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Characteristic functionals of Dirichlet measures*

Lorenzo Dello Schiavo†

Abstract

We compute the characteristic functional of the Dirichlet–Ferguson measure over a locally compact Polish space and prove continuous dependence of the random measure on the parameter measure. In finite dimension, we identify the dynamical symmetry algebra of the characteristic functional of the Dirichlet distribution with a simple Lie algebra of type A. We study the lattice determined by characteristic functionals of categorical Dirichlet posteriors, showing that it has a natural structure of weight Lie algebra module and providing a probabilistic interpretation. A partial generalization to the case of the Dirichlet–Ferguson measure is also obtained.

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1 Introduction and main results

Let X be a locally compact Polish space with Borel σ-algebra $\mathcal{B}$ and let $\mathscr{P}(X)$ be the space of probability measures on $(X, \mathcal{B})$. A $\mathscr{P}(X)$-valued random field P is termed a *Dirichlet–Ferguson process* [16] with *intensity* (*measure*) $\sigma \in \mathscr{P}(X)$ if, for any measurable partition $\mathbf{X} := (X_1, \ldots, X_k)$ of X, the random vector $(PX_1, \ldots, PX_k)$ is distributed according to the Dirichlet distribution with parameter $(\sigma X_1, \ldots, \sigma X_k)$; see Dfn. 2.1 below.

For P as above, we term $\mathcal{D}_\sigma := \operatorname{law} P$ the *Dirichlet–Ferguson measure* with intensity σ. We regard $\mathcal{D}_\sigma$ as a probability measure on the linear space $\mathscr{M}_b(X)$ of finite signed measures over $(X, \mathcal{B})$, supported on $\mathscr{P}(X)$. The Dirichlet distribution and the Dirichlet–Ferguson measure have found a wide range of application, including Bayesian non-parametrics [16, 34, 33], genetics [17, 40], representation theory [58, 56], number theory [11, 12].

The characteristic functional of $\mathcal{D}_\sigma$ is commonly recognized as hardly tractable [22] and approaches to $\mathcal{D}_\sigma$ based on characteristic functional methods appear de facto ruled

*Research supported by the Sonderforschungsbereich 1060 and by the Hausdorff Center for Mathematics.
†Institut für Angewandte Mathematik, Universität Bonn, Germany. E-mail: delloschiavo@iam.uni-bonn.de

out in the literature. Notably, this led to the introduction of different characterizing transforms (e.g. the *Markov–Krein transform* [26, 55] or the c*-transform* [22, 23]), inversion formulas based on characteristic functionals of other random measures, (in particular, the Gamma measure, as in [43]) and, at least in the case $X=\mathbb{R}$, to the celebrated *Markov–Krein identity*; see, e.g., [34].

These investigations are based on complex analysis techniques and integral representations of special functions, in particular the Lauricella hypergeometric function ${}_kF_D$ [31] and Carlson's R function [6]. The novelty in this work consists in the combinatorial/algebraic approach adopted, allowing for broader generality and far reaching connections, especially with Lie algebra theory.

1.1 Fourier analysis

Let $D_{\boldsymbol{\alpha}_k}$ be the *Dirichlet distribution* on the standard simplex Δ^{k-1} with parameter $\boldsymbol{\alpha}_k \in \mathbb{R}^k_+$; see Dfn. 2.1. We regard $D_{\boldsymbol{\alpha}_k}$ as the discretization of $\mathcal{D}_\sigma$ induced by a measurable k-partition $\mathbf{X}_k$ of X. Our first result is the following.

Theorem 1.1 (See Thm. 3.10)**.** *The characteristic functional $\widehat{\mathcal{D}_\sigma}$ of $\mathcal{D}_\sigma$ is — for suitable sequences of partitions $\mathbf{X}_k$ — the limit of the discrete $\mathcal{D}_\sigma$-martingale $(\widehat{D_{\boldsymbol{\alpha}_k}})_k$. For every continuous compactly supported real-valued f, it satisfies*

$$\widehat{\mathcal{D}_\sigma}(f) := \int_{\mathscr{M}_b(X)} \mathrm{d}\mathcal{D}_\sigma(\eta)\, e^{\mathrm{i}\,\eta f} = \sum_{n=0}^{\infty} \frac{\mathrm{i}^n}{n!}\, Z_n(\sigma f^1, \dots, \sigma f^n)\ ,$$

where $\mathrm{i}=\sqrt{-1}$ *is the imaginary unit,* Z_n *is the cycle index polynomial* (2.1) *of the symmetric group* $\mathfrak{S}_n$ *and* f^j *denotes the* j^{th} *power of* f.

Furthermore, the map $\sigma \mapsto \mathcal{D}_\sigma$ is continuous with respect to the narrow topologies.

The characteristic functional representation is new. It provides — in the unified framework of Fourier analysis:

- a new (although non-explicit) construction of $\mathcal{D}_\sigma$ as the unique probability measure on $\mathscr{P}(X)$ satisfying $\widehat{\mathcal{D}_\sigma} = \lim_k \widehat{D_{\boldsymbol{\alpha}_k}}$; see Cor. 3.20. Following [58], we call this construction a *weak Fourier limit*.
- new proofs of known results on the tightness and asymptotics of families of Dirichlet–Ferguson measures, proved, elsewhere in the literature, with *ad hoc* techniques; see Cor.s 3.13 and 3.14, cf. Rmk. 3.12.
- the continuity statement in the Theorem, which strengthens [49, Thm. 3.2] concerned with norm-to-narrow continuity. This last result is sharp, in the sense that the domain topology cannot be relaxed to the vague topology.

1.2 Representations of SL_2-currents

The Dirichlet–Ferguson measure $\mathcal{D}$, the *gamma measure* $\mathcal{G}$ [29, 56] and the *'multiplicative infinite-dimensional Lebesgue measure'* $\mathcal{L}^+$ [56, 58] (both defined below) play an important role in a longstanding program [59, 56, 30] for the study of representations of (measurable) $SL_2(\mathbb{R})$-current groups, i.e., spaces of $SL_2(\mathbb{R})$-valued (bounded measurable) functions on a smooth manifold X. In the following, we shall identify some *special linear* objects naturally acting on Dirichlet measures, and translate probabilistic properties of these measures into the language of Lie algebra theory.

We start by briefly recalling the setting and some motivations, postponing connections with Bayesian non-parametrics to §1.3 below.

Quasi-invariance and representation theory Write $h_\sharp \nu := \nu \circ h^{-1}$ for the push-forward of a measure ν via a measurable function h. Consider now a group G acting measurably, freely and transitively on a measurable space $(\Omega, \mathcal{F})$ and write $g.\omega \in \Omega$ for the action of $g \in G$ on $\omega \in \Omega$.

Definition 1.2 (Invariance properties)**.** *We say that a finite measure ν on $(\Omega, \mathcal{F})$ is*

(i) G-quasi-invariant *if for every $g \in G$ there exists a ($\mathcal{F}$-measurable, ν-a.e. finite) Radon–Nikodým derivative $R_g \colon \Omega \to [0, \infty]$ such that $\mathrm{d}(g.)_\sharp \nu(\omega) = R_g(\omega) \cdot \mathrm{d}\nu(\omega)$;*

(ii) projectively G-invariant *if, additionally, R_g is constant on Ω for every $g \in G$;*

(iii) G-invariant *if, additionally, $R_g \equiv \mathbb{1}$ for every $g \in G$;*

(iv) partially G-quasi-invariant *[30, Dfn. 9] if there exists a filtration $(\mathcal{F}_k)_k$ of $\mathcal{F}$ so that*

- *$\mathcal{F}$ is the minimal σ-algebra generated by $(\mathcal{F}_k)_k$;*
- *for each $g \in G$ and $k \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that $g.\mathcal{F}_k \subset \mathcal{F}_n$;*
- *for each $g \in G$ and $k \in \mathbb{N}$ there exists $R_g^{(k)} \colon \Omega \to [0, \infty]$, $\mathcal{F}_k$-measurable, so that $\mathrm{d}(g.)_\sharp \nu_k(\omega) = R_g^{(k)}(\omega) \cdot \mathrm{d}\nu_k(\omega)$, where ν_k denotes the restriction of ν to $\mathcal{F}_k$.*

Note that (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i) $\Longrightarrow$ (iv).

These properties are related to the theory of representations of G. Indeed, each G-quasi-invariant measure ν on Ω induces a so-called 'quasi-regular' unitary representation of G on the Hilbert space $L^2_\nu(\Omega)$ by setting

$$U_\nu(g) \colon f \mapsto (R_g)^{1/2} \cdot f \circ (g^{-1}.) \ .$$

For the heuristics about partial quasi-invariance, see the Introduction to [30].

We are interested in the invariance properties of $\mathcal{D}$, $\mathcal{G}$ and $\mathcal{L}^+$ under the following actions.

Two groups of transformations For $\sigma \in \mathscr{P}(X)$, we define the Abelian Lie algebra $\mathfrak{m} := \mathcal{C}_c(X)$ and its 'σ-traceless' subalgebra $\mathfrak{m}_\sigma := \{f \in \mathfrak{m} : \sigma f = 0\}$. The corresponding Abelian Lie groups are the *group of multipliers* $\mathfrak{M} := \{e^f : f \in \mathfrak{m}\}$, endowed with the pointwise product, and the subgroup of *σ-traceless multipliers* $\mathfrak{M}_\sigma := \{e^f : f \in \mathfrak{m}_\sigma\}$. Both $\mathfrak{M}$ and $\mathfrak{M}_\sigma$ act on $\mathscr{M}_b^+(X)$ by

$$g. \colon \mu \mapsto g \cdot \mu \ , \qquad g \in \mathfrak{M} \ , \quad \mu \in \mathscr{M}_b^+(X) \ , \tag{1.1}$$

hence the name 'multipliers'. Additionally, we denote by $\mathfrak{S}$ the *group of shifts*, i.e., of bi-measurable transformations of X, and by $\mathfrak{S}_\sigma$ the subgroup of $\mathfrak{S}$ that leaves σ invariant, i.e., $\psi \in \mathfrak{S}_\sigma$ is so that $\psi_\sharp \sigma = \sigma$. Both $\mathfrak{S}$ and $\mathfrak{S}_\sigma$ act naturally on $\mathscr{M}_b^+(X)$, or on $\mathscr{P}(X)$, by

$$\psi. \colon \mu \mapsto \psi_\sharp \mu \ , \qquad \psi \in \mathfrak{S} \ , \quad \mu \in \mathscr{M}_b^+(X) \ . \tag{1.2}$$

Definition 1.3 (Factorizations)**.** *Let $\mathbf{N} \colon \mathscr{M}_b^+(X) \setminus \{0\} \to \mathscr{P}(X)$ be the* normalization map *$\mu \mapsto \overline{\mu} := \mu/\mu X$, and let G be a group acting on $\mathscr{M}_b^+(X)$. We say that the action* factors *over the decomposition*

$$\mathscr{M}_b^+(X) \setminus \{0\} \cong \mathscr{P}(X) \times \mathbb{R}_+ \tag{1.3}$$

if and only if $\mathbf{N} \circ g. = g. \circ \mathbf{N}$ for every $g \in G$. We say that a measure $\mathcal{Q}$ on $\mathscr{M}_b^+(X)$ factors *(over (1.3)) if and only if there exists a Borel measure λ on $\mathbb{R}_+$ such that $\mathcal{Q} = \mathbf{N}_\sharp \mathcal{Q} \otimes \lambda$.*

It was shown in [56, Thm. 3.1] that $\mathcal{G}_\sigma \coloneqq \mathcal{D}_\sigma \otimes e^{-s}\,\mathrm{d}s$ is an $\mathfrak{M}$-quasi-invariant measure on $\mathscr{M}_b^+(X)$, and that $\mathcal{L}_\sigma^+ \coloneqq \mathcal{D}_\sigma \otimes \mathrm{d}s$ is a projectively $\mathfrak{M}$-invariant measure on $\mathscr{M}_b^+(X)$ with Radon–Nikodým derivative $R_g = \exp(-\sigma \ln g)$, and thus it is $\mathfrak{M}_\sigma$-invariant. The importance of $\mathcal{G}$ and $\mathcal{L}^+$ arises from their uniqueness properties w.r.t. these group actions. Indeed, $\mathcal{G}_\sigma$ is the unique measure among the laws of compound Poisson point processes factoring over (1.3), cf. [56, Cor. 4.2] and [16, §4, Thm. 2], while $\mathcal{L}_\sigma^+$ is the unique ergodic ($\mathfrak{S}_\sigma \leftthreetimes \mathfrak{M}_\sigma$)-invariant measure equivalent to $\mathcal{G}_\sigma$; here, $\mathfrak{S}_\sigma \leftthreetimes \mathfrak{M}_\sigma$ is an appropriate semidirect product of $\mathfrak{S}_\sigma$ and $\mathfrak{M}_\sigma$; see [58, Prop. 4].

Whereas $\mathcal{D}_\sigma$, $\mathcal{G}_\sigma$ and $\mathcal{L}_\sigma^+$ are trivially $\mathfrak{S}_\sigma$-invariant (by Thm. 3.9 below), their $\mathfrak{S}$-(quasi-)invariance does not hold. Indeed, let (X,σ) be a smooth Riemannian manifold with normalized volume σ and $\mathfrak{G} < \mathfrak{S}$ be the group of smooth diffeomorphisms of X. It was shown in [30, §2.4] that $\mathcal{G}_\sigma$ is partially $\mathfrak{G}$-quasi-invariant *not* $\mathfrak{G}$-quasi-invariant. Since the action of (every subgroup of) $\mathfrak{S}$ factors over (1.3), the measures $\mathcal{D}_\sigma$ and $\mathcal{L}_\sigma^+$ are *not* $\mathfrak{G}$- (hence not $\mathfrak{S}$-) quasi-invariant as well. However, the partial quasi-invariance does not transfer immediately to $\mathcal{D}_\sigma$ or $\mathcal{L}_\sigma^+$, since the normalization map is not necessarily measurable on the filtration $(\mathcal{F}_k)_k$ in the definition of partial quasi-invariance. In fact, $\mathcal{D}_\sigma$ too is partially $\mathfrak{G}$-quasi-invariant, see [8], hence so is $\mathcal{L}_\sigma^+$.

When X is a smooth manifold, one main application of the (partial) $\mathfrak{G}$- (or $\mathfrak{G} \leftthreetimes \mathfrak{M}$-) quasi-invariance of these and other random measures is the construction of stochastic dynamics on spaces of measures, for which these random measures are invariant or even ergodic. See e.g., [50, 8, 40] for $\mathcal{D}$, [45] for the related *entropic measure* $\mathbb{P}$, [2] for Poisson measures, and [30] for $\mathcal{G}$. Furthermore, $\mathfrak{G}$-quasi-invariance has been proven essential to the study of geometric properties of the Dirichlet forms associated to the aforementioned stochastic dynamics, such as for instance the Rademacher and Sobolev-to-Lipschitz properties. See, e.g., [7] for measures on $\mathscr{P}(X)$, [45] for the entropic measure on $\mathscr{P}(\mathbb{S}^1)$, and [47] for measures on the configuration space over X.

Although inspired by the invariance properties of $\mathcal{L}_\sigma^+$, in the following we will mostly concentrate on $\mathcal{D}_\sigma$. This is in fact not restrictive, since the discretization of the spaces and measures we are interested in factors over (1.3). Whereas Theorem 1.1 allows for Bochner–Minlos and Lévy Continuity related results to come into play, the non-multiplicativity of $\widehat{\mathcal{D}_\sigma}$ (corresponding to the non-infinite-divisibility of the measure) immediately rules out the usual approach to quasi-invariance via Fourier transforms [2, 55, 30, 56]. Other approaches to this problem rely on finite-dimensional approximation techniques, variously concerned with approximating the space [45, 46], the σ-algebra [30] or the acting group [19, 58]. The common denominator here is for the approximation to be a filtration (as, e.g., for partial quasi-invariance) — in order to allow for some kind of martingale convergence — and, possibly, for the approximating groups and/or spaces to be (embedded into) linear structures; cf., e.g., [58, 45]. In the present case, detailing this approach requires however some preparation.

We shall see in §2.3 below how a measurable partition $\mathbf{X}_k \coloneqq (X_{k,1},\dots,X_{k,k})$ of X induces a discretization $D_{\boldsymbol{\alpha}_k}$ of $\mathcal{D}_\sigma$, corresponding to the discretization of X to the space $[k] \coloneqq \{1,\dots,k\}$. Here, $\boldsymbol{\alpha}_k \coloneqq (\alpha_{k,1},\dots,\alpha_{k,k})$ and $\alpha_{k,i} \coloneqq \sigma X_{k,i}$. Again, since this discretization factors over (1.3), the same holds for (the discretizations of) $\mathcal{G}_\sigma$ and $\mathcal{L}_\sigma^+$. Varying $k \in \mathbb{N}$, the family of such discretizations yields the filtration of the $\mathcal{D}_\sigma$-martingale $\left(\widehat{D_{\boldsymbol{\alpha}_k}}\right)_k$ in Theorem 1.1. This is a natural candidate for a filtration $(\mathcal{F}_k)_k$ along which to study the partial $\mathfrak{S}$- or partial $\mathfrak{S} \leftthreetimes \mathfrak{M}_\sigma$-quasi-invariance of $\mathcal{D}_\sigma$. Note however that $\mathscr{P}(X)$ is *not* homogeneous for the action of $\mathfrak{M}_\sigma$, thus the $\mathfrak{M}_\sigma$-quasi-invariance of $\mathcal{D}_\sigma$ should be given a precise meaning.

In the following, we aim to show how the actions of $\mathfrak{S}$ and $\mathfrak{M}$ may be discretized according to the choice of $\mathbf{X}_k$, and to study the quasi-invariance of $D_{\boldsymbol{\alpha}_k}$ under a general action subsuming the two. We start by recalling the analogous framework for the discretizations of $\mathcal{L}^+$.

Discretizations: the case of $\mathcal{L}^+$ The discretization of the action of $\mathfrak{M}_\sigma$ was given in [58], as we briefly recall now. For $r > 0$, define the $(k-1)$-dimensional *affine sphere of radius* r as

$$M_r^{k-1} := \left\{ \mathbf{s} \in \mathbb{R}_+^k : s_1 \cdots s_k = r \right\} .$$

Define the Hadamard product $\mathbf{s} \diamond \mathbf{t} := (s_1 t_1, \dots, s_k t_k)$ and observe that $(M_1^{k-1}, \diamond)$ is a group. Since $\diamond \colon M_1^{k-1} \times M_r^{k-1} \to M_r^{k-1}$, the group M_1^{k-1} acts naturally on M_r^{k-1} for every $r > 0$. It is readily checked that the measure $L_{\boldsymbol{\alpha}_k}$ on M_r^{k-1} with density

$$\mathrm{d}L_{\boldsymbol{\alpha}_k}(\mathbf{y}) = \mathbb{1}_{M_r^{k-1}}(\mathbf{y}) \prod_{i=1}^k \frac{y_i^{\alpha_{k,i}-1}}{\Gamma(\alpha_i)} \,\mathrm{d}y_i \ , \qquad \boldsymbol{\alpha}_k := (\alpha_{k,1}, \dots, \alpha_{k,k}) \ ,$$

is M_1^{k-1}-projectively invariant with Radon–Nikodým derivative

$$R_{\mathbf{s}} := \frac{\mathrm{d}(\mathbf{s}.)_\sharp L_{\boldsymbol{\alpha}_k}}{\mathrm{d}L_{\boldsymbol{\alpha}_k}} = \mathbf{s}^{-\boldsymbol{\alpha}_k} := \prod_{i=1}^k s_i^{-\alpha_{k,i}} \ . \tag{1.4}$$

Indeed, $L_{\boldsymbol{\alpha}_k}$ is the discretization of $\mathcal{L}_\sigma^+$ with $\boldsymbol{\alpha}_k$ as above, see [58, Prop. 2], and, for a suitable sequence of radii $(r_k)_k$, the measure spaces $(M_{r_k}^{k-1}, L_{\boldsymbol{\alpha}_k})$ converge to $(\mathscr{M}_b^+(X), \mathcal{L}_\sigma^+)$ in the weak Fourier sense; see [58, Thm. 2] for a precise statement.

The construction of $\mathcal{L}_\sigma^+$ by the aforementioned limiting procedure draws intuition from a parallel with the Maxwell–Poincaré construction of Gaussian measures on $\mathbb{R}^\infty$; see [58, §2]. In that case, the acting group is the — *non-commutative* — special orthogonal group $SO_k(\mathbb{R})$ and the homogeneous space is the standard sphere $\mathbb{S}_{r_k}^{k-1}$ for some suitable sequence of radii $r_k > 0$.

For $\mathbf{a} \in \mathbb{R}^k$ let now $\operatorname{diag} \mathbf{a}$ be the corresponding diagonal matrix. Let $\mathfrak{h}_{k-1}$ be the diagonal Cartan subalgebra of the real *special linear Lie algebra* $\mathfrak{sl}_k(\mathbb{R})$ of traceless k^2-matrices, and $dSL_k(\mathbb{R})$ be the Abelian Lie group of diagonal matrices with determinant 1. As already noted in [58], the image of $(M_1^{k-1}, \diamond)$ under diag coincides with the group $dSL_k^+(\mathbb{R})$ of positive definite diagonal matrices with determinant 1, i.e. the connected component of the identity in “the” maximal Abelian subgroup $dSL_k(\mathbb{R})$ of the special linear group $SL_k(\mathbb{R})$. In this language, the Abelian Lie algebra $\mathfrak{m}_\sigma$ of σ-traceless functions is discretized to the Abelian Lie algebra $\mathfrak{h}_{k-1}$ of traceless diagonal k^2-matrices. The resulting acting group is the image $dSL_k^+(\mathbb{R})$ of $\mathfrak{h}_{k-1}$ under the Lie exponential of $\mathfrak{sl}_k(\mathbb{R})$. We summarize the actions and discretizations above in Table 1.

In comparison with the Maxwell–Poincaré construction, the following question arises.

Question: Does the action of $M_1^{k-1} \cong dSL_k^+(\mathbb{R})$ on M_r^{k-1} extend to an action of the whole (non-commutative) group $SL_k(\mathbb{R})$? If so, how does the measure $L_{\boldsymbol{\alpha}_k}$ vary under this action?

In the following, we will answer in the affirmative — in the conjugate Fourier picture — an analogous question for the simplicial part $D_{\boldsymbol{\alpha}_k}$ of $L_{\boldsymbol{\alpha}_k}$.

Discretizations: the case of $\mathcal{D}$ For $\boldsymbol{\alpha} \in \mathbb{R}_+^k$, the Fourier transform of $D_{\boldsymbol{\alpha}}$ satisfies $\widehat{D_{\boldsymbol{\alpha}}} = {}_k\Phi_2[\boldsymbol{\alpha}]$, the *second Humbert function* or *confluent hypergeometric Lauricella function of type* D; see Dfn. 2.3. For the purpose of stating our next theorem, let us note that ${}_k\Phi_2[\boldsymbol{\alpha}]$ is well-defined for every $\boldsymbol{\alpha} \in \mathbb{R}^k$ with

$$\boldsymbol{\alpha}_\bullet := \alpha_1 + \cdots + \alpha_k \notin \mathbb{Z}_0^- \ .$$

Table 1: Discretizations of multipliers and currents

		∞-dimensional objects			k-discretizations		
		structured	setwise	hom. sp.	structured	setwise	hom. sp.
maximal toral	alg.s	$\mathfrak{m}_\sigma$		$\mathscr{M}_b^+(X)$	$\mathfrak{h}_{k-1}$		$M_{r_k}^{k-1}$
		$\mathfrak{m}$	$\mathcal{C}_c(X;\mathbb{R})$		$\operatorname{diag}\mathbb{R}^k$	$\mathbb{R}^{[k]}$	
	grp.s	$\mathfrak{M}_\sigma$			$dSL_k^+(\mathbb{R})$	M_1^{k-1}	
		$\mathfrak{M}$	$\mathcal{C}_c(X;\mathbb{R}_+)$		$dGL_k^+(\mathbb{R})$	$\mathbb{R}_+^{[k]}$	
semi-simple	alg.	$\mathcal{C}_c(X,\mathfrak{sl}_2(\mathbb{R}))$		(*)	$\mathfrak{sl}_k(\mathbb{R})$		(*)
	grp.	$\mathcal{C}_c(X,SL_2(\mathbb{R}))$			$SL_k^+(\mathbb{R})$		

** It seems that it is not possible to consistently identify a homogeneous space for this algebra/group.*

Let $\boldsymbol{\alpha} \in \operatorname{int}\Delta^{k-1}$ be an interior point of the standard simplex. Set $\Lambda_{\boldsymbol{\alpha}} := \boldsymbol{\alpha} + \mathbb{Z}^k$ and define $\mathcal{O}_{\Lambda_{\boldsymbol{\alpha}}}$ as the real vector space spanned by ${}_k\Phi_2[\boldsymbol{\epsilon}]$ varying $\boldsymbol{\epsilon} \in \Lambda_{\boldsymbol{\alpha}}$. Finally, let $\mathfrak{l}_k := \mathfrak{sl}_{k+1}(\mathbb{R})$ with diagonal Cartan subalgebra $\mathfrak{h}_k$. Our second main result is the following.

Theorem 1.4 (See Thm. 4.16)**.** *Let $\boldsymbol{\alpha} \in \mathbb{R}^k$ with $\boldsymbol{\alpha}_\bullet \notin \mathbb{Z}_0^-$. Then, there exists a faithful representation $\rho_{\boldsymbol{\alpha}}$ of $\mathfrak{l}_k$ on $\mathcal{O}_{\Lambda_{\boldsymbol{\alpha}}}$ such that*

(i) $\mathcal{O}_{\boldsymbol{\epsilon}} := \mathbb{R}\,{}_k\Phi_2[\boldsymbol{\epsilon}]$ is invariant under the action of $\mathfrak{h}_k$ for every $\boldsymbol{\epsilon} \in \Lambda_{\boldsymbol{\alpha}}$;

(ii) if additionally $\boldsymbol{\alpha} \in \operatorname{int}\Delta^{k-1}$, then $\mathfrak{h}_k$ acts on $\mathcal{O}_{\boldsymbol{\alpha}}$ by weight $\boldsymbol{\alpha}$;

(iii) if additionally $\boldsymbol{\alpha}_\bullet \notin \mathbb{Z}$, the representation $\rho_{\boldsymbol{\alpha}}$ extends to a faithful representation on $\mathcal{O}_{\Lambda_{\boldsymbol{\alpha}}}$ of the universal enveloping algebra $\mathfrak{U}(\mathfrak{l}_k)$;

(iv) the discretization of the action (1.2) of $\mathfrak{G}$ on $\mathscr{M}_b^+(X)$ is — up to a canonical isomorphism independent of $\boldsymbol{\alpha}$ — the action of a subgroup of the Weyl group W_k of $\mathfrak{l}_k$; see the proof for details.

