac{1}{\sqrt{n}} S_{n,m} \xrightarrow{\mathcal{L}} \mathcal{N}(0, \eta_m^2), \quad \text{for some } \eta_m^2 \in [0,\infty), \text{ and} \tag{5.14}$$

$$\eta_m^2 \to \eta^2, \quad \text{as } m \to \infty. \tag{5.15}$$

We show (5.14) first. Set $\theta^{n,m}_j = \mathrm{cov}(f(Y^{n,m}_k), f(Y^{n,m}_{k+j}))$ for $n \in \mathbb{N} \cup \{\infty\}$. Since the sequence $(Y^{n,m}_r)_{r=k,\dots}$ is stationary the variance of $S_{n,m}$ is then given by

$$n^{-1}\mathrm{var}(S_{n,m}) = n^{-1}\bigg\{(n-k+1)\theta^{n,m}_0 + 2\sum_{j=1}^{m}(n-k-j)\theta^{n,m}_j\bigg\}.$$

An application of Lemma 6.5 with $p = 2$ yields that the covariances $\theta^{n,m}_j$ converge to $\theta^{\infty,m}_j$ for all m, j, as $n \to \infty$. Since the sequence $(Y^{n,m}_r)_{r=k,\dots}$ is m-dependent, (5.14) follows now from the central limit theorem for m-dependent sequences, see e.g. [12], with the limiting variance

$$\eta_m^2 = \theta^{\infty,m}_0 + 2\sum_{j=1}^{m} \theta^{\infty,m}_j. \tag{5.16}$$

Next, we argue that η_m^2 is a Cauchy sequence, which then shows (5.15) with $\eta^2 := \lim_{m\to\infty} \eta_m^2$. This is indeed an immediate consequence of (5.13) since

$$\Big||\eta_m| - |\eta_r|\Big| = \lim_{n\to\infty} n^{-1/2}\Big|\|S_{n,m}\|_{L^2} - \|S_{n,r}\|_{L^2}\Big| \leq \limsup_{n\to\infty} n^{-1/2}\Big\|S_{n,m} - S_{n,r}\Big\|_{L^2}$$
$$\leq \limsup_{n\to\infty} n^{-1/2}\Big\|S_{n,m} - S_n\Big\|_{L^2} + \limsup_{n\to\infty} n^{-1/2}\Big\|S_n - S_{n,r}\Big\|_{L^2} \to 0$$

as $m, r \to \infty$ by (5.13). The proof of (2.7) can thus be completed by deriving (5.13), which we do in the following.

We can express S_n and $S_{n,m}$ as the telescoping sums

$$S_n = \sum_{r=k}^{n}\sum_{j=1}^{\infty}(\mathbb{E}[f(Y^n_r)|\mathcal{F}_{r-j+1}] - \mathbb{E}[f(Y^n_r)|\mathcal{F}_{r-j}]),$$
$$S_{n,m} = \sum_{r=k}^{n}\sum_{j=1}^{m}(\mathbb{E}[f(Y^{n,m}_r)|\mathcal{F}_{r-j+1}] - \mathbb{E}[f(Y^{n,m}_r)|\mathcal{F}_{r-j}]).$$

Indeed, the first telescoping sum coincides with S_n almost surely, since by the backwards martingale convergence theorem and Kolmogorov's 0-1 law $\mathbb{E}[f(Y_r^n)|\mathcal{F}_{r-j}] \xrightarrow{\text{a.s.}} \mathbb{E}[f(Y_r^n)]$, as $j \to \infty$. We denote for $n \geq 1$ and $m, r, j \geq 0$

$$\xi_{r,j}^{n,m} = \mathbb{E}[f(Y_r^n) - f(Y_r^{n,m})|\mathcal{F}_{r-j+1}] - \mathbb{E}[f(Y_r^n) - f(Y_r^{n,m})|\mathcal{F}_{r-j}],$$

and obtain

$$S_n - S_{n,m} = \sum_{r=k}^{n} \sum_{j=1}^{\infty} \xi_{r,j}^{n,m}. \tag{5.17}$$

Making the decomposition

$$\begin{aligned} & n^{-1}\mathbb{E}[(S_n - S_{n,m})^2] \\ & \leq 3n^{-1}\mathbb{E}\Big[\Big(\sum_{r=k}^{n}\sum_{j=m+1}^{\infty}\xi_{r,j}^{n,m}\Big)^2\Big] + 3n^{-1}\mathbb{E}\Big[\Big(\sum_{r=k}^{n}\sum_{j=2}^{m}\xi_{r,j}^{n,m}\Big)^2\Big] + 3n^{-1}\mathbb{E}\Big[\Big(\sum_{r=k}^{n}\xi_{r,1}^{n,m}\Big)^2\Big], \end{aligned}$$

we show that each summand on the right hand side converges to 0. Observing that

$$\text{cov}(\xi_{r,j}^{n,m}, \xi_{r',j'}^{n,m}) = 0, \quad \text{unless } r - j = r' - j',$$

an application of Cauchy-Schwarz inequality and Fatou's lemma yields

$$n^{-1}\mathbb{E}[(S_n - S_{n,m})^2] \leq 3n^{-1}Q_{n,1,m} + 3n^{-1}Q_{n,2,m} + 3n^{-1}Q_{n,3,m},$$

where

$$\begin{aligned} Q_{n,1,m} &= \sum_{r=k}^{n}\sum_{j=2}^{m}\sum_{j'=2}^{m} \mathbb{E}[(\xi_{r,j}^{n,m})^2]^{1/2}\mathbb{E}[(\xi_{r',j'}^{n,m})^2]^{1/2}, \\ Q_{n,2,m} &= \sum_{r=k}^{n}\sum_{j=m+1}^{\infty}\sum_{j'=m+1}^{\infty} \mathbb{E}[(\xi_{r,j}^{n,m})^2]^{1/2}\mathbb{E}[(\xi_{r',j'}^{n,m})^2]^{1/2}, \\ Q_{n,3,m} &= \sum_{r=k}^{n} \mathbb{E}[(\xi_{r,1}^{n,m})^2], \end{aligned}$$

and we denoted $r' = r - j + j'$. For the proof of (5.13) it remains to show that

$$\limsup_{n\to\infty} \frac{1}{n} Q_{n,i,m} \to 0, \quad \text{as } m \to \infty, \text{ for } i = 1, 2, 3.$$

Estimation of $Q_{n,1,m}$: We introduce the notation

$$\widetilde{\Phi}_j^n(x) = \mathbb{E}\big[f(x + Y_r^{n,j})\big],$$

which allows us to write $\mathbb{E}[f(Y_r^n)|\mathcal{F}_{r-j}] = \widetilde{\Phi}_j^n(Y_r^{n,[j,\infty]})$, due to the independence of the increments of L. For $2 \leq j \leq m$ we obtain

$$\xi_{r,j}^{n,m} = \widetilde{\Phi}_{j-1}^n\big(Y_r^{n,[j-1,\infty]}\big) - \widetilde{\Phi}_j^n\big(Y_r^{n,[j,\infty]}\big) - \big\{\widetilde{\Phi}_{j-1}^n\big(Y_r^{n,[j-1,m]}\big) - \widetilde{\Phi}_j^n\big(Y_r^{n,[j,m]}\big)\big\}. \tag{5.18}$$

The involved random variables can be decomposed into the sum of independent random variables as

$$\begin{aligned} Y_r^{n,[j-1,\infty]} &= Y_r^{n,[j-1,j]} + Y_r^{n,[j,m]} + Y_r^{n,[m,\infty]} \\ Y_r^{n,[j,\infty]} &= Y_r^{n,[j,m]} + Y_r^{n,[m,\infty]} \\ Y_r^{n,[j-1,m]} &= Y_r^{n,[j-1,j]} + Y_r^{n,[j,m]}. \end{aligned}$$

Denoting by $F^n_{[j-1,j]}, F^n_{[j,m]}$ and $F^n_{[m,\infty]}$ the corresponding distribution functions, we obtain

$$\mathbb{E}[(\xi^{n,m}_{r,j})^2] = \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \big\{\widetilde{\Phi}^n_{j-1}(u+v+w) - \widetilde{\Phi}^n_j(v+w) - \big(\widetilde{\Phi}^n_{j-1}(u+v) - \widetilde{\Phi}^n_j(v)\big)\big\}^2 dF^n_{[j-1,j]}(u)dF^n_{[j,m]}(v)dF^n_{[m,\infty]}(w).$$

