The effect of boundary conditions on mixing of 2D Potts models at discontinuous phase transitions

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Abstract

We study Swendsen–Wang dynamics for the critical q-state Potts model on the square lattice. For $q = 2, 3, 4$, where the phase transition is continuous, the mixing time t_{mix} is expected to obey a universal power-law independent of the boundary conditions. On the other hand, for large q, where the phase transition is discontinuous, the authors recently showed that t_{mix} is highly sensitive to boundary conditions: $t_{\mathrm{mix}} \geq \exp(cn)$ on an $n \times n$ box with periodic boundary, yet under free or monochromatic boundary conditions, $t_{\mathrm{mix}} \leq \exp(n^{o(1)})$.

In this work we classify this effect under boundary conditions that interpolate between these two (torus vs. free/monochromatic). Specifically, if one of the q colors is red, mixed boundary conditions such as red-free-red-free on the 4 sides of the box induce $t_{\mathrm{mix}} \geq \exp(cn)$, yet Dobrushin boundary conditions such as red-red-free-free, as well as red-periodic-red-periodic, induce sub-exponential mixing.

1 Introduction

The q-state Potts model at inverse temperature $\beta > 0$ is a generalization of the Ising model ($q = 2$) to $q \geq 3$ possible states. It is a canonical model of statistical physics and is one of the simplest models exhibiting a discontinuous (first-order) phase transition for some choices of q. Concretely, the model on a graph G is a probability distribution over $\{1, ..., q\}^{V(G)}$ with $\mu(\sigma) \propto \exp(\beta \sum_{ij \in E(G)} \mathbf{1}\{\sigma_i = \sigma_j\})$. Much of the analysis of the Potts model relies heavily on the random cluster (FK) model; the FK model is a model of dependent bond percolation parametrized by (p, q), identified with the q-state Potts model via the Edwards–Sokal coupling [8] when q is integer and $p = 1 - e^{-\beta}$.

On $\mathbb{Z}^2$, substantial recent progress has been made in understanding the Potts and random cluster phase transitions in β and p, respectively. On that geometry, the critical

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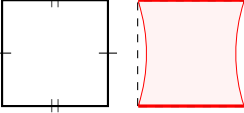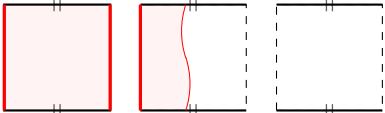
Boundary Conditions	Swendsen–Wang
Periodic/Mixed	$t_{\mathrm{mix}} \geq \exp(cn)$
Dobrushin	$t_{\mathrm{mix}} \leq \exp(O(\sqrt{n \log n}))$
Cylindrical	$t_{\mathrm{mix}} \leq \exp(n^{1/2+o(1)})$

Figure 1: Mixing time bounds for Swendsen–Wang dynamics on $n \times n$ boxes with different sets of boundary conditions. Dashed lines indicate free boundary conditions, the bold red lines denote red boundary conditions, and hash markings $|, \|$ indicate periodic boundary conditions on their respective sides. The torus and all-red boundary conditions were considered in [10]; the other examples are special cases of Theorems 1–3.

$p_c(q) = 1 - e^{-\beta_c}$ was identified [1] for all $q \geq 1$ with the self-dual point $p_{\mathrm{sd}}(q) = \frac{\sqrt{q}}{1+\sqrt{q}}$. It was shown in [7] that for $q \leq 4$, the phase transition is continuous (there is a unique infinite-volume Gibbs measure at $\beta_c(q)$) whereas for $q > 4$, the phase transition is discontinuous [6] (there are $q+1$ extremal infinite-volume Gibbs measures corresponding to q ordered phases and an additional disordered phase). In the latter case, the phase asymmetry at the critical point is expected to induce order-order and order-disorder surface tensions (known rigorously for q large [14, 19]). In the present work we study the relationship between the order-disorder surface tension and the effect of boundary conditions on mixing times (time to reach equilibrium) for the critical 2D Potts model.

Specifically, we study the Swendsen–Wang dynamics [22], a non-local Markov chain suggested in the physics literature as a fast MCMC sampler of the Potts model, as it switches between the different ordered phases (low-temperature bottlenecks) by moving through the FK representation of the Potts model using the Edwards–Sokal coupling. The authors of [10] analyzed the mixing times of the Swendsen–Wang dynamics at $\beta_c(q)$ on subsets of $\mathbb{Z}^2$ as the parameter q varied. In [10], polynomial and quasipolynomial upper bounds independent of the boundary conditions were proved for $q \leq 4$. When $q > 4$, it was shown that $t_{\mathrm{mix}} \geq \exp(cn)$ on the $n \times n$ torus (where the dynamics is slowed by the free energy barrier between the disordered phase and the q ordered phases), while on the $n \times n$ box with free or red boundary conditions and large q, the authors proved that $t_{\mathrm{mix}} \leq \exp(n^{o(1)})$.

This sensitivity to boundary conditions is in analogy to the sensitivity of mixing to boundary conditions in the low-temperature Ising Glauber dynamics, where for low enough temperatures, the plus-minus surface tension is very well understood [5] via cluster expansion. There, in $\mathbb{Z}^2$, the first sub-exponential bound of $t_{\mathrm{mix}} \leq \exp(n^{\frac{1}{2}+o(1)})$ was obtained in [17] for all sufficiently low temperatures under plus boundary. This was improved to $t_{\mathrm{mix}} \leq \exp(n^{o(1)})$ in [18] and subsequently to $t_{\mathrm{mix}} \leq n^{O(\log n)}$ for all $\beta > \beta_c$ in [16]; it is believed to be of order n^2, governed by motion by mean-curvature (cf. [12]). However, at β_c, all known bounds are independent of the boundary conditions; in fact, it is believed that $t_{\mathrm{mix}} \asymp n^z$ for some universal constant z. Such behavior should hold through $q \leq 4$. (See, e.g., [9, 10] for a more extensive account of related literature.)

When $q > 4$ is sufficiently large, similar cluster expansion techniques (large β is now replaced by large q) lead to an emergent order-disorder surface tension at the critical point, destroying the independence of mixing times and boundary conditions. Certain boundary conditions can destabilize the order-disorder phase symmetry, eliminating the exponential bottlenecks in the state space. Thus, when the boundary conditions are monochromatic or free, the mixing time of Swendsen–Wang was shown [10] to be $\exp(n^{o(1)})$ and is believed to be $n^{O(1)}$, matching the above picture for the low temperature Ising model under plus boundary conditions.

In the present paper, we investigate the relationship between mixing times and boundary conditions that interpolate between periodic and free/monochromatic, at the critical point of a discontinuous phase transition (see Figure 1). Our results hold for q large enough, and are expected to hold whenever the order-disorder surface tension (see Def. 2.1) is positive, which in turn should hold for all $q > 4$.

For two sequences f_n, g_n, here and throughout the paper, we say $f_n \lesssim g_n$ if there exists $C > 0$ such that $f_n \leq C g_n$ for all n and analogously define $f_n \gtrsim g_n$. We let $\Lambda_{n,m} = ([0,n] \times [0,m]) \cap \mathbb{Z}^2$ with nearest-neighbor edges. The boundary of $\Lambda_{n,m}$, denoted $\partial \Lambda_{n,m}$, is the set of vertices of $\Lambda_{n,m}$ adjacent to $\mathbb{Z}^2 \setminus \Lambda_{n,m}$; let $\partial_N \Lambda = \partial \Lambda_{n,m} \cap (\mathbb{Z} \times \{m\})$ be the its northern boundary of Λ, and define $\partial_E \Lambda, \partial_W \Lambda, \partial_S \Lambda$ similarly.

Theorem 1 (Mixed b.c.). Let q be large, $\varepsilon > 0$, and let (a_n, b_n, c_n, d_n) be marked vertices on $\partial \Lambda_{n,n}$ such that they are not all within εn of any one side of $\partial \Lambda_{n,n}$ and are all distance greater than εn from each other. There exists $c(\varepsilon, q) > 0$ (independent of n and a_n, b_n, c_n, d_n) such that the Swendsen–Wang dynamics on $\Lambda_{n,n}$ at $\beta = \beta_c(q)$ with boundary conditions that are red on the boundary segments (a_n, b_n) and (c_n, d_n) and free elsewhere, has
$$t_{\text{mix}} \gtrsim \exp(cn).$$

In particular, this holds with red boundary conditions on $\partial_{E,W} \Lambda_{n,n}$ and free elsewhere.

Theorem 2 (Dobrushin b.c.). Let q be large and let (a_n, b_n) be marked vertices on $\partial \Lambda_{n,n}$. There exists $c(q) > 0$ (independent of n and a_n, b_n) so that Swendsen–Wang dynamics at $\beta = \beta_c(q)$ with boundary conditions that are red on the boundary segment (a_n, b_n) and free elsewhere, has
$$t_{\text{mix}} \lesssim \exp(c\sqrt{n \log n}).$$

In particular, this holds with red boundary conditions on $\partial_{S,W} \Lambda_{n,n}$ and free elsewhere.

Theorem 3 (Cylinders). Let q be large. The critical Swendsen–Wang dynamics with periodic boundary conditions on $\partial_{N,S} \Lambda_{n,n}$ and either red or free boundary conditions on each of $\partial_E \Lambda_{n,n}$ and $\partial_W \Lambda_{n,n}$ satisfies
$$t_{\text{mix}} \lesssim \exp(n^{1/2 + o(1)}).$$

Remark 1.1. Theorems 1–3 also hold for the Glauber dynamics for the random cluster (FK) model for q sufficiently large (not necessarily integer). In fact, the proofs proceed by proving the desired result for the FK Glauber dynamics then using a priori estimates comparing t_{mix} for the FK Glauber dynamics to that of Swendsen–Wang dynamics (cf. [23, 24]) for the corresponding Potts model.

Remark 1.2. Theorems 1–3 showed the dependence of mixing times on boundary conditions for the critical Potts model in the phase coexistence regime. If, instead, one were interested in the simpler setting of Glauber dynamics for the 2D Ising model at large β (where the plus and minus phases would assume the role of red and free phases in our theorems), the proofs would carry over and even simplify, via the tools of [5].

2 Preliminaries

2.1 The q-state Potts model

In this section we formally introduce relevant facts about the Potts and random cluster models (for further details, see, e.g., [11]).

The Potts and FK models

Define the q-state Potts model on a graph $G = (V, E)$ as the probability measure $\mu_{G,\beta,q}$ on $\Omega_{\mathrm{P}} = [q]^V := \{1, ..., q\}^V$ as

$$\mu_{G,\beta,q}(\sigma) = \mathcal{Z}_{\mathrm{P}}^{-1} e^{\beta \sum_{(i,j) \in E} \mathbf{1}\{\sigma_i = \sigma_j\}},$$

where the normalizer $\mathcal{Z}_{\mathrm{P}}^{-1}$ is the *partition function*. Define the random cluster (FK) model on a graph G as the probability measure $\pi_{G,p,q}$ on state space $\Omega_{\mathrm{FK}} = \{0, 1\}^E$ as

$$\pi_{G,p,q}(\omega) = \mathcal{Z}_{\mathrm{FK}}^{-1} p^{o(\omega)} (1-p)^{|E| - o(\omega)} q^{k(\omega)},$$

where $o(\omega) = \sum_{e \in E} \omega(e)$ and $k(\omega)$ is the number of connected components (clusters) in the subgraph of G induced by ω (we count singletons as their own clusters). We call edges that have $\omega(e) = 1$ *open*, or *wired*, and edges that have $\omega(e) = 0$, *closed*, or *free*.

Potts and FK boundary conditions

Consider the Potts and FK models on $G = (V, E)$ with boundary $\partial G \subset V$. A Potts boundary condition on $\partial G \subset G$ is an assignment of spin values $\eta \in [q]^{\partial G}$ so that $\mu_{\beta,q,G}^{\eta} = \mu_{\beta,q,G}(\cdot \mid \sigma\!\restriction_{\partial G} = \eta)$.

For a subset $\partial G \subset V$, and FK boundary condition on ∂G is defined as follows: first augment G to a graph $G' = (V, E')$ where E' adds edges between any vertices in ∂G not adjacent in G and let $E'(\partial G)$ be the set of all edges between vertices in ∂G; then an assignment $\xi \in \{0, 1\}^{E'}$ is an FK boundary condition on ∂G. Then ξ can be identified with a partition of the vertices of ∂G, where the partition is given by the connected components of ξ. The FK measure with boundary conditions ξ is denoted by $\pi_{p,q,G}^{\xi}$ and is given by counting $k(\omega)$ with connections from ξ in mind.

The red Potts boundary condition is an assignment of $\sigma(i) = 1$ to all vertices of ∂G where we always call the first state $\sigma(i) = 1$ red, or R. The wired FK boundary condition is that in which all of $V(\partial G)$ is in the same boundary component. The free Potts boundary condition corresponds to $\partial G = \emptyset$ while the free FK boundary condition corresponds to the partition of ∂G consisting only of singletons $\{\{v\} : v \in V(\partial G)\}$. The wired/red boundary conditions are *ordered* as they pick out one of the q ordered phases of the Potts model, whereas the free boundary conditions are *disordered* as they pick out the disordered phase, where the q states are symmetric.

Edwards–Sokal coupling

The Edwards–Sokal coupling [8] is a coupling of the Potts and FK measures on a graph G that enables us to reduce the study of the q-state Potts model, to the study of the FK model at integer q. The joint probability assigned to (σ, ω), where $\sigma \in \Omega_{\mathrm{P}}$ is a q-state Potts configuration at inverse-temperature $\beta > 0$ and $\omega \in \Omega_{\mathrm{FK}}$ is an FK configuration with parameters $(p = 1 - e^{-\beta}, q)$, is proportional to

$$\prod_{xy \in E(G)} \left[(1-p)\mathbf{1}\{\omega(xy) = 0\} + p\mathbf{1}\{\omega(xy) = 1, \sigma(x) = \sigma(y)\} \right].$$

Planar duality

Throughout this paper we are concerned only with the Potts and FK models on planar graphs, and specifically rectangular subsets $\Lambda_{n,m} \subset \mathbb{Z}^2$ with vertices

$$\Lambda_{n,m} = [\![0,n]\!] \times [\![0,m]\!] := \{k \in \mathbb{Z} : 0 \leq k \leq n\} \times \{k \in \mathbb{Z} : 0 \leq k \leq m\}$$

and nearest-neighbor edges. For a general subgraph $G \subset \mathbb{Z}^2$, the boundary ∂G will be the set of vertices in G with neighbors in $\mathbb{Z}^2 - G$. When considering rectangles and other graphs where it makes sense, the southern (bottom) boundary of Λ is denoted $\partial_{\mathrm{S}}\Lambda_{n,m} = [\![0,n]\!] \times \{0\}$ and $\partial_{\mathrm{E,N,W}}$ are analogously defined. Then for rectangles, we observe that $\partial\Lambda_{n,m} = \bigcup_{i \in \{\mathrm{N,S,E,W}\}} \partial_i \Lambda_{n,m}$.

A very useful tool in the study of these models when G is planar is the planar duality of the FK model. For a planar graph G, let G^* be its dual graph. To every FK configuration ω, we associate the dual configuration ω^* given by $\omega^*(e^*) = 1$ if and only if $\omega(e) = 0$ (where e^* is the dual-edge intersecting e). A simple calculation yields that at the self-dual point $p_{\mathrm{sd}}(q) = \frac{\sqrt{q}}{1+\sqrt{q}}$, for $G \subset \mathbb{Z}^2$, we have $\pi_{G,p_{\mathrm{sd}},q}(\omega) = \pi_{G^*,p_{\mathrm{sd}},q}(\omega^*)$.

In the presence of boundary conditions ξ on G whose augmented graph G' is planar, the same holds for the corresponding dual boundary conditions ξ^* on G^*; these are determined on a case by case basis via the planarity of the augmented graph, but importantly, the wired and free boundary conditions are dual to each other.

Throughout the paper, for two vertices x, y we will write $x \xleftrightarrow{D} y$ if they are in the same component in $\omega\!\restriction_D$; when we include an asterisk, we mean x and y are in the same dual-component, i.e. they are in the same component of the dual configuration ω^*.

FKG inequality and monotonicity

When $q \geq 1$, the FK model satisfies positive correlation (FKG) inequalities: if A and B are increasing events in the edge configuration, for any boundary condition ξ, we have $\pi_{G,p,q}^\xi(A \cap B) \geq \pi_{G,p,q}^\xi(A)\pi_{G,p,q}^\xi(B)$.