The theorem provides a rigorous framework for the following *informal* statements. *Up to Fourier transform*:

- assertion (i) is the quasi-invariance of $D_{\boldsymbol{\alpha}}$ under a suitable action of $dSL_{k+1}^+(\mathbb{R})$;
- assertion (ii) specifies how the Radon–Nikodým derivative $R_\cdot$ — in algebraic terms, the *weight* of the representation — depends on $\boldsymbol{\alpha}$; for the dependence on the acting element, see (4.7) below;
- since the action of (non-diagonal elements in) $SL_{k+1}(\mathbb{R})$ leaves $\mathbb{R}\{D_{\boldsymbol{\epsilon}}\}_{\boldsymbol{\epsilon}\in\Lambda_{\boldsymbol{\alpha}}}$ invariant but does not fix $\mathbb{R}\{D_{\boldsymbol{\alpha}}\}$, assertion (iii) specifies iterative applications of the said action;
- assertion (iv) describes the discretization of the action of $\mathfrak{G}$ in terms of the Weyl group of $SL_{k+1}(\mathbb{R})$. We summarize this action in Table 2;

- together with Theorem 1.1, assertion (i) yields the partial quasi-invariance of $\mathcal{D}_\sigma$ under the action of traceless multipliers. The filtration in the definition of partial quasi-invariance is exactly the one generated by the martingale $\left(\widehat{D_{\boldsymbol{\alpha}_k}}\right)_k$, i.e. it is given by the σ-algebras generated by measurable partitions in a monotone null-array of partitions; see §2.3.

Insights about Theorem 1.4 are provided by basic properties of the Fourier transform. Indeed, any discretization of the action of $\mathfrak{M}$ is naturally a *multiplication*. Again informally, the Fourier transform — as opposed to, e.g., the Markov–Krein or the c-transform — maps multiplication by a Lie group element into *differentiation* by the corresponding infinitesimal increment in the Lie algebra of the group. In the case of ${}_k\Phi_2[\boldsymbol{\alpha}] = \widehat{D_{\boldsymbol{\alpha}}}$, we call the minimal semi-simple Lie algebra generated by these increments the *dynamical symmetry algebra* $\mathfrak{g}_k$ of ${}_k\Phi_2[\boldsymbol{\alpha}]$. The terminology originates in the works [36, 37, 38], concerned with the dynamical symmetry algebras of different Lauricella hypergeometric functions.

Finally, let us note here that Theorem 1.1 allows for a partial generalization of Theorem 1.4 to infinite dimension, the ultimate goal thereof is essentially that to "fill the empty block" in Table 2. We shall extensively comment on this point in Remark 4.19 below.

Table 2: Discretizations of shifts

	Objects			k-discretizations			
	acting obj.	hom. sp.	meas.	acting obj.	homogeneous space		meas.(s)
					structured	setwise	
	$\mathfrak{S}$	X	σ	$\mathfrak{S}_k$	$[k]$		$\boldsymbol{\alpha}$ (*)
		$\mathscr{P}(X)$	$\mathcal{D}_\sigma$		$\mathscr{P}([k])$	Δ^{k-1} (*)	$D_{\boldsymbol{\alpha}}$
in Fourier				$\mathfrak{S}_k < W_k$		$[k]$ (***)	
				$\mathfrak{h}_k$	$\mathcal{O}_{\boldsymbol{\alpha}}$	$\mathbb{R}\,{}_k\Phi_2[\boldsymbol{\alpha}]$	$\boldsymbol{\alpha}$ (**)
				$\mathfrak{U}(\mathfrak{l}_k)$	$\mathcal{O}_{\Lambda_{\boldsymbol{\alpha}}}$	$\mathbb{R}^{\mathbb{Z}^k}$	$\Lambda_{\boldsymbol{\alpha}}$ (**)

* *Under the identification of a measure* $\alpha := \sum_{i=1}^k \alpha_i \delta_i$ *on* $[k]$ *with the vector* $\boldsymbol{\alpha} := (\alpha_1, \dots, \alpha_k)$.

** *Under the correspondence between* ϵ *and the weight* $w_\epsilon \in \mathbb{R}^k$ *by which* $\mathfrak{h}_k$ *acts on* $\mathcal{O}_\epsilon$.

*** *Understood as the indexing of a basis* Π *for the root system of* $\mathfrak{f}_k < \mathfrak{l}_k$*; see Lemma 4.8.*

1.3 Bayesian non-parametrics

A *statistical model* on a *sample space* X is any subset $M = \{P_\theta\}_{\theta \in T} \subset \mathscr{P}(X)$. A model M is *parametric* if the *parameter space* T is finite-dimensional, *non-parametric* otherwise. In Bayesian statistics, the parameter θ is modeled as a T-valued random variable Θ. The probability measure $Q := \operatorname{law}\Theta$ is termed a *prior* (*distribution*). Under a Bayesian model, any data W is sampled in two stages, as

$$\Theta \sim Q \; ,$$

$$W_1, W_2, \ldots \mid \Theta \stackrel{\text{iid}}{\sim} P_\Theta \ ,$$

and we aim to determine the conditional distribution of Θ given the data, or *posterior* (*distribution*),

$$Q^{\mathbf{w}} \coloneqq Q[\Theta \in \ \cdot \ \mid W_1 = w_1, \ldots, W_n = w_n] \ .$$

In this framework, one remarkable property of Dirichlet measures is the following; see, e.g., [16, p. 212, property iii°].

Proposition 1.5 (Bayesian property for $D_{\boldsymbol{\alpha}}$)**.** *Let $\boldsymbol{\Theta}$ be a Δ^{k-1}-valued random vector, W be a $[k]$-valued (categorical) random variable, and let $i \in [k]$. If the (prior) distribution of $\boldsymbol{\Theta}$ is $D_{\boldsymbol{\alpha}}$ and if*

$$\mathbb{P}\left[W = i \mid \boldsymbol{\Theta}\right] = \Theta_i \quad \textit{a.s.} \ ,$$

then the posterior distribution of $\boldsymbol{\Theta}$ given $W = i$ is $D_{\boldsymbol{\alpha}+\mathbf{e}_i}$, where $\mathbf{e}_i$ is the i^{th} vector of the canonical basis of $\mathbb{R}^k$.

We term any posterior distribution as in the above proposition a *Dirichlet-categorical posterior*. It is then the content of the proposition that Dirichlet-categorical posteriors are themselves Dirichlet measures with different parameter; that is, Dirichlet measures are *self-conjugate priors*.

This property is implicit in the action of the dynamical symmetry algebra $\mathfrak{g}_k \cong \mathfrak{l}_k$. Indeed, the latter is the minimal semi-simple Lie algebra containing the (nilpotent) *raising* differential operators, see (4.5) and Lemma 4.10,

$$E_{\alpha_i} \colon {}_k\Phi_2[\boldsymbol{\alpha}] \longmapsto \alpha_i \ {}_k\Phi_2[\boldsymbol{\alpha} + \mathbf{e}_i] \ , \qquad i \in [k] \ .$$

These correspond, in the conjugate Fourier picture, to take posteriors of $D_{\boldsymbol{\alpha}}$ given knowledge on the occurrence of categorical random variables in $[k]$ in the sense of Proposition 1.5.

Improper priors Let M be a Bayesian model with parameter Θ and W be some observation sampled from M. It is of high practical interest in statistics to find priors corresponding to known posteriors of Θ given W. By Bayes' formula, any such prior is determined up to a multiplicative constant. If the prior distribution is integrable, then the constant is fixed in such a way that the prior be a probability distribution. If otherwise, the constant is (usually) immaterial, and the prior is termed *improper*.

As a consequence of Theorem 1.4, we are able to identify a family of distinguished (possibly improper, hyper-)priors of Dirichlet measures. Indeed, each element E_{α_i} in the Lie algebra $\mathfrak{g}_k$ is paired with a (nilpotent) *lowering* operator $E_{-\alpha_i}$ in the same $\mathfrak{sl}_2$-triple (see Lem. 4.14) and such that, see Lemma 4.10,

$$E_{-\alpha_i} \colon {}_k\Phi_2[\boldsymbol{\alpha}] \longmapsto (1 - \boldsymbol{\alpha}_\bullet) \ {}_k\Phi_2[\boldsymbol{\alpha} - \mathbf{e}_i] \ , \qquad i \in [k] \ .$$

Let $\boldsymbol{\alpha} \in \Delta^{k-1}$ be an interior point of the standard simplex. Set $\Lambda^+_{\boldsymbol{\alpha}} \coloneqq \{\boldsymbol{\epsilon} \in \Lambda_{\boldsymbol{\alpha}} : \boldsymbol{\epsilon} > 0_\bullet\}$ and define $\mathcal{O}_{\Lambda^+_{\boldsymbol{\alpha}}}$ analogously to $\mathcal{O}_{\Lambda_{\boldsymbol{\alpha}}}$. It is shown in Theorem 4.16 that the action of $\mathfrak{g}_k$ on $\mathcal{O}_{\Lambda_{\boldsymbol{\alpha}}}$ fixes $\mathcal{O}_{\Lambda^+_{\boldsymbol{\alpha}}}$. For every $\boldsymbol{\epsilon} \in \Lambda^+_{\boldsymbol{\alpha}} \setminus \mathbb{R}^k_+$, the function ${}_k\Phi_2[\boldsymbol{\epsilon}]$ is the Fourier transform of a σ-finite (possibly: finite) measure which we identify as a (non-normalized, possibly: improper) hyper-prior of $D_{\boldsymbol{\alpha}}$.

Plan of the work Preliminary results are collected in §2, together with the definition and properties of Dirichlet measures and an account of the discretization procedure that we dwell upon in the following. In §3 we prove Theorem 1.1. As a consequence, by the classical theory of characteristic functionals on linear topological spaces, cf. e.g., [18, §IV.4] or [57, §IV], we recover known asymptotic expressions for $\mathcal{D}_{\beta\sigma}$ when $\beta \to 0$ or ∞ is a real parameter (Cor. 3.14, cf. [49, p. 311]), propose a *Gibbsean* interpretation thereof (Rem. 3.16), and prove analogous expressions for the *entropic measure* $\mathbb{P}^\beta_\sigma$ on compact Riemannian manifolds [53], generalizing the case $X = \mathbb{S}^1$ [45, Prop. 3.14]. In the process of deriving Theorem 1.1 we obtain a moment formula for the Dirichlet distribution in terms of the cycle index polynomials Z_n; see Thm. 3.3. In light of *Pólya Enumeration Theory* we interpret this result by means of a coloring problem, §4.1. This motivates the study of the dynamical symmetry algebra $\mathfrak{l}_k$ of the Humbert function ${}_k\Phi_2$ resulting in the proof of Theorem 1.4. Finally, in §4.2 we study the limiting action of the dynamical symmetry algebra $\mathfrak{l}_k$ when k tends to infinity.

Some preliminary results in topology and measure theory are collected in the Appendix.

2 Definitions and preliminaries

Notation Everywhere in the following let $\mathbb{N}_0 := \{0, 1, \dots\}$, $\mathbb{N}_1 := \mathbb{N}_0 \setminus \{0\}$, $\mathbb{Z}^-_0 := -\mathbb{N}_0$. Denote the imaginary unit by i, as opposed to an index i, by $\mathbf{G}[a_n](t)$ (resp. by $\mathbf{G}_{\exp}[a_n](t)$) the (exponential) generating function of the sequence $(a_n)_n \subset \mathbb{C}$, computed in the variable t, viz.

$$\mathbf{G}[a_n](t) := \sum_{n\in\mathbb{N}_0} a_n t^n \ , \qquad \text{resp.} \quad \mathbf{G}_{\exp}[a_n](t) := \sum_{n\in\mathbb{N}_0} \frac{a_n}{n!} t^n \ .$$

Whenever not otherwise specified, for $a \in \mathbb{N}_0$ set $a' := a + 1$. Let i, k, n be positive integers and set for $1 \le i \le k$ (the position of an element in a vector is stressed by a *left* subscript)

$$\begin{aligned} \mathbf{y} &:= (y_1, \dots, y_k) & \mathbf{e}_i &:= ({}_10, \dots, 0, {}_i1, 0, \dots, {}_k0) \\ \mathbf{1} &:= ({}_11, \dots, {}_k1) & \mathbf{y}_{\hat{i}} &:= (y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_k) \\ \vec{\mathbf{k}} &:= (1, 2, \dots, k) & \mathbf{y}_\bullet &:= y_1 + \dots + y_k \ . \end{aligned}$$

Write $\mathbf{y} > \mathbf{0}$ for $y_1, \dots, y_k > 0$ and analogously for $\mathbf{y} \ge \mathbf{0}$. Further set

$$\begin{aligned} [k] &:= \{1, \dots, k\} & \pi \in \mathfrak{S}_k &:= \{\text{bijections of } [k]\} \\ \mathbf{y}_\pi &:= \big(y_{\pi(1)}, \dots, y_{\pi(k)}\big) & \mathbf{y} \diamond \mathbf{z} &:= (y_1 z_1, \dots, y_k z_k) \\ \mathbf{y}^{\diamond n} &:= \underbrace{\mathbf{y} \diamond \dots \diamond \mathbf{y}}_{n \text{ times}} & \mathbf{y} \cdot \mathbf{z} &:= y_1 z_1 + \dots + y_k z_k \ , \end{aligned}$$

where $\diamond$ denotes the *Hadamard product* and we write $\mathbf{y}^{\diamond\mathbf{z}} = \big(y_1^{z_1}, \dots, y_k^{z_k}\big)$ vs. $\mathbf{y}^{\mathbf{z}} = y_1^{z_1} \cdots y_k^{z_k}$. For $f\colon \mathbb{C} \to \mathbb{C}$, write

$$f(\mathbf{y}) := f(y_1) \cdots f(y_k) \qquad\qquad f^\diamond(\mathbf{y}) := (f(y_1), \dots, f(y_k)) \ .$$

Denote by Γ the *Euler Gamma function*, by $\langle\alpha\rangle_k := \Gamma(\alpha + k)/\Gamma(\alpha)$ the *Pochhammer symbol* of $\alpha \notin \mathbb{Z}^-_0$, by $\mathrm{B}(y, z) := \Gamma(y)\Gamma(z)/\Gamma(y + z)$, resp. $\mathrm{B}(\mathbf{y}) := \Gamma(\mathbf{y})/\Gamma(\mathbf{y}_\bullet)$, the *Euler Beta function*, resp. its multivariate analogue.

2.1 Combinatorial preliminaries

Set and integer partitions For a subset $L \subset [n]$ denote by $\tilde{L}$ the ordered tuple of elements in L in the usual order of $[n]$. An *ordered set partition* of $[n]$ is an ordered tuple $\tilde{\mathbf{L}} := (\tilde{L}_1, \tilde{L}_2 \dots)$ of tuples $\tilde{L}_i$ such that the corresponding sets L_i, termed *clusters* or blocks, satisfy $\varnothing \subsetneq L_i \subset [n]$ and $\sqcup_i L_i = [n]$. (By $\sqcup$ we denote the disjoint union.) The order of the tuples in $\tilde{\mathbf{L}}$ is assumed ascending with respect to the cardinalities of the corresponding subsets and, subordinately, ascending with respect to the first element in each tuple. A *set partition* $\mathbf{L}$ of $[n]$ is the family of subsets corresponding to an ordered set partition. This correspondence is bijective. For any set partition write $\mathbf{L} \vdash [n]$ and $\mathbf{L} \vdash_r [n]$ if $\#\mathbf{L} = r$, i.e. if $\mathbf{L}$ has r clusters. A (*integer*) *partition* $\boldsymbol{\lambda}$ *of* n *into* r *parts*, write: $\boldsymbol{\lambda} \vdash_r n$, is an integer solution $\boldsymbol{\lambda} \geq \mathbf{0}$ of the system, $\vec{\mathbf{n}} \cdot \boldsymbol{\lambda} = n$, $\boldsymbol{\lambda}_\bullet = r$; if the second equality is dropped we term $\boldsymbol{\lambda}$ a (*integer*) *partition of* n. Write: $\boldsymbol{\lambda} \vdash n$. We always regard a partition in its *frequency representation*, i.e. as the tuple of its ordered frequencies; cf. e.g., [3, §1.1]. To a set partition $\mathbf{L} \vdash_r [n]$ one can associate in a unique way a partition $\boldsymbol{\lambda}(\mathbf{L}) \vdash_r n$ by setting $\lambda_i(\mathbf{L}) := \# \{h : \#L_h = i\}$.

Permutations and cycle index A permutation π in $\mathfrak{S}_n$ is said to have *cycle structure* $\boldsymbol{\lambda}$, write $\boldsymbol{\lambda} = \boldsymbol{\lambda}(\pi)$, if λ_i equals the number of cycles in π of length i for each i. Let $\mathfrak{S}_n(\boldsymbol{\lambda}) \subset \mathfrak{S}_n$ be the set of permutations with cycle structure $\boldsymbol{\lambda}$, so that $\mathfrak{S}_n(\boldsymbol{\lambda}(\pi)) = K_\pi$ the conjugacy class of π and $\#\mathfrak{S}_n(\boldsymbol{\lambda}) = M_2(\boldsymbol{\lambda}) := n!/(\boldsymbol{\lambda}!\, \vec{\mathbf{n}}^{\boldsymbol{\lambda}})$ [52, Prop. I.1.3.2].

Let now $G < \mathfrak{S}_n$ be any permutation group. The *cycle index polynomial* of G is defined by

$$Z^G(\mathbf{t}) := \frac{1}{\#G} \sum_{\pi \in G} \mathbf{t}^{\boldsymbol{\lambda}(\pi)} , \quad \mathbf{t} := (t_1, \dots, t_n) .$$

We write $Z_n := Z^{\mathfrak{S}_n}$ for the cycle index polynomial of the group $\mathfrak{S}_n$. For $\mathbf{t} := (t_1, \dots, t_n)$, and $\mathbf{t}_k := (t_1, \dots, t_k)$ with $k \leq n$, it satisfies the identities

$$Z_n(\mathbf{t}) = \frac{1}{n!} \sum_{\boldsymbol{\lambda} \vdash n} M_2(\boldsymbol{\lambda})\, \mathbf{t}^{\boldsymbol{\lambda}} , \qquad Z_n\big((a\,\mathbf{1})^{\diamond \vec{\mathbf{n}}} \diamond \mathbf{t}\big) = a^n Z_n(\mathbf{t}) , \qquad a \in \mathbb{R} . \tag{2.1}$$

and the recurrence relation

$$Z_n(\mathbf{t}) = \frac{1}{n} \sum_{k=0}^{n-1} Z_k(\mathbf{t}_k)\, t_{n-k} , \quad Z_0(\varnothing) := 1 . \tag{2.2}$$

2.2 The Dirichlet distribution

Denote the *standard*, resp. *corner*, $(k-1)$-dimensional simplex by

$$\Delta^{k-1} := \big\{\mathbf{y} \in \mathbb{R}^k : \mathbf{y} \geq \mathbf{0},\, \mathbf{y}_\bullet = 1\big\} , \quad \Delta^{k-1}_* := \big\{\mathbf{z} \in \mathbb{R}^{k-1} : \mathbf{z} \geq \mathbf{0},\, \mathbf{z}_\bullet \leq 1\big\} .$$

Definition 2.1 (Dirichlet distribution)**.** *We denote by* $D_{\boldsymbol{\alpha}}(\mathbf{y})$ *the* Dirichlet distribution with parameter $\boldsymbol{\alpha} \in \mathbb{R}^k_+$ *(e.g., [39]), i.e. the probability measure with density*

$$\mathbb{1}_{\Delta^{k-1}}(\mathbf{y})\, \frac{\mathbf{y}^{\boldsymbol{\alpha}-\mathbf{1}}}{\mathrm{B}(\boldsymbol{\alpha})} \tag{2.3}$$

with respect to the k*-dimensional Lebesgue measure on the hyperplane of equation* $\mathbf{y}_\bullet = 1$ *in* $\mathbb{R}^k$*, concentrated on (the interior of)* Δ^{k-1}*.*

Remark 2.2. Alternatively, for fixed $\boldsymbol{\alpha} \in \mathbb{R}^k_+$ and any measurable $A \subset \mathbb{R}^{k-1}$,

$$D_{\boldsymbol{\alpha}}(A) = \frac{1}{\mathrm{B}(\boldsymbol{\alpha})} \int_{\Delta^{k-1}_*} \mathbb{1}_A(\mathbf{z}) \prod_{i=1}^{k} z_i^{\alpha_i - 1}\, \mathrm{d}\mathbf{z} \quad \text{where} \quad \mathbf{z} := (z_1, \dots, z_{k-1}) , \quad z_k := 1 - \mathbf{z}_\bullet .$$

Whereas this second description is also common in the literature, the first one makes more apparent property (ii) below.

Write '$\sim$' for '*distributed as*' and let $\mathbf{Y}$ be any Δ^{k-1}-valued random vector. The following properties of the Dirichlet distribution are well-known:

(i) *aggregation*, e.g., [16, p. 211, property i°]. For $i \in [k-1]$ set

$$\mathbf{y}_{+i} \coloneqq (y_1, \dots, y_{i-1}, y_i + y_{i+1}, y_{i+2}, \dots, y_k) \ .$$

Then,

$$\mathbf{Y} \sim D_{\boldsymbol{\alpha}} \implies \mathbf{Y}_{+i} \sim D_{\boldsymbol{\alpha}_{+i}} \ . \tag{2.4}$$

(ii) *quasi-exchangeability*, or *symmetry*. For all $\pi \in \mathfrak{S}_k$

$$\mathbf{Y} \sim D_{\boldsymbol{\alpha}} \implies \mathbf{Y}_\pi \sim D_{\boldsymbol{\alpha}_\pi} \ . \tag{2.5}$$

(iii) *Bayesian property.* Iterative generalization of Proposition 1.5. Let $\mathbf{W} \in [k]^r$ be a vector of $[k]$-valued (categorical) random variables and $\mathbf{P} \coloneqq (P_1, \dots, P_k) \in \mathbb{N}_0^k$ be the vector of occurrences

$$P_i \coloneqq \# \{ j \in [r] : W_j = i \} \ , \qquad i \in [k] \ .$$

For $\mathbf{p} \coloneqq (p_1, \dots, p_k) \in \mathbb{N}_0^k$ and $\mathbf{Y}$ a Δ^{k-1}-valued random vector, denote by $D^{\mathbf{p}}$ the distribution of $\mathbf{Y}$ given $\mathbf{P} = \mathbf{p}$. If $\mathbb{P}[P_i = p_i \mid \mathbf{Y}] = Y_i$ for all $i \in [k]$, then

$$\mathbf{Y} \sim D_{\boldsymbol{\alpha}} \implies D^{\mathbf{p}} = D_{\boldsymbol{\alpha}+\mathbf{p}} \ , \tag{2.6}$$

in which case we set $D_{\boldsymbol{\alpha}}^{\mathbf{p}} \coloneqq D^{\mathbf{p}}$, termed the *posterior distribution of $D_{\boldsymbol{\alpha}}$ given atoms with masses p_i at points $i \in [k]$.*

Most properties of the Dirichlet distribution may be inferred from its characteristic functional. We recall its definition below.

Definition 2.3 (*Confluent ${}_kF_D$ or (second) Humbert function ${}_k\Phi_2$* [14, §2.1])**.** *For $\mathbf{b}, \mathbf{s} \in \mathbb{C}^k$, $a \in \mathbb{C}$ and $c \in \mathbb{C} \setminus \mathbb{Z}_0^-$ the k-variate* Lauricella hypergeometric function of type D*, write ${}_kF_D$, is*

$$\begin{aligned} {}_kF_D[a, \mathbf{b}; c; \mathbf{s}] \coloneqq& \sum_{\mathbf{m} \in \mathbb{N}_0^k} \frac{\langle a \rangle_{\mathbf{m}_\bullet} \langle \mathbf{b} \rangle_{\mathbf{m}} \mathbf{s}^{\mathbf{m}}}{\langle c \rangle_{\mathbf{m}_\bullet} \mathbf{m}!} && \|\mathbf{s}\|_\infty < 1 \\ =& \frac{1}{\mathrm{B}(a, c-a)} \int_0^1 t^{a-1} (1-t)^{c-a-1} (\mathbf{1} - t\mathbf{s})^{-\mathbf{b}} \, \mathrm{d}t \ , && \Re c > \Re a > 0 \ . \end{aligned}$$

For $\mathbf{b}, \mathbf{s} \in \mathbb{C}^k$, its confluent *form, or* second k-variate Humbert function*, write ${}_k\Phi_2$, is*

$${}_k\Phi_2[\mathbf{b}; c; \mathbf{s}] \coloneqq \lim_{\varepsilon \to 0^+} {}_kF_D[1/\varepsilon; \mathbf{b}; c; \varepsilon \mathbf{s}] = \sum_{\mathbf{m} \in \mathbb{N}_0^k} \frac{\langle \mathbf{b} \rangle_{\mathbf{m}} \mathbf{s}^{\mathbf{m}}}{\langle c \rangle_{\mathbf{m}_\bullet} \mathbf{m}!} \ , \qquad c \in \mathbb{C} \setminus \mathbb{Z}_0^- \ . \tag{2.7}$$

Note that the distribution $D_{\boldsymbol{\alpha}}$ is moment-determinate for any $\boldsymbol{\alpha} > \mathbf{0}$ by compactness of Δ^{k-1}. Its moments are straightforwardly computed via the multinomial theorem as

$$\mu_n'[\mathbf{s}, \boldsymbol{\alpha}] \coloneqq \int_{\Delta^{k-1}} (\mathbf{s} \cdot \mathbf{y})^n \, \mathrm{d}D_{\boldsymbol{\alpha}}(\mathbf{y}) = \sum_{\substack{\mathbf{m} \in \mathbb{N}_0^k \\ \mathbf{m}_\bullet = n}} \binom{n}{\mathbf{m}} \mathbf{s}^{\mathbf{m}} \tfrac{\mathrm{B}(\boldsymbol{\alpha}+\mathbf{m})}{\mathrm{B}(\boldsymbol{\alpha})} = \frac{n!}{\langle \boldsymbol{\alpha}_\bullet \rangle_n} \sum_{\substack{\mathbf{m} \in \mathbb{N}_0^k \\ \mathbf{m}_\bullet = n}} \frac{\mathbf{s}^{\mathbf{m}}}{\mathbf{m}!} \langle \boldsymbol{\alpha} \rangle_{\mathbf{m}} \ , \tag{2.8}$$

so that the characteristic functional of the distribution indeed satisfies (Cf. [14, §7.4.3])

$$\widehat{D_{\boldsymbol{\alpha}}}(\mathbf{s}) \coloneqq \int_{\Delta^{k-1}} \exp(\mathrm{i}\mathbf{s} \cdot \mathbf{y}) \, \mathrm{d}D_{\boldsymbol{\alpha}}(\mathbf{y}) = \sum_{\mathbf{m} \in \mathbb{N}_0^k} \frac{\langle \boldsymbol{\alpha} \rangle_{\mathbf{m}}}{\mathbf{m}!} \frac{\mathrm{i}^{\mathbf{m}_\bullet} \mathbf{s}^{\mathbf{m}}}{\langle \boldsymbol{\alpha}_\bullet \rangle_{\mathbf{m}_\bullet}} =: {}_k\Phi_2[\boldsymbol{\alpha}; \boldsymbol{\alpha}_\bullet; \mathrm{i}\,\mathbf{s}] \ . \tag{2.9}$$

2.3 The Dirichlet–Ferguson measure

Notation Everywhere in the following let $(X, \tau(X))$ be a *second countable locally compact Hausdorff* topological space with Borel σ-algebra $\mathcal{B}$. We denote respectively by $\operatorname{cl} A$, $\operatorname{int} A$, $\operatorname{bd} A$ the *closure*, *interior* and *boundary* of a set $A \subset X$ with respect to τ. Recall (Prop. 2.4) that any space $(X, \tau(X))$ as above is Polish, i.e. there exists a metric d, metrising τ, such that (X, d) is separable and complete; we denote by $\operatorname{diam} A$ the *diameter* of $A \subset X$ with respect to any such metric d (apparent from context and thus omitted in the notation).

