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A new approach for the construction of a Wasserstein diffusion

Victor Marx*

Abstract

We propose in this paper a construction of a diffusion process on the space $\mathcal{P}_2(\mathbb{R})$ of probability measures with a second-order moment. This process was introduced in several papers by Konarovskyi (see e.g. [12]) and consists of the limit as N tends to $+\infty$ of a system of N coalescing and mass-carrying particles. It has properties analogous to those of a standard Euclidean Brownian motion, in a sense that we will precise in this paper. We also compare it to the Wasserstein diffusion on $\mathcal{P}_2(\mathbb{R})$ constructed by von Renesse and Sturm in [22]. We obtain that process by the construction of a system of particles having short-range interactions and by letting the range of interactions tend to zero. This construction can be seen as an approximation of the singular process of Konarovskyi by a sequence of smoother processes.

Keywords: Wasserstein diffusion; interacting particle system; coalescing particles; modified Arratia flow; Brownian sheet; differential calculus on Wasserstein space; Itô formula for measure-valued processes.
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1 Introduction

This paper introduces a new approach to construct the stochastic diffusion process studied by Konarovskyi (see [10, 11, 12, 13]). It is a close relative to the Wasserstein diffusion, introduced by von Renesse and Sturm [22]. Our interest is to construct an analogous process to the Euclidean Brownian motion taking values on the Wasserstein space $\mathcal{P}_2(\mathbb{R})$, defined as the set of probability measures on $\mathbb{R}$ having a second-order moment.

In [22], von Renesse and Sturm construct a strong Markov process called *Wasserstein diffusion* on $\mathcal{P}_2(M)$, for M equal either to the interval $[0,1]$ or to the circle $\mathbb{S}^1$. Two major features of that process illustrate the analogy with the standard Brownian motion on a Euclidean space. First, the energy of the martingale part of the Wasserstein diffusion

*Université Côte d'Azur, CNRS, Laboratoire J.A. Dieudonné UMR 7351, France.
E-mail: victor.marx@univ-cotedazur.fr

has the same form as that of a k-dimensional standard Brownian motion, up to replacing the Euclidean norm on $\mathbb{R}^k$ by the L_2-Wasserstein distance:

$$d_W(\mu,\nu) = \inf \mathbb{E}\left[|X-Y|^2\right]^{1/2},$$

where the infimum is taken over all couplings of two random variables X and Y such that X (resp. Y) has law μ (resp. ν). It should be noticed that the geometry of $\mathcal{P}_2(M)$, equipped with the Wasserstein distance, for M a Euclidean space, was the subject of fundamental studies conducted by Ambrosio, Gigli, Savare, Villani, Lions and many others (see [1, 5, 15, 20, 21]), which led to important improvements in optimal transport theory. Second, the transition costs of the Wasserstein diffusion are given by a Varadhan formula (see [22], Corollary 7.19). The formula is identical to the Euclidean case, up to the replacement of the Euclidean norm by d_W.

Although the existence of a Wasserstein diffusion was initially proven by von Renesse and Sturm using Dirichlet processes and the theory of Dirichlet forms (see [8]), it can also be obtained as a limit of finite-dimensional systems of interacting particles, see [2, 19]. Nevertheless, we will focus in this paper on a construction of a system of particles which seems more natural and simpler and which is due to Konarovskyi in [10, 12].

1.1 Konarovskyi's model

In [12], Konarovskyi studies a simple system of N interacting and coalescing particles and proves its convergence to an infinite-dimensional process which has the features of a diffusion on the L_2-Wasserstein space of probability measures (see also [10, 11, 13]). However, even if it has common properties with the diffusion of von Renesse and Sturm, there are also important differences between the two processes. An outstanding property of Konarovskyi's process is the fact that, for a large family of initial measures, it takes values in the set of measures with finite support for each time $t>0$ (see [11]), whereas the values of the Wasserstein diffusion of von Renesse and Sturm are probability measures on $[0,1]$ with no absolutely continuous part and no discrete part.

The model introduced by Konarovskyi is a modification of the Arratia flow, also called Coalescing Brownian flow, introduced by Arratia [3] and subject of many interest, among others in [7, 14, 16, 17]. It consists of Brownian particles starting at discrete points of the real line and moving independently until they meet another particle: when they meet, they stick together to form a single Brownian particle.

In his model (see [12]), Konarovskyi adds a mass to every particle: at time $t=0$, N particles, denoted by $(x_k(t))_{k\in\{1,\dots,N\}}$, start from N points regularly distributed on the unit interval $[0,1]$, and each particle has a mass equal to $\frac{1}{N}$. When two particles stick together, they form as in the standard Arratia flow a unique particle, but with a mass equal to the sum of the two incident particles. Furthermore, the quadratic variation process of each particle is assumed to be inversely proportional to its mass. In other words, the heavier a particle is, the smaller its fluctuations are.

Konarovskyi constructs an associated process $(y^N(u,t))_{u\in[0,1],t\in[0,T]}$ in $\mathcal{D}([0,1],\mathcal{C}[0,T])$, the set of càdlàg functions on $[0,1]$ taking values in $\mathcal{C}[0,T]$, by defining:

$$y^N(u,t) := \sum_{k=1}^N x_k(t)\mathbb{1}_{\{u\in[\frac{k-1}{N},\frac{k}{N})\}} + x_N(t)\mathbb{1}_{\{u=1\}}.$$

In other words, $y^N(\cdot,t)$ is the quantile function associated to the empirical measure $\frac{1}{N}\sum_{k=1}^N \delta_{x_k(t)}$. Konarovskyi showed in [12] that the sequence $(y^N)_{N\geqslant 1}$ is tight in $\mathcal{D}([0,1],\mathcal{C}[0,T])$. Hence, by passing to the limit upon a subsequence, there exists a process $(y(u,t))_{u\in[0,1],t\in[0,T]}$ belonging to $\mathcal{D}([0,1],\mathcal{C}[0,T])$ and satisfying the following four properties:

(i_0) for all $u \in [0,1]$, $y(u,0) = u$;

(ii) for all $u \leqslant v$, for all $t \in [0,T]$, $y(u,t) \leqslant y(v,t)$;

(iii) for all $u \in [0,1]$, $y(u,\cdot)$ is a square integrable continuous martingale relatively to the filtration $(\mathcal{F}_t)_{t\in[0,T]} := (\sigma(y(v,s), v \in [0,1], s \leqslant t))_{t\in[0,T]}$;

(iv) for all $u, u' \in [0,1]$,

$$\langle y(u,\cdot), y(u',\cdot)\rangle_t = \int_0^t \frac{\mathbb{1}_{\{\tau_{u,u'} \leqslant s\}}}{m(u,s)} \mathrm{d}s,$$

where $m(u,t) := \int_0^1 \mathbb{1}_{\{\exists s\leqslant t:\, y(u,s)=y(v,s)\}}\mathrm{d}v$; $\tau_{u,u'} := \inf\{t \geqslant 0 : y(u,t) = y(u',t)\} \wedge T$.

By transporting the Lebesgue measure on $[0,1]$ by the map $y(\cdot,t)$, we obtain a measure-valued process $(\mu_t)_{t\in[0,T]}$ defined by: $\mu_t := \mathrm{Leb}\,|_{[0,1]} \circ y(\cdot,t)^{-1}$. In other words, $u \mapsto y(u,t)$ is the quantile function associated to μ_t. An important feature of this process is that for each positive t, μ_t is an atomic measure with a finite number of atoms, or in other words that $y(\cdot,t)$ is a step function.

More generally, Konarovskyi proves in [11] that this construction also holds for a greater family of initial measures μ_0. He constructs a process y^g in $\mathcal{D}([0,1],\mathcal{C}[0,T])$ satisfying $(ii) - (iv)$ and:

(i) for all $u \in [0,1]$, $y^g(u,0) = g(u)$,

for every non-decreasing càdlàg function g from $[0,1]$ into $\mathbb{R}$ such that there exists $p > 2$ satisfying $\int_0^1 |g(u)|^p \mathrm{d}u < \infty$. In other words, he generalizes the construction of a diffusion starting from any probability measure μ_0 satisfying $\int_{\mathbb{R}} |x|^p \mathrm{d}\mu_0(x) < \infty$ for a certain $p > 2$, where $\mu_0 = \mathrm{Leb}\,|_{[0,1]} \circ g^{-1}$, which means that g is the quantile function of the initial measure. The property that $y^g(\cdot,t)$ is a step function for each $t > 0$ remains true for this larger class of functions g.

The process y^g is said to be *coalescent*: almost surely, for every $u, v \in [0,1]$ and for every $t \in (\tau_{u,v}, T]$, we have $y^g(u,t) = y^g(v,t)$ (recall that $\tau_{u,v} = \inf\{t \geqslant 0 : y^g(u,t) = y^g(v,t)\} \wedge T$). This property is a consequence of (ii), (iii) and of the fact that for each $t > 0$, $y^g(\cdot,t)$ is a step function (see [13, p.11]). Therefore, we can rewrite the formula for the mass as follows:

$$m^g(u,t) = \int_0^1 \mathbb{1}_{\{\exists s\leqslant t:\, y^g(u,s)=y^g(v,s)\}}\mathrm{d}v = \int_0^1 \mathbb{1}_{\{y^g(u,t)=y^g(v,t)\}}\mathrm{d}v.$$

Moreover, we can compare the diffusive properties of the process $(\mu_t)_{t\in[0,T]}$ in the Wasserstein space $\mathcal{P}_2(\mathbb{R})$ with the Wasserstein diffusion of von Renesse and Sturm. To that extent and thanks to Lions' differential calculus on $\mathcal{P}_2(\mathbb{R})$ ([15, 5]), we give in Appendix A an Itô formula on $\mathcal{P}_2(\mathbb{R})$ for the process $(\mu_t)_{t\in[0,T]}$ in order to describe the energy of the martingale part of this diffusion. Appendix A also contains a small introduction to the differentiability on $\mathcal{P}_2(\mathbb{R})$ in the sense of Lions.

1.2 Approximation of a Wasserstein diffusion

In this paper, we propose a new method to construct a process y satisfying properties (i)-(iv), by approaching y by a sequence of smooth processes. Finding smooth approximations of processes having singularities has already led to interesting results, typically in the case of the Arratia flow. Piterbarg [17] shows that the Coalescing Brownian flow is the weak limit of isotropic homeomorphic flows in some space of discontinuous functions, and deduces from the properties of the limit process a careful description of contraction and expansion regions of homeomorphic flows. Dorogovtsev's approximation [7] is based on a representation of the Arratia flow with a Brownian sheet.

We propose an adaptation of Dorogovtsev's idea in the case of Wasserstein diffusions. First, we show that a process y satisfying (i)-(iv) admits a representation in terms of a Brownian sheet; we refer to the lectures of Walsh [23] for a complete introduction to Brownian sheet and to Section 2 for the characterization of Brownian sheet which we use in this paper.

Theorem 1.1. *Let $g : [0,1] \to \mathbb{R}$ be a non-decreasing and càdlàg function such that there exists $p > 2$ satisfying $\int_0^1 |g(u)|^p \mathrm{d}u < +\infty$. Let y be a process in $L_2([0,1], \mathcal{C}[0,T])$ that satisfies conditions (i), (ii), (iii) and (iv). There exists a Brownian sheet w on $[0,1] \times [0,T]$ such that for all $u \in [0,1]$ and $t \in [0,T]$:*

$$y(u,t) = g(u) + \int_0^t \int_0^1 \frac{\mathbb{1}_{\{y(u,s)=y(u',s)\}}}{m(u,s)} \mathrm{d}w(u',s), \tag{1.1}$$

where $m(u,s) = \displaystyle\int_0^1 \mathbb{1}_{\{y(u,s)=y(v,s)\}} \mathrm{d}v$.

Remark 1.2. We refer to Appendix A to justify the use of the term "Wasserstein diffusion" for a process satisfying equation (1.1). Indeed, we can write an Itô formula for this process for a smooth function $u : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$. As in the case of the standard Euclidean Brownian motion, the quadratic variation of the martingale term is proportional to the square of the gradient of u, in the sense of Lions' differential calculus on $\mathcal{P}_2(\mathbb{R})$, which is the same as the differential calculus on the Wasserstein space (see [6, Section 5.4]).

The aim of this paper is to construct a sequence of smooth processes approaching y in the space $L_2([0,1], \mathcal{C}[0,T])$. Therefore, we use the representation (1.1) in terms of a Brownian sheet of y and, given a positive parameter σ, we replace in the latter representation the indicator functions by a smooth function φ_σ equal to 1 in the neighbourhood of 0 and whose support is included in the interval $\left[-\frac{\sigma}{2}, \frac{\sigma}{2}\right]$ of small diameter σ. Fix $\sigma > 0$ and $\varepsilon > 0$. Given a Brownian sheet w on $[0,1] \times [0,T]$, we prove the existence of a process $y_{\sigma,\varepsilon}$ satisfying:

$$y_{\sigma,\varepsilon}(u,t) = g(u) + \int_0^t \int_0^1 \frac{\varphi_\sigma(y_{\sigma,\varepsilon}(u,s) - y_{\sigma,\varepsilon}(u',s))}{\varepsilon + m_{\sigma,\varepsilon}(u,s)} \mathrm{d}w(u',s), \tag{1.2}$$

where $m_{\sigma,\varepsilon}(u,s) := \int_0^1 \varphi_\sigma^2(y_{\sigma,\varepsilon}(u,s) - y_{\sigma,\varepsilon}(v,s)) \mathrm{d}v$ can be seen as a kind of mass of particle $y_{\sigma,\varepsilon}(u)$ at time s. Remark that, due to the fact that the support of φ_σ is small, only the particles located at a distance lower than $\frac{\sigma}{2}$ of particle u at time s are taken into account in the computation of the mass $m_{\sigma,\varepsilon}(u,s)$.

The smooth process $(y_{\sigma,\varepsilon}(u,t))_{u \in [0,1], t \in [0,T]}$ offers several advantages. First, we are able to construct a strong solution $(y_{\sigma,\varepsilon}, w)$ to equation (1.2), whereas in equation (1.1), we do not know if, given a Brownian sheet w, there exists an adapted solution y. Second, in Konarovskyi's process, the question of uniqueness of a solution to (1), even in the weak sense, or equivalently the question of uniqueness of a process in $L_2([0,1], \mathcal{C}[0,T])$ satisfying conditions (i)-(iv), remains open. Here, pathwise uniqueness holds for equation (1.2). Moreover, the measure-valued process $(\mu_t^{\sigma,\varepsilon})_{t \in [0,T]}$ associated to the process of quantile functions $(y_{\sigma,\varepsilon}(\cdot,t))_{t \in [0,T]}$ does generally no longer consist of atomic measures. For example, if $g(u) = u$, $(\mu_t^{\sigma,\varepsilon})_{t \in [0,T]}$ is a process of absolutely continuous measures with respect to the Lebesgue measure.

Let $L_2[0,1]$ be the usual space of square integrable functions from $[0,1]$ to $\mathbb{R}$, and $(\cdot,\cdot)_{L_2}$ the usual scalar product. We denote by $L_2^\uparrow[0,1]$ the set of functions $f \in L_2[0,1]$ such that there exists a non-decreasing and therefore *càdlàg* (*i.e.* right-continuous with left limits everywhere) element in the equivalence class of f. Let $\mathcal{D}((0,1), \mathcal{C}[0,T])$ be the space of right-continuous $\mathcal{C}[0,T]$-valued functions with left limits, equipped with the Skorohod metric.

We follow the definition given in [9, p.21]:

Definition 1.3. *An* $(\mathcal{F}_t)_{t\in[0,T]}$*-adapted process* M *is called an* $L_2^{\uparrow}[0,1]$-valued $(\mathcal{F}_t)_{t\in[0,T]}$-martingale *if* M_t *belongs to* $L_2^{\uparrow}[0,1]$ *for each* $t\in[0,T]$*, if* $\mathbb{E}\left[\|M_t\|_{L_2}\right]<\infty$ *and if for each* $h\in L_2[0,1]$*,* $(M_t,h)_{L_2}$ *is a real-valued* $(\mathcal{F}_t)_{t\in[0,T]}$*-martingale. The martingale is said to be* square integrable *if for each* $t\in[0,T]$*,* $\mathbb{E}\left[\|M_t\|_{L_2}^2\right]<+\infty$*, and* continuous *if the process* $t\mapsto M_t$ *is a continuous function from* $[0,T]$ *to* $L_2[0,1]$*.*

Let us denote by $\overline{\mathbb{R}}:=\mathbb{R}\cup\{-\infty,+\infty\}$ and by $\mathcal{L}_{2+}^{\uparrow}[0,1]$ the set of all non-decreasing and càdlàg functions $g:[0,1]\to\overline{\mathbb{R}}$ such that there exists $p>2$ for which $\int_0^1|g(u)|^p\mathrm{d}u<+\infty$. Let $\mathbb{Q}_+=\mathbb{Q}\cap[0,1]$. The following Theorem states the convergence of the mollified sequence $(y_{\sigma,\varepsilon})_{\sigma>0,\varepsilon>0}$ to a limit process satisfying properties $(i)-(iv)$. It uses the framework introduced by Konarovskyi in [11]:

Theorem 1.4. *Let* $g\in\mathcal{L}_{2+}^{\uparrow}[0,1]$*. For each positive* σ *and* ε*, there exists a solution* $y_{\sigma,\varepsilon}$ *to equation* (1.2) *such that* $(y_{\sigma,\varepsilon}(u,t))_{u\in[0,1],t\in[0,T]}$ *belongs to* $L_2([0,1],\mathcal{C}[0,T])$ *and almost surely, for each* $t\in[0,T]$*,* $y_{\sigma,\varepsilon}(\cdot,t)\in L_2^{\uparrow}[0,1]$*.*

Furthermore, up to extracting a subsequence, the sequence $(y_{\sigma,\varepsilon})_{\varepsilon>0}$ *converges in distribution in* $L_2([0,1],\mathcal{C}[0,T])$ *for every* $\sigma\in\mathbb{Q}_+$ *as* ε *tends to* 0 *to a limit* y_σ *and the sequence* $(y_\sigma)_{\sigma\in\mathbb{Q}_+}$ *converges in distribution in* $L_2([0,1],\mathcal{C}[0,T])$ *as* σ *tends to* 0 *to a limit* y*. Let* $Y(t):=y(\cdot,t)$*. Then* $(Y(t))_{t\in[0,T]}$ *is a* $L_2^{\uparrow}[0,1]$*-valued process such that:*

$(C1)$ $Y(0)=g$*;*

$(C2)$ $(Y(t))_{t\in[0,T]}$ *is a square integrable continuous* $L_2^{\uparrow}[0,1]$*-valued* $(\mathcal{F}_t)_{t\in[0,T]}$*-martingale, where* $\mathcal{F}_t:=\sigma(Y(s),s\leqslant t)$*;*

$(C3)$ *almost surely, for every* $t>0$*,* $Y(t)$ *is a step function,* i.e. *there exist* $n\geqslant 1$*,* $0=a_1<a_2<\dots<a_n<a_{n+1}=1$ *and* $z_1<z_2<\dots<z_n$ *such that for all* $u\in[0,1]$

$$Y(t)(u)=y(u,t)=\sum_{k=1}^n z_k\mathbb{1}_{\{u\in[a_k,a_{k+1})\}}+z_n\mathbb{1}_{\{u=1\}};$$

$(C4)$ y *belongs to* $\mathcal{D}((0,1),\mathcal{C}[0,T])$ *and for every* $u\in(0,1)$*,* $y(u,\cdot)$ *is a square integrable and continuous* $(\mathcal{F}_t)_{t\in[0,T]}$*-martingale and*

$$\mathbb{P}\left[\forall u,v\in(0,1),\forall s\in[0,T],y(u,s)=y(v,s)\text{ implies }\forall t\geqslant s,y(u,t)=y(v,t)\right]=1;$$

$(C5)$ *for each* u *and* u' *in* $(0,1)$*,*

$$\langle y(u,\cdot),y(u',\cdot)\rangle_t=\int_0^t\frac{\mathbb{1}_{\{\tau_{u,u'}\leqslant s\}}}{m(u,s)}\mathrm{d}s,$$

where $m(u,s)=\displaystyle\int_0^1\mathbb{1}_{\{y(u,s)=y(v,s)\}}\mathrm{d}v$ *and* $\tau_{u,u'}=\inf\{t\geqslant 0:y(u,t)=y(u',t)\}\wedge T$*.*

Remark 1.5. More precisely, the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ is given by:

$$\mathcal{F}_t=\sigma((Y(s),h)_{L_2},s\leqslant t,h\in L_2[0,1]).$$

Remark 1.6. By property $(C4)$, the limit process y is said to be coalescent: if for a certain time t_0, two particles $y(u,t_0)$ and $y(v,t_0)$ coincide, then they move together forever, *i.e.* $y(u,t)=y(v,t)$ for every $t\geqslant t_0$.

It is interesting to wonder how the coalescence property of the process y translates to its smooth approximation $y_{\sigma,\varepsilon}$: two paths $(y_{\sigma,\varepsilon}(u,t))_{t\in[0,T]}$ and $(y_{\sigma,\varepsilon}(v,t))_{t\in[0,T]}$, starting from two distinct points $g(u)$ and $g(v)$, do not meet, which means that $y_{\sigma,\varepsilon}(\cdot,t)$ is non-decreasing for each fixed t. If $y_{\sigma,\varepsilon}(u,\cdot)$ and $y_{\sigma,\varepsilon}(v,\cdot)$ get close enough, at distance smaller than σ, they begin to interact and to move together, whereas as long as they remain at distance greater than σ, they move "independently": more precisely, the covariation $\langle y_{\sigma,\varepsilon}(u,\cdot),y_{\sigma,\varepsilon}(v,\cdot)\rangle_t$ is equal to zero for every time $t\leqslant\tau_{u,v}^\sigma:=\inf\{s\geqslant 0:|y_{\sigma,\varepsilon}(u,s)-y_{\sigma,\varepsilon}(v,s)|\leqslant\sigma\}$ (see figure 1).

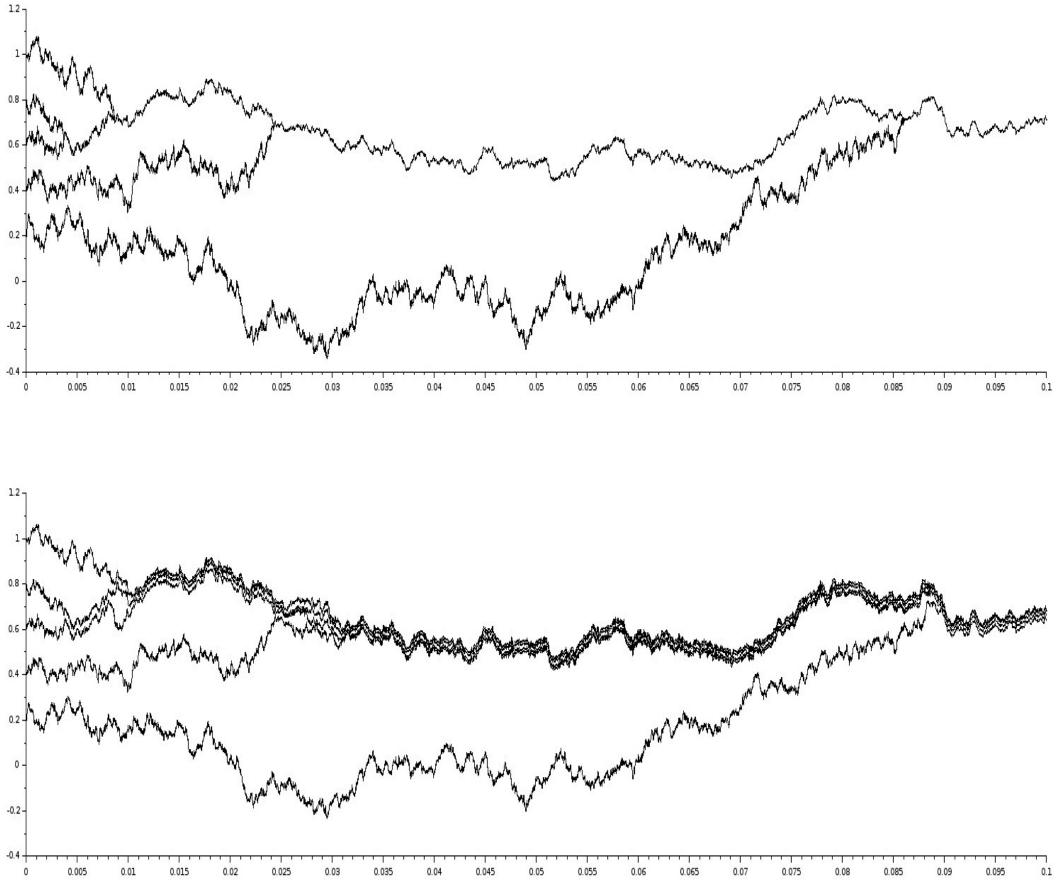

Figure 1: Two simulations, based on the same underlying Brownian sheet, for the limit process $(\mu_t)_{t\in[0,T]}$ *(on top)* and for the process $(\mu_t^{\sigma,\varepsilon})_{t\in[0,T]}$ with positive σ and ε *(on bottom)*. The horizontal axis represents time. On the vertical axis, we put the position of the particles (initially, we took five particles on $[0,1]$).

Organisation of the article

We begin in Section 2 by proving Theorem 1.1, which states that a process y satisfying properties (i)-(iv) admits a representation in terms of a Brownian sheet. In Section 3, given a two-dimensional Brownian sheet, we prove the existence of a smooth process in the space $L_2([0,1],\mathcal{C}[0,T])$ intended to approach Konarovskyi's process of coalescing particles. This smooth process can be seen as a cloud of point-particles interacting with all the particles at a distance smaller than σ, and in which two particles have independent trajectories conditionally to the fact that the distance between them is greater than σ. When the distance becomes smaller than σ, both trajectories are correlated, mimicking the coalescence property.

Section 4 is devoted to the proof of convergence when the parameter ε and the range of interaction σ tend to zero, using a tightness criterion in $L_2([0,1],\mathcal{C}[0,T])$. In Section 5, we study the stochastic properties of the limit process, including the convergence of the mass process. The aim of this final part is to prove that the limit process y satisfies properties $(C1)$-$(C5)$ of Theorem 1.4, in other words that our sequence of short-range interaction processes converges in distribution to the process of coalescing particles.

In Appendix A, we give an Itô formula in the Wasserstein space for the limit process y, after having recalled some basic definitions and properties of Lions' differential calculus on $\mathcal{P}_2(\mathbb{R})$.