This yields monotonicity in boundary conditions at $q \geq 1$: for FK boundary conditions $\xi' \geq \xi$ (ξ is a finer partition than ξ') and every increasing A, $\pi_{G,p,q}^{\xi'}(A) \geq \pi_{G,p,q}^\xi(A)$.

The domain Markov property

The FK model also satisfies the domain Markov property: for a graph G with boundary conditions ξ, and a subgraph $G' \subset G$,

$$\pi_G^\xi(\omega\!\restriction_{G'} \in \cdot \mid \omega\!\restriction_{G-G'} = \eta) \stackrel{d}{=} \pi_{G'}^{\eta,\xi}$$

where (η, ξ) is the boundary conditions induced on G' by connections from η and ξ.

The Potts and FK phase transition

On $\mathbb{Z}^2$, the FK and Potts models undergo a phase transition from—in the FK setting–existence a.s. of an infinite cluster at $p > p_c$ to a.s. no infinite cluster at $p < p_c$. In [1] it was proved that for all $q \geq 1$, $p_c(q) = p_{\mathrm{sd}}(q)$. This corresponds to a Potts phase transition from a unique infinite-volume Gibbs measure when $\beta < \beta_c$ to q different extremal Gibbs measures corresponding to weak-limits of boundary conditions of the q colors when $\beta > \beta_c$.

While for $q \leq 4$, the phase transition described above is continuous [7], when $q > 4$, the phase transition is discontinuous [6]: there are two extremal FK Gibbs measures at

$p = p_c$ corresponding to the wired and free boundary conditions at infinity, $\pi^1_{\mathbb{Z}^2} \neq \pi^0_{\mathbb{Z}^2}$ (resp. at $\beta = \beta_c$, $q + 1$ extremal Potts Gibbs measures corresponding to the q different colors, along with a disordered phase with free boundary conditions at infinity). As a result, we have the following [6, 7]: let $q > 4$ and $p = p_c$; there exists $c(q) > 0$ such that

$$\pi^0_{\mathbb{Z}^2} \left(0 \longleftrightarrow \partial([\![-n, n]\!]^2) \right) \lesssim e^{-cn} . \tag{2.1}$$

Surface tension

The order-disorder surface tension will play a large role in both upper and lower bounds studying the effect of boundary conditions on mixing in the phase coexistence regime. In the low-temperature regime, the metastable phases are the q ordered ones and there is a positive order-order surface tension (see, e.g., the $q = 2$ case); this leads to sensitivity of mixing times to boundary conditions in Ising/Potts Glauber dynamics (cf., e.g., [17, 18]), but not in FK/Swendsen–Wang dynamics (as these are symmetric w.r.t. the q ordered phases). At the critical point, the disordered phase is also metastable and induces similar sensitivity to boundary conditions in FK Glauber and Swendsen–Wang dynamics.

Let $\mathcal{S}_n = [\![0, n]\!] \times [\![-\infty, \infty]\!]$ and let $(1, 0, \phi)$ FK boundary conditions on $\partial \mathcal{S}_n$ denote those that are wired on $\partial \mathcal{S}_n \cap \{(x, y) \in \mathbb{Z}^2 : y \geq x \tan \phi\}$ and free elsewhere on $\partial \mathcal{S}_n$. We will always be taking $\phi \in (-\pi/2, \pi/2)$.

Definition 2.1. *The order-disorder surface tension on $\mathcal{S}_n$ in direction ϕ is given by*

$$\tau_{1,0}(\phi) = \lim_{n \to \infty} \frac{\cos \phi}{\beta_c n} \log \left[\frac{\mathcal{Z}^{1,0,\phi}_{\mathcal{S}_n}}{(\mathcal{Z}^1_{\mathcal{S}_n} \mathcal{Z}^0_{\mathcal{S}_n})^{1/2}} \right]$$

(whenever this limit exists) where $\mathcal{Z}^\xi_{\mathcal{S}_n}$ denotes the FK partition function on $\mathcal{S}_n$ with boundary conditions ξ on $\partial \mathcal{S}_n$.

It was proved first in [14] that for q sufficiently large, at $\beta = \beta_c$ and $\phi \neq \pm \frac{\pi}{2}$, the surface tension $\tau_{1,0}(\phi)$ exists and satisfies $\tau_{1,0}(\phi) > 0$. In §3, we will study consequences of positive surface tension for large deviation estimates of FK interfaces (see Def. 3.1).

2.2 Markov chain mixing times

We introduce the relevant dynamical quantities and techniques in the study of mixing times; for an extensive treatment, see [15].

Mixing times

Consider a Markov chain with finite state space Ω, reversible w.r.t. an invariant measure π. Define the total variation distance between measures μ, ν on Ω as

$$\|\mu - \nu\|_{\mathrm{tv}} = \tfrac{1}{2} \|\mu - \nu\|_{\ell_1} = \sup_{A \subset \Omega} |\mu(A) - \nu(A)| ,$$

also defined as a coupling distance $\|\mu - \nu\|_{\mathrm{tv}} = \inf \left\{ \mathbb{P}(X_t \neq Y_t) : X \sim \mu, Y \sim \nu \right\}$, where the infimum is over all couplings (X, Y). Then for a discrete time Markov chain with transition kernel $P(\cdot, \cdot)$, we define the total variation mixing time of the chain as

$$t_{\mathrm{mix}} = \inf \left\{ t : \max_{x \in \Omega} \|P^t(x, \cdot) - \pi\|_{\mathrm{tv}} < 1/(2e) \right\} .$$

Spectral gap and Dirichlet form

One commonly used technique to bound the mixing time of a Markov chain with transition kernel P is to bound the *spectral gap* of P. A transition matrix P reversible w.r.t., π has largest eigenvalue 1 by Perron–Frobenius, and has real spectrum. Thus we can enumerate its spectrum $1 > \lambda_1 \geq \lambda_2 \geq ...$ and define its spectral gap as $\mathsf{gap} = 1 - \lambda_1$; the following relation with t_{mix} is well known:

$$\mathsf{gap}^{-1} - 1 \leq t_{\mathrm{mix}} \leq \log(\tfrac{2e}{\pi_{min}})\mathsf{gap}^{-1} \,, \tag{2.2}$$

where $\pi_{min} = \min_{x \in \Omega} \pi(x)$. The variational form of the spectral gap is

$$\mathsf{gap} = \inf_{f:f\not\equiv 0,\, \mathbb{E}f=0} \frac{\mathcal{E}(f,f)}{\mathbb{E}_\pi(f^2)} \tag{2.3}$$

where the Dirichlet form $\mathcal{E}(f,f)$ is given by $\sum_{x,y \in \Omega} \pi(x)P(x,y)(f(y) - f(x))^2$.

Though our main results are stated for the Swendsen–Wang dynamics for the Potts model, as remarked earlier, the proofs all analyze instead the mixing time of the Glauber dynamics for the FK model. We formally define both of these dynamics in the sequel.

Heat-bath Glauber dynamics

Discrete time heat-bath Glauber dynamics for the FK model (FK Glauber dynamics) on a finite graph G with boundary conditions ξ is the Markov chain $(X_t)_{t\geq 0}$ defined as follows: at time t it picks an edge e uniformly at random, and resamples $\omega(e)$ via $\pi_G^\xi(\omega\restriction_{\{e\}} \mid \omega\restriction_{E-\{e\}} = X_{t-1}\restriction_{E-\{e\}})$ to obtain X_{t+1}. Though this paper does not consider it, the heat-bath Potts Glauber dynamics $(X_t')_{t\geq 0}$ is defined similarly, where at each time step a site v is picked uniformly at random and $\sigma(v)$ is resampled according to $\mu_G^\eta(\sigma\restriction_{\{v\}} \mid \sigma\restriction_{V-\{v\}} = X_{t-1}'\restriction_{V-\{v\}})$.

Swendsen–Wang dynamics

Theorems 1–3 all treat the Swendsen–Wang dynamics [22], which, for the q-state Potts model on a finite graph $G = (V,E)$ at inverse temperature β, is the following discrete-time Markov chain. Given that the Markov chain is at state $\sigma \in [q]^V$ at time t, generate state $\sigma' \in [q]^V$ at time $t+1$ as follows.

1. To σ, assign an (random) FK configuration $(\omega(e))_{e\in E}$ defined as follows: for every edge $e = (x,y)$, independently, set $\omega(e)$ to be closed with probability 1 if $\sigma_x \neq \sigma_y$ and with probability $e^{-\beta}$ if $\sigma_x = \sigma_y$.

2. For every connected component of ω, reassign all the vertices in the cluster, collectively, an i.i.d. color in $[q]$, to obtain the new configuration σ'.

One can check using the Edwards–Sokal coupling of the FK and Potts models, that the Swendsen–Wang dynamics is reversible with respect to $\mu_{G,\beta}$.

Remark 2.2. The definition of the Swendsen–Wang dynamics indicates why it has fast mixing at low-temperature whereas the Potts Glauber dynamics slows down on $(\mathbb{Z}/n\mathbb{Z})^2$ (see Theorem 4 of [10]). When $\beta > \beta_c$, the metastable states are the q ordered phases, corresponding to the q colors; but step 2) above recolors all FK clusters, and in particular, reassigns the large macroscopic cluster of the ordered phase an i.i.d. color, allowing the dynamics to easily jump between the q metastable states. However, as it relies heavily on the FK representation of the model, the order-disorder energy barrier is still hard to overcome when both ordered and disordered phases are metastable.

Comparison between cluster and Glauber dynamics

The following estimates allow us to reduce the analysis of the Swendsen–Wang dynamics to the analysis of FK Glauber dynamics, and demonstrate why our results do not carry over to Glauber dynamics for the Potts model.

Theorem 2.3 ([23, 24])**.** *Let $q \geq 2$ be integer. Let $\mathsf{gap}_{\mathrm{P}}$ and $\mathsf{gap}_{\mathrm{FK}}$ be the spectral gaps of discrete-time Glauber dynamics for Potts and FK model on a graph on m edges and maximum degree Δ, resp., and let $\mathsf{gap}_{\mathrm{SW}}$ be the spectral gap of Swendsen–Wang. Then*

$$\mathsf{gap}_{\mathrm{P}} \leq 2q^2 (qe^{2\beta})^{4\Delta} \mathsf{gap}_{\mathrm{SW}}, \tag{2.4}$$

$$\mathsf{gap}_{\mathrm{FK}} \leq \mathsf{gap}_{\mathrm{SW}} \leq 16 \mathsf{gap}_{\mathrm{FK}} \, m \log m. \tag{2.5}$$

Theorem 2.3 implies, in particular, that the mixing time of the FK Glauber dynamics and Swendsen–Wang dynamics on a graph G are comparable up to polynomial factors in the number of vertices. However, the Potts Glauber dynamics can be much slower than both, e.g., on the $n \times n$ torus when $\beta > \beta_c(q)$.

Potts boundary conditions and mixing

The comparison estimates above are only valid in the absence of boundary conditions, while the focus of this paper is the influence of various fixed boundary conditions. In particular, the estimates of Theorem 2.3 hold immediately for $\Lambda_{n,n}$ with free or periodic (the torus) boundary conditions. For other boundary conditions, we can deform $\Lambda_{n,n}$ as in the following remark; this could, however, distort Δ by an order n factor, leading to an exponential in n cost in (2.4).

Remark 2.4. For any FK boundary condition ξ on G, we can define a (not necessarily planar) graph $\tilde{G}$ by identifying all vertices of every boundary component of ξ with a single vertex in $\tilde{G}$, and keeping the same edge structure. Then the FK Glauber dynamics on $\tilde{G}$ with free boundary conditions is the same as that on G with boundary conditions ξ. In such a case, let $\partial\tilde{G}$ denote the set of vertices in $\tilde{G}$ that arise from the boundary components of G.

Remark 2.5. The exponential dependence on Δ in Eq. (2.4) can be improved to exponential in the maximum degree of all but one vertex (see [23, Theorem 1']), whence Δ in Eq. (2.4) can be replaced with the second largest degree of G. This implies that Theorem 1 in fact also holds for the Potts Glauber dynamics.

The following is a consequence of spin symmetry of the Swendsen–Wang dynamics.

Fact 2.6. *Consider Swendsen–Wang dynamics on G with Potts boundary conditions η, by considering the graph $\tilde{G}$, where boundary vertices of each color are identified as single vertices. Let $\mathsf{gap}_{\mathrm{SW},\eta,G}$ be the spectral gap of Swendsen–Wang dynamics on G with b.c., η, and let $\mathsf{gap}_{\mathrm{SW},\eta,\tilde{G}}$ be the spectral gap of Swendsen–Wang dynamics on $\tilde{G}$ with $\partial\tilde{G}$ assigned the colors given by η. If $\partial\tilde{G}$ consists of at most one vertex, then*

$$\mathsf{gap}_{\mathrm{SW},\eta,G} = \mathsf{gap}_{\mathrm{SW},\eta,\tilde{G}} = \mathsf{gap}_{\mathrm{SW},0,\tilde{G}}.$$

Remark 2.5 and Fact 2.6 imply that for $\Lambda_{n,n}$ with FK boundary conditions ξ with at most one nontrivial boundary component, corresponding to Potts b.c., η (an assignment of a color, e.g., red, to the nontrivial component of ξ and no other color assignments),

$$\mathsf{gap}_{\mathrm{FK}} \lesssim \mathsf{gap}_{\mathrm{SW}} \lesssim n^2 \log n \cdot \mathsf{gap}_{\mathrm{FK}}.$$

Notice that all of the boundary conditions considered in Theorems 1–3 are of this form.

Thus, it suffices to prove Theorems 1–3 for the FK Glauber dynamics. From now on, when we write `gap` with no subscript we mean it to refer to the FK Glauber dynamics.

Monotonicity and the grand coupling

A discrete-time Markov chain with state space Ω and transition kernel P is *monotone* if $\mu P \succeq \nu P$ for every two probability distributions μ, ν on Ω such that $\mu \succeq \nu$; we say that a spin system is monotone whenever single-site Glauber dynamics for it is monotone (cf. [15, §22.3]).

The random mapping representation of the discrete time FK Glauber dynamics on a graph $G = (V, E)$ views the updates as a sequence $(J_i, U_i)_{i \geq 1}$, in which J_i's are i.i.d. uniform edges (the updated locations), and the U_i's are i.i.d. uniform on $[0, 1]$: starting from an initial configuration ω_0, at time T_i, writing $J_i = (x, y)$, the dynamics replaces the value of $\omega(J_i)$ by $\mathbf{1}\{U_i \leq p\}$ if $x \longleftrightarrow y$ in $E - \{J_i\}$ and by $\mathbf{1}\{U_i \leq \frac{p}{p+q(1-p)}\}$ otherwise. The *grand coupling* for FK Glauber dynamics is a coupling of the chains from all initial configurations on G which, via the above random mapping representation, uses the same update sequence $(J_i, U_i)_{i \geq 1}$ for each one of these chains. Using this representation and the FKG inequality, one sees that heat-bath Glauber dynamics for the FK model at $q \geq 1$ is monotone: for every two FK configurations $\omega_1 \geq \omega_2$ and every $t \geq 0$, we have $P^t(\omega_1, \cdot) \succeq P^t(\omega_2, \cdot)$.

Thus, for $q \geq 1$, this coupling preserves the partial ordering of the Markov chains started from all possible initial configurations, at all times $t \geq 0$. In particular, under the grand coupling, the value of an edge e in Glauber dynamics at time t from an arbitrary initial state ω_0, is sandwiched between the corresponding values from the free and wired initial states; thus, by a union bound over all edges,

$$\max_{x \in \Omega} \|P^t(x, \cdot) - \pi\|_{\mathrm{tv}} \leq \max_{x,y \in \Omega} \|P^t(x, \cdot) - P^t(y, \cdot)\|_{\mathrm{tv}} \leq |E(\Lambda)| \, \|P^t(1, \cdot) - P^t(0, \cdot)\|_{\mathrm{tv}} . \quad (2.6)$$

Censoring inequalities

The Peres–Winkler censoring inequalities [20] for monotone spin systems allow one to "guide" the dynamics to equilibrium, using that prohibiting updates at various sites for periods of time can only slow down mixing.