Denote by $\mathcal{C}_c(X)$ (resp. $\mathcal{C}_b(X)$) the space of continuous compactly supported (resp. continuous bounded) functions on $(X, \tau(X))$, (both) endowed with the topology of uniform convergence; by $\mathcal{C}_0(X)$ the completion of $\mathcal{C}_c(X)$, i.e. the space of continuous functions on X vanishing at infinity; by $\mathscr{M}_b(X)$ (resp. $\mathscr{M}_b^+(X)$) the space of *finite*, signed (resp. non-negative) *Radon* measures on $(X, \mathcal{B})$ — the topological dual of $\mathcal{C}_c(X)$ and $\mathcal{C}_0(X)$ — endowed with the the vague topology $\tau_v(\mathscr{M}_b(X))$, i.e. the weak* topology, and the induced Borel σ-algebra. Denote further by $\mathscr{P}(X) \subset \mathscr{M}_b^+(X)$ (Cf. Cor. 5.3) the space of probability measures on $(X, \mathcal{B})$. If not otherwise stated, we assume $\mathscr{P}(X)$ to be endowed with the vague topology $\tau_v(\mathscr{P}(X))$ and σ-algebra $\mathcal{B}_v(\mathscr{P}(X))$. On $\mathscr{M}_b^+(X)$ (resp. on $\mathscr{P}(X)$) we additionally consider the narrow topology $\tau_n(\mathscr{M}_b^+(X))$ (resp. $\tau_n(\mathscr{P}(X))$), i.e. the topology induced by duality with $\mathcal{C}_b(X)$. Finally, given any measure $\nu \in \mathscr{M}_b(X)$ and any bounded measurable function g on $(X, \mathcal{B})$, denote by νg the expectation of g with respect to ν and by $g^*\colon \nu \mapsto \nu g$ the linear functional induced by g on $\mathscr{M}_b(X)$ via integration.

The following statement is well-known; see e.g., [25, Thm. 5.3]. A proof is sketched to establish further notation.

Proposition 2.4. *A topological space $(X, \tau(X))$ is second countable locally compact Hausdorff if and only if it is locally compact Polish, i.e. such that $\tau(X)$ is a locally compact separable completely metrizable topology on X. Moreover, if $(X, \mathcal{B})$ additionally admits a fully supported diffuse measure ν, then $(X, \tau(X))$ is perfect, i.e. it has no isolated points.*

Sketch of proof. Let $(\alpha X, \tau(\alpha X))$ denote the Alexandrov compactification of $(X, \tau(X))$ and $\alpha\colon X \to \alpha X$ denote the associated embedding. Note that αX is Hausdorff, for X is locally compact Hausdorff; hence αX is metrizable, for it is second countable compact Hausdorff, and separable, for it is second countable metrizable, thus Polish by compactness. Finally, recall that X is (homeomorphic via α to) a G_δ-set in αX and every G_δ-set in a Polish space is itself Polish. The converse and the statement on perfectness are trivial. $\square$

Partitions Fix $\sigma \in \mathscr{P}(X)$. We denote by $\mathfrak{P}_k(X)$ the family of measurable non-trivial k-partitions of $(X, \mathcal{B}, \sigma)$, i.e. the set of tuples $\mathbf{X} := (X_1, \ldots, X_k)$ such that

$$X_i \in \mathcal{B}\ , \quad \sigma X_i > 0\ , \quad X_i \cap X_j = \varnothing \quad i, j \in [k], i \neq j\ , \quad \cup_{i \in [k]} X_i = X\ .$$

Given $\mathbf{X} \in \mathfrak{P}_k(X)$ we say that it *refines* A in $\mathcal{B}$ if $X_i \subset A$ whenever $X_i \cap A \neq \varnothing$, respectively that it is a *continuity partition for* σ if $\sigma(\operatorname{bd} X_i) = 0$ for all $i \in [k]$. We denote by $\mathfrak{P}_k(A \subset X)$, resp. $\mathfrak{P}_k(X, \tau(X), \sigma)$ the family of all such partitions. Given $\mathbf{X}_1 \in \mathfrak{P}_{k_1}(X)$ and $\mathbf{X}_2 \in \mathfrak{P}_{k_2}(X)$ with $k_1 < k_2$ we say that $\mathbf{X}_2$ *refines* $\mathbf{X}_1$, write $\mathbf{X}_1 \preceq \mathbf{X}_2$, if for every $i \in [k_2]$ there exists $j_i \in [k_1]$ such that $X_{2,i} \subset X_{1,j_i}$. A sequence $(\mathbf{X}_h)_h$ of partitions $\mathbf{X}_h \in \mathfrak{P}_{k_h}(X)$ is termed a *monotone null-array* if $\mathbf{X}_h \preceq \mathbf{X}_{h+1}$ and $\lim_h \max_{i \in [k_h]} \operatorname{diam} X_{h,i} = 0$. Recall that $\operatorname{diam} X_{h,i}$ vanishes independently of the chosen metric on $(X, \tau(X))$, cf. [24, §2.1]. We denote the family of all such null-arrays by $\mathfrak{Na}(X)$. Analogously to partitions, we write with obvious meaning of the notation $\mathfrak{Na}(A \subset X)$ and $\mathfrak{Na}(X, \tau(X), \sigma)$. If σ is *diffuse* (i.e. atomless), then $\lim_h \sigma X_{h,i_h} = 0$ for every choice of $X_{h,i_h} \in \mathbf{X}_h$ with $(\mathbf{X}_h)_h \in \mathfrak{Na}(X)$.

Given a (real-valued) simple function f and a partition $\mathbf{X} \in \mathfrak{P}_k(X)$, we say that f is *locally constant on* $\mathbf{X}$ *with values* s if $f\big|_{X_i} \equiv s_i$ constantly for every $X_i \in \mathbf{X}$. Given a function f in $\mathcal{C}_c(X)$ we say that a sequence of (measurable) simple functions $(f_h)_h$ is a *good approximation* of f if $|f_h| \uparrow_h |f|$ and $\lim_h f_h = f$ pointwise. The existence of good approximations is standard; see e.g., [10, Prop. III.3.1].

The Dirichlet–Ferguson measure By a *random probability* over $(X, \mathcal{B})$ we mean any probability measure on $\mathscr{P}(X)$. For $\mathbf{X} \in \mathfrak{P}_k(X)$ and η in $\mathscr{P}(X)$ set $\eta^\diamond \mathbf{X} := (\eta X_1, \dots, \eta X_k)$ and

$$\begin{aligned} \mathrm{ev}^{\mathbf{X}} \colon \mathscr{P}(X) &\longrightarrow \Delta^{k-1} \subset \mathbb{R}^k \\ \eta &\longmapsto \eta^\diamond \mathbf{X} \; . \end{aligned}$$

Recall, cf. [51], that, if $\sigma \in \mathscr{P}(X)$ is diffuse, then for every $k \in \mathbb{N}_1$ and $\mathbf{y} \in \operatorname{int} \Delta^{k-1}$ there exists $\mathbf{X} \in \mathfrak{P}_k(X)$ such that $\sigma^\diamond \mathbf{X} = \mathbf{y}$.

Definition 2.5 (Dirichlet–Ferguson measure)**.** *Fix $\beta > 0$ and $\sigma \in \mathscr{P}(X)$. The* Dirichlet–Ferguson measure $\mathcal{D}_{\beta\sigma}$ with intensity $\beta\sigma$ *[16, §1, Def. 1] (also:* Dirichlet *[33],* Poisson–Dirichlet *[58],* Fleming–Viot with parent-independent mutation *[15]; see e.g., [48, §2] for an explicit construction) is the unique random probability over $(X, \mathcal{B})$ such that*

$$\mathrm{ev}^{\mathbf{X}}_\sharp \mathcal{D}_{\beta\sigma} = D_{\beta\, \mathrm{ev}^{\mathbf{X}} \sigma} \; , \quad \mathbf{X} \in \mathfrak{P}_k(X) \; , \quad k \in \mathbb{N}_1 \; . \tag{2.10}$$

Recall that $\sigma^\diamond \mathbf{X} > \mathbf{0}$. More explicitly, for every bounded measurable $u \colon \Delta^{k-1} \to \mathbb{R}$,

$$\int_{\mathscr{P}(X)} u(\eta^\diamond \mathbf{X}) \, \mathrm{d}\mathcal{D}_{\beta\sigma}(\eta) = \int_{\Delta^{k-1}} u(\mathbf{y}) \, \mathrm{d}D_{\beta\sigma^\diamond \mathbf{X}}(\mathbf{y}) \; . \tag{2.11}$$

Existence was originally proved in [16] by means of Kolmogorov's Extension Theorem, using the aggregation property of Dirichlet distributions to establish the consistency condition; cf. Fig. 1 below. A construction on spaces more general than in our assumptions is given in [28]. Other characterizations are available; see e.g., [48, 9]. Since X is Polish (Prop. 2.4), in (2.11) it is in fact sufficient to consider u continuous with $|u| < 1$ and, by the Portmanteau Theorem, $\mathbf{X} \in \mathfrak{P}_k(X, \tau(X), \sigma)$; cf. e.g., [53, p. 15].

Let P be a $\mathscr{P}(X)$-valued random field on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and recall the following properties of $\mathcal{D}_\sigma$, to be compared with those of $D_{\boldsymbol{\alpha}}$,

(i) *realization properties*: If $P \sim \mathcal{D}_{\beta\sigma}$, then $P(\omega) = \sum_{i \in J} \eta_i(\omega) \delta_{x_i(\omega)}$ is $\mathbb{P}$-a.s. purely atomic (here: $J \subset \mathbb{N}_0$. See [16, §4, Thm. 2]) with $\operatorname{supp} P(\omega) = \operatorname{supp} \sigma$; see [16, §3, Prop. 1] or [35]. In particular, if σ is diffuse and fully supported, then J is countable and $\{x_i\}_i$ is $\mathbb{P}$-a.s. dense in X. The sequence $(\eta_i)_i$ is distributed according to the *stick-breaking process*. In particular, $\mathbb{E}\eta_i = \beta^{i-1}/(1+\beta)^i$; see [20]. The r.v.'s x_i's are i.i.d., independent also of the η_i's [13], and σ-distributed.

(ii) *σ-symmetry*: for every measurable σ-preserving map $\psi \colon X \to X$, i.e. with $\psi_\sharp \sigma = \sigma$,

$$P \sim \mathcal{D}_\sigma \implies \psi_\sharp P \sim \mathcal{D}_\sigma \; . \tag{2.12}$$

(Consequence of [24, Lem. 9.0] together with (2.10) and the quasi-exchangeability of $D_{\boldsymbol{\alpha}}$.)

(iii) *Bayesian property* [16, §3, Thm. 1]: Let $\mathbf{W} := (W_1, \dots, W_r)$ be a sample of size r from P, conditionally i.i.d., and denote by $\mathcal{D}_\sigma^{\mathbf{W}}$ the distribution of P given $\mathbf{W}$, termed the *posterior distribution of $\mathcal{D}_\sigma$ given atoms* $\mathbf{W}$. Then,

$$P \sim \mathcal{D}_\sigma \implies (P \mid \mathbf{W}) \sim \mathcal{D}_{\sigma + \sum_{j=1}^r \delta_{W_j}} \; .$$

Discretizations In order to consider finite-dimensional marginalizations of $\mathcal{D}_{\beta\sigma}$, we introduce the following discretization procedure; cf. [44] for a similar construction. Any partition $\mathbf{X}\in\mathfrak{P}_k(X)$ induces a discretization of X to $[k]$ by collapsing $X_i\in\mathbf{X}$ to an arbitrary point in X_i, uniquely identified by its index $i\in[k]$, i.e. via the map $\mathrm{pr}^{\mathbf{X}}\colon X\supset X_i\ni x\mapsto i\in[k]$. The finite σ-algebra $\sigma_0(\mathbf{X})$ generated by $\mathbf{X}$ induces then a discretization of $\mathscr{P}(X)$ to the space $\mathscr{P}([k])$ via the mapping $\mu\mapsto\sum_{i=1}^k \mu X_i\,\delta_i$. Since the latter space is in turn homeomorphic to the standard simplex Δ^{k-1} via the mapping $\sum_{i=1}^k y_i\delta_i\mapsto\mathbf{y}$, every choice of $\mathbf{X}\in\mathfrak{P}_k(X)$ induces a discretization of $\mathscr{P}(X)$ to Δ^{k-1} via the resulting composition $\mathrm{ev}^{\mathbf{X}}=\mathrm{pr}^{\mathbf{X}}_\sharp$. It is then precisely the content of (2.10) that any partition $\mathbf{X}$ as above induces a discretization of the tuple $\big((X,\sigma),(\mathscr{P}(X),\mathcal{D}_{\beta\sigma})\big)$ to the tuple $\big(([k],\boldsymbol{\alpha}),(\Delta^{k-1},D_{\boldsymbol{\alpha}})\big)$, where $\boldsymbol{\alpha}\coloneqq\beta\,\mathrm{ev}^{\mathbf{X}}\,\sigma$ is identified with the measure $\sum_{i=1}^k\alpha_i\delta_i$ on $[k]$; cf. Fig. 1 below.

Moving further in this fashion, the subgroup $\mathfrak{S}_{\mathbf{X}}$ of bi-measurable isomorphisms ψ of $(X,\mathcal{B})$ respecting $\mathbf{X}$, i.e. such that $\psi^\diamond(\mathbf{X})\coloneqq(\psi(X_1),\dots,\psi(X_k))$ coincides with $\mathbf{X}$ up to reordering, is naturally isomorphic to the symmetric group $\mathfrak{S}_k$, the bi-measurable isomorphism group $\mathfrak{S}([k])$ of $[k]$. The canonical action of $\mathfrak{S}_{\mathbf{X}}$ on X, corresponding to the canonical action of $\mathfrak{S}_k$ on $[k]$, lifts to the action of $\mathfrak{S}_k$ on Δ^{k-1} by permutation of its vertices, that is, to the action on $\mathscr{P}([k])$ defined by $\pi.\mathbf{y}\coloneqq\pi_\sharp\mathbf{y}$ under the identification of $\mathbf{y}$ with the measure $\sum_{i=1}^k y_i\delta_i$.

3 Proof of Theorem 1.1 and accessory results

3.1 Finite-dimensional statements

Thinking of $\boldsymbol{\alpha}$ as a measure on $[k]$ as in §2, the aggregation property (2.4) may be given a measure-theoretical interpretation too. Indeed with the same notation of §2.2, for $i\in[k-1]$ let additionally $\mathfrak{s}^i\colon[k]\to[k-1]$ denote the i^{th} degeneracy map of $[k]$, i.e. the unique weakly order preserving surjection such that $\#(\mathfrak{s}^i)^{-1}(i)=2$. Then, up to the usual identification of Δ^{k-1} with $\mathscr{P}([k])$, it holds that $\mathfrak{s}^i_\sharp\mathbf{y}=\mathbf{y}_{+i}$ and one has $\mathfrak{s}^i_\sharp\mathbf{Y}\sim\mathbf{Y}_{+i}$. Thus, choosing $\mathbf{Y}\sim D_{\boldsymbol{\alpha}}$, the aggregation property reads $(\mathfrak{s}^i_\sharp)_\sharp D_{\boldsymbol{\alpha}}=D_{\mathfrak{s}^i_\sharp\boldsymbol{\alpha}}$.

The following result is a rather obvious generalization of the latter fact, obtained by substituting degeneracy maps with arbitrary maps. We provide a proof for completeness.

Proposition 3.1 (Mapping Theorem for $D_{\boldsymbol{\alpha}}$)**.** *Fix $\boldsymbol{\alpha}\in\mathbb{R}^k_+$. Then, for every $g\colon[k]\to[k]$*

$$(g_\sharp)_\sharp D_{\boldsymbol{\alpha}}=D_{g_\sharp\boldsymbol{\alpha}}\ .$$

Proof. Define the *additive contraction* $\mathbf{y}_{+\boldsymbol{\lambda}}$ of a vector $\mathbf{y}$ with respect to $\boldsymbol{\lambda}\vdash k$ as

$$\begin{aligned}\mathbf{y}_{+\boldsymbol{\lambda}}\coloneqq\big(&\underbrace{y_1,\dots,y_{\lambda_1}}_{\lambda_1},\underbrace{y_{\lambda_1+1}+y_{\lambda_1+2},\dots,y_{\lambda_1+2\lambda_2-1}+y_{\lambda_1+2\lambda_2}}_{2\lambda_2},\dots,\\ &\underbrace{y_{\vec{\mathbf{k}}\cdot\boldsymbol{\lambda}-k\lambda_k+1}+\dots+y_{\vec{\mathbf{k}}\cdot\boldsymbol{\lambda}-(k-1)\lambda_k},\dots,y_{\vec{\mathbf{k}}\cdot\boldsymbol{\lambda}-\lambda_k+1}+\dots+y_{\vec{\mathbf{k}}\cdot\boldsymbol{\lambda}}}_{k\lambda_k}\big)\ ,\end{aligned}\tag{3.1}$$

whence inductively applying (2.4) to any Δ^{k-1}-valued random variable $\mathbf{Y}$ yields $\mathbf{Y}\sim D_{\boldsymbol{\alpha}}\implies\mathbf{Y}_{+\boldsymbol{\lambda}}\sim D_{\boldsymbol{\alpha}_{+\boldsymbol{\lambda}}}$ for $\boldsymbol{\lambda}\vdash k$. Combining the latter with the quasi-exchangeability in (2.5), $D_{\boldsymbol{\alpha}}$ satisfies

$$\mathbf{Y}\sim D_{\boldsymbol{\alpha}}\implies(\mathbf{Y}_\pi)_{+\boldsymbol{\lambda}}\sim D_{(\boldsymbol{\alpha}_\pi)_{+\boldsymbol{\lambda}}}\qquad\pi\in\mathfrak{S}_k\ ,\quad\boldsymbol{\lambda}\vdash k\ .\tag{3.2}$$

For $\boldsymbol{\lambda}\vdash k$ set $\lambda_0\coloneqq 0$ and define the map $\star\boldsymbol{\lambda}\colon[k]\to[|\boldsymbol{\lambda}|]$ by

$$\star\boldsymbol{\lambda}\colon i\mapsto\lambda_{j-1}+\lceil i/j\rceil\quad\text{if}\quad i\in\{(j-1)\lambda_{j-1}+1,\dots,j\lambda_j\}$$

varying j in $[k]$, where $\lceil\alpha\rceil$ denotes the ceiling of α. It is readily checked that $(\star\boldsymbol{\lambda}\circ\pi)_\sharp\boldsymbol{\alpha} = (\boldsymbol{\alpha}_\pi)_{+\boldsymbol{\lambda}}$ for any π in $\mathfrak{S}_k$. The proof is completed by exhibiting, for fixed $g\colon [k]\to[k]$, the unique partition $\boldsymbol{\lambda}_g\vdash k$ and some permutation $\pi_g\in\mathfrak{S}_k$ such that $g=\star\boldsymbol{\lambda}_g\circ\pi_g$. To this end set $L_{g,(i)} \coloneqq g^{-1}(i)$ and

$\mathbf{L}_g \coloneqq \big(L_{g,(1)},\dots,L_{g,(k)}\big)$, where it is understood that $L_{g,(i)}$ is omitted if empty;
$\tilde{\mathbf{L}}_g \coloneqq (\tilde L_{1,1},\tilde L_{1,2},\dots,\tilde L_{2,1},\dots)$ the ordered set partition associated to $\mathbf{L}_g$, where
$\tilde L_{j,r}\coloneqq(\ell_{j,r,1},\dots,\ell_{j,r,j})$ denotes the r^{th} tuple of cardinality j in $\tilde{\mathbf{L}}_g$;

moreover, varying j in $[k]$ and r in $\lfloor k/\lambda_j\rfloor$, where $\lfloor\alpha\rfloor$ denotes the floor of α, define π in $\mathfrak{S}_k$ by

$$\pi\colon i\mapsto \ell_{j,r,(i-\lambda_{j-1}-1 \mod j)+1} \quad\text{if}\quad \begin{cases} i\in\{(j-1)\lambda_{j-1}+1,\dots,j\lambda_j\} \\ \lceil (i-\lambda_{j-1}-1)/\lambda_j\rceil = r\end{cases} .$$

Finally set $\pi_g\coloneqq\pi^{-1}$ and $\boldsymbol{\lambda}_g\coloneqq\boldsymbol{\lambda}(\mathbf{L}_g)$. □

Remark 3.2. Assuming the point of view of conditional expectations rather than that of marginalizations, (2.10) may be restated as

$$\mathbb{E}_{\mathcal{D}_{\beta\sigma}}[\,\cdot\,|\sigma_0(\mathbf{X})] = \mathbb{E}_{D_{\beta\sigma^\diamond\mathbf{X}}}[\,\cdot\,] ,$$

where $\sigma_0(\mathbf{X})$ denotes as before the σ-algebra generated by some partition $\mathbf{X}\in\mathfrak{P}_k(X)$. The aggregation property (2.4) is but an instance of the tower property of conditional expectations, whereas its generalization (3.2) is a consequence of the σ-symmetry of $\mathcal{D}_\sigma$.

Theorem 3.3 (Moments of $D_{\boldsymbol{\alpha}}$)**.** *Fix $\boldsymbol{\alpha}>\mathbf{0}$ and $\mathrm{s}\in\mathbb{R}^k$. Then, the following identity holds*

$$\mu_n'[\mathbf{s},\boldsymbol{\alpha}] = \frac{n!}{\langle\boldsymbol{\alpha}_\bullet\rangle_n}\sum_{\substack{\mathbf{m}\in\mathbb{N}_0^k\\ \mathbf{m}_\bullet=n}}\frac{\mathbf{s}^{\mathbf{m}}}{\mathbf{m}!}\langle\boldsymbol{\alpha}\rangle_{\mathbf{m}} = \frac{n!}{\langle\boldsymbol{\alpha}_\bullet\rangle_n} Z_n(\mathrm{s}^{\diamond1}\cdot\boldsymbol{\alpha},\dots,\mathrm{s}^{\diamond n}\cdot\boldsymbol{\alpha}) \eqqcolon \zeta_n[\mathbf{s},\boldsymbol{\alpha}] . \tag{3.3}$$

Proof. Let

$$\tilde\mu_n[\mathbf{s},\boldsymbol{\alpha}]\coloneqq\langle\boldsymbol{\alpha}_\bullet\rangle_n (n!)^{-1}\mu_n'[\mathbf{s},\boldsymbol{\alpha}] , \qquad \tilde\zeta_n[\mathbf{s},\boldsymbol{\alpha}]\coloneqq\langle\boldsymbol{\alpha}_\bullet\rangle_n (n!)^{-1}\zeta_n[\mathbf{s},\boldsymbol{\alpha}] .$$

The statement is equivalent to $\tilde\mu_n=\tilde\zeta_n$, which we prove in two steps.