2 Singular representation of the process y

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let us consider on $(\Omega, \mathcal{F}, \mathbb{P})$ a random process $y \in L_2((0,1), \mathcal{C}[0,T])$ satisfying properties (i)-(iv). We refer to [11] for a comprehensive construction of y. We will give another one later in this paper.

The aim of this Paragraph is to prove Theorem 1.1. Before that, we recall the definition of a Brownian sheet given by Walsh in [23, p.269]. Let $(E, \mathcal{E}, \nu)$ be a Euclidean space equipped with Lebesgue measure. A white noise based on ν is a random set function W on the sets $A \in \mathcal{E}$ of finite ν-measure such that

- $W(A)$ is a $\mathcal{N}(0, \nu(A))$ random variable,
- if $A \cap B = \emptyset$, then $W(A)$ and $W(B)$ are independent and $W(A \cap B) = W(A) + W(B)$.

Let $T > 0$. Consider $E = [0,1] \times [0,T]$ and ν the associated Lebesgue measure. The *Brownian sheet* w on $[0,1] \times [0,T]$ associated to the white noise W is the process $(w(u,t))_{u \in [0,1] \times [0,T]}$ defined by $w(u,t) := W((0,u] \times (0,t])$.

Define the filtration $(\mathcal{G}_t)_{t \in [0,T]}$ by $\mathcal{G}_t := \sigma(w(u,s), u \in [0,1], s \leqslant t)$. Then in particular,

(i) for each $(\mathcal{G}_t)_{t \in [0,T]}$-progressively measurable function f defined on $[0,1] \times [0,T]$ such that $\int_0^T \int_0^1 f^2(u,s) \mathrm{d}u \mathrm{d}s < +\infty$ almost surely, the process $\left(\int_0^t \int_0^1 f(u,s) \mathrm{d}w(u,s) \right)_{t \in [0,T]}$ is a local martingale (we often write $\mathrm{d}w(u,s)$ instead of $w(\mathrm{d}u, \mathrm{d}s)$);

(ii) for each f_1 and f_2 satisfying the same conditions as f,

$$\langle \int_0^\cdot \int_0^1 f_1(u,s) \mathrm{d}w(u,s), \int_0^\cdot \int_0^1 f_2(u,s) \mathrm{d}w(u,s) \rangle_t = \int_0^t \int_0^1 f_1(u,s) f_2(u,s) \mathrm{d}u \mathrm{d}s.$$

By Lévy's characterization of the Brownian motion, a process w satisfying (i) and (ii) is a Brownian sheet. Let us now prove Theorem 1.1.

Proof (Theorem 1.1). We take a Brownian sheet η on $[0,1] \times [0,T]$ independent of the process y, constructed by possibly extending the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then, we define $(w(u,t))_{u \in [0,1], t \in [0,T]}$ by $w(0, \cdot) \equiv 0$, $w(\cdot, 0) \equiv 0$ and:

$$w(\mathrm{d}u, \mathrm{d}t) = \eta(\mathrm{d}u, \mathrm{d}t) + y(u, \mathrm{d}t)\mathrm{d}u - \frac{1}{m(u,t)} \int_0^1 \mathbb{1}_{\{y(u,t) = y(u',t)\}} \eta(\mathrm{d}u', \mathrm{d}t) \mathrm{d}u.$$

We denote by $\mathcal{H}_t$ the filtration $\sigma((y(u,s))_{u \in [0,1], s \leqslant t}, (\eta(u,s))_{u \in [0,1], s \leqslant t})$.

In order to prove that w is an $(\mathcal{H}_t)_{t \in [0,T]}$-Brownian sheet on $[0,1] \times [0,T]$, let us consider two $(\mathcal{H}_t)_{t \in [0,T]}$-progressively measurable functions f_1 and f_2 and compute, using independence of η and y:

$$\langle \int_0^\cdot \int_0^1 f_1(u,s) \mathrm{d}w(u,s), \int_0^\cdot \int_0^1 f_2(v,s) \mathrm{d}w(v,s) \rangle_t = V_1 + V_2 - V_3 - V_4 + V_5,$$

where

$$V_1 := \langle \int_0^\cdot \int_0^1 f_1(u,s) \mathrm{d}\eta(u,s), \int_0^\cdot \int_0^1 f_2(v,s) \mathrm{d}\eta(v,s) \rangle_t = \int_0^t \int_0^1 f_1(u,s) f_2(u,s) \mathrm{d}u \mathrm{d}s,$$

since η is an $(\mathcal{H}_t)_{t \in [0,T]}$-Brownian sheet;

$$\begin{aligned} V_2 &:= \langle \int_0^\cdot \int_0^1 f_1(u,s) \mathrm{d}y(u,s) \mathrm{d}u, \int_0^\cdot \int_0^1 f_2(v,s) \mathrm{d}y(v,s) \mathrm{d}v \rangle_t \\ &= \int_0^t \int_0^1 \int_0^1 f_1(u,s) f_2(v,s) \frac{\mathbb{1}_{\{y(u,s) = y(v,s)\}}}{m(u,s)} \mathrm{d}u \mathrm{d}v \mathrm{d}s, \end{aligned}$$

using property (iv) of process y;

$$V_3 := \langle \int_0^\cdot \int_0^1 f_1(u,s)\mathrm{d}\eta(u,s), \int_0^\cdot \int_0^1 \frac{f_2(v,s)}{m(v,s)} \int_0^1 \mathbb{1}_{\{y(v,s)=y(v',s)\}}\mathrm{d}\eta(v',s)\mathrm{d}v\rangle_t$$
$$= \int_0^t \int_0^1 \int_0^1 \frac{f_1(u,s)f_2(v,s)}{m(v,s)} \mathbb{1}_{\{y(v,s)=y(u,s)\}}\mathrm{d}u\mathrm{d}v\mathrm{d}s = V_2,$$

since $m(u,s) = m(v,s)$ whenever $y(u,s)$ is equal to $y(v,s)$. By similar computations,

$$V_4 := \langle \int_0^\cdot \int_0^1 \frac{f_1(u,s)}{m(u,s)} \int_0^1 \mathbb{1}_{\{y(u,s)=y(u',s)\}}\mathrm{d}\eta(u',s)\mathrm{d}u, \int_0^\cdot \int_0^1 f_2(v,s)\mathrm{d}\eta(v,s)\rangle_t = V_2,$$

and

$$V_5 := \langle \int_0^\cdot \int_0^1 \frac{f_1(u,s)}{m(u,s)} \int_0^1 \mathbb{1}_{\{y(u,s)=y(u',s)\}}\mathrm{d}\eta(u',s)\mathrm{d}u,$$
$$\int_0^\cdot \int_0^1 \frac{f_2(v,s)}{m(v,s)} \int_0^1 \mathbb{1}_{\{y(v,s)=y(v',s)\}}\mathrm{d}\eta(v',s)\mathrm{d}v\rangle_t$$
$$= \int_0^t \int_0^1 \int_0^1 \int_0^1 \frac{f_1(u,s)f_2(v,s)}{m(u,s)m(v,s)} \mathbb{1}_{\{y(u,s)=y(u',s)\}}\mathbb{1}_{\{y(v,s)=y(u',s)\}}\mathrm{d}u'\mathrm{d}u\mathrm{d}v\mathrm{d}s$$
$$= \int_0^t \int_0^1 \int_0^1 \frac{f_1(u,s)f_2(v,s)}{m(u,s)^2} \left(\int_0^1 \mathbb{1}_{\{y(u,s)=y(u',s)\}}\mathrm{d}u'\right) \mathbb{1}_{\{y(u,s)=y(v,s)\}}\mathrm{d}u\mathrm{d}v\mathrm{d}s$$
$$= \int_0^t \int_0^1 \int_0^1 \frac{f_1(u,s)f_2(v,s)}{m(u,s)} \mathbb{1}_{\{y(u,s)=y(v,s)\}}\mathrm{d}u\mathrm{d}v\mathrm{d}s = V_2.$$

To sum up,

$$\langle \int_0^\cdot \int_0^1 f_1(u,s)\mathrm{d}w(u,s), \int_0^\cdot \int_0^1 f_2(v,s)\mathrm{d}w(v,s)\rangle_t = V_1 = \int_0^t \int_0^1 f_1(u,s)f_2(u,s)\mathrm{d}u\mathrm{d}s,$$

whence w is an $(\mathcal{H}_t)_{t\in[0,T]}$-Brownian sheet. Finally, we show that (y,w) satisfies equation (1.1):

$$\int_0^t \int_0^1 \frac{\mathbb{1}_{\{y(u,s)=y(u',s)\}}}{m(u,s)}\mathrm{d}w(u',s) = \int_0^t \int_0^1 \frac{\mathbb{1}_{\{y(u,s)=y(u',s)\}}}{m(u,s)}\mathrm{d}\eta(u',s) \qquad (=: W_1)$$
$$+ \int_0^t \int_0^1 \frac{\mathbb{1}_{\{y(u,s)=y(u',s)\}}}{m(u,s)}\mathrm{d}y(u',s)\mathrm{d}u' \qquad (=: W_2)$$
$$- \int_0^t \int_0^1 \frac{\mathbb{1}_{\{y(u,s)=y(u',s)\}}}{m(u,s)} \int_0^1 \frac{\mathbb{1}_{\{y(u',s)=y(v,s)\}}}{m(u',s)}\mathrm{d}\eta(v,s)\mathrm{d}u'. \qquad (=: W_3)$$

The result follows from the two below equalities:

$$W_2 = \int_0^t \int_0^1 \frac{\mathbb{1}_{\{y(u,s)=y(u',s)\}}}{m(u,s)}\mathrm{d}y(u,s)\mathrm{d}u' = \int_0^t \mathrm{d}y(u,s) = y(u,t) - y(u,0) = y(u,t) - g(u);$$
$$W_3 = \int_0^t \int_0^1 \frac{\mathbb{1}_{\{y(u,s)=y(u',s)\}}}{m(u,s)} \int_0^1 \frac{\mathbb{1}_{\{y(u',s)=y(v,s)\}}}{m(v,s)}\mathrm{d}\eta(v,s)\mathrm{d}u'$$
$$= \int_0^t \int_0^1 \frac{\mathbb{1}_{\{y(u,s)=y(v,s)\}}}{m(u,s)m(v,s)} \left(\int_0^1 \mathbb{1}_{\{y(u',s)=y(v,s)\}}\mathrm{d}u'\right)\mathrm{d}\eta(v,s)$$
$$= \int_0^t \int_0^1 \frac{\mathbb{1}_{\{y(u,s)=y(v,s)\}}}{m(u,s)}\mathrm{d}\eta(v,s),$$

which implies that $W_3 = W_1$ and consequently equation (1.1). □

Therefore, every solution of the martingale problem (i)-(iv) has a representation in terms of a Brownian sheet. In the next Section, we will construct, given a Brownian sheet, an approximation of the process y.

3 Construction of a process with short-range interactions

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, on which we define a Brownian sheet w on $[0,1] \times [0,T]$. We associate to that process the filtration $\mathcal{G}_t := \sigma(w(u,s), u \in [0,1], s \leqslant t)$. Up to completing the filtration, we assume that $\mathcal{G}_0$ contains all the $\mathbb{P}$-null sets of $\mathcal{F}$ and that the filtration $(\mathcal{G}_t)_{t\in[0,T]}$ is right-continuous.

Fix $\sigma > 0$ and $\varepsilon > 0$. Let φ_σ denote a smooth and even function, bounded by 1, equal to 1 on $[0, \frac{\sigma}{3}]$ and equal to 0 on $[\frac{\sigma}{2}, +\infty)$. Recall that $\mathcal{L}^{\uparrow}_{2+}[0,1]$ represents the set of non-decreasing and càdlàg functions $g : [0,1] \to \mathbb{R}$ such that there exists $p > 2$ satisfying $\int_0^1 |g(u)|^p \mathrm{d}u < +\infty$. The aim of this Section is to construct, for each initial quantile function $g \in \mathcal{L}^{\uparrow}_{2+}[0,1]$, a square integrable random variable $y^g_{\sigma,\varepsilon}$ taking values in $L_2([0,1], \mathcal{C}[0,T])$ such that almost surely, for every $t \in [0,T]$, the following equality holds in $L_2[0,1]$:

$$y^g_{\sigma,\varepsilon}(\cdot,t) = g + \int_0^t \int_0^1 \frac{\varphi_\sigma(y^g_{\sigma,\varepsilon}(\cdot,s) - y^g_{\sigma,\varepsilon}(u',s))}{\varepsilon + \int_0^1 \varphi^2_\sigma(y^g_{\sigma,\varepsilon}(\cdot,s) - y^g_{\sigma,\varepsilon}(v,s))\mathrm{d}v} \mathrm{d}w(u',s). \tag{3.1}$$

Remark 3.1. We add the parameter ε to the denominator in order to ensure that it is bounded by below. We also point out that relation (3.1) has to be compared with equation (1.1), where $x \mapsto \mathbb{1}_{\{x=0\}}$ is replaced by the function φ_σ.

More precisely, we will prove the following Proposition. Recall that $L^{\uparrow}_2[0,1]$ represents the set of functions $f \in L_2[0,1]$ such that there is a non-decreasing and càdlàg element in the equivalence class of f.

Proposition 3.2. *Let $g \in \mathcal{L}^{\uparrow}_{2+}[0,1]$. There is an $L^{\uparrow}_2[0,1]$-valued process $(Y^g_{\sigma,\varepsilon}(t))_{t\in[0,T]} = (y^g_{\sigma,\varepsilon}(\cdot,t))_{t\in[0,T]}$ such that:*

$(A1)$ $Y^g_{\sigma,\varepsilon}(0) = g$;

$(A2)$ $Y^g_{\sigma,\varepsilon}$ *is a square integrable continuous $L^{\uparrow}_2[0,1]$-valued $(\mathcal{F}^{\sigma,\varepsilon}_t)_{t\in[0,T]}$-martingale, where* $\mathcal{F}^{\sigma,\varepsilon}_t := \sigma(Y^g_{\sigma,\varepsilon}(s), s \leqslant t)$;

$(A3)$ *for every $h, k \in L_2[0,1]$,*

$$\langle (Y^g_{\sigma,\varepsilon}, h)_{L_2}, (Y^g_{\sigma,\varepsilon}, k)_{L_2} \rangle_t = \int_0^t\int_0^1\int_0^1 h(u)k(u') \frac{m^g_{\sigma,\varepsilon}(u,u',s)}{(\varepsilon + m^g_{\sigma,\varepsilon}(u,s))(\varepsilon + m^g_{\sigma,\varepsilon}(u',s))} \mathrm{d}u\mathrm{d}u'\mathrm{d}s,$$

where $m^g_{\sigma,\varepsilon}(u,u',s) = \int_0^1 \varphi_\sigma(y^g_{\sigma,\varepsilon}(u,s) - y^g_{\sigma,\varepsilon}(v,s))\varphi_\sigma(y^g_{\sigma,\varepsilon}(u',s) - y^g_{\sigma,\varepsilon}(v,s))\mathrm{d}v$ *and* $m^g_{\sigma,\varepsilon}(u,s) = \int_0^1 \varphi^2_\sigma(y^g_{\sigma,\varepsilon}(u,s) - y^g_{\sigma,\varepsilon}(v,s))\mathrm{d}v$.

3.1 Existence of an approximate solution

Denote by $\mathcal{M}$ the set of random variables $z \in L_2(\Omega, \mathcal{C}([0,T], L_2(0,1)))$ such that $(z(\omega,\cdot,t))_{t\in[0,T]}$ is a $(\mathcal{G}_t)_{t\in[0,T]}$-progressively measurable process with values in $L_2(0,1)$. We consider the following norm on $\mathcal{M}$:

$$\|z\|_{\mathcal{M}} = \mathbb{E}\left[\sup_{t\leqslant T} \int_0^1 |z(u,t)|^2 \mathrm{d}u\right]^{1/2}.$$

Throughout this Section, σ and ε are two *fixed* positive numbers. To begin, we want to prove that the map $\psi : \mathcal{M} \to \mathcal{M}$, defined below, admits a unique fixed point. Fix $g \in \mathcal{L}^{\uparrow}_{2+}[0,1]$ an initial quantile function. For all $z \in \mathcal{M}$, define:

$$\psi(z)(\omega,u,t) := g(u) + \int_0^t \int_0^1 \frac{\varphi_\sigma(z(\omega,u,s) - z(\omega,u',s))}{\varepsilon + m_\sigma(\omega,u,s)} \mathrm{d}w(\omega,u',s), \tag{3.2}$$

where $m_\sigma(\omega,u,s) = \int_0^1 \varphi^2_\sigma(z(\omega,u,s) - z(\omega,v,s))\mathrm{d}v$. We start by making sure that ψ is well-defined.

Proposition 3.3. *For all $z \in \mathcal{M}$, $\psi(z)$ belongs to $\mathcal{M}$. Furthermore, $(\psi(z)(\cdot,t))_{t\in[0,T]}$ is an $L_2(0,1)$-valued continuous $(\mathcal{G}_t)_{t\in[0,T]}$-martingale.*

Remark 3.4. The definition of an $L_2^\uparrow[0,1]$-valued martingale was given in Definition 1.3. Up to replacing $L_2^\uparrow$ by L_2, the definition of an $L_2(0,1)$-valued martingale is exactly the same.

Proof. We want to prove that $(\psi(z)(\cdot,t))_{t\in[0,T]}$ is an $L_2(0,1)$-valued $(\mathcal{G}_t)_{t\in[0,T]}$-martingale. Since z belongs to $\mathcal{M}$, the process $(z(\cdot,t))_{t\in[0,T]}$ is $(\mathcal{G}_t)_{t\in[0,T]}$-progressively measurable. Therefore $(m_\sigma(\cdot,t))_{t\in[0,T]}$ is also $(\mathcal{G}_t)_{t\in[0,T]}$-progressively measurable and we deduce that $(\psi(z)(\cdot,t))_{t\in[0,T]}$ is $(\mathcal{G}_t)_{t\in[0,T]}$-progressively measurable.

Then, we check that for each $t\in[0,T]$, $\psi(z)(\cdot,t)\in L_2(0,1)$ and $\mathbb{E}\left[\|\psi(z)(\cdot,t)\|_{L_2}\right]<\infty$. We deduce this statement by recalling that $\|g\|_{L_2}<+\infty$, because $g\in\mathcal{L}_{2+}^\uparrow[0,1]$, and by computing:

$$\begin{aligned}
&\mathbb{E}\left[\left\|\int_0^t\int_0^1 \frac{\varphi_\sigma(z(\cdot,s)-z(u',s))}{\varepsilon+m_\sigma(\cdot,s)}\mathrm{d}w(u',s)\right\|_{L_2}\right]^2\\
&\qquad\leqslant \mathbb{E}\left[\left\|\int_0^t\int_0^1 \frac{\varphi_\sigma(z(\cdot,s)-z(u',s))}{\varepsilon+m_\sigma(\cdot,s)}\mathrm{d}w(u',s)\right\|_{L_2}^2\right]\\
&\qquad= \mathbb{E}\left[\int_0^1\left|\int_0^t\int_0^1 \frac{\varphi_\sigma(z(u,s)-z(u',s))}{\varepsilon+m_\sigma(u,s)}\mathrm{d}w(u',s)\right|^2\mathrm{d}u\right]\\
&\qquad= \int_0^1\mathbb{E}\left[\left|\int_0^t\int_0^1 \frac{\varphi_\sigma(z(u,s)-z(u',s))}{\varepsilon+m_\sigma(u,s)}\mathrm{d}w(u',s)\right|^2\right]\mathrm{d}u\\
&\qquad= \int_0^1\mathbb{E}\left[\int_0^t\int_0^1 \left(\frac{\varphi_\sigma(z(u,s)-z(u',s))}{\varepsilon+m_\sigma(u,s)}\right)^2\mathrm{d}u'\mathrm{d}s\right]\mathrm{d}u\\
&\qquad\leqslant \frac{\|\varphi_\sigma\|_\infty^2 t}{\varepsilon^2}=\frac{t}{\varepsilon^2}<+\infty.
\end{aligned} \tag{3.3}$$

Furthermore, for each $h\in L_2[0,1]$,

$$(\psi(z)(\cdot,t),h)_{L_2}=(g,h)_{L_2}+\int_0^t\int_0^1\int_0^1 h(u)\frac{\varphi_\sigma(z(u,s)-z(u',s))}{\varepsilon+m_\sigma(u,s)}\mathrm{d}u\mathrm{d}w(u',s)$$

is a $(\mathcal{G}_t)_{t\in[0,T]}$-local martingale. Then, we compute the quadratic variation:

$$\begin{aligned}
&\mathbb{E}\left[\langle(\psi(z),h)_{L_2},(\psi(z),h)_{L_2}\rangle_t\right]\\
&\quad=\int_0^t\int_0^1\int_0^1\int_0^1 h(u_1)h(u_2)\frac{\varphi_\sigma(z(u_1,s)-z(u',s))\varphi_\sigma(z(u_2,s)-z(u',s))}{(\varepsilon+m_\sigma(u_1,s))(\varepsilon+m_\sigma(u_2,s))}\mathrm{d}u_1\mathrm{d}u_2\mathrm{d}u'\mathrm{d}s\\
&\quad\leqslant\frac{t}{\varepsilon^2}\|h\|_{L_2}^2.
\end{aligned}$$

Since it is finite, the local martingale is actually a martingale.

Moreover, by Doob's inequality (see Theorem 2.2 in [9, p.22])

$$\begin{aligned}
\|\psi(z)\|_{\mathcal{M}}&=\mathbb{E}\left[\sup_{t\leqslant T}\int_0^1|\psi(z)(u,t)|^2\mathrm{d}u\right]^{1/2}\\
&\leqslant\|g\|_{L_2}+\mathbb{E}\left[\sup_{t\leqslant T}\int_0^1\left|\int_0^t\int_0^1\frac{\varphi_\sigma(z(u,s)-z(u',s))}{\varepsilon+m_\sigma(u,s)}\mathrm{d}w(u',s)\right|^2\mathrm{d}u\right]^{1/2}\\
&\leqslant\|g\|_{L_2}+2\mathbb{E}\left[\int_0^1\left|\int_0^T\int_0^1\frac{\varphi_\sigma(z(u,s)-z(u',s))}{\varepsilon+m_\sigma(u,s)}\mathrm{d}w(u',s)\right|^2\mathrm{d}u\right]^{1/2}.
\end{aligned}$$

The last term is finite by (3.3). Thus $\|\psi(z)\|_{\mathcal{M}}$ is finite and $\psi(z)$ belongs to $\mathcal{M}$, which concludes the proof. □

Let us now prove that ψ has a unique fixed point:

Proposition 3.5. *Let $\sigma > 0$ and $\varepsilon > 0$. Then the map $\psi : \mathcal{M} \to \mathcal{M}$ defined by (3.2) has a unique fixed point in $\mathcal{M}$, denoted by $y^g_{\sigma,\varepsilon}$.*

Proof. For all $n \in \mathbb{N}$, denote by ψ^n the n-fold composition of ψ, where ψ^0 denotes the identity function of $\mathcal{M}$. We want to prove that ψ^n is a contraction for n large enough.

Let z_1 and z_2 be two elements of $\mathcal{M}$. We define

$$h_n(t) := \mathbb{E}\left[\sup_{s\leqslant t}\int_0^1 |\psi^n(z_1)(u,s) - \psi^n(z_2)(u,s)|^2 \mathrm{d}u\right].$$

Let us remark that $h_n(T) = \|\psi^n(z_1) - \psi^n(z_2)\|^2_{\mathcal{M}}$ and recall that, by Proposition 3.3, $(\psi(z_1)(\cdot,t) - \psi(z_2)(\cdot,t))_{t\in[0,T]}$ is a $(\mathcal{G}_t)_{t\in[0,T]}$-martingale. We denote by $m_{\sigma,1}$ and $m_{\sigma,2}$ the masses associated respectively to z_1 and z_2. By Doob's inequality, we have:

$$\begin{aligned} h_1(t) &= \mathbb{E}\left[\sup_{s\leqslant t}\int_0^1 |\psi(z_1)(u,s) - \psi(z_2)(u,s)|^2 \mathrm{d}u\right] \\ &= \mathbb{E}\left[\sup_{s\leqslant t}\int_0^1 \left|\int_0^s\int_0^1 \left(\frac{\varphi_\sigma(z_1(u,r) - z_1(u',r))}{\varepsilon + m_{\sigma,1}(u,r)} - \frac{\varphi_\sigma(z_2(u,r) - z_2(u',r))}{\varepsilon + m_{\sigma,2}(u,r)}\right)\mathrm{d}w(u',r)\right|^2 \mathrm{d}u\right] \\ &\leqslant 4\mathbb{E}\left[\int_0^1\int_0^t\int_0^1 \left|\frac{\varphi_\sigma(z_1(u,s) - z_1(u',s))}{\varepsilon + m_{\sigma,1}(u,s)} - \frac{\varphi_\sigma(z_2(u,s) - z_2(u',s))}{\varepsilon + m_{\sigma,2}(u,s)}\right|^2 \mathrm{d}u'\mathrm{d}s\mathrm{d}u\right]. \end{aligned}$$

Furthermore, we compute:

$$\begin{aligned} &\left|\frac{\varphi_\sigma(z_1(u,s) - z_1(u',s))}{\varepsilon + m_{\sigma,1}(u,s)} - \frac{\varphi_\sigma(z_2(u,s) - z_2(u',s))}{\varepsilon + m_{\sigma,2}(u,s)}\right|^2 \\ &\qquad\leqslant 2\Bigg(\left|\frac{\varphi_\sigma(z_1(u,s) - z_1(u',s)) - \varphi_\sigma(z_2(u,s) - z_2(u',s))}{\varepsilon + m_{\sigma,1}(u,s)}\right|^2 \\ &\qquad\qquad + \left|\frac{\varphi_\sigma(z_2(u,s) - z_2(u',s))}{(\varepsilon + m_{\sigma,1}(u,s))(\varepsilon + m_{\sigma,2}(u,s))}\left(m_{\sigma,1}(u,s) - m_{\sigma,2}(u,s)\right)\right|^2\Bigg). \end{aligned}$$

Moreover, we have:

$$\begin{aligned} |m_{\sigma,1}(u,s) - m_{\sigma,2}(u,s)| &\leqslant \int_0^1 |\varphi_\sigma^2(z_1(u,s) - z_1(v,s)) - \varphi_\sigma^2(z_2(u,s) - z_2(v,s))|\mathrm{d}v \\ &\leqslant \mathrm{Lip}(\varphi_\sigma^2)\int_0^1 |(z_1(u,s) - z_1(v,s)) - (z_2(u,s) - z_2(v,s))|\mathrm{d}v \\ &\leqslant \mathrm{Lip}(\varphi_\sigma^2)\left(|z_1(u,s) - z_2(u,s)| + \int_0^1 |z_1(v,s) - z_2(v,s)|\mathrm{d}v\right). \end{aligned}$$

We obtain the following upper bound:

$$\begin{aligned} &\left|\frac{\varphi_\sigma(z_1(u,s) - z_1(u',s))}{\varepsilon + m_{\sigma,1}(u,s)} - \frac{\varphi_\sigma(z_2(u,s) - z_2(u',s))}{\varepsilon + m_{\sigma,2}(u,s)}\right|^2 \\ &\qquad\leqslant \left(4\left(\frac{\mathrm{Lip}\,\varphi_\sigma}{\varepsilon}\right)^2 + 4\left(\frac{\mathrm{Lip}(\varphi_\sigma^2)}{\varepsilon^2}\right)^2\right)\Bigg(|z_1(u,s) - z_2(u,s)|^2 \\ &\qquad\qquad + |z_1(u',s) - z_2(u',s)|^2 + \int_0^1 |z_1(v,s) - z_2(v,s)|^2\mathrm{d}v\Bigg). \end{aligned}$$

Finally, we deduce that there is a constant $C_{\sigma,\varepsilon}$ depending only on σ and ε such that

$$\begin{aligned} h_1(t) &\leqslant C_{\sigma,\varepsilon}\mathbb{E}\left[\int_0^t\int_0^1 |z_1(u,s)-z_2(u,s)|^2\mathrm{d}u\mathrm{d}s\right] \\ &\leqslant C_{\sigma,\varepsilon}\int_0^t \mathbb{E}\left[\sup_{r\leqslant s}\int_0^1 |z_1(u,r)-z_2(u,r)|^2\mathrm{d}u\right]\mathrm{d}s = C_{\sigma,\varepsilon}\int_0^t h_0(s)\mathrm{d}s. \end{aligned}$$

Applied to $\psi^n(z_1)$ and $\psi^n(z_2)$ instead of z_1 and z_2, those computations show that for every $t\in[0,T]$, $h_{n+1}(t)\leqslant C_{\sigma,\varepsilon}\int_0^t h_n(s)\mathrm{d}s$. Using the fact that h_0 is non-decreasing with respect to t, it follows that $h_n(T)\leqslant \frac{(C_{\sigma,\varepsilon}T)^n}{n!}h_0(T)$, whence we have:

$$\|\psi^n(z_1)-\psi^n(z_2)\|^2_{\mathcal{M}} \leqslant \frac{(C_{\sigma,\varepsilon}T)^n}{n!}\|z_1-z_2\|^2_{\mathcal{M}}.$$

Thus, for n large enough, the map ψ^n is a contraction. By completeness of $\mathcal{M}$ under the norm $\|\cdot\|_{\mathcal{M}}$ (remark that $\mathcal{M}$ is a closed subset of $L_2(\Omega,\mathcal{C}([0,T],L_2(0,1)))$, it follows that ψ has a unique fixed point in $\mathcal{M}$. □

We denote by $y^g_{\sigma,\varepsilon}$ the unique fixed point of ψ. Remark that by construction it satisfies equation (3.1) almost surely and for every $t\in[0,T]$.

