Electronic Journal of Probability

Electron. J. Probab. **24** (2019), no. 64, 1–21.
ISSN: 1083-6489 https://doi.org/10.1214/18-EJP171

Existence of a phase transition of the interchange process on the Hamming graph

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Abstract

The interchange process on a finite graph is obtained by placing a particle on each vertex of the graph, then at rate 1, selecting an edge uniformly at random and swapping the two particles at either end of this edge. In this paper we develop new techniques to show the existence of a phase transition of the interchange process on the 2-dimensional Hamming graph. We show that in the subcritical phase, all of the cycles of the process have length $O(\log n)$, whereas in the supercritical phase a positive density of vertices lies in cycles of length at least $n^{2-\varepsilon}$ for any $\varepsilon > 0$.

Keywords: random permutation; phase transition; Hamming graph.
AMS MSC 2010: 60G99; 81S99; 82B99.
Submitted to EJP on May 19, 2017, final version accepted on April 19, 2018.
Supersedes arXiv:1605.03548.

1 Introduction and main results

The interchange process $\sigma = (\sigma_t : t \geq 0)$ is defined on a finite, connected, undirected graph $G = (V, E)$ as follows. For each $t \geq 0$, $\sigma_t : V \to V$ is a permutation on the vertices of the graph and initially σ_0 is the identity permutation. Then at rate 1, an edge $e \in E$ is selected uniformly at random and the vertices on either end of e are swapped in the permutation. Under the assumption of bounded degree, this can be defined similarly on infinite graphs as well.

The interchange process was introduced by [11] in the case of $\mathbb{Z}^d$. Later this model was modified by [14] to derive results about the quantum Heisenberg ferromagnetic model. Since then, this has been studied on infinite regular trees [2, 9, 10] and the complete graph [13, 4, 5]. Recently, [12] extended the techniques of [4] to show some results about the interchange process on the hypercube. Kozma and Sidoravicius have announced that they have computed the expected cycle size on $\mathbb{Z}$ and also similar questions have been answered for a different, but related model by [8].

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The interchange process is of interest in both physics and mathematics. The prior results on trees and the complete graph use very particular properties of these graphs which make them rather hard to extend to other graphs. It is widely believed that similar results should hold for a large variety of graphs. An important open conjecture is that for the interchange process on $\mathbb{Z}^d$ for $d \geq 3$ there exists a time $t_c = t_c(d) < +\infty$ such that for $t < t_c$, every cycle of σ_t has finite length and for $t > t_c$, σ_t has at least one cycle of infinite length. The situation for $\mathbb{Z}^2$ is not clear, recent results [3] suggest that in cycles are always finite.

In this paper we consider the 2–dimensional Hamming graph $H(2,n) = (V,E)$ for which the vertices $V = \{0,\dots,n-1\}^2$ are given by the square lattice and an edge is present between any pair of vertices which either are on the same row or the same column. We will often denote this graph by H. The Hamming graph is the product of two complete graphs and has a relatively simple geometric structure so it is surprising that applying the techniques on the complete graph directly gives results that are far from optimal (one can only prove the existence of cycles of size n). In this paper we obtain results that are almost optimal. In order to do this we have to develop a lot of new ideas, which, we believe, are not straightforward. Further, we speculate that this work is a step in moving away from the complete graph, which is well understood, towards other graphs with different geometry, and in particular, towards $\mathbb{Z}^d$.

Let

$$\mathrm{orb}_t(v) = \{\underbrace{\sigma_t \circ \cdots \circ \sigma_t}_{\ell}(v) : \ell = 0, 1, \dots\},$$

be the cycle of σ_t, $t \geq 0$ containing $v \in V$. Further,

$$V_t(k) = \{v \in V : |\mathrm{orb}_t(v)| \geq k\},$$

are vertices belonging to cycles longer than k.

Our main result is the occurrence of a phase transition for the interchange process on H.

Theorem 1.1. *Consider the random interchange process σ on the 2–dimensional Hamming graph $H(2,n)$. For $\beta < 1/2$ we are in the subcritical phase where there exist a constant $C > 0$ such that*

$$\lim_{n\to\infty} \mathbb{P}(|V_{\beta n^2}(C \log n)| = 0) = 1.$$

For $\beta > 1/2$ we are in the supercritical phase where for any $\varepsilon > 0$ there exists a constant $C > 0$ such that

$$\lim_{n\to\infty} \mathbb{P}\left(\left|V_{\beta n^2}(n^{2-\varepsilon})\right| \geq Cn^2\right) = 1.$$

We expect that our result is almost sharp and conjecture that for $\beta > 1/2$ with high probability there exists cycles of macroscopic size.

The main technical contribution of our paper, instrumental to show Theorem 1.1, is to show that cycles cannot concentrate too much on any row or column of the graph $H(2,n)$. Roughly speaking it is expected that the cycles of the interchange process should resemble is many aspects a random walk on the graph. To the best of our knowledge, we are the first to successfully implement this idea for the interchange model.

Refining the estimates in this paper and adapting the argument of [4], we can show that in supercritical phase

$$\lim_{n\to\infty} \mathbb{P}\left(\sup_{u\in[t-\Delta_n,t]} \left|V_u(n^2/\log^4 n)\right| \geq Cn^2\right) = 1, \tag{1.1}$$

for any $\Delta_n \geq n^{1/2} \log^3 n$. We predict that the $\log^4 n$ above could also be replaced by polylog(n) with sharper estimates. To replace the $\log^4 n$ by a constant, thereby showing the existence of macroscopic cycles, would require some new ideas.

We discuss the above questions in more detail in Section 6.

1.1 Heuristics and the outline of the paper

The subcritical phase will follow from an easy coupling with percolation on H and so we focus on explaining the heuristics behind the supercritical phase.

In the supercritical phase we proceed iteratively proving the existence of larger and larger cycles. Let $t = \beta n^2$ with $\beta > 1/2$. It is easy to show that for small $\alpha_1 > 0$ cycles of size n^{α_1} occupy a positive density of vertices. We introduce a random graph process $G^t = (G^t_s : s \geq 0)$. Initially G^t_0 is a graph whose connected components are the cycles of σ_t. Next, whenever $(\sigma_{t+s} : s \geq 0)$ swaps a pair of particles across an edge e, we add e to the graph process. As we increase s, the largest component in G^t_s becomes giant rather quickly; in fact, an easy sprinkling argument shows that this happens after $n^{2-\alpha_1} \log n$ units of time. In this short time, there cannot be too many splits which result in cycles of size less than n^{α_2} for some $\alpha_2 > 0$. In other words, almost every vertex lying inside of the giant component of G^t belongs to a cycle bigger than n^{α_2}.

Clearly, the bigger α_2 the better, and the main difficulty is that there might be cycles which "split too easily". A pathological example is a cycle which has its support on a single row or a single column. We show is that with high probability most cycles do not behave in this pathological way. In some aspects the cycle of σ_t resembles the trace of a simple random walk on H, which in turn resembles a set of i.i.d. uniformly chosen points. In the last case it is not hard to observe that the vertices cannot cluster in any row or column. Formalising the above is an important technical result and turns out to be rather delicate.

In the proof as an input we use information that cycles of size n^{α_1} are common. Once we obtain cycles of size n^{α_2} we can repeat the analysis obtaining better estimates and proving the existence of even longer cycles of size n^{α_3}, for $\alpha_3 > \alpha_2$. Our methods are sharp enough to continue inductively with a sequence α_k converging to 2.

Outline of the paper

The paper is organised as follows. In Section 2.1 we introduce our notion of isoperimetry and show some basic results. In Section 2.2 we introduce the cyclic random walk, which plays an important role in our proof. Next in Section 3 we use the cyclic random walk to give bounds on the isoperimetry of the cycles of σ_t. In Section 4 we obtain results which give lower bounds for the cycle lengths, under the assumption of good isoperimetry. In Section 5 we combine the previous results to show Theorem 1.1. Finally, Section 6 contains open questions and further discussion.

2 Definitions and preliminary results

2.1 Isoperimetry

Let $H = H(2, n)$ be the 2–dimensional Hamming graph. Let V denote the vertices and E denote the edges of H. Recall that the vertices $V = \{0, \dots, n-1\}^2$. For $i \in \{0, \dots, n-1\}$ sets $L_i = \{0, \dots, n-1\} \times \{i\}$ will be called rows and $D_i = \{i\} \times \{0, \dots, n-1\}$ columns. An edge $e \in E$ is placed between any two distinct vertices on the same row or column. One can check that $|V| = n^2$ and $|E| = n^2(n-1)$. In the whole paper, we assume implicitly that $n \geq 2$.

In the proofs, we will extensively use isoperimetric properties of sets. Let us fix the notation. Given $A, B \subset V$ by $E(A, B)$ denote the set of edges $(v, w) \in E$ such that $v \in A$

and $w \in B$. Next we define a notion of isoperimetry for a set $A \subset V$ by setting

$$\iota(A) := \max\left\{ \max_{i\in\{0,\dots,n-1\}} |L_i \cap A| \,,\, \max_{i\in\{0,\dots,n-1\}} |D_i \cap A| \right\}. \tag{2.1}$$

We think of $\iota(A)$ as an isoperimetric constant of A. Notice that ι is sub-additive in the sense that for $A, B \subset V$, $\iota(A \cup B) \leq \iota(A) + \iota(B)$.

The next lemma shows how to bound $|E(A, A)|$ using $\iota(A)$.

Lemma 2.1. *Let $A \subset V$ then*

$$|E(A, A)| \leq |A|\iota(A).$$

Proof. Fix $A \subset V$. For each $v \in A$, v has at most $2\iota(A)$ many neighbours in A, thus the total number of edges from v to vertices of A is at most $(2\iota(A)|A|)/2$, where the division by 2 comes from the fact that each edge is counted twice. □

Given $v, w \in V$ we set $v + w$ to be component-wise addition modulo $n - 1$. Further, for $A \subset V$ and $v \in V$ we put $v + A = \{v + w : w \in A\}$. The next lemma follows easily from the definition and thus we leave the proof out.