Using the relation $\widetilde{\Phi}^n_j(x) = \mathbb{E}f\big(x + Y^{n,j-1}_r + Y^{n,[j-1,j]}_r\big) = \int_{\mathbb{R}} \widetilde{\Phi}^n_{j-1}(x+z)dF^n_{[j-1,j]}(z)$, we obtain

$$\begin{aligned}\mathbb{E}[(\xi^{n,m}_{r,j})^2] &= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \Big(\int_{\mathbb{R}} D_{n,j}(u,v,w,z)dF^n_{[j-1,j]}(z)\Big)^2 dF^n_{[j-1,j]}(u)dF_{[j,m]}(v)dF^n_{[m,\infty]}(w)\\ &\le \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} D^2_{n,j}(u,v,w,z)dF^n_{[j-1,j]}(z)dF^n_{[j-1,j]}(u)dF^n_{[j,m]}(v)dF^n_{[m,\infty]}(w), \qquad (5.19)\end{aligned}$$

where

$$\begin{aligned}D_{n,j}(u,v,w,z) &= \widetilde{\Phi}^n_{j-1}(u+v+w) - \widetilde{\Phi}^n_{j-1}(v+w+z) - \big(\widetilde{\Phi}^n_{j-1}(u+v) - \widetilde{\Phi}^n_{j-1}(v+z)\big)\\ &= \Phi_{\rho^n_{j-1}}(u+v+w) - \Phi_{\rho^n_{j-1}}(v+w+z) - \big(\Phi_{\rho^n_{j-1}}(u+v) - \Phi_{\rho^n_{j-1}}(v+z)\big),\end{aligned}$$

and ρ^n_{j-1} is the scale parameter of the $S\beta S$ random variable $Y^{n,j-1}_r$. It follows from Lemma 6.1 that $D_{n,j}$ satisfies the estimate

$$D^2_{n,j}(u,v,w,z) \le C\big(|u-z|^{2p} \wedge (u-z)^2\big)(|w|^{2p} \wedge w^2), \quad \text{for all } j \ge 2, n \in \mathbb{N}, \qquad (5.20)$$

where p is as in (2.5), provided $\{\rho^n_{j-1}\}_{j\ge 2, n\in\mathbb{N}}$ is bounded away from 0 and ∞. This is indeed the case, as follows from the estimates

$$\begin{aligned}(\rho^n_{j-1})^\beta &= \int_{r-j+1}^r |\phi^n_r(s)|^\beta ds \;\le\; \|\phi^n_r\|^\beta_{L^\beta(\mathbb{R})} \;=\; \rho^n, \quad \text{and}\\ (\rho^n_{j-1})^\beta &\ge \int_{r-1}^r |\phi^n_r(s)|^\beta ds \;\to\; \int_0^1 s^{\alpha\beta}ds \;>\; 0 \quad \text{as } n\to\infty,\end{aligned}$$

where the convergence follows by the dominated convergence theorem, since Assumption (A) implies the existence of a $C>0$ such that $|\phi^n_r(s)| \le C|r-s|^\alpha$ for all $s \in [r-1,r]$ and all $n \ge 1$.

Applying (5.20) on the right hand side of (5.19) yields the estimate

$$\mathbb{E}[(\xi^{n,m}_{r,j})^2] \le C\Big(\int_{\mathbb{R}^2} |u-z|^{2p} \wedge (u-z)^2\, dF^n_{[j-1,j]}(u)dF^n_{[j-1,j]}(z)\Big)\int_{\mathbb{R}} |w|^{2p} \wedge w^2\, dF^n_{[m,\infty]}(w).$$

It follows now from (5.4) and (5.3) that $\mathbb{E}[(\xi^{n,m}_{r,j})^2] \le C(\rho^n_{[j-1,j]}\rho^n_{[m,\infty]})^\beta$, where $\rho^n_{[j-1,j]}$ and $\rho^n_{[m,\infty]}$ are the scale parameters of the stable distributions $F^n_{[j-1,j]}$ and $F^n_{[m,\infty]}$, respectively. By (3.1) and (5.4) the scale parameters satisfy $\rho^n_{[j-1,j]} = \rho_L\|\phi^n_j\|_{L^\beta([0,1])} \le Cj^{\alpha-k}$, and

$$(\rho^n_{[m,\infty]})^\beta = \rho_L\int_{-\infty}^{r-m} |\phi^n_r(s)|^\beta ds = \rho_L \sum_{l=m+1}^\infty \|\phi^n_l\|^\beta_{L^\beta([0,1])} \le C\sum_{l=m+1}^\infty l^{\beta(\alpha-k)}.$$

It follows that

$$\mathbb{E}[(\xi^{n,m}_{r,j})^2] \le Cj^{\beta(\alpha-k)}\sum_{l=m+1}^\infty l^{\beta(\alpha-k)},$$

for all $j \in \{2, ..., m\}$ and we obtain

$$\limsup_{n\to\infty} \frac{1}{n} Q_{n,1,m} \leq C\bigg(\sum_{j=2}^{m} j^{\frac{\beta}{2}(\alpha-k)} \bigg)^2 \bigg(\sum_{l=m+1}^{\infty} l^{\beta(\alpha-k)} \bigg),$$

which converges to 0, as $m \to \infty$ since $\beta(\alpha - k) < -2$.

Estimation of $Q_{n,2,m}$*:* This term is estimated by similar, and in fact easier, arguments as used for the estimation of $Q_{n,1,m}$ which we do not repeat here.

Estimation of $Q_{n,3,m}$*:* Using the inequality $\mathbb{E}\big\{\mathbb{E}[X|\mathcal{F}] - \mathbb{E}[Y|\mathcal{F}]\big\}^2 \leq 2\mathbb{E}X^2 + 2\mathbb{E}Y^2$ we obtain

$$\frac{1}{n} Q_{n,3,m} \leq \frac{4}{n} \sum_{r=k}^{n} \mathbb{E}[(f(Y_r^n) - f(Y_r^{n,m}))^2] = \frac{n-k+1}{n} \mathbb{E}[(f(Y_1^n) - f(Y_1^{n,m}))^2],$$

and it is sufficient to argue that $\limsup_{n\to\infty} \mathbb{E}[(f(Y_1^n) - f(Y_1^{n,m}))^2] \to 0$ as $m \to \infty$. However, this follows by Lemma 6.5 with $p = 2$, and completes the proof of (5.13), and thus of Theorem 2.5. □

5.2 Proof of Theorem 2.6(i)

In the following section we set for all $n \in \mathbb{N}$

$$\tilde{S}_n = \Phi'_{\rho^n}(0) \sum_{r=k}^{n} Y_r^n, \qquad \text{where} \quad \Phi'_\rho(x) = \frac{\partial}{\partial x} \Phi_\rho(x).$$

To prove Theorem 2.6(i), it is enough to show that the following (5.21) and (5.22) hold, where

$$n^{k-\alpha-1/\beta-1}(S_n - \tilde{S}_n) \xrightarrow{\mathbb{P}} 0, \tag{5.21}$$

$$n^{k-\alpha-1/\beta-1} \tilde{S}_n \xrightarrow{\mathcal{L}} S\beta S(\sigma), \qquad \text{with } \sigma := \rho_L \Phi'_{\rho^\infty}(0) c_0^{1/\beta}. \tag{5.22}$$

Proof of (5.21): We show a stronger statement than (5.21), namely convergence in L^γ for a suitable $\gamma \geq 1$. Let $\tilde{f}_\rho(x) = f(x) - \Phi'_\rho(0)x$, and set

$$\tilde{\Phi}_\rho(x) := \mathbb{E}[\tilde{f}_\rho(x + S)] - \mathbb{E}[\tilde{f}_\rho(S)] = \Phi_\rho(x) - \Phi'_\rho(0)x,$$

for all $x \in \mathbb{R}$, $n \in \mathbb{N}$ and $S \sim S\beta S(\rho)$. For all $\epsilon > 0$ there exists $C > 0$ such that $|\tilde{\Phi}'_\rho(x)| \leq C$ and $|\tilde{\Phi}''_\rho(x)| \leq C$ for all $x \in \mathbb{R}$ and $\rho \in [\epsilon, \epsilon^{-1}]$, and since $\tilde{\Phi}_\rho(0) = \tilde{\Phi}'_\rho(0) = 0$ it follows that

$$|\tilde{\Phi}_\rho(x)| \leq C_\epsilon(|x| \wedge |x|^2), \tag{5.23}$$

which will be crucial for the following estimates. We set

$$\zeta_{r,j}^n = \mathbb{E}[\tilde{f}_{\rho^n}(Y_r^n)|\mathcal{F}_{r-j+1}] - \mathbb{E}[\tilde{f}_{\rho^n}(Y_r^n)|\mathcal{F}_{r-j}] - \mathbb{E}[\tilde{f}_{\rho^n}(Y_r^n)|\mathcal{F}_{r-j}^1] + \mathbb{E}[f(Y_r^n)],$$

and decompose $S_n - \tilde{S}_n$ as follows

$$S_n - \tilde{S}_n = \sum_{r=k}^{n} \Big(\sum_{j=1}^{\infty} \zeta_{r,j}^n \Big) + \sum_{r=k}^{n} \Big(\sum_{j=1}^{\infty} \big(\mathbb{E}[\tilde{f}_{\rho^n}(Y_r^n)|\mathcal{F}_{r-j}^1] - \mathbb{E}[f(Y_r^n)] \big) \Big) =: V_n + W_n. \tag{5.24}$$

In the following we will estimate W_n and V_n separately.