Theorem 2.7 ([20, Theorem 1.1]). *Let μ_T be the law of the discrete-time Glauber dynamics at time T for a monotone spin system with state space $\Omega = S^\Lambda$ and stationary distribution π, whose initial distribution μ_0 is such that $\omega \mapsto \mu_0(\omega)/\pi(\omega)$ is increasing w.r.t. the partial order on Ω. Set $0 = t_0 < t_1 < \ldots < t_k = T$ for some k, let $(\Lambda_i)_{i=1}^k$ be subsets of Λ, and let $\tilde{\mu}_T$ be the law at time T of the censored dynamics, started at μ_0, where only updates in Λ_i are kept during the times $\{t_{i-1} + 1, \ldots, t_i\}$. Then $\|\mu_T - \pi\|_{\mathrm{tv}} \leq \|\tilde{\mu}_T - \pi\|_{\mathrm{tv}}$ and $\mu_T \preceq \tilde{\mu}_T$; moreover, both μ_T/π and $\tilde{\mu}_T/\pi$ are increasing.*

Theorem 2.7 implies in particular that if we consider two chains—the standard FK Glauber dynamics $X_t \sim P^t(1, \cdot)$, and the modified dynamics $\tilde{X}_t \sim \tilde{P}^t(1, \cdot)$ which at certain times sets all bonds in predetermined subsets of G to be wired—then $\tilde{P}^t(1, \cdot)$ is always farther from π and from $P^t(0, \cdot)$ in total-variation than $P^t(1, \cdot)$ is.

3 Consequences of positive surface tension

3.1 Cluster expansion

We first introduce the cluster expansion framework used to prove that the FK order-disorder surface tension $\tau_{1,0}(\phi)$ as defined in Definition 2.1 is positive for all large q at $p = p_c(q)$. We skip many of the details here as understanding them is not necessary to

the equilibrium estimates we require (§3.2). This approach was extensively developed in [5] for the low-temperature Ising model when $\beta \gg \beta_c$, and then extended to the critical Potts/FK model for large q (at the discontinuous phase transition point) in [19] where one can find more details on the below. Though these cluster expansion techniques only go through at $p = p_c$ when q is sufficiently large, all of the interface estimates are expected to hold at p_c whenever $\tau_{1,0}(\phi) > 0$, so in particular, for *every* $q > 4$.

In what follows, two (primal) edges e, f are adjacent if they share a vertex. Two primal edges e, f are co-adjacent if $e^\star$ is adjacent to $f^\star$. Connectedness and co-connectedness are defined naturally with respect to adjacency and co-adjacency. For an edge subset $A \subset E(\mathbb{Z}^2)$, the (edge)-*boundary of* A is the set of all edges in A that are co-adjacent to $E(\mathbb{Z}^2) \setminus A$. The *co-boundary of* A is the set of edges in $E(\mathbb{Z}^2) \setminus A$ that are adjacent to A.

Definition 3.1. *Let D be a connected subgraph of $\mathbb{Z}^2$ and let a, b be two marked boundary points on ∂D. Consider Dobrushin boundary conditions that are wired on the clockwise segment (a, b) and free on (b, a) (where if D is infinite, simply connected, we define these boundary arcs in the natural way). Then for an FK realization ω on D, the* primal-FK interface *(or simply the FK interface), $\mathcal{I} = \mathcal{I}(\omega)$, is defined as follows:*

1. *Consider the dual-component of the boundary arc (b, a) in ω, and consider its co-boundary (a set of closed dual-edges in ω).*

2. *This co-boundary has a unique co-connected component, call it $\Gamma^\star(\omega)$, that is incident to the boundary arc (a, b). The primal-FK interface $\mathcal{I}$ is the set of all primal edges that are dual to edges in $\Gamma^\star(\omega)$.*

(Notice that this interface is not necessarily a simple path, but it is connected, and will have no cycles.) Define the dual-FK interface *analogously as follows: consider the connected component of open edges touching the boundary arc (a, b) and consider its co-boundary (a set of closed primal edges). This co-boundary has a unique co-connected component that is incident the boundary arc (b, a); the set of dual-edges that are dual to edges in this co-connected component will be the dual-FK interface. For more general boundary conditions, there will be a compatible collection of interfaces between all boundary segments that are wired.*

Recall that we defined the infinite strip $\mathcal{S}_n = [\![0, n]\!] \times [\![-\infty, \infty]\!]$.

Definition 3.2. *For any angle $\phi \in (-\frac{\pi}{2}, \frac{\pi}{2})$, an edge-cluster weight function $\Phi(\mathcal{C}, \mathcal{I})$ is any real-valued function with first argument that is a connected set of edges in $\mathcal{S}_n$ and second argument that is a possible realization of an FK interface with Dobrushin boundary conditions that are wired on the clockwise arc between $(0, 0)$ to $(n, \lfloor n \tan \phi \rfloor)$, such that for some $\lambda > 0$ and every $\mathcal{C}, \mathcal{I}$,*

1. $\Phi(\mathcal{C}, \mathcal{I}) = 0$ *when* $\mathcal{C} \cap \mathcal{I} = \emptyset$,

2. $\Phi(\mathcal{C}, \mathcal{I}_1) = \Phi(\mathcal{C}, \mathcal{I}_2)$ *when* $\Pi_{\mathcal{C}} \cap \mathcal{I}_1 = \Pi_{\mathcal{C}} \cap \mathcal{I}_2$,

3. $\Phi(\mathcal{C}, \mathcal{I}) = \Phi(\mathcal{C} + (0, s), \mathcal{I}_1)$ *when* $\mathcal{I}_1 \cap \Pi_{\mathcal{C}} = (0, s) + \mathcal{I} \cap \Pi_{\mathcal{C}}$,

4. $|\Phi(\mathcal{C}, \mathcal{I})| \leq \exp(-\lambda \ell(\mathcal{C}))$,

where

$$\Pi_{\mathcal{C}} = \{(x, y) \in \mathbb{R}^2 : \exists y' \text{ s.t. } (x, y') \in \mathcal{C}\},$$

and $\ell(\mathcal{C})$ is the minimum number of edges in a connected subset of $E(\mathbb{Z}^2)$ that contains all the boundary edges of $\mathcal{C}$. More generally, for an edge-cluster weight function Φ and a domain V_n with Dobrushin boundary conditions between $(0, 0)$ and $(n, n \lfloor \tan \phi \rfloor)$, its

partition function is given by

$$\mathcal{Z}_\Phi = \mathcal{Z}_\Phi(V_n, \phi) = \sum_{\mathcal{I}} \lambda^{|\mathcal{I}| + \sum_{\mathcal{C}:\mathcal{C} \cap \mathcal{I} = \emptyset} \Phi(\mathcal{C}, \mathcal{I})}, \qquad (3.1)$$

where the sum runs over all possible FK order-disorder interfaces in V_n.

There exists a unique edge-cluster weight function $\Phi_{o/d}$—called the FK order-disorder weight function—such that the probability that the FK interface in $\mathcal{S}_n$ under the $(1, 0, \phi)$ boundary condition is $\mathcal{I}$ is given by

$$\pi_{\mathcal{S}_n}^{(1,0,\phi)}(\mathcal{I}) = \mathcal{Z}_{\Phi_{o/d}}^{-1} \lambda^{|\mathcal{I}| + \sum_{\mathcal{C}:\mathcal{C} \cap \mathcal{I} \neq \emptyset} \Phi_{o/d}(\mathcal{C}, \mathcal{I})}.$$

The FK order-disorder weight function $\Phi_{o/d}$ is made explicit in [19, Proposition 5].

By adapting the methods of [5] to the FK cluster expansion, the following large deviation estimate on order-disorder interface fluctuations was obtained in [19].

Proposition 3.3 ([19, Proposition 5]). *Consider the critical FK model on $\mathcal{S}_n$ and fix a $\delta > 0$. There exist some q_0 and $c(\delta) > 0$ such that for all $\phi \in [-\frac{\pi}{2} + \delta, \frac{\pi}{2} - \delta]$ and every $q \geq q_0$, every $h \geq 1$ and $n \geq 1$,*

$$\pi_{\mathcal{S}_n}^{1,0,\phi}\big(\mathcal{I} \not\subset \{(x,y) : y \in [x \tan \phi - h, x \tan \phi + h]\}\big) \lesssim n^2 \exp(-ch^2/n).$$

The following is a finer result, that reformulates results of [5] in the FK setting. Define the *cigar-shaped region* $U_{\kappa,d,\phi}$ for every $d > 0$ and $\kappa > 0$ by

$$U_{\kappa,d,\phi} = \mathcal{S}_n \cap \left\{ (x,y) \in \mathbb{Z}^2 : |y - x \tan \phi| \leq d \left| \frac{x(n-x)}{n} \right|^{\frac{1}{2} + \kappa} \right\}, \qquad (3.2)$$

Proposition 3.4. *Consider the critical FK model on a domain $V_n \supset U_{\kappa,d,\phi}$, let $\Phi_{o/d}$ be the FK order-disorder weight function, and let $\tilde{\Phi}$ be any function satisfying*

$$\tilde{\Phi}(\mathcal{C}, \mathcal{I}) = \Phi_{o/d}(\mathcal{C}, \mathcal{I}) \text{ when } \mathcal{C} \subset U_{\kappa,d,\phi}, \qquad |\tilde{\Phi}(\mathcal{C}, \mathcal{I})| \leq \exp(-\lambda \ell(\mathcal{C})) \qquad \text{for every } \mathcal{C}, \mathcal{I}$$

(e.g., $\tilde{\Phi} = \Phi_{o/d} \mathbf{1}\{\mathcal{C} \subset U_{\kappa,d,\phi}\}$). There exist $q_0 > 0$ and $f(\kappa) = O(\kappa^{-1})$ such that for all $q \geq q_0$, all $\phi \in [-\frac{\pi}{2} + \delta, \frac{\pi}{2} - \delta]$, there exists $C(q, \delta) > 0$, such that

$$|\log \mathcal{Z}_{\tilde{\Phi}}(V_n, \phi) - \log \mathcal{Z}_{\Phi_{o/d}}(\mathcal{S}_n, \phi)| \leq C(\log n)^{f(\kappa)}. \qquad (3.3)$$

Moreover, the order-disorder surface tension $\tau_{1,0}(\phi)$ (given in Def. 2.1) satisfies

$$|\log \mathcal{Z}_{\tilde{\Phi}}(V_n, \phi) - n \log(1 + \sqrt{q})(\cos \phi)^{-1} \tau_{1,0}(\phi)| \leq C(\log n)^{f(\kappa)}, \qquad (3.4)$$

and the large deviation estimate of Proposition 3.3 holds for V_n and $d/2$.

3.2 Surface tension estimates on subsets of $\Lambda_{n,n}$

In this section, we extend the surface tension estimate of Proposition 3.3 to tilted half-infinite and finite strips. We first define subsets of $\mathcal{S}_n = [\![0, n]\!] \times [\![-\infty, \infty]\!]$ we consider in obtaining the desired sub-exponential upper bounds of Theorems 1–2 (see Figure 2).

Definition 3.5. *For every n, which will be understood by context and omitted from the following notation, define a tilted strip $\mathcal{S}_{b,h,\phi}$ as (for $b \in \mathbb{R}$, $h \in \mathbb{R}_+$, $\phi \in (-\frac{\pi}{2}, \frac{\pi}{2})$),*

$$\mathcal{S}_{b,h,\phi} = \{(x,y) \in \mathcal{S}_n : y \in [\![b - h + x \tan \phi, b + h + x \tan \phi]\!]\}.$$
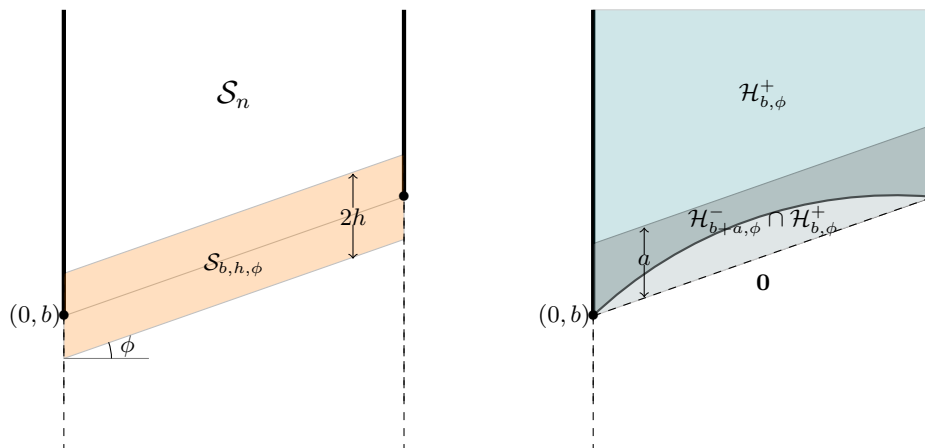
Figure 2: An illustration of some of the sets and interfaces considered in Definition 3.5 and Proposition 3.6. The $(1, 0, \phi)$ boundary conditions are depicted with bold lines being wired and dashed lines being free.

Also, define the half-infinite strips,

$$\mathcal{H}_{b,\phi}^{+} = \{(x, y) \in \mathcal{S}_n : y \geq b + x \tan \phi\}, \qquad \mathcal{H}_{b,\phi}^{-} = \{(x, y) \in \mathcal{S}_n : y \leq b + x \tan \phi\}.$$

The following estimate—extending an estimate of [19], which in turn adapts [5] to the FK model, to half-infinite strips—is a consequence of monotonicity of the FK model and Proposition 3.3.

Proposition 3.6. *Fix $\delta > 0$ and let q be sufficiently large. For any $b \in \mathbb{R}$ and $\phi \in [-\frac{\pi}{2} + \delta, \frac{\pi}{2} - \delta]$, consider the critical FK model on $\mathcal{H}_{b,\phi}^{+} \subset \mathcal{S}_n$ with boundary conditions $(1, 0, b, \phi)$ denoting wired boundary conditions on $\partial\mathcal{H}_{b,\phi}^{+} \cap \mathcal{H}_{b+1,\phi}^{+}$ and free boundary elsewhere on $\partial\mathcal{H}_{b,\phi}^{+}$. There exist constants $A > 0$ and $c(\delta, q) > 0$ such that the order-disorder interface $\mathcal{I}$, satisfies*

$$\pi_{\mathcal{H}_{b,\phi}^{+}}^{1,0,b,\phi}(\mathcal{I} \not\subset \mathcal{H}_{b+a,\phi}^{-} \cap \mathcal{H}_{b,\phi}^{+}) \leq An^2 \exp\left(-ca^2/n\right).$$

Proof. The proposition was proven in the case $\phi = 0$ in Proposition 4.2 of [10] combining monotonicity of the FK model in boundary conditions with Proposition 3.3. The same proof carries over to the case $\phi \neq 0$ as long as ϕ is uniformly bounded away from $\pm\frac{\pi}{2}$ as the surface tension estimate of Proposition 3.3 on $\mathcal{S}_n$ is expressed in that setup. $\square$

The following is the main equilibrium estimate we will use in the proofs of sub-exponential mixing for general Dobrushin boundary conditions.

Proposition 3.7. *Fix $\delta > 0$ and let q be sufficiently large. For any $b \in \mathbb{R}$, $h \leq m \leq n$, and $\phi \in [-\frac{\pi}{2} + \delta, \frac{\pi}{2} - \delta]$, consider the critical FK model on $\mathcal{S}_{b,m,\phi} \cap \Lambda_{n,n}$ with $(1, 0, b, \phi)$ boundary conditions denoting wired on $\partial(\mathcal{S}_{b,m,\phi} \cap \Lambda_{n,n}) \cap \mathcal{H}_{b+1,\phi}^{+}$ and free elsewhere on $\partial(\mathcal{S}_{b,m,\phi} \cap \Lambda_{n,n})$. Then there exists $c(\delta, q) > 0$ such that*

$$\pi_{\mathcal{S}_{b,m,\phi} \cap \Lambda_{n,n}}^{1,0,b,\phi}(\mathcal{I} \not\subset \mathcal{S}_{b,h,\phi} \cap \Lambda_{n,n}) \lesssim n^2 \exp(-ch^2/n).$$

We need the following preliminary estimate (see Figure 3 as a guide).