Step 1. The following identity holds

$$\tilde\mu_{n-1}[\mathbf{s},\boldsymbol{\alpha}+\mathbf{e}_\ell] = \sum_{h=1}^n s_\ell^{h-1}\tilde\mu_{n-h}[\mathbf{s},\boldsymbol{\alpha}] . \tag{3.4}$$

By induction on n with trivial (i.e. $1=1$) base step $n=1$. *Inductive step.* Assume for every $\boldsymbol{\alpha}>\mathbf{0}$ and s in $\mathbb{R}^k$

$$\tilde\mu_{n-2}[\mathbf{s},\boldsymbol{\alpha}+\mathbf{e}_\ell] = \sum_{h=1}^{n-1} s_\ell^{h-1}\tilde\mu_{n-1-h}[\mathbf{s},\boldsymbol{\alpha}] . \tag{3.5}$$

Let $\partial_j\coloneqq\partial_{s_j}$ and note that

$$\begin{aligned}\partial_j\tilde\mu_n[\mathbf{s},\boldsymbol{\alpha}] &= \sum_{\substack{\mathbf{m}\in\mathbb{N}_0^k\\ \mathbf{m}_\bullet=n}}\frac{m_j\,\mathbf{s}^{\mathbf{m}-\mathbf{e}_j}}{\mathbf{m}!}\langle\boldsymbol{\alpha}\rangle_{\mathbf{m}} = \sum_{\substack{\mathbf{m}\in\mathbb{N}_0^k\\ \mathbf{m}_\bullet=n}}\frac{\mathbf{s}^{\mathbf{m}-\mathbf{e}_j}}{(\mathbf{m}-\mathbf{e}_j)!}\alpha_j\langle\boldsymbol{\alpha}+\mathbf{e}_j\rangle_{\mathbf{m}-\mathbf{e}_j} \\ &=\alpha_j\sum_{\substack{\mathbf{m}\in\mathbb{N}_0^k\\ \mathbf{m}_\bullet=n-1}}\frac{\mathbf{s}^{\mathbf{m}}}{\mathbf{m}!}\langle\boldsymbol{\alpha}+\mathbf{e}_j\rangle_{\mathbf{m}} = \alpha_j\tilde\mu_{n-1}[\mathbf{s},\boldsymbol{\alpha}+\mathbf{e}_j] .\end{aligned} \tag{3.6}$$

If $k \geq 2$, we can choose $j \neq \ell$. Applying (3.6) to both sides of (3.4) yields

$$\begin{aligned}\partial_j \tilde{\mu}_{n-1}[\mathbf{s}, \boldsymbol{\alpha} + \mathbf{e}_\ell] =& \alpha_j \tilde{\mu}_{n-2}[\mathbf{s}, \boldsymbol{\alpha} + \mathbf{e}_j + \mathbf{e}_\ell] \\ \partial_j \sum_{h=1}^{n} s_\ell^{h-1} \tilde{\mu}_{n-h}[\mathbf{s}, \boldsymbol{\alpha}] =& \sum_{h=1}^{n} s_\ell^{h-1} \alpha_j \tilde{\mu}_{n-h-1}[\mathbf{s}, \boldsymbol{\alpha} + \mathbf{e}_j] \\ =& \alpha_j \sum_{h=1}^{n-1} s_\ell^{h-1} \tilde{\mu}_{n-h-1}[\mathbf{s}, \boldsymbol{\alpha} + \mathbf{e}_j] \ ,\end{aligned}$$

where the latter equality holds by letting $\tilde{\mu}_{-1} \coloneqq 0$. Letting now $\boldsymbol{\alpha}' \coloneqq \boldsymbol{\alpha} + \mathbf{e}_j$ and applying the inductive hypothesis (3.5) with $\boldsymbol{\alpha}'$ in place of $\boldsymbol{\alpha}$ yields

$$\partial_j \left(\tilde{\mu}_{n-1}[\mathbf{s}, \boldsymbol{\alpha} + \mathbf{e}_\ell] - \sum_{h=1}^{n} s_\ell^{h-1} \tilde{\mu}_{n-h}[\mathbf{s}, \boldsymbol{\alpha}] \right) = 0$$

for every $j \neq \ell$. By arbitrariness of $j \neq \ell$, the bracketed quantity is a polynomial in the sole variables s_ℓ and $\boldsymbol{\alpha}$ of degree at most $n-1$. Obviously, the same holds also in the case $k = 1$. As a consequence, or trivially if $k = 1$, every monomial not in the sole variable s_ℓ cancels out by arbitrariness of $\mathbf{s}$, yielding

$$\tilde{\mu}_{n-1}[\mathbf{s}, \boldsymbol{\alpha} + \mathbf{e}_\ell] - \sum_{h=1}^{n} s_\ell^{h-1} \tilde{\mu}_{n-h}[\mathbf{s}, \boldsymbol{\alpha}] = \frac{s_\ell^{n-1} \langle \alpha_\ell + 1 \rangle_{n-1}}{(n-1)!} - \sum_{h=1}^{n} s_\ell^{h-1} \frac{s_\ell^{n-h}}{(n-h)!} \langle \alpha_\ell \rangle_{n-h} \ .$$

The latter quantity is proved to vanish as soon as

$$\frac{\langle \alpha + 1 \rangle_{n-1}}{(n-1)!} = \sum_{h=1}^{n} \frac{\langle \alpha \rangle_{n-h}}{(n-h)!} \ , \quad \text{or equivalently} \qquad \langle \alpha + 1 \rangle_{n-1} = \sum_{h=0}^{n-1} \frac{\langle \alpha \rangle_h (n-1)!}{h!} \ ,$$

in fact a particular case of the well-known *Chu–Vandermonde identity*

$$\langle \alpha + \beta \rangle_n = \sum_{k=0}^{n} \binom{n}{k} \langle \alpha \rangle_k \langle \beta \rangle_{n-k} \ . \tag{3.7}$$

Step 2. It holds that $\tilde{\mu}_n = \tilde{\zeta}_n$. By strong induction on n with trivial (i.e. $1 = 1$) base step $n = 0$. *Inductive step.* Assume for every $\boldsymbol{\alpha} > \mathbf{0}$ and $\mathbf{s}$ in $\mathbb{R}^k$ that $\tilde{\mu}_{n-1}[\mathbf{s}, \boldsymbol{\alpha}] = \tilde{\zeta}_{n-1}[\mathbf{s}, \boldsymbol{\alpha}]$. Then

$$\begin{aligned}\partial_j \tilde{\zeta}_n[\mathbf{s}, \boldsymbol{\alpha}] =& \sum_{\boldsymbol{\lambda} \vdash n} \frac{M_2(\boldsymbol{\lambda})}{n!} \sum_{h=1}^{n} \frac{\partial_j (\mathbf{s}^{\diamond h} \cdot \boldsymbol{\alpha})^{\lambda_h}}{(\mathbf{s}^{\diamond h} \cdot \boldsymbol{\alpha})^{\lambda_h}} \prod_{i=1}^{n} (\mathbf{s}^{\diamond i} \cdot \boldsymbol{\alpha})^{\lambda_i} \\ =& \sum_{\boldsymbol{\lambda} \vdash n} \frac{M_2(\boldsymbol{\lambda})}{n!} \sum_{h=1}^{n} \frac{h \lambda_h s_j^{h-1} \alpha_j}{\mathbf{s}^{\diamond h} \cdot \boldsymbol{\alpha}} \prod_{i=1}^{n} (\mathbf{s}^{\diamond i} \cdot \boldsymbol{\alpha})^{\lambda_i} \\ =& \alpha_j \sum_{h=1}^{n} s_j^{h-1} \sum_{\boldsymbol{\lambda} \vdash n} \frac{h \lambda_h}{1^{\lambda_1} \lambda_1! \dots h^{\lambda_h} \lambda_h! \dots n^{\lambda_n} \lambda_n!} \frac{1}{\mathbf{s}^{\diamond h} \cdot \boldsymbol{\alpha}} \prod_{i=1}^{n} (\mathbf{s}^{\diamond i} \cdot \boldsymbol{\alpha})^{\lambda_i} \\ =& \alpha_j \sum_{h=1}^{n} s_j^{h-1} \sum_{\boldsymbol{\lambda} \vdash n-h} \frac{M_2(\boldsymbol{\lambda})}{(n-h)!} \prod_{i=1}^{n-h} (\mathbf{s}^{\diamond i} \cdot \boldsymbol{\alpha})^{\lambda_i} \\ =& \alpha_j \sum_{h=1}^{n} s_j^{h-1} \tilde{\zeta}_{n-h}[\mathbf{s}, \boldsymbol{\alpha}] \ .\end{aligned}$$

The inductive hypothesis, (3.4) and (3.6) yield

$$\partial_j \tilde{\zeta}_n[\mathbf{s}, \boldsymbol{\alpha}] = \alpha_j \sum_{h=1}^{n} s_j^{h-1} \tilde{\mu}_{n-h}[\mathbf{s}, \boldsymbol{\alpha}] = \partial_j \tilde{\mu}_n[\mathbf{s}, \boldsymbol{\alpha}] \ .$$

By arbitrariness of j this implies that $\tilde{\zeta}_n[\mathrm{s},\boldsymbol{\alpha}]-\tilde{\mu}_n[\mathrm{s},\boldsymbol{\alpha}]$ is constant as a function of s (for fixed $\boldsymbol{\alpha}$), hence vanishing by choosing $\mathrm{s}=\mathbf{0}$. □

Remark 3.4. Here, we gave an elementary combinatorial proof of the moment formula for $D_{\boldsymbol{\alpha}}$, independently of any property of the distribution. Note for further purposes that, *defining* $\mu'_n[\mathrm{s},\boldsymbol{\alpha}]$ as in (3.3), the statement holds with identical proof for all $\boldsymbol{\alpha}$ in $\mathbb{C}^k$ such that $\boldsymbol{\alpha}_\bullet\notin\mathbb{Z}_0^-$. For further representations of the moments see Remark 3.11 below. Also, note that expanding $\langle\boldsymbol{\alpha}+\mathrm{e}_\ell\rangle_{\mathbf{m}}$ via the Chu–Vandermonde identity yields a simpler proof of (3.4). We opted for the given proof, since we shall need (3.6) for future comparison.

Proposition 3.5. *The function* ${}_k\Phi_2[\boldsymbol{\alpha};1;t\mathrm{s}]$ *is the exponential generating function of the polynomials* Z_n*, in the sense that, for all* $\boldsymbol{\alpha}\in\Delta^{k-1}$*,*

$$ {}_k\Phi_2[\boldsymbol{\alpha};1;t\mathrm{s}]=\mathbf{G}_{\exp}\left[Z_n\left(\mathrm{s}^{\diamond 1}\cdot\boldsymbol{\alpha},\dots,\mathrm{s}^{\diamond n}\cdot\boldsymbol{\alpha}\right)\right](t)\,,\qquad \mathrm{s}\in\mathbb{R}^k\,,\quad t\in\mathbb{R}\,. $$

More generally,

$$ {}_k\Phi_2[\boldsymbol{\alpha};\boldsymbol{\alpha}_\bullet;t\mathrm{s}]=\mathbf{G}_{\exp}\left[\frac{n!}{\langle\boldsymbol{\alpha}_\bullet\rangle_n}Z_n\left(\mathrm{s}^{\diamond 1}\cdot\boldsymbol{\alpha},\dots,\mathrm{s}^{\diamond n}\cdot\boldsymbol{\alpha}\right)\right](t)\,,\qquad \mathrm{s}\in\mathbb{R}^k\,,\quad t\in\mathbb{R}\,. $$

Proof. Recalling that ${}_k\Phi_2[\boldsymbol{\alpha};\boldsymbol{\alpha}_\bullet;\mathrm{i}\mathrm{s}]=\widehat{D_{\boldsymbol{\alpha}}}(\mathrm{s})$ by (2.9) and noticing that $\boldsymbol{\alpha}_\bullet=1$, Theorem 3.3 provides an exponential series representation for the characteristic functional of the Dirichlet distribution in terms of the cycle index polynomials of symmetric groups, viz.

$$ \widehat{D_{\boldsymbol{\alpha}}}(\mathrm{s})=\sum_{n=0}^{\infty}\frac{1}{n!}Z_n\left((\mathrm{i}\mathrm{s})^{\diamond 1}\cdot\boldsymbol{\alpha},\dots,(\mathrm{i}\mathrm{s})^{\diamond n}\cdot\boldsymbol{\alpha}\right)\,. $$

Replacing s with $-\mathrm{i}t\mathrm{s}$ above and using (2.1) to extract the term t^n from each summand, the conclusion follows. The second statement has a similar proof. □

Remark 3.6. It is well-known that the characteristic functional of a measure μ on $\mathbb{R}^d$ (or, more generally, on a nuclear space) is always *positive definite*, i.e. it holds that

$$ \sum_{h,k=1}^{n}\widehat{\mu}(\mathrm{s}_h-\mathrm{s}_k)\,\xi_h\bar{\xi}_k\geq 0\,,\qquad n\in\mathbb{N}_0\,,\quad \mathrm{s}_1,\dots,\mathrm{s}_n\in\mathbb{R}^d\,,\quad \xi_1,\dots,\xi_n\in\mathbb{C}\,, \tag{3.8} $$

where $\bar{\xi}$ denotes the complex conjugate of $\xi\in\mathbb{C}$. Thus, the functional $\mathrm{s}\mapsto{}_k\Phi_2[\boldsymbol{\alpha};\boldsymbol{\alpha}_\bullet;\mathrm{i}\mathrm{s}]$ is positive definite by (2.9) for all $\boldsymbol{\alpha}\in\mathbb{R}^k_+$.

The following Lemma also appeared in [32, Eqn.'s (2), (3)].

Lemma 3.7. *There exist the narrow limits*

$$ \lim_{\beta\to 0^+}D_{\beta\boldsymbol{\alpha}}=\boldsymbol{\alpha}_\bullet^{-1}\sum_{i=1}^{k}\alpha_i\delta_{\mathrm{e}_i}\qquad\text{and}\qquad\lim_{\beta\to+\infty}D_{\beta\boldsymbol{\alpha}}=\delta_{\boldsymbol{\alpha}_\bullet^{-1}\boldsymbol{\alpha}}\,. $$

Proof. Since $D_{\boldsymbol{\alpha}}$ is moment determinate, it suffices — by compactness of Δ^{k-1} and Stone–Weierstraß Theorem — to show the convergence of its moments. By Theorem 3.3, cf. also (2.1),

$$ \begin{aligned}\mu'_n[\mathrm{s},\beta\boldsymbol{\alpha}]:=&\frac{n!}{\langle\beta\boldsymbol{\alpha}_\bullet\rangle_n}Z_n\left(\beta\mathrm{s}^{\diamond 1}\cdot\boldsymbol{\alpha},\dots,\beta\mathrm{s}^{\diamond n}\cdot\boldsymbol{\alpha}\right)=\frac{1}{\langle\beta\boldsymbol{\alpha}_\bullet\rangle_n}\sum_{r=1}^{n}\sum_{\boldsymbol{\lambda}\vdash_r n}M_2(\boldsymbol{\lambda})\prod_i^n(\beta\,\mathrm{s}^{\diamond i}\cdot\boldsymbol{\alpha})^{\lambda_i}\\ =&\frac{1}{\langle\beta\boldsymbol{\alpha}_\bullet\rangle_n}\sum_{r=1}^{n}\sum_{\boldsymbol{\lambda}\vdash_r n}M_2(\boldsymbol{\lambda})\,\beta^{\boldsymbol{\lambda}_\bullet}\prod_i^n(\mathrm{s}^{\diamond i}\cdot\boldsymbol{\alpha})^{\lambda_i}\end{aligned} $$

$$
\begin{aligned}
&= \frac{1}{\langle\beta\boldsymbol{\alpha}_\bullet\rangle_n}\sum_{r=1}^n \beta^r \sum_{\boldsymbol{\lambda}\vdash_r n} M_2(\boldsymbol{\lambda})\prod_i^n (\mathbf{s}^{\diamond i}\cdot\boldsymbol{\alpha})^{\lambda_i}\\
&\underset{\beta\ll 1}{\approx} \frac{1}{\beta\boldsymbol{\alpha}_\bullet\,\Gamma(n)}\,\beta M_2(\mathbf{e}_n)\,(\mathbf{s}^{\diamond n}\cdot\boldsymbol{\alpha})^1 = \boldsymbol{\alpha}_\bullet^{-1}\boldsymbol{\alpha}\cdot\mathbf{s}^{\diamond n}\ ,\\
&\underset{\beta\gg 1}{\approx} \frac{1}{\beta^n\boldsymbol{\alpha}_\bullet^n}\,\beta^n M_2(n\mathbf{e}_1)\,(\mathbf{s}^{\diamond 1}\cdot\boldsymbol{\alpha})^n = \boldsymbol{\alpha}_\bullet^{-n}(\mathbf{s}\cdot\boldsymbol{\alpha})^n\ . \qquad \square
\end{aligned}
$$

As a consequence of the Lemma further confluent forms of ${}_k\Phi_2$ may be computed:

Corollary 3.8 (Confluent forms of ${}_k\Phi_2$)**.** *There exist the limits*

$$
\lim_{\beta\to 0^+} {}_k\Phi_2[\beta\boldsymbol{\alpha};\beta\boldsymbol{\alpha}_\bullet;\mathbf{s}] = \boldsymbol{\alpha}_\bullet^{-1}\boldsymbol{\alpha}\cdot\exp^\diamond(\mathbf{s})\ ,\qquad \lim_{\beta\to+\infty} {}_k\Phi_2[\beta\boldsymbol{\alpha};\beta\boldsymbol{\alpha}_\bullet;\mathbf{s}] = \exp(\boldsymbol{\alpha}_\bullet^{-1}\boldsymbol{\alpha}\cdot\mathbf{s})\ .
$$

3.2 Infinite-dimensional statements

Together with the introductory discussion, Proposition 3.1 suggests the following Mapping Theorem for $\mathcal{D}_\sigma$, to be compared with the analogous result for the Poisson random measure $\mathcal{P}_\sigma$ over $(X,\mathcal{B})$; see e.g., [27, §2.3 and *passim*]. The σ-symmetry of $\mathcal{D}_{\beta\sigma}$ and the quasi-exchangeability and aggregation property of $D_{\boldsymbol{\alpha}}$ are trivially recovered from the Theorem by (2.10).

Theorem 3.9 (Mapping theorem for $\mathcal{D}_\sigma$)**.** *Let $(X,\tau(X),\mathcal{B})$ and $(X',\tau(X'),\mathcal{B}')$ be second countable locally compact Hausdorff spaces, ν a non-negative finite measure on $(X,\mathcal{B})$ and $f\colon(X,\mathcal{B})\to(X',\mathcal{B}')$ be any measurable map. Then,*

$$
(f_\sharp)_\sharp\mathcal{D}_\nu = \mathcal{D}_{f_\sharp\nu}\ .
$$

Proof. For measurable $g\colon X\to[k]$ set $\mathbf{X}:=(g^{-1}(1),\ldots,g^{-1}(k))$. The characterization in (2.11) is equivalent to the requirement that $(g_\sharp)_\sharp\mathcal{D}_\nu = D_{g_\sharp\nu}$ for any $g\colon X\to[k]$ such that $\nu g^{-1}(i)>0$ for all $i\in[k]$. Denote by $\mathcal{S}(X,\nu,k)$ the family of such functions and note that if $h\in\mathcal{S}(X',f_\sharp\nu,k)$, then $g:=h\circ f\in\mathcal{S}(X,\nu,k)$. The proof is now merely typographical:

$$
(h_\sharp)_\sharp(f_\sharp)_\sharp\mathcal{D}_\nu = (g_\sharp)_\sharp\mathcal{D}_\nu = D_{g_\sharp\nu} = D_{h_\sharp(f_\sharp\nu)}\ ,
$$

where the second equality suffices to establish that $(f_\sharp)_\sharp\mathcal{D}_\nu$ is a Dirichlet–Ferguson measure by arbitrariness of h, while the third one characterizes its intensity as $f_\sharp\nu$. $\square$

We denote by $\mathscr{P}(\mathscr{P}(X))$ the space of probability measures on $(\mathscr{P}(X),\mathcal{B}_n(\mathscr{P}(X)))$, endowed with the narrow topology $\tau_n(\mathscr{P}(\mathscr{P}(X)))$ induced by duality with $\mathcal{C}_b(\mathscr{P}(X))$. We are now able to prove the following more general version of Theorem 1.1.

Theorem 3.10 (Characteristic functional of $\mathcal{D}_{\beta\sigma}$)**.** *Let $(X,\tau(X),\mathcal{B})$ be a second countable locally compact Hausdorff Borel measurable space, σ a probability measure on $(X,\mathcal{B})$ and fix $\beta>0$. Then,*

$$
\widehat{\mathcal{D}_{\beta\sigma}}(tf^*) = \mathbf{G}_{\exp}\left[n!\,\langle\beta\rangle_n^{-1}\,Z_n\left(\beta\sigma f^1,\ldots,\beta\sigma f^n\right)\right](\mathrm{i}t)\ ,\quad t\in\mathbb{R}\ ,\quad f\in\mathcal{C}_c(X)\ . \tag{3.9}
$$

Moreover, the map $\nu\mapsto\mathcal{D}_\nu$ is narrowly continuous on $\mathscr{M}_b^+(X)$.

Proof. Characteristic functional. Fix f in $\mathcal{C}_c(X)$ and let $(f_h)_h$ be a good approximation of f, locally constant on $\mathbf{X}_h:=(X_{h,1},\ldots,X_{h,k_h})$ with values $\mathbf{s}_h$ for some $(\mathbf{X}_h)_h\in\mathfrak{Na}(X)$. Fix $n>0$ and set $\boldsymbol{\alpha}_h:=\beta\sigma^\diamond\mathbf{X}_h$. Choosing $u\colon\Delta^{k_h-1}\to\mathbb{R}$, $u\colon\mathbf{y}\mapsto(\mathbf{s}_h\cdot\mathbf{y})^n$ in (2.11) yields

$$
{\mu'_n}^{\mathcal{D}_{\beta\sigma}}[f_h^*] := \int_{\mathscr{P}(X)} (f_h^*\eta)^n\,\mathrm{d}\mathcal{D}_{\beta\sigma}(\eta) = \int_{\Delta^{k_h-1}} (\mathbf{s}_h\cdot\mathbf{y})^n\,\mathrm{d}D_{\beta\,\mathrm{ev}^{\mathbf{X}_h}\sigma}(\mathbf{y}) = \mu'_n[\mathbf{s}_h,\boldsymbol{\alpha}_h]\ ,
$$

hence, by Theorem 3.3,

$$\mu_n'^{\mathcal{D}_{\beta\sigma}}[f_h^*] = n!\,\langle\beta\rangle_n^{-1}\, Z_n\left(\mathbf{s}_h^{\diamond 1}\cdot\boldsymbol{\alpha}_h,\dots,\mathbf{s}_h^{\diamond n}\cdot\boldsymbol{\alpha}_h\right) = n!\,\langle\beta\rangle_n^{-1}\, Z_n\left(\beta\sigma f_h^1,\dots,\beta\sigma f_h^n\right) \ ,$$

thus, by Dominated Convergence Theorem, continuity of Z_n and arbitrariness of f,

$$\mu_n'^{\mathcal{D}_{\beta\sigma}}[tf^*] = n!\,\langle\beta\rangle_n^{-1}\, Z_n\left(t^1\beta\sigma f^1,\dots,t^n\beta\sigma f^n\right) \ , \quad t\in\mathbb{R} \ , \quad f\in\mathcal{C}_c \ .$$

Using (2.1) to extract the term t^n from Z_n and substituting t with $\mathrm{i}\,t$ on the right-hand side, the conclusion follows by definition of exponential generating function.

Continuity. Assume first that $(X,\tau(X))$ is compact. By compactness of $(X,\tau(X))$, the narrow and vague topology on $\mathscr{P}(X)$ coincide and $\mathscr{P}(X)$ is compact as well by Prokhorov Theorem. Let $(\nu_h)_{h\in\mathbb{N}}$ be a sequence of finite non-negative measures narrowly convergent to ν_∞. Again by Prokhorov Theorem and by compactness of $\mathscr{P}(X)$ there exists some $\tau_n(\mathscr{P}(\mathscr{P}(X)))$-cluster point $\mathcal{D}_\infty$ for the family $\{\mathcal{D}_{\nu_h}\}_h$. By narrow convergence of ν_h to ν_∞, continuity of Z_n and absolute convergence of $\widehat{\mathcal{D}_\cdot}(f)$, it follows that $\lim_h \widehat{\mathcal{D}_{\nu_h}} = \widehat{\mathcal{D}_{\nu_\infty}}$ pointwise on $\mathcal{C}_c(X)$, hence, by Corollary 5.3, it must be $\mathcal{D}_\infty = \mathcal{D}_{\nu_\infty}$.

In the case when X is not compact, recall the notation established in Proposition 2.4, denote by $\mathcal{B}(\alpha X)$ the Borel σ-algebra of $(\alpha X,\tau(\alpha X))$ and by $\mathscr{P}(\alpha X)$ the space of probability measures on $(\alpha X,\mathcal{B}(\alpha X))$. By the Continuous Mapping Theorem there exists the narrow limit $\tau_n(\mathscr{P}(X))\text{-}\lim_h \alpha_\sharp\nu_h = \alpha_\sharp\nu_\infty$, thus, by the result in the compact case applied to the space $(\alpha X,\mathcal{B}_\alpha)$ together with the sequence $\alpha_\sharp\nu_h$,

$$\tau_n\big(\mathscr{P}(\mathscr{P}(X))\big)\text{-}\lim_h \mathcal{D}_{\alpha_\sharp\nu_h} = \mathcal{D}_{\alpha_\sharp\nu_\infty} \ . \tag{3.10}$$

The narrow convergence of ν_h to ν_∞ implies that $\alpha_\sharp\nu_\infty$ does not charge the point at infinity in αX, hence the measure spaces $(X,\mathcal{B},\nu_*)$ and $(\alpha X,\mathcal{B}(\alpha X),\alpha_\sharp\nu_*)$ are isomorphic for $*=h,\infty$ via the map α, with inverse α^{-1} defined on $\operatorname{im}\alpha\subsetneq\alpha X$. The continuity of α^{-1} and the Continuous Mapping Theorem together yield the narrow continuity of the map $({\alpha^{-1}}_\sharp)_\sharp$. The conclusion follows by applying $({\alpha^{-1}}_\sharp)_\sharp$ to (3.10) and using the Mapping Theorem 3.9. □

Remark 3.11. Different representations of the univariate moments of the Dirichlet–Ferguson measure have also appeared, without mention to Z_n, in [42, Eq. (17)] (in terms of incomplete Bell polynomials, solely in the case when $X\Subset\mathbb{R}_+$ and $f=\mathrm{id}_\mathbb{R}$) and in [32, proof of Prop. 3.3] (in implicit recursive form). Representations of the multi-variate moments have also appeared in [26, Prop. 7.4] (in terms of summations over 'color-respecting' permutations, in the case $\beta=1$), in [13, (4.20)] and [15, Lem. 5.2] (in terms of summations over constrained set partitions).

Remark 3.12. In the case when ν_h converges to ν_∞ in total variation, the continuity statement in the Theorem and the asymptotics for $\beta\to 0$ in Corollary 3.14 below were first shown in [49, Thm. 3.2], relying on Sethuraman's stick-breaking representation. The asymptotic expressions in Corollary 3.14 have been subsequently rediscovered many times in different simplified settings: Lastly, in the case $X=\mathbb{R}^d$, in [32, Prop. 3.4 and Thm. 3.5]. The following result was also obtained, again with different methods, in [49].

Corollary 3.13 (Tightness of Dirichlet–Ferguson measures [49, Thm. 3.1])**.** *Under the same assumptions of Theorem 3.10, let $M\subset\mathscr{M}_b^+(X)\setminus\{0\}$ be such that $\overline{M}\coloneqq\{\overline{\nu}:\nu\in M\}$ is a tight, resp. narrowly compact, family of finite non-negative measures. Then, the family $\{\mathcal{D}_\nu\}_{\nu\in M}$ is itself tight, resp. narrowly compact.*

Corollary 3.14 (Asymptotic expressions)**.** *Under the same assumptions of Theorem 3.10, for all f in $\mathcal{C}_c$ and complex t there exist the limits*

$$\lim_{\beta\downarrow 0}\widehat{\mathcal{D}_{\beta\sigma}}(tf^*) = \sigma\exp(\mathrm{i}\,tf) \qquad \textit{and} \qquad \lim_{\beta\to\infty}\widehat{\mathcal{D}_{\beta\sigma}}(tf^*) = \exp(\mathrm{i}\,t\,\sigma f) \tag{3.11}$$

corresponding to the narrow limits

$$\mathcal{D}^0_\sigma \coloneqq \lim_{\beta\downarrow 0} \mathcal{D}_{\beta\sigma} = \delta_\sharp \sigma \qquad \text{and} \qquad \mathcal{D}^\infty_\sigma \coloneqq \lim_{\beta\to\infty} \mathcal{D}_{\beta\sigma} = \delta_\sigma \;, \tag{3.12}$$

where, in the first case, $\delta\colon X \to \mathscr{P}(X)$ *denotes the Dirac embedding* $x \mapsto \delta_x$.