Estimation of W_n*:* By the substitution $s = r - j$ we obtain the representation

$$W_n = \sum_{s=-\infty}^{n-1} \Big(\sum_{j=(k-s)\vee 1}^{n-s} \big(\mathbb{E}[\tilde{f}_{\rho^n}(Y_{s+j}^n)|\mathcal{F}_s^1] - \mathbb{E}[f(Y_r^n)] \big) \Big) = \sum_{s=-\infty}^{n-1} D_s^n, \qquad \text{where}$$

$$D_s^n := \sum_{j=(k-s)\vee 1}^{n-s} \big(\mathbb{E}[\tilde{f}_{\rho^n}(Y_{s+j}^n)|\mathcal{F}_s^1] - \mathbb{E}[f(Y_{s+j}^n)] \big).$$

Since $\{D_s^n : s \in \mathbb{Z}\}$ is a martingale difference sequence, the von Bahr–Esseen inequality [37, Theorem 1] yields that for any $\gamma \in (1, \beta)$

$$\begin{aligned}\mathbb{E}[|W_n|^\gamma] &\leq 2 \sum_{s=-\infty}^{n-1} \mathbb{E}[|D_s^n|^\gamma] \\ &\leq 2 \sum_{s=-\infty}^{n-1} \Big(\sum_{j=(k-s)\vee 1}^{n-s} \big(\mathbb{E}\big[\big|\mathbb{E}[\tilde{f}_{\rho^n}(Y_{s+j}^n)|\mathcal{F}_s^1] - \mathbb{E}[f(Y_{s+j}^n)]\big|^\gamma\big]\big)^{1/\gamma}\Big)^\gamma, \end{aligned} \tag{5.25}$$

where the second inequality follows by Minkowski's inequality. We have that

$$|\Phi'_{\rho^n}(0) - \Phi'_{\rho_j^n}(0)| \leq C|\rho^n - \rho_j^n| \leq C\Big||\rho^n|^\beta - |\rho_j^n|^\beta\Big| = C\|\phi_j^n\|_{L^\beta([0,1])}^\beta \leq Cj^{\beta(\alpha-k)} \tag{5.26}$$

where the first inequality follows by boundedness of $\frac{\partial^2}{\partial\rho\partial x}\Phi_\rho(x)$, for the second inequality we use that ρ^n, ρ_j^n are bounded away from 0 and ∞, cf. Lemma 6.3, and the last inequality is (5.4). By a calculation similar to (5.18) we obtain the identity

$$\mathbb{E}[\tilde{f}_{\rho^n}(Y_{s+j}^n)|\mathcal{F}_s^1] - \mathbb{E}[f(Y_{s+j}^n)] = \Phi_{\rho_j^n}(U_{j+s,s}^n) - \mathbb{E}[\Phi_{\rho_j^n}(U_{j+s,s}^n)] - \Phi'_{\rho^n}(0)U_{j+s,s}^n,$$

and hence for all $r \in (1,2)$ with $r\gamma < \beta$, we have

$$\begin{aligned}&\mathbb{E}\Big[\Big|\mathbb{E}[\tilde{f}_{\rho^n}(Y_{s+j}^n)|\mathcal{F}_s^1] - \mathbb{E}[f(Y_{s+j}^n)]\Big|^\gamma\Big] \\ &\quad\leq\ C\Big(\mathbb{E}[|\tilde{\Phi}_{\rho_j^n}(U_{j+s,s}^n)|^\gamma] + |\Phi'_{\rho^n}(0) - \Phi'_{\rho_j^n}(0)|^\gamma \mathbb{E}[|U_{s+j,s}^n|^\gamma]\Big) \\ &\quad\leq\ C\Big(\mathbb{E}[|U_{j+s,s}^n|^{r\gamma}] + |\Phi'_{\rho^n}(0) - \Phi'_{\rho_j^n}(0)|^\gamma j^{\gamma(\alpha-k)}\Big)\ \leq\ C\Big(j^{\gamma r(\alpha-k)} + j^{\gamma(\alpha-k)(1+\beta)}\Big) \\ &\quad\leq\ Cj^{\gamma r(\alpha-k)}, \end{aligned} \tag{5.27}$$

where the estimate $|\tilde{\Phi}_{\rho_j^n}(x)| \leq C|x|^r$ is used in the second inequality (cf. (5.23)), and (5.26) is used in the third inequality. From (5.25) and (5.27) we deduce

$$\begin{aligned}\mathbb{E}[|W_n|^\gamma] &\leq C \sum_{s=-\infty}^{n-1} \Big(\sum_{j=(k-s)\vee 1}^{n-s} j^{r(\alpha-k)}\Big)^\gamma \\ &= C\Big(\sum_{s=-\infty}^{-n} \Big(\sum_{j=k-s}^{n-s} j^{r(\alpha-k)}\Big)^\gamma + \sum_{s=-n+1}^{k-2} \Big(\sum_{j=k-s}^{n-s} j^{r(\alpha-k)}\Big)^\gamma + \sum_{s=k-1}^{n} \Big(\sum_{j=1}^{n-s} j^{r(\alpha-k)}\Big)^\gamma\Big) \\ &=: C\big(A_n' + A_n'' + A_n'''\big). \end{aligned}$$

We may and do choose r and β such that $r(\alpha-k) \neq -1$ and $-\beta < r\gamma(\alpha-k) < -1$. Recall that $-\beta < \beta(\alpha-k) < -1$ by assumption, and $r, \gamma > 1$ satisfies $r\gamma < \beta$. We start by estimating A_n' as follows

$$A_n' \leq \sum_{s=n}^{\infty} (ns^{r(\alpha-k)})^\gamma \leq Cn^{\gamma r(\alpha-k)+1+\gamma} \tag{5.28}$$

where we have used $r\gamma(\alpha-k) < -1$ in the last inequality. By Jensen's inequality we have

$$A_n'' \leq n^{\gamma-1} \sum_{s=1}^{n} \Big(\sum_{j=k+s}^{n+s} j^{r\gamma(\alpha-k)}\Big) \leq Cn^{\gamma-1} \sum_{s=1}^{n} s^{r\gamma(\alpha-k)+1} \leq Cn^{r\gamma(\alpha-k)+\gamma+1}, \tag{5.29}$$

where we have used $r\gamma(\alpha-k) < -1$ in the second inequality, and $r\gamma(\alpha-k) > -2$ in the last inequality. For $\gamma < \beta$ close enough to β we have that $r(\alpha-k) > -1$ for all $r \in (1, \beta/\gamma)$, by the assumption $\alpha > k-1$. The substitution $v = n-s$ yields that

$$A_n''' \leq C \sum_{v=1}^{n} \Big(\sum_{j=1}^{v} j^{r(\alpha-k)} \Big)^{\gamma} \leq C \sum_{v=1}^{n} v^{\gamma r(\alpha-k)+\gamma} \leq n^{\gamma r(\alpha-k)+\gamma+1}, \tag{5.30}$$

where we have used $r(\alpha-k) > -1$ in the second inequality, and $\gamma r(\alpha-k) > -2$ in the last inequality. The above three estimates (5.28)–(5.30) show the bound

$$\mathbb{E}[|W_n|^{\gamma}] \leq C n^{\gamma r(\alpha-k)+\gamma+1}. \tag{5.31}$$

Estimation of V_n*:* By the substitution $s = r-j$ we have that

$$V_n = \sum_{r=k}^{n} \Big(\sum_{j=1}^{\infty} \zeta_{r,j}^n \Big) = \sum_{s=-\infty}^{n-1} \Big(\sum_{r=k\vee(s+1)}^{n} \zeta_{r,r-s}^n \Big) = \sum_{s=-\infty}^{n-1} M_s^n,$$

where $M_s^n := \sum_{r=k\vee(s+1)}^{n} \zeta_{r,r-s}^n$. Since $(M_s^n)_{s\in\mathbb{Z}}$ is a martingale difference for all fixed $n \in \mathbb{N}$, we have by the von Bahr–Esseen inequality [37, Theorem 1] for all $\gamma \in [1,2]$ with $\gamma < \beta$ that

$$\mathbb{E}[|V_n|^{\gamma}] \leq 2 \sum_{s=-\infty}^{n-1} \mathbb{E}[|M_s^n|^{\gamma}] \leq \sum_{s=-\infty}^{n-1} \Big(\sum_{r=k\vee(s+1)}^{n} \|\zeta_{r,r-s}^n\|_{\gamma} \Big)^{\gamma}, \tag{5.32}$$

where the last inequality follows from the Minkowski inequality. In the following we define the random variables $\vartheta_{r,j,l}^n$, $l \geq j$, by

$$\begin{aligned} \vartheta_{r,j,l}^n &= \mathbb{E}[\zeta_{r,j}^n \,|\, \mathcal{F}_{r-j}^1 \vee \mathcal{F}_{r-l}] - \mathbb{E}[\zeta_{r,j}^n \,|\, \mathcal{F}_{r-j}^1 \vee \mathcal{F}_{r-l-1}] \\ &= \mathbb{E}[f(Y_r^n) \,|\, \mathcal{F}_{r-j}^1 \vee \mathcal{F}_{r-l}] - \mathbb{E}[f(Y_r^n) \,|\, \mathcal{F}_{r-j}^1 \vee \mathcal{F}_{r-l-1}] \\ &\quad - \big\{ \mathbb{E}\big[\mathbb{E}[f(Y_r^n) \,|\, \mathcal{F}_{r-j}] \,|\, \mathcal{F}_{r-j}^1 \vee \mathcal{F}_{r-l}\big] - \mathbb{E}\big[\mathbb{E}[f(Y_r^n) \,|\, \mathcal{F}_{r-j}] \,|\, \mathcal{F}_{r-j}^1 \vee \mathcal{F}_{r-l-1}\big] \big\}. \end{aligned} \tag{5.33}$$

By a telescoping sum argument similar to (5.17), we obtain the representation

$$\zeta_{r,j}^n = \sum_{l=j}^{\infty} \vartheta_{r,j,l}^n.$$

Since $\{\vartheta_{r,j,l}^n : l = j, j+1, \dots\}$ is a martingale difference sequence, the von Bahr–Esseen inequality [37, Theorem 1] yields that

$$\mathbb{E}[|\zeta_{r,j}^n|^{\gamma}] \leq 2 \sum_{l=j}^{\infty} \mathbb{E}[|\vartheta_{r,j,l}^n|^{\gamma}] \leq C \sum_{l=j}^{\infty} j^{(\alpha-k)\gamma} l^{(\alpha-k)\gamma} \leq C j^{2(\alpha-k)\gamma+1} \tag{5.34}$$

for all $\gamma \in (1,\beta)$ such that $(\alpha-k)\gamma < -1$. Here we have used Lemma 6.2 in the second inequality, and the third inequality follows, since $(\alpha-k)\gamma < -1$, from comparison with the integral

$$\sum_{l=j}^{\infty} l^{(\alpha-k)\gamma} \leq \int_j^{\infty} (x-1)^{(\alpha-k)\gamma}\, dx = \frac{1}{(\alpha-k)\gamma-1} (j-1)^{(\alpha-k)\gamma+1}.$$