Lemma 3.8. *Fix $\delta > 0$ and let q be sufficiently large. For any $b \in \mathbb{R}$ and $\phi \in [-\frac{\pi}{2} + \delta, \frac{\pi}{2} - \delta]$, consider the critical FK model on $\mathcal{H}_{b,\phi}^{+} \cap \Lambda_{n,n}$ with $(1, 0, b, \phi)$ boundary conditions denoting*
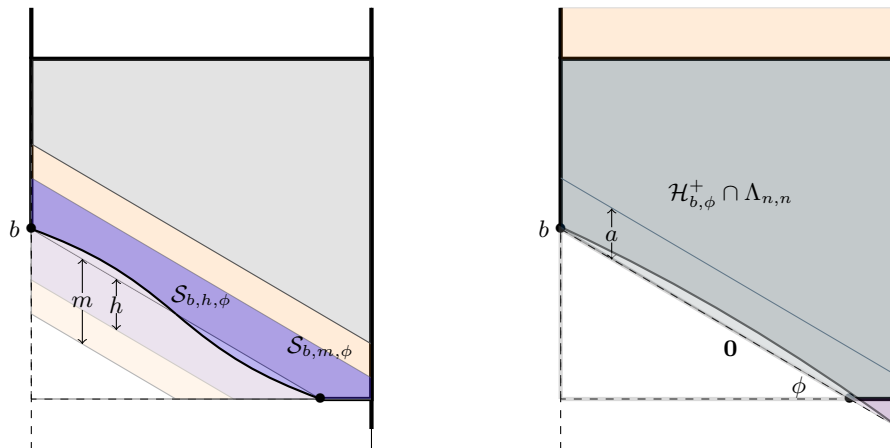
Figure 3: Left: the sets considered in Proposition 3.7, where we bound the probability of the interface exceeding height $\pm h$. Right: the proof of Lemma 3.8 uses monotonicity in b.c., to go from bounding the probability of the order-disorder interface exceeding height a in the gray set to the union of the gray and purple sets, to all shaded regions.

wired on $\partial(\mathcal{H}^+_{b,\phi} \cap \Lambda_{n,n}) \cap \mathcal{H}^+_{b+1,\phi}$ and free elsewhere. There exists $c(\delta, q) > 0$ such that

$$\pi^{1,0,b,\phi}_{\mathcal{H}^+_{b,\phi} \cap \Lambda_{n,n}}(\mathcal{I} \not\subset \mathcal{H}^-_{b+a,\phi} \cap \mathcal{H}^+_{b,\phi} \cap \Lambda_{n,n}) \lesssim n^2 \exp(-ca^2/n).$$

Proof. Let $B = \mathcal{H}^-_{b+a,\phi} \cap \mathcal{H}^+_{b,\phi} \cap \Lambda_{n,n}$ and consider the region $S' = \mathcal{H}^+_{b,\phi} \cap \mathcal{H}^-_{n,0}$.

For a general domain $D \subset \mathcal{S}_n$ let the boundary conditions $(1, 0, b, \phi)$ on it denote wired on $\partial D \cap \mathcal{H}^+_{b+1,\phi}$ and free elsewhere on ∂D. By monotonicity in boundary conditions, and then inclusion of events,

$$\pi^{1,0,b,\phi}_{\mathcal{H}^+_{b,\phi} \cap \Lambda_{n,n}}(\mathcal{I} \not\subset B) \leq \pi^{1,0,b,\phi}_{S'}(\mathcal{I}\!\restriction_{\Lambda_{n,n}} \not\subset B)$$
$$\leq \pi^{1,0,b,\phi}_{S'}(\mathcal{I} \not\subset \mathcal{H}^-_{b+a,\phi} \cap \mathcal{H}^+_{b,\phi}).$$

The application of the FKG inequality here is valid as under $(1, 0, b, \phi)$ boundary conditions, events of the sort "$\mathcal{I}$ exceeds some height a", are decreasing events, as adding edges only pushes the order-disorder interface down. This property will be used throughout the paper. By monotonicity in boundary conditions again,

$$\pi^{1,0,b,\phi}_{S'}(\mathcal{I} \not\subset \mathcal{H}^-_{b+a,\phi} \cap \mathcal{H}^+_{b,\phi}) \leq \pi^{1,0,b,\phi}_{\mathcal{H}^+_{b,\phi}}(\mathcal{I} \not\subset \mathcal{H}^-_{b+a,\phi} \cap \mathcal{H}^+_{b,\phi}).$$

By Proposition 3.6, there exists a $c(\delta, q) > 0$ such that

$$\pi^{1,0,b,\phi}_{\mathcal{H}^+_{b,\phi}}(\mathcal{I} \not\subset \mathcal{H}^-_{b+a,\phi} \cap \mathcal{H}^+_{b,\phi}) \lesssim n^2 e^{-ca^2/n}. \qquad \square$$

Proof of Proposition 3.7. As before, let $(1, 0, b, \phi)$ boundary conditions on $D \subset \mathbb{Z}^2$ denote those that are wired on $\partial D \cap \mathcal{H}^+_{b+1,\phi}$ and free on $\partial D \cap \mathcal{H}^-_{b+1,\phi}$. By a union bound, write

$$\pi^{1,0,b,\phi}_{\mathcal{S}_{b,m,\phi} \cap \Lambda_{n,n}}(\mathcal{I} \not\subset \mathcal{S}_{b,h,\phi}) \leq \pi^{1,0,b,\phi}_{\mathcal{S}_{b,m,\phi} \cap \Lambda_{n,n}}(\mathcal{I} \not\subset \mathcal{H}^+_{b-h,\phi}) + \pi^{1,0,b,\phi}_{\mathcal{S}_{b,m,\phi} \cap \Lambda_{n,n}}(\mathcal{I} \not\subset \mathcal{H}^-_{b+h,\phi}) \qquad (3.5)$$

and consider the two quantities independently. Observe that the first event on the

right-hand side is an increasing event, while the second event is a decreasing event. By reflection symmetry and self-duality, if we prove the desired bound on the latter, for general b, h, ϕ, it implies the former also. By monotonicity in boundary conditions,

$$
\pi^{1,0,b,\phi}_{\mathcal{S}_{b,m,\phi} \cap \Lambda_{n,n}} (\mathcal{I} \not\subset \mathcal{H}^-_{b+h,\phi}) \leq \pi^{1,0,b,\phi}_{\mathcal{S}_{b,m,\phi} \cap \mathcal{H}^+_{b-1,\phi} \cap \Lambda_{n,n}} (\mathcal{I} \not\subset \mathcal{H}^-_{b+h,\phi})
$$
$$
\leq \pi^{1,0,b,\phi}_{\mathcal{H}^+_{b-1,\phi} \cap \Lambda_{n,n}} (\mathcal{I} \not\subset \mathcal{H}^-_{b+h,\phi}) .
$$

The right-hand side above is exactly the probability bounded in Lemma 3.8, from which, along with the symmetry noted above and (3.5), the desired upper bound follows. □

We will also need the following bound in order to prove the mixing time upper bounds on cylinders in §4.3. It is an adaptation of the proof of [18, Lemma A.6] from the Ising model to the FK model via Propositions 3.3–3.4, and we omit some details.

Proposition 3.9. *Fix q to be large enough and $\varepsilon > 0$, and consider the critical FK model on $\Lambda_{n,h}$ for $n^{\frac{1}{2}+\varepsilon} \leq h \leq n$ with $1, 0$ boundary conditions denoting wired on $\partial_{\mathrm{S}} \Lambda_{n,h}$ and free elsewhere. For $\rho \in (0,1)$ and δ small enough, there exists $c(\varepsilon) > 0$ such that*

$$
\pi^{1,0}_{\Lambda_{n,h}} (\mathcal{I} \cap [\![\lfloor \tfrac{n}{2} \rfloor - \delta n, \lfloor \tfrac{n}{2} \rfloor + \delta n]\!] \times [\![0, \rho h]\!] = \emptyset) \gtrsim e^{-c(\rho h)^2 / n} .
$$

Proof. Denote by $B = [\![\lfloor \tfrac{n}{2} \rfloor - \delta n, \lfloor \tfrac{n}{2} \rfloor + \delta n]\!] \times [\![0, \rho h]\!]$. Recall the definition of the cigar-shaped region $U_{\kappa,d,\phi}$ in (3.2). Following [18], for every $-\log_2 n + 2 \leq i \leq \log_2 n - 2$, let z_i be the nearest vertex to

$$
\left(\tilde{x}_i, d \left| \frac{\tilde{x}_i(n - \tilde{x}_i)}{n} \right|^{\frac{1}{2}+\kappa} \right) \qquad \text{where} \qquad \tilde{x}_i = \frac{n}{2} + \frac{\mathrm{sgn}\, i}{4} \sum_{j=1}^{|i|-1} 2^{-j} .
$$

Let $U_{z_i, z_{i+1}}$ be the cigar shaped region $z_i + U_{\varepsilon/2, (1-\rho) \wedge \rho, \phi_{z_i, z_{i+1}}}$ where $d = (1 + \rho) h / n^{\frac{1}{2}+\varepsilon}$ and $\phi_{z_i, z_{i+1}}$ is the angle of the vector from z_i to z_{i+1} (see, e.g., [18, Fig. 8]).

By monotonicity in boundary conditions and $\{\mathcal{I} \cap B = \emptyset\}$ being an increasing event,

$$
\pi^{1,0}_{\Lambda_{n,h}} (\mathcal{I} \cap B = \emptyset) \geq \pi^{1,0}_{\mathcal{H}^-_{h,0}} (\mathcal{I} \cap B = \emptyset) ,
$$

where $(1,0)$ boundary conditions on $\partial \mathcal{H}^-_{h,0}$ are wired on $\partial \mathcal{H}^-_{h,0} \cap \mathcal{H}^-_{0,0}$ and free elsewhere. Since $U_{z_i, z_{i+1}} \cap B = \emptyset$ for all i, if $\mathcal{I} \subset \bigcup_{|i| \leq \log_2 n} U_i$, then the desired property holds.

Lemma A.6 of [18] gives a lower bound on the partition function restricted to interfaces contained in $\bigcup U_i$ as defined above in the setting of the Ising model; the same proof extends that lower bound to the partition function of such FK interfaces, noting that Proposition 3.4 is an analogue of [5, Theorem 4.16]: there exists $c > 0$ such that

$$
\sum_{\mathcal{I} \subset \mathcal{H}^-_{h,0}} \lambda^{|\mathcal{I}| + \sum_{\mathcal{C} \cap \mathcal{I} \neq \emptyset} \Phi(\mathcal{C}, \mathcal{I})} \mathbf{1}\{ \mathcal{I} \subset \bigcup U_i \} \geq \exp(-\beta_c \tau_{1,0}(0) n - c(\rho h)^2 / n) ,
$$

where an error of $(\log n)^{O(1)}$ was absorbed into the term $c(\rho h)^2 / n$ via the assumption that $h \geq n^{1/2+\varepsilon}$ and a choice of a suitable constant $c > 0$.

Furthermore, Proposition 3.4, in particular (3.4), implies there exists $c > 0$ so that

$$
\sum_{\mathcal{I} \subset \mathcal{H}^-_{h,0}} \lambda^{|\mathcal{I}| + \sum_{\mathcal{C} \cap \mathcal{I} \neq \emptyset} \Phi(\mathcal{C}, \mathcal{I})} \leq \exp(-\beta_c \tau_{1,0}(0) n + c(\log n)^c) .
$$

Then writing the desired probability as in [18, (A.36)] as

$$\pi^{1,0}_{\mathcal{H}^-_{h,0}}(\mathcal{I} \cap B \neq \emptyset) \geq \pi^{1,0}_{\mathcal{H}^-_{h,0}}\left(\mathcal{I} \subset \bigcup_{|i| \leq \log_2 n - 2} U_i\right)$$

$$= \frac{\sum_{\mathcal{I} \subset \mathcal{H}^-_{h,0}} \lambda^{|\mathcal{I}| + \sum_{\mathcal{C} \cap \mathcal{I} \neq \emptyset} \Phi(\mathcal{C}, \mathcal{I})} \mathbf{1}\{\mathcal{I} \subset \bigcup U_i\}}{\sum_{\mathcal{I} \subset \mathcal{H}^-_{h,0}} \lambda^{|\mathcal{I}| + \sum_{\mathcal{C} \cap \mathcal{I} \neq \emptyset} \Phi(\mathcal{C}, \mathcal{I})}},$$

and plugging in the above lower bound on the numerator and upper bound on the denominator, concludes the proof. $\qquad\square$

4 Sub-exponential mixing with symmetry-breaking boundary

In this section we prove sub-exponential upper bounds on the inverse spectral gap of Swendsen–Wang dynamics for general Dobrushin boundary conditions. Here, there is a single high probability minimizer of the surface tension for the model, breaking the order-disorder phase symmetry that induces slow mixing in Section 5. In the proofs in this section, we will need to consider random boundary conditions on $\Lambda_{n,m}$.

Definition 4.1 (wired/free-at-infinity b.c.)**.** *Define the* wired-at-infinity *boundary conditions as the distribution over boundary conditions on $\Delta \subset \partial\Lambda_{n,m}$ given by the distribution on partitions of Δ induced by $\pi^1_{\mathbb{Z}^2}(\cdot\lceil_{\mathbb{Z}^2 - \Lambda_{n,m}})$, so that in general, the connections interior to $\Lambda_{n,m}$ do not count towards the induced boundary conditions. Define the* free-at-infinity *boundary conditions analogously as the distribution given by $\pi^0_{\mathbb{Z}^2}(\cdot\lceil_{\mathbb{Z}^2 - \Lambda_{n,m}})$.*

We say that a distribution $\mathbf{P}$ on boundary conditions on Δ dominates $\mathbf{P}'$ if $\xi' \sim \mathbf{P}'$ is a stochastically finer partition than $\xi \sim \mathbf{P}$. Moreover, if boundary conditions are given by different distributions on different subsets of $\partial\Lambda_{n,m}$, then define the overall distribution on boundary conditions by sampling the boundary partitions independently on each of the boundary subsets (with no connections between the subsets). We say such a boundary condition piecewise *dominates wired-at-infinity and is dominated by free-at-infinity.*

A particular example of random boundary conditions that we will be useful later on is the following: if $R \subset D$ are two concentric rectangles, then the boundary conditions induced by the connections from $\omega\lceil_{D-R}$, where ω is sampled from π^1_R, dominate the wired-at-infinity boundary conditions.

4.1 A canonical paths estimate for 2D random cluster models

The canonical paths technique has yielded (see [3, 4, 13, 21]) a very useful upper bound on the mixing time of spin-systems on general graphs; namely, the mixing time is at most exponential in the cut-width of the underlying graph (so that on rectangular subsets of $\mathbb{Z}^2$, it is at most exponential in the shorter side length). In the case of the random cluster model, the long-range interactions complicate this for general boundary condition, but we prove a modified version of such an estimate for a wide class of boundary conditions.

Proposition 4.2. *Let $q > 4$, $p = p_c(q)$, and $\phi \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Consider the FK model on $S = \mathcal{S}_{b,h,\phi} \cap \Lambda_{n,n}$ with boundary conditions $\xi \sim \mathbf{P}$, where $\mathbf{P}$ is arbitrary on $\partial S \cap \partial_{E,W}\Lambda_{n,n}$, and piecewise dominates wired-at-infinity or is dominated by free-at-infinity on each of the other sides of ∂S. There exists $C_q > 0$ such that for every sequence $f_n \to \infty$, the FK Glauber dynamics have*

$$\mathbf{P}\left(\xi : t^\xi_{\mathrm{mix}} \geq |E(S)|^2 \exp[2(4h + f_n)\log q]\right) \leq O(n^2 e^{-C_q f_n}).$$

The proof will use the following straightforward comparison estimate.

Lemma 4.3. *Fix any $p \in (0,1)$ and $q \geq 1$ and consider the FK model on $S = \mathcal{S}_{b,h,\phi} \cap \Lambda_{n,n}$ with arbitrary boundary conditions ξ. For a connected edge subset $F \subset E(S)$, we have for every $\omega \in \{0,1\}^{E(S)}$ and every $\zeta, \zeta' \in \{0,1\}^F$*

$$\pi_S^\xi(\omega \mid \omega\!\restriction_F = \zeta) \leq q^{|\partial F - \partial S| + \tilde{k}(\xi,F) - 1} \pi_S^\xi(\omega \mid \omega\!\restriction_F = \zeta')$$

where $\tilde{k}(\xi, F)$ is the number of components of ξ that intersect both ∂F and $\partial S - \partial F$.