Lemma 2.2. *Let $A, B \subset V$ then*

$$\sum_{v\in V} |(v + A) \cap B| = |A||B|, \quad \sum_{v\in L_0} |(v + A) \cap B| = \sum_{i=0}^{n-1} |A \cap L_i||B \cap L_i|.$$

2.2 Poissonian construction and the cyclic random walk

We think of the interchange process as a Poisson point process on the edges E of H and construct it as follows. Consider a Poisson point process $\mathcal{M}$ on $E \times [0, \infty)$ with intensity measure given by $|E|^{-1}\#(\cdot) \otimes \mathrm{Leb}$ where $\#(\cdot)$ is the counting measure and Leb is the Lebesgue measure. For each $t \geq 0$ we set

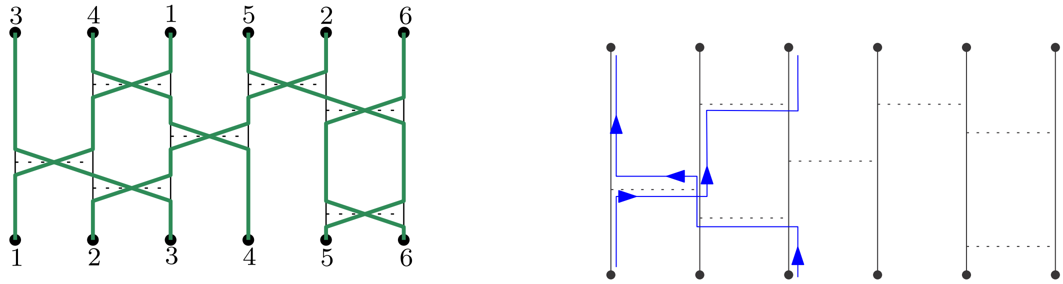
$$\mathcal{B}_t := \{(e, z) \in \mathcal{M} : z \leq t\}$$

to be the restriction of $\mathcal{M}$ to $E \times [0, t]$. Below we will implicitly implicitly identify $z = 0$ and $z = t$. Note that this is not a problem as the Poisson point process almost surely does not have any point for fixed times.

We call each $b \in \mathcal{B}_t$ a *bridge* and call $\{v\} \times [0, t]$ the *bar* at vertex v. We think of a bridge $((v, w), z)$ as going across two bars from vertex v to vertex w at time $z \in [0, t]$. We let $\mathcal{B}_t(A, B)$ be the set of bridges $b = (e, t) \in \mathcal{B}$ such that $e \in E(A, B)$. In cases when t is fixed we will often drop it from the notation by writing $\mathcal{B} = \mathcal{B}_t$.

Let $v \in V$, then we obtain $\sigma_t(v)$ from the following procedure (see Figure 1). We start at $(v, 0) \in H \times [0, t]$ and follow the bar $\{v\} \times [0, t]$ until we reach the first bridge $(e, z) \in \mathcal{B}_t$, if it exists, such that $e = (v, w)$ for some $w \in V$. Then we jump to (w, z) and then again follow the interval $\{w\} \times [z, t]$ until the next bridge. Repeating this procedure, we eventually end up at some $(v', t) \in H \times [0, t]$ and we have that $\sigma_t(v) = v'$.

Now we present the *cyclic random walk* (CRW), where we take a slight modification from the original introduction in [2], which will explore the set of bridges $\mathcal{B}$ in a convenient way. Let $v \in V$ and $z \in [0, t]$, the CRW $\mathcal{X}(v, z) = (\mathcal{X}_s(v, z) : s \geq 0)$ is a continuous time process which takes values in $V \times [0, t]$ and is defined as follows (see Figure 1). Initially, $\mathcal{X}_0(v, z) = (v, z)$, then the CRW moves upwards on the bar of the vertex v, starting at height z, at unit speed, until it encounters a bridge. It does this periodically, so that if it gets to height t, then the CRW goes to the bottom of the bar, at height 0. If a bridge has been encountered, then the CRW jumps to the other end of the bridge and repeats the same procedure. We will be assuming that $\mathcal{X}(v, z)$ is a right continuous function.



(a) Bridges (b) Cyclic random walk

Figure 1: The dotted lines represent the bridges $\mathcal{B}$. The picture on the left shows how to obtain σ_t, the labels at the bottom are the labels of the vertices and at the top we have put where they map to under σ_t. The figure on the right is the path of the CRW which is in blue and the direction that the CRW travels is indicated by the arrows.

Notice that the CRW is periodic. For example in the case there are no bridges coming out of v, $\mathcal{X}(v,z)$ will be given by $\mathcal{X}_s(v,z) = (v, z+s \mod t)$. In general, once $\mathcal{X}(v,z)$ reaches the point (v,z) again (which it will do so in a finite time), then it will repeat itself. Crucially, the paths of $\mathcal{X}(v,z)$ encode the permutation σ_t. Indeed note that the vertices $\mathrm{orb}_t(v)$ in the cycle of v can be obtained from $\mathcal{X}(v,z)$ as follows:

$$\mathrm{orb}_t(v) = \{w : \mathcal{X}_s(v,z) = (w,t) \text{ for some } s \geq 0\}.$$

We are interested in the path of $\mathcal{X}(v,z)$, more precisely the set of sites that are visited before time s, which we define as

$$\mathcal{Z}_s(v,z) := \{w \in V : \mathcal{X}_{s'} = (v,z') \text{ for some } s' \leq s \text{ and } z' \in [0,t]\}.$$

For $x > 0$ we let $T_x(v,z) := \inf\{s \geq 0 : |\mathcal{Z}_s(v,z)| \geq x\}$ where we use the convention that $\inf \emptyset = \infty$. We define $(Z_k(v,z) : k = 0,1,\dots)$ and $(X_k(v,z) : k = 0,1,\dots)$ by $Z_k(v,z) := \mathcal{Z}_{T_k(v,z)}(v,z)$ and

$$X_k(v,z) := \begin{cases} Z_1(v,z) & \text{if } k = 1 \\ Z_k(v,z) \backslash Z_{k-1}(v,z) & \text{if } k \geq 2. \end{cases}$$

In words, if the CRW visits at least k vertices, then $Z_k(v,z)$ is the first k vertices it visits and $X_k(v,z)$ is the k-th vertex that it visited. Note that $X_k(v,z) = \emptyset$ if $T_k(v,z) = \infty$. When $X_k(v,z) \neq \emptyset$ we will ignore the fact that $X_k(v,z)$ is a set and write $X_k(v,z) = w$ instead of $X_k = \{w\}$.

Let $\mathcal{F}(v,z) = (\mathcal{F}_s(v,z) : s \geq 0)$ denote the natural filtration of the CRW $\mathcal{X}(v,z)$. We also set $\mathcal{G}_k(v,z) := \mathcal{F}_{T_k}(v,z)$. it is important to note that $\mathcal{G}_k(v,z)$ is finer than the natural filtration of $X(v,z)$ as it records all the bridges that $\mathcal{X}(v,z)$ has crossed, prior to and including time T_k.

The four processes $\mathcal{X}(v,z)$, $\mathcal{Z}(v,z)$, $X(v,z)$ and $Z(v,z)$ are all measurable with respect to the set of bridges $\mathcal{B}$. This means that we only have one source of randomness, nevertheless these processes are a useful tool to explore the set of bridges $\mathcal{B}$.

The homogeneity of the Poisson point process gives that

$$(\mathcal{X}_s(v,z), s \geq 0) \stackrel{d}{=} ((v,z) + \mathcal{X}_s(0,0), s \geq 0) \tag{2.2}$$

where $+$ is applied component-wise. In particular this implies similar equalities in distribution for $\mathcal{X}, \mathcal{Z}, X, Z$ and T, for example

$$(Z_k(v,z), k = 0,1\dots) \stackrel{d}{=} (v + Z_k(0,0), k = 0,1,\dots).$$

In most cases the argument (v, z) will be obvious for the context and will will omit it writing $\mathcal{Z}_s$ instead of $\mathcal{Z}_s(v, z)$, X_k instead of $X_k(v, z)$, etc.

3 Isoperimetric properties of cycles

In this section we fix $t > 0$ and in most cases we drop it from the notation.

The goal of this section is to show that if for some $k \in \mathbb{N}$, $\liminf_{n\to\infty} \mathbb{P}(T_k < \infty) > 0$, then Z_k has good isoperimetry, where we think of good isoperimetry as saying that $\iota(Z_k) \leq \log^2 n$ with high probability. We first give the heuristics of this section.

Heuristics of the section

Imagine first that after ℓ steps the CRW is restarted from a point v chosen uniformly from V. Assuming that $\ell \ll n$ most of the graph has not been explored thus after the restart, then if $T_{k-\ell}(v, 0) < \infty$, the next $k - \ell \ll n$ steps of the restarted CRW looks like an independent CRW started from a uniformly chosen vertex. Since the number of vertices visited $\ell \ll n$, the CRW started from a uniformly chosen point is very unlikely to intersect any given row, say L_0, thus we see that on $T_{k-\ell}(v, 0) < \infty$ after the restart, the original CRW will not hit L_0 again with high probability.

Next step is deciding an event after which the CRW ends up in a roughly uniform vertex. We choose it to be an L shaped jump, i.e. X_ℓ to $X_{\ell+1}$ is a horizontal jump and $X_{\ell+1}$ to $X_{\ell+2}$ is a vertical jump. We observe that each time the CRW visits L_0 it has a positive chance of jumping out via an L shaped jump and further doing an excursion of length $k - \ell$ without touching L_0. This leads to the conclusion, stated in Lemma 3.5, that the tail of $|Z_k \cap L_0|$ has exponential decay. This result is strong enough to allow the use of a union bound to show that with high probability the intersection of Z_k with any row or column is small.

In Lemma 3.6 we transfer this result to cycles of the permutation σ_t. This is perhaps the result easiest to understand. For reader's convenience, we rephrased it in (3.8) to a simple form, which intuitively states that in the interesting regime of parameters, whenever it is likely that the random interchange permutation has cycles of length k then they do not concentrate in any row or column. In concluding Lemma 3.7 we restate our results in a form applicable in further sections.

We begin with the central technical lemma of our argument.