From (5.32) and (5.34) we have

$$\begin{aligned} \mathbb{E}[|V_n|^{\gamma}] &\leq C \sum_{s=-\infty}^{n-1} \Big(\sum_{r=k\vee(s+1)}^{n} (r-s)^{2(\alpha-k)+1/\gamma} \Big)^{\gamma} \\ &= C \Big\{ \sum_{s=-\infty}^{-n} \Big(\sum_{r=k}^{n} (r-s)^{2(\alpha-k)+1/\gamma} \Big)^{\gamma} + \sum_{s=-n+1}^{k-1} \Big(\sum_{r=k}^{n} (r-s)^{2(\alpha-k)+1/\gamma} \Big)^{\gamma} \\ &\qquad + \sum_{s=k}^{n-1} \Big(\sum_{r=s+1}^{n} (r-s)^{2(\alpha-k)+1/\gamma} \Big)^{\gamma} \Big\} =: \{B_n' + B_n'' + B_n'''\}. \end{aligned}$$

We estimate B_1', B_n'' and B_n''' in a similar fashion as in (5.28)–(5.30), but need to divide into several cases depending on the value of $\gamma(\alpha-k)$. As, ultimately, we will choose γ such that $\mathbb{E}[|V_n|^\gamma] \to 0$, we may and do exclude in the following the cases $\gamma(\alpha-k) = (-1-\gamma)/2$ and $\gamma(\alpha-k) = -3/2$. We find the following estimates

$$B_n' \le Cn^{2\gamma(\alpha-k)+\gamma+2}, \qquad B_n'' \le \begin{cases} Cn^{2\gamma(\alpha-k)+\gamma+2} & \text{for } \gamma(\alpha-k) > -3/2, \\ Cn^{\gamma-1} & \text{for } \gamma(\alpha-k) < -3/2, \end{cases}$$

$$B_n''' \le \begin{cases} Cn^{2\gamma(\alpha-k)+\gamma+2} & \text{for } \gamma(\alpha-k) > (-1-\gamma)/2, \\ Cn & \text{for } \gamma(\alpha-k) < (-1-\gamma)/2, \end{cases}$$

which implies

$$\mathbb{E}[|V_n|^\gamma] \le C\Big(n^{2\gamma(\alpha-k)+\gamma+2} + n\Big). \tag{5.35}$$

Combining (5.24) with the estimates (5.31) and (5.35) yields

$$\mathbb{E}\Big[\Big|n^{k-\alpha-1/\beta-1}(S_n - \tilde{S}_n)\Big|^\gamma\Big] \le C\Big(n^{-\gamma/\beta+\gamma(r-1)(\alpha-k)+1} + n^{-\gamma/\beta+\gamma(\alpha-k)+2} + n^{\gamma(k-\alpha-1/\beta-1)+1}\Big). \tag{5.36}$$

The three terms on the right-hand side of (5.36) converge to zero as $n \to \infty$. Indeed, it follows that the first term converges to zero, by choosing $\gamma \in (1,\beta)$ close enough to β and then choose $r \in (1, \beta/\gamma)$ close enough to β/γ, which can be done under the above restrictions on r and γ. The second term converges to zero due to the assumption $\gamma(\alpha-k) < -1$ and the third term converges to 0 for γ close enough to β by the assumption $\alpha > k-1$. Hence, (5.36) completes the proof of (5.21).

Proof of (5.22): In the following we write $g_{i,n,k}$ for $g_{i,k}^n$, given in (3.2), to stress the dependence of the order of increments $k \ge 1$. We have

$$n^{k-\alpha-1/\beta-1}\tilde{S}_n \overset{\mathrm{d}}{=} \Phi'_{\rho^n}(0) n^{k-1} \sum_{r=k}^{n} \Delta_{r,k}^n X = \Phi'_{\rho^n}(0) n^{k-1}\Big(\Delta_{n,k-1}^n X - \Delta_{k-1,k-1}^n X\Big), \tag{5.37}$$

where the last equality follows by the telescoping sum structure. According to the mean value theorem there exists $\theta_1, \theta_2 \in [-k/n, 0]$ (depending on n and s) such that

$$\begin{aligned} \Big|n^{k-1}\Big(g_{n,n,k-1}(s) - g_{k-1,n,k-1}(s)\Big)\Big| &\le C\Big|g^{(k-1)}(1-s+\theta_1) - g^{(k-1)}(-s+\theta_2)\Big| \\ &\le C\Big(\mathbb{1}_{\{|s|\le 1\}} + \mathbb{1}_{\{s<-1\}}|s|^{\alpha-k}\Big) =: c(s), \end{aligned} \tag{5.38}$$

where the last inequality follows by Assumption (A2) and the mean value theorem for $s < -1$, and by the assumption $\alpha > k-1$ for the case $|s| \le 1$. The function c in (5.38) is in $L^\beta(ds)$, due to the fact that $\alpha < k - 1/\beta$. Hence, by the dominated convergence theorem, we have

$$\int_{\mathbb{R}} \Big|n^{k-1}\Big(g_{n,n,k-1}(s) - g_{k-1,n,k-1}(s)\Big)\Big|^\beta ds \to \int_{\mathbb{R}} \Big|g^{(k-1)}(1-s) - g^{(k-1)}(-s)\Big|^\beta ds =: c_0 < \infty, \tag{5.39}$$

as $n \to \infty$. By [10, Lemma 5.3], $\rho^n \to \rho^\infty$ which implies that $\Phi'_{\rho^n}(0) \to \Phi'_{\rho^\infty}(0)$ by continuity of $\rho \mapsto \Phi'_\rho(0)$ on $(0,\infty)$. Therefore, by (5.37) and (5.39) we conclude that

$$n^{k-\alpha-1/\beta-1}\tilde{S}_n \xrightarrow{d} S\beta S(\sigma), \qquad \text{with } \sigma := \rho_L \Phi'_{\rho^\infty}(0) c_0^{1/\beta}, \tag{5.40}$$

which completes the proof of Theorem 2.6(i).

5.3 Proof of Theorem 2.6(ii)

Before we start the proof of Theorem 2.6(ii) we will deduce some estimates on $\Phi_\rho(x)$ relying on the assumption of Appell rank greater or equal to 2 in this theorem. Let $\epsilon \in (0,1)$ be fixed. The mean value theorem, together with assumptions (2.5) and (2.6) and the Appell rank greater or equal to 2 condition, $\frac{\partial}{\partial x}\Phi_\rho(0) = 0$ for all $\rho > 0$, implies that

$$|\Phi_\rho(x) - \Phi_\rho(y)| \le C\Big((1 \wedge |x| + 1 \wedge |y|)|x-y|\mathbb{1}_{\{|x-y|\le 1\}} + |x-y|^p\mathbb{1}_{\{|x-y|>1\}}\Big) \tag{5.41}$$

for all $x, y \in \mathbb{R}$ and $\rho \in [\epsilon, \epsilon^{-1}]$. Specializing (5.41) to $y = 0$ yields that

$$|\Phi_\rho(x)| \le C(|x|^p \wedge |x|^2), \qquad x \in \mathbb{R},\ \rho \in [\epsilon, \epsilon^{-1}]. \tag{5.42}$$

Next let $x \in \mathbb{R}$ and $\rho_1, \rho_2 \in [\epsilon, \epsilon^{-1}]$. From an application of the mean value theorem in the ρ variable it follows that there exists $\tilde\rho \in [\epsilon, \epsilon^{-1}]$ such that

$$\begin{aligned}|\Phi_{\rho_1}(x) - \Phi_{\rho_2}(x)| &\le C|\rho_1-\rho_2| \cdot \Big|\frac{\partial}{\partial\rho}\Phi_{\tilde\rho}(x)\Big| \\ &\le C|\rho_1-\rho_2|\Big(1 \wedge |x|^2\Big) \le C|\rho_1-\rho_2|\Big(|x|^p \wedge |x|^2\Big)\end{aligned}$$

where in the second inequality we use that $|\frac{\partial^3}{\partial x^2\partial\rho}\Phi_\rho(x)| \le C$, $|\frac{\partial}{\partial\rho}\Phi_\rho(x)| \le C$, $\frac{\partial^2}{\partial x\partial\rho}\Phi_\rho(0) = 0$ and $\frac{\partial}{\partial\rho}\Phi_\rho(0) = 0$; the latter fact follows since $\Phi_\rho(0) = 0$ for all $\rho > 0$.