Proof. By the domain Markov property, the difference between $\pi_S^\xi(\cdot \mid \omega\!\restriction_F = \zeta)$ and $\pi_S^\xi(\cdot \mid \omega\!\restriction_F = \zeta')$ is in the boundary conditions that $\zeta \cup \xi$ and $\zeta' \cup \xi$ induce on $S - F$. The boundary partitions induced can differ arbitrarily on $\partial F - \partial S$ contributing a factor of $q^{|\partial F - \partial S|}$; on the other hand, since both configurations use ξ boundary conditions on S, the boundary partitions on the rest of $\partial(S - F)$ can only differ if ζ induces additional boundary connections between distinct boundary components of ξ that reach $\partial S - \partial F$; this accounts for the factor of $q^{\tilde{k}(\xi,F)-1}$. $\qquad\square$

Proof of Proposition 4.2. We modify the proof of the canonical path estimate for spin-systems with short-range interactions to the present setup. Partition every segment L of $\partial S \setminus \partial_{E,W} \Lambda_{n,n}$ on which ξ is independently sampled, into $\lfloor \frac{|L|}{f_n} \rfloor$ sub-segments ℓ_i of f_n vertices each, and possibly an additional exceptional segment ℓ_0. For every edge $e \in L$, let Γ^e be the set of components of ξ that contain vertices on both sides of $L - \{e\}$, which we hereafter refer to as *bridges*. Let

$$\mathcal{E}_{f_n} = \left\{ \xi : \bigcap_L \bigcap_{i \geq 1} \{\exists e \in \ell_i \text{ s.t. } |\Gamma^e| \leq 1\} \right\}. \tag{4.1}$$

We first prove that for every $\xi \in f_n$, the FK Glauber dynamics on S has

$$\mathsf{gap}_{\xi,S} \leq |E(S)|(1 + \sqrt{q}) \exp[2(4h + f_n)\log q]. \tag{4.2}$$

This will follow from a standard application of the canonical paths argument for spin-systems with short-range interactions. Namely, label the edges in S lexicographically in their midpoint, first by horizontal coordinate, then by vertical coordinate, and define the path $\gamma_{\omega,\omega'}$ (identified with a sequence of edges in Ω_{FK} between FK configurations, ω and ω') as follows: let $e_{l_1}, ..., e_{l_k}$ be the sequence of edges on which $\omega(\tilde{e}_{l_i}) \neq \omega'(\tilde{e}_{l_i})$, labeled in their lexicographic ordering. The i'th edge in $\gamma_{\omega,\omega'}$ will then be between configuration

$$\eta = \omega'\!\restriction_{\{e_1,...,e_{l_i-1}\}} \cup \omega\!\restriction_{\{e_{l_i},...,e_{|E(S)|}\}}$$

and its neighbor η' which also has $\eta'(\tilde{e}_i) = \omega'(\tilde{e}_i)$. Also, let η^* be the configuration that is instead given by $\eta^* = \omega\!\restriction_{\{e_1,...,e_{l_i-1}\}} \cup \omega'\!\restriction_{\{e_{l_i},...,e_{|E(S)|}\}}$. Then by Lemma 4.3 applied with the choice of F being $\{e_1, ..., e_{l_i-1}\}$ or $\{e_{l_i}, ..., e_{|E(S)|}\}$,

$$\pi_S^\xi(\omega)\pi_S^\xi(\omega') \leq \pi_S^\xi(\eta)\pi_S^\xi(\eta^*) q^{2(4h+f_n)}$$

as $|\partial F - \partial S| \leq 2h$ and $\tilde{k}(\xi, F) \leq 2h + f_n$. This follows from the fact that $\xi \in \mathcal{E}_{f_n}$ and the nested structure of boundary bridges, and the fact that the sides with arbitrary boundary conditions have height at most $2h$. By construction, for every transition (η, η'), the map $(\omega, \omega') \mapsto (\eta, \eta^*)$ is injective. Moreover, the probability of making any transition in Ω_{FK} is bounded below by $\frac{1}{1+\sqrt{q}}$. Putting all this together, by the path method we see that for every $\xi \in \mathcal{E}_{f_n}$, Eq. (4.2) holds and by (2.2), the corresponding bound with an extra factor of $O(|E(S)|)$ also holds for the mixing time.

It remains to bound the $\mathbf{P}$-probability that $\xi \in \mathcal{E}_{f_n}$. Fix a segment L of $\partial S \setminus \partial_{\mathrm{E,W}} \Lambda_{n,n}$ on which $\mathbf{P}$ is piecewise sampled, and fix a sub-segment ℓ_i, then take a union bound over all such segments and all ℓ_i. Moreover, suppose that the boundary conditions on the segment L are dominated by free-at-infinity (the estimate for the case when the distribution of $\mathbf{P}$ on $\xi\restriction_L$ dominates wired-at-infinity follows by similar reasoning). Since $|\Gamma^e| \geq 1$ is an increasing event, it suffices to show

$$\pi^0_{\mathbb{Z}^2}(\exists e \in \ell_i \text{ s.t. } |\Gamma^e| = 0) \geq 1 - O(n^2 e^{-C_q f_n}) \,.$$

However, by planarity of boundary conditions induced by $\pi^0_{\mathbb{Z}^2}$ on ∂S, the complement of the left-hand side is the event that there exist x, y in the two parts of $L - \ell_i$ such that $x \longleftrightarrow y$, which, if C_q is the constant from (2.1), has probability at most

$$|L - \ell_i| e^{-C_q f_n} \leq 2n e^{-C_q f_n} \,.$$

(For the wired-at-infinity boundary conditions, observe that in order for $|\Gamma^e| > 1$ for every $e \in \ell_i$, there must exist x, y in the two parts of $L - \ell_i$ such that $x \overset{*}{\longleftrightarrow} y$: this is in turn a decreasing event with exponentially decaying probability under $\pi^1_{\mathbb{Z}^2}$.) Taking a union bound over at most $4n$ sub-segments ℓ_i of various segments L, we obtain that

$$\mathbf{P}(\xi \in \mathcal{E}_C) \leq 8n^2 e^{-C_q f_n} \,. \qquad \square$$

Remark 4.4. By standard comparison estimates, one could allow *arbitrary* boundary conditions on any boundary segment of size $O(h)$ of ∂S, paying a cost in the spectral gap of at most $\exp(ch \log q)$. This would follow from bounding the ratio of the Dirichlet forms and the Radon-Nikodym derivative between the two (see e.g., [17, Lemma 2.8] and [10, Eq. (5.1)] for details).

4.2 Dobrushin boundary conditions

In this section, we consider the mixing time of Swendsen–Wang dynamics with boundary conditions that are free on a subset of $\partial \Lambda$ and red elsewhere. While §5 demonstrates that such boundary conditions can induce a slow mixing (at least $\exp(cn)$) by respecting the order-disorder phase symmetry, this section will establish that the mixing time is faster (at most $\exp(n^{1/2+o(1)})$) under boundary conditions that have a single order-disorder interface.

Define a general class of *order-disorder Dobrushin boundary conditions*, whose FK representation is wired on one connected boundary arc and free elsewhere. Let a_n, b_n be two distinct points on $\partial \Lambda_{n,n}$. For marked boundary points (a_n, b_n), FK Dobrushin boundary conditions are those that are wired on the clockwise (starting from the origin) arc (a_n, b_n) and free on (b_n, a_n).

Sketch of proof

Our proof of Theorem 2 adapts the proof of $t_{\mathrm{mix}} \lesssim \exp(c\sqrt{n \log n})$ in [17] for the low-temperature Ising model with plus boundary conditions, but using the censoring inequality Theorem 2.7 instead of the block dynamics approach of [17]. For Dobrushin boundary conditions between (a, b), we sequentially (at times t_i) censor all updates except those in the strip B_i of height $\sqrt{n \log n}$ parallel to the line segment $\langle a, b \rangle$ such that B_i and B_{i+1} overlap on half their height. On the one hand, the canonical paths estimate Proposition 4.2 bounds the mixing time of B_i by exponential in its height, so that we take $t_i - t_{i-1} = O(e^{c\sqrt{n \log n}})$. On the other hand, by Theorem 2.7, if the censored chains started from all wired and all free are coupled with high probability, that bounds

the mixing time of the original chain. To couple these two chains, we systematically push the interfaces of the chains started from these initial configurations down (resp., up), until they are within $O(\sqrt{n \log n})$ of each other. This is possible because with high probability (see Lemma 3.8) the interface of the bottom boundary of B_i (where the free initial configuration is seen) never reaches the top of B_{i-1} and the censored dynamics continues pushing the interface down to $O(\sqrt{n \log n})$ distance of $\langle a, b \rangle$ (see Figure 5).

Proof of Theorem 2. If a_n, b_n are on the same side of $\partial \Lambda_{n,n}$, rotate $\Lambda_{n,n}$ so that they are both on $\partial_S \Lambda_{n,n}$ and the angle of the interface between them will be zero. Then the proof below when a_n and b_n are on different sides applies identically; the only difference is the boundary conditions in the last step of recursion, where the identity coupling still couples all FK chains with probability 1 in the mixing time of that last block.

Now suppose $a_n = (a_n^1, a_n^2)$ and $b_n = (b_n^1, b_n^2)$ are on different sides of $\partial \Lambda_{n,n}$; by rotational and reflective symmetry and self-duality, we can take $a_n \in \partial_W \Lambda_{n,n}$ to be the first point encountered clockwise from the origin, and ensure that $\phi_n = \tan^{-1}(\frac{a_n^2 - b_n^2}{a_n^1 - b_n^1})$ is such that $\phi_n \in [-\frac{\pi}{4}, \frac{\pi}{4}]$. Fix any such choice of $a_n, b_n \in \partial \Lambda_{n,n}$ and let $\phi = \phi_n$.

We establish the theorem for FK Glauber dynamics with (a_n, b_n) Dobrushin boundary conditions. Throughout this proof, let $c_1 = c_1(q) > 0$ be a large enough constant (e.g., $5/C_q$ for C_q from Proposition 4.2 would suffice). Define the overlapping blocks

$$B_i := \mathcal{S}_{a_n + (N-i)\ell, \ell, \phi} \cap \Lambda_{n,n} \qquad (i = 1, \ldots, N) \qquad \text{and}$$
$$B_i := \mathcal{S}_{a_n - (N-i)\ell, \ell, \phi} \cap \Lambda_{n,n} \qquad (i = -1, \ldots, -N+1) \qquad \text{for } N = \left\lceil \frac{n}{\ell} \right\rceil,$$

where we choose $\ell = c_3 \sqrt{n \log n}$ for $c_3 = 4/\sqrt{c_2}$, with $c_2(q)$ as given by Proposition 3.6, so $N \sim c_3^{-1} \sqrt{n/\log n}$ (see Figure 4). Because of our choice of $(B_{\pm j})_{j=1}^N$, as many as N of the B_j may be empty, and henceforth if B_j is empty, we say any associated events hold trivially. Let

$$t_i = i \cdot K \log n \cdot N^2 \cdot |E(B_i \cup B_{-i})|^2 e^{4(4\ell + c_1 \log n) \log q} \qquad \text{for} \qquad 0 \leq i \leq N,$$

for K to be chosen large later. Define the censored chain $\bar{X}_t$: between times t_{i-1} and t_i censor all updates except those in $B_i \cup B_{-i}$. Let $\bar{X}_t^1$ be the censored chain started from $\bar{X}_0 = 1$ and $\bar{X}_t^0$ be the censored chain started from $\bar{X}_0 = 0$. By Theorem 2.7 and (2.6), it suffices to show that there exists a coupling of $\bar{X}_t^0$ and $\bar{X}_t^1$ such that

$$\mathbb{P}(\bar{X}_{t_N}^1 \neq \bar{X}_{t_N}^0) = o(n^{-2}), \tag{4.3}$$

since for any K, c_1 fixed, we have $t_N \lesssim \exp(O(\sqrt{n \log n}))$ as desired.

We now define a monotone coupling of $\bar{X}_t^1$ and $\bar{X}_t^0$ which satisfies the above. For each $i = 1, \ldots, N$, define the event

$$A_i^{\pm} = \left\{ \bar{X}_{t_i}^0 \!\restriction_{E(R_i^{\pm})} \neq \bar{X}_{t_i}^1 \!\restriction_{E(R_i^{\pm})} \right\}, \qquad \text{where} \qquad R_i^{\pm} = \bigcup_{j=1}^i B_{\pm j} \setminus B_{\pm(i+1)}.$$

We can then write under our coupling,

$$\mathbb{P}(\bar{X}_{t_N}^0 \neq \bar{X}_{t_N}^1) \leq \mathbb{P}\left(\bigcup_{i=1}^N (A_i^+ \cup A_i^-) \right)$$
$$\leq \mathbb{P}(A_{N-1}^+) + \mathbb{P}(A_{N-1}^-) + \mathbb{P}(A_N \mid (A_{N-1}^+)^c, (A_{N-1}^-)^c).$$

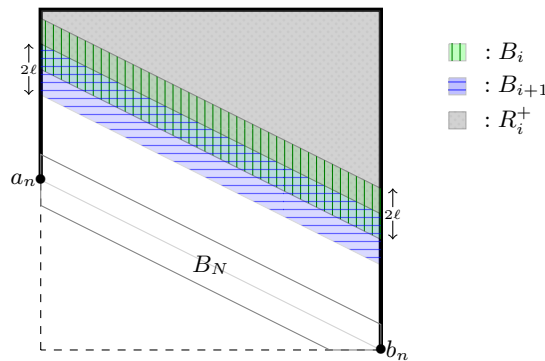The bounds on $\mathbb{P}(A_{N-1}^+)$ and $\mathbb{P}(A_{N-1}^-)$ are analogous (using the duality of the FK model)

Figure 4: The blocks B_i in Λ, each of which are wired on all sides except possibly $\partial_S B_i$ and the region R_i^+ on which we have coupled.

and therefore we only bound the former. Abusing notation slightly, when we consider the restriction of the chain $\bar{X}_t^{1/0}$ to a boundary ∂S, we mean the boundary conditions induced on that line by $\bar{X}_t^{1/0}\!\restriction_{\Lambda-S}$. We will prove the following inductively.

Claim 4.5. *There exists $c(q) > 3$ so that, for large enough K, c_1, the following holds.*

(1) For every $1 \le i \le N-1$, there exists an event F_i measurable w.r.t. $(\bar{X}_t^0, \bar{X}_t^1)_{t \le t_i}$ such that $\mathbb{P}(F_i) \ge 1 - O(in^{-c})$ and $\mathbb{P}(\bar{X}_{t_i}^0\!\restriction_{\partial_N B_{i+1}} \in \cdot \mid F_i)$ dominates the boundary conditions induced by $\pi^1_{R_{N-1}^+}(\cdot\!\restriction_{R_i^+})$ on $\partial_N B_{i+1}$.

(2) For every $1 \le i \le N-1$,

$$\mathbb{P}(A_i^+) = \mathbb{P}(\bar{X}_{t_i}^1\!\restriction_{R_i^+} \ne \bar{X}_{t_i}^0\!\restriction_{R_i^+}) \lesssim in^{-c}.$$

Proof of Claim 4.5. For the base case, (1) holds trivially because the outer boundary on B_1 is just given by the boundary of $\partial\Lambda$ which will be all-wired there. The base case proof of (2) is just a simplification of the proof of the inductive step for (2) so we do not repeat it here. Now suppose both (1) and (2) hold for some $i-1$ and prove they hold for i for a $c(q) > 0$ we will pick later. By the inductive hypothesis, with probability $1 - O((i-1)n^{-c})$, we have $\bar{X}_{t_i}^0\!\restriction_{R_{i-1}^+} = \bar{X}_{t_i}^1\!\restriction_{R_{i-1}^+}$, and the boundary conditions on $\partial_N B_i$ dominate wired-at-infinity. In particular, since on the other sides of B_i both chains' boundary conditions are either all-free or all-wired, by Proposition 4.2, with probability $1 - O((i-1)n^{-c}) - O(n^{-C_q c_1 + 2})$ (where $C_q > 0$ is the constant from that proposition) the mixing time of FK Glauber dynamics with boundary conditions given by $\bar{X}_{t_i}^1\!\restriction_{\partial B_i}$ is at most $|E(B_i \cup B_{-i})|^2 e^{4(4\ell + c_1 \log n)\log q}$. Now suppose that both of these events hold and consider the probability that $\bar{X}_{t_i}^1\!\restriction_{R_i^+} \ne \bar{X}_{t_i}^0\!\restriction_{R_i^+}$.