3.2 Non-decreasing property

Define, for each $t\in[0,T]$, $Y^g_{\sigma,\varepsilon}(t):=y^g_{\sigma,\varepsilon}(\cdot,t)$. So far, by Proposition 3.5, we have established that $(Y^g_{\sigma,\varepsilon}(t))_{t\in[0,T]}$ is an $L_2[0,1]$-valued process, satisfying property $(A1)$ of Proposition 3.2. Since $Y^g_{\sigma,\varepsilon}$ belongs to $\mathcal{M}$, and by Proposition 3.3, $(Y^g_{\sigma,\varepsilon}(t))_{t\in[0,T]}$ is a square integrable continuous $L_2[0,1]$-valued martingale, with respect to the filtration $(\mathcal{G}_t)_{t\in[0,T]}$. Therefore, it is also an $(\mathcal{F}^{\sigma,\varepsilon}_t)_{t\in[0,T]}$-martingale, where $\mathcal{F}^{\sigma,\varepsilon}_t:=\sigma(Y^g_{\sigma,\varepsilon}(s),s\leqslant t)$.

In order to obtain property $(A2)$, it remains to prove the following statement:

Proposition 3.6. *$(Y^g_{\sigma,\varepsilon}(t))_{t\in[0,T]}$ is an $L_2^{\uparrow}[0,1]$-valued process.*

We will start by proving three Lemmas and then we will conclude the proof of Proposition 3.6. For every $x\in\mathbb{R}$, we consider the following stochastic differential equation:

$$z(x,t)=x+\int_0^t\int_0^1 \frac{\varphi_\sigma(z(x,s)-y^g_{\sigma,\varepsilon}(u',s))}{\varepsilon+\int_0^1\varphi^2_\sigma(z(x,s)-y^g_{\sigma,\varepsilon}(v,s))\mathrm{d}v}\mathrm{d}w(u',s), \tag{3.4}$$

where $y^g_{\sigma,\varepsilon}$ is the unique solution of equation (3.1).

Lemma 3.7. *Let $x\in\mathbb{R}$. For almost every $\omega\in\Omega$, equation* (3.4) *has a unique solution in $\mathcal{C}[0,T]$, denoted by $(z(\omega,x,t))_{t\in[0,T]}$. Moreover, $(z(x,t))_{t\in[0,T]}$ is a real-valued $(\mathcal{G}_t)_{t\in[0,T]}$-martingale.*

Proof. We get existence and uniqueness of the solution by applying a fixed-point argument. The proof is the same as the proof of Proposition 3.5. We obtain the martingale property by the same argument as in Proposition 3.3. □

Then, take x_1, $x_2\in\mathbb{R}$. After some computations similar to those of the proof of Proposition 3.5, we have for every $t\in[0,T]$:

$$\mathbb{E}\left[\sup_{s\leqslant t}|z(x_1,s)-z(x_2,s)|^2\right]\leqslant 2|x_1-x_2|^2+C_{\sigma,\varepsilon}\int_0^t\mathbb{E}\left[\sup_{r\leqslant s}|z(x_1,r)-z(x_2,r)|^2\right]\mathrm{d}s.$$

By Gronwall's Lemma, we deduce that:

$$\mathbb{E}\left[\sup_{t\leqslant T}|z(x_1,t)-z(x_2,t)|^2\right]\leqslant C_{\sigma,\varepsilon}|x_1-x_2|^2.$$

By Kolmogorov's Lemma, there is a modification $\widetilde{z}$ of z in $\mathcal{C}(\mathbb{R}\times[0,T])$. We define $\widetilde{y}^g_{\sigma,\varepsilon}(u,t) := \widetilde{z}(g(u),t)$. In particular, $u\mapsto \widetilde{y}^g_{\sigma,\varepsilon}(u,\cdot)$ is measurable and, since g is a càdlàg function, $\widetilde{y}^g_{\sigma,\varepsilon}$ belongs to $\mathcal{D}((0,1),\mathcal{C}[0,T])$.

Remark 3.8. In the case where g is continuous, it is straightforward to see that $\widetilde{y}^g_{\sigma,\varepsilon}$ belongs to $\mathcal{C}([0,1]\times[0,T])$.

Furthermore, $\widetilde{y}^g_{\sigma,\varepsilon}$ belongs to $\mathcal{M}$. Indeed,

$$\mathbb{E}\left[\sup_{t\leqslant T}\int_0^1 \left|\widetilde{y}^g_{\sigma,\varepsilon}(u,t)\right|^2 \mathrm{d}u\right] \leqslant \mathbb{E}\left[\int_0^1 \sup_{t\leqslant T}\left|\widetilde{y}^g_{\sigma,\varepsilon}(u,t)\right|^2 \mathrm{d}u\right] = \int_0^1 \mathbb{E}\left[\sup_{t\leqslant T}\left|\widetilde{y}^g_{\sigma,\varepsilon}(u,t)\right|^2\right]\mathrm{d}u.$$

By Lemma 3.7, for every $u\in[0,1]$, $(\widetilde{y}^g_{\sigma,\varepsilon}(u,t))_{t\in[0,T]}$ is a martingale, we have by Doob's inequality:

$$\begin{aligned}\mathbb{E}\left[\sup_{t\leqslant T}\left|\widetilde{y}^g_{\sigma,\varepsilon}(u,t)\right|^2\right] &\leqslant C\mathbb{E}\left[\left|\widetilde{y}^g_{\sigma,\varepsilon}(u,T)\right|^2\right]\\ &\leqslant 2Cg(u)^2 + 2C\mathbb{E}\left[\int_0^T\!\!\int_0^1 \left|\frac{\varphi_\sigma(\widetilde{y}^g_{\sigma,\varepsilon}(u,s)-y^g_{\sigma,\varepsilon}(u',s))}{\varepsilon+\int_0^1\varphi_\sigma^2(\widetilde{y}^g_{\sigma,\varepsilon}(u,s)-y^g_{\sigma,\varepsilon}(v,s))\mathrm{d}v}\right|^2 \mathrm{d}u'\mathrm{d}s\right]\\ &\leqslant 2Cg(u)^2 + 2C\frac{T}{\varepsilon^2}.\end{aligned}$$

Therefore, $\|\widetilde{y}^g_{\sigma,\varepsilon}\|_{\mathcal{M}} \leqslant 2C\|g\|^2_{L_2} + 2C\frac{T}{\varepsilon^2} < +\infty$. Moreover, $(\widetilde{y}^g_{\sigma,\varepsilon}(\cdot,t))_{t\in[0,T]}$ is an $L_2[0,1]$-valued $(\mathcal{G}_t)_{t\in[0,T]}$-martingale. Indeed, for every $h\in L_2[0,1]$, for every $t\in[0,T]$, the expectation $\mathbb{E}\left[(\widetilde{y}^g_{\sigma,\varepsilon}(\cdot,t),h)_{L_2}\right]$ is finite. Fix $0\leqslant s\leqslant t\leqslant T$, and $A_s\in\mathcal{G}_s$. We have:

$$\begin{aligned}&\mathbb{E}\left[\left(\int_0^1 \widetilde{y}^g_{\sigma,\varepsilon}(u,t)h(u)\mathrm{d}u - \int_0^1 \widetilde{y}^g_{\sigma,\varepsilon}(u,s)h(u)\mathrm{d}u\right)\mathbb{1}_{A_s}\right]\\ &\qquad = \int_0^1 \mathbb{E}\left[(\widetilde{y}^g_{\sigma,\varepsilon}(u,t)-\widetilde{y}^g_{\sigma,\varepsilon}(u,s))\mathbb{1}_{A_s}\right]h(u)\mathrm{d}u = 0.\end{aligned}$$

Lemma 3.9. *We have* $\mathbb{E}\left[\sup_{t\leqslant T}\int_0^1\left|\widetilde{y}^g_{\sigma,\varepsilon}(u,t)-y^g_{\sigma,\varepsilon}(u,t)\right|^2\mathrm{d}u\right]=0$. *Therefore,* $\widetilde{y}^g_{\sigma,\varepsilon} = y^g_{\sigma,\varepsilon}$ *in* $\mathcal{M}$.

Proof. Since $(\widetilde{y}^g_{\sigma,\varepsilon}(\cdot,t)-y^g_{\sigma,\varepsilon}(\cdot,t))_{t\in[0,T]}$ is an $L_2[0,1]$-valued martingale, then by [9, p.21-22] $\int_0^1\left|\widetilde{y}^g_{\sigma,\varepsilon}(u,t)-y^g_{\sigma,\varepsilon}(u,t)\right|^2\mathrm{d}u$ is a real-valued submartingale. By Doob's inequality,

$$\begin{aligned}\mathbb{E}\left[\sup_{s\leqslant t}\int_0^1\left|\widetilde{y}^g_{\sigma,\varepsilon}(u,s)-y^g_{\sigma,\varepsilon}(u,s)\right|^2\mathrm{d}u\right] &\leqslant C\mathbb{E}\left[\int_0^1\left|\widetilde{y}^g_{\sigma,\varepsilon}(u,t)-y^g_{\sigma,\varepsilon}(u,t)\right|^2\mathrm{d}u\right]\\ &\leqslant C\mathbb{E}\left[\int_0^1\left|\int_0^t\!\!\int_0^1(\theta_{\sigma,\varepsilon}(\widetilde{y}^g_{\sigma,\varepsilon}(u,s),u',s)-\theta_{\sigma,\varepsilon}(y^g_{\sigma,\varepsilon}(u,s),u',s))\mathrm{d}w(u',s)\right|^2\mathrm{d}u\right]\\ &\leqslant C\mathbb{E}\left[\int_0^1\!\!\int_0^t\!\!\int_0^1\left|\theta_{\sigma,\varepsilon}(\widetilde{y}^g_{\sigma,\varepsilon}(u,s),u',s)-\theta_{\sigma,\varepsilon}(y^g_{\sigma,\varepsilon}(u,s),u',s)\right|^2\mathrm{d}u'\mathrm{d}s\mathrm{d}u\right],\end{aligned}$$

where $\theta_{\sigma,\varepsilon}(x,u',s) = \frac{\varphi_\sigma(x-y^g_{\sigma,\varepsilon}(u',s))}{\varepsilon+\int_0^1\varphi_\sigma^2(x-y^g_{\sigma,\varepsilon}(v,s))\mathrm{d}v}$. Using the same constant $C_{\sigma,\varepsilon}$ as in the proof of Proposition 3.5, we have:

$$\begin{aligned}\mathbb{E}\left[\sup_{s\leqslant t}\int_0^1\left|\widetilde{y}^g_{\sigma,\varepsilon}(u,s)-y^g_{\sigma,\varepsilon}(u,s)\right|^2\mathrm{d}u\right] &\leqslant C_{\sigma,\varepsilon}\mathbb{E}\left[\int_0^1\!\!\int_0^t\left|\widetilde{y}^g_{\sigma,\varepsilon}(u,s)-y^g_{\sigma,\varepsilon}(u,s)\right|^2\mathrm{d}s\mathrm{d}u\right]\\ &\leqslant C_{\sigma,\varepsilon}\int_0^t\mathbb{E}\left[\sup_{r\leqslant s}\int_0^1\left|\widetilde{y}^g_{\sigma,\varepsilon}(u,r)-y^g_{\sigma,\varepsilon}(u,r)\right|^2\mathrm{d}u\right]\mathrm{d}s.\end{aligned}$$

By Gronwall's Lemma, we deduce that $\mathbb{E}\left[\sup_{s\leqslant t}\int_0^1 \left|\widetilde{y}^g_{\sigma,\varepsilon}(u,s)-y^g_{\sigma,\varepsilon}(u,s)\right|^2 \mathrm{d}u\right]=0$ for every $t\in[0,T]$. This implies the statement of the Lemma. □

Lemma 3.10. *Almost surely, for every $u_1,u_2\in\mathbb{Q}$ such that $u_1<u_2$, we have for every $t\geqslant 0$, $\widetilde{y}^g_{\sigma,\varepsilon}(u_1,t)\leqslant \widetilde{y}^g_{\sigma,\varepsilon}(u_2,t)$. Furthermore, if $g(u_1)<g(u_2)$ (resp. $g(u_1)=g(u_2)$), then for every $t\geqslant 0$, $\widetilde{y}^g_{\sigma,\varepsilon}(u_1,t)<\widetilde{y}^g_{\sigma,\varepsilon}(u_2,t)$ (resp. $\widetilde{y}^g_{\sigma,\varepsilon}(u_1,t)=\widetilde{y}^g_{\sigma,\varepsilon}(u_2,t)$).*

Proof. Let $(u_1,u_2)\in\mathbb{Q}^2$ such that $0\leqslant u_1<u_2\leqslant 1$. For $u=u_1,u_2$, we have:

$$\widetilde{y}^g_{\sigma,\varepsilon}(u,t)=g(u)+\int_0^t\int_0^1\theta_{\sigma,\varepsilon}(\widetilde{y}^g_{\sigma,\varepsilon}(u,s),u',s)\mathrm{d}w(u',s),$$

where $\theta_{\sigma,\varepsilon}(x,u',s)=\frac{\varphi_\sigma(x-y^g_{\sigma,\varepsilon}(u',s))}{\varepsilon+\int_0^1\varphi_\sigma^2(x-y^g_{\sigma,\varepsilon}(v,s))\mathrm{d}v}$. Therefore, we have (writing $\widetilde{y}$ instead of $\widetilde{y}^g_{\sigma,\varepsilon}$ and θ instead of $\theta_{\sigma,\varepsilon}$):

$$\begin{aligned}\widetilde{y}(u_2,t)-\widetilde{y}(u_1,t)&=g(u_2)-g(u_1)+\int_0^t\int_0^1(\theta(\widetilde{y}(u_2,s),u',s)-\theta(\widetilde{y}(u_1,s),u',s))\mathrm{d}w(u',s)\\&=g(u_2)-g(u_1)+\int_0^t(\widetilde{y}(u_2,s)-\widetilde{y}(u_1,s))\mathrm{d}M_s\end{aligned}\tag{3.5}$$

where $M_t=\int_0^t\int_0^1\mathbb{1}_{\{\widetilde{y}(u_2,s)\neq\widetilde{y}(u_1,s)\}}\frac{\theta(\widetilde{y}(u_2,s),u',s)-\theta(\widetilde{y}(u_1,s),u',s)}{\widetilde{y}(u_2,s)-\widetilde{y}(u_1,s)}\mathrm{d}w(u',s)$. Observe that:

$$\theta(\widetilde{y}(u_2,s),u',s)-\theta(\widetilde{y}(u_1,s),u',s)=\int_{\widetilde{y}(u_1,s)}^{\widetilde{y}(u_2,s)}\partial_x\theta(x,u',s)\mathrm{d}x,$$

and that $\partial_x\theta(x,u',s)=\frac{\varphi'_\sigma(x-y(u',s))}{\varepsilon+\int_0^1\varphi_\sigma^2(x-y(v,s))\mathrm{d}v}-\frac{\varphi_\sigma(x-y(u',s))\int_0^1(\varphi_\sigma^2)'(x-y(v,s))\mathrm{d}v}{(\varepsilon+\int_0^1\varphi_\sigma^2(x-y(v,s))\mathrm{d}v)^2}$. Therefore, $\partial_x\theta$ is bounded uniformly in $(x,u',s)\in\mathbb{R}\times[0,1]\times[0,T]$ by $C_{\sigma,\varepsilon}:=\frac{\|\varphi'_\sigma\|_{L_\infty}}{\varepsilon}+\frac{\|\varphi_\sigma\|_{L_\infty}\|(\varphi_\sigma^2)'\|_{L_\infty}}{\varepsilon^2}$. We deduce that

$$\begin{aligned}\mathbb{E}\left[\langle M,M\rangle_T\right]&=\mathbb{E}\left[\int_0^T\int_0^1\mathbb{1}_{\{\widetilde{y}(u_2,s)\neq\widetilde{y}(u_1,s)\}}\left(\frac{\theta(\widetilde{y}(u_2,s),u',s)-\theta(\widetilde{y}(u_1,s),u',s)}{\widetilde{y}(u_2,s)-\widetilde{y}(u_1,s)}\right)^2\mathrm{d}u'\mathrm{d}s\right]\\&\leqslant\mathbb{E}\left[\int_0^T\int_0^1(C_{\sigma,\varepsilon})^2\mathrm{d}u'\mathrm{d}s\right]\leqslant T\,(C_{\sigma,\varepsilon})^2,\end{aligned}$$

and thus M is a $(\mathcal{G}_t)_{t\in[0,T]}$-martingale on $[0,T]$. We resolve the stochastic differential equation (3.5): $\widetilde{y}^g_{\sigma,\varepsilon}(u_2,t)-\widetilde{y}^g_{\sigma,\varepsilon}(u_1,t)=(g(u_2)-g(u_1))\exp\left(M_t-\frac{1}{2}\langle M,M\rangle_t\right)$. If $g(u_1)<g(u_2)$ (resp. $g(u_1)=g(u_2)$), then almost surely for every $t\in[0,T]$, $\widetilde{y}^g_{\sigma,\varepsilon}(u_1,t)<\widetilde{y}^g_{\sigma,\varepsilon}(u_2,t)$ (resp. $=$). Thus it is true almost surely for every $(u_1,u_2)\in\mathbb{Q}^2$ such that $u_1<u_2$. □

Therefore the proof of Proposition 3.6 is complete:

Proof (Proposition 3.6). For each $t\in[0,T]$, $Y^g_{\sigma,\varepsilon}(t)=y^g_{\sigma,\varepsilon}(\cdot,t)$ has a modification $\widetilde{y}^g_{\sigma,\varepsilon}(\cdot,t)$ belonging to $L_2^{\uparrow}[0,1]$. □

We precise the properties of $\widetilde{y}^g_{\sigma,\varepsilon}$ in the following Corollary, which derives directly from Proposition 3.6. From now on, we will always use this version of the process.

Corollary 3.11. *The following two statements hold:*

- *for almost every $u\in(0,1)$, $(\widetilde{y}^g_{\sigma,\varepsilon}(\omega,u,t))_{t\in[0,T]}$ is a $(\mathcal{F}^{\sigma,\varepsilon}_t)_{t\in[0,T]}$-martingale, and it is continuous for almost every $(u,\omega)\in(0,1)\times\Omega$.*
- *almost surely, for every $t\in[0,T]$, $u\mapsto\widetilde{y}^g_{\sigma,\varepsilon}(u,t)$ is càdlàg and non-decreasing.*

We complete the proof of Proposition 3.2.

Proof (Proposition 3.2). Thanks to Proposition 3.6, the proof of properties $(A1)$ and $(A2)$ has been completed. It remains to compute the quadratic variation. Recall that for every $u \in [0,1]$, $(\widetilde{y}^g_{\sigma,\varepsilon}(u,t))_{t\in[0,T]}$ is a $(\mathcal{G}_t)_{t\in[0,T]}$-martingale and that

$$\widetilde{y}^g_{\sigma,\varepsilon}(u,t) = g(u) + \int_0^t \int_0^1 \theta_{\sigma,\varepsilon}(\widetilde{y}^g_{\sigma,\varepsilon}(u,s),u',s)\mathrm{d}w(u',s).$$

Therefore, for every $u,u' \in [0,1]$,

$$\begin{aligned}\langle \widetilde{y}^g_{\sigma,\varepsilon}(u,\cdot),\widetilde{y}^g_{\sigma,\varepsilon}(u',\cdot)\rangle_t &= \langle \int_0^{\cdot}\int_0^1 \theta_{\sigma,\varepsilon}(\widetilde{y}^g_{\sigma,\varepsilon}(u,s),v,s)\mathrm{d}w(v,s), \int_0^{\cdot}\int_0^1 \theta_{\sigma,\varepsilon}(\widetilde{y}^g_{\sigma,\varepsilon}(u',s),v,s)\mathrm{d}w(v,s)\rangle_t \\ &= \int_0^t\int_0^1 \theta_{\sigma,\varepsilon}(\widetilde{y}^g_{\sigma,\varepsilon}(u,s),v,s)\theta_{\sigma,\varepsilon}(\widetilde{y}^g_{\sigma,\varepsilon}(u',s),v,s)\mathrm{d}v\mathrm{d}s.\end{aligned}$$

Therefore, for every $h,k \in L_2[0,1]$,

$$\begin{aligned}&\langle (Y^g_{\sigma,\varepsilon},h)_{L_2},(Y^g_{\sigma,\varepsilon},k)_{L_2}\rangle_t \\ &= \int_0^t\int_0^1\int_0^1 h(u)k(u')\int_0^1 \theta_{\sigma,\varepsilon}(\widetilde{y}^g_{\sigma,\varepsilon}(u,s),v,s)\theta_{\sigma,\varepsilon}(\widetilde{y}^g_{\sigma,\varepsilon}(u',s),v,s)\mathrm{d}v\mathrm{d}u\mathrm{d}u'\mathrm{d}s \\ &= \int_0^t\int_0^1\int_0^1 h(u)k(u')\int_0^1 \theta_{\sigma,\varepsilon}(y^g_{\sigma,\varepsilon}(u,s),v,s)\theta_{\sigma,\varepsilon}(y^g_{\sigma,\varepsilon}(u',s),v,s)\mathrm{d}v\mathrm{d}u\mathrm{d}u'\mathrm{d}s \\ &= \int_0^t\int_0^1\int_0^1 h(u)k(u')\frac{m^g_{\sigma,\varepsilon}(u,u',s)}{(\varepsilon+m^g_{\sigma,\varepsilon}(u,s))(\varepsilon+m^g_{\sigma,\varepsilon}(u',s))}\mathrm{d}u\mathrm{d}u'\mathrm{d}s,\end{aligned}$$

which completes the proof. □

We conclude this Section with a property on the quadratic variation of two fixed particles, which will be useful to obtain lower bounds on the mass in the next Section.

Corollary 3.12. *For almost every* $u,u' \in [0,1]$*,*

$$\langle \widetilde{y}^g_{\sigma,\varepsilon}(u,\cdot),\widetilde{y}^g_{\sigma,\varepsilon}(u',\cdot)\rangle_t = \int_0^t\int_0^1 \frac{m^g_{\sigma,\varepsilon}(u,u',s)}{(\varepsilon+m^g_{\sigma,\varepsilon}(u,s))(\varepsilon+m^g_{\sigma,\varepsilon}(u',s))}\mathrm{d}v\mathrm{d}s. \tag{3.6}$$

Proof. This statement follows clearly from the proof of Proposition 3.2, from the fact that for almost every $u \in (0,1)$, $(\widetilde{y}^g_{\sigma,\varepsilon}(u,t))_{t\in[0,T]}$ is a continuous martingale. □

4 Convergence of the process $(y^g_{\sigma,\varepsilon})_{\sigma,\varepsilon\in\mathbb{Q}_+}$

From now on, for the sake of simplicity, we fix a function g in $\mathcal{L}^{\uparrow}_{2+}[0,1]$ and $y_{\sigma,\varepsilon}$ will denote the version $\widetilde{y}^g_{\sigma,\varepsilon}$ starting from g. We denote by p a number such that $p > 2$ and $g \in L_p(0,1)$.

We begin by proving the tightness of the sequence $(y_{\sigma,\varepsilon})_{\sigma,\varepsilon\in\mathbb{Q}_+}$ in $L_2([0,1],\mathcal{C}[0,T])$ in Paragraph 4.1. We will then pass to the limit in distribution, first when $\varepsilon \to 0$ and then when $\sigma \to 0$ and prove, in Paragraph 4.3, that the limit process is also a martingale.