Lemma 3.1. *Let $\ell, k \in \mathbb{N}$ with $\ell + 2 \leq k$, and let $\mathcal{L}_{\ell+2}$ be an event which is $\mathcal{G}_{\ell+2}$ measurable such that*

$$\mathcal{L}_{\ell+2} \subset \{X_{\ell+2} \notin L_0\} \cap \{T_{\ell+2} < \infty\}.$$

Then

$$\begin{aligned}&\mathbb{P}(|(Z_k \backslash Z_\ell) \cap L_0| \leq 1 | \mathcal{G}_\ell) \\ &\qquad \geq \left(\mathbb{P}(T_k < \infty) - k(\ell + n) \max_{v\in V} \mathbb{P}(X_{\ell+2} = v | \mathcal{L}_{\ell+2}; \mathcal{G}_\ell) - \frac{k}{n-1}\right) \mathbb{P}(\mathcal{L}_{\ell+2} | \mathcal{G}_\ell),\end{aligned}$$

where by $\mathbb{P}(\cdot | \mathcal{L}_{\ell+2}; \mathcal{G}_\ell)$ we understand $\mathbb{P}(\cdot \cap \mathcal{L}_{\ell+2} | \mathcal{G}_\ell)/\mathbb{P}(\mathcal{L}_{\ell+2} | \mathcal{G}_\ell)$.

Remark 3.2. The statement of the lemma might be hard to grasp at first. To make reader's life easier we present its "simplified version". Assume $\ell + 2 < k = o(n)$ then

$$\mathbb{P}(|(Z_k \backslash Z_\ell) \cap L_0| \leq 1 | \mathcal{G}_\ell) \geq \big(\mathbb{P}(T_k < \infty) - o(1)\big)\mathbb{P}(\mathcal{L}_{\ell+2} | \mathcal{G}_\ell).$$

Further, imagine that $Z_\ell \in L_0$ and $\mathcal{L}_{\ell+2}$ is an L shaped jump. The story behind the condition $|(Z_k \backslash Z_\ell) \cap L_0| \leq 1$ is that in its first step the L jump is inside L_0, the second step leads outside and later the CRW does not return to L_0 for "a long time", typically $k - \ell = n^{\alpha}$, $\alpha \in (0,1)$. Using other arguments later we prove that $\mathbb{P}(T_k < \infty)$, being the probability that the CRW visits at least k vertices before closing, is bounded away from 0. Crucially, there is no conditioning in $\mathbb{P}(T_k < \infty)$, which roughly speaking, is equivalent to "forgetting" the history. Imagine decomposition of the CRW into excursions between subsequent visits to L_0. The forgetting implies that the excursions are approximately i.i.d. moreover, they are likely to be longer than k. These ideas are essential to prove that the CRW cannot visit L_0 too often and similarly does not concentrate in any row or column.

The proof of Lemma 3.1 is based on the fact that the evolution $s \mapsto \mathcal{X}_{T_{\ell+2}+s}$ can be coupled with another CRW independent of $\mathcal{G}_{\ell+2}$ except for the starting point. Such a coupling is constructed in the following lemma.

Lemma 3.3. *For $\ell \in \mathbb{N}$ there exists a process $\tilde{\mathcal{X}} = (\tilde{\mathcal{X}}_s : s \geq 0)$ with the following properties:*

1. *$\tilde{\mathcal{X}}$ is a CRW viz. $\tilde{\mathcal{X}} =^d \mathcal{X}$ and conditionally on $\mathcal{X}_{T_{\ell+2}} = (v,z)$ the process $\tilde{\mathcal{X}}$ and $\mathcal{G}_{\ell+2}$ are independent.*
2. *On the event $\tilde{\mathcal{E}} := \{\tilde{T}_k(v,z) < \infty; \tilde{Z}_k(v,z) \subset Z^c_{\ell+1}\}, k \geq 0$ we have*

$$\mathcal{X}_{T_{\ell+2}+s} = \tilde{\mathcal{X}}_s, \text{ for } s \leq \tilde{T}_k(v,z).$$

In the formulation above and the proof below the quantities with $\tilde{}$ like $\tilde{T}_k(v,z)$ are defined as in Section 2.2 for $\tilde{\mathcal{X}}$.

Proof. Let $\mathbb{Q}$ to be regular conditional probability with respect to $\mathcal{G}_{\ell+2}$. To shorten the notation we denote $A = Z^c_{\ell+1}$, $(v,z) := \mathcal{X}_{T_{\ell+2}}$ and $(v',z) := \mathcal{X}_{T_{\ell+2}-}$.

In the first step we construct a new set of bars $\tilde{\mathcal{B}}$, which will be used to define $\tilde{\mathcal{X}}$. It will have the following properties:

(i) $\tilde{\mathcal{B}}$ with respect to $\mathbb{Q}$ is a Poisson point process on $E \times [0,t]$ with intensity $|E|^{-1}\#(\cdot) \otimes \mathrm{Leb}$ (the same law as of the unconditional law of $\mathcal{B}$)

(ii) $\tilde{\mathcal{B}}(A,A) = \mathcal{B}(A,A)$,

(iii) let $b = ((v',z),(v,z))$ be the bridge that was traversed by $\mathcal{X}$ at time $T_{\ell+2}$, then $(\mathcal{B}(A,A^c)\backslash\{b\}) \subset \tilde{\mathcal{B}}(A,A^c)$.

We will define $\tilde{\mathcal{B}}$ separately on disjoint sets of edges. We start by setting $\tilde{\mathcal{B}}(A,A) = \mathcal{B}(A,A)$, and note that under $\mathbb{Q}$, $\tilde{\mathcal{B}}(A,A)$ is a Poisson point process on $E(A,A) \times [0,t]$ with the correct uniform intensity. Next we set $\tilde{\mathcal{B}}(A^c,A^c) = \hat{\mathcal{B}}(A^c,A^c)$, where $\hat{\mathcal{B}}$ is an independent copy of $\mathcal{B}$. By the independence we have that $\tilde{\mathcal{B}}(A^c,A^c)$ is a Poisson process with the desired intensity. Constructing the bridges $\tilde{\mathcal{B}}(A,A^c)$ requires more work. Let $w \in A^c\backslash\{v'\}$ and I_w be the segments of the bar $w \times [0,t]$ that $\mathcal{X}$ has visited prior to time $T_{\ell+2}$:

$$I_w := \{u \in [0,t] : \mathcal{X}_s = (w,u) \text{ for some } s \leq T_{\ell+2}\}.$$

Notice that for any $u \in I_w$, the conditioning dictates that there is no bridge in $\mathcal{B}(w,A)$ connecting w to A with one end point at (w,u) for $u \in I_w$, otherwise the CRW would visited w (see Figure 2). On the other hand, the set of bridges in $\mathcal{B}(w,A)$ with one end point at (w,u), for some $u \in [0,t]\backslash I_w$, forms a Poisson point process of bridges. We set

$$\tilde{\mathcal{B}}(w,A) = \mathcal{B}(w,A) \cup \{b' \in \hat{\mathcal{B}}(w,A) : b' \text{ has an end point } (w,u) \text{ with } u \in I_w\}.$$

Thus $\mathcal{B}(w,A) \subset \tilde{\mathcal{B}}(w,A)$ and further $\tilde{\mathcal{B}}(w,A)$ under $\mathbb{Q}$ is a Poisson point process on $E(w,A) \times [0,t]$ with intensity $|E|^{-1}\#(\cdot) \otimes \mathrm{Leb}$. In the case when $w = v'$, we have that

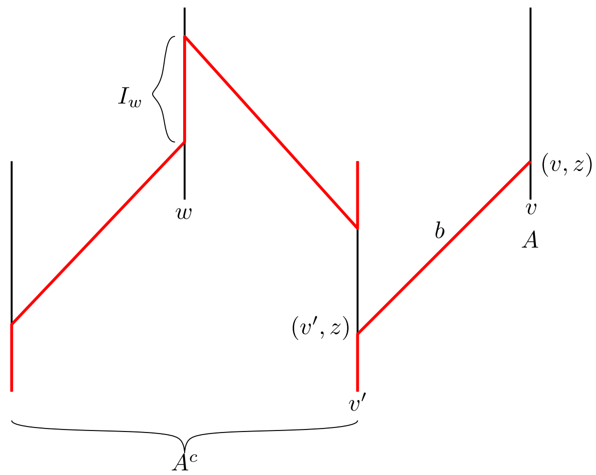


Figure 2: Figure showing the various quantities in the construction of $\tilde{\mathcal{B}}$. The red line represent the path of the CRW. Notice that there cannot be any bridges connecting to I_w since if this were the case, the CRW would instead cross that bridge.

the bridge $b = ((v', z), (v, z))$ has the property that $z \in I_{v'}$. Similar to before, there is no bridge in $\mathcal{B}(v', A)\backslash\{b\}$ with one end point (v', u) with $u \in I_{v'}$. Again as before, the set of bridges in $\mathcal{B}(v', A)\backslash\{b\}$ with one end point at (v', u), for some $u \in [0, t]\backslash I_{v'}$, forms an independent Poisson process of bridges. In this case we set

$$\tilde{\mathcal{B}}(v', A) = (\mathcal{B}(v', A)\backslash\{b\}) \cup \{b' \in \hat{\mathcal{B}}(v', A) : b' \text{ has an end point } (v', u) \text{ with } u \in I_w\}.$$

The cases above cover all edges E and thus we have constructed $\tilde{\mathcal{B}}$ on satisfying (i),(ii) and (iii) above.