By submultiplicativity of total-variation distance, with probability $1 - O(n^{-K})$ there exists a coupling of $\bar{X}_{t_i}^{1/0}\!\restriction_{B_i}$ to $\pi_{B_i}^{\bar{X}_{t_{i-1}}^{1/0}}$. It remains to compute the probability of (1) obtaining boundary conditions under $\pi_{B_i}^{\bar{X}_{t_{i-1}}^0}$ that dominate those induced by $\pi^1_{R_{N-1}^+}(\cdot\!\restriction_{R_i^+})$ on $\partial_N B_{i+1}$, and (2) succeeding in coupling $\pi_{B_i}^{\bar{X}_{t_{i-1}}^1}$ to $\pi_{B_i}^{\bar{X}_{t_{i-1}}^0}$ on $B_i - B_{i+1}$ (and therefore on all of R_i^+).

Under the above events, let $(\zeta, 0)$ be the boundary conditions induced on ∂B_i by $\bar{X}_{t_{i-1}}^0$ and $(\zeta, 1)$ be those induced by $\bar{X}_{t_{i-1}}^1$ where ζ is a random boundary condition sampled from a distribution dominating $\pi^1_{R_{N-1}^+}(\cdot\!\restriction_{R_{i-1}^+})$ on $\partial_N B_i$. Then, the monotone coupling of $\pi_{B_i}^{\zeta,0}$ to $\pi_{B_i}^{\zeta,1}$ couples the two on $B_i - B_{i+1}$ whenever the bottom-most horizontal crossing
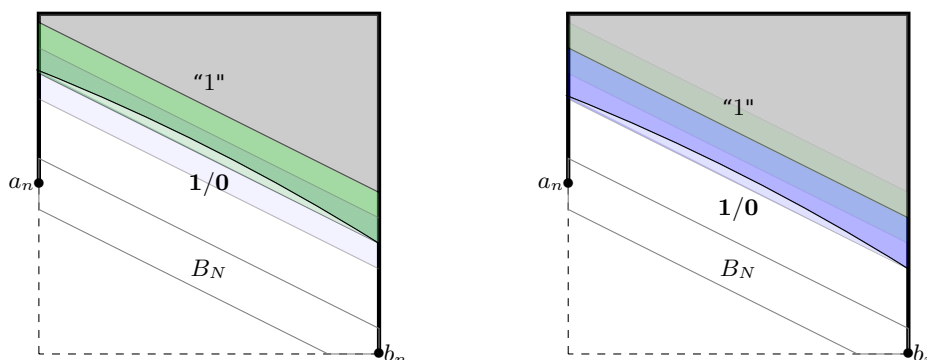
Figure 5: The green and blue blocks B_k, B_{k+1} are updated by the censored dynamics in two consecutive steps; with high probability, the chain $(\bar{X}^0_{t_k})_k$ pushes its interface down toward $\langle a, b \rangle$ by ℓ at every step, and is subsequently coupled to $(\bar{X}^1_{t_k})_k$ on the growing gray region.

$\mathcal{I}$ in the sample from $\pi^{\zeta,0}_{B_i}$ has $\mathcal{I} \subset B_{i+1}$ (see Figure 5). In that case, by revealing dual-edges from the bottom up, the configurations from $\pi^{\zeta,0}_{B_i}$ and $\pi^{\zeta,1}_{B_i}$ could be coupled above that interface and in particular on all of $B_i - B_{i+1}$. (Observe that conditioning on the configuration below the interface, in order to reveal $\mathcal{I}$, cannot affect the boundary conditions above it because on each side of $\partial(R^+_i \cup B_{i+1})$ the boundary conditions are all-wired or all-free and additional connections cannot be induced (cf. the boundary bridges of [9]).)

Observe, also, that when considering ζ dominating the boundary conditions induced by $\pi^1_{R^+_{N-1}}(\cdot\restriction_{R^+_{i-1}})$, since the boundary on $\partial B_{i+1} \cap \partial\Lambda$ is wired, by the domain Markov property, the boundary conditions induced by $\bar{X}^0_{t_{i+1}}$ on $\partial_N B_{i+1}$ when $\mathcal{I} \subset B_{i+1}$ holds will dominate $\pi^1_{R^+_{N-1}}(\cdot\restriction_{R^+_i})$. This implies part (1) if we can bound $\pi^{\zeta,0}_{\partial_N B_i}(\mathcal{I} \subset B_{i+1})$.

Thus, for both (1) and (2) of the induction, it only remains to bound the probability

$$\mathbb{E}\left[\pi^{\zeta,0}_{B_i}(\mathcal{I} \not\subset B_{i+1}) \mid F_{i-1}\right] \leq \mathbb{E}_{\pi^1_{R^+_{N-1}}}\left[\pi^{\omega\restriction_{R^+_{i-1}},0}_{B_i}(\mathcal{I} \not\subset B_{i+1})\right]$$
$$\leq \pi^{1,0}_{R^+_i \cup B_i}(\mathcal{I} \not\subset B_{i+1}), \tag{4.4}$$

where $(1,0)$ boundary conditions on $R^+_i \cup B_i$ are free on $\partial_S B_i$ and wired elsewhere.

In that case, Lemma 3.8 (noting that the estimate there was independent of b) with the choice of $a = \frac{1}{2} c_3 \sqrt{n \log n}$ implies that the probability in (4.4) is at most $O(n^{-6})$. Combining all of the above, the probability that items (1) and (2) hold is at least

$$1 - O((i-1)n^{-c}) + O(n^{-C_q c_1 + 2}) + O(n^{-K}) + O(n^{-6}),$$

which concludes the proof of the induction as long as we take c_1, K large enough that that the latter three terms are all $o(n^{-c})$. $\qquad\square$

By Claim 4.5, we see that both $\mathbb{P}(A^+_{N-1})$ and $\mathbb{P}(A^-_{N-1})$ have probability at most $O(Nn^{-3})$ which is $o(n^{-2})$. It remains to bound $\mathbb{P}(A_N \mid (A^+_{N-1})^c, (A^-_{N-1})^c)$ using similar reasoning to the above. First of all, by part (2), with probability $1 - o(n^{-2})$ the chains $\bar{X}^0_{t_{N-1}}$ and $\bar{X}^1_{t_{N-1}}$ are coupled on both $\partial_N B_N$ and $\partial_S B_N$. Moreover, by part (1), the boundary conditions they induce dominates wired-at-infinity on $\partial_N B_N$ with probability at least $1 - o(n^{-2})$ and likewise, are dominated by free-at-infinity on $\partial_S B_N$ with similar

probability. Therefore, by time t_N, by submultiplicativity of total-variation distance and Proposition 4.2, we have

$$\mathbb{P}(A_N \mid (A_{N-1}^+)^c, (A_{N-1}^-)^c) \leq 1 - o(n^{-2}) - O(n^{-C_q c_1 + 2}) - O(n^{-K}),$$

and for c_1, K large enough, the right-hand side is $1 - o(n^{-2})$. $\qquad\square$

4.3 Sub-exponential mixing on cylinders

For the rectangle $\Lambda_{n,n}$ define boundary conditions (p, R) (resp. $(p, 0)$ or $(p, R, 0)$ boundary conditions) to be periodic boundary conditions on $\partial_{N,S}\Lambda_{n,n}$ and red boundary conditions on $\partial_{E,W}\Lambda_{n,n}$ (resp. free on $\partial_{E,W}\Lambda_{n,n}$ or red on $\partial_W\Lambda_{n,n}$ and free on $\partial_E\Lambda_{n,n}$). We prove the mixing time upper bounds on cylinders with the above boundary conditions (Theorem 3) at the same time. In what follows, we use $c > 0$ to denote the existence of a constant (possibly depending on q), where different appearances of c at different places may refer to different values.

The proof builds on the proof of Theorem 2 in that we use the censoring inequalities to push the FK order-disorder across $\Lambda_{n,n}$ in order to couple the chains $\bar{X}_t^1$ and $\bar{X}_t^0$. We consider the censored dynamics that sequentially update $N = O(n^{\frac{1}{2}-\varepsilon})$ overlapping vertical strips of width $n^{\frac{1}{2}+\varepsilon}$, ordered from left to right. However, unlike the case in Theorem 2, our strips do not have wired boundary conditions on three sides (their boundary conditions on the top and bottom are periodic), and therefore, the interface is pushed to the next strip to be updated with probability $\exp(-cn^{2\varepsilon})$ (rather than $1 - o(1)$): see Figure 6. Thus, with probability $\exp(-cn^{\frac{1}{2}+\varepsilon})$ the interface moves to next strip in N consecutive time steps, so that one will succeed, with high probability, at pushing the interface completely across $\Lambda_{n,n}$ after $\exp(n^{\frac{1}{2}+2\varepsilon})$ attempts.

Proof of Theorem 3. We again prove the upper bound for the FK Glauber dynamics which translates to an upper bound on Swendsen–Wang dynamics by Theorem 2.3. Fix any $\varepsilon, \delta > 0$ small and consider blocks B_i for $i = 1, ..., N$ where $N = \lceil \frac{n}{\ell} \rceil - 1$, given by

$$\left\{ \begin{array}{l} B_{2i-1} = [\![(i-1)\ell, (i+1)\ell]\!] \times [\![\delta n, (1-\delta)n]\!] \\ B_{2i} = [\![(i-1)\ell, (i+1)\ell]\!] \times [\![0, \lfloor \frac{n}{2} \rfloor - \delta n]\!] \cup [\![\lfloor \frac{n}{2} \rfloor + \delta n, n]\!] \end{array} \right\} \quad \text{where} \quad \ell = n^{\frac{1}{2}+\varepsilon},$$

and $B_{2N+1} = B_0 = \Lambda - \cup_{i=1}^N B_{2i-1} \cup B_{2i}$. Since the boundary conditions on $\partial_{N,S}\Lambda_{n,n}$ are periodic, each B_{2i} can be viewed as a single connected rectangle with boundary

$$\partial_{N,S}B_{2i} = [\![(i-1)\ell, (i+1)\ell]\!] \times \left\{ \lfloor \tfrac{n}{2} \rfloor + \delta n, \lfloor \tfrac{n}{2} \rfloor - \delta n \right\}.$$

We prove the mixing time upper bound for $(p, 1, 0)$ boundary conditions. We will pause to comment where the $(p, 1)$ boundary conditions would behave differently (namely only when updating block B_{2N+1}), and on why this does not affect the proof. The $(p, 0)$ boundary conditions can be treated by the dual version of the argument we present.

We will cycle through the blocks B_i periodically, so define

$$B_j = B_{j \bmod (2N+1)} \qquad \text{for all } j > 2N.$$

Define the following censored Markov chain $\bar{X}_t^1$ (resp., $\bar{X}_t^0$) started from initial configuration 1 (resp., 0): for all $i \geq 0$, Let $f_n = n^{\frac{1}{2}+3\varepsilon}$ and let

$$t_i = i \cdot n \cdot N^2 \cdot |E(B_i)|^2 e^{2(4\ell + f_n)\log q};$$

during times $[t_{i-1}, t_i)$, censor all updates outside block B_i. Let

$$T := t_{2N+1} \exp(n^{\frac{1}{2}+2\varepsilon}) = \exp(O(n^{\frac{1}{2}+3\varepsilon})) \, ; \tag{4.5}$$

by Theorem 2.7 and (2.6), it will suffice to show that

$$\mathbb{P}(\bar{X}_T^1 \neq \bar{X}_T^0) = o(n^{-2}) \, .$$

We begin with a uniform upper bound on the mixing times of B_i.

Claim 4.6. Let $m \leq (2N+1)\exp(n^{\frac{1}{2}+2\varepsilon})$ and, for every $i \leq m$, define the event

$$\Upsilon_i = \left\{ \left(t_{\text{mix}}^{\bar{X}_{t_{i-1}}^0}(B_i) \vee t_{\text{mix}}^{\bar{X}_{t_{i-1}}^1}(B_i) \right) \leq |E(B_i)|^2 e^{2(4\ell+f_n)\log q} \right\} , \tag{4.6}$$

where the superscript $\bar{X}_{t_{i-1}}^{\omega_0}$ denotes boundary conditions induced by $\bar{X}_{t_{i-1}}^{\omega_0}$ on B_i. Then

$$\mathbb{P}\left(\bigcup_{i \leq m} \Upsilon_i^c \right) \lesssim m(n^2 e^{-C_q f_n} + e^{-n/2}) \, , \tag{4.7}$$

where $C_q > 0$ is the constant given by Proposition 4.2.

Proof of Claim 4.6. Let Ξ_i be the event that the law of the boundary conditions on B_i under $\bar{X}_{t_{i-1}}^0$ piecewise dominate/are dominated by wired/free-at-infinity, resp., and likewise for $\bar{X}_{t_{i-1}}^1$. We prove inductively that for every $m \leq (2N+1)\exp(n^{\frac{1}{2}+2\varepsilon})$,

$$\mathbb{P}\left(\bigcup_{i \leq m} (\Upsilon_i^c \cup \Xi_i^c) \right) \lesssim m(n^2 e^{-C_q f_n} + e^{-n/2}) \, . \tag{4.8}$$

The base case, $m = 1$, has boundary conditions that are wired on $\partial_{\text{W}} B_1$ and free/wired on $\partial_{\text{N,S,E}} B_1$, and thus $\Upsilon_1 \cap \Xi_1$ holds with probability 1 by a canonical paths estimate (there are no distinct bridges). Suppose now that (4.8) holds for some m; to show that it holds for $m + 1$, it suffices to show that the boundary conditions induced by $\bar{X}_{t_m}^0$, i.e., the chain $\bar{X}_t^0$ (the bound for the chain $\bar{X}_t^1$ follows symmetrically).

Assume that the event $\bigcap_{i \leq m}(\Upsilon_i \cap \Xi_i)$ holds. First of all, we notice that for any i satisfying Υ_i, by the sub-multiplicativity of total-variation distance and definition of t_i,

$$\|\mathbb{P}(\bar{X}_{t_i}^0 \!\restriction_{B_i} \in \cdot) - \pi_{B_i}^{\bar{X}_{t_{i-1}}^0}\|_{\text{tv}} \leq e^{-n} \, ; \tag{4.9}$$

thus, a union bound over all $i \leq m = O(N\exp(n^{\frac{1}{2}+2\varepsilon}))$ implies that we may construct a coupling of $(\bar{X}_t^0)$ and some random variables $\bar{Z}_1, \ldots, \bar{Z}_m$ such that

$$\mathbb{P}\left(\bigcap_{i \leq m} \{\bar{X}_{t_i}^0 \!\restriction_{B_i} = \bar{Z}_i\} \right) \geq 1 - e^{-n/2} \quad \text{where } \bar{Z}_i \sim \pi_{B_i}^{\bar{X}_{t_{i-1}}^0} \text{ for each } i \, .$$

We now claim that the boundary conditions induced by $\bar{X}_{t_m}^0$ on B_{m+1} are such that they piecewise dominate/are dominated by wired/free-at-infinity on ∂B_{m+1}. Consider the case where m is even (the case m is odd follows analogously). According to Ξ_{m-1}, if we denote by ζ the boundary conditions induced by $\bar{X}_{t_{m-2}}^0$ on ∂B_{m-1}, then ζ piecewise dominates/is dominated by wired/free-at-infinity. Hence, when sampling from $\pi_{B_{m-1}}^\zeta$, there would be a well-defined FK order-disorder interface $\mathcal{I}$ between the boundary subsets that are alternately wired and free. For every such interface, by the domain Markov property, the marginal under $\pi_{B_{m-1}}^\zeta$ on each of the connected components of $E(B_m) \setminus \mathcal{I}$ either

dominates wired-at-infinity or is dominated by free-at-infinity.

As a consequence, the boundary conditions on $\partial_{N,S,W} B_m$ are piecewise sampled from distributions dominating/dominated by wired/free-at-infinity, as are the boundary conditions on the vertical bisector of $(B_{m-1} \setminus B_m)$, a subset of $\partial_W B_{m+1}$. Repeating this reasoning for the update on B_m, we see that the boundary conditions on $\partial_{N,S,W} B_{m+1}$ are all sampled from distributions that piecewise dominate/are dominated by wired/free-at-infinity. Finally, the same is true of $\partial_E B_{m+1}$ as it is either completely free if $m \leq 2N + 1$ or it is similarly sampled from $\pi_{B_j}^{\bar{X}^0_{t_{j-1}}}$ for some $j < m$ which likewise satisfied Ξ_j. Thus we deduce that, except with probability $e^{-n/2}$, Ξ_{m+1} holds. Then by Proposition 4.2, the boundary condition on B_{m+1} induced by $\bar{X}^0_{t_m}$ is such that

$$\mathbb{P}(t_{\mathrm{mix}}^{\bar{X}^0_{t_m}}(B_{m+1}) \geq |E(B_{m+1})|^2 e^{2(4\ell + f_n)\log q}) \leq O(n^2 e^{-C_q f_n}).$$

A union bound over the above errors concludes the proof. □

Henceforth, we suppose that the event Υ_i holds for all $i \leq (2N+1)\exp(n^{\frac{1}{2}+2\varepsilon})$, which is the case with probability $1 - \exp(-\Omega(n^{\frac{1}{2}+3\varepsilon}))$ since $f_n = n^{\frac{1}{2}+3\varepsilon}$.