Proof. The existence of $\mathcal{D}^0_\sigma$ and $\mathcal{D}^\infty_\sigma$ as narrow cluster points for $\{\mathcal{D}_{\beta\sigma}\}_{\beta>0}$ follows by Corollary 3.13. Retaining the notation established in Theorem 3.10, Corollary 3.8 yields for all k

$$\lim_{\beta\downarrow 0} \widehat{\mathcal{D}_{\beta\sigma}}(f_k^*) = \sigma \exp(\mathrm{i} f_k) \qquad \text{and} \qquad \lim_{\beta\to\infty} \widehat{\mathcal{D}_{\beta\sigma}}(f_k^*) = \exp(\mathrm{i}\, \sigma f_k) \;,$$

hence, by Dominated Converge,

$$\lim_k \lim_{\beta\downarrow 0} \widehat{\mathcal{D}_{\beta\sigma}}(f_k^*) = \sigma \exp(\mathrm{i} f) \qquad \text{and} \qquad \lim_k \lim_{\beta\to\infty} \widehat{\mathcal{D}_{\beta\sigma}}(f_k^*) = \exp(\mathrm{i}\, \sigma f) \;. \tag{3.13}$$

Furthermore, recalling that $|f_k| \leq |f|$ one has

$$\begin{aligned} \left|\widehat{\mathcal{D}_{\beta\sigma}}(f^*) - \widehat{\mathcal{D}_{\beta\sigma}}(f_k^*)\right| \leq & e^{\|f\|} \int_{\mathscr{P}(X)} \mathrm{d}\mathcal{D}_{\beta\sigma}(\eta)\, |f - f_k|^*\, \eta \\ = & e^{\|f\|}\, \|f - f_k\|_{L^1_\sigma} \leq e^{\|f\|}\, \|f_k - f\| \;, \end{aligned} \tag{3.14}$$

where the equality follows by [16, §3 Prop. 1]. As a consequence, the order of the limits in each left-hand side of (3.13) may be exchanged, for the convergence in k is uniform with respect to β. This shows (3.11). □

Remark 3.15. By Theorem 3.10, $\beta\sigma$ may be substituted with any sequence $(\beta_h \sigma_h)_h$ with $\lim_h \beta_h = 0, \infty$ and $\{\sigma_h\}_h$ a tight family. Despite the similarity with Lemma 3.7, Corollary 3.14 is not a direct consequence of the former, since the evaluation map $\mathrm{ev}^{\mathbf{X}}$ is not continuous.

Remark 3.16 (A Gibbsean interpretation)**.** Corollary 3.14 states that, varying $\beta \in [0, \infty]$, the map $\mathcal{D}_{\beta\,\cdot}\colon \mathscr{P}(X) \to \mathscr{P}(\mathscr{P}(X))$ is a (continuous) interpolation between the two extremal maps $\mathcal{D}^0_\cdot = \delta^{(0)}_\sharp$ and $\mathcal{D}^\infty_\cdot = \delta^{(1)}$, where $\delta^{(0)} \coloneqq \delta\colon X \to \mathscr{P}(X)$ and $\delta^{(1)} \coloneqq \delta\colon \mathscr{P}(X) \to \mathscr{P}(\mathscr{P}(X))$. These asymptotic distributions may be interpreted — at least formally — in the framework of statistical mechanics. In order to establish some lexicon, consider a physical system at *inverse temperature* β, driven by a *Hamiltonian* H.

Let $Z^H_\beta \coloneqq \langle \exp(-\beta H)\rangle$, $F_\beta \coloneqq -\beta^{-1} \ln Z^H_\beta$ and $G_\beta \coloneqq (Z^H_\beta)^{-1} \exp(-\beta H)$ respectively denote the *partition function*, the *Helmholtz free energy* and (the distribution of) the *Gibbs measure* of the system. It was heuristically argued in [45, §3.1] that — at least in the case when $(X, \mathcal{B}, \sigma)$ is the unit interval —

$$\mathrm{d}\mathcal{D}_{\beta\sigma}(\eta) = \frac{e^{-\beta\, S(\eta)}}{Z_\beta}\, \mathrm{d}\mathcal{D}^*_\sigma(\eta) \;,$$

where: S is now an *entropy functional* (rather than an energy functional), Z_β is a normalization constant and β plays the role of the inverse temperature. Here, $\mathcal{D}^*_\sigma$ denotes a *non-existing* (!) uniform distribution on $\mathscr{P}(X)$. Borrowing again the terminology, this time in full generality: for small β (i.e. large *temperature*) the system *thermalizes* towards the "uniform" distribution $\delta_\sharp\sigma$ induced by the reference measure σ on the base space; for large β it *crystallizes* to δ_σ, so that all randomness is lost. Consistently with property i of $\mathcal{D}_\sigma$, we see that $\mathbb{E}_{\mathcal{D}^\infty_\sigma} \eta_i = 0$ and $\mathbb{E}_{\mathcal{D}^0_\sigma} \eta_i = \delta_{i1}$ for all i, where δ_{ab} denotes the Kronecker symbol; both statements hold in fact with probability 1.

It is worth noticing that a different interpretation for the parameter β has been given in [32], where the latter is regarded as a 'time' parameter in the definition of a *Processus Croissant pour l'Ordre Convexe* (PCOC).

Remark 3.17. By the Continuous Mapping Theorem, both the continuity statement in Theorem 3.10 and the asymptotic expressions in Corollary 3.14 hold, mutatis mutandis, for every narrowly continuous image of $\mathcal{D}_{\beta\sigma}$, hence, for instance, for the *entropic measure* $\mathbb{P}^{\beta}_{\sigma}$ [45, 53]. This generalizes [45, Prop. 3.14] and the discussion for the entropic measure thereafter.

Corollary 3.18. *Let* $(X, \mathcal{B}, \sigma)$ *and* β *be as in Theorem 3.10 and let* $h \in \mathcal{B}_b(X; \mathbb{R})$. *Then, Equation* (3.9) *holds with* h *in place of* f. *In particular,* $\widehat{\mathcal{D}_{\beta\sigma}}(h)$ *does not depend on the choice of the representative of* $h \in L^{\infty}_{\sigma}(X, \mathcal{B})$.

Proof. Since h is bounded, $\widehat{\mathcal{D}_{\beta\sigma}}(h^*)$ is well-defined. Let $(f_k)_k \subset \mathcal{C}_c(X)$ be such that $h = L^1_\sigma\text{-}\lim_k f_k$. Observe that we can choose $(f_k)_k$ so that $\sup_k |f_k| \vee \|h\| \le M$ for some finite $M > 0$. Analogously to (3.14) (with e^M in place of $e^{\|f\|}$), we have $\lim_k \widehat{\mathcal{D}_{\beta\sigma}}(f_k^*) = \widehat{\mathcal{D}_{\beta\sigma}}(h^*)$. By the same reasoning, $\widehat{\mathcal{D}_{\beta\sigma}}(h^*)$ does not depend on the L^1_σ-representative of h. The L^1_σ-continuity of the right-hand side of (3.9) in f is straightforward. Thus, the assertion follows by replacing f with f_k in (3.9) and letting $k \to \infty$. ☐

Remark 3.19 (Some alternative proofs)**.** Applying [57, §IV.2.2, Prop. 2.4, p. 204] and Corollary 3.18 together yields a different proof of the Mapping Theorem 3.9, not relying on the marginal distributions of $\mathcal{D}_{\beta\sigma}$. As an immediate consequence, the aggregation property is also recovered by choosing a purely atomic intensity measure.

Corollary 3.20 (Alternative construction of $\mathcal{D}_{\beta\sigma}$)**.** *Assume there exists a nuclear function space* $\mathcal{S} \subset \mathcal{C}_0(X)$, *continuously embedded into* $\mathcal{C}_0(X)$ *and such that* $\mathcal{S} \cap \mathcal{C}_c(X)$ *is norm-dense in* $\mathcal{C}_0(X)$ *and dense in* $\mathcal{S}$. *Then, there exists a unique Borel probability measure on the dual space* $\mathcal{S}'$, *namely* $\mathcal{D}_{\beta\sigma}$, *whose characteristic functional is given by the extension of* (3.9) *to* $\mathcal{S}$.

Proof. By the classical Bochner–Minlos Theorem, e.g., [18, §4.2, Thm. 2], it suffices to show that the extension to $\mathcal{S}$, say χ, of the functional (3.9) is a characteristic functional. By the convention in (2.2), $\chi(\mathbf{0}_{\mathcal{S}}) = \chi(\mathbf{0}_{\mathcal{C}_c(X)}) = 1$. The (sequential) continuity of χ on $\mathcal{S}$ follows by that on $\mathcal{C}_0(X)$ and the continuity of the embedding $\mathcal{S} \subset \mathcal{C}_0(X)$. It remains to show the positivity (3.8) of χ, which can be checked only on $\mathcal{S} \cap \mathcal{C}_c(X)$ by $\|\cdot\|$-density of the inclusions $\mathcal{S} \cap \mathcal{C}_c(X) \subset \mathcal{C}_0(X)$. The positivity of χ restricted to $\mathcal{C}_c(X)$ follows from the positivity of ${}_k\Phi_2$ in Remark 3.6 by approximation of f with simple functions as in the proof of Theorem 3.10. ☐

Remark 3.21. Let us note that the assumption of Corollary 3.20 is satisfied, whenever X is (additionally) either finite (trivially), or a differentiable manifold, or a topological group (by the main result in [1]). In particular, when $X = \mathbb{R}^d$, we can choose $\mathcal{S} = \mathcal{S}(\mathbb{R}^d)$, the space of Schwartz functions on $\mathbb{R}^d$.

Consider now the map $\mathscr{G} \colon \mathscr{P}(\mathscr{P}(X)) \to \mathscr{P}(X)$ defined by

$$(\mathscr{G}(\mu))A = \int_{\mathscr{P}(X)} \mathrm{d}\mu(\eta)\, \eta A \,, \qquad A \in \mathcal{B} \,, \quad \mu \in \mathscr{P}(\mathscr{P}(X)) \,.$$

Since f^* is $\tau_n(\mathscr{P}(X))$-continuous for every $f \in \mathcal{C}_b(X)$ and bounded by $\|f\|$, the map $\mathscr{G}$ is continuous.

Corollary 3.22. *Let* $(X, \tau(X), \mathcal{B})$ *be a locally compact second countable Hausdorff Borel measurable space. For fixed* $\beta \in (0, \infty)$, *the map* $\mathcal{D}_{\beta\,\cdot} \colon \mathscr{P}(X) \to \mathscr{P}(\mathscr{P}(X))$ *is a homeomorphism onto its image, with inverse* $\mathscr{G}$.

Proof. The continuity of $\mathcal{D}_{\beta\,\cdot}$ is proven in Theorem 3.10. By e.g., [16, Thm. 3] for all $f \in \mathcal{C}_c(X)$ one has $\mathcal{D}_{\beta\sigma} f^* = \beta\sigma f$, hence $\mathscr{G}$ inverts $\mathcal{D}_{\beta\,\cdot}$ on its image. ☐

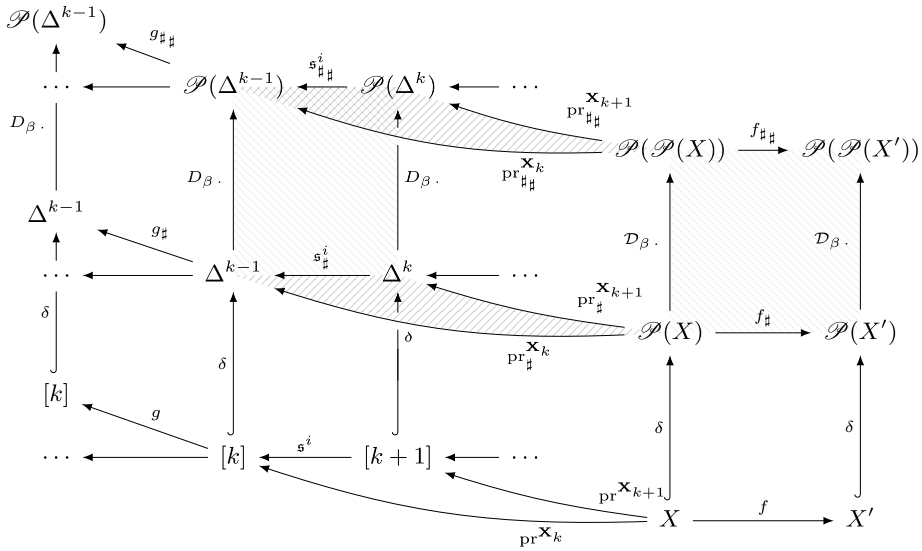

Figure 1: Many properties of Dirichlet(–Ferguson) measures can be phrased in terms of the commutation of some diagrams. The commutation of dashed squares of the diagram above, from left to right, respectively corresponds to

- the symmetry property (2.5) when $g = \pi \in \mathfrak{S}_k$ and, more generally, Proposition 3.1;
- the aggregation property (2.4);
- the marginalization (2.10) (recall that $\mathrm{pr}^{\mathbf{x}}_\sharp = \mathrm{ev}^{\mathbf{x}}$);
- the symmetry property (2.12) when $f = \psi$ is measure preserving and, more generally, Theorem 3.9;

the commutation of the solid sub-diagram delimited by the two dashed triangles corresponds to the requirement of Kolmogorov consistency; cf. [16, p. 214].

As a further application of Theorem 3.10, we show how the continuity Theorem extends to *hierarchical Dirichlet processes* [54]. Such processes arise as a natural counterpart to Dirichlet–Ferguson processes in a non-parametric Bayesian approach to the modeling of *grouped* data.

Definition 3.23 (Hierarchical Dirichlet Process [54, Eqn.s (13)–(14)])**.** *Under the same assumptions as in Theorem 3.10, given $\beta > 0$ and η a $\mathscr{P}(X)$-valued $\mathcal{D}_\nu$-distributed random field, a* hierarchical Dirichlet process *is any $\mathscr{P}(\mathscr{P}(X))$-valued random field distributed as $\mathcal{D}_{\beta\eta}$.*

As usual, we consider rather the laws of hierarchical Dirichlet processes, namely the measures $\mathcal{D}_{\beta\mathcal{D}_\nu}$ on $\mathscr{P}(\mathscr{P}(X))$. The construction is easily iterated, as in the following definition.

Definition 3.24 (Dirichlet–Ferguson-measured Vershik tower)**.** *Let $(X, \tau(X), \mathcal{B})$ be a locally compact second countable Hausdorff Borel measurable space. Define inductively for $n \in \mathbb{N}_0$*

$$\mathscr{P}^{(0)}(X) := X\,, \qquad \mathscr{P}^{(n)}(X) := \mathscr{P}\big(\mathscr{P}^{(n-1)}(X)\big)\,,$$

always endowed with the corresponding narrow topologies, and let $\delta^{(n)}\colon \mathscr{P}^{(n)}(X) \to \mathscr{P}^{(n+1)}(X)$ be the corresponding Dirac embeddings for $n \in \mathbb{N}_0$. Following [5, p. 796] we term the family $\big\{\big(\mathscr{P}^{(n)}(X), \delta^{(n)}\big)\big\}_{n\in\mathbb{N}_0}$ the Vershik tower *over X. Note that, since X*

is Polish, so is $\mathscr{P}^{(n)}(X)$ for every $n \in \mathbb{N}_0$. Analogously, if X is additionally compact, then $\mathscr{P}^{(n)}(X)$ is compact as well, for every $n \in \mathbb{N}_0$.

Let now $\boldsymbol{\beta} \in \mathbb{R}_+^{\mathbb{N}_0}$ and σ be a probability measure on X. Define inductively for $n \in \mathbb{N}_0$

$$\mathcal{D}^{(0)}_{\boldsymbol{\beta},\sigma} := \sigma \ , \qquad \mathcal{D}^{(n)}_{\boldsymbol{\beta},\sigma} := \mathcal{D}_{\beta_{n-1}\mathcal{D}^{(n-1)}_{\boldsymbol{\beta},\sigma}} \ ,$$

and observe that $\mathcal{D}^{(n)}_{\boldsymbol{\beta},\sigma}$ is a Borel probability measure over $\mathscr{P}^{(n)}(X)$ for every $n \in \mathbb{N}_1$. We term the family $\left\{\left(\mathscr{P}^{(n)}(X), \mathcal{D}^{(n)}_{\boldsymbol{\beta},\sigma}, \delta^{(n)}\right)\right\}_{n\in\mathbb{N}_0}$ the $(\boldsymbol{\beta},\sigma)$-Dirichlet–Ferguson-measured Vershik tower *over X.*

Corollary 3.25. *Let $(X, \tau(X), \mathcal{B})$ be a compact Hausdorff Borel measurable space and let $\mathbb{R}_+^{\mathbb{N}_0}$ be endowed with the uniform topology. Then, the map $\mathcal{D}^{(n)}_{\cdot}\colon \mathbb{R}_+^{\mathbb{N}_0} \times \mathscr{P}(X) \to \mathscr{P}^{(n)}(X)$ given by $(\boldsymbol{\beta},\sigma) \mapsto \mathcal{D}^{(n)}_{\boldsymbol{\beta},\sigma}$ is continuous for every $n \in \mathbb{N}_1$.*

Proof. Since $\mathscr{P}^{(n)}$ is a (locally) compact second countable Hausdorff for every $n \in \mathbb{N}_1$, it is sufficient to iteratively apply the continuity statement in Theorem 3.10. □

Remark 3.26. By resorting to the Alexandrov compactification of X, analogously to the proof of Theorem 3.10, Corollary 3.25 holds even if X is merely *locally* compact. Rigorously, this is however beyond our framework, since $\mathscr{P}(X)$ (hence $\mathscr{P}^{(n)}(X)$, $n \in \mathbb{N}_1$) is locally compact if and only if X is compact. Thus, if X is not compact, $\mathcal{D}_{\beta_0\sigma}$ does not satisfy our definition of intensity measure as a finite measure on a locally compact second countable Hausdorff Borel measurable space.

4 Proof of Theorem 1.4 and accessory results

4.1 Finite-dimensional statements

Multisets Given a set S, a (*finite integer-valued*) S*-multi-set* is any function $f\colon S \to \mathbb{N}_0$ such that its *cardinality* $\# f := \sum_{s\in S} f(s)$ is finite. We denote any such multiset by $[\![\mathbf{s}_{\boldsymbol{\alpha}}]\!]$, where $\mathrm{s} \in S^{\times k}$ has mutually different entries and $\boldsymbol{\alpha} := f^{\diamond}(\mathrm{s}) \in \mathbb{N}_1^k$. We term the set $[\mathrm{s}] := \{s_1, \ldots, s_k\}$ the *underlying set* to $[\![\mathbf{s}_{\boldsymbol{\alpha}}]\!]$ and put

$$[\mathbf{s}_{\boldsymbol{\alpha}}] := \{(s_1, 1), \ldots, (s_1, \alpha_1), \ldots, (s_k, 1), \ldots, (s_k, \alpha_k)\} \ .$$

4.1.1 A coloring problem

An interpretation of the moments formula (3.3) may be given in enumerative combinatorics, by means of *Pólya Enumeration Theory*, PET, see e.g., [41]. A minimal background is as follows. Let $G < \mathfrak{S}_n$ be a permutation group acting on $[n]$ and $[\mathrm{s}] := \{s_1, \ldots, s_k\}$ denote a set of (distinct) colors.

Definition 4.1 (Colorings)**.** *A* k-coloring *of $[n]$ is any function f in $[\mathbf{s}]^{[n]}$, where we understand the elements $s_1, \ldots, s_k$ of $[\mathbf{s}]$ as placeholders for different colors. Whenever these are irrelevant, given a k-coloring f of $[n]$ we denote by $\tilde{f}$ the unique function in $[k]^{[n]}$ such that $s_{\tilde{f}(\,\cdot\,)} = f(\,\cdot\,)$. We say that two k-colorings f_1, f_2 of $[n]$ are* G-equivalent *if $f_1 \circ \pi = f_2$ for all π in G. We denote the family of $[k]$-colorings of $[n]$ by $\mathcal{C}^k_n(\mathbf{s})$.*

Theorem 4.2 (Pólya [41, §4])**.** *Let $G < \mathfrak{S}_n$ be a permutation group acting on $[n]$ and $a_{h_1,\ldots,h_k}$ be the number of G-inequivalent k-colorings of $[n]$ into k colors with exactly h_i occurrences of the i^{th} color. Then, the (multivariate) generating function $\mathbf{G}[a_{h_1,\ldots,h_k}](\mathbf{t})$ satisfies*

$$\mathbf{G}[a_{h_1,\ldots,h_k}](\mathbf{t}) = Z^G\left(p_{k,1}[\mathbf{t}], \ldots, p_{k,n}[\mathbf{t}]\right) \ , \tag{4.1}$$

where $p_{k,i}[\mathbf{t}] \coloneqq \mathbf{1} \cdot \mathbf{t}^{\diamond i}$ *with* $\mathbf{1} \in \mathbb{R}^k$ *denotes the* i^{th} k-variate power-sum symmetric polynomial.

In the following we consider an extension of PET to multisets of colors, and explore its connections — arising in the case $G = \mathfrak{S}_n$ — with the Dirichlet distribution $D_{\boldsymbol{\alpha}}$. A different approach in terms of colorings, limited to the case $\boldsymbol{\alpha}_{\bullet} = 1$, was briefly sketched in [26, §7]. The purpose of this section is that to revisit the key idea of *'color-respecting' permutations* in [26, p. 112] in the well-established framework of PET.

Let now $[\![\mathbf{s}_{\boldsymbol{\alpha}}]\!]$ be an integer-valued multiset with $\boldsymbol{\alpha} \in \mathbb{N}_1^k$, henceforth a *palette*. As before, we understand the elements $s_1, \ldots, s_k$ of its underlying set $[\mathbf{s}]$ as placeholders for different colors, and the elements $(s_i, 1), \ldots, (s_i, \alpha_i)$ of $[\mathbf{s}_{\boldsymbol{\alpha}}]$ as placeholders for different *shades* of the same color s_i.