For all $r \ge k$ we define Z_r by

$$Z_r := \sum_{j=1}^{\infty}\big\{\Phi_{\rho_j^\infty}(U_{j+r,r}^\infty) - \mathbb{E}[\Phi_{\rho_j^\infty}(U_{j+r,r}^\infty)]\big\}, \tag{5.43}$$

where the sum is almost surely absolutely convergent. Indeed, this fact follows by the same arguments as in [10, (5.19) b], where this statement is derived in the context of power variation (the proof relies on the estimate (5.42)). Since for all $j \ge 0$ the sequence $(U_{j+r,r}^\infty)_{r\ge k}$ is i.i.d., the random variables $Z_r, r \ge k$ are i.i.d. as well. For $n \ge 1, m, r \ge 0$ we denote

$$\zeta_{r,j}^n := \mathbb{E}[V_r^n|\mathcal{F}_{r-j+1}] - \mathbb{E}[V_r^n|\mathcal{F}_{r-j}] - \mathbb{E}[V_r^n|\mathcal{F}_{r-j}^1],$$

$$R_r^n := \sum_{j=1}^{\infty}\zeta_{r,j}^n \qquad \text{and} \qquad Q_r^n := \sum_{j=1}^{\infty}\mathbb{E}[V_r^n \mid \mathcal{F}_{r-j}^1].$$

The sums R_r^n and Q_r^n converge almost surely, which follows by the arguments of [10, (5.21)] and thereafter. We obtain the following important decomposition

$$S_n = \sum_{r=k}^{n} R_r^n + \sum_{r=k}^{n}(Q_r^n - Z_r) + \sum_{r=k}^{n} Z_r, \tag{5.44}$$

where we will argue that the first two sums in (5.44) are negligible in probability. In order to derive

$$n^{\frac{1}{(\alpha-k)\beta}}\sum_{r=k}^{n} R_r^n \xrightarrow{\mathbb{P}} 0,$$

we may argue along the lines of the proof of (5.22) in [10, Proposition 5.2] where this statement is derived in the context of power variation (note that R_r^n corresponds to $R_r^{n,0}$

in their notation). Key to the proof is the estimate [10, Lemma 5.7], which we generalize to our setting in Lemma 6.2. Similarly, we obtain

$$n^{\frac{1}{(\alpha-k)\beta}} \sum_{r=k}^{n} (Q_r^n - Z_r) \xrightarrow{\mathbb{P}} 0$$

by arguing along the lines of the proof of (5.24) in [10, Proposition 5.2]. The proof relies on the estimates (5.3)–(5.5) and [10, Eq. (5.15), (5.18)], as well as on Lemma 6.4. The estimate [10, Eq. (5.15)] is in our context replaced by (5.41), where we need to argue that for sufficiently large N the set $\{\rho_j^n : n \in \{N, ..., \infty\}, j \in \mathbb{N}\}$ is bounded away from 0 and ∞, which is done in Lemma 6.3.

It therefore remains to show that Z_r is in the domain of attraction of a $(k-\alpha)\beta$-stable random variable, which we do in two steps. First we define the random variable

$$Q := \overline{\Phi}(L_{k+1} - L_k) - \mathbb{E}[\overline{\Phi}(L_{k+1} - L_k)], \quad \text{where} \quad \overline{\Phi}(x) := \sum_{j=1}^{\infty} \Phi_{\rho_j^\infty}(\phi_j^\infty(0)x)$$

and show that it is in the domain of attraction of a $(k-\alpha)\beta$-stable random variable S. Thereafter we argue that we can find $r > (k-\alpha)\beta$ such that

$$\mathbb{P}(|Z_k - Q| > x) \le Cx^{-r}, \quad \text{for all } x \ge 1, \tag{5.45}$$

which yields that Z_k is in the domain of attraction of S as well, and an application of [32, Theorem 1.8.1] concludes the proof.

Let us first remark that the function $\overline{\Phi}$ and the random variable Q are well-defined. Indeed, since $\rho_j^\infty \to \rho^\infty \in (0,\infty)$, the set $\{\rho_j^\infty\}_{j\in\mathbb{N}}$ is bounded away from 0 and ∞ and by (5.42) it follows for any $\gamma \in (p,\beta)$ that

$$\sum_{j=1}^{\infty} |\Phi_{\rho_j^\infty}(\phi_j^\infty(0)x)| \le C \sum_{j=1}^{\infty} |\phi_j^\infty(0)x|^\gamma \le C|x|^\gamma \sum_{j=1}^{\infty} j^{\gamma(\alpha-k)}.$$

By choosing $\gamma > 1/(k-\alpha)$ it follows that $\overline{\Phi}$ and Q are well-defined. Moreover, an application of the dominated convergence theorem shows that $\overline{\Phi}$ is continuous.

In order to show that Q is in the domain of attraction of a $(k-\alpha)\beta$-stable random variable we next determine constants c_-, c_+ such that

$$\lim_{x\to\infty} x^{(k-\alpha)\beta}\mathbb{P}(Q < -x) = c_-, \quad \lim_{x\to\infty} x^{(k-\alpha)\beta}\mathbb{P}(Q > x) = c_+.$$

Then it follows by [32, Theorem 1.8.1] that the random variable Q is in the domain of attraction of a $(k-\alpha)\beta$-stable random variable with scale parameter ρ_1 and skewness parameter η_1, given by

$$\rho_1 := \left(\frac{c_+ + c_-}{\tau_{(k-\alpha)\beta}}\right)^{1/(k-\alpha)\beta}, \qquad \text{and} \qquad \eta_1 := \frac{c_+ - c_-}{c_+ + c_-}. \tag{5.46}$$

Here the constant τ_γ, $\gamma \in (0,2)$, is defined as

$$\tau_\gamma := \begin{cases} \frac{1-\gamma}{\Gamma(2-\gamma)\cos(\pi\gamma/2)} & \text{if } \gamma \neq 1, \\ \pi/2 & \text{if } \gamma = 1. \end{cases} \tag{5.47}$$

In the following we derive explicit expressions for c_+ and c_-, which are stated in (5.51) and (5.53) below. For $x > 0$ it holds by substituting $t = (x/u)^{1/(k-\alpha)}$ that

$$x^{1/(\alpha-k)}\overline{\Phi}(x) = x^{1/(\alpha-k)} \int_0^\infty \Phi_{\rho^\infty_{1+[t]}}(\phi^\infty_{1+[t]}(0)x)\,dt$$
$$= \frac{1}{k-\alpha}\int_0^\infty \Phi_{\rho^\infty_{1+[(x/u)^{1/(k-\alpha)}]}}\big(\phi^\infty_{1+[(x/u)^{1/(k-\alpha)}]}(0)x\big)u^{-1+1/(\alpha-k)}\,du \tag{5.48}$$
$$\to \frac{1}{k-\alpha}\int_0^\infty \Phi_{\rho^\infty}(k_\alpha u)u^{-1+1/(\alpha-k)}\,du := \kappa_+, \quad \text{as } x\to\infty, \tag{5.49}$$

where $k_\alpha = \alpha(\alpha-1)\dots(\alpha-k+1)$. The convergence as well as the existence of the integral follow from the estimate (5.42) and the dominated convergence theorem, where we use that $\{\rho^\infty_j\}$ is bounded away from 0 and ∞. The convergence of the integrand from (5.48) as $x\to\infty$ follows since by the mean value theorem for all $t\in\mathbb{R}$ there is a $\xi_t \in [t-k-1,t]$ such that

$$\phi^\infty_{[t]}(0) = h_k([t]) = k_\alpha(\xi_t)_+^{\alpha-k},$$

which implies the convergence

$$\phi^\infty_{1+[(x/u)^{1/(k-\alpha)}]}(0)x \to k_\alpha u, \quad \text{as } x\to\infty.$$

Similarly we obtain for $x<0$ that

$$|x|^{1/(\alpha-k)}\overline{\Phi}(x) \to \frac{1}{k-\alpha}\int_{-\infty}^0 \Phi_{\rho^\infty}(k_\alpha u)|u|^{-1+1/(\alpha-k)}\,du := \kappa_-, \quad \text{as } x\to-\infty. \tag{5.50}$$

We argue next that

$$\lim_{x\to\infty} x^{(k-\alpha)\beta}\mathbb{P}(Q>x) = \tau_\beta\rho_L\big(\kappa_+^{k-\alpha}\mathbb{1}_{\{\kappa_+>0\}} + \kappa_-^{k-\alpha}\mathbb{1}_{\{\kappa_->0\}}\big) := c_+, \tag{5.51}$$

where τ_β was defined in (5.47) and ρ_L denotes the scale parameter of the Lévy process L. To this end we make the decomposition

$$\mathbb{P}(Q>x) = \mathbb{P}(Q>x, L_{k+1}-L_k>0) + \mathbb{P}(Q>x, L_{k+1}-L_k<0), \tag{5.52}$$

and analyse the two summands separately. Consider the first summand and assume $\kappa_+>0$. By (5.49) it follows that $\overline{\Phi}(y)\to\infty$ as $y\to\infty$ and we have for sufficiently large x that

$$\mathbb{P}(\overline{\Phi}(L_{k+1}-L_k)>x, L_{k+1}-L_k>0) = \mathbb{P}(|\overline{\Phi}(L_{k+1}-L_k)|>x, L_{k+1}-L_k>0).$$

Applying Lemma 6.6 with $\xi(x) = \overline{\Phi}(x)$ and $\psi(x) = x^{1/(k-\alpha)}\kappa_+$, we deduce from (5.49) that

$$\lim_{x\to\infty} x^{(k-\alpha)\beta}\mathbb{P}(Q>x, L_{k+1}-L_k>0) = \lim_{x\to\infty} x^{(k-\alpha)\beta}\mathbb{P}\big(\kappa_+^{k-\alpha}(L_{k+1}-L_k)>x^{k-\alpha}\big)$$
$$= \tau_\beta\rho_L^\beta\kappa_+^{(k-\alpha)\beta},$$

where the second identity follows from [32, Property 1.2.15]. If $\kappa_+<0$, it follows from (5.49) that $\limsup_{x\to\infty}\overline{\Phi}(x)\le 0$ and therefore that $\overline{\Phi}(x)$ is bounded for $x\ge 0$. We obtain

$$\lim_{x\to\infty} x^{(k-\alpha)\beta}\mathbb{P}(Q>x, L_{k+1}-L_k>0) = 0.$$

The same identity holds for $\kappa_+=0$, as follows from Lemma 6.6, (5.49), and the estimate

$$\mathbb{P}(\overline{\Phi}(L_{k+1}-L_k)>x, L_{k+1}-L_k>0) \le \mathbb{P}(|\overline{\Phi}(L_{k+1}-L_k)|>x, L_{k+1}-L_k>0).$$