We pass to the second part of the proof. Using $\tilde{\mathcal{B}}$ as a set of bridges we define the CRW $\tilde{\mathcal{X}}$ started from (v, z), which have the desired properties. Crucially, by (i) the process $\tilde{\mathcal{X}}$ is a "fresh CRW" starting from (v, z) (unaffected by conditioning $\mathcal{G}_{\ell+2}$), which proves property *1.* Suppose for a contradiction that property *2* is not true namely that

$$\tau = \inf\{u \geq 0 : \tilde{\mathcal{X}}_u \neq \mathcal{X}_{T_{\ell+2}+u}\} < \tilde{T}_k(v, z).$$

Let $(w, z_0) = \tilde{\mathcal{X}}_{\tau-}$. By assumption in the event $\tilde{\mathcal{E}}$ we have $w \in A$. Suppose first that $(w, z_0) \neq (v, z)$, then by properties (ii) and (iii), there exists a bridge from (w, z_0) to A^c in $\tilde{\mathcal{B}}$ (which is not in $\mathcal{B}$). This however contradicts the event $\{\tilde{Z}_k(v, z) \subset A\}$. Suppose now that $(w, z_0) = (v, z)$, then $\tilde{\mathcal{X}}$ closes into a cycle and behaves periodically after τ, contradicting the event $\{\tilde{T}_k(v, z) < \infty\}$. □

Proof of Lemma 3.1. Let $\tilde{\mathcal{X}}$ be as in Lemma 3.3. We set

$$\mathcal{E} = \tilde{\mathcal{E}} \cap \{\tilde{Z}_k(\mathcal{X}_{T_{\ell+2}}) \cap L_0 = \emptyset\} = \{\tilde{T}_k(\mathcal{X}_{T_{\ell+2}}) < \infty; \tilde{Z}_k(\mathcal{X}_{T_{\ell+2}}) \cap (Z_{\ell+1} \cup L_0) = \emptyset\}.$$

Since the paths $(\tilde{\mathcal{X}}_s : s \leq \tilde{T}_k)$ and $(\mathcal{X}_{T_{\ell+2}+s} : s \leq \tilde{T}_k)$ agree on the event $\tilde{\mathcal{E}}$, it follows that

$$\mathbb{P}(|(Z_k \backslash Z_\ell) \cap L_0| \leq 1 | \mathcal{G}_{\ell+2}) \geq \mathbb{P}((Z_k \backslash Z_{\ell+1}) \cap L_0 = \emptyset | \mathcal{G}_{\ell+2}) \geq \mathbb{P}(\mathcal{E} | \mathcal{G}_{\ell+2}).$$

Using a union bound, and independence of $\tilde{\mathcal{X}}$ we get

$$\begin{aligned}&\mathbb{P}(|(Z_k \backslash Z_\ell) \cap L_0| \leq 1 | \mathcal{G}_{\ell+2})\\ &\qquad \geq \mathbb{P}(T_k < \infty) - \mathbb{P}(\tilde{Z}_k(\mathcal{X}_{T_{\ell+2}}) \cap (Z_{\ell+1} \cup L_0) \neq \emptyset | \mathcal{G}_{\ell+2}),\end{aligned}$$

where we also used $\mathbb{P}(T_k < \infty) = \mathbb{P}(\tilde{T}_k < \infty)$. By the independence of $\tilde{Z}$ we deduce that $k \mapsto \tilde{Z}_k(\mathcal{X}_{T_{\ell+2}})$ has the same law as $k \mapsto \mathcal{X}_{T_{\ell+2}} + \tilde{Z}_k$ (recall that by convention $\tilde{Z}$ is shortcut for $\tilde{Z}(0,0)$) and $+$ corresponds to the natural group structure of the Hamming graph.

By the law of total expectation we see that

$$\begin{aligned}\mathbb{P}(|(Z_k \backslash Z_\ell) \cap L_0| \leq 1 | \mathcal{L}_{\ell+2}; \mathcal{G}_\ell) &\geq \mathbb{E}\Big[\mathbb{P}(|(Z_k \backslash Z_\ell) \cap L_0| \leq 1 | T_{\ell+2} < \infty; \mathcal{G}_{\ell+2}) \mathbb{1}_{\mathcal{L}_{\ell+2}} | \mathcal{G}_\ell\Big] \\ &\geq \Big(\mathbb{P}(T_k < \infty) - \mathbb{P}((\mathcal{X}_{T_{\ell+2}} + \tilde{Z}_k) \cap (Z_{\ell+1} \cup L_0) \neq \emptyset | \mathcal{L}_{\ell+2}; \mathcal{G}_\ell)\Big) \mathbb{P}(\mathcal{L}_{\ell+2} | \mathcal{G}_\ell).\end{aligned}$$

Now it suffices to show the claimed upper bound for $\mathbf{P}((\mathcal{X}_{T_{\ell+2}} + \tilde{Z}_k) \cap (Z_{\ell+1} \cup L_0) \neq \emptyset)$, where for convenience we set $\mathbf{P}(\cdot) := \mathbb{P}(\cdot | \mathcal{L}_{\ell+2}; \mathcal{G}_\ell)$. We get

$$\begin{aligned}&\mathbf{P}\left((\mathcal{X}_{T_{\ell+2}} + \tilde{Z}_k) \cap (Z_{\ell+1} \cup L_0) \neq \emptyset\right) \\ &\qquad \leq \mathbf{P}\left((\mathcal{X}_{T_{\ell+2}} + \tilde{Z}_k) \cap (Z_\ell \cup L_0) \neq \emptyset\right) + \mathbf{P}\left((\mathcal{X}_{T_{\ell+2}} + \tilde{Z}_k) \cap \{X_{\ell+1}\} \neq \emptyset\right). \qquad (3.1)\end{aligned}$$

Further, by Markov's inequality

$$\begin{aligned}\mathbf{P}\left((\mathcal{X}_{T_{\ell+2}} + \tilde{Z}_k) \cap (Z_\ell \cup L_0) \neq \emptyset\right) &\leq \mathbf{E}\left(|(\mathcal{X}_{T_{\ell+2}} + \tilde{Z}_k) \cap (Z_\ell \cup L_0)|\right) \qquad (3.2) \\ &= \sum_{v \in V} \mathbf{P}(X_{\ell+2} = v) \mathbf{E}\left(|(v + \tilde{Z}_k) \cap (Z_\ell \cup L_0)|\right) \\ &\leq \max_{v \in V} \mathbf{P}(X_{\ell+2} = v) \sum_{v \in V} \mathbf{E}\left(|(v + \tilde{Z}_k) \cap (Z_\ell \cup L_0)|\right).\end{aligned}$$

Lemma 2.2 gives that

$$\mathbf{E}\left(\sum_{v \in V} |(v + \tilde{Z}_k) \cap (Z_\ell \cup L_0)|\right) = \mathbf{E}\left(|\tilde{Z}_k||(Z_\ell \cup L_0)|\right) \leq k(\ell + n)$$

and hence we bound the first term of (3.1)

$$\mathbf{P}\left((\mathcal{X}_{T_{\ell+2}} + \tilde{Z}_k) \cap (Z_\ell \cup L_0) \neq \emptyset\right) \leq k(\ell + n) \max_{v \in V} \mathbf{P}(X_{\ell+2} = v).$$

To deal with the second term, we distinguish two cases depending on if $X_{\ell+1}$ and $X_{\ell+2}$ are in the same row/column or not. For the first case we assume without loss of generality that $X_{\ell+1}$ and $X_{\ell+2}$ are in the same row L_i. Using the independence asserted in Lemma 3.3 and (2.2) we conclude that for any $v \in L_0 \backslash \{(0,0)\}$,

$$\mathbf{P}\left((\mathcal{X}_{T_{\ell+2}} + \tilde{Z}_k) \cap \{X_{\ell+1}\} \neq \emptyset\right) = \mathbb{P}(Z_k(v,0) \cap \{(0,0)\} \neq \emptyset),$$

Applying Lemma 2.2 we obtain

$$\begin{aligned}\mathbf{P}\left((\mathcal{X}_{T_{\ell+2}} + \tilde{Z}_k) \cap \{X_{\ell+1}\} = \emptyset\right) &= \frac{1}{n-1} \sum_{v \in L_0 \backslash \{(0,0)\}} \mathbb{E}[|Z_k(v,0) \cap \{(0,0)\}|] \\ &\leq \frac{1}{n-1} \mathbb{E}\left[\sum_{v \in L_0} |(v + Z_k) \cap \{(0,0)\}|\right] \\ &= \frac{1}{n-1} \mathbb{E}[|Z_k \cap L_0|] \leq \frac{k}{n-1}.\end{aligned}$$

The second case follows similarly to (3.2) and is skipped. The proof is thus concluded. □

The quality of the estimate provided by Lemma 3.1 depends on the choice of the event $\mathcal{L}_{\ell+2}$. We will set the event $\mathcal{L}_{\ell+2}$ to be the aforementioned L shaped jump, that is, we want it to be the event that X_ℓ and $X_{\ell+1}$ are on the same column, and $X_{\ell+1}$ and $X_{\ell+2}$ are on the same row. The rationale behind this definition is that after an L shaped jump the position of a CRW is approximately uniformly-distributed over the graph (see (3.3) below). Additionally we require that $\mathcal{X}$ discovers $X_{\ell+2}$ by jumping directly from $X_{\ell+1}$. Note that this is not always the case, $\mathcal{X}$ can discover $X_{\ell+1}$, then retrace some of its steps back to X_j for some $j \leq \ell$ and then jump to $X_{\ell+2}$ from X_j. To be precise, we let

$$\begin{aligned}\mathcal{L}_{\ell+2} :=& \big\{X_\ell, X_{\ell+1} \text{ are on the same row, and } X_{\ell+1}, X_{\ell+2} \text{ are on the same column}\big\}\\ &\cap \big\{|\mathcal{B}(X_\ell, V)| = 2\big\} \cap \big\{|\mathcal{B}(X_\ell, X_{\ell+1})| = 1\big\}\\ &\cap \big\{|\mathcal{B}(X_{\ell+1}, V)| = 2\big\} \cap \big\{|\mathcal{B}(X_{\ell+1}, X_{\ell+2})| = 1\big\}.\end{aligned}$$

Notice that on the event $\mathcal{L}_{\ell+2}$, $X_{\ell+1}$ has two bridges. Further, $\mathcal{X}$ arrives at $X_{\ell+1}$ from X_ℓ using one of the two bridges and $\mathcal{X}$ arrives at $X_{\ell+2}$ from $X_{\ell+1}$ by crossing the second bridge.

Lemma 3.4. *Suppose that $\ell \leq n/2$ and $\mathcal{L}_{\ell+2}$ is defined as above. Then*

$$\mathbb{P}(\mathcal{L}_{\ell+2}|\mathcal{G}_\ell) \geq \frac{t^2}{4n^2(n-1)^2} e^{-\frac{5t}{n(n-1)}}$$

and

$$\max_{v\in V} \mathbb{P}(X_{\ell+2} = v|\mathcal{G}_\ell; \mathcal{L}_{\ell+2}) \leq \frac{4}{n^2} e^{-\frac{3t}{n(n-1)}}. \tag{3.3}$$

Before presenting the proof, let us remark that in applying the above lemma it will be the case that t has the same order as n^2. For such t, Lemma 3.4 gives that as $n \to \infty$, $\mathbb{P}(\mathcal{L}_{\ell+2}|\mathcal{G}_\ell)$ is bounded below and

$$\max_{v\in V} \mathbb{P}(X_{\ell+2} = v|\mathcal{G}_\ell; \mathcal{L}_{\ell+2}) = O(n^{-2}).$$

Proof. Throughout we work conditionally on $\mathcal{G}_\ell$, so that every expression appearing throughout this proof is conditionally on $\mathcal{G}_\ell$.