4.1 Tightness of the collection $(y_{\sigma,\varepsilon})_{\sigma>0,\varepsilon>0}$ in $L_2([0,1],\mathcal{C}[0,T])$

Recall that for all $\sigma > 0$, the map φ_σ is smooth, even, bounded by 1, equal to 1 on $\left[0,\frac{\sigma-\eta}{2}\right]$ and equal to 0 on $\left[\frac{\sigma}{2},+\infty\right)$, where η is chosen so that $\eta < \frac{\sigma}{3}$. Recall that $y_{\sigma,\varepsilon}$ is solution of the following equation:

$$y_{\sigma,\varepsilon}(u,t) = g(u) + \int_0^t\int_0^1 \frac{\varphi_\sigma(y_{\sigma,\varepsilon}(u,s)-y_{\sigma,\varepsilon}(u',s))}{\varepsilon+\int_0^1\varphi^2_\sigma(y_{\sigma,\varepsilon}(u,s)-y_{\sigma,\varepsilon}(v,s))\mathrm{d}v}\mathrm{d}w(u',s).$$

We begin by proving that the collection $(y_{\sigma,\varepsilon})_{\sigma>0,\varepsilon>0}$ satisfies a compactness criterion in the space $L_2([0,1],\mathcal{C}[0,T])$. We recall the following criterion (see [18, Theorem 1, p.71]):

Proposition 4.1. *Let K be a subset of $L_2([0,1],\mathcal{C}[0,T])$.*
K is relatively compact in $L_2([0,1],\mathcal{C}[0,T])$ if and only if:

$(H1)$ *for every $0\leqslant u_1<u_2\leqslant 1$, $\left\{\int_{u_1}^{u_2} f(u,\cdot)\mathrm{d}u, f\in K\right\}$ is relatively compact in $C[0,T]$,*

$(H2)$ $\lim_{h\to 0^+}\sup_{f\in K}\int_0^{1-h}\|f(u+h,\cdot)-f(u,\cdot)\|^2_{\mathcal{C}[0,T]}\mathrm{d}u=0$.

By Ascoli's Theorem, $(H1)$ is satisfied if and only if for every $0\leqslant u_1<u_2\leqslant 1$,

- for every $t\in[0,T]$, $\int_{u_1}^{u_2} f(u,t)\mathrm{d}u$ is uniformly bounded,
- $\lim_{\eta\to 0^+}\sup_{f\in K}\sup_{|t_2-t_1|<\eta}\left|\int_{u_1}^{u_2}(f(u,t_2)-f(u,t_1))\mathrm{d}u\right|=0$.

In order to prove tightness for the collection $(y_{\sigma,\varepsilon})_{\sigma>0,\varepsilon>0}$, we will prove the following Proposition:

Proposition 4.2. *Let $\delta>0$. The following statements hold:*

$(K1)$ *there exists $M>0$ such that for all $\sigma,\varepsilon>0$, $\mathbb{P}\left[\int_0^1\|y_{\sigma,\varepsilon}(u,\cdot)\|^2_{\mathcal{C}[0,T]}\mathrm{d}u\leqslant M\right]\geqslant 1-\delta$,*

$(K2)$ *for all $k\geqslant 1$, there exists $\eta_k>0$ such that for all $\sigma>0$, $\varepsilon>0$,*

$$\mathbb{P}\left[\int_0^1\sup_{|t_2-t_1|<\eta_k}|y_{\sigma,\varepsilon}(u,t_2)-y_{\sigma,\varepsilon}(u,t_1)|\mathrm{d}u\leqslant\frac{1}{k}\right]\geqslant 1-\frac{\delta}{2^k},$$

$(K3)$ *for all $k\geqslant 1$, there exists $h_k>0$ such that for all $\sigma>0$, $\varepsilon>0$,*

$$\mathbb{P}\left[\forall h\in(0,h_k),\int_0^{1-h}\|y_{\sigma,\varepsilon}(u+h,\cdot)-y_{\sigma,\varepsilon}(u,\cdot)\|^2_{\mathcal{C}[0,T]}\mathrm{d}u\leqslant\frac{1}{k}\right]\geqslant 1-\frac{\delta}{2^k}.$$

Proposition 4.2 will be proved in Paragraph 4.1.2. It implies tightness of $(y^g_{\sigma,\varepsilon})_{\sigma>0,\varepsilon>0}$ in $L_2([0,1],\mathcal{C}[0,T])$:

Corollary 4.3. *For all $g\in\mathcal{L}^{\uparrow}_{2+}[0,1]$, the collection $(y^g_{\sigma,\varepsilon})_{\sigma>0,\varepsilon>0}$ is tight in $L_2([0,1],\mathcal{C}[0,T])$.*

Proof (Corollary 4.3). Let $\delta>0$. Let M, $(h_k)_{k\geqslant 1}$, $(\eta_k)_{k\geqslant 1}$ be such that the statements of Proposition 4.2 hold for δ.

Denote K_δ the closed set of all functions $f\in L_2([0,1],\mathcal{C}[0,T])$ satisfying:

$(L1)$ $\displaystyle\int_0^1\|f(u,\cdot)\|^2_{\mathcal{C}[0,T]}\mathrm{d}u\leqslant M$.

$(L2)$ for all $k\geqslant 1$, $\displaystyle\int_0^1\sup_{|t_2-t_1|<\eta_k}|f(u,t_2)-f(u,t_1)|\mathrm{d}u\leqslant\frac{1}{k}$.

$(L3)$ for all $k\geqslant 1$, $\forall h\in(0,h_k)$, $\displaystyle\int_0^{1-h}\|f(u+h,\cdot)-f(u,\cdot)\|^2_{\mathcal{C}[0,T]}\mathrm{d}u\leqslant\frac{1}{k}$.

Let $0\leqslant u_1<u_2\leqslant 1$. We deduce from $(L1)$ that for every $t\in[0,T]$, and every $f\in K_\delta$, $\left|\int_{u_1}^{u_2}f(u,t)\mathrm{d}u\right|\leqslant\left(\int_{u_1}^{u_2}f(u,t)^2\mathrm{d}u\right)^{1/2}\leqslant\left(\int_0^1\|f(u,\cdot)\|^2_{\mathcal{C}[0,T]}\mathrm{d}u\right)^{1/2}\leqslant\sqrt{M}$. We deduce from $(L2)$ that for every $k\geqslant 1$,

$$\sup_{f\in K_\delta}\sup_{|t_2-t_1|<\eta_k}\left|\int_{u_1}^{u_2}(f(u,t_2)-f(u,t_1))\mathrm{d}u\right|\leqslant\sup_{f\in K_\delta}\int_0^1\sup_{|t_2-t_1|<\eta_k}|f(u,t_2)-f(u,t_1)|\mathrm{d}u\leqslant\frac{1}{k}.$$

Therefore, by Ascoli's Theorem, condition $(H1)$ of Proposition 4.1 is satisfied.

Furthermore, by $(L3)$, condition $(H2)$ is also satisfied uniformly for $f\in K_\delta$. Therefore, K_δ is compact in $L_2([0,1],\mathcal{C}[0,T])$. By Proposition 4.2, for all $\sigma>0$, $\varepsilon>0$, $\mathbb{P}[y_{\sigma,\varepsilon}\in K_\delta]\geqslant 1-3\delta$. This concludes the proof. □

To prove Proposition 4.2, we will first give in the next Paragraph an estimation of the inverse of the mass function (see Lemma 4.6). This Lemma is an equivalent in our case of short-range interacting particles of Lemma 2.16 in [12], stated in the case of a system of coalescing particles.

4.1.1 Estimation of the inverse of mass

Recall that $m_{\sigma,\varepsilon}(u,t) = \int_0^1 \varphi_\sigma^2(y_{\sigma,\varepsilon}(u,t) - y_{\sigma,\varepsilon}(v,t))\mathrm{d}v$. We define a modified mass

$$M_{\sigma,\varepsilon}(u,t) := \begin{cases} \dfrac{(\varepsilon + m_{\sigma,\varepsilon})^2}{m_{\sigma,\varepsilon}}(u,t) \text{ if } m_{\sigma,\varepsilon}(u,t) > 0, \\ +\infty \text{ otherwise.} \end{cases}$$

Clearly, $M_{\sigma,\varepsilon}(u,t) \geqslant m_{\sigma,\varepsilon}(u,t)$ for every $u \in [0,1]$ and $t \in [0,T]$.

By Corollary 3.11, there exists a (non-random) Borel set $\mathcal{A}$ in $[0,1]$, $\mathrm{Leb}(\mathcal{A}) = 1$, such that for all $u \in \mathcal{A}$, $(y_{\sigma,\varepsilon}(u,t))_{t\in[0,T]}$ is almost surely a continuous $(\mathcal{F}_t^{\sigma,\varepsilon})_{t\in[0,T]}$-martingale. Recall also that almost surely, for every $t \in [0,T]$, $u \mapsto y_{\sigma,\varepsilon}(u,t)$ is càdlàg and non-decreasing. Moreover, we assume that for every $u, u' \in \mathcal{A}$, equality (3.6) holds.

Lemma 4.4. *There exist $C > 0$ and $\gamma \in (0,1)$ such that for each $\sigma, \varepsilon > 0$, $t \in (0,T]$ and for every $u \in \mathcal{A}$ and every $h > 0$ satisfying $u - h \in (0,1)$,*

$$\mathbb{P}\left[\int_0^T \mathbb{1}_{\{M_{\sigma,\varepsilon}(u,s)<\gamma h\}}\mathrm{d}s \geqslant t\right] \leqslant C\,[g(u) - g(u-h)]\sqrt{\frac{h}{t}}. \tag{4.1}$$

Proof. Fix $\sigma > 0$ and $\varepsilon > 0$. Let $h > 0$ be such that $u - h$ belongs to $\mathcal{A}$. If $g(u-h) = g(u)$, then for every $t \in [0,T]$, $y_{\sigma,\varepsilon}(u-h,t) = y_{\sigma,\varepsilon}(u,t)$. By the non-decreasing and càdlàg property, for every $v \in (u-h,u)$, we have $y_{\sigma,\varepsilon}(v,t) = y_{\sigma,\varepsilon}(u,t)$. We deduce that $m_{\sigma,\varepsilon}(u,t) \geqslant \int_{u-h}^u \varphi_\sigma^2(y_{\sigma,\varepsilon}(u,t) - y_{\sigma,\varepsilon}(v,t))\mathrm{d}v = \int_{u-h}^u \varphi_\sigma^2(0)\mathrm{d}v = h$. Therefore, $M_{\sigma,\varepsilon}(u,t) \geqslant h \geqslant \gamma h$ for every $t \in [0,T]$, and (4.1) is satisfied.

Consider now the case where $g(u-h) < g(u)$. Choose k in $(\frac{h}{3}, \frac{2h}{3})$ such that $u - k \in \mathcal{A}$. Denote by N and $\widetilde{N}$ the two following $(\mathcal{F}_t^{\sigma,\varepsilon})_{t\in[0,T]}$-martingales:

$$N_t = y_{\sigma,\varepsilon}(u,t) - y_{\sigma,\varepsilon}(u-h,t),$$
$$\widetilde{N}_t = y_{\sigma,\varepsilon}(u,t) - y_{\sigma,\varepsilon}(u-k,t).$$

Denote by G_s and H_s respectively the events $\{M_{\sigma,\varepsilon}(u,s) < \frac{h}{2^6}\}$ and $\{\widetilde{N}_s > \frac{\sigma+\eta}{2}\}$. We want to prove the existence of a constant C_1 independent of h and u such that for all $\sigma > 0$, $\varepsilon > 0$ and $t > 0$,

$$\mathbb{P}\left[\int_0^T \mathbb{1}_{\{G_s\}}\mathrm{d}s \geqslant t\right] \leqslant C_1\,[g(u) - g(u-h)]\sqrt{\frac{h}{t}}. \tag{4.2}$$

Decompose this probability in two terms:

$$\mathbb{P}\left[\int_0^T \mathbb{1}_{\{G_s\}}\mathrm{d}s \geqslant t\right] \leqslant \mathbb{P}\left[\int_0^T \mathbb{1}_{\{G_s\cap H_s\}}\mathrm{d}s \geqslant \frac{t}{2}\right] + \mathbb{P}\left[\int_0^T \mathbb{1}_{\{G_s\cap H_s^{\complement}\}}\mathrm{d}s \geqslant \frac{t}{2}\right], \tag{4.3}$$

where $H_s^{\complement}$ denotes the complement of the event H_s.

- **First step:** Study of $G_s \cap H_s$.

  Fix $s \in [0,T]$. Under $G_s \cap H_s$, we have $M_{\sigma,\varepsilon}(u,s) < \frac{h}{2^6}$ and $\widetilde{N}_s > \frac{\sigma+\eta}{2}$. We want to show that it implies the following inequality:

$$\frac{2m_{\sigma,\varepsilon}(u,u-h,s)}{(\varepsilon + m_{\sigma,\varepsilon}(u,s))(\varepsilon + m_{\sigma,\varepsilon}(u-h,s))} \leqslant \frac{1}{M_{\sigma,\varepsilon}(u,s)^{3/4} M_{\sigma,\varepsilon}(u-h,s)^{1/4}}. \tag{4.4}$$

  Suppose, by contradiction, that (4.4) is false. Using Cauchy-Schwarz inequality, $m_{\sigma,\varepsilon}(u,u-h,s) \leqslant m_{\sigma,\varepsilon}(u,s)^{1/2} m_{\sigma,\varepsilon}(u-h,s)^{1/2}$, and we would deduce that:

$$\frac{1}{M_{\sigma,\varepsilon}(u,s)^{3/4} M_{\sigma,\varepsilon}(u-h,s)^{1/4}} \leqslant \frac{2}{M_{\sigma,\varepsilon}(u,s)^{1/2} M_{\sigma,\varepsilon}(u-h,s)^{1/2}},$$

  and thus $M_{\sigma,\varepsilon}(u-h,s) \leqslant 2^4 M_{\sigma,\varepsilon}(u,s)$. Using the fact that $M_{\sigma,\varepsilon} \geqslant m_{\sigma,\varepsilon}$, we can deduce that

$$m_{\sigma,\varepsilon}(u,s) + m_{\sigma,\varepsilon}(u-h,s) \leqslant M_{\sigma,\varepsilon}(u,s) + M_{\sigma,\varepsilon}(u-h,s) \leqslant (1+2^4)\frac{h}{2^6} < \frac{h}{3}. \tag{4.5}$$

  We distinguish three cases depending on the value of $N_s = y_{\sigma,\varepsilon}(u,s) - y_{\sigma,\varepsilon}(u-h,s)$.

  - $N_s \leqslant \sigma - \eta$**:** For each $v \in [u-h,u]$, one of the two terms $y_{\sigma,\varepsilon}(u,s) - y_{\sigma,\varepsilon}(v,s)$ and $y_{\sigma,\varepsilon}(v,s) - y_{\sigma,\varepsilon}(u-h,s)$ is lower than $\frac{\sigma-\eta}{2}$, which means that one of those terms belongs to the preimage of 1 by the function φ_σ. Hence

$$\begin{aligned} &m_{\sigma,\varepsilon}(u,s) + m_{\sigma,\varepsilon}(u-h,s) \\ &\qquad = \int_0^1 \left(\varphi_\sigma^2(y_{\sigma,\varepsilon}(u,s) - y_{\sigma,\varepsilon}(v,s)) + \varphi_\sigma^2(y_{\sigma,\varepsilon}(u-h,s) - y_{\sigma,\varepsilon}(v,s))\right) \mathrm{d}v \\ &\qquad \geqslant \int_{u-h}^u \mathrm{d}v = h. \end{aligned}$$

    This is in contradiction with (4.5). Therefore inequality (4.4) is satisfied in this case.

  - $N_s \in (\sigma-\eta, \sigma)$**:** Introduce $\mathrm{Med} := \{v : y_{\sigma,\varepsilon}(u,s) - y_{\sigma,\varepsilon}(v,s) \in [\frac{\sigma-\eta}{2}, \frac{\sigma+\eta}{2}]\}$, which is a set of particles more or less at half distance between particle u and particle $u-h$. Since $\eta < \frac{\sigma}{3}$, we have $N_s > \sigma - \eta \geqslant \frac{\sigma+\eta}{2}$ and thus $\mathrm{Med} \subset [u-h,u]$. Let $v \in [u-h,u]$. We distinguish three new cases:

    - if $y_{\sigma,\varepsilon}(u,s) - y_{\sigma,\varepsilon}(v,s) < \frac{\sigma-\eta}{2}$, then $\varphi_\sigma(y_{\sigma,\varepsilon}(u,s) - y_{\sigma,\varepsilon}(v,s)) = 1$.
    - if $y_{\sigma,\varepsilon}(u,s) - y_{\sigma,\varepsilon}(v,s) > \frac{\sigma+\eta}{2}$, and since $N_s \leqslant \sigma$, $y_{\sigma,\varepsilon}(v,s) - y_{\sigma,\varepsilon}(u-h,s)$ is lower than $\frac{\sigma-\eta}{2}$ and thus $\varphi_\sigma(y_{\sigma,\varepsilon}(u-h,s) - y_{\sigma,\varepsilon}(v,s)) = 1$.
    - otherwise, v belongs to Med.

    It follows that:

$$\begin{aligned} h &= \int_{u-h}^u (\mathbb{1}_{\{y_{\sigma,\varepsilon}(u,s) - y_{\sigma,\varepsilon}(v,s) < \frac{\sigma-\eta}{2}\}} + \mathbb{1}_{\{y_{\sigma,\varepsilon}(u,s) - y_{\sigma,\varepsilon}(v,s) > \frac{\sigma+\eta}{2}\}} + \mathbb{1}_{\{v \in \mathrm{Med}\}}) \mathrm{d}v \\ &\leqslant \int_{u-h}^u (\varphi_\sigma^2(y_{\sigma,\varepsilon}(u,s) - y_{\sigma,\varepsilon}(v,s)) + \varphi_\sigma^2(y_{\sigma,\varepsilon}(u-h,s) - y_{\sigma,\varepsilon}(v,s)) + \mathbb{1}_{\{v \in \mathrm{Med}\}}) \mathrm{d}v \\ &\leqslant m_{\sigma,\varepsilon}(u,s) + m_{\sigma,\varepsilon}(u-h,s) + \mathrm{Leb}(\mathrm{Med}). \end{aligned}$$

    By inequality (4.5), we deduce that $\mathrm{Leb}(\mathrm{Med}) > \frac{2h}{3}$. As Med is an interval included in $[u-h,u]$ and since $k \in (\frac{h}{3}, \frac{2h}{3})$ we deduce that $u-k \in \mathrm{Med}$, *i.e.* $\widetilde{N}_s \in [\frac{\sigma-\eta}{2}, \frac{\sigma+\eta}{2}]$, which is in contradiction with the hypothesis $\widetilde{N}_s > \frac{\sigma+\eta}{2}$. Thus inequality (4.4) is also true in this case.

- $N_s \geqslant \sigma$**:** In this case, the two particles u and $u-h$ do not have any interaction. In other words, since the support of φ_σ is included in $[-\frac{\sigma}{2}, \frac{\sigma}{2}]$, $\varphi_\sigma(y_{\sigma,\varepsilon}(u,s) - y_{\sigma,\varepsilon}(v,s))$ and $\varphi_\sigma(y_{\sigma,\varepsilon}(u-h,s) - y_{\sigma,\varepsilon}(v,s))$ can not be simultaneously non-zero, whence we deduce that $m_{\sigma,\varepsilon}(u,u-h,s) = 0$. Inequality (4.4) follows clearly.

Therefore, inequality (4.4) is proved. By Corollary 3.12, it follows that, on $G_s \cap H_s$:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\langle N,N\rangle_s &= \frac{1}{M_{\sigma,\varepsilon}(u,s)} + \frac{1}{M_{\sigma,\varepsilon}(u-h,s)} - \frac{2m_{\sigma,\varepsilon}(u,u-h,s)}{(\varepsilon + m_{\sigma,\varepsilon}(u,s))(\varepsilon + m_{\sigma,\varepsilon}(u-h,s))} \\ &\geqslant \frac{1}{M_{\sigma,\varepsilon}(u,s)} + \frac{1}{M_{\sigma,\varepsilon}(u-h,s)} - \frac{1}{M_{\sigma,\varepsilon}(u,s)^{3/4} M_{\sigma,\varepsilon}(u-h,s)^{1/4}} \\ &\geqslant \frac{1}{4M_{\sigma,\varepsilon}(u,s)} + \frac{3}{4M_{\sigma,\varepsilon}(u-h,s)} \geqslant \frac{1}{4M_{\sigma,\varepsilon}(u,s)} \geqslant \frac{2^4}{h},\end{aligned}$$

where we have applied a convexity inequality: $\forall a,b>0, a^{3/4}b^{1/4} \leqslant \frac{3a}{4} + \frac{b}{4}$.

To sum up, we showed that $G_s \cap H_s$ implies $\frac{\mathrm{d}}{\mathrm{d}s}\langle N,N\rangle_s \geqslant \frac{2^4}{h}$. If $\int_0^T \mathbb{1}_{\{G_s\cap H_s\}}\mathrm{d}s \geqslant \frac{t}{2}$, we get

$$\langle N,N\rangle_T = \int_0^T \frac{\mathrm{d}}{\mathrm{d}s}\langle N,N\rangle_s \mathrm{d}s \geqslant \int_0^T \frac{\mathrm{d}}{\mathrm{d}s}\langle N,N\rangle_s \mathbb{1}_{\{G_s\cap H_s\}}\mathrm{d}s \geqslant \frac{2^4}{h}\int_0^T \mathbb{1}_{\{G_s\cap H_s\}}\mathrm{d}s \geqslant \frac{2^3 t}{h}.$$

Hence, since N is a continuous square integrable $(\mathcal{F}_t^{\sigma,\varepsilon})_{t\in[0,T]}$-martingale, there exists a standard $(\mathcal{F}_t^{\sigma,\varepsilon})_{t\in[0,T]}$-Brownian motion β such that we have the relation $N_t = g(u) - g(u-h) - \beta(\langle N,N\rangle_t)$. Since N remains positive on $[0,T]$ by Lemma 3.10 (because $g(u-h) < g(u)$), we deduce that $\sup_{[0,\langle N,N\rangle_T]}\beta \leqslant g(u) - g(u-h)$. Therefore,

$$\begin{aligned}\mathbb{P}\left[\int_0^T \mathbb{1}_{\{G_s\cap H_s\}}\mathrm{d}s \geqslant \frac{t}{2}\right] &\leqslant \mathbb{P}\left[\sup_{[0,\frac{2^3 t}{h}]}\beta \leqslant g(u) - g(u-h)\right] \\ &= \mathbb{P}\left[\sqrt{\frac{2^3}{h}}\sup_{[0,t]}\widehat{\beta} \leqslant g(u) - g(u-h)\right] \\ &\leqslant C_2\,[g(u) - g(u-h)]\sqrt{\frac{h}{t}}, \qquad (4.6)\end{aligned}$$

where $\widehat{\beta}$ is a rescaled Brownian motion and C_2 does not depend on u, h, σ, ε and t.

- **Second step:** Study of $G_s \cap H_s^{\complement}$.

  Under this event, we have $M_{\sigma,\varepsilon}(u,s) < \frac{h}{2^6}$ and $\widetilde{N}_s \leqslant \frac{\sigma+\eta}{2}$. In particular, by the assumption $\eta < \frac{\sigma}{3}$, we have $\widetilde{N}_s \leqslant \sigma - \eta$. We claim that the following inequality holds true:

$$\frac{2m_{\sigma,\varepsilon}(u,u-k,s)}{(\varepsilon + m_{\sigma,\varepsilon}(u,s))(\varepsilon + m_{\sigma,\varepsilon}(u-k,s))} \leqslant \frac{1}{M_{\sigma,\varepsilon}(u,s)^{3/4} M_{\sigma,\varepsilon}(u-k,s)^{1/4}}. \qquad (4.7)$$

  To prove it, it is sufficient to imitate the proof of the case $N_s \leqslant \sigma - \eta$ of the previous step. We should notice that we did not use the hypothesis $\widetilde{N}_s > \frac{\sigma+\eta}{2}$ in that case.

  Using inequality (4.7) as in the first step, we show that $\frac{\mathrm{d}}{\mathrm{d}s}\langle \widetilde{N},\widetilde{N}\rangle_s \geqslant \frac{2^4}{h}$. Therefore, $\mathbb{P}\left[\int_0^T \mathbb{1}_{\{G_s\cap H_s^{\complement}\}}\mathrm{d}s \geqslant \frac{t}{2}\right] \leqslant \mathbb{P}\left[\langle \widetilde{N},\widetilde{N}\rangle_T \geqslant \frac{2^3 t}{h}\right]$. There exists a $(\mathcal{F}_t^{\sigma,\varepsilon})_{t\in[0,T]}$-Brownian motion $\widetilde{\beta}$ such that $\widetilde{N}_t = g(u) - g(u-k) - \widetilde{\beta}(\langle \widetilde{N},\widetilde{N}\rangle_t)$. Finally, we obtain the existence of a constant C_3 independent of u, h, k, σ, ε and t such that:

$$\begin{aligned}\mathbb{P}\left[\int_0^T \mathbb{1}_{\{G_s\cap H_s^{\complement}\}}\mathrm{d}s \geqslant \frac{t}{2}\right] &\leqslant \mathbb{P}\left[\sup_{[0,\frac{2^3 t}{h}]} \widetilde{\beta} \leqslant g(u)-g(u-k)\right]\\ &\leqslant C_3\left[g(u)-g(u-k)\right]\sqrt{\frac{h}{t}}\\ &\leqslant C_3\left[g(u)-g(u-h)\right]\sqrt{\frac{h}{t}}. \end{aligned} \tag{4.8}$$

Putting together inequality (4.3) and inequalities (4.6) and (4.8), we conclude the proof of inequality (4.2). Thus inequality (4.1) is proved for every h such that $u-h\in\mathcal{A}$. Let $h>0$ be such that $u-h\in(0,1)$. Let $h_1\in(\frac{h}{2},h)$ be such that $u-h_1\in\mathcal{A}$.

$$\begin{aligned}\mathbb{P}\left[\int_0^T \mathbb{1}_{\left\{M_{\sigma,\varepsilon}(u,s)<\frac{\gamma h}{2}\right\}}\mathrm{d}s \geqslant t\right] &\leqslant \mathbb{P}\left[\int_0^T \mathbb{1}_{\{M_{\sigma,\varepsilon}(u,s)<\gamma h_1\}}\mathrm{d}s \geqslant t\right]\\ &\leqslant C\left[g(u)-g(u-h_1)\right]\sqrt{\frac{h_1}{t}}\\ &\leqslant C\left[g(u)-g(u-h)\right]\sqrt{\frac{h}{t}}.\end{aligned}$$

Up to replacing γ by $\frac{\gamma}{2}$, inequality (4.1) follows for every $h>0$ such that $u-h\in(0,1)$. □

Remark 4.5. Similarly, there exist $C>0$ and $\gamma\in(0,1)$ such that for each $\sigma,\varepsilon>0$, $t\in(0,T]$ and for every $u\in\mathcal{A}$ and every $h>0$ satisfying $u+h\in(0,1)$,

$$\mathbb{P}\left[\int_0^T \mathbb{1}_{\{M_{\sigma,\varepsilon}(u,s)<\gamma h\}}\mathrm{d}s \geqslant t\right] \leqslant C\left[g(u+h)-g(u)\right]\sqrt{\frac{h}{t}}.$$

Thanks to Lemma 4.4 and to the above remark, we obtain the following result, which has to be compared with Proposition 4.3 in [11]:

Lemma 4.6. *Let $g\in L_p(0,1)$. For all $\beta\in(0,\frac{3}{2}-\frac{1}{p})$, there is a constant $C>0$ depending only on β and $\|g\|_{L_p}$ such that for all $\sigma,\varepsilon>0$ and $0\leqslant s<t\leqslant T$, we have the following inequality:*

$$\mathbb{E}\left[\int_s^t\int_0^1 \frac{1}{M^{\beta}_{\sigma,\varepsilon}(u,r)}\mathrm{d}u\mathrm{d}r\right] \leqslant C\sqrt{t-s}. \tag{4.9}$$

Remark 4.7. Observe that by the assumption $p>2$, made at the beginning of Section 4, there exists some $\beta>1$ such that (4.9) holds.