We conclude that the first summand of (5.52) satisfies

$$\lim_{x\to\infty} x^{(k-\alpha)\beta}\mathbb{P}(Q > x, L_{k+1} - L_k > 0) = \tau_\beta\rho_L\kappa_+^{k-\alpha}\mathbb{1}_{\{\kappa_+>0\}}.$$

By similar arguments, applying Lemma 6.6 on the function $\xi(x) = \overline{\Phi}(-x)$ and using (5.50), we obtain for the second summand of (5.52) the convergence

$$\lim_{x\to\infty} x^{(k-\alpha)\beta}\mathbb{P}(Q > x, L_{k+1} - L_k < 0) = \tau_\beta\rho_L\kappa_-^{k-\alpha}\mathbb{1}_{\{\kappa_->0\}},$$

which completes the proof of (5.51). Arguing similarly for $\mathbb{P}(Q < -x)$ we derive that

$$\lim_{x\to\infty} x^{(k-\alpha)\beta}\mathbb{P}(Q < -x) = \tau_\beta\rho_L\big(|\kappa_+|^{k-\alpha}\mathbb{1}_{\{\kappa_+<0\}} + |\kappa_-|^{k-\alpha}\mathbb{1}_{\{\kappa_-<0\}}\big) := c_-. \tag{5.53}$$

This shows that Q is in the domain of attraction of a $(k-\alpha)\beta$-stable random variable with location parameter 0, and scale and skewness parameters as given in (5.46).

Now the proof of the theorem is completed by showing (5.45). To this end it is by Markov's inequality sufficient to show that $\mathbb{E}[|Z_k - Q|^r] < \infty$ for some $r > (k-\alpha)\beta$. Indeed $(k-\alpha)\beta > 1$, and an application of Minkowski's inequality yields

$$\|Z_k - Q\|_r \le \sum_{j=1}^{\infty} \big\|\Phi_{\rho_j^\infty}(U_{j+k,k}^\infty) - \Phi_{\rho_j^\infty}\big(\phi_j^\infty(0)(L_{k+1} - L_k)\big)\big\|_r. \tag{5.54}$$

We remark that by the mean value theorem there exists a constant $C > 0$ such that for all $x \in [0,1]$ and $j \in \mathbb{N}$ it holds that

$$|\phi_{j+k}^\infty(x) - \phi_j^\infty(0)| = |h_k(j+k+x) - h_k(j)| \le Cj^{\alpha-k-1}.$$

Since $\{\rho_j^\infty\}_{j\in\mathbb{N}}$ is bounded away from 0, there is a $\delta > 0$ with $\delta < \rho_j^\infty$ for all j. Letting $r_\varepsilon = (k-\alpha)\beta + \varepsilon$ with $\varepsilon \in (0,\delta)$, an application of Lemma 6.4 yields

$$\begin{aligned}
&\big\|\Phi_{\rho_j^\infty}(U_{j+k,k}^\infty) - \Phi_{\rho_j^\infty}(\phi_j^\infty(0)(L_{k+1} - L_k))\big\|_{r_\varepsilon} \\
&\le C\big(\|\phi_{j+k}^\infty - \phi_j^\infty(0)\|_{L^\beta([0,1])}^{1-\varepsilon} + \|\phi_{j+k}^\infty - \phi_j^\infty(0)\|_{L^\beta([0,1])}^{\frac{1}{k-\alpha+\varepsilon/\beta}}\big) \le C(j^{(\alpha-k-1)(1-\varepsilon)} + j^{\frac{\alpha-k-1}{k-\alpha+\varepsilon/\beta}}).
\end{aligned} \tag{5.55}$$

For sufficiently small $\varepsilon > 0$, both powers of j on the right-hand side of (5.55) are smaller than -1, which together with (5.54) implies $\|Z_k - Q\|_r < \infty$, and thus (5.45). Since Q is in the domain of attraction of a $(k-\alpha)\beta$-stable random variable with scale parameter ρ_1 and skewness parameter η_1, and $r > (k-\alpha)\beta$, so is Z_k. This completes the proof of Theorem 2.6(ii). ☐

6 Auxiliary results

In this section we show some technical results used in the proofs of Theorems 2.5 and 2.6.

Lemma 6.1. *Let p be as in (2.5). For any $\varepsilon > 0$ there exists a finite constant $C_\varepsilon > 0$ such that for all $\rho \in [\varepsilon, \varepsilon^{-1}]$, $a \in \mathbb{R}$ and $x, y > 0$ we have that*

$$F(a,x,y) := \left|\int_0^y \int_0^x \Phi_\rho''(a+u+v)\,du\,dv\right| \le C(x^p \wedge x)(y^p \wedge y).$$

Proof. Let us first remark that $x^p \wedge x = x\mathbb{1}_{\{x\le1\}} + x^p\mathbb{1}_{\{x>1\}}$ since $p < 1$. By assumption, $\Phi_\rho'(x)$ and $\Phi_\rho''(x)$ are uniformly bounded for $\rho \in [\varepsilon, \varepsilon^{-1}]$ and $x \in \mathbb{R}$. Boundedness of Φ_ρ'' immediately implies $F(a,x,y) \le Cxy$. Moreover, it holds that

$$\begin{aligned}
\int_0^y \int_0^x \Phi_\rho''(a+u+v)\,du\,dv &= \int_0^y \Phi_\rho'(a+x+v) - \Phi_\rho'(a+v)\,dv \\
&= \big(\Phi_\rho(a+x+y) - \Phi_\rho(a+y)\big) - \big(\Phi_\rho(a+x) - \Phi_\rho(a)\big).
\end{aligned}$$

The first equality and boundedness of Φ'_ρ implies $F(a,x,y) \le Cy$, and consequently $F(a,x,y)\mathbb{1}_{\{x>1\}} \le Cx^p y \mathbb{1}_{\{x>1\}}$, and similarly $F(a,x,y)\mathbb{1}_{\{y>1\}} \le Cxy^p\mathbb{1}_{\{y>1\}}$. Finally, the second equality together with (2.5) implies that $F(a,x,y)\mathbb{1}_{\{x>1,y>1\}} \le Cx^p\mathbb{1}_{\{x>1,y>1\}} \le Cx^py^p\mathbb{1}_{\{x>1,y>1\}}$, completing the proof. ☐

Lemma 6.2. *For all $\gamma \in [1,2]$ there exists a $C>0$ such that for all $n\in\mathbb{N}$, $r\in\{k,\dots,n\}$, $j\in\mathbb{N}$ and $l\ge j$ it holds that*

$$\mathbb{E}[|\vartheta^n_{r,j,l}|^\gamma] \le \begin{cases} Cj^{(\alpha-k)\beta}l^{(\alpha-k)\beta} & \text{for } \beta<\gamma<\beta/p, \\ Cj^{(\alpha-k)\gamma}l^{(\alpha-k)\gamma} & \text{for } \gamma<\beta, \end{cases}$$

where $\vartheta^n_{r,j,l}$ was defined in (5.33).

Proof. It is sufficient to consider the case $r=1$, since for fixed j,l,n the sequence $(\vartheta^n_{r,j,l})_{r\in\mathbb{N}}$ is stationary. Without loss of generality we may assume that $l\ge 2\vee j$ since the case $l=j=1$ can be covered by choosing a larger constant. To this end we remark that $(\mathbb{E}[|\vartheta^n_{1,1,1}|^\gamma])_{n\in\mathbb{N}}$ is bounded, since $Y^n_r \sim \mathrm{S}\beta\mathrm{S}(\rho^n)$ with ρ^n (which was introduced in (5.2)) bounded away from 0 and ∞ by [10, Lemma 5.3]. By definition of ϑ it holds that

$$\begin{aligned}\vartheta^n_{1,j,l} &= \mathbb{E}[f(Y^n_1)\,|\,\mathcal{F}^1_{1-j}\vee\mathcal{F}_{1-l}] - \mathbb{E}[f(Y^n_1)\,|\,\mathcal{F}_{1-l}] \\ &\quad - \big\{\mathbb{E}[f(Y^n_1)\,|\,\mathcal{F}^1_{1-j}\vee\mathcal{F}_{-l}] - \mathbb{E}[f(Y^n_1)\,|\,\mathcal{F}_{-l}]\big\}.\end{aligned}$$