Definition 4.3 (Shadings)**.** *An* $\boldsymbol{\alpha}$-shading *of* $[n]$ *is any function* ϕ *in* $[\mathbf{s}_{\boldsymbol{\alpha}}]^{[n]}$*, where* $\boldsymbol{\alpha} \in \mathbb{N}_1^k$*. To each* $\boldsymbol{\alpha}$*-shading of* $[n]$ *we associate uniquely a* $[k]$*-coloring of* $[n]$ *by letting* $f(\,\cdot\,) \coloneqq \phi(\,\cdot\,)_1$*. Here, by* $\phi(\,\cdot\,)_1$ *we mean the first element of the pair* $\phi(\,\cdot\,)$*. This association (trivially surjective) just amounts to forget information about the shade and only retain information about the color. We say that two* $\boldsymbol{\alpha}$*-shadings* ϕ_1, ϕ_2 *of* $[n]$ *are* G-equivalent *if so are the corresponding* $[k]$*-colorings* f_1, f_2 *of* $[n]$*. We denote the family of* $\boldsymbol{\alpha}$*-shadings of* $[n]$ *by* $\mathcal{S}_n^k(\mathbf{s}_{\boldsymbol{\alpha}})$*.*

Corollary 4.4 (Counting shadings)**.** *Let* $G < \mathfrak{S}_n$ *be a permutation group acting on* $[n]$ *and* $b^{\boldsymbol{\alpha}}_{h_1,\ldots,h_k}$ *be the number of* G*-inequivalent* $\boldsymbol{\alpha}$*-shadings of* $[n]$ *with exactly* h_i *occurrences of the* i^{th} *color. Then,*

$$\mathbf{G}[b^{\boldsymbol{\alpha}}_{h_1,\ldots,h_k}](\mathbf{s}) = Z^G\left(\boldsymbol{\alpha}\cdot\mathbf{s}^{\diamond 1}, \ldots, \boldsymbol{\alpha}\cdot\mathbf{s}^{\diamond n}\right) \ , \quad \mathbf{s} \in \mathbb{R}^k \ .$$

Proof. For each $i \leq k$ set $r_i \coloneqq (\alpha_1, \ldots, \alpha_i)_{\bullet}$ and $r_0 \coloneqq 0$. Note that $r_k = \boldsymbol{\alpha}_{\bullet}$. For every r_k-coloring g of $[n]$ let

$$\phi_{\boldsymbol{\alpha}}[g](x) \coloneqq (s_i, \tilde{g}(x) - r_{i-1}) \quad \text{if} \quad \tilde{g}(x) \in \{r_{i-1}+1, \ldots, r_i\}$$

varying $i \in [k]$ and $x \in [n]$. It is readily seen that, for every fixed $\boldsymbol{\alpha} \in \mathbb{N}_1^k$, the map

$$\begin{aligned} Q_{\boldsymbol{\alpha}} \colon \mathcal{C}_n^{\boldsymbol{\alpha}_{\bullet}}(\mathbf{s}) &\longrightarrow \mathcal{S}_n^k(\mathbf{s}_{\boldsymbol{\alpha}}) \\ g &\longmapsto \phi_{\boldsymbol{\alpha}}[g] \end{aligned}$$

is bijective and preserves G-equivalence. Thus, the number $a_{h_{1,1},\ldots,h_{1,\alpha_1},\ldots,h_{k,1},\ldots,h_{k,\alpha_k}}$ of G-inequivalent r_k-colorings of $[n]$ with exactly $h_{i,j}$ occurrences of the $(r_{i-1}+j)^{\text{th}}$ color is also the number of G-inequivalent $\boldsymbol{\alpha}$-shadings of $[n]$ with exactly $h_{i,j}$ occurrences of the j^{th} shade of the i^{th} color. By Theorem 4.2 this is the coefficient of the monomial $t_1^{h_{1,1}} \cdots t_{r_1}^{h_{1,\alpha_1}} \cdots t_{r_{k-1}+1}^{h_{k,1}} \cdots t_{r_k}^{h_{k,\alpha_k}}$ in $Z^G\left(\mathbf{1}\cdot\mathbf{t}, \ldots, \mathbf{1}\cdot\mathbf{t}^{\diamond n}\right)$ with $\mathbf{1} \in \mathbb{R}^{r_k}$. By definition,

$$b^{\boldsymbol{\alpha}}_{h_1,\ldots,h_k} = \sum_{\substack{h_{1,1},\ldots,h_{1,\alpha_1},\ldots,h_{k,1},\ldots,h_{k,\alpha_k} \\ \sum_{j=1}^{\alpha_i} h_{i,j} = h_i}} a_{h_{1,1},\ldots,h_{1,\alpha_1},\ldots,h_{k,1},\ldots,h_{k,\alpha_k}} \ ,$$

which equals the coefficient of the monomial $s_1^{h_1} \ldots s_k^{h_k}$ in

$$Z^G\left(\mathbf{1}\cdot\mathbf{t}^{\diamond 1}, \ldots, \mathbf{1}\cdot\mathbf{t}^{\diamond n}\right) = Z^G\left(\boldsymbol{\alpha}\cdot\mathbf{s}^{\diamond 1}, \ldots, \boldsymbol{\alpha}\cdot\mathbf{s}^{\diamond n}\right) \ , \quad \mathbf{t} \coloneqq (\underbrace{s_1, \ldots, s_1}_{\alpha_1}, \ldots, \underbrace{s_k, \ldots, s_k}_{\alpha_k}) \ . \qquad \square$$

Corollary 4.5. *For fixed* $\boldsymbol{\alpha} \in \mathbb{N}_1^k$*, denote by* $\mathcal{S}_n^{\boldsymbol{\alpha}}$ *the set of* $\mathfrak{S}_n$*-equivalence classes* $\phi^{\bullet}$ *of* $\boldsymbol{\alpha}$*-shadings of* $[n]$*. Then, the probability* $p^{\boldsymbol{\alpha}}_{h_1,\ldots,h_k}$ *of some* $\phi^{\bullet}$ *uniformly drawn from* $\mathcal{S}_n^{\boldsymbol{\alpha}}$ *having exactly* h_i *occurrences of the* i^{th} *color satisfies*

$$\mathbf{G}[p^{\boldsymbol{\alpha}}_{h_1,\ldots,h_k}](\mathbf{s}) = \mu'_n[\mathbf{s}, \boldsymbol{\alpha}] \ .$$

Proof. By Corollary 4.4,

$$\#\mathcal{S}_n^{\boldsymbol{\alpha}} = \sum_{\substack{\mathbf{h}\in\mathbb{N}_0^k \\ \mathbf{h}_\bullet = n}} b_{\mathbf{h}}^{\boldsymbol{\alpha}} = \mathbf{G}[b_{h_1,\ldots,h_k}^{\boldsymbol{\alpha}}](\mathbf{1}) = \langle\boldsymbol{\alpha}_\bullet\rangle_n / n! \tag{4.2}$$

By definition, $p_{h_1,\ldots,h_k}^{\boldsymbol{\alpha}} = (\#\mathcal{S}_n^{\boldsymbol{\alpha}})^{-1} b_{h_1,\ldots,h_k}^{\boldsymbol{\alpha}}$, hence, by homogeneity,

$$\mathbf{G}[p_{h_1,\ldots,h_k}^{\boldsymbol{\alpha}}](\mathbf{s}) = (\#\mathcal{S}_n^{\boldsymbol{\alpha}})^{-1}\,\mathbf{G}[b_{h_1,\ldots,h_k}^{\boldsymbol{\alpha}}](\mathbf{s})\ .$$

The conclusion follows by (4.2), Corollary 4.4 and Theorem 3.3. □

The study of $D_{\boldsymbol{\alpha}}$ in the case when $\boldsymbol{\alpha}_\bullet = 1$ is singled out as computationally easiest (as suggested by Theorem 3.3, noticing that $\langle 1\rangle_n = n!$), $\boldsymbol{\alpha}$ representing in that case a *probability* on $[k]$, as detailed in §2. For these reasons, this is often the only case considered; cf. e.g., [26]. On the other hand though, the general case when $\boldsymbol{\alpha} > \mathbf{0}$ is the one relevant in Bayesian non-parametrics, since posterior distributions of Dirichlet-categorical and Dirichlet-multinomial priors do not have probability intensity. The above coloring problem suggests that the case when $\boldsymbol{\alpha} \in \mathbb{N}_1^k$ is interesting from the point of view of PET, since it allows for some natural operations on palettes, corresponding to functionals of the distribution.

Indeed, we can change the number of colors and shades in a palette $[\![\mathbf{s}_{\boldsymbol{\alpha}}]\!]$ by composing any permutation of the indices $[k]$ with the following elementary operations:

- (i) 'widen', resp. (ii) 'narrow', 'the color spectrum', by adding a color, say s_{k+1}, resp. removing a color, say s_k. That is, we consider new palettes $[\![(\mathbf{s}\oplus s_{k+1})_{\boldsymbol{\alpha}\oplus\alpha_{k+1}}]\!]$, resp. $[\![(s_1,\ldots,s_{k-1})_{(\alpha_1,\ldots,\alpha_{k-1})}]\!]$;
- (iii) 'reduce color resolution' by regarding two different colors, say s_i and s_{i+1}, as the same, relabeled s_i. In so doing we regard the shades of the former colors as distinct shades of the new one, so that it has $\alpha_i + \alpha_{i+1}$ shades. That is, we consider the new palette $[\![(\mathbf{s}_{\hat{\imath}})_{\boldsymbol{\alpha}_{+i}}]\!]$;
- (iv) 'enlarge', resp. (v) 'reduce the color depth', by adding a shade, say the α_{i+1}^{th}, to the color s_i, resp. removing a shade, say the α_i^{th}, to the color s_i. This latter operation we allow only if $\alpha_i > 1$, so to make it distinct from removing the color s_i from the palette. That is, we consider the new palettes $[\![\mathbf{s}_{\boldsymbol{\alpha}+\mathbf{e}_i}]\!]$, resp. $[\![\mathbf{s}_{\boldsymbol{\alpha}-\mathbf{e}_i}]\!]$ when $\alpha_i > 1$.

Increasing the color resolution of a multi-shaded color, say s_k with $\alpha_k > 1$ shades, by splitting it into two colors, say s_k^* and s_{k+1} with $\alpha_k^* > 0$ and $\alpha_{k+1} > 0$ shades respectively and such that $\alpha_k^* + \alpha_{k+1} = \alpha_k$, is not an elementary operation. It can be obtained by widening the spectrum of the palette by adding a color s_{k+1} with α_{k+1} shades and reducing the color depth of the color s_k to α_k^*. Thus, this operation is not listed above. We do not allow for the number of shades of a color to be reduced to zero: although this is morally equivalent to removing that color, the latter operation amounts more rigorously to remove the color placeholder from the palette.

Remark 4.6. The said elementary operations are of two distinct kinds: (i)–(iii) alter the number k of colors in a palette, while (iv)–(v) fix it. Operations (i)–(iii) may be understood as ways to move across some projective system of two-level towers, visualized as the first two levels of the towers in Figure 1. Understanding as usual $\boldsymbol{\alpha} = \sum_{i=1}^k \alpha_i\delta_i$ as a measure on $[k]$, then: operation (iii) corresponds to the pair of degeneracy maps $(\mathfrak{s}^i, \mathfrak{s}_\sharp^i)$. It is the only operation among (i)–(iii) that keeps the total mass of $\boldsymbol{\alpha}$ fixed. Operation (ii) corresponds to $\left(\mathfrak{s}^k, \cdot\big|_{[k-1]}\right)$, where $\cdot\big|_{[k-1]}$ denotes the restriction of a measure on $[k]$ to a measure on $[k-1]$; finally, operation (i) corresponds to $\left(\mathfrak{d}^k, \alpha_k\delta_k + \mathfrak{d}_\sharp^k\right)$, where $\mathfrak{d}^k\colon [k-1] \to [k]$ is the k^{th} *face map*, the adjoint to $\mathfrak{s}^k$.

We restrict now our attention to the operations (iv)–(v) and ask how the probability $p^{\boldsymbol\alpha}_{h_1,\dots,h_k}$ changes under them. By Corollary 4.5 this is equivalent to study the corresponding functionals of the n^{th} moment of the Dirichlet distribution. For fixed k, we address all the moments at once, by studying the moment generating function

$$ {}_k\Phi_2[\boldsymbol\alpha;\boldsymbol\alpha_\bullet;t\mathbf{s}] = \mathbf{G}_{\exp}[\mathbf{G}[p^{\boldsymbol\alpha}_{h_1,\dots,h_k}](\mathbf{s})](t) \ . $$

Namely, we seek natural transformations yielding the mappings $E_{\pm i}$ defined by

$$ E_{\pm i}\ {}_k\Phi_2[\boldsymbol\alpha;\boldsymbol\alpha_\bullet;\mathbf{s}] = C_{\boldsymbol\alpha}\ {}_k\Phi_2[\boldsymbol\alpha\pm\mathbf{e}_i;\boldsymbol\alpha_\bullet\pm 1;\mathbf{s}] \ , \tag{4.3} $$

where $C_{\boldsymbol\alpha}$ is some constant, possibly dependent on $\boldsymbol\alpha$. Here 'natural' means that we only allow for meaningful *linear* operations on generating functions: addition, scalar multiplication by variables or constants, differentiation and integration. For practical reasons, it is convenient to consider the following construction.

Definition 4.7 (Dynamical symmetry algebra of ${}_k\Phi_2$)**.** *Denote by $\mathfrak{g}_k$ the minimal semi-simple Lie algebra containing the linear span of the operators $E_{\pm 1},\dots,E_{\pm k}$ in* (4.3) *and endowed with the bracket induced by their composition. Following [36], we term the Lie algebra $\mathfrak{g}_k$ the* dynamical symmetry algebra *of the function ${}_k\Phi[\boldsymbol\alpha;\mathbf{s}] \coloneqq {}_k\Phi_2[\boldsymbol\alpha;\boldsymbol\alpha_\bullet;\mathbf{s}]$, characterized below.*

4.1.2 Dynamical symmetry algebras

We compute now the dynamical symmetry algebra of the function ${}_k\Phi[\boldsymbol\alpha;\mathbf{s}] \coloneqq {}_k\Phi_2[\boldsymbol\alpha;\boldsymbol\alpha_\bullet;\mathbf{s}]$, in this section always regarded as the meromorphic extension (2.9) of the Fourier transform of $\widehat{D_{\boldsymbol\alpha}}(\mathbf{s})$ in the *complex* variables $\boldsymbol\alpha,\mathbf{s}\in\mathbb{C}^k$. The choice of complex variables is merely motivated by this identification and every result in the following concerned with complex Lie algebras holds verbatim for their split real form. For dynamical symmetry algebras of Lauricella hypergeometric functions see [37, 36] and references therein; we refer to [21] for the general theory of Lie algebra (representations) and for Weyl groups' theory.

Notation and definitions Denote by $\mathbf{E}_{i,j}$ varying $i,j\in[k+1]$ the canonical basis of $\mathrm{Mat}_{k+1}(\mathbb{C})$, with $[\mathbf{E}_{i,j}]_{m,n}=\delta_{mi}\,\delta_{nj}$, where δ_{ab} is the Kronecker delta, and by A^* the conjugate transpose of a matrix A. In the following Lemma, we set $i'\coloneqq i+1$, and analogously for all other variables.

Lemma 4.8 (A presentation of $\mathfrak{sl}_{k+1}(\mathbb{C})$)**.** *For $i,j=0,\dots,k$ with $j>i$ let*

$$ e_{i,j}\coloneqq\mathbf{E}_{i',j'} \ , \quad h_{i,j}\coloneqq\mathbf{E}_{i',i'}-\mathbf{E}_{j',j'} \ , \quad f_{j,i}\coloneqq e^*_{i,j} \ . $$

Then, the complex Lie sub-algebra $\mathfrak{l}_k$ of $\mathfrak{gl}_{k+1}(\mathbb{C})$ generated by these vectors is $\mathfrak{l}_k=\mathfrak{sl}_{k+1}(\mathbb{C})$, with generators the $\mathfrak{sl}_2$-triples $\{e_{i,i'},h_{i,i'},f_{i',i}\}_{i=0,\dots,k-1}$. Denote further by $\mathfrak{f}_k<\mathfrak{l}_k$ the sub-algebra spanned by $\{e_{i,j},f_{j,i},h_{i,j}\}_{i,j\in[k]}$. Then, $\mathfrak{f}_k\cong\mathfrak{sl}_k(\mathbb{C})$.

Proof. It straightforward to verify Serre's relations of type A; see e.g., [21, §18.1]. □

Everywhere in the following we regard $\mathfrak{l}_k$ together with the distinguished Cartan sub-algebra $\mathfrak{h}_k<\mathfrak{l}_k$ of diagonal traceless matrices spanned by the basis $\{h_{0,j}\}_{j\in[k]}$; the root system Ψ_k induced by $\mathfrak{h}_k$, with simple roots γ_j corresponding to the $\mathfrak{sl}_2$-triples of the vectors $e_{i,i'}$ for $i\in[k]$; positive, resp. negative, roots $\Psi^\pm_k$ corresponding to the spaces of strictly upper, resp. strictly lower, triangular matrices $\mathfrak{n}^\pm_k$. The inclusion $\mathfrak{f}_k<\mathfrak{l}_k$ induces the decomposition of vector spaces (*not* of algebras)

$$ \mathfrak{l}_k=\mathfrak{r}^-_k\oplus\mathfrak{h}_1\oplus\mathfrak{f}_k\oplus\mathfrak{r}^+_k \ , \quad \text{where} \quad \mathfrak{r}^+_k\coloneqq\mathbb{C}\{e_{0,j}\}_{j\in[k]} \ , \quad \mathfrak{r}^-_k\coloneqq\mathbb{C}\{f_{j,0}\}_{j\in[k]} \ , \quad \mathfrak{h}_1=\mathbb{C}\{h_{0,1}\} \, . $$

For fixed $\boldsymbol\alpha\in\mathbb{C}^k$ regard ${}_k\Phi[\boldsymbol\alpha;\,\cdot\,]$ as a formal power series and let $f_{\boldsymbol\alpha}\colon \mathbb{C}^{2k+1}_{\mathbf{s},\mathbf{u},t}\longrightarrow\mathbb{C}$ be

$$f_{\boldsymbol\alpha}=f_{\boldsymbol\alpha}(\mathbf{s},\mathbf{u},t)\coloneqq{}_k\Phi[\boldsymbol\alpha;\mathbf{s}]\mathbf{u}^{\boldsymbol\alpha}t^{\boldsymbol\alpha_\bullet}\ . \tag{4.4}$$

Let $\mathrm{A}\subset\mathbb{C}^k$. It is readily seen that the functions $\{f_{\boldsymbol\alpha}\}_{\boldsymbol\alpha\in\mathrm{A}}$ are (finitely) linearly independent, since so are the functions $\{f_{\boldsymbol\alpha}(\mathbf{1},\mathbf{u},1)\propto\mathbf{u}^{\boldsymbol\alpha}\}_{\boldsymbol\alpha\in\mathrm{A}}$. Set

$$\mathcal{O}_{\mathrm{A}}\coloneqq\bigoplus_{\boldsymbol\alpha\in\mathrm{A}}\mathbb{C}\{f_{\boldsymbol\alpha}\}\ ,\quad \mathcal{O}_{\boldsymbol\alpha}\coloneqq\mathcal{O}_{\{\boldsymbol\alpha\}}\ ,\quad \mathcal{O}\coloneqq\mathcal{O}_{\mathbb{C}^k}$$

and define the following differential operators, acting formally on $\mathcal{O}$,

$$\begin{aligned} E_{\alpha_i}\coloneqq&u_it(s_i\partial_{s_i}+u_i\partial_{u_i}-(\mathbf{s}\cdot\nabla^{\mathbf{s}})\partial_{s_i})\ , & E_{\alpha_i,-\alpha_j}\coloneqq&u_iu_j^{-1}\left((u_i-u_j)\partial_{s_i}+u_i\partial_{u_i}\right)\ ,\\ E_{-\alpha_i}\coloneqq&(u_it)^{-1}(s_i-\mathbf{s}\cdot\nabla^{\mathbf{s}}-t\partial_t+1)\ , & J_{\alpha_i}\coloneqq&t\partial_t+u_i\partial_{u_i}-1\,, \end{aligned}\tag{4.5}$$

where $i,j\in[k]$, $i\neq j$, and $\nabla^{\mathbf{y}}\coloneqq(\partial_{y_1},\dots,\partial_{y_k})$ for $\mathbf{y}=\mathbf{u},\mathbf{s}$.

Definition 4.9. *We term the operators E_{α_i}, resp. $E_{-\alpha_i}$ defined in* (4.5)*,* raising*, resp.* lowering, operators*, and define $\mathfrak{g}_k$ to be the complex linear span of all the operators* (4.5) *endowed with the bracket induced by their composition.*

The operators $E_{\pm\alpha_i}$ are a concrete realization of the operators $E_{\pm i}$ defined in (4.3), as differential operators acting on basis vectors of the form (4.4). We will verify in Lemma 4.14 below that the vector space $\mathfrak{g}_k$ above is a Lie algebra, and a concrete representation of the dynamical symmetry algebra introduced in Definition 4.7.

Actions on spaces of holomorphic functions For $\boldsymbol\alpha\in\mathbb{C}^k$ set $\Lambda_{\boldsymbol\alpha}\coloneqq\boldsymbol\alpha+\mathbb{Z}^k$ and, for every $\ell\in\mathbb{R}_+$,

$$\Lambda^+_{\boldsymbol\alpha}\coloneqq\{\boldsymbol\epsilon\in\Lambda_{\boldsymbol\alpha}:\boldsymbol\epsilon_\bullet>0\}\ ,\quad H^\pm_{\boldsymbol\alpha}\coloneqq\boldsymbol\alpha\pm\mathbb{N}_0^k\ ,\quad M_{\boldsymbol\alpha,\ell}\coloneqq\left\{\boldsymbol\epsilon\in\Lambda^+_{\boldsymbol\alpha}:\boldsymbol\epsilon_\bullet=\ell\right\}\ .$$

If $\Re^\diamond\boldsymbol\alpha>\mathbf{0}$, the space $\mathcal{O}_{\Lambda^+_{\boldsymbol\alpha}}$ is a space of holomorphic functions $\mathcal{O}\big(\mathbb{C}^k_{\mathbf{s}}\times(\mathbb{C}\setminus\mathbb{R}_0^-)^{k+1}_{\mathbf{u},t}\big)$, where we choose $\mathbb{R}^-_0$ as branch cut for the complex logarithm in the variables $\mathbf{u}$ and t. The same holds for $\mathcal{O}_{\Lambda_{\boldsymbol\alpha}}$ if $\boldsymbol\alpha_\bullet\notin\mathbb{Z}$.

Lemma 4.10 (Raising/lowering actions)**.** *The operators* (4.5) *satisfy, for $i,j\in[k]$, $j\neq i$,*

$$\begin{aligned} E_{\alpha_i}f_{\boldsymbol\alpha}=&\alpha_if_{\boldsymbol\alpha+\mathbf{e}_i}\ , & E_{-\alpha_i}f_{\boldsymbol\alpha}=&(1-\boldsymbol\alpha_\bullet)f_{\boldsymbol\alpha-\mathbf{e}_i}\ ,\\ E_{\alpha_i,-\alpha_j}f_{\boldsymbol\alpha}=&\alpha_if_{\boldsymbol\alpha+\mathbf{e}_i-\mathbf{e}_j}\ , & J_{\alpha_i}f_{\boldsymbol\alpha}=&(\boldsymbol\alpha_\bullet+\alpha_i-1)f_{\boldsymbol\alpha}\ . \end{aligned}\tag{4.6}$$

Proof. The statement on J_{α_i} is straightforward. Moreover,

$$\begin{aligned} E_{\alpha_i,-\alpha_j}f_{\boldsymbol\alpha}=&\mathbf{u}^{\boldsymbol\alpha+\mathbf{e}_i-\mathbf{e}_j}t^{\boldsymbol\alpha_\bullet}\left(\sum_{\mathbf{m}\geq\mathbf{0}}\frac{\langle\boldsymbol\alpha\rangle_{\mathbf{m}}(m_i+\alpha_i)\mathbf{s}^{\mathbf{m}}}{\langle\boldsymbol\alpha_\bullet\rangle_{\mathbf{m}_\bullet}\mathbf{m}!}-\frac{\langle\boldsymbol\alpha\rangle_{\mathbf{m}}m_i\mathbf{s}^{\mathbf{m}-\mathbf{e}_i+\mathbf{e}_j}}{\langle\boldsymbol\alpha_\bullet\rangle_{\mathbf{m}_\bullet}\mathbf{m}!}\right)\\ =&\mathbf{u}^{\boldsymbol\alpha+\mathbf{e}_i-\mathbf{e}_j}t^{\boldsymbol\alpha_\bullet}\left(\sum_{\mathbf{m}\geq\mathbf{0}}\frac{\langle\boldsymbol\alpha\rangle_{\mathbf{m}}(m_i+\alpha_i)\mathbf{s}^{\mathbf{m}}}{\langle\boldsymbol\alpha_\bullet\rangle_{\mathbf{m}_\bullet}\mathbf{m}!}-\frac{\langle\boldsymbol\alpha\rangle_{\mathbf{m}+\mathbf{e}_i-\mathbf{e}_j}(m_i+1)\mathbf{s}^{\mathbf{m}}}{\langle\boldsymbol\alpha_\bullet\rangle_{\mathbf{m}_\bullet}(\mathbf{m}+\mathbf{e}_i-\mathbf{e}_j)!}\right)\\ =&\mathbf{u}^{\boldsymbol\alpha+\mathbf{e}_i-\mathbf{e}_j}t^{\boldsymbol\alpha_\bullet}\frac{\alpha_i}{\alpha_j-1}\times\\ &\times\left(\sum_{\mathbf{m}\geq\mathbf{0}}\frac{\langle\boldsymbol\alpha+\mathbf{e}_i-\mathbf{e}_j\rangle_{\mathbf{m}-\mathbf{e}_i+\mathbf{e}_j}(m_i+\alpha_i)\mathbf{s}^{\mathbf{m}}}{\langle\boldsymbol\alpha_\bullet\rangle_{\mathbf{m}_\bullet}\mathbf{m}!}-\frac{\langle\boldsymbol\alpha+\mathbf{e}_i-\mathbf{e}_j\rangle_{\mathbf{m}}\mathbf{s}^{\mathbf{m}}}{\langle\boldsymbol\alpha_\bullet\rangle_{\mathbf{m}_\bullet}(\mathbf{m}-\mathbf{e}_j)!}\right)\\ =&\mathbf{u}^{\boldsymbol\alpha+\mathbf{e}_i-\mathbf{e}_j}t^{\boldsymbol\alpha_\bullet}\frac{\alpha_i}{\alpha_j-1}\times \end{aligned}$$