Proof. By Fubini-Tonelli Theorem, we have:

$$\begin{aligned}\mathbb{E}\left[\int_s^t\int_0^1 \frac{\mathrm{d}u\mathrm{d}r}{M^{\beta}_{\sigma,\varepsilon}(u,r)}\right] &= \int_0^1 \mathbb{E}\left[\int_s^t\int_0^{+\infty} \mathbb{1}_{\{M^{-\beta}_{\sigma,\varepsilon}(u,r)>x\}}\mathrm{d}x\mathrm{d}r\right]\mathrm{d}u\\ &\leqslant 2^{\beta}(t-s)+\int_0^1\int_{2^{\beta}}^{+\infty}\mathbb{E}\left[\int_s^t \mathbb{1}_{\{M_{\sigma,\varepsilon}(u,r)<x^{-1/\beta}\}}\mathrm{d}r\right]\mathrm{d}x\mathrm{d}u\\ &\leqslant 2^{\beta}\sqrt{T}\sqrt{t-s}+\int_0^1\int_{2^{\beta}\gamma^{\beta}}^{+\infty}\mathbb{E}\left[\int_s^t \mathbb{1}_{\{M_{\sigma,\varepsilon}(u,r)<\gamma x^{-1/\beta}\}}\mathrm{d}r\right]\gamma^{-\beta}\mathrm{d}x\mathrm{d}u.\end{aligned}$$

Furthermore, we compute:

$$\begin{aligned}\mathbb{E}\left[\int_s^t \mathbb{1}_{\{M_{\sigma,\varepsilon}(u,r)<\gamma x^{-1/\beta}\}}\mathrm{d}r\right] &= \int_0^{t-s} \mathbb{P}\left[\int_s^t \mathbb{1}_{\{M_{\sigma,\varepsilon}(u,r)<\gamma x^{-1/\beta}\}}\mathrm{d}r > \alpha\right]\mathrm{d}\alpha \\ &\leqslant \int_0^{t-s} \mathbb{P}\left[\int_0^T \mathbb{1}_{\{M_{\sigma,\varepsilon}(u,r)<\gamma x^{-1/\beta}\}}\mathrm{d}r > \alpha\right]\mathrm{d}\alpha.\end{aligned}$$

Using Lemma 4.4, we obtain a constant C_1 independent of σ and ε such that for all $x > 2^\beta$:

$$\begin{aligned}\int_{\frac{1}{2}}^1 \mathbb{E}\left[\int_s^t \mathbb{1}_{\{M_{\sigma,\varepsilon}(u,r)<\gamma x^{-1/\beta}\}}\mathrm{d}r\right]\mathrm{d}u &\leqslant \int_{\frac{1}{2}}^1\int_0^{t-s} C_1\left[g(u)-g(u-x^{-1/\beta})\right]\sqrt{\frac{x^{-1/\beta}}{\alpha}}\mathrm{d}\alpha\mathrm{d}u \\ &\leqslant 2C_1\frac{\int_{1/2}^1(g(u)-g(u-x^{-1/\beta}))\mathrm{d}u}{x^{1/(2\beta)}}\sqrt{t-s}.\end{aligned}$$

Moreover, we have for each $x > 2^\beta$, using Hölder's inequality:

$$\begin{aligned}\int_{\frac{1}{2}}^1\left(g(u)-g(u-x^{-1/\beta})\right)\mathrm{d}u &= \int_0^1\left(\mathbb{1}_{[\frac{1}{2},1]}(u)-\mathbb{1}_{[\frac{1}{2}-x^{-1/\beta},1-x^{-1/\beta}]}(u)\right)g(u)\mathrm{d}u \\ &\leqslant \|g\|_{L_p}(2x^{-1/\beta})^{1-\frac{1}{p}}.\end{aligned} \tag{4.10}$$

Therefore,

$$\begin{aligned}\int_{\frac{1}{2}}^1\int_{2^\beta\gamma^\beta}^{+\infty}\mathbb{E}\left[\int_s^t \mathbb{1}_{\{M_{\sigma,\varepsilon}(u,r)<x^{-1/\beta}\}}\mathrm{d}r\right]\mathrm{d}x\mathrm{d}u &\leqslant C_2\int_{2^\beta\gamma^\beta}^{+\infty}\frac{\|g\|_{L_p}\sqrt{t-s}}{x^{\frac{1}{2\beta}}x^{\frac{1}{\beta}(1-\frac{1}{p})}}\mathrm{d}x \\ &\leqslant C_3\|g\|_{L_p}\sqrt{t-s},\end{aligned}$$

where C_2 and C_3 are independent of σ, ε, and t. The last inequality holds because $\frac{1}{\beta}\left(\frac{3}{2}-\frac{1}{p}\right) > 1$.

We conclude the proof of the Lemma by using a similar argument for u belonging to $[0,\frac{1}{2}]$ and using $g(u+x^{-1/\beta})-g(u)$ instead of $g(u)-g(u-x^{-1/\beta})$. □

Corollary 4.8. *There is a constant C such that for every $t \in [0,T]$ and for every $\sigma,\varepsilon > 0$,*

$$\mathbb{E}\left[\int_0^1 y^2_{\sigma,\varepsilon}(u,t)\mathrm{d}u\right] \leqslant C.$$

Proof. We have:

$$\mathbb{E}\left[\int_0^1 y^2_{\sigma,\varepsilon}(u,t)\mathrm{d}u\right]^{1/2} \leqslant \mathbb{E}\left[\int_0^1 g(u)^2\mathrm{d}u\right]^{1/2} + \mathbb{E}\left[\int_0^1(y_{\sigma,\varepsilon}(u,t)-g(u))^2\mathrm{d}u\right]^{1/2}.$$

Since g belongs to $L_2(0,1)$, the first term of the right hand side is bounded. Furthermore, by Corollary 3.12 and Fubini-Tonelli Theorem:

$$\begin{aligned}\mathbb{E}\left[\int_0^1(y_{\sigma,\varepsilon}(u,t)-g(u))^2\mathrm{d}u\right] = \int_0^1\mathbb{E}\left[\langle y_{\sigma,\varepsilon}(u,\cdot),y_{\sigma,\varepsilon}(u,\cdot)\rangle_t\right]\mathrm{d}u &= \int_0^1\mathbb{E}\left[\int_0^t\frac{1}{M_{\sigma,\varepsilon}(u,s)}\mathrm{d}s\right]\mathrm{d}u \\ &\leqslant C\sqrt{t},\end{aligned}$$

by Lemma 4.6. □

4.1.2 Proof of Proposition 4.2

We will now use Lemma 4.6 and its Corollary 4.8 to prove Proposition 4.2. We start by $(K1)$:

Proposition 4.9. *Let $g \in \mathcal{L}_{2+}^{\uparrow}[0,1]$ and δ be positive. Then there exists $M > 0$ such that for all $\sigma > 0$ and $\varepsilon > 0$, $\mathbb{P}\left[\int_0^1 \|y_{\sigma,\varepsilon}(u,\cdot)\|^2_{\mathcal{C}[0,T]}\mathrm{d}u \geqslant M\right] \leqslant \delta$.*

Proof. Using again Fubini-Tonelli Theorem,

$$\mathbb{E}\left[\int_0^1 \sup_{t\leqslant T} |y_{\sigma,\varepsilon}(u,t)|^2 \mathrm{d}u\right] = \int_0^1 \mathbb{E}\left[\sup_{t\leqslant T} |y_{\sigma,\varepsilon}(u,t)|^2\right] \mathrm{d}u.$$

Moreover, for almost every $u \in [0,1]$, $y_{\sigma,\varepsilon}(u,\cdot)$ is a $(\mathcal{F}_t^{\sigma,\varepsilon})_{t\in[0,T]}$-martingale. Hence by Doob's inequality, there is a constant C_1 independent of u, σ and ε such that:

$$\mathbb{E}\left[\sup_{t\leqslant T} |y_{\sigma,\varepsilon}(u,t)|^2\right] \leqslant C_1 \mathbb{E}\left[|y_{\sigma,\varepsilon}(u,T)|^2\right].$$

Therefore, by Corollary 4.8,

$$\mathbb{E}\left[\int_0^1 \sup_{t\leqslant T} |y_{\sigma,\varepsilon}(u,t)|^2 \mathrm{d}u\right] \leqslant C_1 \int_0^1 \mathbb{E}\left[|y_{\sigma,\varepsilon}(u,T)|^2\right] \mathrm{d}u \leqslant C_2, \tag{4.11}$$

where C_2 is independent of σ and ε. We conclude by Markov's inequality: there is a constant $C > 0$ such that for all $\sigma, \varepsilon > 0$,

$$\mathbb{P}\left[\int_0^1 \|y_{\sigma,\varepsilon}(u,\cdot)\|^2_{\mathcal{C}[0,T]}\mathrm{d}u \geqslant M\right] \leqslant \frac{\mathbb{E}\left[\int_0^1 \sup_{t\leqslant T} |y_{\sigma,\varepsilon}(u,t)|^2 \mathrm{d}u\right]}{M} \leqslant \frac{C}{M}.$$

For M large enough, that last quantity is smaller than δ. □

Then, we show criterion $(K2)$:

Proposition 4.10. *Let $g \in L_p[0,1]$ and $\delta > 0$. Then for all $k \geqslant 1$, there exists $\eta_k > 0$ such that for every $\sigma, \varepsilon > 0$,*

$$\mathbb{P}\left[\int_0^1 \sup_{|t_2-t_1|<\eta_k} |y_{\sigma,\varepsilon}(u,t_2) - y_{\sigma,\varepsilon}(u,t_1)|\mathrm{d}u \geqslant \frac{1}{k}\right] \leqslant \frac{\delta}{2^k}.$$

Proof. By Markov's inequality, it is sufficient to prove that:

$$\lim_{\eta\to 0^+} \sup_{\sigma>0,\varepsilon>0} \mathbb{E}\left[\int_0^1 \sup_{|t_2-t_1|<\eta} |y_{\sigma,\varepsilon}(u,t_2) - y_{\sigma,\varepsilon}(u,t_1)|\mathrm{d}u\right] = 0. \tag{4.12}$$

Fix $\delta > 0$ and $\beta \in (1, \frac{3}{2} - \frac{1}{p})$. For every $u \in (0,1)$, define

$$K_1(u) := \mathbb{E}\left[\|y_{\sigma,\varepsilon}(u,\cdot)\|_{\mathcal{C}[0,T]}\right],$$
$$K_2(u) := \mathbb{E}\left[\int_0^T \frac{1}{M^\beta_{\sigma,\varepsilon}(u,s)}\mathrm{d}s\right].$$

Since $y_{\sigma,\varepsilon}$ is uniformly bounded for $\sigma > 0$ and $\varepsilon > 0$ in $L_2([0,1],\mathcal{C}[0,T])$ (see inequality (4.11)) and by Lemma 4.6, $\int_0^1 K_1(u)\mathrm{d}u$ and $\int_0^1 K_2(u)\mathrm{d}u$ are uniformly bounded for

$\sigma > 0$ and $\varepsilon > 0$. Therefore, there exists $C > 0$ such that $\int_0^1 \mathbb{1}_{\{K_1(u) \geqslant C\}} \mathrm{d}u \leqslant \delta$ and $\int_0^1 \mathbb{1}_{\{K_2(u) \geqslant C\}} \mathrm{d}u \leqslant \delta$. We define:

$$\mathcal{K}_1 := \{u \in (0,1) : K_1(u) \leqslant C\},$$
$$\mathcal{K}_2 := \{u \in (0,1) : K_2(u) \leqslant C\}.$$

The collection $(y_{\sigma,\varepsilon}(u,\cdot))_{\sigma>0,\varepsilon>0,u\in\mathcal{K}_1\cap\mathcal{K}_2}$ is tight in $\mathcal{C}[0,T]$. We use Aldous' tightness criterion to prove this claim (see [4, Theorem 16.10]). We prove the two following statements:

- $\lim_{a\to\infty} \sup_{\sigma>0,\varepsilon>0,u\in\mathcal{K}_1\cap\mathcal{K}_2} \mathbb{P}\left[\|y_{\sigma,\varepsilon}(u,\cdot)\|_{\mathcal{C}[0,T]} \geqslant a\right] = 0$.
- for all $\alpha > 0$ and $r > 0$, there is η_0 such that for all $\eta \in (0,\eta_0)$, for all $\sigma > 0$, $\varepsilon > 0$ and $u \in \mathcal{K}_1 \cap \mathcal{K}_2$, if τ is a stopping time for $y_{\sigma,\varepsilon}(u,\cdot)$ such that $\tau \leqslant T$, then $\mathbb{P}\left[|y_{\sigma,\varepsilon}(u,\tau+\eta) - y_{\sigma,\varepsilon}(u,\tau)| \geqslant r\right] \leqslant \alpha$.

By Markov's inequality, for all $a > 0$, $\sigma > 0$, $\varepsilon > 0$ and $u \in \mathcal{K}_1 \cap \mathcal{K}_2$,

$$\mathbb{P}\left[\|y_{\sigma,\varepsilon}(u,\cdot)\|_{\mathcal{C}[0,T]} \geqslant a\right] \leqslant \frac{1}{a}\mathbb{E}\left[\|y_{\sigma,\varepsilon}(u,\cdot)\|_{\mathcal{C}[0,T]}\right] = \frac{K_1(u)}{a} \leqslant \frac{C}{a},$$

whence we obtain the first statement. Moreover, for all $u \in \mathcal{K}_1 \cap \mathcal{K}_2$, by Hölder's inequality,

$$\mathbb{E}\left[|y_{\sigma,\varepsilon}(u,\tau+\eta) - y_{\sigma,\varepsilon}(u,\tau)|^2\right] = \mathbb{E}\left[\int_\tau^{\tau+\eta} \frac{1}{M_{\sigma,\varepsilon}(u,s)} \mathrm{d}s\right] \leqslant K_2(u)^{\frac{1}{\beta}} \eta^{1-\frac{1}{\beta}} \leqslant C^{\frac{1}{\beta}} \eta^{1-\frac{1}{\beta}},$$

whence we obtain the second statement.

By Aldous' tightness criterion, there exists a compact L of the set $\mathcal{D}[0,T]$ of càdlàg functions on $[0,T]$ such that for all $\sigma > 0$, $\varepsilon > 0$ and $u \in \mathcal{K}_1 \cap \mathcal{K}_2$, $\mathbb{P}\left[y_{\sigma,\varepsilon}(u,\cdot) \in L\right] \geqslant 1 - \delta$. Since $\mathcal{C}[0,T]$ is closed in $\mathcal{D}[0,T]$ with respect to Skorohod's topology, and since $y_{\sigma,\varepsilon}(u,\cdot) \in \mathcal{C}[0,T]$ almost surely, we may suppose that L is a compact set of $\mathcal{C}[0,T]$.

Back to (4.12), we have:

$$\begin{aligned}
\mathbb{E}\left[\int_0^1 \sup_{|t_2-t_1|<\eta} |y_{\sigma,\varepsilon}(u,t_2) - y_{\sigma,\varepsilon}(u,t_1)| \mathrm{d}u\right] &= \int_0^1 \mathbb{E}\left[\sup_{|t_2-t_1|<\eta} |y_{\sigma,\varepsilon}(u,t_2) - y_{\sigma,\varepsilon}(u,t_1)|\right] \mathrm{d}u \\
&= \int_0^1 \mathbb{E}\left[\mathbb{1}_{\{u\in\mathcal{K}_1\cap\mathcal{K}_2, y_{\sigma,\varepsilon}(u,\cdot)\in L\}^\complement} \sup_{|t_2-t_1|<\eta} |y_{\sigma,\varepsilon}(u,t_2) - y_{\sigma,\varepsilon}(u,t_1)|\right] \mathrm{d}u \\
&\quad + \int_0^1 \mathbb{E}\left[\mathbb{1}_{\{u\in\mathcal{K}_1\cap\mathcal{K}_2, y_{\sigma,\varepsilon}(u,\cdot)\in L\}} \sup_{|t_2-t_1|<\eta} |y_{\sigma,\varepsilon}(u,t_2) - y_{\sigma,\varepsilon}(u,t_1)|\right] \mathrm{d}u.
\end{aligned} \tag{4.13}$$

The first term on the right hand side of (4.13) is bounded by:

$$\left(\int_0^1 \mathbb{E}\left[\mathbb{1}_{\{u\in\mathcal{K}_1\cap\mathcal{K}_2, y_{\sigma,\varepsilon}(u,\cdot)\in L\}^\complement}\right] \mathrm{d}u\right)^{1/2} \left(\int_0^1 \mathbb{E}\left[\sup_{|t_2-t_1|<\eta} |y_{\sigma,\varepsilon}(u,t_2) - y_{\sigma,\varepsilon}(u,t_1)|^2\right] \mathrm{d}u\right)^{1/2}.$$

We have:

$$\begin{aligned}
\int_0^1 \mathbb{E}\left[\mathbb{1}_{\{u\in\mathcal{K}_1\cap\mathcal{K}_2, y_{\sigma,\varepsilon}(u,\cdot)\in L\}^\complement}\right] \mathrm{d}u &\leqslant \int_0^1 \mathbb{1}_{\{u\in\mathcal{K}_1\cap\mathcal{K}_2\}} \mathbb{P}\left[y_{\sigma,\varepsilon}(u,\cdot) \notin L\right] \mathrm{d}u \\
&\quad + \int_0^1 \mathbb{1}_{\{K_1(u)\geqslant C\}} \mathrm{d}u + \int_0^1 \mathbb{1}_{\{K_2(u)\geqslant C\}} \mathrm{d}u \\
&\leqslant 3\delta.
\end{aligned}$$

Moreover,

$$\int_0^1 \mathbb{E}\left[\sup_{|t_2-t_1|<\eta} |y_{\sigma,\varepsilon}(u,t_2) - y_{\sigma,\varepsilon}(u,t_1)|^2\right] \mathrm{d}u \leqslant 4\int_0^1 \mathbb{E}\left[\sup_{t\leqslant T} |y_{\sigma,\varepsilon}(u,t)|^2\right] \mathrm{d}u \leqslant 4M,$$

where M is a constant independent of $\sigma > 0$ and $\varepsilon > 0$ by inequality (4.11).

It remains to handle the second term on the right hand side of (4.13). Since L is a compact set of $\mathcal{C}[0,T]$, there exists $\eta > 0$ such that for every $f \in L$, $\omega_f(\eta) := \sup_{|t-s|<\eta} |f(t) - f(s)| < \delta$. Therefore, there exists $\eta > 0$ such that:

$$\int_0^1 \mathbb{E}\left[\mathbb{1}_{\{u\in K_1\cap K_2, y_{\sigma,\varepsilon}(u,\cdot)\in L\}} \sup_{|t_2-t_1|<\eta} |y_{\sigma,\varepsilon}(u,t_2) - y_{\sigma,\varepsilon}(u,t_1)|\right] \mathrm{d}u \leqslant \delta.$$

Back to equality (4.13), we have proved that there is $\eta > 0$ such that for every $\sigma > 0$ and $\varepsilon > 0$:

$$\mathbb{E}\left[\int_0^1 \sup_{|t_2-t_1|<\eta} |y_{\sigma,\varepsilon}(u,t_2) - y_{\sigma,\varepsilon}(u,t_1)| \mathrm{d}u\right] \leqslant \delta + \sqrt{12\delta M}.$$

This proves convergence (4.12) and thus concludes the proof of the Proposition. □

Then, to obtain criterion $(K3)$, we state the following Proposition:

Proposition 4.11. *Let $g \in \mathcal{L}^{\uparrow}_{2+}[0,1]$ and $\delta > 0$. Then for all $k \geqslant 1$, there is $h_k > 0$ such that for all $\sigma, \varepsilon > 0$,*

$$\mathbb{P}\left[\int_0^{1-h_k} \|y_{\sigma,\varepsilon}(u+h_k,\cdot) - y_{\sigma,\varepsilon}(u,\cdot)\|^2_{\mathcal{C}[0,T]} \mathrm{d}u \geqslant \frac{1}{k}\right] \leqslant \frac{\delta}{2^k}.$$

If $\int_0^{1-h_k} \|y_{\sigma,\varepsilon}(u+h_k,\cdot) - y_{\sigma,\varepsilon}(u,\cdot)\|^2_{\mathcal{C}[0,T]} \mathrm{d}u \leqslant \frac{1}{k}$, we deduce by monotonicity of $u \mapsto y_{\sigma,\varepsilon}(u,t)$ for every $t \in [0,T]$ that for every $h \in (0,h_k)$,

$$\begin{aligned}
&\int_0^{1-h} \|y_{\sigma,\varepsilon}(u+h,\cdot) - y_{\sigma,\varepsilon}(u,\cdot)\|^2_{\mathcal{C}[0,T]} \mathrm{d}u \\
&\leqslant \int_0^{1-h_k} \|y_{\sigma,\varepsilon}(u+h,\cdot) - y_{\sigma,\varepsilon}(u,\cdot)\|^2_{\mathcal{C}[0,T]} \mathrm{d}u + \int_{1-2h_k+h}^{1-h_k} \|y_{\sigma,\varepsilon}(u+h_k,\cdot) - y_{\sigma,\varepsilon}(u+h_k-h,\cdot)\|^2_{\mathcal{C}[0,T]} \mathrm{d}u \\
&\qquad \leqslant 2\int_0^{1-h_k} \|y_{\sigma,\varepsilon}(u+h_k,\cdot) - y_{\sigma,\varepsilon}(u,\cdot)\|^2_{\mathcal{C}[0,T]} \mathrm{d}u \leqslant \frac{2}{k}.
\end{aligned}$$

Therefore, the latter Proposition implies the following Corollary, which is equivalent to criterion $(K3)$:

Corollary 4.12. *Let $g \in \mathcal{L}^{\uparrow}_{2+}[0,1]$ and $\delta > 0$. Then for all $k \geqslant 1$, there is $h_k > 0$ such that for all $\sigma, \varepsilon > 0$,*

$$\mathbb{P}\left[\forall h \in (0,h_k), \int_0^{1-h} \|y_{\sigma,\varepsilon}(u+h,\cdot) - y_{\sigma,\varepsilon}(u,\cdot)\|^2_{\mathcal{C}[0,T]} \mathrm{d}u \leqslant \frac{2}{k}\right] \geqslant 1 - \frac{\delta}{2^k}.$$

Proof (Proposition 4.11). Let $h \in (0,1)$. By Corollary 3.11, for almost every $u \in (0,1-h)$, $N_{u,t} := y_{\sigma,\varepsilon}(u+h,t) - y_{\sigma,\varepsilon}(u,t)$ is a martingale. By Fubini-Tonelli Theorem and Doob's inequality, we have:

$$\mathbb{E}\left[\int_0^{1-h} \|N_{u,\cdot}\|^2_{\mathcal{C}[0,T]} \mathrm{d}u\right] = \int_0^{1-h} \mathbb{E}\left[\|N_{u,\cdot}\|^2_{\mathcal{C}[0,T]}\right] \mathrm{d}u \leqslant C\int_0^{1-h} \mathbb{E}\left[N^2_{u,T}\right] \mathrm{d}u. \qquad (4.14)$$

Let us split $\mathbb{E}\left[N^2_{u,T}\right]$ in two terms $\mathbb{E}\left[N^2_{u,T}\mathbb{1}_{\{N_{u,T}\leqslant 1\}}\right] + \mathbb{E}\left[N^2_{u,T}\mathbb{1}_{\{N_{u,T}>1\}}\right]$.