We say that a pair of vertices (v, w) is *eligible* if

(i) $v, w \in Z_\ell^c$ with $v \neq w$,

(ii) v is on the same row as X_ℓ,

(iii) w is on the same column as v.

Let $\mathcal{E}$ denote set of eligible pair of vertices, then notice that $\mathbb{P}((X_\ell, X_{\ell+1}) = (v, w); \mathcal{L}_{\ell+2}) > 0$ if and only if $(v, w) \in \mathcal{E}$.

Fix $(v, w) \in \mathcal{E}$, then

$$\begin{aligned}&\big\{(X_{\ell+1}, X_{\ell+2}) = (v, w)\big\} \cap \mathcal{L}_{\ell+2}\\ &= \big\{|\mathcal{B}(X_\ell, V\backslash\{v\})| = 1\big\} \cap \big\{|\mathcal{B}(X_\ell, v)| = 1\big\} \cap \big\{|\mathcal{B}(v, w)| = 1\big\} \cap \big\{|\mathcal{B}(v, V\backslash\{w, X_\ell\})| = 0\big\}\end{aligned}$$

and note that all of the events on the right hand side are independent of each other. We further break down the last event by writing

$$\big\{|\mathcal{B}(v, V\backslash\{w, X_\ell\})| = 0\big\} = \big\{|\mathcal{B}(v, Z_\ell^c\backslash\{w\})| = 0\big\} \cap \big\{|\mathcal{B}(v, Z_{\ell-1})| = 0\big\}$$

both of which are again independent.

First, $|\mathcal{B}(X_\ell, v)|$, $|\mathcal{B}(v, w)|$ and $|\mathcal{B}(v, Z_\ell^c\backslash\{w\})|$ are independent Poisson random variables where the first two have means $t/|E|$, $t/|E|$ respectively. The mean of $|\mathcal{B}(v, Z_\ell^c\backslash\{w\})|$

depends on how many elements of Z_ℓ are on the same row as v. Nevertheless, this mean is at most $2tn/|E|$ and at least $2t(n-\ell-1)/|E| \geq t(n-2)/|E|$ since $\ell \leq n/2$. Thus

$$\frac{t^2}{|E|^2} e^{-t\frac{n+2}{|E|}} \leq \mathbb{P}(|\mathcal{B}(X_\ell, v)| = 1; |\mathcal{B}(v,w)| = 1; |\mathcal{B}(v, Z_\ell^c \backslash \{w\})| = 0) \leq \frac{t^2}{|E|^2} e^{-2t\frac{n}{|E|}}.$$

Next, $|\mathcal{B}(v, Z_{\ell-1})|$ is a thinned Poisson random variable and is stochastically dominated by a Poisson random variable with mean $2tn/|E|$, hence

$$\mathbb{P}(|\mathcal{B}(v, Z_{\ell-1})| = 0) \geq e^{-\frac{2tn}{|E|}}.$$

Finally, $\mathcal{B}(X_\ell, V\backslash\{v\})$ is a thinned Poisson random variable conditioned to be non-empty (since we know that there is at least one bridge to X_ℓ from somewhere). The segments of bars that have not been visited by $\mathcal{X}$ is an independent Poisson process, thus $|\mathcal{B}(X_\ell, V\backslash\{v\})| - 1$ is stochastically dominated by a Poisson random variable with mean $2t(n-1)/|E|$ and hence

$$\mathbb{P}(|\mathcal{B}(X_\ell, Z_\ell^c\backslash\{v\})| = 1) \geq e^{-2t\frac{n-1}{|E|}}.$$

Combining the estimates together with the fact that $|E| = n^2(n-1)$, we have that for each $(v,w) \in \mathcal{E}$,

$$\frac{t^2}{n^4(n-1)^2} e^{-\frac{5t}{n(n-1)}} \leq \mathbb{P}\big((X_{\ell+1}, X_{\ell+2}) = (v,w); \mathcal{L}_{\ell+2}\big) \leq \frac{t^2}{n^4(n-1)^2} e^{-\frac{2t}{n(n-1)}}. \tag{3.4}$$

Since there are at most $\ell \leq n/2$ many elements of Z_ℓ on any row or column, an easy counting argument shows that

$$\frac{n^2}{4} \leq |\mathcal{E}| \leq n^2.$$

Thus summing over (3.4),

$$\mathbb{P}(\mathcal{L}_{\ell+2}) \geq \frac{t^2}{4n^2(n-1)^2} e^{-\frac{5t}{n(n-1)}} \tag{3.5}$$

which shows the first claim.

Next, for $w \in V$, there is at most one $v \in V$ such that (v,w) is eligible. Thus dividing (3.4) by (3.5), we see that

$$\mathbb{P}(X_{\ell+2} = w | \mathcal{L}_{\ell+2}) \leq \frac{4}{n^2} e^{-\frac{3t}{n(n-1)}}$$

which shows the second claim. □

Now we combine Lemma 3.5 and Lemma 3.4 to deduce if it is likely that the CRW makes excursions of length k (quantifed by $\mathbb{P}(T_k < \infty)$) than its intersection with L_0 is small.

Lemma 3.5. *Let $k, M \in \mathbb{N}$ with $k \leq n/2$, then*

$$\mathbb{P}(|Z_k \cap R| \geq M) \leq \left(1 - \frac{t^2}{4n^2(n-1)^2} e^{-4t/n^2} \left(\mathbb{P}(T_k < \infty) - \frac{10k}{n} e^{2t/n^2}\right)\right)^{\lfloor M/2 \rfloor},$$

for R being any line L_i or column D_i.

We will use the lemma when $t = O(n^2)$, $k = o(n)$ and $\mathbb{P}(T_k < \infty)$ is bounded away from 0. Then there exists $c > 0$ such that

$$\mathbb{P}(|Z_k \cap R| \geq M) \leq (1 - c \cdot \mathbb{P}(T_k < \infty))^{M/2}.$$

Intuitively, this result states that during its first k steps a CRW intersects with a given row/column only a few times.

Proof. Fix $k, M \in \mathbb{N}$ with $k \leq n/2$ and let

$$p := 1 - \frac{t^2}{4n^2(n-1)^2} e^{-4t/n^2} \left(\mathbb{P}(T_k < \infty) - \frac{10k}{n} e^{2t/n^2} \right).$$

By symmetry of the graph we can assume that $R = L_0$. Then by Lemma 3.1, Lemma 3.4 we have that for any $\ell \leq k-2$,

$$\mathbb{P}(|Z_k \backslash Z_\ell \cap L_0| > 1 | \mathcal{G}_\ell) \leq p. \tag{3.6}$$

For a sequence of natural numbers $u_1 < \cdots < u_\ell \leq k$, consider the event

$$\mathcal{A}(u_1, \ldots, u_\ell) := \{X_j \in L_0, \forall j \in \{u_1, \ldots, u_\ell\}; X_j \notin L_0, \forall j \in \{1, \ldots, u_\ell\} \backslash \{u_1, \ldots, u_\ell\}\}.$$

In words, this is the event that the set of times that X intersects L_0 at times $u_1, \ldots, u_\ell$ before time u_ℓ. In other words, we only observe the first ℓ times that X intersects L_0. Now let

$$\mathcal{A}_\ell := \bigcup_{u_1 < \cdots < u_\ell \leq k} \mathcal{A}(u_1, \ldots, u_\ell).$$

The event $\mathcal{A}_\ell$ is the event that X intersects L_0 at least ℓ times before time k and hence $\{|Z_k \cap L_0| \geq M\} = \mathcal{A}_M$. Importantly, the terms in the union above are disjoint.

We will estimate the probability of $\mathcal{A}_\ell$ by induction.

Notice that if $\{\mathcal{A}(u_1, \ldots, u_{\ell-2}); |(Z_k \backslash Z_{u_{\ell-2}}) \cap L_0| \geq 2\}$ holds, then this means that there are is at least two more visits to L_0 between the times $u_{\ell-1}$ and k. Thus

$$\begin{aligned}
\mathbb{P}(\mathcal{A}_\ell) &= \mathbb{P}\left(\bigcup_{u_1 < \cdots < u_{\ell-2} \leq k} \mathcal{A}(u_1, \ldots, u_{\ell-2}); |(Z_k \backslash Z_{u_{\ell-2}}) \cap L_0| \geq 2 \right) \\
&= \sum_{u_1 < \cdots < u_{\ell-2} \leq k} \mathbb{P}(\mathcal{A}(u_1, \ldots, u_{\ell-2}); |(Z_k \backslash Z_{u_{\ell-2}}) \cap L_0| \geq 2) \\
&= \sum_{u_1 < \cdots < u_{\ell-2} \leq k} \mathbb{E}\big[\mathbb{1}_{\mathcal{A}(u_1, \ldots, u_{\ell-2})} \mathbb{P}(|(Z_k \backslash Z_{u_{\ell-2}}) \cap L_0| \geq 2 | \mathcal{G}_{u_{\ell-2}})\big],
\end{aligned}$$

where in the second equality we have used disjointness, in the third equality we have used the fact that $\mathcal{A}(u_1, \ldots, u_{\ell-2})$ is $\mathcal{G}_{u_{\ell-2}}$ measurable. Using (3.6) we get that

$$\mathbb{P}(\mathcal{A}_\ell) \leq p \sum_{u_1 < \cdots < u_{\ell-2} \leq k} \mathbb{P}(\mathcal{A}(u_1, \ldots, u_{\ell-2})) = p \mathbb{P}(\mathcal{A}_{\ell-2}).$$

The result now follows by induction. □

Roughly speaking the previous lemma states that excursions of the CRW cannot have large intersection with any row or column. In other words the isoperimetric constant defined in (2.1) is likely to be small. We transfer this result to the cycles of the interchange process. For $k \in \mathbb{N}$ and $v \in V$ we define

$$\mathrm{orb}^k(v) = \{\underbrace{\sigma \circ \cdots \circ \sigma}_{\ell}(v) : \ell = 0, \ldots, k\} \tag{3.7}$$

be the first k elements of the cycle of containing v. We write orb for orb^∞.