$$
\begin{aligned}
&\times\left(\sum_{\mathbf{m}\geq\mathbf{0}} \frac{\langle\boldsymbol{\alpha}+\mathbf{e}_i-\mathbf{e}_j\rangle_{\mathbf{m}}\,(m_j+\alpha_j-1)\mathbf{s}^{\mathbf{m}}}{\langle\boldsymbol{\alpha}_\bullet\rangle_{\mathbf{m}_\bullet}\,\mathbf{m}!} - \frac{\langle\boldsymbol{\alpha}+\mathbf{e}_i-\mathbf{e}_j\rangle_{\mathbf{m}}\,m_j\mathbf{s}^{\mathbf{m}}}{\langle\boldsymbol{\alpha}_\bullet\rangle_{\mathbf{m}_\bullet}\,\mathbf{m}!}\right)\\
=&\alpha_i f_{\boldsymbol{\alpha}+\mathbf{e}_i-\mathbf{e}_j}\ ,\\
E_{\alpha_i} f_{\boldsymbol{\alpha}} =&\mathbf{u}^{\boldsymbol{\alpha}+\mathbf{e}_i} t^{\boldsymbol{\alpha}_\bullet+1}\left(\sum_{\mathbf{m}\geq\mathbf{0}} \frac{\langle\boldsymbol{\alpha}\rangle_{\mathbf{m}}\,(m_i+\alpha_i)\mathbf{s}^{\mathbf{m}}}{\langle\boldsymbol{\alpha}_\bullet\rangle_{\mathbf{m}_\bullet}\,\mathbf{m}!} - \frac{\langle\boldsymbol{\alpha}\rangle_{\mathbf{m}}\,m_i(\mathbf{m}_\bullet-1)\mathbf{s}^{\mathbf{m}-\mathbf{e}_i}}{\langle\boldsymbol{\alpha}_\bullet\rangle_{\mathbf{m}_\bullet}\,\mathbf{m}!}\right)\\
=&\mathbf{u}^{\boldsymbol{\alpha}+\mathbf{e}_i} t^{\boldsymbol{\alpha}_\bullet+1}\left(\sum_{\mathbf{m}\geq\mathbf{0}} \frac{\langle\boldsymbol{\alpha}\rangle_{\mathbf{m}+\mathbf{e}_i}\,\mathbf{s}^{\mathbf{m}}}{\langle\boldsymbol{\alpha}_\bullet\rangle_{\mathbf{m}_\bullet}\,\mathbf{m}!} - \frac{\langle\boldsymbol{\alpha}\rangle_{\mathbf{m}}\,(\mathbf{m}_\bullet-1)\mathbf{s}^{\mathbf{m}-\mathbf{e}_i}}{\langle\boldsymbol{\alpha}_\bullet\rangle_{\mathbf{m}_\bullet}\,(\mathbf{m}-\mathbf{e}_i)!}\right)\\
=&\mathbf{u}^{\boldsymbol{\alpha}+\mathbf{e}_i} t^{\boldsymbol{\alpha}_\bullet+1}\left(\sum_{\mathbf{m}\geq\mathbf{0}} \frac{\langle\boldsymbol{\alpha}\rangle_{\mathbf{m}+\mathbf{e}_i}\,\mathbf{s}^{\mathbf{m}}}{\langle\boldsymbol{\alpha}_\bullet\rangle_{\mathbf{m}_\bullet}\,\mathbf{m}!} - \frac{\langle\boldsymbol{\alpha}\rangle_{\mathbf{m}+\mathbf{e}_i}\,\mathbf{m}_\bullet\mathbf{s}^{\mathbf{m}}}{\langle\boldsymbol{\alpha}_\bullet\rangle_{\mathbf{m}_\bullet+1}\,\mathbf{m}!}\right)\\
=&\mathbf{u}^{\boldsymbol{\alpha}+\mathbf{e}_i} t^{\boldsymbol{\alpha}_\bullet+1}\frac{\alpha_i}{\boldsymbol{\alpha}_\bullet}\left(\sum_{\mathbf{m}\geq\mathbf{0}} \frac{\langle\boldsymbol{\alpha}+\mathbf{e}_i\rangle_{\mathbf{m}}\,\mathbf{s}^{\mathbf{m}}}{\langle\boldsymbol{\alpha}_\bullet+1\rangle_{\mathbf{m}_\bullet-1}\,\mathbf{m}!} - \frac{\langle\boldsymbol{\alpha}+\mathbf{e}_i\rangle_{\mathbf{m}}\,\mathbf{m}_\bullet\mathbf{s}^{\mathbf{m}}}{\langle\boldsymbol{\alpha}_\bullet+1\rangle_{\mathbf{m}_\bullet}\,\mathbf{m}!}\right)\\
=&\mathbf{u}^{\boldsymbol{\alpha}+\mathbf{e}_i} t^{\boldsymbol{\alpha}_\bullet+1}\frac{\alpha_i}{\boldsymbol{\alpha}_\bullet}\left(\sum_{\mathbf{m}\geq\mathbf{0}} \frac{\langle\boldsymbol{\alpha}+\mathbf{e}_i\rangle_{\mathbf{m}}\,\mathbf{s}^{\mathbf{m}}(\boldsymbol{\alpha}_\bullet+\mathbf{m}_\bullet)}{\langle\boldsymbol{\alpha}_\bullet+1\rangle_{\mathbf{m}_\bullet}\,\mathbf{m}!} - \frac{\langle\boldsymbol{\alpha}+\mathbf{e}_i\rangle_{\mathbf{m}}\,\mathbf{m}_\bullet\mathbf{s}^{\mathbf{m}}}{\langle\boldsymbol{\alpha}_\bullet+1\rangle_{\mathbf{m}_\bullet}\,\mathbf{m}!}\right)\\
=&\alpha_i f_{\boldsymbol{\alpha}+\mathbf{e}_i}\ ,\\
E_{-\alpha_i} f_{\boldsymbol{\alpha}} =&\mathbf{u}^{\boldsymbol{\alpha}-\mathbf{e}_i} t^{\boldsymbol{\alpha}_\bullet-1}\left(\sum_{\mathbf{m}\geq\mathbf{0}} \frac{\langle\boldsymbol{\alpha}\rangle_{\mathbf{m}}\,\mathbf{s}^{\mathbf{m}+\mathbf{e}_i}}{\langle\boldsymbol{\alpha}_\bullet\rangle_{\mathbf{m}_\bullet}\,\mathbf{m}!} - \frac{\langle\boldsymbol{\alpha}\rangle_{\mathbf{m}}\,\mathbf{s}^{\mathbf{m}}}{\langle\boldsymbol{\alpha}_\bullet\rangle_{\mathbf{m}_\bullet}\,\mathbf{m}!}(\mathbf{m}_\bullet+\boldsymbol{\alpha}_\bullet-1)\right)\\
=&\mathbf{u}^{\boldsymbol{\alpha}-\mathbf{e}_i} t^{\boldsymbol{\alpha}_\bullet-1}\left(\sum_{\mathbf{m}\geq\mathbf{0}} \frac{\langle\boldsymbol{\alpha}\rangle_{\mathbf{m}-\mathbf{e}_i}\,m_i\mathbf{s}^{\mathbf{m}}}{\langle\boldsymbol{\alpha}_\bullet\rangle_{\mathbf{m}_\bullet-1}\,\mathbf{m}!} - \frac{\langle\boldsymbol{\alpha}\rangle_{\mathbf{m}}\,\mathbf{s}^{\mathbf{m}}}{\langle\boldsymbol{\alpha}_\bullet\rangle_{\mathbf{m}_\bullet}\,\mathbf{m}!}(\mathbf{m}_\bullet+\boldsymbol{\alpha}_\bullet-1)\right)\\
=&\mathbf{u}^{\boldsymbol{\alpha}-\mathbf{e}_i} t^{\boldsymbol{\alpha}_\bullet-1}\frac{\boldsymbol{\alpha}_\bullet-1}{\alpha_i-1}\times\\
&\times\left(\sum_{\mathbf{m}\geq\mathbf{0}} \frac{\langle\boldsymbol{\alpha}-\mathbf{e}_i\rangle_{\mathbf{m}}\,m_i\mathbf{s}^{\mathbf{m}}}{\langle\boldsymbol{\alpha}_\bullet-1\rangle_{\mathbf{m}_\bullet}\,\mathbf{m}!} - \frac{\langle\boldsymbol{\alpha}-\mathbf{e}_i\rangle_{\mathbf{m}}\,(\alpha_i+m_i-1)\mathbf{s}^{\mathbf{m}}}{\langle\boldsymbol{\alpha}_\bullet-1\rangle_{\mathbf{m}_\bullet}\,(\boldsymbol{\alpha}_\bullet+\mathbf{m}_\bullet-1)\mathbf{m}!}(\mathbf{m}_\bullet+\boldsymbol{\alpha}_\bullet-1)\right)\\
=&(1-\boldsymbol{\alpha}_\bullet)f_{\boldsymbol{\alpha}-\mathbf{e}_i}\ . \qquad \square
\end{aligned}
$$

Remark 4.11. The variables $\mathbf{u}$ and t are merely auxiliary; cf. [38, §1]. The operators do not depend on the parameter $\boldsymbol{\alpha}$, rather, the subscripts indicate which indices they affect. Heuristically, the action of the operators (4.5) given in Lemma 4.10 may be deduced from that [36, Eqn. (1.5)] of operators in the dynamical symmetry algebra of ${}_kF_D$ by a formal contraction procedure [36, p. 1398], letting (in the notation of [36]) $\alpha = 0$, $\boldsymbol{\beta} = \boldsymbol{\alpha}$, $\gamma = \boldsymbol{\alpha}_\bullet$ and dropping redundancies. Finally, note that the action of the operator E_{α_j} on $f_{\boldsymbol{\alpha}}$ corresponds to a differentiation in the variable s_j of the moment $\mu'_n[\mathbf{s}, \boldsymbol{\alpha}]$, as in (3.6).

Remark 4.12. If $\boldsymbol{\alpha}_\bullet = 1$, the action of the lowering operators $E_{-\alpha_i}$ vanishes on $\mathcal{O}_{\boldsymbol{\alpha}}$. This is natural when regarding $f_{\boldsymbol{\alpha}}$ as a formal power series, whereas it is conventional when regarding $f_{\boldsymbol{\alpha}}$ as a meromorphic function, for the functions $(1-\boldsymbol{\alpha}_\bullet)f_{\boldsymbol{\alpha}-\mathbf{e}_i}$ are in fact — after cancellations — well-defined, not identically vanishing, and holomorphic in s even for $\boldsymbol{\alpha}_\bullet = 1$. The convention here reads $0 \times \infty = 0$, which is consistent with the usual convention in measure theory when we identify $\boldsymbol{\alpha}_\bullet - 1$ with the quantity $(\sigma - \delta_y)X$ for any y in X; the reason for such identification will be apparent in §4.2 below.

Corollary 4.13. *The operators* (4.5) *fix* $\mathcal{O}_{\Lambda_{\boldsymbol{\alpha}}}$ *for any* $\boldsymbol{\alpha} \in \mathbb{C}^k$.

In the statement of the next Lemma and in the diagrams in Fig.s 2 and 3 we write for simplicity E_i in place of E_{α_i} and analogously for all other operators.

Lemma 4.14. *For $\boldsymbol{\alpha} \in \mathbb{C}^k$ consider the operators in $\mathfrak{g}_k$ as restricted to $\mathcal{O}_{\Lambda_{\boldsymbol{\alpha}}}$. The following commutation relations hold:*

$$= \begin{cases} 2E_{p,-q} & \text{if } j=p, i=q \\ -2E_{p,-q} & \text{if } j=q, i=p \\ E_{p,-q} & \text{if } j=p, i\neq q \\ & \quad \text{or } j\neq p, i=q \\ -E_{p,-q} & \text{if } j=q, i\neq p \\ & \quad \text{or } j\neq q, i=p \\ \mathbf{0} & \text{otherwise} \end{cases} \qquad = \begin{cases} J_i & \text{if } i=p \\ E_{i,-p} & \text{otherwise} \end{cases},$$

$$[E_i, E_{p,-q}] = \begin{cases} -E_p & \text{if } i=q \\ \mathbf{0} & \text{otherwise} \end{cases}, \qquad [E_{-i}, E_{p,-q}] = \begin{cases} E_{-q} & \text{if } i=p \\ \mathbf{0} & \text{otherwise} \end{cases},$$

$$[J_i, E_{\pm p}] = \begin{cases} \pm 2E_{\pm p} & \text{if } i=p \\ \pm E_{\pm p} & \text{otherwise} \end{cases}, \qquad [E_{j,-i}, E_{p,-q}] = \begin{cases} J_j - J_i & \text{if } j=q, i=p \\ -E_{p,-i} & \text{if } j=q, i\neq p \\ E_{j,-q} & \text{if } j\neq q, i=p \\ \mathbf{0} & \text{otherwise} \end{cases}$$

$$[J_i, J_p] = \mathbf{0}\ , \qquad [E_i, E_p] = \mathbf{0}\ ,$$

where $i, j, p, q = 1, \ldots, k$ with $i \neq j$, $p \neq q$.

Proof. Straightforward. □

Proposition 4.15. *Let $\rho\colon \mathfrak{l}_k \to \mathrm{End}(\mathcal{O})$ be the linear map defined by*

$$e_{0,i} \mapsto E_{\alpha_i}\ , \quad e_{i,j} \mapsto E_{\alpha_j,-\alpha_i}\ , \quad h_{0,i} \mapsto J_{\alpha_i}\ , \quad f_{i,0} \mapsto E_{-\alpha_i}\ , \quad f_{j,i} \mapsto E_{\alpha_i,-\alpha_j}$$

where $i, j \in [k]$ with $j > i$. Then, for any fixed $\boldsymbol{\alpha} \in \mathbb{C}^k$, the pair $\rho_{\boldsymbol{\alpha}} := (\rho(\,\cdot\,)\big|_{\mathcal{O}_{\Lambda_{\boldsymbol{\alpha}}}}, \mathcal{O}_{\Lambda_{\boldsymbol{\alpha}}})$ is a faithful Lie algebra representation of $\mathfrak{l}_k$ with image $\mathfrak{g}_k\big|_{\mathcal{O}_{\Lambda_{\boldsymbol{\alpha}}}}$. Furthermore, the functions $f_{\boldsymbol{\alpha}}$ transform as basis vectors for $\rho_{\boldsymbol{\alpha}}$, in the sense that for every v in the basis for $\mathfrak{l}_k$ and every $\boldsymbol{\epsilon}$ in $\Lambda_{\boldsymbol{\alpha}}$ there exists a unique $\bar{\boldsymbol{\epsilon}} = \bar{\boldsymbol{\epsilon}}(\boldsymbol{\epsilon}, v)$ in $\Lambda_{\boldsymbol{\alpha}}$ such that $(\rho_{\boldsymbol{\alpha}} v)\, f_{\boldsymbol{\epsilon}} \propto f_{\bar{\boldsymbol{\epsilon}}}$.

Proof. By Corollary 4.13, $\rho_{\boldsymbol{\alpha}}$ is a well-defined linear morphism into $\mathrm{End}(\mathcal{O}_{\Lambda_{\boldsymbol{\alpha}}})$. The fact that $f_{\boldsymbol{\alpha}}$ transforms as a basis vector of $\mathcal{O}_{\Lambda_{\boldsymbol{\alpha}}}$ is an immediate consequence of Lemma 4.10. For $\boldsymbol{\epsilon} \in \Lambda_{\boldsymbol{\alpha}}$ such that $\Re^\diamond \boldsymbol{\epsilon} > \mathbf{1}$, the actions of operators in (4.5) on $\mathcal{O}_{\boldsymbol{\epsilon}}$ are mutually different again by Lemma 4.10, hence $\rho_{\boldsymbol{\alpha}}$ is injective. In order to show that $\rho_{\boldsymbol{\alpha}}\mathfrak{l} = \mathfrak{g}\big|_{\mathcal{O}_{\Lambda_{\boldsymbol{\alpha}}}}$ is a Lie algebra of type A_k and that $\rho_{\boldsymbol{\alpha}}$ is a Lie algebra representation, it suffices to verify that the morphism $\rho\colon \mathfrak{l}_k \to \mathfrak{g}_k$ is a morphism of Lie algebras. Note that $h_{i,i'} = h_{0,i'} - h_{0,i}$, hence $\rho(h_{i,i'}) = J_{\alpha_{i'}} - J_{\alpha_i}$. Thus, the assertion follows by direct comparison of the commutators in Lemma 4.14 (for the choice $j > i$, $p > q$) with those of the presentation of $\mathfrak{l}_k$ given in Lemma 4.8. □

Theorem 4.16. *Let $k \in \mathbb{N}_1$. For $\boldsymbol{\alpha}$ in $\operatorname{int}\Delta^{k-1}$ and $\mathbf{p} \in \mathbb{N}_0^k$ denote by $D^{\mathbf{p}}_{\boldsymbol{\alpha}}$ the posterior distribution of $D_{\boldsymbol{\alpha}}$ given atoms of mass p_i at point $i \in [k]$. Then,*

(i) the semi-lattice $\mathcal{O}_{\Lambda^+_{\boldsymbol{\alpha}}}$ is a weight $\mathfrak{l}_k$-module and $\mathfrak{U}(\mathfrak{l}_k)$-module;

(ii) the space $\mathcal{O}_{M_{\boldsymbol{\alpha},\ell}}$ in invariant under the action of the universal enveloping algebra $\mathfrak{U}(\mathfrak{f}_k) < \mathfrak{U}(\mathfrak{l}_k)$ for all $\ell \in \mathbb{N}_1$, while $\mathcal{O}_{H^+_{\boldsymbol{\alpha}}}$ is invariant under the action of $\mathfrak{U}(\mathfrak{h}_k) \oplus \mathfrak{U}(\mathfrak{r}^+_k)$;

(iii) for every $\mathbf{p} \in \mathbb{N}_0^k$ there exists a unique $v = v(\mathbf{p}) \in \mathfrak{U}(\mathfrak{r}^+_k)$ such that $v.\mathcal{O}_{\boldsymbol{\alpha}} \cong \mathbb{C}\widehat{D^{\mathbf{p}}_{\boldsymbol{\alpha}}}$;

(iv) the canonical action of $\mathfrak{S}_k$ on $\mathscr{P}([k])$ corresponds to the natural action of the unique subgroup (isomorphic to) $\mathfrak{S}_k$ of the Weyl group of $\mathfrak{l}_k$ permuting roots corresponding to basis elements in $\mathfrak{r}^+_k$.

Proof. By (4.6), the operators $\rho\,\mathfrak{sl}_{k+1}(\mathbb{C})$ fix $\mathcal{O}_{\Lambda^+_{\boldsymbol\alpha}} \subset \mathcal{O}$, thus $\rho_{\boldsymbol\alpha}$ is a (faithful) Lie algebra representation by Proposition 4.15, hence $\mathcal{O}_{\Lambda^+_{\boldsymbol\alpha}} \subset \mathcal{O}$ is an $\mathfrak{l}_k$-module for the linear extension of the action $v.f_{\boldsymbol\epsilon} \coloneqq (\rho_{\boldsymbol\alpha} v) f_{\boldsymbol\epsilon}$ varying v in the basis of $\mathfrak{l}_k$. The extension to a representation of $\mathfrak{U}(\mathfrak{l}_k)$ is standard from the universal property of universal enveloping algebras; see, e.g., [21, §17.2].

In order to prove (i)-(ii) it suffices to show that, for all $\boldsymbol\epsilon \in \Lambda^+_{\boldsymbol\alpha}$ and $\ell \in \mathbb{N}_1$, one has

$$h_{0,i}.f_{\boldsymbol\epsilon} = (\boldsymbol\epsilon_\bullet - 1 + \varepsilon_i) f_{\boldsymbol\epsilon}\ , \quad v.\mathcal{O}_{M_{\boldsymbol\epsilon,\ell}} \subset \mathcal{O}_{M_{\boldsymbol\epsilon,\ell}}\ , \quad w.\mathcal{O}_{H^+_{\boldsymbol\alpha}} \subset \mathcal{O}_{H^+_{\boldsymbol\alpha}}$$

for all $i \in [k]$, v in the basis of $\mathfrak{f}_k$, $\ell \in \mathbb{N}_1$ and w in the basis for $\mathfrak{h}_k \oplus \mathfrak{r}^+_k$. All of the above follow immediately from Lemma 4.10. Notably $\mathfrak{h}_k$ acts on $\mathcal{O}_{\boldsymbol\alpha}$ precisely by weight $\boldsymbol\alpha$ since $\boldsymbol\alpha_\bullet = 1$.

Since $\boldsymbol\alpha \in \Delta^{k-1}$, then $f_{\boldsymbol\alpha+\mathbf{p}}(\,\cdot\,, \mathbf{1}, 1) = \widehat{D_{\boldsymbol\alpha+\mathbf{p}}}(\,\cdot\,)$. By the Bayesian property of $D_{\boldsymbol\alpha}$ the space $\mathcal{O}_{H^+_{\boldsymbol\alpha}}$ is spanned precisely by the Fourier transforms of the form $\widehat{D^{\mathbf{p}}_{\boldsymbol\alpha}}$. It remains to show that $\mathfrak{U}(\mathfrak{r}^+_k).\mathcal{O}_{\boldsymbol\alpha} = \mathcal{O}_{H^+_{\boldsymbol\alpha}}$. Setting $v = e^{p_1}_{0,1} \cdots e^{p_k}_{0,k} \in \mathfrak{U}(\mathfrak{r}^+_k)$ yields $v.\mathcal{O}_{\boldsymbol\alpha} = \mathcal{O}_{\boldsymbol\alpha+\mathbf{p}}$ as required. The uniqueness of v follows by the fact that, since $\mathfrak{r}^+_k$ is Abelian, $\mathfrak{U}(\mathfrak{r}^+_k)$ coincides with the (Abelian) symmetric algebra generated by $\mathfrak{r}^+_k$; see [21, §17.2]. This proves (iii).

For $k \in \mathbb{N}_1$ denote by $\Pi_k < GL_k(\mathbb{C})$ the group of permutation matrices, i.e. matrices with exactly one entry equal to 1 in each row and column and all other entries equal to 0. In order to show (iv), recall (e.g., [21, §12.1]) that the Weyl group W_k of Ψ_k is isomorphic to $\mathfrak{S}_{k+1}$ and its action on Ψ_k may be canonically identified as dual to the action of $\mathfrak{S}_{k+1}$ on $\mathfrak{h}_k$ via conjugation by permutation matrices in $\Pi_{k+1} \cong \mathfrak{S}_{k+1} < GL(\mathfrak{h}_k) \cong GL_{k+1}(\mathbb{C})$. Let $\Pi_{2:k+1} < \Pi_{k+1}(\mathbb{C})$ denote the subgroup of permutations matrices whose action on $\mathrm{Mat}_{k+1}(\mathbb{C})$ fixes the first row and column, i.e. the subgroup of those permutation matrices $\mathbf{A}$ in Π_{k+1} with $\mathbf{A}_{11} = 1$. Clearly $\mathfrak{S}_k \cong \Pi_{2:k+1} < \Pi_{k+1}$. Composing the isomorphism $\rho_{\boldsymbol\alpha}$ with the identification of the action of Π_{k+1} above completes the proof. □

The commutative action of $\mathfrak{h}_k$ It is the content of Theorem 4.16(i) that the characteristic functionals of the measures $D_{\boldsymbol\alpha}$, varying $\boldsymbol\alpha \in \operatorname{int}\Delta^{k-1}$, are projectively invariant under the action of the maximal toral subalgebra $\mathfrak{h}_k < \mathfrak{l}_k$ in the representation $\rho_{\boldsymbol\alpha}$. Since $\mathfrak{h}_k$ acts on $\mathcal{O}_{\boldsymbol\alpha}$ by weight $\boldsymbol\alpha$ (See the proof of Thm. 4.16(i).), for arbitrary $J_{\mathbf{t}} \coloneqq t_1 J_{\alpha_1} + \cdots + t_k J_{\alpha_k} \in \rho_{\boldsymbol\alpha}\mathfrak{h}_k$ one has

$$J_{\mathbf{t}} f_{\boldsymbol\alpha} = (\mathbf{t}\cdot\boldsymbol\alpha) f_{\boldsymbol\alpha}\ , \qquad \mathbf{t} \in \mathbb{R}^k\ . \tag{4.7}$$

This is to be compared with the case of $L_{\boldsymbol\alpha}$. Indeed, let $\mathbf{t} \in \mathbb{R}^k$ be such that $\mathbf{t}_\bullet = 0$ and set $\mathbf{s} \coloneqq \exp^\diamond \mathbf{t} \in M^{k-1}_1$. Then, $\mathbf{s}^{-\boldsymbol\alpha} = \exp(-\boldsymbol\alpha \cdot \ln^\diamond \mathbf{s}) = \exp(-\mathbf{t}\cdot\boldsymbol\alpha)$. Thus, by (1.4),

$$\mathrm{d}\big((\exp^\diamond \mathbf{t}).\big)_\sharp L_{\boldsymbol\alpha} = \exp(-\mathbf{t}\cdot\boldsymbol\alpha)\ \mathrm{d}L_{\boldsymbol\alpha}\ , \qquad \mathbf{t} \in \mathbb{R}^k\ , \quad \mathbf{t}_\bullet = 0\ .$$

Improper hyper-priors Before commenting on the non-commutative action of $\mathfrak{l}_k$, let us introduce a family of distinguished (possibly improper) hyper-priors of the Dirichlet distribution.

Definition 4.17 (Dirichlet-categorical hyper-priors)**.** *Let $\boldsymbol\alpha_0 \in \Delta^{k-1}$ and fix $\boldsymbol\alpha \in \Lambda^+_{\boldsymbol\alpha_0}$. For $\boldsymbol\epsilon \in \Lambda^+_{\boldsymbol\alpha_0} \cap H^-_{\boldsymbol\alpha}$ we denote by $\tilde D_{\boldsymbol\epsilon}$ the (possibly non-finite) definite (i.e., positive or negative,* not *signed) measure with density*

$$\mathbb{1}_{\Delta^{k-1}}(\mathbf{y})\,\frac{\mathbf{y}^{\boldsymbol\epsilon-\mathbf{1}}}{\mathrm{B}(\boldsymbol\epsilon)}$$

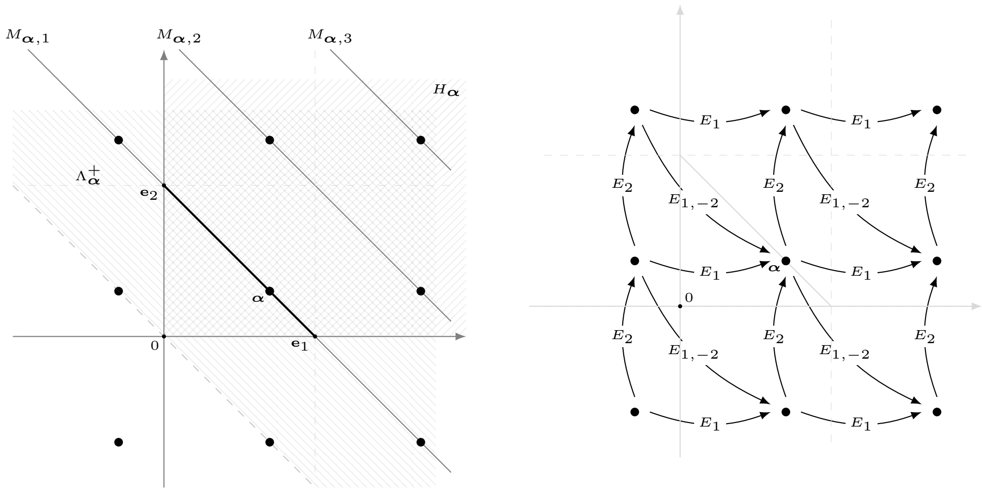

Figure 2: *(both)* Each marked point corresponds to some $\boldsymbol{\epsilon} \in \Lambda_{\boldsymbol{\alpha}}$ for fixed $\boldsymbol{\alpha}$, and is chosen to indicate the one-dimensional vector space $\mathcal{O}_{\boldsymbol{\epsilon}}$. *(left)* The gray anti-diagonal lines denote the *isoplethic surfaces*: marked points $\boldsymbol{\epsilon}$ lying on these surfaces belong to $M_{\boldsymbol{\alpha},\ell}$, i.e. they have fixed length $\boldsymbol{\epsilon}_\bullet = \ell \in \mathbb{N}_1$. The simplex Δ^1 is marked as a thick black segment. Analogously, marked points lying in the North-West dashed region delimited by the hyper-plane of equation $\mathbf{y}_\bullet = 0$ belong to the semi-lattice $\Lambda^+_{\boldsymbol{\alpha}}$, whereas marked points lying in the first hyper-octant (in the figure: the North-East dashed quadrant) belong to $H^+_{\boldsymbol{\alpha}}$. *(right)* The action of operators in $\rho_{\boldsymbol{\alpha}}(\mathfrak{n}^+_2)$ on the lattice $\mathcal{O}_{\Lambda_{\boldsymbol{\alpha}}}$ for $\boldsymbol{\alpha} = \left(\frac{2}{3}, \frac{1}{3}\right)$ is shown.

with respect to the k-dimensional Lebesgue measure on the hyperplane of equation $\mathbf{y}_\bullet = 1$ *in* $\mathbb{R}^k$*, concentrated on (the interior of)* Δ^{k-1}*. We term this measure the* Dirichlet-categorical hyper-prior *of parameter* $\boldsymbol{\epsilon}$*. The measure* $\tilde{D}_{\boldsymbol{\epsilon}}$ *has sign given by*

$$\operatorname{sgn}(\mathrm{B}(\boldsymbol{\epsilon})) = \operatorname{sgn}(\Gamma(\boldsymbol{\epsilon})) = \begin{cases} 1 & \textit{if } \boldsymbol{\epsilon} \in H^+_{\boldsymbol{\alpha}_0} \\ (-1)^{\lceil \varepsilon_1 \rceil + \cdots + \lceil \varepsilon_k \rceil} & \textit{otherwise} \end{cases} .$$

The non-commutative action of $\mathfrak{l}_k$ If $\boldsymbol{\alpha}_0 \in \operatorname{int}\Delta^{k-1}$ and $\boldsymbol{\alpha} \in H^+_{\boldsymbol{\alpha}_0}$, then

(a) the action of basis elements in $\mathfrak{r}^+_k$ amounts to take (characteristic functionals of) Dirichlet-categorical *posteriors*; it fixes the space $\mathcal{O}_{H^+_{\boldsymbol{\alpha}}}$ of (characteristic functionals of) such posteriors;

(b) the action of basis elements in $\mathfrak{r}^-_k$ amounts to take (characteristic functionals of) Dirichlet-categorical (hyper-)priors. The action of $\mathfrak{r}^-_k$ fixes the space $\mathcal{O}_{\Lambda^+_{\boldsymbol{\alpha}_0} \cap H^-_{\boldsymbol{\alpha}}}$ of (characteristic functionals of) all such (hyper-)priors and vanishes on the line $M_{\boldsymbol{\alpha}_0,1}$, since $M_{\boldsymbol{\alpha}_0,0}$ is the singular set of the normalization constant $\mathrm{B}[\boldsymbol{\epsilon}]^{-1}$;

(c) the action of basis elements in $\mathfrak{f}_k$ contains every non-trivial combination of the actions (a) and (b), and fixes isoplethic hypersurfaces $M_{\boldsymbol{\alpha}_0,\ell}$, i.e. those where the intensity $\boldsymbol{\epsilon}$ has constant total mass $\boldsymbol{\epsilon}_\bullet = \ell$.