Study of $\int_0^{1-h} \mathbb{E}\left[N_{u,T}^2 \mathbb{1}_{\{N_{u,T}\leqslant 1\}}\right] \mathrm{d}u$. Let $u \in (0, 1-h)$ be such that $N_{u,\cdot}$ is a martingale. By Lemma 3.10, if $g(u+h) - g(u) = 0$, then $N_{u,T} = 0$ almost surely, thus $\mathbb{E}\left[N_{u,T}^2 \mathbb{1}_{N_{u,T}\leqslant 1}\right] = 0$. From now on, we suppose that $g(u+h) - g(u) > 0$. $N_{u,\cdot}$ is a square integrable continuous martingale, starting from $g(u+h) - g(u) > 0$ and positive by Lemma 3.10. Therefore, there exists a standard Brownian motion β_u such that $N_{u,t} = N_{u,0} + \beta_u(\langle N_{u,\cdot}, N_{u,\cdot}\rangle_t)$. Recall that $N_{u,0} = g(u+h) - g(u)$ is a deterministic quantity. If $N_{u,0} \geqslant 1$, then the inequality $\mathbb{E}\left[N_{u,T}^2 \mathbb{1}_{\{N_{u,T}\leqslant 1\}}\right] \leqslant N_{u,0}$ is obvious. Otherwise, we have

$$\mathbb{E}\left[N_{u,T}^2 \mathbb{1}_{\{N_{u,T}\leqslant 1\}}\right] = \int_0^{+\infty} \mathbb{P}\left[N_{u,T}^2 \mathbb{1}_{\{N_{u,T}\leqslant 1\}} \geqslant \lambda\right] \mathrm{d}\lambda \leqslant \int_0^1 \mathbb{P}\left[N_{u,T}^2 \geqslant \lambda\right] \mathrm{d}\lambda$$
$$\leqslant N_{u,0}^2 + \int_{N_{u,0}^2}^1 \mathbb{P}\left[N_{u,T} \geqslant \lambda^{1/2}\right] \mathrm{d}\lambda. \quad (4.15)$$

Let us estimate $\mathbb{P}\left[N_{u,T} \geqslant \kappa\right]$ for a real number $\kappa > N_{u,0}$. We define the following stopping times:

$$\tau_{-N_{u,0}} := \inf\{t \geqslant 0 : N_{u,0} + \beta_u(t) \leqslant 0\};$$
$$\tau_{\kappa-N_{u,0}} := \inf\{t \geqslant 0 : N_{u,0} + \beta_u(t) \geqslant \kappa\};$$
$$\tau := \inf\{t \geqslant 0 : N_{u,t} \geqslant \kappa\} \wedge T.$$

On the first hand, we know that almost surely, for all $t \in [0, T]$, $N_{u,t} > 0$, hence $\tau_{-N_{u,0}} \geqslant \langle N_{u,\cdot}, N_{u,\cdot}\rangle_T$. On the other hand, if $N_{u,T} \geqslant \kappa$, $N_{u,\tau}$ is equal to κ by continuity of $N_{u,\cdot}$, hence $\langle N_{u,\cdot}, N_{u,\cdot}\rangle_\tau \geqslant \tau_{\kappa-N_{u,0}}$. It follows from both inequalities that $\tau_{\kappa-N_{u,0}} \leqslant \tau_{-N_{u,0}}$. Therefore,

$$\mathbb{P}\left[N_{u,T} \geqslant \kappa\right] \leqslant \mathbb{P}\left[\tau_{\kappa-N_{u,0}} \leqslant \tau_{-N_{u,0}}\right] = \frac{N_{u,0}}{\kappa}, \quad (4.16)$$

by a usual martingale equality. Using inequality (4.15) and $N_{u,0} \leqslant 1$, we obtain:

$$\mathbb{E}\left[N_{u,T}^2 \mathbb{1}_{\{N_{u,T}\leqslant 1\}}\right] \leqslant N_{u,0}^2 + \int_{N_{u,0}^2}^1 \frac{N_{u,0}}{\lambda^{1/2}} \mathrm{d}\lambda \leqslant N_{u,0}^2 + 2N_{u,0} \leqslant 3N_{u,0}.$$

Therefore, we have: $\int_0^{1-h} \mathbb{E}\left[N_{u,T}^2 \mathbb{1}_{\{N_{u,T}\leqslant 1\}}\right] \mathrm{d}u \leqslant 3\int_0^{1-h} N_{u,0} \mathrm{d}u$.

Study of $\int_0^{1-h} \mathbb{E}\left[N_{u,T}^2 \mathbb{1}_{\{N_{u,T}> 1\}}\right] \mathrm{d}u$. Recall that g belongs to $L_p(0,1)$ for some $p > 2$. Fix $\beta \in (1, \frac{3}{2} - \frac{1}{p})$. We compute:

$$\int_0^{1-h} \mathbb{E}\left[N_{u,T}^2 \mathbb{1}_{\{N_{u,T}> 1\}}\right] \mathrm{d}u$$
$$\leqslant 2\int_0^{1-h} \mathbb{E}\left[(N_{u,T} - N_{u,0})^2 \mathbb{1}_{\{N_{u,T}> 1\}}\right] \mathrm{d}u + 2\int_0^{1-h} \mathbb{E}\left[N_{u,0}^2 \mathbb{1}_{\{N_{u,T}> 1\}}\right] \mathrm{d}u$$
$$\leqslant 2\left(\int_0^{1-h} \mathbb{E}\left[(N_{u,T} - N_{u,0})^{2\beta}\right] \mathrm{d}u\right)^{\frac{1}{\beta}} \left(\int_0^{1-h} \mathbb{P}\left[N_{u,T} > 1\right] \mathrm{d}u\right)^{1-\frac{1}{\beta}} + 2\int_0^{1-h} N_{u,0}^2 \mathrm{d}u.$$

Furthermore, we have $\mathbb{P}\left[N_{u,T} > 1\right] \leqslant N_{u,0}$: that inequality is obvious if $N_{u,0} \geqslant 1$ and otherwise, it is a consequence of inequality (4.16).

Then, we are willing to give an upper bound for $\mathbb{E}\left[(N_{u,T} - N_{u,0})^{2\beta}\right]$. Using Burkholder-Davis-Gundy inequality, there exists C_β such that $\mathbb{E}\left[(N_{u,T} - N_{u,0})^{2\beta}\right] \leqslant$

$C_\beta \mathbb{E}\left[\langle N_{u,\cdot}, N_{u,\cdot}\rangle_T^{\beta}\right]$. We compute the quadratic variation of the martingale $N_{u,t} = y_{\sigma,\varepsilon}(u+h,t) - y_{\sigma,\varepsilon}(u,t)$:

$$\mathbb{E}\left[\langle N_{u,\cdot}, N_{u,\cdot}\rangle_T^{\beta}\right] = \mathbb{E}\left[\left|\int_0^T \left(\frac{1}{M_{\sigma,\varepsilon}(u,s)} + \frac{1}{M_{\sigma,\varepsilon}(u+h,s)} \right.\right.\right.$$
$$\left.\left.\left. - \frac{2m_{\sigma,\varepsilon}(u,u+h,s)}{(\varepsilon + m_{\sigma,\varepsilon}(u,s))(\varepsilon + m_{\sigma,\varepsilon}(u+h,s))}\right)\mathrm{d}s\right|^{\beta}\right].$$

By Cauchy-Schwarz inequality $m_{\sigma,\varepsilon}(u,u+h,s) \leqslant m_{\sigma,\varepsilon}^{1/2}(u,s)m_{\sigma,\varepsilon}^{1/2}(u+h,s)$, we deduce that the sum of the three terms in the integral is non-negative and thus that it is bounded by $\frac{1}{M_{\sigma,\varepsilon}(u,s)} + \frac{1}{M_{\sigma,\varepsilon}(u+h,s)}$, whence we obtain:

$$\begin{aligned}\mathbb{E}\left[\langle N_{u,\cdot}, N_{u,\cdot}\rangle_T^{\beta}\right] &\leqslant T^{\beta-1}\mathbb{E}\left[\int_0^T \left|\frac{1}{M_{\sigma,\varepsilon}(u,s)} + \frac{1}{M_{\sigma,\varepsilon}(u+h,s)}\right|^{\beta} \mathrm{d}s\right] \\ &\leqslant C_{\beta,T}\left(\mathbb{E}\left[\int_0^T \frac{\mathrm{d}s}{M^{\beta}_{\sigma,\varepsilon}(u,s)}\right] + \mathbb{E}\left[\int_0^T \frac{\mathrm{d}s}{M^{\beta}_{\sigma,\varepsilon}(u+h,s)}\right]\right).\end{aligned}$$

By Lemma 4.6, we deduce that $\int_0^{1-h} \mathbb{E}\left[\langle N_{u,\cdot}, N_{u,\cdot}\rangle_T^{\beta}\right] \mathrm{d}u$ is bounded, because $\beta < \frac{3}{2} - \frac{1}{p}$. Therefore, we can conclude that there is a constant $C_{T,\beta}$ such that:

$$\int_0^{1-h} \mathbb{E}\left[N_{u,T}^2 \mathbb{1}_{\{N_{u,T}>1\}}\right]\mathrm{d}u \leqslant 2C_{T,\beta}\left(\int_0^{1-h} N_{u,0}\,\mathrm{d}u\right)^{1-1/\beta} + 2\int_0^{1-h} N_{u,0}^2\,\mathrm{d}u.$$

Conclusion: Putting together the studies of both cases, we have proved that there is a positive constant C satisfying, for all σ, ε and $h \in (0,1)$:

$$\int_0^{1-h} \mathbb{E}\left[N_{u,T}^2\right]\mathrm{d}u \leqslant C\int_0^{1-h} N_{u,0}\,\mathrm{d}u + C\left(\int_0^{1-h} N_{u,0}\,\mathrm{d}u\right)^{1-1/\beta} + C\int_0^{1-h} N_{u,0}^2\,\mathrm{d}u. \tag{4.17}$$

Recall that there is $p > 2$ such that $g \in L_p(0,1)$. As for inequality (4.10), we get:

$$\int_0^{1-h} N_{u,0}\,\mathrm{d}u = \int_0^{1-h} (g(u+h) - g(u))\mathrm{d}u \leqslant \|g\|_{L_p}(2h)^{1-\frac{1}{p}}.$$

Furthermore, define $\alpha := \frac{p-2}{p-1} \in (0,1)$. We have

$$\begin{aligned}\int_0^{1-h} N_{u,0}^2\,\mathrm{d}u &= \int_0^{1-h} (g(u+h)-g(u))^{\alpha}(g(u+h)-g(u))^{2-\alpha}\mathrm{d}u \\ &\leqslant \left(\int_0^{1-h}(g(u+h)-g(u))\mathrm{d}u\right)^{\alpha}\left(\int_0^{1-h}(g(u+h)-g(u))^{\frac{2-\alpha}{1-\alpha}}\mathrm{d}u\right)^{1-\alpha} \\ &\leqslant \left(\|g\|_{L_p}(2h)^{1-\frac{1}{p}}\right)^{\alpha}\left(C_p\|g\|_{L_p}\right)^{1-\alpha},\end{aligned}$$

because $\frac{2-\alpha}{1-\alpha} = p$. Therefore

$$\int_0^{1-h} N_{u,0}^2\,\mathrm{d}u = \int_0^{1-h}(g(u+h)-g(u))^2\mathrm{d}u \leqslant C_p^{1-\alpha}\|g\|_{L_p}h^{\frac{p-2}{p}}. \tag{4.18}$$

It follows from (4.17) that there is C_β such that for each $\sigma, \varepsilon > 0$,

$$\int_0^{1-h} \mathbb{E}\left[(y_{\sigma,\varepsilon}(u+h,T) - y_{\sigma,\varepsilon}(u,T))^2\right] \mathrm{d}u \leqslant C_\beta \|g\|_{L_p} \left(h^{\frac{p-1}{p}} + h^{\frac{p-1}{p}(1-\frac{1}{\beta})} + h^{\frac{p-2}{p}}\right),$$

for every $\beta < \frac{3}{2} - \frac{1}{p}$, *i.e.* such that $0 < 1 - \frac{1}{\beta} < \frac{p-2}{3p-2}$. Thus, there is $q > 0$ depending on p (e.g. $q = \frac{(p-1)(p-2)}{2p(3p-2)}$ by choosing $1 - \frac{1}{\beta} = \frac{p-2}{2(3p-2)}$) and a constant C such that for each $\sigma, \varepsilon > 0$,

$$\int_0^{1-h} \mathbb{E}\left[(y_{\sigma,\varepsilon}(u+h,T) - y_{\sigma,\varepsilon}(u,T))^2\right] \mathrm{d}u \leqslant C \|g\|_{L_p} h^q. \tag{4.19}$$

Therefore, by (4.14) and Markov's inequality, there is C such that for each $\sigma, \varepsilon > 0$,

$$\mathbb{P}\left[\int_0^{1-h} \|y_{\sigma,\varepsilon}(u+h,\cdot) - y_{\sigma,\varepsilon}(u,\cdot)\|^2_{\mathcal{C}[0,T]} \mathrm{d}u \geqslant \frac{1}{k}\right] \leqslant kC\|g\|_{L_p} h^q,$$

whence it is sufficient to choose h_k so that $kC\|g\|_{L_p} h_k^q < \frac{\delta}{2^k}$. □

4.2 Convergence when $\varepsilon \to 0$

Fix $\sigma \in \mathbb{Q}_+$. By Prokhorov's Theorem, it follows from Corollary 4.3 that the collection of laws of the sequence $(y_{\sigma,\varepsilon})_{\varepsilon \in \mathbb{Q}_+}$ is relatively compact in $\mathcal{P}(L_2([0,1], \mathcal{C}[0,T]))$. In particular, up to extracting a subsequence, we may suppose that $(y_{\sigma,\varepsilon})_{\varepsilon \in \mathbb{Q}_+}$ converges in distribution in $L_2([0,1], \mathcal{C}[0,T])$ to a limit, denoted by y_σ.

For every $t \in [0,T]$, let us denote by $e_t(f) := f(\cdot, t)$ the continuous evaluation function: $L_2([0,1], \mathcal{C}[0,T]) \to L_2[0,1]$. We define $Y_\sigma(t) := e_t(y_\sigma) = y_\sigma(\cdot, t)$. Under the same model as Proposition 3.2, we obtain:

Proposition 4.13. *Fix $\sigma \in \mathbb{Q}_+$. Suppose that $g \in \mathcal{L}^\uparrow_{2+}[0,1]$. $(Y_\sigma(t))_{t\in[0,T]}$ is a $L_2^\uparrow[0,1]$-valued process such that:*

- $(B1)$ *$Y_\sigma(0) = g$;*
- $(B2)$ *$(Y_\sigma(t))_{t\in[0,T]}$ is a square integrable continuous $L_2^\uparrow[0,1]$-valued martingale relatively to the filtration $(\mathcal{F}^\sigma_t)_{t\in[0,T]}$, where $\mathcal{F}^\sigma_t = \sigma(Y_\sigma(s), s \leqslant t)$;*
- $(B3)$ *for every $h, k \in L_2[0,1]$,*

$$\langle (Y_\sigma, h)_{L_2}, (Y_\sigma, k)_{L_2} \rangle_t = \int_0^t \int_0^1 \int_0^1 h(u)k(u') \frac{m_\sigma(u,u',s)}{m_\sigma(u,s) m_\sigma(u',s)} \mathrm{d}u\mathrm{d}u'\mathrm{d}s,$$

where $m_\sigma(u,u',s) = \int_0^1 \varphi_\sigma(y_\sigma(u,s) - y_\sigma(v,s))\, \varphi_\sigma(y_\sigma(u',s) - y_\sigma(v,s))\, \mathrm{d}v$ and $m_\sigma(u,s) = \int_0^1 \varphi^2_\sigma(y_\sigma(u,s) - y_\sigma(v,s))\mathrm{d}v$.

Proof. Fix $t \in [0,T]$. We want to prove that $Y_\sigma(t)$ belongs to $L_2^\uparrow[0,1]$. For each $\varepsilon \in \mathbb{Q}_+$, $Y_{\sigma,\varepsilon}(t)$ belongs with probability 1 to the set $\mathcal{K} :=$

$$\left\{ f \in L_2(0,1) : \forall u, u', \forall r, r', \text{if } 0 < u < u+r < u' < u'+r' < 1, \text{then } \frac{1}{r}\int_u^{u+r} f \leqslant \frac{1}{r'}\int_{u'}^{u'+r'} f \right\}$$

which is closed in $L_2(0,1)$. Recall that the sequence $(y_{\sigma,\varepsilon})_{\varepsilon\in\mathbb{Q}_+}$ converges in distribution to y_σ in $L_2([0,1], \mathcal{C}[0,T])$. Therefore, $(Y_{\sigma,\varepsilon}(t))_{\varepsilon\in\mathbb{Q}_+}$ converges in distribution to $Y_\sigma(t)$ in $L_2[0,1]$. Because $\mathcal{K}$ is closed, the limit $Y_\sigma(t)$ also belongs to $\mathcal{K}$ with probability 1.

Therefore, almost surely, for every $t \in [0,T] \cap \mathbb{Q}$, $Y_\sigma(t) \in \mathcal{K}$. Let $\omega \in \Omega'$, where Ω' is such that $\mathbb{P}[\Omega'] = 1$ and for every $\omega \in \Omega'$, $\int_0^1 \sup_{s\leqslant T} |y_\sigma(v,s)|^2(\omega)\mathrm{d}v < +\infty$ and for every

$t \in [0,T] \cap \mathbb{Q}$, $Y_\sigma(t)(\omega) \in \mathcal{K}$. Let $t \in [0,T]$ and (t_n) be a sequence in $[0,T] \cap \mathbb{Q}$ tending to t. For every $n \in \mathbb{N}$ and each u, u', r, r' such that $0 < u < u+r < u' < u'+r' < 1$, $\frac{1}{r}\int_u^{u+r} y_\sigma(v,t_n)(\omega)\mathrm{d}v \leqslant \frac{1}{r'}\int_{u'}^{u'+r'} y_\sigma(v,t_n)(\omega)\mathrm{d}v$. Since $y_\sigma(\omega)$ belongs to $L_2([0,1],\mathcal{C}[0,T])$, and since $\int_u^{u+r} y_\sigma(v,t_n)^2(\omega)\mathrm{d}v \leqslant \int_0^1 \sup_{s\leqslant T}|y_\sigma(v,s)|^2(\omega)\mathrm{d}v < +\infty$, $\frac{1}{r}\int_u^{u+r} y_\sigma(v,t_n)(\omega)\mathrm{d}v$ tends to $\frac{1}{r}\int_u^{u+r} y_\sigma(v,t)(\omega)\mathrm{d}v$ (and the same is true for u' and r'). Thus almost surely $Y_\sigma(t)$ belongs to $\mathcal{K}$ for every $t \in [0,T]$. It remains to prove that it implies that $Y_\sigma(t)$ belongs to $L_2^\uparrow[0,1]$.

Let $f \in \mathcal{K}$. Define, for each $u \in (0,1)$, $\widehat{f}(u) := \liminf_{h\to 0^+} \frac{1}{h}\int_u^{(u+h)\wedge 1} f(v)\mathrm{d}v$. First, remark that $\widehat{f}$ is non-decreasing. Then, since $h \mapsto \frac{1}{h}\int_u^{u+h} f$ is non-increasing, we have $\widehat{f}(u) = \lim_{h\to 0^+}\frac{1}{h}\int_u^{(u+h)\wedge 1} f(v)\mathrm{d}v$. Choose a sequence $(u_n) \searrow u$. By monotonicity, $\widehat{f}(u) \leqslant \widehat{f}(u_n)$. Fix $\delta > 0$. There exists $h > 0$ such that $u + h < 1$ and $|\widehat{f}(u) - \frac{1}{h}\int_u^{u+h} f| < \delta$. Since $f \in L_2$, there exists N such that for all $n \geqslant N$, $|\frac{1}{h}\int_{u_n}^{u_n+h} f - \frac{1}{h}\int_u^{u+h} f| < \delta$. Therefore, $\widehat{f}(u_n) \leqslant \frac{1}{h}\int_{u_n}^{u_n+h} f \leqslant \widehat{f}(u) + 2\delta$ for all $n \geqslant N$. Thus $\widehat{f}(u_n) \to \widehat{f}(u)$. In addition, $\widehat{f}$ has left limits because of its monotonicity. Hence $\widehat{f}$ is a càdlàg function.

Furthermore, $\widehat{f} = f$ almost everywhere. Indeed, for every $\delta > 0$, there exists $F \in \mathcal{C}[0,1]$ such that $\|f - F\|_{L_1(0,1)} < \delta$. Define $\widehat{F}(u) = \lim_{h\to 0^+}\frac{1}{h}\int_u^{(u+h)\wedge 1} F(v)\mathrm{d}v$. By continuity of F, $F(u) = \widehat{F}(u)$ for every $u \in (0,1)$. Thus we have:

$$\begin{aligned}\int_0^1 |f(u) - \widehat{f}(u)|\mathrm{d}u &\leqslant \int_0^1 |f(u) - F(u)|\mathrm{d}u + \int_0^1 |\widehat{f}(u) - \widehat{F}(u)|\mathrm{d}u \\ &\leqslant \delta + \int_0^1 \lim_{h\to 0^+}\frac{1}{h}\int_u^{(u+h)\wedge 1} |f(v) - F(v)|\mathrm{d}v\mathrm{d}u \\ &\leqslant \delta + \liminf_{h\to 0^+}\int_0^1 |f(v) - F(v)|\mathrm{d}v \leqslant 2\delta,\end{aligned}$$

where we used Fatou's Lemma to obtain the last line. Thus $\int_0^1 |f(u) - \widehat{f}(u)|\mathrm{d}u = 0$, whence $\widehat{f} = f$ almost everywhere. Thus f belongs to $L_2^\uparrow[0,1]$: Y_σ is a $L_2^\uparrow[0,1]$-valued process.

Property $(B1)$**.** $(Y_{\sigma,\varepsilon}(0))_{\varepsilon\in\mathbb{Q}_+}$ converges in law to $Y_\sigma(0)$ in $L_2[0,1]$. Therefore, $Y_\sigma(0) = g$.

Property $(B2)$**.** By inequality (4.11), $\mathbb{E}\left[\|Y_{\sigma,\varepsilon}\|^2_{L_2([0,1],\mathcal{C}[0,T])}\right]$ is bounded uniformly in $\varepsilon \in \mathbb{Q}_+$. We deduce that for every $t \in [0,T]$, $\mathbb{E}\left[\|Y_\sigma(t)\|^2_{L_2([0,1])}\right] < +\infty$, thus the process Y_σ is square integrable.

Furthermore, Y_σ is a continuous $L_2^\uparrow[0,1]$-valued process. Indeed, for each sequence $(t_n)_{n\geqslant 0}$ converging to a time t, $\|Y_\sigma(t_n) - Y_\sigma(t)\|^2_{L_2} = \int_0^1 (y_\sigma(u,t_n) - y_\sigma(u,t))^2\mathrm{d}u \underset{n\to\infty}{\longrightarrow} 0$ by dominated convergence Theorem, since for almost every $u \in (0,1)$, $y_\sigma(u,\cdot)$ is continuous at time t, and $(y_\sigma(u,t_n) - y_\sigma(u,t))^2 \leqslant 4\sup_{t\leqslant T}|y_\sigma(u,t)|^2$ which is almost surely integrable.

Moreover, we know from property $(A2)$ that for each $h \in L_2(0,1)$, for each $l \geqslant 1$, $0 \leqslant s_1 \leqslant s_2 \leqslant \ldots \leqslant s_l \leqslant s \leqslant t$ and for each bounded and continuous function $f_l : (L_2(0,1))^l \to \mathbb{R}$:

$$\mathbb{E}\left[\int_0^1 h(u)(y_{\sigma,\varepsilon}(u,t) - y_{\sigma,\varepsilon}(u,s))\mathrm{d}u\, f_l(y_{\sigma,\varepsilon}(\cdot,s_1),\ldots,y_{\sigma,\varepsilon}(\cdot,s_l))\right] = 0. \tag{4.20}$$

Since $\left|\int_0^1 h(u)b(u,t)\mathrm{d}u\right| \leqslant \|h\|_{L_2}\left(\int_0^1 \sup_{[0,T]}|b(u,\cdot)|^2\mathrm{d}u\right)^{1/2}$ for every $b \in L_2([0,1],\mathcal{C}[0,T])$, the function $\varphi : b \in L_2([0,1],\mathcal{C}[0,T]) \mapsto \int_0^1 h(u)(b(u,t) - b(u,s))\mathrm{d}u\, f_l(b(\cdot,s_1),\ldots,b(\cdot,s_l))$ is continuous. Furthermore, we prove that $(\varphi(y_{\sigma,\varepsilon}))_{\varepsilon\in\mathbb{Q}_+}$ is bounded in L_2:

$$\begin{aligned}\mathbb{E}\left[\varphi(y_{\sigma,\varepsilon})^2\right] &\leqslant \|f_l\|_\infty^2\|h\|_{L_2}^2\mathbb{E}\left[\int_0^1 (y_{\sigma,\varepsilon}(u,t)-y_{\sigma,\varepsilon}(u,s))^2\mathrm{d}u\right]\\ &\leqslant C\|f_l\|_\infty^2\|h\|_{L_2}^2,\end{aligned}$$

where C is independent of ε by Corollary 4.8. We deduce that $(\varphi(y_{\sigma,\varepsilon}))_{\varepsilon\in\mathbb{Q}_+}$ is uniformly integrable. By continuity of φ and since $(y_{\sigma,\varepsilon})_{\varepsilon\in\mathbb{Q}_+}$ converges in law to y_σ in $L_2([0,1],\mathcal{C}[0,T])$, we get: $\mathbb{E}\left[\varphi(y_{\sigma,\varepsilon})\right]\underset{\varepsilon\to0}{\longrightarrow}\mathbb{E}\left[\varphi(y_\sigma)\right]$. Since by equality (4.20), $\mathbb{E}\left[\varphi(y_{\sigma,\varepsilon})\right]=0$ for each $\varepsilon\in\mathbb{Q}_+$, we have:

$$\mathbb{E}\left[\int_0^1 h(u)(y_\sigma(u,t)-y_\sigma(u,s))\mathrm{d}u f_l(Y_\sigma(s_1),\dots,Y_\sigma(s_l))\right]=0. \tag{4.21}$$

Therefore, $Y_\sigma(\cdot)$ is a square integrable continuous $(\mathcal{F}_t^\sigma)_{t\in[0,T]}$-martingale.