Lemma 3.6. *Let $k, M \in \mathbb{N}$ set $K = \lceil e^{2t/n^2} k/2 \rceil$ and assume that $K \leq n/2$. Then*

$$\begin{aligned}
\mathbb{P}\left(\max_{v \in V} \iota(\mathrm{orb}^k(v)) \geq M \right) \leq & n^4 \left(1 - \frac{t^2 e^{-4t/n^2}}{4n^2(n-1)^2} \left(\mathbb{P}(T_K < \infty) - \frac{10K}{n} e^{2t/n^2} \right) \right)^{\lfloor M/2 \rfloor} \\
& + n^2 e^{-e^{-t/n^2} k}.
\end{aligned}$$

We will use the lemma when $t = O(n^2)$, $k = o(n)$, $k = O(\log^2 n)$, $\mathbb{P}(T_k < \infty)$ is bounded away from 0 and $M \geq \log^2 n$. Then there exist constants $c, \kappa > 0$ such that

$$\mathbb{P}\left(\max_{v\in V} \iota(\mathrm{orb}^k(v)) \geq M\right) \leq n^4(1 - c\cdot\mathbb{P}(T_k < \infty))^{M/2} + o(e^{-\kappa\log^2 n}) = o(e^{-\kappa\log^2 n}). \quad (3.8)$$

Proof. Fix $v \in V$, $k \in \mathbb{N}$ and set $K = \lceil e^{2t/n^2}k/2\rceil$. Consider the CRW started at $(v, 0)$ and for $\ell \in \mathbb{N}$, define the event

$$\mathcal{A}_\ell := \{|\mathcal{B}(X_\ell, V)| = 1; T_\ell < \infty\} \cup \{T_\ell = \infty\}. \quad (3.9)$$

As in the proof of Lemma 3.4, we have that conditionally on $\mathcal{G}_\ell; T_\ell < \infty$, $|\mathcal{B}(X_\ell, V)| - 1$ is stochastically dominated by a Poisson random variable with mean $2t(n-1)/|E|$. Since $|E| = n^2(n-1)$ we have that

$$\mathbb{P}(|\mathcal{B}(X_\ell, V)| = 1|\mathcal{G}_\ell) \geq e^{-2t/n^2}\mathbb{1}_{T_\ell<\infty}.$$

Hence it follows that

$$\mathbb{P}(\mathcal{A}_\ell|\mathcal{G}_\ell) \geq e^{-2t/n^2}\mathbb{P}(T_\ell < \infty|\mathcal{G}_\ell) + \mathbb{P}(T_\ell = \infty|\mathcal{G}_\ell) \geq e^{-2t/n^2}. \quad (3.10)$$

The rationale behind condition $|\mathcal{B}(X_\ell, V)| = 1$ in (3.9) is that it implies that $X_\ell \in \mathrm{orb}(v)$. Indeed, after entering the vertex X_l the CRW traverses its whole bar and exits by the same unique bar. On the other hand if $T_\ell = \infty$ then $\mathrm{orb}(v) \subset Z_\ell(v, 0)$. If $\mathcal{A}_\ell$ occurs at least k times before time K, then it must be the case that $\mathrm{orb}^k(v) \subset Z_K(v, 0)$. In other words,

$$\mathbb{P}\left(\mathrm{orb}^k(v) \subset Z_K\right) \geq \mathbb{P}\left(\sum_{\ell=0}^{K}\mathbb{1}_{\mathcal{A}_\ell} \geq k\right).$$

Next let $\xi_0, \ldots, \xi_K$ be a sequence of i.i.d. $\{0, 1\}$–valued Bernoulli random variables with parameter e^{-2t/n^2}. Then from (3.10), $\sum_{\ell=0}^{K}\mathbb{1}_{\mathcal{A}_\ell}$ stochastically dominates $\sum_{\ell=0}^{K}\xi_i$ and hence

$$\mathbb{P}\left(\mathrm{orb}_t^k(v) \subset Z_K\right) \geq \mathbb{P}\left(\sum_{\ell=0}^{K}\xi_i > k\right) \geq 1 - e^{-e^{-t/n^2}k}$$

where the final inequality follows from Hoeffding's inequality. On the event $\{\mathrm{orb}^k(v) \subset Z_K\}$ we have

$$\iota(\mathrm{orb}^k(v)) \leq \iota(Z_K).$$

Applying Lemma 3.5 and a union bound we get that

$$\mathbb{P}(\iota(Z_K) \geq M) \leq n^2\left(1 - \frac{t^2}{4n^2(n-1)^2}e^{-4t/n^2}\left(\mathbb{P}(T_K < \infty) - \frac{10K}{n}e^{2t/n^2}\right)\right)^{\lfloor M/2\rfloor}.$$

This lemma now follows by taking a union bound over $v \in V$. □

Recall that t is a parameter of definition of the CRW (see Section 2.2). So far in this section t was fixed. In the concluding lemma we make it variable. We write T_k^t to indicate the dependence of T_k on t etc.

Lemma 3.7. *Suppose that there exists a constant $c > 0$ such that $t \in [c^{-1}n^2, cn^2]$ and let $\Delta = \Delta(n) \geq 0$ be a sequence with the property that $\Delta(n) \leq n/\log n$. For some $k \leq n/\log n$ suppose that*

$$\liminf_{n\to\infty}\inf_{s\in[t-\Delta,t]}\mathbb{P}(T_k^s < \infty) > 0.$$

Then there exist two constants $\kappa, C > 0$ such that

$$\mathbb{P}\left(\sup_{s\in[t-\Delta,t]}\max_{v\in V}\iota(\mathrm{orb}_s^k(v)) \geq \log^2 n\right) \leq Ce^{-\kappa\log^2 n}.$$

Proof. By Lemma 3.6 there exists two constant $C, \kappa > 0$ such that,

$$\sup_{s\in[t-\Delta,t]} \mathbb{P}\left(\max_{v\in V} \iota(\mathrm{orb}_s^k(v)) \geq \log^2 n\right) \leq Ce^{-\kappa \log^2 n}. \tag{3.11}$$

Thus it remains to see how to pull the supremum inside of the probability.

Set $m := \lceil \Delta e^{(\kappa/2)\log^2 n} \rceil$ and let $I_1, \dots, I_m$ be any sequence of closed intervals of length $|I_i| \leq e^{-(\kappa/2)/\log^2 n}$ such that $\bigcup_i I_i = [t-\Delta, t]$. Then

$$\mathbb{P}\left(\sup_{s\in[t-\Delta,t]} \max_{v\in V} \iota(\mathrm{orb}_s^k(v)) \geq \log^2 n\right) \leq \sum_{i=1}^m \mathbb{P}\left(\sup_{s\in I_i} \max_{v\in V} \iota(\mathrm{orb}_s^k(v)) \geq \log^2 n\right). \tag{3.12}$$

The permutation σ_t changes when a point of $\mathcal{M}$ is included as a bridge in $\mathcal{B}_t$. For each $i \leq m$, let J_i be the event that σ_t has two or more new points inside of the interval I_i. Since $|I_i| \leq e^{-(\kappa/2)\log^2 n}$, we have that there exists a constant $C' > 0$ such that $\mathbb{P}(J_i) \leq C' e^{-\kappa \log^2 n}$. Let $a_i = \inf I_i$ and $b_i = \sup I_i$, then on the event J_i^c, we have

$$\sup_{s\in I_i} \max_{v\in V} \iota(\mathrm{orb}_s^k(v)) = \max\left\{\max_{v\in V} \iota(\mathrm{orb}_{a_i}^k(v)), \max_{v\in V} \iota(\mathrm{orb}_{b_i}^k(v))\right\}.$$

Hence we see that

$$\begin{aligned}\mathbb{P}\left(\sup_{s\in I_i} \max_{v\in V} \iota(\mathrm{orb}_s^k(v)) \geq \log^2 n\right) &\leq \mathbb{P}\left(\sup_{s\in I_i} \max_{v\in V} \iota(\mathrm{orb}_s^k(v)) \geq \log^2 n; J_i^c\right) + \mathbb{P}(J_i)\\ &\leq \mathbb{P}\left(\max_{v\in V} \iota(\mathrm{orb}_{a_i}^k(v))\right) + \mathbb{P}\left(\max_{v\in V} \iota(\mathrm{orb}_{b_i}^k(v))\right)\\ &\quad + C' e^{-\kappa\log^2 n} \leq C'' e^{-\kappa \log^2 n}\end{aligned}$$

for some constants $C', C'' > 0$, where in the final inequality we have used (3.11). Plugging this into (3.12) we see that

$$\mathbb{P}\left(\sup_{s\in[t-\Delta,t]} \max_{v\in V} \iota(\mathrm{orb}_t^k(v)) \geq \log^2 n\right) \leq C'' m e^{-\kappa \log^2 n}.$$

The result now follows from the fact that $m = O(e^{(\kappa/2)\log^2 n})$. □

4 Cycle lengths under good isoperimetry

In this section we prove results about the cycle sizes when we assume that we have good isoperimetry. The arguments are based on adaptations of the techniques in [4]. In a nutshell the arguments there are based on coupling of the interchange process with the random graph model. The coupling is used to deduce that the number of new edges which split cycles must be large and this may happen only when cycles are big. There are two crucial differences which we make here. One is that we couple the interchange process with the random graph process at some time t, rather than just at $t = 0$. The advantage of this is that the coupling runs for less time which minimises the error terms of the coupling. The second difference is that we incorporating isoperimetry. This means that our bounds become sharper as we are able to show better bounds on the isoperimetry of the cycles of σ_t.

Recall (3.7), for an interval $I \subset [0, \infty)$ and $k \in \mathbb{N}$ let

$$\mathcal{I}_k(I) := \left\{\sup_{t\in I} \max_{v\in V} \iota(\mathrm{orb}_t^k(v)) \leq \log^2 n\right\}$$

denote the event that any fragment of a cycle of length k, started from any vertex, has good isoperimetric properties and further that this holds uniformly for $t \in I$.

We begin with a simple lemma about the probability of splitting into small cycles.