Note that every time increment of t_{2N+1} we make an independent attempt at coupling $\bar{X}^1_t$ to $\bar{X}^0_t$, albeit with initial configurations induced by the chains at the end of the last sweep, and once the two chains are coupled on all of Λ, they will remain coupled for all subsequent times. We will show that there exists $c(\delta, q) > 0$ such that for every k and every two configurations $\omega^1_k = \bar{X}^1_{t_{(k-1)(2N+1)}}$ and $\omega^0_k = \bar{X}^0_{t_{(k-1)(2N+1)}}$,

$$\mathbb{P}(\bar{X}^0_{t_{k(2N+1)}} = \bar{X}^1_{t_{k(2N+1)}} \mid \omega^1_k, \omega^0_k) \geq \mathbb{P}(A^c_{2kN+k-1} \mid \omega^1_k, \omega^0_k)\mathbb{P}(A^c_{k(2N+1)} \mid A^c_{2kN+k-1}, \omega^1_k, \omega^0_k)$$
$$\gtrsim \exp(-cn^{\frac{1}{2}+\varepsilon}), \tag{4.10}$$

where, in analogy to the proof of Theorem 2, if $R_i = \bigcup_{j \leq i \bmod (2N+1)} B_j$,

$$A_{2i} = \left\{ X^0_{t_{2i}} \restriction_{R_{2i} - B_{2i+1} - B_{2i+2}} \neq X^1_{t_{2i}} \restriction_{R_{2i} - B_{2i+1} - B_{2i+2}} \right\}.$$

Eq. (4.10) is sufficient because the probability of not coupling $\bar{X}^0_T$ and $\bar{X}^1_T$ by time T (as defined in (4.5)) would then be bounded by

$$\mathbb{P}\left(\mathrm{Bin}(\exp(n^{\frac{1}{2}+2\varepsilon}), \exp(-cn^{\frac{1}{2}+\varepsilon})) = 0\right) = o(n^{-2}).$$

In order to lower bound the probability in (4.10), we will construct a monotone coupling of the two chains; therefore it suffices to consider the wired and free initial configurations; by the Markov property, it also suffices to only consider the first sweep $k = 1$.

Recall from (4.1) that for any rectangle $\mathcal{E}_{f_n}$ is the set of boundary conditions on B_i such that in every boundary segment of length f_n, there is an edge with at most one boundary component containing vertices on both sides of that edge (bridge).

Claim 4.7. *There exists $c(\delta, q) > 0$ such that, for large K and c_1, the following holds.*

(1) *For every $1 \leq i \leq N$, there exists an event F_i measurable w.r.t. $(\bar{X}^0_t, \bar{X}^1_t)_{t \leq t_{2i}}$ such that $\mathbb{P}(\bar{X}^0_{t_{2i}} \restriction_{\partial_W(B_{2i+1} \cup B_{2i+2})} \in \cdot \mid F_i)$ is in $\mathcal{E}_{f_n}$ and dominates the wired-at-infinity distribution on boundary conditions on $\partial_W(B_{2i+1} \cup B_{2i+2})$.*

(2) *For every $1 \leq i \leq N$, the above defined event F_i satisfies*

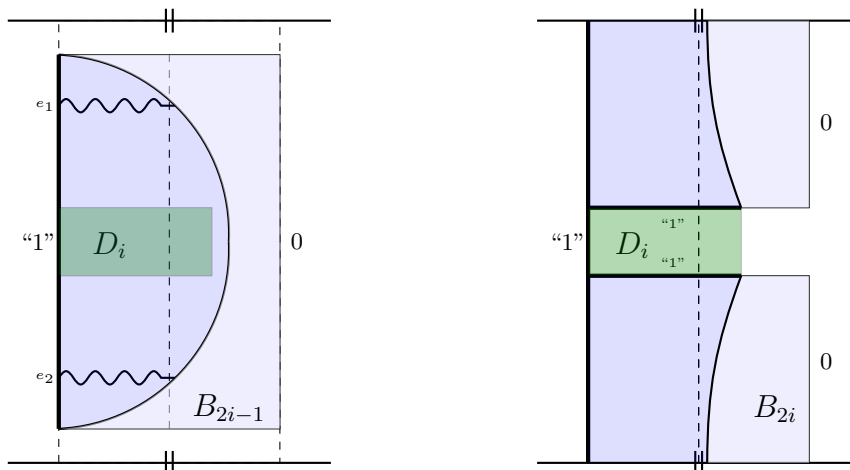$$\mathbb{P}(F_i, A^c_{2i}) \gtrsim \exp(-cin^{2\varepsilon}).$$

Figure 6: Left: the event Γ_{2i-1} where the interface of the component of $\partial_W B_{2i-1}$ does not intersect D_i, and is connected to two edges e_1, e_2 that have no bridges. Right: the event Γ_{2i} where the interface of the component of $\partial_W B_{2i} \cup \partial_{N,S} D_i$ does not intersect the dashed line. Overall, the intersection $\Gamma_{2i-1} \cap \Gamma_{2i}$ pushes the interface forward by ℓn.

Proof of Claim 4.7. For the base case of (1), observe that the boundary conditions on $\partial_W(B_1 \cup B_2)$ dominate wired-at-infinity as they are all-wired; the base case of (2) follows as the proof of the inductive step does, so we do not repeat it here. Now assume that items (1) and (2) hold for $i-1$ (for $c(q) > 0$ to be determined later) and show that they hold for i for the same choice of c. Consider the middle rectangle

$$D_i = [\![(i-1)\ell, (i+\tfrac{1}{2})\ell]\!] \times [\![\lfloor \tfrac{n}{2} \rfloor - \delta n, \lfloor \tfrac{n}{2} \rfloor + \delta n]\!]\,,$$

and let $\mathcal{I}$ be the interface bounding the cluster(s) of $\bar{X}^0_{t_j}\!\restriction_{\partial_W B_j}$. Define the events

$$\Gamma_{2i} = \left\{ \bar{X}^0_{t_{2i}}\!\restriction_{B_{2i}} : \mathcal{I} \cap (B_{2i} - B_{2i+2}) = \emptyset \right\}\,,$$

and

$$\Gamma_{2i-1} = \left\{ \bar{X}^0_{t_{2i-1}}\!\restriction_{B_{2i-1}} : \mathcal{I} \cap D_i = \emptyset \right\} \cap \tilde{\Gamma}_{2i-1}\,,$$

where $\tilde{\Gamma}_{2i-1}$ is the event that $\mathcal{I}$ is connected to e_1, e_2 in $B_{2i-1} - D_i$, where e_1, e_2 are a pair of edges in either side of $\partial_W B_{2i-1} - \partial_W D_i$, with at most one bridge over them (such a pair of edges exist since by assumption (1), the boundary conditions on B_{2i-1} are in $\mathcal{E}_{f_n}$). It is clear that both Γ_{2i-1} and Γ_{2i} are increasing events. Then, we can lower bound

$$\mathbb{P}(A^c_{2i}, F_i) \geq \mathbb{P}(A^c_{2i-2}, F_{i-1})\mathbb{P}\left(A^c_{2i} \cap F_i \cap \Gamma_{2i-1} \cap \Gamma_{2i} \mid A^c_{2i-2}, F_{i-1} \right). \tag{4.11}$$

By the inductive hypothesis, $\mathbb{P}(A^c_{2i-2}, F_{i-1}) \gtrsim \exp(-c(i-1)n^{2\varepsilon})$ and from now on work in the probability space conditioned on $F_{i-1} \cap A^c_{2i-2}$. Under $F_{i-1} \cap A^c_{2i-2}$, the boundary conditions $\bar{X}^0_{t_{2i-2}}\!\restriction_{\partial_W(B_{2i-1} \cup B_{2i})}$ and $\bar{X}^1_{t_{2i-2}}\!\restriction_{\partial_W(B_{2i-1} \cup B_{2i})}$ are coupled and dominate wired-at-infinity (let ζ denote that random boundary condition). In the following time increment $[t_{2i-2}, t_{2i-1})$, only updates on B_{2i-1} are permitted; since we are working on the event Υ_{2i-1}, we can couple $\bar{X}^0_{t_{2i-1}}\!\restriction_{B_{2i-1}}$ to $\pi^{\zeta,0}_{B_{2i-1}}$ and $\bar{X}^1_{t_{2i-1}}\!\restriction_{B_{2i-1}}$ to $\pi^{\zeta,1}_{B_{2i-1}}$ with probability $1 - O(e^{-n})$ by (4.9).

Thus, by monotonicity, we consider the probability of the event $\mathcal{I} \cap D_i = \emptyset$ in Γ_{2i-1}

holding for a sample from $\pi_{B_{2i-1}}^{\zeta,0}$. We claim that for some $c(\delta, q) > 0$,

$$\mathbb{E}[\pi_{B_{2i-1}}^{\zeta,0}(\mathcal{I} \cap D_i = \emptyset) \mid F_{i-1}] \geq \mathbb{E}_{\pi_{\mathbb{Z}^2}^1}[\pi_{B_{2i-1}}^{\xi,0}(\mathcal{I} \cap D_i = \emptyset)]$$
$$\gtrsim \pi_{\Lambda_{(1-2\delta)n,3\ell}}^{1,0}(\mathcal{I} \cap [\![\tfrac{n}{2} - \delta n, \tfrac{n}{2} + \delta n]\!] \times [\![0, \tfrac{5\ell}{2}]\!] = \emptyset) - e^{-c\ell}$$

where the expectation is over all boundary conditions ξ induced by $\pi_{\mathbb{Z}^2}^1$ on $\partial_W B_{2i-1}$ and $(1,0)$ boundary conditions denote wired on $\partial_S \Lambda_{(1-2\delta)n,3\ell}^{1,0}$. Indeed, the second inequality follows from considering the ℓ-enlargement $E_{\ell,2i-1}$ of B_{2i-1} which is its concentric rectangle with extra side length ℓ. If there is a wired circuit in $E_{\ell,2i-1} - B_{2i-1}$ under $\pi_{\mathbb{Z}^2}^1$ (by (2.1) this has probability $1 - e^{-c\ell}$), we can replace the expectation over b.c. induced by $\pi_{\mathbb{Z}^2}^1$ with an expectation over b.c. induced by $\pi_{E_{\ell,2i-1}}^1$. Then extending the free boundary conditions on the other three sides of B_{2i-1} all the way to $\partial_W E_{\ell,2i-1}$ and rotating yields the second inequality. By Proposition 3.9 with the choices $h = 3\ell$ and $\rho = \tfrac{5}{6}$, there exists $c(\delta, q) > 0$ so that the probability in the right-hand side above is at least order $e^{-cn^{2\varepsilon}}$ (see e.g., [10, §5 and Fig. 7] for a similar monotonicity argument).

Moreover, by the exponential decay of dual-connectivities, it is clear that for any $\zeta \in \mathcal{E}_{f_n}$, we have $\pi_{B_{2i-1}}^{\zeta,0}(\tilde{\Gamma}_{2i-1}) \geq \eta$ for some $\eta(q) > 0$. Thus by the FKG inequality,

$$\mathbb{E}[\pi_{B_{2i-1}}^{\zeta,0}(\Gamma_{2i-1})] \gtrsim e^{-cN^{2\varepsilon}}.$$

Let $\mathcal{I}_{2i-1}$ be the interface revealed by the component of $\partial_W B_{2i-1}$. Observe that because Γ_{2i-1} is an increasing event, conditioned on Γ_{2i-1}, if we reveal $\mathcal{I}_{2i-1}$ from east to west, under the monotone coupling of $\pi_{B_{2i-1}}^{\zeta,1}$ to $\pi_{B_{2i-1}}^{\zeta,0}$ the same edges would also be open under $\pi_{B_{2i-1}}^{\zeta,1}$; the same is also true of the edges that constitute $\tilde{\Gamma}_{2i-1}$. Having revealed these sets of open edges under both $\pi_{B_{2i-1}}^{\zeta,1}$ and $\pi_{B_{2i-1}}^{\zeta,0}$, by the domain Markov property (there can not be distinct bridges over the interface we have revealed), $\bar{X}_{t_{2i-1}}^1 \restriction_{D_i} = \bar{X}_{t_{2i-1}}^0 \restriction_{D_i}$ with probability at least $(1 - O(e^{-n}))e^{-cn^{2\varepsilon}}$.

Now consider the next time increment $[t_{2i-1}, t_{2i})$ on B_{2i}. Under the above events, the configuration $\bar{X}_{t_{2i-1}}^0 \restriction_{D_i} \succeq \pi_{\mathbb{Z}^2}^1(\cdot \restriction_{D_i})$, whence by (2.1), with probability at least $1 - e^{-c\delta n}$, there is a pair of primal horizontal crossings of the top and bottom halves of D_i connecting $\partial_W B_{2i-1}$ to $\mathcal{I}_{2i-1}$. In that case, the distribution on boundary conditions induced by $\bar{X}_{t_{2i-1}}^0$ (as well as $\bar{X}_{t_{2i-1}}^1$) on $\partial_{N,S} B_{2i} \cap D_i$ dominates wired-at-infinity. Again, since we are working under the event Υ_{2i}, we just consider the event in Γ_{2i} under $\pi_{B_{2i}}^{\zeta,0}$. By applying (2.1) and enlarging the domains under consideration as in the earlier bound on Γ_{2i-1}, we obtain for some $c(\delta, q) > 0$,

$$\mathbb{P}\left(\Gamma_{2i} \mid A_{2i-2}^c, \Gamma_{2i-1}, F_{i-1}\right) \geq (1 - O(e^{-n}))\mathbb{E}_{\pi_{\mathbb{Z}^2}^1}\left[\pi_{B_{2i}}^{\xi,0,(i+1/2)\ell}(\Gamma_{2i})\right]$$
$$\geq (1 - O(e^{-n}))\mathbb{E}_{\pi_{\mathbb{Z}^2}^1}\left[\pi_{\Lambda_{(1-2\delta)n,3\ell/2}}^{0,\xi}(\mathcal{I} \cap \Lambda_{(1-2\delta)n,\ell} = \emptyset)\right]$$
$$\geq (1 - O(e^{-n}))(\pi_{\Lambda_{n,2\ell}}^{0,1}(\mathcal{I} \cap \Lambda_{n,3\ell/2} = \emptyset) - e^{-c\ell}).$$

Here, the boundary conditions $(\xi, 0)$ in the first line denote ξ (over which we take an expectation) induced on $\partial B_{2i} \cap \{(x,y) : x = (i + \tfrac{1}{2})\ell\}$, and free elsewhere on ∂B_{2i}, and the boundary conditions in the second and third lines denote free on ∂_N of the boundary and respectively ξ and wired elsewhere. The second inequality is a simple consequence of monotonicity in boundary conditions and the third inequality follows from enlarging B_{2i} by $\ell/2$ up to an error of $e^{-c\ell}$ coming from (2.1). By Lemma 3.8 with $\phi = 0$ and $b = 1$, there exists $c(\delta, q) > 0$ such that the probability on the right-hand side above is bounded below by $(1 - O(e^{-n}))(1 - e^{-cn^{2\varepsilon}})$. In that case, revealing the interface from east to west,

we can couple $\bar{X}_{t_{2i}}^0$ to $\bar{X}_{t_{2i}}^1$ beyond the interface (see also Fig. 6), so

$$\mathbb{P}(A_{2i}^c \mid A_{2i-2}, F_{i-1}) \geq (1 - O(e^{-n}))(1 - e^{-cn^{2\varepsilon}})e^{-cn^{2\varepsilon}} \gtrsim \exp(-cn^{2\varepsilon}). \qquad (4.12)$$