Define for $-\infty\le a<b\le 1$ the random variable

$$U^n_{[a,b]} = \int_a^b \phi^n_1(s)\,dL_s.$$

Let in the following $\widetilde{L}$ be an independent copy of L and define $\widetilde{U}^n_{[a,b]}$ accordingly, and denote by $\widetilde{\mathbb{E}}$ the expectation with respect to $\widetilde{L}$ only. Moreover, we denote by $\rho^n_{j,l} = \|\phi^n_1\|_{L^\beta([1-l,1-j]\cup[2-j,1])}$, i.e. the scale parameter of $\int_{1-l}^{1-j}\phi^n_1\,dL_s + \int_{2-j}^1\phi^n_1\,dL_s$. Then, decomposing $\int_{-\infty}^1\phi^n_1\,dL_s$ into the independent integrals

$$\int_{-\infty}^1 \phi^n_1\,dL_s = \int_{-\infty}^{-l}\phi^n_1\,dL_s + \int_{-l}^{1-l}\phi^n_1\,dL_s + \int_{1-l}^{1-j}\phi^n_1\,dL_s + \int_{1-j}^{2-j}\phi^n_1\,dL_s + \int_{2-j}^1\phi^n_1\,dL_s$$

we obtain the expression

$$\begin{aligned}\vartheta^n_{1,j,l} &= \widetilde{\mathbb{E}}\Big[\Phi_{\rho^n_{j,l}}(U^n_{[-\infty,-l]} + U^n_{[-l,1-l]} + U^n_{[1-j,2-j]}) - \Phi_{\rho^n_{j,l}}(U^n_{[-\infty,-l]} + U^n_{[-l,1-l]} + \widetilde{U}^n_{[1-j,2-j]}) \\ &\qquad - \Phi_{\rho^n_{j,l}}(U^n_{[-\infty,-l]} + \widetilde{U}^n_{[-l,1-l]} + U^n_{[1-j,2-j]}) + \Phi_{\rho^n_{j,l}}(U^n_{[-\infty,-l]} + \widetilde{U}^n_{[-l,1-l]} + \widetilde{U}^n_{[1-j,2-j]})\Big] \\ &= \widetilde{\mathbb{E}}\Big[\int_{\widetilde{U}^n_{[-l,1-l]}}^{U^n_{[-l,1-l]}}\int_{\widetilde{U}^n_{[1-j,2-j]}}^{U^n_{[1-j,2-j]}} \Phi''_{\rho^n_{j,l}}(U^n_{[-\infty,-l]}+u+v)\,du\,dv\Big],\end{aligned}$$

and by substitution there is a random variable $\widetilde{W}^n_{j,l}$ such that

$$|\vartheta^n_{1,j,l}| \le \widetilde{\mathbb{E}}\bigg[\bigg|\int_0^{|\widetilde{U}^n_{[-l,1-l]}-U^n_{[-l,1-l]}|}\int_0^{|\widetilde{U}^n_{[1-j,2-j]}-U^n_{[1-j,2-j]}|}\Phi''_{\rho^n_{j,l}}(\widetilde{W}^n_{j,l}+u+v)\,du\,dv\bigg|\bigg].$$

We denote $\varphi_p(x) := |x|^p\wedge|x|$. Suppose in the following that $\gamma>\beta$. Using Lemma 6.1, Jensen's inequality, the inequality $\varphi_p(|x-y|)\le 2(\varphi_p(|x|)+\varphi_p(|y|))$ and the independence of U and $\widetilde{U}$, we obtain that

$$\begin{aligned}\mathbb{E}[|\vartheta^n_{1,j,l}|^\gamma] &\le C\mathbb{E}[\widetilde{\mathbb{E}}[\varphi_p^\gamma(|\widetilde{U}^n_{[-l,1-l]} - U^n_{[-l,1-l]}|)\varphi_p^\gamma(|\widetilde{U}^n_{[1-j,2-j]} - U^n_{[1-j,2-j]}|)]] \\ &\le C\mathbb{E}[\widetilde{\mathbb{E}}[\varphi_p^\gamma(|\widetilde{U}^n_{[-l,1-l]}|) + \varphi_p^\gamma(|U^n_{[-l,1-l]}|)]]\mathbb{E}[\widetilde{\mathbb{E}}[\varphi_p^\gamma(|\widetilde{U}^n_{[1-j,2-j]}|) + \varphi_p^\gamma(|U^n_{[1-j,2-j]}|)]] \\ &\le C\|\phi^n_1\|^\beta_{L^\beta([-l,1-l])}\|\phi^n_1\|^\beta_{L^\beta([1-j,2-j])} \le Cl^{(\alpha-k)\beta}j^{(\alpha-k)\beta}.\end{aligned}$$

In the third inequality we used the estimate (5.3), where we remark that by assumption $\gamma > \beta$ and $p\gamma < \beta$, and the expression (3.1) for the scale parameter of integrals with respect to a stable Lévy process. The last inequality follows from (5.4). For $\gamma < \beta$ we use the same arguments above, however, due to the fact that (5.3) gives at different estimate in this case we obtain the bound $\mathbb{E}[|\vartheta^n_{1,j,l}|^\gamma] \le C l^{(\alpha-k)\gamma} j^{(\alpha-k)\gamma}$, which concludes the proof. □

Lemma 6.3. *The set $\{\rho^n_j : n \in \{N, ..., \infty\}, j \in \mathbb{N}\}$ is bounded away from 0 and ∞ for sufficiently large $N \in \mathbb{N}$.*

Proof. Choose $\varepsilon > 0$ such that $\varepsilon < \rho^\infty < \varepsilon^{-1}$ and $\varepsilon < \rho^\infty_j < \varepsilon^{-1}$ for all $j \in \mathbb{N}$. It follows from [10, Lemma 5.3] that $\rho^n \to \rho^\infty$ and we can choose N sufficiently large such that $|\rho^n - \rho^\infty| < \varepsilon/3$ for all $n > N$, implying that $2\varepsilon/3 < \rho^n < \varepsilon^{-1} + \varepsilon/3$. Moreover, ρ^n_j converges to ρ^n uniformly in n by the estimate

$$|(\rho^n_j)^\beta - (\rho^n)^\beta| = \|\phi^n_j\|^\beta_{L^\beta([0,1])} \le C j^{\beta(\alpha-k)},$$

where we used (5.4), and that the function $x \mapsto |x|^\beta$, restricted to a compact set, is uniformly continuous. Consequently, we can find a $J > 0$ such that for all $j > J$ and all n it holds that $|\rho^n_j - \rho^n| < \varepsilon/3$, implying that $\varepsilon/3 < \rho^n_j < \varepsilon^{-1} + 2\varepsilon/3$ for all $j > J$ and $n > N$. For $j \in \{1, ..., J\}$ we use that $\rho^n_j \to \rho^\infty_j \in (\varepsilon, \varepsilon^{-1})$ as $n \to \infty$, which follows similarly from (5.5). Therefore, choosing N larger if necessary, we obtain $\varepsilon/3 < \rho^n_j < \varepsilon^{-1} + 1$ for all $j \in \mathbb{N}$ and $n > N$. □

The following auxiliary result was derived in [10] in the context of power variation. The proof relies only the estimate (5.41) on Φ_ρ.

Lemma 6.4. *([10, Lemma 5.4]). Under the setting of Theorem 2.6(ii), we have for any $q \ge 1$ with $q \ne \beta$ that there exists $\delta > 0$ and a finite constant C such that for all $\varepsilon \in (0, \delta)$, $\rho > \delta$ and $\kappa, \tau \in L^\beta([0,1])$ satisfying $\|\kappa\|_{L^\beta([0,1])}, \|\tau\|_{L^\beta([0,1])} \le 1$ and*

$$\left\| \Phi_\rho\left(\int_0^1 \kappa(s)\, dL_s \right) - \Phi_\rho\left(\int_0^1 \tau(s)\, dL_s \right) \right\|_q \le \begin{cases} C\|\kappa - \tau\|^{\beta/q}_{L^\beta([0,1])} & \beta < q \\ C\Big\{ \big(\|\kappa\|^{(\beta-q)/q-\varepsilon}_{L^\beta([0,1])} + \|\tau\|^{(\beta-q)/q-\varepsilon}_{L^\beta([0,1])}\big) \|\kappa-\tau\|^{1-\varepsilon}_{L^\beta([0,1])} + \|\kappa-\tau\|^{\beta/q}_{L^\beta([0,1])} \Big\} & \beta > q. \end{cases}$$

We will need the following minor extension of [27, Lemma 2.1]:

Lemma 6.5. *([27, Lemma 2.1]). Let $\{X_n : n \in \mathbb{N}_0\}$ denote symmetric β-stable random variables such that $X_n \to X_0$ in probability. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a measurable function such that $\mathbb{E}[|f(X_0)|^p] < \infty$ for some $p \ge 1$. Then, $\mathbb{E}[|f(X_n) - f(X_0)|^p] \to 0$.*

Note that Lemma 6.5 relies heavily on the β-stable assumption, and a similar result (with no continuity assumptions on f) does not hold for e.g. discrete random variables.

Proof. If f is bounded, $p = 2$ and $X_n \to X_0$ almost surely, Lemma 6.5 is [27, Lemma 2.1]. However, going through the proof of [27, Lemma 2.1] shows that it also holds for a general $p \ge 1$ and if $X_n \to X$ in probability, by using the same arguments. To extend Lemma 6.5 from bounded f, to unbounded f satisfying $\mathbb{E}[|f(X)|^p] < \infty$, it is enough to show tightness of $\{|f(X_n)|^p : n \ge 1\}$, due to a truncation argument. The density of X_n satisfies

$$f_{X_n}(x) = \rho_n^{-1} g_\beta(x/\rho_n), \qquad x \in \mathbb{R}, \tag{6.1}$$

where g_β is the density of a standard symmetric β-stable random variable and ρ_n is the scale parameter for X_n for $n \in \mathbb{N}_0$. Since $\mathbb{E}[|f(X_0)|^p] < \infty$ and $\rho_n \to \rho$ (follows since

$X_n \to X$ in distribution), we deduce tightness of $\{|f(X_n)|^p : n \geq 1\}$ from (6.1). This completes the proof. $\square$