Lemma 4.1. *Suppose that for some $k \in \mathbb{N}$,*

$$\max_{v \in V} \iota(\mathrm{orb}_\sigma^k(v)) \le \log^2 n \tag{4.1}$$

and let $e = (v, w)$ be an edge chosen uniformly at random. Then for $\ell \ge k$ the probability that a cycle of σ is split in $(v, w) \circ \sigma$ into two cycles, one of which has size smaller than ℓ is at most

$$\frac{4\ell}{kn} \log^2 n.$$

Proof. For a given vertex $v \in V$, the number of $w \in V$ such that (v, w) is an edge so that a cycle of σ is split in $(v, w) \circ \sigma$ into two cycles, one of which has size smaller than ℓ, is

$$|\mathrm{orb}_\sigma^\ell(v) \cup \mathrm{orb}_\sigma^{-\ell}(v) \cap (D \cup L \setminus \{v\})| \le 2\iota(\mathrm{orb}_\sigma^\ell(v) \cup \mathrm{orb}_\sigma^{-\ell}(v)),$$

where L, D are respectively the row and column containing v and $\mathrm{orb}^{-\ell}$ corresponds to composition of σ_t^{-1} in (3.7). Then by the sub-additivity of ι we get that,

$$\iota(\mathrm{orb}_\sigma^\ell(v) \cup \mathrm{orb}_\sigma^{-\ell}(v)) \le \left\lceil \frac{2\ell}{k} \right\rceil \max_{v' \in V} \iota(\mathrm{orb}_\sigma^k(v')) \le \left\lceil \frac{2\ell}{k} \right\rceil \log^2 n.$$

Now, suppose that $e = (v, w)$ is a uniformly chosen edge. Considering v fixed, there are $2(n-1)$ many vertices that are neighbouring v. Thus we see that the probability that a cycle of σ is split in $\sigma \circ (v, w)$ into two cycles, one of which has size smaller than ℓ, is at most

$$\frac{1}{n-1} \left\lceil \frac{2\ell}{k} \right\rceil \log^2 n \le \frac{4\ell}{kn} \log^2 n. \qquad \square$$

Next we present a coupling between the interchange process and a random graph process. Let $t \ge 0$ and consider a process $G^t = (G_s^t : s \ge 0)$ of random graphs on the vertex set V, defined as follows. Initially G_0^t is a graph whose connected components are precisely the cycles of σ_t. There may be several graphs that satisfy this and for our purpose it will not matter which one is chosen. Next, whenever $(\sigma_{t+s} : s \ge 0)$ swaps a pair of particles across an edge e, we add e to the graph process.

Recall that $V_t(\ell)$ is the set of vertices of H which belong to cycles of length at least ℓ. Let $V_{t,s}^G(\ell)$ denote the vertices of G_s^t which belong to connected components of size at least ℓ. One important property of this coupling is that every cycle of σ_{t+s} is contained in a connected component G_s^t. Hence it follows that $V_{t+s}(\ell) \subset V_{t,s}^G(\ell)$ for every $t, s \ge 0$ and $\ell \in \mathbb{N}$. We will now estimate $|V_{t,s}^G(\ell) \backslash V_{t+s}(\ell)|$ using a similar argument to that in [13, Lemma 2.2].

Lemma 4.2. *Let $t, \Delta \ge 0$ and suppose that $\ell, k \in \mathbb{N}$ are such that $k \le \ell$. Then*

$$\mathbb{E}\left[\sup_{s \in [0,\Delta]} |V_{t,s}^G(\ell) \backslash V_{t+s}(\ell)|\right] \le \frac{4\ell^2 \Delta}{kn} \log^2 n + \ell\Delta\, \mathbb{P}(\mathcal{I}_k[t, t+\Delta]^c).$$

Note that for $k = \log^2 n$, $\mathbb{P}(\mathcal{I}_k[t, t+\Delta]) = 1$ and this gives the bound $4\ell^2\Delta/n$ which also follows from a straight forward adaptation of [13, Lemma 2.2]. Starting from this bound, we will later see that the term $\ell\Delta\, \mathbb{P}(\mathcal{I}_k[t, t+\Delta]^c)$ becomes negligible for any $k = o(n)$. In this way we obtain a much better bound, which is crucial for proving our result.

Proof. Let I be the set of $s \in [0, \Delta]$ such that σ experiences a fragmentation at time $t + s$ which splits a cycle and at least one of the resulting cycles has length less than ℓ.

From Lemma 4.1 we obtain that at time u the rate of fragmentations where one piece is smaller than ℓ is at most

$$\frac{4\ell}{kn}\log^2 n + \mathbb{1}_{\{\mathcal{I}_k(\{u\})^c\}}.$$

Hence we see that

$$\mathbb{E}[|I|] \leq \frac{4\ell\Delta}{kn}\log^2 n + \Delta\,\mathbb{P}(\mathcal{I}_k[t,t+\Delta]^c). \tag{4.2}$$

Let $s \in [0,\Delta]$ and suppose that a cycle $\gamma \subset V^G_{t,s}(\ell)\backslash V_{t+s}(\ell)$, that is, γ is contained in a component of $G_{t,s}$ of size at least ℓ and $|\gamma| < \ell$. Then it follows that there must have been a vertex $v \in \gamma$ such that the cycle containing v must have fragmented between at some time $u \in [t,t+s]$ producing a cycle of size smaller than ℓ.

For $t' \in [t,t+s]$, let $\gamma^{(v)}_{t'}$ be the cycle of $\sigma_{t'}$ containing v. Let $u \in [t,t+s]$ be the maximal time such that the size of $\gamma^{(v)}_{t'}$ jumps downwards, that is, the cycle containing v experiences a fragmentation. Then at this time u, σ experiences a fragmentation which splits a cycle into two and at least one of the resulting cycles has length less than ℓ. Hence it follows that $u \in I$ and consequently $|V^G_{t,s}(\ell)\backslash V_{t+s}(\ell)| \leq \ell|I|$. Taking supremums and using (4.2) we obtain the desired result. □

Our final argument is recursive and at each iteration we will know a priori for a certain $\ell \in \mathbb{N}$, $V^G_{t,0}(\ell) \geq Cn^2$ for some constant $C > 0$. Given this, we wish to know how long it takes until G^t components of size comparable to n^2. We do this in the next lemma by using a sprinkling argument first introduced by [1].

Lemma 4.3. *Let $t \geq 0$ and $\ell \in \mathbb{N}$ such that $\ell \leq n^2$. Then for any constant $\delta \in (0,1)$ and any $s \geq (n^2/\ell)\log n$*

$$\lim_{n\to\infty}\mathbb{P}\left(|V^G_{t,s}(\delta n^2/8)| \geq \delta n^2/8 \big| |V^G_{t,0}(\ell)| \geq \delta n^2\right) = 1.$$

Proof. To shorten the notation we denote $\mathbb{Q}(\cdot) = \mathbb{P}\left(\cdot \big| |V^G_{t,0}(\ell)| \geq \delta n^2\right)$. Since $V^G_{t,s}(\ell) \subset V^G_{t,s'}(\ell)$ for $s \leq s'$, it suffices to consider only the case when $s = (n^2/\ell)\log n$.

The event $\{V^G_{t,s}(\delta n^2/8) < \delta n^2/8\}$ implies that $V^G_{t,0}$ can be partitioned into two sets A and B, each of size at least $\delta n^2/4$, such that the vertices in A and B are not connected in G^t_s. Now consider two fixed sets A and B which partition $V^G_{t,0}$ and let $\mathcal{C}(A,B)$ be the event that the vertices in A and B are not connected in G^t_s.

Let

$$D := \{v \in V : E(\{v\},A) \geq \delta^2 n/64 \text{ and } E(\{v\},B) \geq \delta^2 n/64\}$$

be the set of vertices that have at least $\delta^2 n/64$ many neighbours in both A and B. Notice that

$$\begin{aligned}\mathbb{Q}(\mathcal{C}(A,B)) &\leq \prod_{v\in D}\mathbb{P}(v \text{ is not connected to } A \text{ or } B)\\ &= \left(1-\left(1-e^{-s\frac{\delta^2 n}{64|E|}}\right)^2\right)^{|D|}.\end{aligned} \tag{4.3}$$

Now we bound $|D|$. Notice that there at least $\delta^2 n^4/16$ many paths of length 2 between A and B. On the other hand, for every $v \notin D$, there are at most $\delta^2 n^2/32$ many paths of length 2 between A and B with v as the mid-point. Every $v \in D$ can create at most $4n^2$ many paths of length 2 between A and B with v as the mid-point. Hence the total number of paths of length 2 between A and B is bounded from above by $(\delta^2 n^2/32)|D^c| + 4n^2|D|$. Combining with the lower bound we get that

$$\frac{\delta^2 n^2}{32}(n^2 - |D|) + 4n^2|D| \geq \frac{\delta^2 n^4}{16}$$

and thus there exists a constant $\rho > 0$ such that $|D| \geq \rho n^2$. Plugging this into (4.3) we see that

$$\mathbb{Q}(\mathcal{C}(A,B)) \leq \left(1 - \left(1 - e^{-s\frac{\delta^2 n}{64|E|}}\right)^2\right)^{\rho n^2}.$$

Next notice that there are at most $2^{n^2/\ell}$ many partitions A and B of the set $V^G_{t,0}(\ell)$, hence

$$\mathbb{Q}(V^G_{t,s}(\delta n^2/8) < \delta n^2/8) \leq 2^{\frac{n^2}{\ell}} \left(1 - \left(1 - e^{-s\frac{\delta^2 n}{64|E|}}\right)^2\right)^{\rho n^2}.$$

Since $s = (n^2/\ell)\log n$ and $|E| = n^2(n-1)$, we have that there exist constants $C, c > 0$ such that

$$2^{\frac{n^2}{\ell}} \left(1 - \left(1 - e^{-s\frac{\delta^2 n}{16|E|}}\right)^2\right)^{\rho n^2} \leq C\exp\left\{\frac{n^2}{\ell}\log 2 - c\frac{n^2\log n}{\ell}\right\}.$$

Since $\ell \leq n^2$ and $n \to \infty$, we have that the above converges to zero. $\square$

Finally we combine the last two lemmas with Lemma 3.7 to show a result that will allow us recursively to obtain better bounds on the lengths of cycles.

Lemma 4.4. *Suppose that there exists a constant $c > 0$ such that $t \in [c^{-1}n^2, cn^2]$. Let $\ell = \ell(n)$ and $\Delta = (n^2/\ell)\log n$ and $k = \min\{\ell, n/\log n\}$ and $\delta > 0$ be such that*

$$\lim_{n\to+\infty} \inf_{s\in[t-2\Delta,t]} \mathbb{P}(V_s(\ell) > \delta n^2) = 1.$$

Then there exists a $\rho > 0$ such that

$$\lim_{n\to\infty} \mathbb{P}\left(\inf_{s\in[t-\Delta,t]} \left|V_s\left(\frac{\sqrt{\ell k n}}{\log^2 n}\right)\right| > \rho n^2\right) = 1.$$

Proof. By Lemma 4.3 we have that

$$\lim_{n\to\infty} \mathbb{P}\left(\inf_{s\in[t-\Delta,t]} |V^G_{t,s}(\delta n^2/8)| \geq \delta n^2/8\right) = 1 \tag{4.4}$$

where we have also used the fact that $V^G_{t,s}(\ell) \subset V^G_{t,s'}(\ell)$ whenever $s \leq s'$.