Property $(B3)$**.** We know, by property $(A3)$, that for every $l\geqslant1$, for every $0\leqslant s_1\leqslant s_2\leqslant\dots\leqslant s_l\leqslant s\leqslant t$, for every bounded and continuous $f_l:(L_2(0,1))^l\to\mathbb{R}$ and for every h and k in $L_2(0,1)$:

$$\begin{aligned}&\mathbb{E}\bigg[\int_0^1\int_0^1 h(u)k(u')[(y_{\sigma,\varepsilon}(u,t)-g(u))(y_{\sigma,\varepsilon}(u',t)-g(u'))\\ &\qquad -(y_{\sigma,\varepsilon}(u,s)-g(u))(y_{\sigma,\varepsilon}(u',s)-g(u'))]\mathrm{d}u\mathrm{d}u' f_l(Y_{\sigma,\varepsilon}(s_1),\dots,Y_{\sigma,\varepsilon}(s_l))\bigg]\\ &=\mathbb{E}\bigg[\int_0^1\int_0^1 h(u)k(u')\int_s^t \frac{m_{\sigma,\varepsilon}(u,u',r)\ \mathrm{d}r\mathrm{d}u\mathrm{d}u'}{(\varepsilon+m_{\sigma,\varepsilon}(u,r))(\varepsilon+m_{\sigma,\varepsilon}(u',r))} f_l(Y_{\sigma,\varepsilon}(s_1),\dots,Y_{\sigma,\varepsilon}(s_l))\bigg].\end{aligned} \tag{4.22}$$

First, we want to obtain the convergence of the left hand side of (4.22). We proceed in the same way as for the proof of equality (4.21); to get a uniform integrability property, we have now to prove the existence of $\beta>1$ such that

$$\sup_{\varepsilon\in\mathbb{Q}_+}\mathbb{E}\left[\left(\int_0^1 h(u)(y_{\sigma,\varepsilon}(u,t)-g(u))\mathrm{d}u\int_0^1 k(u')(y_{\sigma,\varepsilon}(u',t)-g(u'))\mathrm{d}u'\right)^\beta\right] \tag{4.23}$$

is finite. Therefore, it is sufficient to prove the existence of $\beta>1$ such that

$$\sup_{\varepsilon\in\mathbb{Q}_+}\mathbb{E}\left[\left(\int_0^1 h(u)(y_{\sigma,\varepsilon}(u,t)-g(u))\mathrm{d}u\right)^{2\beta}\right]$$

is finite for every $h\in L_2[0,1]$. By Cauchy-Schwarz inequality,

$$\begin{aligned}\mathbb{E}\left[\left(\int_0^1 h(u)(y_{\sigma,\varepsilon}(u,t)-g(u))\mathrm{d}u\right)^{2\beta}\right] &\leqslant \mathbb{E}\left[\|h\|_{L_2}^{2\beta}\left(\int_0^1 (y_{\sigma,\varepsilon}(u,t)-g(u))^2\mathrm{d}u\right)^\beta\right]\\ &\leqslant \|h\|_{L_2}^{2\beta}\mathbb{E}\left[\int_0^1 (y_{\sigma,\varepsilon}(u,t)-g(u))^{2\beta}\mathrm{d}u\right].\end{aligned} \tag{4.24}$$

We deduce by Burkholder-Davis-Gundy inequality and Fubini's Theorem that there are some constants independent of ε such that

$$\begin{aligned}\mathbb{E}\left[\int_0^1 (y_{\sigma,\varepsilon}(u,t)-g(u))^{2\beta}\mathrm{d}u\right] &\leqslant C_1\int_0^1 \mathbb{E}\left[\langle y_{\sigma,\varepsilon}(u,\cdot),y_{\sigma,\varepsilon}(u,\cdot)\rangle_t^\beta\right]\mathrm{d}u\\ &\leqslant C_2\mathbb{E}\left[\int_0^1\int_0^t \frac{1}{M_{\sigma,\varepsilon}^\beta(u,r)}\mathrm{d}r\mathrm{d}u\right].\end{aligned}$$

By Lemma 4.6, there exists $\beta > 1$ such that $\mathbb{E}\left[\int_0^1 \int_0^t \frac{1}{M^\beta_{\sigma,\varepsilon}(u,r)} \mathrm{d}r\mathrm{d}u\right]$ is bounded uniformly for $\varepsilon \in \mathbb{Q}_+$. Thus (4.23) is finite. It is also finite if we replace t by s.

To obtain the convergence of the right hand side of (4.22), we start by using Skorohod's representation Theorem[1]: there exists a sequence $(\widehat{y}_{\sigma,\varepsilon})_{\varepsilon\in\mathbb{Q}_+}$ defined on a common probability space $(\widehat{\Omega}, \widehat{\mathbb{P}})$ that converges to $\widehat{y}_\sigma$ in $L_2([0,1], \mathcal{C}[0,T])$ almost surely, where $\widehat{y}_{\sigma,\varepsilon}$ (resp. $\widehat{y}_\sigma$) has same distribution as $y_{\sigma,\varepsilon}$ (resp. y_σ). We denote by $\widehat{m}_{\sigma,\varepsilon}$ (resp. $\widehat{m}_\sigma$) the mass associated to $\widehat{y}_{\sigma,\varepsilon}$ (resp. $\widehat{y}_\sigma$).

Furthermore, on the probability space $(\widehat{\Omega} \times [0,1], \widehat{\mathbb{P}} \otimes \mathrm{Leb}\,|_{[0,1]})$, $\widehat{y}_{\sigma,\varepsilon}$ converges in probability in the space $\mathcal{C}[0,T]$ to $\widehat{y}_\sigma$. Indeed, for every $\delta > 0$, we have:

$$\begin{aligned}
\widehat{\mathbb{P}} \otimes \mathrm{Leb}\,|_{[0,1]}&\{(\omega,u) : \|(\widehat{y}_{\sigma,\varepsilon} - \widehat{y}_\sigma)(\omega,u)\|_{\mathcal{C}[0,T]} \geqslant \delta\} \\
&= \widehat{\mathbb{E}}\left[\mathrm{Leb}\{u : \|(\widehat{y}_{\sigma,\varepsilon} - \widehat{y}_\sigma)(\omega,u)\|_{\mathcal{C}[0,T]} \geqslant \delta\}\right] \\
&\leqslant \widehat{\mathbb{E}}\left[1 \wedge \frac{1}{\delta^2}\int_0^1 \|(\widehat{y}_{\sigma,\varepsilon} - \widehat{y}_\sigma)(\omega,u)\|^2_{\mathcal{C}[0,T]}\mathrm{d}u\right].
\end{aligned}$$

We know that, for every fixed $\delta > 0$, $1 \wedge \frac{1}{\delta^2}\int_0^1 \|(\widehat{y}_{\sigma,\varepsilon} - \widehat{y}_\sigma)(\omega,u)\|^2_{\mathcal{C}[0,T]}\mathrm{d}u$ converges to 0 almost surely, and it is bounded by 1, so we deduce that the latter term tends to 0. We deduce from the convergence in probability that there exists a subsequence $(\varepsilon_n)_n$, $\varepsilon_n \to 0$, such that for almost every $(\omega,u) \in \widehat{\Omega} \times [0,1]$, $\|(\widehat{y}_{\sigma,\varepsilon_n} - \widehat{y}_\sigma)(\omega,u)\|_{\mathcal{C}[0,T]} \to 0$.

We want to prove that,

$$\begin{aligned}
&\mathbb{E}\left[\int_0^1\int_0^1 h(u)k(u')\int_s^t \frac{\widehat{m}_{\sigma,\varepsilon_n}(u,u',r)\ \mathrm{d}r\mathrm{d}u\mathrm{d}u'}{(\varepsilon_n + \widehat{m}_{\sigma,\varepsilon_n}(u,r))(\varepsilon_n + \widehat{m}_{\sigma,\varepsilon_n}(u',r))} f_l(\widehat{Y}_{\sigma,\varepsilon_n}(s_1),\dots,\widehat{Y}_{\sigma,\varepsilon_n}(s_l))\right] \\
&\underset{n\to\infty}{\longrightarrow} \mathbb{E}\left[\int_0^1\int_0^1 h(u)k(u')\int_s^t \frac{\widehat{m}_\sigma(u,u',r)}{\widehat{m}_\sigma(u,r)\widehat{m}_\sigma(u',r)}\mathrm{d}r\mathrm{d}u\mathrm{d}u' f_l(\widehat{Y}_\sigma(s_1),\dots,\widehat{Y}_\sigma(s_l))\right]. \quad (4.25)
\end{aligned}$$

On the one hand, almost surely and for almost every $u \in (0,1)$, $\widehat{y}_{\sigma,\varepsilon_n}(u,\cdot) \to \widehat{y}_\sigma(u,\cdot)$ in $\mathcal{C}[0,T]$. Then for almost every $u,u' \in (0,1)$,

$$\begin{aligned}
\widehat{m}_{\sigma,\varepsilon_n}(u,u',r) &= \int_0^1 \varphi_\sigma(\widehat{y}_{\sigma,\varepsilon_n}(u,r) - \widehat{y}_{\sigma,\varepsilon_n}(v,r))\varphi_\sigma(\widehat{y}_{\sigma,\varepsilon_n}(u',r) - \widehat{y}_{\sigma,\varepsilon_n}(v,r))\mathrm{d}v \\
&\underset{n\to\infty}{\longrightarrow} \widehat{m}_\sigma(u,u',r), \qquad (4.26)
\end{aligned}$$

$$\varepsilon_n + \widehat{m}_{\sigma,\varepsilon_n}(u,r) = \varepsilon_n + \int_0^1 \varphi^2_\sigma(\widehat{y}_{\sigma,\varepsilon_n}(u,r) - \widehat{y}_{\sigma,\varepsilon_n}(v,r))\mathrm{d}v \underset{n\to\infty}{\longrightarrow} \widehat{m}_\sigma(u,r). \qquad (4.27)$$

Therefore, in order to obtain (4.25), it remains to justify that there exists $\beta > 1$ such that:

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left[\left(\int_0^1\int_0^1 h(u)k(u')\int_s^t \frac{\widehat{m}_{\sigma,\varepsilon_n}(u,u',r)}{(\varepsilon_n + \widehat{m}_{\sigma,\varepsilon_n}(u,r))(\varepsilon_n + \widehat{m}_{\sigma,\varepsilon_n}(u',r))}\mathrm{d}r\mathrm{d}u\mathrm{d}u'\right)^\beta\right]$$

is finite. By Cauchy-Schwarz inequality, $\widehat{m}_{\sigma,\varepsilon_n}(u,u',r) \leqslant \widehat{m}^{1/2}_{\sigma,\varepsilon_n}(u,r)\widehat{m}^{1/2}_{\sigma,\varepsilon_n}(u',r)$, so that it is sufficient to prove that there is $\beta > 1$ such that

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left[\left(\int_0^1\int_0^1 h(u)k(u')\int_s^t \frac{1}{\widehat{M}^{1/2}_{\sigma,\varepsilon_n}(u,r)\widehat{M}^{1/2}_{\sigma,\varepsilon_n}(u',r)}\mathrm{d}r\mathrm{d}u\mathrm{d}u'\right)^\beta\right]$$

[1] $L_2([0,1], \mathcal{C}[0,T])$ is a Polish space. Its separability can be proved using the separability of $C([0,1] \times [0,T])$ and the density of $C([0,1] \times [0,T])$ in $L_2([0,1], \mathcal{C}[0,T])$.

is finite, and thus that $\sup_{n\in\mathbb{N}} \mathbb{E}\left[\int_0^1\int_s^t \frac{1}{\widehat{M}^\beta_{\sigma,\varepsilon_n}(u,r)}\mathrm{d}r\mathrm{d}u\right]$ is finite, using Cauchy-Schwarz inequality as in the proof of (4.24). By Lemma 4.6, this statement holds. We conclude that we have the following equality:

$$\begin{aligned}\mathbb{E}\bigg[\int_0^1\int_0^1 & h(u)k(u')[(y_\sigma(u,t)-g(u))(y_\sigma(u',t)-g(u')) \\ & -(y_\sigma(u,s)-g(u))(y_\sigma(u',s)-g(u'))]\mathrm{d}u\mathrm{d}u' f_l(Y_\sigma(s_1),\dots,Y_\sigma(s_l))\bigg] \\ &= \mathbb{E}\bigg[\int_0^1\int_0^1 h(u)k(u')\int_s^t \frac{m_\sigma(u,u',r)}{m_\sigma(u,r)m_\sigma(u',r)}\mathrm{d}r\mathrm{d}u\mathrm{d}u' f_l(Y_\sigma(s_1),\dots,Y_\sigma(s_l))\bigg], \quad (4.28)\end{aligned}$$

whence we obtain property $(B3)$, since $\int_0^1\int_0^1 h(u)k(u')\int_0^t \frac{m_\sigma(u,u',r)\,\mathrm{d}r\mathrm{d}u\mathrm{d}u'}{m_\sigma(u,r)m_\sigma(u',r)}$ is $(\mathcal{F}^\sigma_t)_{t\in[0,T]}$-measurable. □

Property $(B3)$ implies the following Corollary:

Corollary 4.14. *Let ψ be a non-negative and bounded map: $[0,1]\to\mathbb{R}$. Then for every $l\in\mathbb{N}\backslash\{0\}$, $0\leqslant s_1\leqslant s_2\leqslant\dots\leqslant s_l\leqslant s\leqslant t$ and for every bounded and continuous function $f_l : L_2[0,1]^l\to\mathbb{R}$, we have:*

$$\begin{aligned}\mathbb{E}\bigg[\int_0^1\psi(u)\left((y_\sigma(u,t)-g(u))^2-(y_\sigma(u,s)-g(u))^2-\int_s^t\frac{1}{m_\sigma(u,r)}\mathrm{d}r\right)\mathrm{d}u \\ f_l(Y_\sigma(s_1),\dots,Y_\sigma(s_l))\bigg]=0.\end{aligned}$$

Proof. We use the following notations. First $z(u,\cdot):=y_\sigma(u,\cdot)-g(u)$ and second $F_l = f_l(Y_\sigma(s_1),\dots,Y_\sigma(s_l))$. Let us consider an orthonormal basis $(e_i)_{i\geqslant1}$ in the Hilbert space $L_2(\psi(x)\mathrm{d}x)$. We denote by $[\cdot,\cdot]_{L_2(\psi)}$ the scalar product of $L_2(\psi(x)\mathrm{d}x)$: $[h,k]_{L_2(\psi)}=\int_0^1 hk\psi$. By Parseval's formula, we have:

$$\begin{aligned}\mathbb{E}\left[\int_0^1\psi(u)(z(u,t)^2-z(u,s)^2)\mathrm{d}uF_l\right] &= \mathbb{E}\left[\sum_{i\geqslant1}([z(\cdot,t),e_i]^2_{L_2(\psi)}-[z(\cdot,s),e_i]^2_{L_2(\psi)})F_l\right] \\ &= \sum_{i\geqslant1}\mathbb{E}\left[((z(\cdot,t),e_i\psi)^2_{L_2}-(z(\cdot,s),e_i\psi)^2_{L_2})F_l\right] \\ &= \sum_{i\geqslant1}\mathbb{E}\left[\int_0^1\int_0^1 e_i(u)\psi(u)e_i(u')\psi(u')\int_s^t\frac{m_\sigma(u,u',r)}{m_\sigma(u,r)m_\sigma(u',r)}\mathrm{d}r\mathrm{d}u\mathrm{d}u'F_l\right],\end{aligned}$$

by applying equality (4.28) with $h=k=e_i$. By definition of $m_\sigma(u,u',r)$, we have:

$$\begin{aligned}\mathbb{E}\left[\int_0^1\psi(u)(z(u,t)^2-z(u,s)^2)\mathrm{d}uF_l\right] &= \mathbb{E}\left[\int_s^t\int_0^1\sum_{i\geqslant1}\left[\frac{\varphi_\sigma(y_\sigma(\cdot,r)-y_\sigma(v,r))}{m_\sigma(\cdot,r)},e_i\right]^2_{L_2(\psi)}\mathrm{d}v\mathrm{d}rF_l\right] \\ &= \mathbb{E}\left[\int_s^t\int_0^1\int_0^1\frac{\varphi^2_\sigma(y_\sigma(u,r)-y_\sigma(v,r))}{m^2_\sigma(u,r)}\psi(u)\mathrm{d}u\mathrm{d}v\mathrm{d}rF_l\right] \\ &= \mathbb{E}\left[\int_0^1\int_s^t\frac{1}{m_\sigma(u,r)}\mathrm{d}r\psi(u)\mathrm{d}uF_l\right],\end{aligned}$$

since $m_\sigma(u,r)=\int_0^1\varphi^2_\sigma(y_\sigma(u,r)-y_\sigma(v,r))\mathrm{d}v$. □

We deduce the following estimation, by analogy with Lemma 4.6:

Lemma 4.15. *For all $\beta \in (0, \frac{3}{2} - \frac{1}{p})$, there is a constant $C > 0$ such that for all $\sigma > 0$ and $0 \leqslant s < t \leqslant T$, we have the following inequality:*

$$\mathbb{E}\left[\int_s^t \int_0^1 \frac{1}{m_\sigma^\beta(u,r)} \mathrm{d}u \mathrm{d}r\right] \leqslant C\sqrt{t-s}.$$

Proof. We use again the sequence $(\widehat{y}_{\sigma,\varepsilon_n})_{n\in\mathbb{N}}$ obtained by Skorohod's representation Theorem, as in the proof of convergence (4.25). Therefore, by Fatou's Lemma,

$$\mathbb{E}\left[\int_s^t \int_0^1 \frac{1}{\widehat{m}_\sigma^\beta(u,r)} \mathrm{d}u \mathrm{d}r\right] \leqslant \liminf_{n\to\infty} \ \mathbb{E}\left[\int_s^t \int_0^1 \frac{1}{\widehat{M}_{\sigma,\varepsilon_n}^\beta(u,r)} \mathrm{d}u \mathrm{d}r\right] \leqslant C\sqrt{t-s},$$

where C is obtained thanks to Lemma 4.6. □

By Burkholder-Davis-Gundy inequality, we obtain immediately the following Corollary:

Corollary 4.16. *For each $\beta \in (0, \frac{3}{2} - \frac{1}{p})$, $\sup_{\sigma\in\mathbb{Q}_+} \sup_{t\leqslant T} \mathbb{E}\left[\int_0^1 (y_\sigma(u,t) - g(u))^{2\beta} \mathrm{d}u\right] < +\infty.$*

4.3 Convergence when $\sigma \to 0$

Recall that by Corollary 4.3 and Prokhorov's Theorem, the collection of laws of the sequence $(y_{\sigma,\varepsilon})_{\sigma,\varepsilon\in\mathbb{Q}_+}$ is relatively compact in $\mathcal{P}(L_2([0,1],\mathcal{C}[0,T]))$. By construction, the collection of laws of the sequence $(y_\sigma)_{\sigma\in\mathbb{Q}_+}$ inherits the same property.

Thus, up to extracting a subsequence, we may suppose that $(y_\sigma)_{\sigma\in\mathbb{Q}_+}$ converges in distribution to a limit, denoted by y, in $L_2([0,1],\mathcal{C}[0,T])$. As before, we define $Y(t) := y(\cdot,t)$. We state the first part of Theorem 1.4 in the following Proposition:

Proposition 4.17. *Suppose that $g \in \mathcal{L}_{2+}^\uparrow[0,1]$. $(Y(t))_{t\in[0,T]}$ is a $L_2^\uparrow[0,1]$-valued process such that:*

$(C1)$ $Y(0) = g$;

$(C2)$ $(Y(t))_{t\in[0,T]}$ *is a square integrable continuous $L_2^\uparrow[0,1]$-valued $(\mathcal{F}_t)_{t\in[0,T]}$-martingale, where $\mathcal{F}_t = \sigma(Y(s), s \leqslant t)$.*

Proof. We refer to the proof of Proposition 4.13. □

Remark 4.18. It should be noticed at this point that a new difficulty arises when we want to obtain a property analogous to $(B3)$. Indeed, whereas it was straightforward to prove (4.26) and (4.27), the convergence of $m_\sigma(u,t) = \int_0^1 \varphi_\sigma^2(y_\sigma(u,t) - y_\sigma(v,t))\mathrm{d}v$ to $m(u,t) = \int_0^1 \mathbb{1}_{\{y(u,t)=y(v,t)\}}\mathrm{d}v$ is not obvious, due to the singularity of the indicator function. It will be the main goal of the next Section to prove this convergence.

In Section 5, we will study the martingale properties of the limit process Y and compute its quadratic variation (property $(C5)$ of Theorem 1.4). To obtain this, we will first prove that for every *positive* t, $Y(t)$ is a step function (see property $(C3)$). It implies that y has a version in $\mathcal{D}((0,1),\mathcal{C}[0,T])$ (see property $(C4)$) by an argument given in ([11, Proposition 2.3]).

5 Properties of the limit process Y

The aim of this Section is to complete the proof of Theorem 1.4. Properties $(C3)$ and $(C4)$ will be proved in Paragraph 5.1 and property $(C5)$ will be proved in two steps in Paragraph 5.2 and Paragraph 5.3.

5.1 Coalescence properties and step functions

In this Paragraph, we will prove the following Proposition:

Proposition 5.1. *Almost surely, for every $t > 0$, $Y(t)$ is a step function.*

Recall that $Y(0) = g$ is not necessarily a step function, since g can be chosen arbitrarily in $\mathcal{L}_{2+}^{\uparrow}[0,1]$. If we denote for each $t \in [0,T]$ by μ_t the measure associated to the quantile function $Y(t)$, that is $\mu_t = \mathrm{Leb}\,|_{[0,1]} \circ Y(t)^{-1}$, Proposition 5.1 means that for every *positive* time t, μ_t is a finite weighted sum of Dirac measures. We begin by the following Lemma. Recall the definition of the mass: $m(u,t) = \int_0^1 \mathbb{1}_{\{y(u,t)=y(v,t)\}} \mathrm{d}v$.

Lemma 5.2. *There exists a probability space $(\widetilde{\Omega}, \widetilde{\mathbb{P}})$ on which the sequence $(\widetilde{y}_\sigma)_{\sigma \in \mathbb{Q}_+}$ converges almost surely to $\widetilde{y}$ in $L_2([0,1], \mathcal{C}[0,T])$ and where, for each $\sigma \in \mathbb{Q}_+$, $\widetilde{y}_\sigma$ (resp. $\widetilde{y}$) has same law as y_σ (resp. y). Furthermore, there is a subsequence $(\sigma_n)_n$, $\sigma_n \to 0$, such that for almost every $(\omega, u) \in \Omega \times (0,1)$ and for every time $t \in [0,T]$,*

$$\limsup_{n \to \infty} \widetilde{m}_{\sigma_n}(u,t) \leqslant \widetilde{m}(u,t).$$

Proof. Recall that $(y_\sigma)_{\sigma \in \mathbb{Q}_+}$ converges in distribution in $L_2([0,1], \mathcal{C}[0,T])$ to y. By Skorohod's representation Theorem, we deduce that there exists a sequence $(\widetilde{y}_\sigma)_{\sigma \in \mathbb{Q}_+}$ and a random variable $\widetilde{y}$ defined on a common probability space $(\widetilde{\Omega}, \widetilde{\mathbb{P}})$ such that for every $\sigma \in \mathbb{Q}_+$, the laws of $\widetilde{y}_\sigma$ and y_σ are the same, the laws of $\widetilde{y}$ and y are also equal and the sequence $(\widetilde{y}_\sigma)_{\sigma \in \mathbb{Q}_+}$ converges almost surely to $\widetilde{y}$ in $L_2([0,1], \mathcal{C}[0,T])$.

For every $\varepsilon > 0$, we get by Markov's inequality:

$$\begin{aligned} \widetilde{\mathbb{P}} \otimes \mathrm{Leb}\{(\omega,u) : \|(\widetilde{y}_\sigma - \widetilde{y})(\omega,u)\|_{\mathcal{C}[0,T]} \geqslant \varepsilon\} &= \widetilde{\mathbb{E}}\left[\mathrm{Leb}\{u : \|(\widetilde{y}_\sigma - \widetilde{y})(\omega,u)\|_{\mathcal{C}[0,T]} \geqslant \varepsilon\}\right] \\ &\leqslant \widetilde{\mathbb{E}}\left[1 \wedge \frac{1}{\varepsilon^2} \int_0^1 \|(\widetilde{y}_\sigma - \widetilde{y})(\omega,u)\|^2_{\mathcal{C}[0,T]} \mathrm{d}u\right]. \end{aligned} \tag{5.1}$$

Since $(\widetilde{y}_\sigma)_{\sigma \in \mathbb{Q}_+}$ converges almost surely to $\widetilde{y}$ in $L_2([0,1], \mathcal{C}[0,T])$, the right hand side tends to 0. Therefore, $(\widetilde{y}_\sigma)_{\sigma \in \mathbb{Q}_+}$ converges in probability to $\widetilde{y}$ in $\mathcal{C}[0,T]$ on the probability space $(\widetilde{\Omega} \times [0,1], \widetilde{\mathbb{P}} \otimes \mathrm{Leb})$. Thus there exists a subsequence $(\sigma_n)_n$ tending to 0 along which $\widetilde{y}_{\sigma_n}$ converges on an almost sure event of $\widetilde{\Omega} \times [0,1]$ to $\widetilde{y}$ in $\mathcal{C}[0,T]$. Therefore, there is Ω', $\widetilde{\mathbb{P}}[\Omega'] = 1$, such that for every $\omega \in \Omega'$, there exists a Borel set $\mathcal{A} = \mathcal{A}(\omega)$ in $[0,1]$, $\mathrm{Leb}(\mathcal{A}) = 1$, such that for all $u \in \mathcal{A}$, $\|\widetilde{y}_{\sigma_n}(u,\cdot) - \widetilde{y}(u,\cdot)\|_{\mathcal{C}[0,T]}$ tends to zero. Remark that the extraction $(\sigma_n)_n$ does not depend on ω. From now on, we forget the tildes and the extraction in our notation.