In this framework, the case $\boldsymbol{\alpha} \in \operatorname{bd}\Delta^{k-1}$ is spurious, since the intensity measure $\boldsymbol{\alpha}$ should always be assumed fully supported.

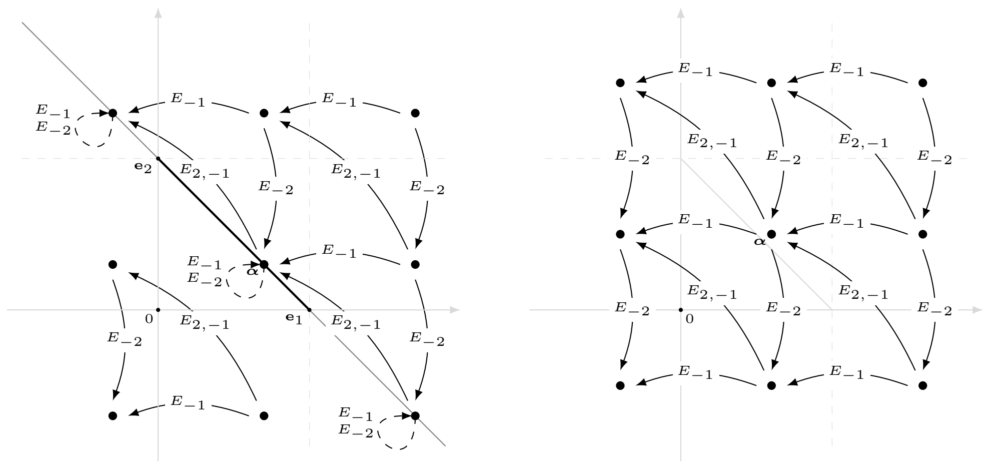

Figure 3: *(both)* Each marked point corresponds to some $\boldsymbol{\epsilon} \in \Lambda_{\boldsymbol{\alpha}}$ and is chosen to indicate the one-dimensional vector space $\mathcal{O}_{\boldsymbol{\epsilon}}$. *(left)* The action of operators in $\rho_{\boldsymbol{\alpha}}(\mathfrak{n}_2^-)$ on the lattice $\mathcal{O}_{\Lambda_{\boldsymbol{\alpha}}}$ for $\boldsymbol{\alpha} = \left(\frac{2}{3}, \frac{1}{3}\right)$ is shown. Since $\boldsymbol{\alpha}_\bullet \in \mathbb{Z}$, the lowering operators $E_{-\alpha_1}, E_{-\alpha_2}$ *(left)* vanish identically on the lowest positive isoplethic line (in gray), containing the standard simplex (the thick segment): their action is here represented by a dashed loop. *(right)* The action of operators in $\rho_{\boldsymbol{\alpha}}(\mathfrak{n}_2^-)$ on the lattice $\mathcal{O}_{\Lambda_{\boldsymbol{\alpha}}}$ for $\boldsymbol{\alpha} = \left(\frac{3}{5}, \frac{1}{2}\right)$ is shown. Since $\boldsymbol{\alpha}_\bullet \in \mathbb{R} \setminus \mathbb{Z}$, the lowering operators $E_{-\alpha_1}, E_{-\alpha_2}$ never vanish.

4.2 Infinite-dimensional statements

As it is well-known, when $\operatorname{supp} \sigma = [k]$, the Dirichlet–Ferguson measure reduces to a Dirichlet distribution. Motivated by this observation and by Theorem 3.10, one might ask about the existence of an infinite-dimensional Lie algebra $\mathfrak{g}_\sigma$ playing the role of the dynamical symmetry algebra of the characteristic functional of $\mathcal{D}_\sigma$.

In this section we extend results in Section 4.1.2 to infinite dimension, providing a partially negative answer to the above question. Namely, we show that, whenever σ is diffuse, a natural counterpart $\mathfrak{g}_\sigma$ to $\mathfrak{g}_k$ would necessarily be an Abelian Lie algebra. We shall comment extensively about the heuristics to this fact in Remark 4.19 below. Before so doing, let us introduce some further notation.

Notation For $a \in \mathbb{R}$ we denote by $\mathscr{M}_b^{>a}(X)$ the space of finite *signed* measures ν in $\mathscr{M}_b(X)$ such that $\nu X > a$. Further set, for any bounded measurable $f\colon X \to \mathbb{R}$,

$$\Phi[\nu, f] := \sum_{n=0}^{\infty} \frac{\mathrm{i}^n}{\langle \nu X \rangle_n} Z_n(\nu f^1, \dots, \nu f^n)$$

and

$$\begin{aligned} E_A \Phi[\nu, f] &:= \int_A \mathrm{d}\nu(y)\, \Phi[\nu + \delta_y, f] \ , \\ E_{A,-B} \Phi[\nu, f] &:= \int_{A \setminus B} \mathrm{d}\nu(y)\, \Phi[\nu + \delta_y, f] + \int_{B \setminus A} \mathrm{d}\nu(y)\, \Phi[\nu - \delta_y, f] \ . \end{aligned} \tag{4.8}$$

It is clear from the definition that $A \mapsto E_A$ is additive in the disjoint union, in the sense that $E_{A_1 \cup A_2} = E_{A_1} + E_{A_2}$ for disjoint $A_1, A_2 \in \mathcal{B}$. An analogous statement holds for $E_{A,-B}$.

The operators E_A and $E_{A,-B}$ are a counterpart to the operators E_{α_i} and $E_{\alpha_i,-\alpha_j}$ respectively, in the following sense. Suppose $\nu = \sum_{i=1}^k \alpha_i \delta_{x_i}$ for some $\boldsymbol{\alpha} \in \mathbb{R}_+^k$ and pairwise different x_i's in X. Then, for every $\mathbf{X} = (X_1, \dots, X_k) \in \mathfrak{P}_k(X)$ with $x_i \in X_i$, and for every f locally constant on $\mathbf{X}$ with values s,

$$E_{X_i}\Phi[\nu, f] = \left(E_{\alpha_i}\left({}_k\Phi[\boldsymbol{\alpha};\mathbf{s}]\,\mathbf{u}^{\boldsymbol{\alpha}}t^{\boldsymbol{\alpha}_\bullet}\right)\right)\Big|_{\mathbf{u}=\mathbf{1},t=1}\ , \qquad i \in [k]\ ;$$

cf. (4.4) and Remark 4.11. Analogous statements hold for $E_{X_i,-X_j}$. In the following theorem, we show how to approximate the operators E_A and $E_{A,-B}$ by their finite-dimensional counterparts when ν has diffuse part.

Theorem 4.18. *Let $(X, \tau(X), \mathcal{B})$ be a second countable locally compact Hausdorff space, ν a diffuse fully supported non-negative finite measure on X. Then,*

(i) $(\nu, f) \mapsto \Phi[\nu, f]$ is a well-defined extension of $(\nu, f) \mapsto \widehat{\mathcal{D}_\nu}(f^*)$ on $\mathscr{M}_b^{>0}(X) \times \mathcal{C}_c(X)$;

(ii) for every ν in $\mathscr{M}_b^{>1}(X)$, every f in $\mathcal{C}_c(X)$, every A, B in $\mathcal{B}$, and every good approximation $(f_h)_h$ of f locally constant on $\mathbf{X}_h$ with values $\mathbf{s}_h$ for some $(\mathbf{X}_h)_h \in \mathfrak{Na}(A, B \subset X)$, one has

$$E_A\Phi[\nu, f] = \lim_h \left(\sum_{i:X_{h,i}\subset A} E_{\alpha_{h,i}}\right) {}_{k_h}\Phi[\nu^\diamond\mathbf{X}_h, \mathbf{s}_h]\ ,$$

$$E_{A,-B}\Phi[\nu, f] = \lim_h \left(\sum_{\substack{i:X_{h,i}\subset A\setminus B\\ j:X_{h,j}\subset B\setminus A}} E_{\alpha_{h,i},-\alpha_{h,j}}\right) {}_{k_h}\Phi[\nu^\diamond\mathbf{X}_h, \mathbf{s}_h]\ ,$$

where $\boldsymbol{\alpha}_h \coloneqq \nu^\diamond\mathbf{X}_h$ and $E_{\alpha_{h,i}}$, $E_{\alpha_{h,i},-\alpha_{h,j}} \in \mathfrak{g}_{k_h}$;

(iii) Let σ be a diffuse probability on $(X, \mathcal{B})$ with full support, and $(\mathbf{X}_h)_h \in \mathfrak{Na}(X, \tau(X), \sigma)$. For σ-a.e. x, such that $X_{h,i_h} \downarrow_h \{x\}$, and for every good approximation $(f_h)_h$ of f, locally constant on $\mathbf{X}_h$ and uniformly convergent to f, there exist the pointwise limiting rescaled actions

$$\begin{aligned}\lim_h \alpha_{h,i_h}^{-1} E_{\alpha_{i_h}} \widehat{\mathcal{D}_\sigma}(f_h^*) &= \widehat{\mathcal{D}_\sigma^x}(f^*)\ ,\\ \lim_h \alpha_{h,i_h}^{-1} J_{\alpha_{i_h}} &= \mathrm{id}\ ,\\ \lim_h \alpha_{h,i_h}^{-1} E_{-\alpha_{i_h}} &= 0\ .\end{aligned} \tag{4.9}$$

Proof. The functional $\Phi[\nu, f]$ is well-defined in the first place since $\nu X > 0$. For $c, t > 0$ denote by $P_{c,t} \subset \mathbb{R}^n$ the polidisk $\{\mathbf{y} \in \mathbb{R}^n : |y_i| \leq c\,t^i\}$. By induction and (2.2) it is not difficult to show that $\max_{P_{c,t}} |Z_n| = Z_n[c(t\,\mathbf{1})^{\diamond\vec{\mathbf{n}}}]$; moreover, by (2.1) and Theorem 3.3, also cf. Rmk. 3.4, the latter equals $t^n \langle c\rangle_n /n!$. As a consequence, for arbitrary ν in $\mathscr{M}_b^{>0}(X)$ and $f \in \mathcal{C}_c(X)$, letting $y_i \coloneqq \nu f^i$ above,

$$|\Phi[\nu, f]| \leq \sum_{n=0}^\infty \langle \nu X\rangle_n^{-1} \max_{P_{\|\nu\|,\|f\|}} |Z_n| = \sum_{n=0}^\infty \frac{\langle \|\nu\|\rangle_n}{n!\,\langle \nu X\rangle_n} \|f\|^n = {}_1F_1\left[\,\|\nu\|\,; \nu X; \|f\|\,\right]\ ,$$

which is finite since $\nu X > 0$. This shows (i). Notably, if ν is positive, then $|\Phi[\nu, f]| \leq \exp\|f\|$ independently of $\|\nu\|$.

Let now A be in $\mathcal{B}$ and $(\mathbf{X}_h)_h$ as in (ii). Fix f in $\mathcal{C}_c(X)$, set $\boldsymbol{\alpha}_h \coloneqq \nu^\diamond\mathbf{X}_h$ and let $(f_h)_h$ be a good approximation of f, locally constant on $\mathbf{X}_h$ with values $\mathbf{s}_h$. Equation (4.6) yields by summation

$$\left(\sum_{i:X_{h,i}\subset A} E_{\alpha_i}\right) {}_{k_h}\Phi\left[\boldsymbol{\alpha}_h; \mathbf{s}_h\right] = \sum_{i:X_{h,i}\subset A} \alpha_{h,i}\ {}_{k_h}\Phi\left[\boldsymbol{\alpha}_h + \mathbf{e}_i; \mathbf{s}_h\right]\ . \tag{4.10}$$

More explicitly, since f_h is constant on each $X_{h,i}$ with value $s_{h,i}$, Proposition 3.5 yields

$$\begin{aligned}
&\left(\sum_{i:X_{h,i}\subset A} E_{\alpha_i}\right)_{k_h}\Phi\left[\boldsymbol{\alpha}_h;\mathbf{s}_h\right]= \\
&=\sum_{i:X_{h,i}\subset A}\nu X_{h,i}\sum_{n=0}^{\infty}\frac{1}{\langle\nu X+1\rangle_n}Z_n\left(\nu f_h+\frac{\nu(f_h\,\mathbb{1}_{X_{h,i}})}{\nu X_{h,i}},\dots,\nu f_h^n+\frac{\nu(f_h^n\,\mathbb{1}_{X_{h,i}})}{\nu X_{h,i}}\right) \\
&=\sum_{n=0}^{\infty}\frac{\mathrm{i}^n}{\langle\nu X+1\rangle_n}\times \\
&\quad\times\sum_{i:X_{h,i}\subset A}\int_{X_{h,i}}\mathrm{d}\nu(y)\,Z_n\left(\nu f_h+\frac{\nu(f_h\,\mathbb{1}_{X_{h,i}})}{\nu X_{h,i}}\,\mathbb{1}_{X_{h,i}}(y),\dots,\nu f_h^n+\frac{\nu(f_h^n\,\mathbb{1}_{X_{h,i}})}{\nu X_{h,i}}\,\mathbb{1}_{X_{h,i}}(y)\right) \\
&=\sum_{n=0}^{\infty}\frac{\mathrm{i}^n}{\langle\nu X+1\rangle_n}\sum_{i:X_{h,i}\subset A}\int_{X_{h,i}}\mathrm{d}\nu(y)\,Z_n\left(\nu f_h+f_h(y),\dots,\nu f_h^n+f_h(y)^n\right) \\
&=\sum_{n=0}^{\infty}\frac{\mathrm{i}^n}{\langle\nu X+1\rangle_n}\int_A\mathrm{d}\nu(y)\,Z_n\left(\nu f_h+f_h(y),\dots,\nu f_h^n+f_h(y)^n\right)\ . \\
&=\int_A\mathrm{d}\nu(y)\sum_{n=0}^{\infty}\frac{\mathrm{i}^n}{\langle(\nu+\delta_y)X\rangle_n}Z_n\left(\nu f_h+f_h(y),\dots,\nu f_h^n+f_h(y)^n\right)\ . \qquad (4.11)
\end{aligned}$$

Since $|f_h|\leq|f|$ pointwise, the sequence $\left(f_h^i\right)_h$ converges strongly in L^1_ν for every $i\leq n$ for every $n\in\mathbb{N}_1$, thus by continuity of Z_n, there exists the limit

$$\lim_h\int_A\mathrm{d}\nu(y)\sum_{n=0}^{\infty}\frac{\mathrm{i}^n}{\langle\nu X+1\rangle_n}Z_n\left(\nu f_h+f_h(y),\dots,\nu f_h^n+f_h(y)^n\right)=E_A\Phi[\nu,f]\ .$$

The proof of the statement for $E_{A,-B}$ is analogous. This completes the proof of (ii). The requirement that $\nu X>1$ is necessary to the convergence of $\Phi[\nu-\delta_y,f]$ for $y\in X$ in the definition of $E_{A,-B}$, whereas it may be relaxed to $\nu X>0$ in the case of E_A. We will make use of this fact in the proof of (iii).

Fix now x in X and let $i_h\coloneqq i_h(x)$ be such that $X_{h,i_h}\downarrow_h\{x\}$. By Lemma 5.1, the sequence $(i_h)_h$ is unique for σ-a.e. x. With the same notation of (ii), let now $A=X_{h,i_h}$ in (4.11). Then,

$$\alpha_{i_h}^{-1}E_{\alpha_{i_h}}{}_{k_h}\Phi[\boldsymbol{\alpha}_h;\mathbf{s}_h]=\alpha_{i_h}^{-1}E_{\alpha_{i_h}}\widehat{\mathcal{D}_\sigma}(f_h^*)=\frac{1}{\sigma X_{h,i_h}}\int_{X_{h,i_h}}\mathrm{d}\sigma(y)\,\widehat{\mathcal{D}_{\sigma+\delta_y}}(f_h^*)\ . \qquad (4.12)$$

By (3.14) and uniform convergence of the approximation

$$\lim_h\left|\tfrac{1}{\sigma X_{h,i_h}}E_{X_{h,i_h}}\Phi[\sigma,f_h]-\tfrac{1}{\sigma X_{h,i_h}}E_{X_{h,i_h}}\Phi[\sigma,f]\right|\leq\lim_h e^{\|f\|}\,\|f-f_h\|=0\ , \qquad (4.13)$$

thus, (4.12) and (4.13) yield, together with the continuity of $y\mapsto\widehat{\mathcal{D}_{\sigma+\delta_y}}(f^*)$ for fixed f and σ,

$$\lim_h\alpha_{i_h}^{-1}E_{\alpha_{i_h}}\widehat{\mathcal{D}_\sigma}(f_h^*)=\lim_h\frac{1}{\sigma_{X_{h,i_h}}}\int_{X_{h,i_h}}\mathrm{d}\sigma(y)\,\widehat{\mathcal{D}_{\sigma+\delta_y}}(f^*)=\widehat{\mathcal{D}_{\sigma+\delta_x}}(f^*)\ .$$

By the Bayesian property $\mathcal{D}_\sigma^x=\mathcal{D}_{\sigma+\delta_x}$, this yields the conclusion for the limiting raising action. Finally, since σ is a probability measure, $(\boldsymbol{\alpha}_h)_\bullet=1$ for all h, thus by (4.6),

$$\lim_h\alpha_{i_h}^{-1}J_{\alpha_{i_h}}\widehat{\mathcal{D}_\sigma}(f_h^*)=\lim_h\widehat{\mathcal{D}_\sigma}(f_h^*)=\widehat{\mathcal{D}_\sigma}(f^*)\ ,$$

$$\lim_h \alpha_{i_h}^{-1} E_{-\alpha_{i_h}} \widehat{\mathcal{D}_\sigma}(f_h^*) = \lim_h 0 = 0 \ ,$$

where the second equality for the first limiting action follows by (3.14). In all three cases, independence of the limits from the chosen (good) approximation is straightforward. $\square$

Remark 4.19. The above theorem shows how the existence of a semisimple dynamical symmetry algebra $\mathfrak{g}_\sigma$ for the characteristic functional of $\mathcal{D}_\sigma$ when σ is a *diffuse* measure should *not* be expected. Indeed, as a consequence of (4.9), the limiting action of the standard negative Borel subalgebra would be trivial and, in particular, the limiting action of the standard Cartan subalgebra would collapse to the identity. As a consequence, the limiting action of the standard positive Borel subalgebra would be Abelian. Assuming this is the case, we provide a conjectural statement for the structure of $\mathfrak{g}_\sigma$ in the general case when σ is any probability measure.

Let $\sigma = \sigma^{\mathrm{a}} + \sigma^{\mathrm{d}}$, where σ^{a}, resp. σ^{d}, is the purely atomic, resp. diffuse, part of σ. By the Mapping Theorem 3.9, we main assume without loss of generality that the atoms of σ^{a} are isolated points in $X = \operatorname{supp}\sigma^{\mathrm{d}} \sqcup \operatorname{supp}\sigma^{\mathrm{a}}$. Together with the infinite-dimensional analogue of [48, Lem. 3.1], this suggests that one might in fact have $\mathfrak{g}_\sigma = \mathfrak{g}_{\sigma^{\mathrm{d}}} \oplus \mathfrak{g}_{\sigma^{\mathrm{a}}}$. If we assume further, as heuristically argued before, that the dynamical symmetry algebra $\mathfrak{g}_{\sigma^{\mathrm{d}}}$ is Abelian, then $\mathfrak{g}_\sigma = \mathfrak{g}_{\sigma^{\mathrm{d}}} \oplus \mathfrak{g}_{\sigma^{\mathrm{a}}}$ would be reductive with semisimple part $\mathfrak{g}_{\sigma^{\mathrm{a}}}$. Thus, we may consider, without loss of generality, the case of purely atomic intensity measures $\sigma = \sigma^{\mathrm{a}}$.

Finally, if σ^{a} has support $[k]$, then clearly $\mathfrak{g}_{\sigma^{\mathrm{a}}} = \mathfrak{g}_k$ by Theorem 3.10, since $\mathcal{D}_{\sigma^{\mathrm{a}}} = D_{\boldsymbol{\alpha}}$ for some $\boldsymbol{\alpha}$. The case $\#\operatorname{supp}\sigma^{\mathrm{a}} = \infty$ remains open, although we expect that $\mathfrak{g}_{\sigma^{\mathrm{a}}} \cong \mathfrak{fsl}_\infty(\mathbb{C})$, the finitary special linear Lie algebra of traceless infinite matrices with finitely many non-zero entries.

5 Appendix

We collect here some results in topology and measure theory.

Lemma 5.1. *Let $(X, \tau(X), \mathcal{B}, \sigma)$ be a second countable locally compact Hausdorff Borel measure space of finite diffuse fully supported measure. Then, for every $(\mathbf{X}_h)_h \in \mathfrak{Na}(X, \tau(X), \sigma)$ for σ-a.e. x in X there exists a unique sequence $(X_{h,i_h})_h$, with $i_h \coloneqq i_h(x)$, such that $\mathbf{X}_h \ni X_{h,i_h} \downarrow_h \{x\}$.*

Proof. Proposition 2.4 justifies well-posedness of the requirements in the definition of $(\mathbf{X}_h)_h$. Without loss of generality, each $X_{h,i}$ may be chosen to be closed by replacing it with its closure $\operatorname{cl} X_{h,i} = X_{h,i} \cup \operatorname{bd} X_{h,i}$. Hence $\mathbf{X}_h$ may be chosen to be consisting of closed sets (disjoint up to a σ-negligible set) with non-empty interior. It follows by the finite intersection property that every decreasing sequence of sets $(X_{h,i_h})_h$ such that $X_{h,i_h} \in \mathbf{X}_h$ admits a non-empty limit, which is a singleton because of the vanishing of diameters. Vice versa, however chosen $(\mathbf{X}_h)_h$, for every point x in X it is not difficult to construct a (possibly non-unique) sequence X_{h,i_h} (with $i_h \coloneqq i_h(x)$) convergent to x and such that $X_{h,i_h} \in \mathbf{X}_h$. Furthermore, letting x be a point for which there exists more than one such sequence, we see that for every h the point x belongs to some intersection $X_{h,i_1} \cap X_{h,i_2} \cap \dots$, hence, since every partition has disjoint interiors by construction, $x \in \operatorname{bd} X_{h,i_1} \cap \operatorname{bd} X_{h,i_2} \cap \dots$. Since for every h and $i \leq k_h$ each set $X_{h,i}$ is a continuity set for σ, the whole union $\cup_{h\geq 0} \cup_{i\in[k_h]} \operatorname{bd} X_{h,i}$ is σ-negligible, thus so is the set of points x considered above, so that for σ-a.e. x there exists a unique sequence $(X_{h,i_h})_h$ such that $X_{h,i_h} \in \mathbf{X}^h$ and $\lim_h X_{h,i_h} = \{x\}$ and x belongs to each X_{h,i_h} in the sequence. $\square$

Finally, recall the following form of Lévy's Continuity Theorem.

Theorem 5.2 ([57, Thm. 3.1, p. 224])**.** *Let $(Y, \tau(Y))$ be a completely regular Hausdorff topological space, V be a linear subspace of $\mathcal{C}(Y)$ separating points in Y and χ be a complex-valued functional on V. If $(\mu_\gamma)_\gamma$ is a narrowly precompact net of Radon probability measures on $(Y, \mathcal{B}(Y))$ and $\lim_\gamma \widehat{\mu_\gamma}(v) = \chi(v)$ for every v in V, then $(\mu_\gamma)_\gamma$ converges narrowly to a Radon probability measure μ, the characteristic functional thereof coincides with χ.*

Corollary 5.3. *Let $(\mu_\gamma)_\gamma$ be a narrowly precompact net of random probabilities over $(X, \mathcal{B})$. If $\lim_\gamma \widehat{\mu_\gamma}(f^*) = \chi(f^*)$ for every f in $\mathcal{C}_c(X)$, then $(\mu_\gamma)_\gamma$ converges narrowly to a random probability μ, the characteristic functional thereof coincides with χ.*

Proof. By Proposition 2.4 the space $(X, \tau(X))$ is Polish, hence so is $\mathscr{M}_b^+(X)$ [24, 15.7.7], thus the space $\mathscr{M}_{\leq 1}^+(X) := \left\{\mu \in \mathscr{M}_b^+(X) : \mu X \leq 1\right\}$ is too, being closed, and $\mathscr{P}(X)$, being a G_δ-set in $\mathscr{M}_{\leq 1}^+(X)$. Since every finite measure on a Polish space is Radon [4, Thm. 7.1.7], each μ_γ is Radon. Consider $\mathscr{M}_b(X)$ endowed with the vague topology. The dense subset $\mathcal{C}_c(X)$ of the topological dual $(\mathscr{M}_b(X), \tau_v(\mathscr{M}_b(X)))' = \mathcal{C}_0(X)$ separates points in $\mathscr{M}_b(X)$, hence it separates points in $\mathscr{P}(X) \subset \mathscr{M}_b(X)$. The conclusion follows now by the Theorem choosing $Y = \mathscr{P}(X)$ and $V = \mathcal{C}_c(X)$. □

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