Next set $k = \min\{\ell, n/\log n\}$. As $\{T_\ell < \infty\} \subset \{T_k < \infty\}$ we have that by the vertex transitivity of the graph for any $s \in [t-\Delta, t]$ we have

$$\mathbb{P}(T^s_k < \infty) \geq \mathbb{P}(T^s_\ell < \infty) = \frac{1}{|V|}\sum_{v\in V}\mathbb{P}(T^s_\ell(v,0) < \infty) \geq \frac{1}{n^2}\mathbb{E}(V_s(\ell)).$$

Thus by the assumption

$$\liminf_{n\to\infty} \inf_{s\in[t-2\Delta,t]} \mathbb{P}(T^s_k < \infty) \geq \delta.$$

By Lemma 3.7, there exist two constants $C, \kappa > 0$ such that

$$\mathbb{P}(\mathcal{I}_k[t-2\Delta,t]^c) = \mathbb{P}\left(\sup_{s\in[t-2\Delta,t]} \max_{v\in V} \iota(\mathrm{orb}^k_s(v)) \geq \log^2 n\right) \leq Ce^{-\kappa\log^2 n}$$

Thus by Lemma 4.2,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{s\in[0,2\Delta]}|V^G_{t-2\Delta,s}(\sqrt{\ell kn}/\log^2 n)\backslash V_{t-2\Delta+s}(\sqrt{\ell kn}/\log^2 n)|\right]\\
&\le \frac{2(\sqrt{\ell kn}/\log^2 n)^2\Delta}{kn}\log^2 n + \ell\Delta\,\mathbb{P}(\mathcal{I}_k[t,t+\Delta]^c)\\
&= O\left(\frac{n^2}{\log n}\right).
\end{aligned}$$

Using the above, together with (4.4) and Markov's inequality gives the desired result. □

5 Proof of Theorem 1.1

5.1 Subcritical phase

We recall that $(G^0_s, s \ge 0)$ is a random graph process in which a uniformly chosen edge is added at rate 1. Let $\beta < 1/2$ and set $t = \beta n^2$. Then by using standard branching arguments (see for example the proof of [7, Theorem 2.3.1]) it is possible to show that there exists a $C > 0$ such that

$$\lim_{n\to\infty}\mathbb{P}(|V^G_{0,t}(C\log n)| = 0) = 1.$$

Now the subcritical phase of Theorem 1.1 is a consequence of the fact that under our coupling $V_t(C\log n) \subset V^G_{0,t}(C\log n)$.

5.2 Supercritical phase

Throughout we let $\beta > 1/2$ and set $t = \beta n^2$. We will proceed inductively applying Lemma 4.4 repeatedly to obtain larger and larger cycles. Formally, fix a sequence of positive real numbers $(\alpha_h, h \in \mathbb{N})$ such that:

1. $\alpha_1 \in (0, 1/2)$,
2. for any $h \ge 1$, $\alpha_{h+1} < \frac{1}{2}(1 + \alpha_h + \min\{1, \alpha_h\})$,
3. $\lim_{h\to\infty}\alpha_h = 2$.

We will show by induction that there exists a sequence of positive real numbers $(\rho_h, h \in \mathbb{N})$ for which

$$\lim_{n\to\infty}\mathbb{P}\left(\inf_{s\in[t-\Delta_h,t]}|V_s(n^{\alpha_h})| > \rho_h n^2\right) = 1, \tag{5.1}$$

where $\Delta_h = n^{2-\alpha_h}\log n$, which implies the statement in Theorem 1.1 in the supercritical phase.

We begin showing the base case $h = 1$ by estimating the sizes of the connected components of G^0_s.

Lemma 5.1. *Let $\beta > 1/2$ and set $t = \beta n^2$. Then there exists a constant $\delta > 0$ such that*

$$\lim_{n\to\infty}\mathbb{P}\left(\inf_{s\in[t-\Delta_1,t]}|V_{0,s}(\delta n^2)| \ge \delta n^2\right) = 1.$$

Proof. Fix $\beta > 1/2$ and $\beta' \in (1/2, \beta)$ and set $t = \beta n^2$ and $s = \beta' n^2$. Using standard branching techniques (see for example [7, Theorem 2.3.2] and [7, Lemma 2.3.4]) one can show that there exists a $\gamma > 0$ such that

$$\lim_{n\to\infty}\mathbb{P}\left(|V_{0,s}(\log^2 n)| \ge \gamma n^2\right) = 1.$$

The result now follows from Lemma 4.3. □

Next we note that for $k = \log^2 n$, we have that trivially $\mathcal{I}_k[0,\infty)$ holds. Thus applying Lemma 4.2 and noting that $\alpha_1 < 1/2$,

$$\mathbb{E}\left[\sup_{s\in[0,t]} |V^G_{0,s}(n^{\alpha_1})\backslash V_s(n^{\alpha_1})|\right] \le \frac{4n^{2\alpha_1}t}{n} = o(n^2).$$

Now an application of Markov's inequality and using the above together with Lemma 5.1 gives that there exists a $\rho_1 > 0$ such that

$$\lim_{n\to\infty} \mathbb{P}\left(\inf_{s\in[t-\Delta_1,t]} |V_s(n^{\alpha_1})| > \rho_1 n^2\right) = 1$$

which shows the case $h = 1$ in (5.1).

Now we induct. Suppose that (5.1) holds for some $h \ge 1$. We apply Lemma 4.4 with $\ell(n) = n^{\alpha_h}$ and $\delta = \rho_h$ to get that there exists a $\rho_{h+1} > 0$ such that

$$\lim_{n\to\infty} \mathbb{P}\left(\inf_{s\in[t-\Delta_{h+1},t]} |V_s(n^{\alpha_{h+1}})| > \rho_{h+1} n^2\right) = 1$$

which finishes the proof.

6 Open questions and discussion

The results in this paper imply that for $t = \beta n^2$ with $\beta > 1/2$, for any $\ell = \ell(n)$, there exists two constants $C, \kappa > 0$ such that

$$\mathbb{P}\left(\max_{v\in V} \iota(\mathrm{orb}^{n^\ell}_t(v)) \ge \frac{\ell}{n}\log^3 n\right) \le Ce^{-\kappa \log^2 n}. \tag{6.1}$$

We conjecture that sharper bounds can be obtained, namely, there exists a constant $C > 0$ such that for any $\ell \ge n\log^2 n$

$$\mathbb{P}\left(\max_{v\in V} \iota(\mathrm{orb}^{\ell}_t(v)) \ge C\ell/n\right) \le f(n), \tag{6.2}$$

for some fast decaying function $f(n)$. In other words (6.1) is sharp up to logarithmic terms. We believe that showing (6.2) and adapting the techniques of [4], one can obtain the existence of cycles of macroscopic length during some small time interval around $t = \beta n^2$ for any $\beta > 1/2$. Let us describe some of the difficulties that arise in proving (6.2). The result of Lemma 3.5 is sharp and can be used to show that for any i and v the tail of $X_i(v) = |\mathrm{orb}^n_t(v) \cap L_i|$ decays exponentially. In order to deduce (6.1) we use a union bound over i and v, which results in logarithmic errors. One way to avoid this would be to prove with high probability $|\mathrm{orb}^{n\log^2 n}_t(v) \cap L_i| \le C\log^2 n$. To obtain such a result we would need to understand what happens when the CRW intersects its past (at the moment we avoid this difficulty by assuming $k \le n/2$ in Lemma 3.5). Another way, would be to find a notion of isoperimetry different than ι. Essentially one needs to be able show that for some constant $C > 0$, with high probability $\max_v \sum_{i=0}^{n-1} X_i(v)^2 \le Cn$ (using Lemma 2.1 and (6.1) we only obtain the bound $n\log^3 n$). However, it is far from clear how to handle the dependence of the random variables $X_i(v)$.

We believe the proof in this paper applies to other Hamming graphs. For $d, k \in \mathbb{N}$, the (d,k)-Hamming graph $H(d,k)$ is a graph on the vertices $\{0,\dots,k-1\}^d$ where an edge is present between any two vertices which differ on exactly one co-ordinate. We believe that the proof in this paper should easily adapt to every $H(d,n)$ where d is fixed.

It would be very interesting to investigate the case when d is varying. Arguably, the most interesting case is $H(n,2)$ which is the hypercube. We make the following conjecture.

Conjecture 6.1. *Consider the interchange process $\sigma = (\sigma_t : t \geq 0)$ on $H(d,k)$ where at least one (or possibly both) of d,k is increasing with n. Let $\beta > 1$ and set $t = \beta dk^d/(2d-2)$. Then there exists a constant $C > 0$ such that*

$$\lim_{\delta \downarrow 0} \lim_{n\to\infty} \mathbb{P}\big(|V_t(\delta k^d)| \geq Ck^d\big) = 1.$$

We also believe that many of the results that hold on the complete graph also hold on 2-dimensional Hamming graph. For example, [13] shows that in the supercritical phase on the complete graph, the cycle lengths suitably rescaled converge in distribution to a Poisson-Dirichlet random variable with parameter 1.

The interchange process is related to the loop representation of the correlation on the $1/2$ quantum Heisenberg ferromagnet. The model in question is defined by the measure $\mathbb{Q}$ given by

$$\mathbb{Q}(\sigma_t = \sigma) = \frac{1}{Z} 2^{\#\text{cycles}(\sigma)} \mathbb{P}(\sigma_t = \sigma)$$

with some normalising constant $Z > 0$. Recently, [6] proved the existence of macroscopic cycles on the complete graph for the model with any weight $\theta > 1$ in place of 2. It would be interesting to see if it is possible to combine our proof with his methods to obtain an analog of Theorem 1.1 for the measure $\mathbb{Q}$.

Acknowledgments. We would like to thank Nathanaël Berestycki, Roman Kotecký and Daniel Ueltschi for useful discussions. The research of PM was supported by the National Science Centre UMO-2014/15/B/ST1/02165 grant. BŞ acknowledges support from EPSRC grant number EP/L002442/1. We thank a anonymous reviewer for constructive remarks which led to improvement of the paper.

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