Lemma 6.6. *Let ψ, ξ be continuous functions on $\mathbb{R}$ with $\psi(x) \sim \xi(x)$ for $x \to \infty$. Let X be a random variable taking values in $\mathbb{R}_+$ and $\gamma \geq 0$ such that*

$$\lim_{x \to \infty} x^\gamma \mathbb{P}(|\psi(X)| > x) = \kappa$$

where $\kappa \in [0, \infty)$. Then it holds that

$$\lim_{x \to \infty} x^\gamma \mathbb{P}(|\xi(X)| > x) = \kappa.$$

Proof. Denote $\psi(x) = \xi(x)\varphi(x)$ with $\varphi(x) \to 1$ for $x \to \infty$. Let $\varepsilon > 0$. By continuity of ψ and ξ we can choose x sufficiently large such that $\varphi(y) \in (1-\varepsilon, 1+\varepsilon)$ whenever $\min(|\psi(y)|, |\xi(y)|) > x$ and $y \geq 0$. Since X takes values in $\mathbb{R}_+$, this implies that $\varphi(X) \in (1-\varepsilon, 1+\varepsilon)$ whenever $|\psi(X)| > x$ or $|\xi(X)| > x$. It follows that

$$\begin{aligned}
x^\gamma |\mathbb{P}(|\psi(X)| > x) - \mathbb{P}(|\xi(X)| > x)| &= \mathbb{E}\big[x^\gamma \big(\mathbb{1}_{\{|\psi(X)| > x > |\xi(X)|\}} + \mathbb{1}_{\{|\psi(X)| < x < |\xi(X)|\}}\big)\big] \\
&\leq 2\mathbb{E}\big[x^\gamma \mathbb{1}_{\{\frac{x}{1+\varepsilon} < |\psi(X)| < \frac{x}{1-\varepsilon}\}}\big] = 2\mathbb{E}\big[x^\gamma \mathbb{1}_{\{\frac{x}{1+\varepsilon} < |\psi(X)|\}} - x^\gamma \mathbb{1}_{\{\frac{x}{1-\varepsilon} \leq |\psi(X)|\}}\big] \\
&\to 2\kappa((1+\varepsilon)^\gamma - (1-\varepsilon)^\gamma), \quad \text{as } x \to \infty.
\end{aligned}$$

The lemma follows by letting $\varepsilon \to 0$. $\square$

Proof of Remark 2.4. (i): We will start by verifying (B) for any bounded measurable function f. Let g_β denote the density of a standard symmetric β-stable random variable. By substitution we have

$$\Phi_\rho(x) = \int_{\mathbb{R}} f(y) g_\beta((y-x)/\rho)\, dy - \int_{\mathbb{R}} f(y) g_\beta(y/\rho)\, dy. \tag{6.2}$$

Recall from (5.7) that $g_\beta \in C^\infty(\mathbb{R})$, and for all $r \geq 1$, the rth derivative of g_β satisfies

$$|g_\beta^{(r)}(x)| \leq C(1 \wedge |x|^{-1-\beta-r}), \qquad x \in \mathbb{R}. \tag{6.3}$$

By the (6.2), (6.3) and using that f is bounded, it follows that $\rho \mapsto \Phi_\rho(x)$ is $C^1((0,\infty))$ and

$$\begin{aligned}
\frac{\partial}{\partial \rho} \Phi_\rho(x) = &\ -\rho^{-2}\Big(\int_{\mathbb{R}} \Big(f(y) g_\beta'((y-x)/\rho)(y-x)\Big)\, dy - \int_{\mathbb{R}} \Big(f(y) g_\beta'(y/\rho) y\Big)\, dy\Big) \\
= &\ -\int_{\mathbb{R}} \Big(f(x+\rho y) g_\beta'(y) y\Big)\, dy + \int_{\mathbb{R}} \Big(f(\rho y) g_\beta'(y) y\Big)\, dy,
\end{aligned}$$

which implies existence of $C > 0$ such that $|\frac{\partial}{\partial\rho} \Phi_\rho(x)| \leq C$ for all $\rho \in [\epsilon, \epsilon^{-1}]$ and $x \in \mathbb{R}$. By similar arguments one can verify the remaining conditions of (B).

(ii): Next we suppose that $f \in L^1_{\text{loc}}(\mathbb{R})$ and there exists $K > 0$ and $q \leq 1$ such that $f \in C^3([-K,K]^c)$ and $|f'(x)|, |f''(x)|, |f'''(x)| \leq C$ and $|f'(x)| \leq C|x|^{q-1}$ for $|x| > K$. In the following we will verify that f satisfies (B) with $p = q$ when $q > 0$, and $p = 0$ when $q < 0$. Let $\xi \in C_c^\infty(\mathbb{R})$ be a function such that $\xi = 1$ on $[-K, K]$. By the equality $1 = \xi + (1-\xi)$ and substitution we have

$$\begin{aligned}
\Phi_\rho(x) - \int f(\rho y) g_\beta(y)\, dy &= \int f(x+\rho y) g_\beta(y)\, dy \\
&= \int f(y)\xi(y) g_\beta((y-x)/\rho)\, dy + \int f(x+y\rho)\big(1 - \xi(x+y\rho)\big) g_\beta(y)\, dy \\
&=: \bar{\Phi}_\rho(x) + \tilde{\Phi}_\rho(x).
\end{aligned}$$

Since f is locally integrable and ξ has compact support we have $f\xi \in L^1(\mathbb{R})$, and due to the fact that $|g'|$ is bounded

$$\Big|\frac{\partial}{\partial x}\bar{\Phi}_\rho(x)\Big| = \Big|\rho^{-1}\int f(y)\xi(y)g'_\beta((y-x)/\rho)\,dy\Big| \leq C\rho^{-1}\int |f(s)\xi(s)|\,ds < \infty. \qquad (6.4)$$

On the other hand, it follows that $(f(1-\xi))'$ is bounded. Indeed, since $f(1-\xi)=0$ on $[-K,K]$ it is enough to show that $(f(1-\xi))'$ is bounded for $|x|>K$. For $|x|>K$ we have $(f(1-\xi))' = f'(1-\xi) - f\xi'$ which is bounded due to the fact that f' is bounded and f is continuous for $|x|>K$, and ξ' has compact support. Therefore,

$$\Big|\frac{\partial}{\partial x}\tilde{\Phi}_\rho(x)\Big| = \Big|\int (f(1-\xi))'(x+y\rho)g_\beta(y)\,dy\Big| \leq C\int |g_\beta(y)|\,dy < \infty. \qquad (6.5)$$

From (6.4) and (6.5) it follows that $\frac{\partial}{\partial x}\Phi_\rho(x)$ is bounded. By similar arguments one can verify the remaining conditions of (2.6). To verify (2.5) we will use that g_β is both Lipschitz continuous and bounded, and hence for any $p\in[0,1]$, $\rho\in[\epsilon,\epsilon^{-1}]$

$$\begin{aligned}|\bar{\Phi}_\rho(x)-\bar{\Phi}_\rho(y)| &\leq \int |f(u)\xi(u)(g_\beta((u-x)/\rho)-g_\beta((u-y)/\rho)|\,du \\ &\leq C\Big(1\wedge|x-y|\Big)\int |f(u)\xi(u)|\,du \leq C|x-y|^p.\end{aligned}$$

For $0<q\leq 1$ and $x\neq 0$ we have that $|(f(1-\xi))'(x)| \leq C|x|^{q-1}$ which implies that $f(1-\xi)$ is q-Hölder continuous, and therefore

$$\begin{aligned}|\tilde{\Phi}_\rho(x)-\tilde{\Phi}_\rho(y)| &\leq \int \Big|(f(1-\xi))(x+u)-(f(1-\xi))(y+u)\Big|g_\beta(u)\,du \\ &\leq C\Big(\int g_\beta(u)\,du\Big)|x-y|^q.\end{aligned}$$

This concludes the proof of (2.5) with $p=q$ when $0<q\leq 1$. For $q<0$, we have that $f\in L^1(\mathbb{R})$, and hence it follows by (6.2) and boundedness of g_β that $|\Phi_\rho(x)|\leq C$ for all $x\in\mathbb{R}$ and $\rho\in[\epsilon,\epsilon^{-1}]$, which shows (2.5) with $p=0$. ☐

Remark 6.7. In the following we prove the statements on the Appell rank at the begining of Subsection 2.2. Suppose first that f is an even function. Since S is a symmetric random variable and $\Phi_\rho(x) = \mathbb{E}[f(x+\rho S)] - \mathbb{E}[f(\rho S)]$, we have that $x\mapsto\Phi_\rho(x)$ is an even function for all ρ. Hence, $\frac{\partial}{\partial x}\Phi_\rho(0)=0$ and $\frac{\partial^2}{\partial x\partial\rho}\Phi_\rho(0)=0$. Next consider the function $f(x)=\sin(ux)$ for all $x\in\mathbb{R}$, where $u\neq 0$. We have that

$$\Phi_\rho(x) = \mathbb{E}[\sin(u(x+\rho S))] - \mathbb{E}[\sin(u\rho S)] = \Im\Big(\mathbb{E}[e^{iu(x+\rho S)}]\Big) = \sin(ux)e^{-|\rho u|^\beta},$$

and hence $\frac{\partial}{\partial x}\Phi_\rho(0) = ue^{-|\rho u|^\beta} \neq 0$. Finally, we let $f(x) = \mathbb{1}_{(-\infty,u]}(x)$ for all $x\in\mathbb{R}$, where $u\in\mathbb{R}$. Then

$$\Phi_\rho(x) = \mathbb{P}(S\leq(u-x)/\rho) - \mathbb{P}(S\leq u/\rho),$$

and hence $\frac{\partial}{\partial x}\Phi_\rho(0) = -\rho g_\beta(u/\rho)$, where g_β denotes the density of a standard $S\beta S$ random variable. Since $g_\beta(x)\neq 0$ for all $x\in\mathbb{R}$ (see e.g. Theorem 1.2 in [38]), it follows that $\frac{\partial}{\partial x}\Phi_\rho(0)\neq 0$, which completes the proofs of the statements.

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