Finally, we claim that under the intersection of all the above events, with probability $1 - O(e^{-c\ell}) - O(n^2 e^{-C_q f_n})$, the boundary conditions induced by $\bar{X}_{t_{2i}}^{1/0}$ on $\partial_W B_{2i+1}$ and $\partial_W B_{2i+2}$ are in $\mathcal{E}_{f_n}$ and dominate wired-at-infinity, which combined with (4.12) defines the desired set set F_i such that

$$\mathbb{P}(A_{2i}^c, F_i \mid A_{2i-2}, F_{i-1}) \gtrsim e^{-cn^{2\varepsilon}}.$$

Recall that the configuration on D_i under Γ_{2i-1} and F_{i-1} dominates $\pi_{\mathbb{Z}^2}^1 {\restriction}_{D_i}$. Then, with probability $1 - 2e^{-c\delta n^{\frac{1}{2}+\varepsilon}}$, D_i contains two horizontal crossings connecting $\partial_W D_i$ to $\mathcal{I}_{2i-1}$; since we are also conditioning on Γ_{2i}, averaging over configurations on D_i, with probability $1 - 3e^{-c\delta n^{\frac{1}{2}+\varepsilon}}$, $\partial_W B_{2i+2}$ is surrounded by a wired circuit in $\bar{X}_{t_{2i}}^0$. Similarly, under $\bar{X}_{t_{2i}}^0$, conditional on Γ_{2i-1}, Γ_{2i} and F_{i-1}, the configuration on

$$D_i' = [\![i\ell, (i+2)\ell]\!] \times [\![0, \delta n]\!] \cup [\![(1-\delta)n, n]\!]$$

below the interface revealed by Γ_{2i} dominates $\pi_{\mathbb{Z}^2}^1$ so that with probability $1 - 2e^{-c\delta n^{\frac{1}{2}+\varepsilon}}$, there are horizontal primal connections to that interface in both halves of D_i'. In that case, averaging over configurations in D_i' with probability $1 - 3e^{-c\delta n^{\frac{1}{2}+\varepsilon}}$, there is a wired circuit around $\partial_W B_{2i+1}$ in $\bar{X}_{t_{2i}}^0$ so that the distribution over boundary conditions induced on $\partial_W B_{2i+1}$ also dominates wired-at-infinity. Moreover, as seen in the proof of Proposition 4.2, a boundary condition dominating wired-at-infinity is in $\mathcal{E}_{f_n}$ with probability $1 - O(n^2 e^{-C_q f_n})$. A union bound over the above concludes the proof. $\qquad \square$

As a result, by item (2) of Claim 4.7, there exists some $c(\delta, q) > 0$ for which

$$\mathbb{P}(A_{2N}^c) \gtrsim \exp\left(-cNn^{2\varepsilon}\right) \gtrsim \exp\left(-cn^{\frac{1}{2}+\varepsilon}\right).$$

Moreover, on that event, with high probability, the boundary conditions on $\partial_W B_{2N+1}$ induced by both $\bar{X}_{2N}^1$ and $\bar{X}_{2N}^0$ dominate wired-at-infinity. On the event Υ_{2N+1}, with probability $1 - O(e^{-n})$ one can couple $\bar{X}_{t_{2N+1}}^1$ and $\bar{X}_{t_{2N+1}}^0$ to agree on B_{2N+1}, leading the two chains to be coupled on all of Λ. (It is only at this final step where there is a difference between the $(p, 1, 0)$ and $(p, 1)$ boundary conditions; clearly, if the coupling on B_{2N+1} succeeds in the former situation, it also succeeds in the latter.) $\qquad \square$

5 Slow mixing with phase-symmetric boundary conditions

For a reversible chain with transition kernel $P(x, y)$ and stationary distribution π, define the edge measure Q between $A, B \subset \Omega$ and conductance of the chain, Φ, by

$$Q(A, B) = \sum_{\omega \in A} \pi(\omega) \sum_{\omega' \in B} P(\omega, \omega'), \qquad \text{and} \qquad \Phi = \max_{\mathcal{A} \subset \Omega} \frac{Q(\mathcal{A}, \mathcal{A}^c)}{\pi(\mathcal{A})\pi(\mathcal{A}^c)}.$$

The Cheeger inequality relates these to the gap (see, e.g., [15, §7]), by stating that

$$2\Phi \geq \mathsf{gap} \geq \Phi^2/2. \qquad (5.1)$$

The torus

In Theorem 2 of [10], the authors used the above to construct an exponential bottleneck relying heavily on the topology of the torus and the exponential decay of correlations under $\pi_{\mathbb{Z}^2}^0$ at a discontinuous phase transition point. We restate the result for the critical Swendsen–Wang dynamics for all $q > 4$, which follows from the sharp identification in [6] of the discontinuity of the phase transition for all $q > 4$.

Theorem 5.1 ([10, Theorem 3], given the result of [6])**.** *Let $q > 4$, and consider the Swendsen–Wang dynamics on $(\mathbb{Z}/n\mathbb{Z})^2$ at $\beta = \beta_c(q)$. There exists $c(q) > 0$ such that*

$$t_{\mathrm{mix}} \gtrsim \exp(cn).$$

Observe that exploiting the topology of the torus, unlike the other results in this paper, the above requires neither validity of the cluster expansion nor positivity of surface tension, and therefore holds up through $q > 4$. On the other hand, for q that is sufficiently large, slow mixing at $\beta = \beta_c$ was previously shown in [2] (in any dimension).

Order-disorder mixed boundary conditions

Although the proof of slow mixing on the torus at a discontinuous phase transition relies heavily on the topology of the torus (see proof of Theorem 2 in [10]) we can—at least for sufficiently large q—use a similar approach to prove slow mixing in the presence of mixed wired-free boundary conditions. Exploiting the self-duality, we see that such boundary conditions still exhibit an exponential bottleneck, slowing down the Swendsen–Wang dynamics.

Definition 5.2. *Let $a_n, b_n, c_n, d_n \in \partial\Lambda_{n,n}$ be a set of marked vertices ordered clockwise from the origin around $\partial\Lambda_{n,n}$ (by rotational symmetry, without loss of generality assume $a_n \in \partial_W \Lambda_{n,n}$). The mixed boundary conditions on (a_n, b_n, c_n, d_n) are those that are red on the clockwise boundary arcs (a_n, b_n) and (c_n, d_n) and free on (b_n, c_n) and (d_n, a_n)—all connected subsets of $\partial\Lambda_{n,n}$. We say that (a_n, b_n, c_n, d_n) are ε-separated if $a_n \in \partial_W \Lambda_{n,n}$, at least one of $\{b_n, c_n, d_n\}$ is not contained in $[\![0, \varepsilon n]\!] \times [\![0, n]\!]$, and*

$$\min_{i,j \in a,b,c,d;\, i \neq j} \|i_n - j_n\|_\infty \geq \varepsilon n.$$

Remark 5.3. The requirement of ε-separation in our consideration of mixed boundary conditions arises from the fact that if (a_n, b_n, c_n, d_n) were all on $\partial_W \Lambda_{n,n}$ repeating the proof of Theorem 2 with such boundary conditions would yield that the mixing time is in fact sub-exponential. Clearly, if the four marked vertices are sufficiently close to being on one side or to each other, a similar picture would emerge. The requirement of macroscopic separation ensures that the bottleneck is exponential in n.

With Definition 5.2 in hand, to prove Theorem 1, by rotational symmetry, we wish to prove the following: let q be large, $\varepsilon > 0$, and consider the Swendsen–Wang dynamics for the critical Potts model on $\Lambda = \Lambda_{n,n}$ with mixed boundary on (a_n, b_n, c_n, d_n) that are ε-separated. Then there exists $c(\varepsilon, q) > 0$ such that

$$t_{\mathrm{mix}} \gtrsim \exp(cn).$$

Proof of Theorem 1. By (2.5) and Fact 2.6, it suffices to prove the bound for the FK Glauber dynamics with mixed FK boundary conditions that are wired on the boundary arcs (a_n, b_n) and $(c_n, d_n) \subset \partial\Lambda_{n,n}$ (and the two boundary arcs are wired together) and free elsewhere (denoted by $\pi_\Lambda^{\mathrm{mixed}}$). Observe that by planarity,

$$\left\{ (a_n, b_n) \longleftrightarrow (c_n, d_n) \right\} = \left\{ (b_n, c_n) \overset{*}{\longleftrightarrow} (d_n, a_n) \right\}^c,$$

and therefore, either

$$\pi_\Lambda^{\text{mixed}}\big((a_n, b_n) \longleftrightarrow (c_n, d_n)\big) \leq \frac{1}{2}\,, \quad \text{or} \quad \pi_\Lambda^{\text{mixed}}\big((b_n, c_n) \overset{*}{\longleftrightarrow} (d_n, a_n)\big) \leq \frac{1}{2}\,.$$

By self-duality of the class of ε-separated, mixed boundary conditions, we can suppose without loss of generality that we are in the former case.

Recall the definition of the strips $\mathcal{S}_{b,h,\phi}$, $\mathcal{H}_{b,\phi}^\pm$ in Definition 3.5, and let $a_n = (a_n^1, a_n^2)$, and likewise for b_n, c_n, d_n. Then let $\phi_{a,d} = \tan^{-1}\big(\frac{d_n^2 - a_n^2}{d_n^1 - a_n^1}\big)$ and $\phi_{b,c} = \tan^{-1}\big(\frac{c_n^2 - b_n^2}{c_n^1 - b_n^1}\big)$. Observe that by the ε-separation of (a_n, b_n, c_n, d_n), one of $\phi_{a,d}$ and $\phi_{b,c}$ is in $[-\frac{\pi}{2} + \delta, \frac{\pi}{2} - \delta]$ for some small enough $\delta > 0$ depending only on ε. Suppose without loss of generality that for some $\delta(\varepsilon) > 0$, $\phi_{a,d} \in [-\frac{\pi}{2} + \delta, \frac{\pi}{2} - \delta]$ and consider the strip

$$S = \mathcal{H}_{a_n^1, \phi_{a,d}}^+ \cap \mathcal{H}_{a_n^1 + \varepsilon^2 n, \phi_{a,d}}^- \cap \Lambda\,,$$

Geometrically, by definition of ε-separation, S satisfies $S \cap \partial\Lambda \subset (a_n, b_n) \cup (c_n, d_n)$. There is some $x, h, \phi = \phi_{a,d}$ such that $S = \mathcal{S}_{x, h, \phi_{a,d}} \cap \Lambda$; fix that $x \in \mathbb{R}_+$, $h = \varepsilon^2/2$. Define $\partial_{\text{N}} S = S \cap \mathcal{H}_{x+h-1, \phi}^+$ and $\partial_{\text{S}} S = S \cap \mathcal{H}_{x-h+1, \phi}^-$, and let

$$\mathcal{A} = \Big\{ (a_n, b_n) \overset{S}{\longleftrightarrow} (c_n, d_n) \Big\}$$

be the bottleneck set whose conductance $Q(\mathcal{A}, \mathcal{A}^c)/(\pi(\mathcal{A})\pi(\mathcal{A}^c))$ we bound. Since

$$\mathcal{A} \subset \{ (a_n, b_n) \longleftrightarrow (c_n, d_n) \}\,,$$

we have that $\pi_\Lambda^{\text{mixed}}(\mathcal{A}^c) > \frac{1}{2}$. Therefore, we can write

$$\mathsf{gap} \leq 2\Phi \leq \frac{2Q(\mathcal{A}, \mathcal{A}^c)}{\pi_\Lambda^{\text{mixed}}(\mathcal{A})\pi_\Lambda^{\text{mixed}}(\mathcal{A}^c)} \leq 4\pi_\Lambda^{\text{mixed}}(\partial\mathcal{A} \mid \mathcal{A})\,,$$

(where $\partial\mathcal{A} := \{\omega : P(\omega, \mathcal{A}^c) > 0\}$ and we used a worst-case bound of 1 on the transition rates in $Q(\mathcal{A}, \mathcal{A}^c)$), in which case it suffices to prove that for some $c(q) > 0$,

$$\pi_\Lambda^{\text{mixed}}(\partial\mathcal{A} \mid \mathcal{A}) \lesssim \exp(-c\varepsilon^4 n)\,.$$

For $\omega \in \mathcal{A}$, in order for $P(\omega, \mathcal{A}^c)$ to be positive ($\omega \in \partial\mathcal{A}$), there must exist an edge e in $\mathcal{S}$ that is *pivotal* to $\mathcal{A}$, i.e., $\omega(e) = 1$ and $\omega' = \omega - \{e\} \notin \mathcal{A}$. We estimate the probability $\pi_\Lambda^{\text{mixed}}(\partial\mathcal{A} \mid \mathcal{A})$ by taking a union bound over the probability of any edge, e, in $E(S)$ being pivotal to S.

First examine whether e is closer in its y coordinate to $\partial_{\text{N}} S$ or $\partial_{\text{S}} S$. Suppose without loss of generality, we are in the former case, whence we expose the north-most primal crossing of S, under $\pi_\Lambda^{\text{mixed}}(\omega \mid \mathcal{A})$ (revealing, first, the configuration on $\Lambda \cap \mathcal{H}_{x+h-1, \phi}^+$, then the dual-components of $\partial_{\text{N}} S$ in S). Such a crossing exists by conditioning on $\mathcal{A}$.

Denote by ζ the horizontal crossing we have revealed as such. By the conditioning on S, it is clear that ζ must connect (a_n, b_n) to (c_n, d_n) in S. In order for e to be pivotal to S, e must be an open edge in ζ and there must exist a dual crossing connecting e to $\partial_{\text{S}} S$. Let D be the southern connected component of $E(\Lambda) - \zeta$; we wish to bound

$$\pi_\Lambda^{\text{mixed}}\left(e \overset{D^*}{\longleftrightarrow} \partial_{\text{S}} S \mid \mathcal{A}, \zeta, \omega\!\restriction_\zeta = 1\right)\,.$$

By monotonicity in boundary conditions, if we let $R = D \cup S$, for every such ζ,

$$\pi_\Lambda^{\text{mixed}}(\omega\!\restriction_D \mid \mathcal{A}, \zeta, \omega\!\restriction_\zeta = 1) = \pi_D^{1,0} \succeq \pi_R^{1,0}(\omega\!\restriction_D)\,,$$

where $(1,0)$ boundary conditions denote free on $\partial R - S$ and wired elsewhere.

We can decompose the probability

$$\pi_R^{1,0}\left(e \xrightarrow{D^*} \partial_{\mathsf{S}} S\right) \leq \pi_R^{1,0}\left(e \xleftrightarrow{*} \partial_{\mathsf{S}} S\right)$$

into the event Γ_1 that the dual-component of $\partial_{\mathsf{S}} S$ (and thus the interface of R with $(1,0)$ boundary conditions) is a subset of $S \cap \mathcal{H}_{x-h/2,\phi}^-$, and Γ_1^c. Under Γ_1^c, since e is closer in its y-coordinate to $\partial_{\mathsf{N}} S$, the vertical distance between e and $\partial_{\mathsf{S}} S$ is at least $h/2$ so that $e \notin S \cap \mathcal{H}_{x-h/2,\phi}^-$ and e cannot be dual-connected to $\partial_{\mathsf{S}} S$.

Bounding the probability of Γ_1 by monotonicity in boundary conditions and Proposition 3.7, there exists $c(q) > 0$ such that for every ζ,

$$\pi_\Lambda^{\mathrm{mixed}}(\Gamma_1 \mid \mathcal{A}, \zeta, \omega\!\restriction_\zeta = 1) \leq \pi_{\mathcal{S}_{a_n^1,h/2,\phi} \cap \Lambda}^{1,0,a_n^1,\phi}(\mathcal{I} \not\subset \mathcal{S}_{a_n^1,h/4,\phi} \cap \Lambda) \lesssim n^2 \exp(-ch^2 n)\,.$$

Under Γ_1 we can take a worst case bound of one on the probability of $e \xleftrightarrow{D^*} \partial_{\mathsf{S}} S$. Therefore, for some $c(q) > 0$, we have $\pi_\Lambda^{\mathrm{mixed}}(\partial\mathcal{A} \mid \mathcal{A}) \lesssim \exp(-c\varepsilon^4 n)$.

Using the above as a bound on $Q(\mathcal{A}, \mathcal{A}^c)/\pi_\Lambda^{\mathrm{mixed}}(\mathcal{A})$ in Eq. (5.1) and plugging into Eq. (5.1) implies for the FK Glauber dynamics and, by Eq. (2.4), Swendsen–Wang dynamics with mixed order-disorder boundary conditions, $\mathtt{gap}^{-1} \gtrsim \exp(c\varepsilon^4 n)$. □

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