Let $\omega \in \Omega$. Fix $u \in \mathcal{A}(\omega)$ and $t \in [0,T]$. We set $v \in \mathcal{A}$ such that $y(v,t) \neq y(u,t)$. Then there exist $\sigma_0 > 0$ and $\delta > 0$ such that for all $\sigma \in (0, \sigma_0) \cap \mathbb{Q}_+$, $|y_\sigma(v,t) - y_\sigma(u,t)| \geqslant \delta$. For all $\sigma \leqslant \min(\sigma_0, \delta)$, we have $|y_\sigma(v,t) - y_\sigma(u,t)| \geqslant \sigma$ and thus $\varphi_\sigma(y_\sigma(v,t) - y_\sigma(u,t)) = 0$. Hence, $\lim_{\sigma \to 0} \left(1 - \varphi_\sigma^2(y_\sigma(v,t) - y_\sigma(u,t))\right) = 1$. Thus we have shown that for all $v \in \mathcal{A}$,

$$\mathbb{1}_{\{y(v,t) \neq y(u,t)\}} \leqslant \liminf_{\sigma \to 0} \left(1 - \varphi_\sigma^2(y_\sigma(v,t) - y_\sigma(u,t))\right),$$

since $1 - \varphi_\sigma^2$ is non-negative. By Fatou's Lemma and since $\mathrm{Leb}(\mathcal{A}) = 1$, we deduce that:

$$1 - m(u,t) = \int_0^1 \mathbb{1}_{\{y(v,t) \neq y(u,t)\}} \mathrm{d}v \leqslant \liminf_{\sigma \to 0} \int_0^1 \left(1 - \varphi_\sigma^2(y_\sigma(v,t) - y_\sigma(u,t))\right) \mathrm{d}v,$$

whence for all $u \in \mathcal{A}$ and $t \in [0,T]$, $\limsup_{n \to \infty} m_{\sigma_n}(u,t) \leqslant m(u,t)$. □

We deduce from Lemma 5.2 the following Corollary. Set $N(t) := \int_0^1 \frac{\mathrm{d}u}{m(u,t)}$. By a classical combinatorial argument, $N(t)$ is the number of equivalence classes at time t

relatively to the equivalence relation $u \underset{t}{\sim} v \iff y(u,t) = y(v,t)$. In other words, if $N(t) < \infty$, $Y(t)$ is a càdlàg step function taking $N(t)$ distinct values: there exist $0 = a_1 < a_2 < \dots < a_{N(t)} < a_{N(t)+1} = 1$ and $y_1 < y_2 < \dots < y_{N(t)}$ such that for all $u \in [0,1]$

$$Y(t)(u) = \sum_{k=1}^{N(t)} y_k \mathbb{1}_{\{u \in [a_k, a_{k+1})\}} + y_{N(t)} \mathbb{1}_{\{u=1\}}.$$

Corollary 5.3. *For every time $t \in [0,T]$, $\mathbb{E}\left[\int_0^t N(s)\mathrm{d}s\right]$ is finite.*

Proof. By Lemma 5.2, there is a subsequence (σ_n) such that almost surely, for every $t \in [0,T]$ and for almost every $u \in [0,1]$, $\limsup_{n\to\infty} m_{\sigma_n}(u,t) \leqslant m(u,t)$. Therefore, $\frac{1}{m(u,t)} \leqslant \liminf_{n\to\infty} \frac{1}{m_{\sigma_n}(u,t)}$. By Fatou's Lemma, we deduce that:

$$\mathbb{E}\left[\int_0^t N(s)\mathrm{d}s\right] \leqslant \mathbb{E}\left[\int_0^t \int_0^1 \liminf_{n\to\infty} \frac{1}{m_{\sigma_n}(u,t)} \mathrm{d}u\mathrm{d}s\right] \leqslant \liminf_{n\to\infty} \mathbb{E}\left[\int_0^t \int_0^1 \frac{\mathrm{d}u\mathrm{d}s}{m_{\sigma_n}(u,s)}\right] \leqslant C\sqrt{t},$$

by Lemma 4.15. □

Corollary 5.4. *Almost surely, for every $t > 0$, $N(t)$ is finite and $t \mapsto N(t)$ is non-increasing on $(0,T]$.*

Proof. We begin by proving the coalescence property. Let u_1, u_2, $h \in \mathbb{Q}$ be such that $0 < u_1 < u_1 + h < u_2 < u_2 + h < 1$. Define $y^h(u_1,t) = \frac{1}{h}\int_{u_1}^{u_1+h} y(v,t)\mathrm{d}v = (Y(t), \frac{1}{h}\mathbb{1}_{(u_1,u_1+h)})_{L_2}$ and $y^h(u_2,t) = (Y(t), \frac{1}{h}\mathbb{1}_{(u_2,u_2+h)})_{L_2}$. By Proposition 4.17, $Z(t) = y^h(u_2,t) - y^h(u_1,t)$ is a continuous $\mathbb{R}$-valued $(\mathcal{F}_t)_{t\in[0,T]}$-martingale, almost surely non-negative. As a consequence, $Z(t) = 0$ for every $t \geqslant \tau_0 = \inf\{s \geqslant 0, Z(s) = 0\}$. In other terms, the following coalescence property holds: for every u_1, u_2, $h \in \mathbb{Q}$ such that $0 < u_1 < u_1 + h < u_2 < u_2 + h < 1$, $y^h(u_1,t_0) = y^h(u_2,t_0)$ implies $y^h(u_1,t) = y^h(u_2,t)$ for every $t \geqslant t_0$ almost surely.

On a full event Ω' of $(\Omega, \mathbb{P})$, the latter statement is true and $\int_0^T N(s)\mathrm{d}s$ is finite (by Corollary 5.3). Fix $\omega \in \Omega'$. In particular, for almost every $t \in (0,T)$, $N(t)$ is finite. Let $t_0 \in (0,T)$ be such that $N(t_0) < +\infty$. There exist $0 = a_1 < a_2 < \dots < a_{N(t_0)} < a_{N(t_0)+1} = 1$ and $z_1 < z_2 < \dots < z_{N(t_0)}$, depending on ω, such that for all $u \in [0,1]$,

$$Y(t_0)(u) = \sum_{k=1}^{N(t_0)} z_k \mathbb{1}_{\{u \in [a_k, a_{k+1})\}} + z_{N(t_0)} \mathbb{1}_{\{u=1\}}.$$

Fix $k \in \{1, \dots, N(t_0)\}$. By the coalescence property, almost surely, for all u_1, u_2, $h \in \mathbb{Q}$ such that $a_k < u_1 < u_1 + h < u_2 < u_2 + h < a_{k+1}$, since $y^h(u_1,t_0) = z_k = y^h(u_2,t_0)$, we have $y^h(u_1,t) = y^h(u_2,t)$ for every $t \geqslant t_0$. Fix $t \geqslant t_0$. By monotonicity of $Y(t)$, we deduce that $Y(t)$ is constant on $(u_1, u_2 + h)$. Thus $Y(t)$ is constant on (a_k, a_{k+1}). Therefore, since $Y(t)$ is càdlàg, there exist $\widetilde{z}_1 \leqslant \widetilde{z}_2 \leqslant \dots \leqslant \widetilde{z}_{N(t_0)}$, depending on ω, such that for all $u \in [0,1]$,

$$Y(t)(u) = \sum_{k=1}^{N(t_0)} \widetilde{z}_k \mathbb{1}_{\{u \in [a_k, a_{k+1})\}} + \widetilde{z}_{N(t_0)} \mathbb{1}_{\{u=1\}}.$$

We deduce that $N(t) \leqslant N(t_0) < +\infty$, for every $t \geqslant t_0$. Therefore, for every $\omega \in \Omega'$, $t \mapsto N(t)$ is finite and non-increasing on $(0,T]$. This concludes the proof of the Lemma. □

Therefore, Corollary 5.4 concludes the proof of Proposition 5.1. Then, Proposition 4.17 and Proposition 5.1 imply the following property, by applying Proposition 2.3 of [11]:

Proposition 5.5. *There exists a modification $\widetilde{y}$ of y in $L_2([0,1],\mathcal{C}[0,T])$ such that $\widetilde{y}$ belongs to $\mathcal{D}((0,1),\mathcal{C}[0,T])$. In particular, for every $t\in[0,T]$, $y(\cdot,t)$ and $\widetilde{y}(\cdot,t)$ are equal in $L_2[0,1]$ almost surely. Moreover, for every $u\in(0,1)$, $\widetilde{y}(u,\cdot)$ is a square integrable and continuous $(\mathcal{F}_t)_{t\in[0,T]}$-martingale and*

$$\mathbb{P}\left[\forall u,v\in(0,1),\forall s\in[0,T],\widetilde{y}(u,s)=\widetilde{y}(v,s) \text{ implies } \forall t\geqslant s,\widetilde{y}(u,t)=\widetilde{y}(v,t)\right]=1.$$

From now on, we denote by y (instead of $\widetilde{y}$) the version of the limit process in $\mathcal{D}((0,1),\mathcal{C}[0,T])$.

Remark 5.6. The proof can be found in Appendix B of [11]. It should be noticed that the difficult part of the proof relies on the construction of a version $\widetilde{y}$ such that for every $u\in(0,1)$, $\widetilde{y}(u,\cdot)$ is continuous at time $t=0$.

This concludes the proof of properties $(C3)$ and $(C4)$ of Theorem 1.4. The aim of the next two Paragraphs is to prove property $(C5)$, in two steps.

5.2 Quadratic variation of $y(u,\cdot)$

The following Proposition shows that the quadratic variation of a particle is proportional to the inverse of its mass:

Proposition 5.7. *Let y be the version in $\mathcal{D}((0,1),\mathcal{C}[0,T])$ of the limit process given by Proposition 5.5. For every $u\in(0,1)$,*

$$\langle y(u,\cdot),y(u,\cdot)\rangle_t=\int_0^t\frac{1}{m(u,s)}\mathrm{d}s,$$

where $m(u,s)=\int_0^1\mathbb{1}_{\{y(u,s)=y(v,s)\}}\mathrm{d}v$.

Proof. By Corollary 4.14, for every positive $\psi\in L_\infty(0,1)$, we have:

$$\begin{aligned}\mathbb{E}\left[\int_0^1\psi(u)[(y_\sigma(u,t)-g(u))^2-(y_\sigma(u,s)-g(u))^2]f_l(Y_\sigma(s_1),\dots,Y_\sigma(s_l))\mathrm{d}u\right]\\=\mathbb{E}\left[\int_0^1\psi(u)\int_s^t\frac{1}{m_\sigma(u,r)}\mathrm{d}r\;f_l(Y_\sigma(s_1),\dots,Y_\sigma(s_l))\mathrm{d}u\right].\end{aligned}\tag{5.2}$$

To obtain the convergence of the left hand side of (5.2), we proceed in the same way as for the proof of equality (4.28). The uniform integrability property follows from Corollary 4.16. Therefore, the left hand side of (5.2) converges when $\sigma\to0$ to

$$\mathbb{E}\left[\int_0^1\psi(u)[(y(u,t)-g(u))^2-(y(u,s)-g(u))^2]f_l(Y(s_1),\dots,Y(s_l))\mathrm{d}u\right].$$

We also get a uniform integrability property for the right hand side of (5.2) by the same argument as in the proof of property $(B3)$ (see Proposition 4.13). Assume that there exists a sequence (σ_n) of rational numbers tending to 0, a probability space $(\widehat{\Omega},\widehat{\mathbb{P}})$, a modification $(\widehat{m}_{\sigma_n},\widehat{y}_{\sigma_n})_{n\in\mathbb{N}}$ of $(m_{\sigma_n},y_{\sigma_n})_{n\in\mathbb{N}}$ on $L_1([0,1],\mathcal{C}[0,T])\times L_2([0,1],\mathcal{C}[0,T])$ and a modification $(\widehat{m},\widehat{y})$ of (m,y) on the same space such that for almost each $\omega\in\Omega$ and almost every $(u,t)\in[0,1]\times[0,T]$, the sequence $(\widehat{m}_{\sigma_n}(\omega,u,t),\widehat{y}_{\sigma_n}(\omega))_{n\in\mathbb{N}}$ converges to $(\widehat{m}(\omega,u,t),\widehat{y}(\omega))$ in $\mathbb{R}\times L_2([0,1],\mathcal{C}[0,T])$. This will be proved in Lemma 5.8.

It follows that for every $\psi\in L_\infty(0,1)$:

$$\mathbb{E}\left[\int_0^1\psi(u)\left[(\widehat{y}(u,t)-g(u))^2-(\widehat{y}(u,s)-g(u))^2-\int_s^t\frac{\mathrm{d}r}{\widehat{m}(u,r)}\right]f_l(\widehat{Y}(s_1),\dots,\widehat{Y}(s_l))\mathrm{d}u\right]=0.$$

By Fubini's Theorem, we deduce that for almost every $u \in (0,1)$,

$$\mathbb{E}\left[\left((\widehat{y}(u,t)-g(u))^2-(\widehat{y}(u,s)-g(u))^2-\int_s^t \frac{\mathrm{d}r}{\widehat{m}(u,r)}\right) f_l(\widehat{Y}(s_1),\dots,\widehat{Y}(s_l))\right]=0. \quad (5.3)$$

We want to prove that (5.3) holds for every $u \in (0,1)$. Let $u \in (0,1)$. Choose $\delta > 0$ such that $u \in (\delta, 1-\delta)$. Let $(u_p)_{p\in\mathbb{N}}$ be a decreasing sequence in $(\delta,1-\delta)$ converging to u such that for every $p \in \mathbb{N}$, equality (5.3) holds at point u_p, $(y_{\sigma_n,\varepsilon}(u_p,t))_{t\in[0,T]}$ is a square integrable continuous $(\mathcal{F}_t^{\sigma_n,\varepsilon})_{t\in[0,T]}$-martingale for every $n \in \mathbb{N}$ and $\varepsilon \in \mathbb{Q}_+$ and $\limsup_{n\to\infty} \widehat{m}_{\sigma_n}(u_p,t) \leqslant \widehat{m}(u_p,t)$ almost surely for all $t \in [0,T]$. Such a sequence exists by Corollary 3.11 and Lemma 5.2. We will use these different properties later in this proof.

Almost surely, for every $r \in (0,T]$, $\widehat{y}(\cdot,r)$ is right-continuous at point u and is a step function. Therefore, $\widehat{m}(\cdot,r)=\int_0^1 \mathbb{1}_{\{\widehat{y}(\cdot,r)=\widehat{y}(v,r)\}}\mathrm{d}v$ is also right continuous at point u for every positive time r. In order to prove (5.3) at point u, it is thus sufficient to show the following uniform integrability property: there exists $\beta > 1$ such that

$$\sup_{p\in\mathbb{N}} \mathbb{E}\left[\left((\widehat{y}(u_p,t)-g(u_p))^2-(\widehat{y}(u_p,s)-g(u_p))^2-\int_s^t \frac{\mathrm{d}r}{\widehat{m}(u_p,r)}\right)^\beta\right]<+\infty. \quad (5.4)$$

First, by monotonicity, for all $p \in \mathbb{N}$, $\mathbb{E}\left[g(u_p)^{2\beta}\right] \leqslant g(\delta)^{2\beta}+g(1-\delta)^{2\beta}$. Then, the following statement holds: there exists $\beta > 1$ such that for every $t \in [0,T]$, $\sup_{p\in\mathbb{N}} \mathbb{E}\left[\widehat{y}(u_p,t)^{2\beta}\right] < +\infty$. Indeed, for every $p \in \mathbb{N}$, by monotonicity,

$$\frac{1}{\delta}\int_0^\delta \widehat{y}(v,t)\mathrm{d}v \leqslant \widehat{y}(u_p,t) \leqslant \frac{1}{\delta}\int_{1-\delta}^1 \widehat{y}(v,t)\mathrm{d}v.$$

Therefore, we have:

$$\begin{aligned}\mathbb{E}\left[\widehat{y}(u_p,t)^{2\beta}\right] &\leqslant \mathbb{E}\left[\left(\frac{1}{\delta}\int_0^\delta \widehat{y}(v,t)\mathrm{d}v\right)^{2\beta}\right]+\mathbb{E}\left[\left(\frac{1}{\delta}\int_{1-\delta}^1 \widehat{y}(v,t)\mathrm{d}v\right)^{2\beta}\right] \\ &\leqslant \frac{2}{\delta}\mathbb{E}\left[\int_0^1 \widehat{y}(v,t)^{2\beta}\mathrm{d}v\right], \end{aligned} \quad (5.5)$$

by Hölder's inequality. By Fatou's Lemma

$$\mathbb{E}\left[\int_0^1 \widehat{y}(v,t)^{2\beta}\mathrm{d}v\right] \leqslant \liminf_{n\to\infty} \mathbb{E}\left[\int_0^1 \widehat{y}_{\sigma_n}(v,t)^{2\beta}\mathrm{d}v\right],$$

which is finite by Corollary 4.16, for a β chosen in $(1, \frac{3}{2}-\frac{1}{p})$.

Let us keep the same exponent $\beta \in (1,\frac{3}{2}-\frac{1}{p})$. It remains to show that for every $t \in [0,T]$, $\sup_{p\in\mathbb{N}} \mathbb{E}\left[\left(\int_0^t \frac{\mathrm{d}r}{\widehat{m}(u_p,r)}\right)^\beta\right] < +\infty$. Since $\limsup_{n\to\infty} \widehat{m}_{\sigma_n}(u_p,t) \leqslant \widehat{m}(u_p,t)$ and by Fatou's Lemma,

$$\begin{aligned}\mathbb{E}\left[\left|\int_0^t \frac{\mathrm{d}r}{\widehat{m}(u_p,r)}\right|^\beta\right] \leqslant \mathbb{E}\left[\left|\int_0^t \liminf_{n\to\infty}\frac{\mathrm{d}r}{\widehat{m}_{\sigma_n}(u_p,r)}\right|^\beta\right] &\leqslant \liminf_{n\to\infty}\mathbb{E}\left[\left|\int_0^t \frac{\mathrm{d}r}{\widehat{m}_{\sigma_n}(u_p,r)}\right|^\beta\right] \\ &\leqslant \liminf_{\substack{n\to\infty \\ \varepsilon\in\mathbb{Q}_+}}\mathbb{E}\left[\left|\int_0^t \frac{\mathrm{d}r}{\widehat{M}_{\sigma_n,\varepsilon}(u_p,r)}\right|^\beta\right].\end{aligned}$$

Because $(\widehat{y}_{\sigma_n,\varepsilon}(u_p,t))_{t\in[0,T]}$ is a square integrable martingale relatively to $(\mathcal{F}_t^{\sigma_n,\varepsilon})_{t\in[0,T]}$ and $\langle \widehat{y}_{\sigma_n,\varepsilon}(u_p,\cdot),\, \widehat{y}_{\sigma_n,\varepsilon}(u_p,\cdot)\rangle_t = \int_0^t \frac{\mathrm{d}r}{\widehat{M}_{\sigma_n,\varepsilon}(u_p,r)}$, we obtain by Burkholder-Davis-Gundy inequality:

$$\mathbb{E}\left[\left(\int_0^t \frac{\mathrm{d}r}{\widehat{M}_{\sigma_n,\varepsilon}(u_p,r)}\right)^\beta\right] \leqslant C\mathbb{E}\left[(\widehat{y}_{\sigma_n,\varepsilon}(u_p,t)-g(u_p))^{2\beta}\right].$$

We have already seen that $\mathbb{E}\left[g(u_p)^{2\beta}\right]$ is uniformly bounded for $p \in \mathbb{N}$. By the same argument as for inequality (5.5), $\mathbb{E}\left[\widehat{y}_{\sigma_n,\varepsilon}(u_p,t)^{2\beta}\right] \leqslant \frac{2}{\delta}\mathbb{E}\left[\int_0^1 \widehat{y}_{\sigma_n,\varepsilon}(v,t)^{2\beta}\mathrm{d}v\right]$, which is uniformly bounded for $n \in \mathbb{N}$ and $\varepsilon \in \mathbb{Q}_+$. This concludes the proof of (5.4).

Therefore, equality (5.3) holds for every $u \in (0,1)$, for every bounded and continuous f_l and for every $0 \leqslant s_1 \leqslant \ldots \leqslant s_l \leqslant s \leqslant t$. Thus for every $u \in (0,1)$, the process $\left((\widehat{y}(u,t)-g(u))^2 - \int_0^t \frac{\mathrm{d}s}{\widehat{m}(u,s)}\right)_{t\in[0,T]}$ is an $(\mathcal{F}_t)_{t\in[0,T]}$-martingale. This concludes the proof of the Proposition. □

In the proof of Proposition 5.7, we used the following Lemma:

Lemma 5.8. *There exists a sequence (σ_n) of rational numbers tending to 0, a sequence of processes $(\widehat{m}_{\sigma_n}, \widehat{y}_{\sigma_n})_{n\in\mathbb{N}}$ and a process $(\widehat{m}, \widehat{y})$ defined on the same probability space such that*

- *for all $n \in \mathbb{N}$, $(\widehat{m}_{\sigma_n}, \widehat{y}_{\sigma_n})$ and $(m_{\sigma_n}, y_{\sigma_n})$ (resp. $(\widehat{m}, \widehat{y})$ and (m, y)) have same law on $L_1([0,1], \mathcal{C}[0,T]) \times L_2([0,1], \mathcal{C}[0,T])$.*
- *for almost each $\omega \in \Omega$ and for almost every (u,t) in $[0,1]\times[0,T]$, the sequence $(\widehat{m}_{\sigma_n}(\omega,u,t), \widehat{y}_{\sigma_n}(\omega))_{n\in\mathbb{N}}$ converges to $(\widehat{m}(\omega,u,t), \widehat{y}(\omega))$ in $\mathbb{R}\times L_2([0,1],\mathcal{C}[0,T])$.*

Remark 5.9. The Borel subset of $[0,1]\times[0,T]$ on which we have the convergence can depend on ω.

Before giving the proof of Lemma 5.8, we give the following definition and state the following Lemma, which will be useful in the proof. Let us define in $L_1([0,1]\times[0,1], \mathcal{C}[0,T])$:

$$C_\sigma(u_1,u_2,t) := \int_0^t \left(\frac{1}{m_\sigma(u_1,s)} + \frac{1}{m_\sigma(u_2,s)} - \frac{2m_\sigma(u_1,u_2,s)}{m_\sigma(u_1,s)m_\sigma(u_2,s)}\right)\mathrm{d}s.$$

Lemma 5.10. *There exists a sequence (σ_n) in $\mathbb{Q}_+$ tending to 0 such that $(y_{\sigma_n}, C_{\sigma_n})_{n\in\mathbb{N}}$ converges in distribution to (y, C) in $L_2([0,1],\mathcal{C}[0,T]) \times L_1([0,1]\times[0,1],\mathcal{C}[0,T])$. For almost every $u_1, u_2 \in [0,1]$, the limit process $C(u_1,u_2,\cdot)$ is the quadratic variation of $y(u_1,\cdot) - y(u_2,\cdot)$ relatively to the filtration generated by Y and C.*

We start by giving the proof of Lemma 5.8 and then we give the proof of Lemma 5.10.

Proof (Lemma 5.8). By Skorohod's representation Theorem, it follows from Lemma 5.10 that there exists a sequence $(\widehat{y}_{\sigma_n}, \widehat{C}_{\sigma_n})_n$ and a random variable $(\widehat{y}, \widehat{C})$ defined on the same probability space such that

- for all $n \in \mathbb{N}$, $(\widehat{y}_{\sigma_n}, \widehat{C}_{\sigma_n})$ and $(y_{\sigma_n}, C_{\sigma_n})$ (resp. $(\widehat{y}, \widehat{C})$ and (y, C)) have same law,
- the sequence $(\widehat{y}_{\sigma_n}, \widehat{C}_{\sigma_n})_n$ converges almost surely to $(\widehat{y}, \widehat{C})$ in $L_2([0,1],\mathcal{C}[0,T]) \times L_1([0,1]\times[0,1],\mathcal{C}[0,T])$.

We apply to $(\widehat{y}_{\sigma_n})_n$ the argument in the proof of Lemma 5.2 and we prove that, up to extracting another subsequence (independent of ω), for almost every $u \in [0,1]$ and almost surely, $\limsup_{n\to\infty} \widehat{m}_{\sigma_n}(u,t) \leqslant \widehat{m}(u,t)$ for every $t \in [0,T]$.

For each $t \in [0,T]$, we may suppose that for each $n \in \mathbb{N}$, $\widehat{y}_{\sigma_n}(\cdot,t)$ is a càdlàg function, so that for every $u \in (0,1)$,

$$\begin{aligned}\widehat{m}_{\sigma_n}(u,t) &= \int_0^1 \varphi_{\sigma_n}^2(\widehat{y}_{\sigma_n}(u,t)-\widehat{y}_{\sigma_n}(v,t))\mathrm{d}v \\ &= \lim_{p\to\infty} p\int_u^{(u+\frac{1}{p})\wedge 1}\int_0^1 \varphi_{\sigma_n}^2(\widehat{y}_{\sigma_n}(u',t)-\widehat{y}_{\sigma_n}(v,t))\mathrm{d}v\mathrm{d}u'\end{aligned}$$

is a measurable function with respect to $\widehat{y}_{\sigma_n}(\cdot,t)$. We deduce that $(\widehat{m}_{\sigma_n}(u,t),\widehat{y}_{\sigma_n})$ has the same law as $(m_{\sigma_n}(u,t),y_{\sigma_n})$ for every $u \in (0,1)$.

From now on, we forget the hats in our notation. We may suppose that y is the version in $\mathcal{D}((0,1),\mathcal{C}[0,T])$ given by Proposition 5.5. Let Ω' be such that $\mathbb{P}[\Omega']=1$ and for all $\omega \in \Omega'$, we have the following convergences in $\mathbb{R}$:

$$\int_0^1 \sup_{t\leqslant T}|y_{\sigma_n}(u,t)-y(u,t)|^2(\omega)\mathrm{d}u \underset{n\to\infty}{\longrightarrow} 0, \tag{5.6}$$

$$\int_0^1\int_0^1 \sup_{t\leqslant T}|C_{\sigma_n}(u_1,u_2,t)-C(u_1,u_2,t)|(\omega)\mathrm{d}u_1\mathrm{d}u_2 \underset{n\to\infty}{\longrightarrow} 0. \tag{5.7}$$

Fix $\omega \in \Omega'$. Thanks to (5.6), we already have the convergence of $(y_{\sigma_n}(\omega))_n$ to $y(\omega)$ in $L_2([0,1],\mathcal{C}[0,T])$. It remains to show that for almost every $(u,t) \in [0,1]\times[0,T]$, $(m_{\sigma_n}(\omega,u,t))_n$ converges to $m(\omega,u,t)=\int_0^1 \mathbb{1}_{\{y(u,t)=y(v,t)\}}(\omega)\mathrm{d}v$. We already know that for every $\omega\in\Omega'$, every $t\in[0,T]$ and almost every $u\in(0,1)$, $\limsup_{n\to\infty} m_{\sigma_n}(\omega,u,t) \leqslant m(\omega,u,t)$.

Proof of inequality: $\liminf_{n\to\infty} m_{\sigma_n}(\omega,u,t) \geqslant m(\omega,u,t)$.

By the coalescence property given by Proposition 5.5, for every u_1,u_2 and for all $t>\tau_{u_1,u_2}$, $y(u_1,t)=y(u_2,t)$. Therefore, since $C(u_1,u_2,\cdot)$ is the quadratic variation of $y(u_1,\cdot)-y(u_2,\cdot)$, $t\mapsto C(u_1,u_2,t)$ remains constant on (τ_{u_1,u_2},T). Thus we obtain:

$$\begin{aligned}&\left|\int_0^1\int_0^1\int_{\tau_{u_1,u_2}}^T \left(\frac{1}{m_{\sigma_n}(u_1,t)}+\frac{1}{m_{\sigma_n}(u_2,t)}-\frac{2m_{\sigma_n}(u_1,u_2,t)}{m_{\sigma_n}(u_1,t)m_{\sigma_n}(u_2,t)}\right)\mathrm{d}t\mathrm{d}u_1\mathrm{d}u_2\right| \\ &=\left|\int_0^1\int_0^1 (C_{\sigma_n}(u_1,u_2,T)-C_{\sigma_n}(u_1,u_2,\tau_{u_1,u_2}))\mathrm{d}u_1\mathrm{d}u_2\right| \\ &=\left|\int_0^1\int_0^1 (C_{\sigma_n}(u_1,u_2,T)-C(u_1,u_2,T)+C(u_1,u_2,\tau_{u_1,u_2})-C_{\sigma_n}(u_1,u_2,\tau_{u_1,u_2}))\mathrm{d}u_1\mathrm{d}u_2\right| \\ &\leqslant 2\int_0^1\int_0^1 \sup_{t\leqslant T}|C_{\sigma_n}(u_1,u_2,t)-C(u_1,u_2,t)|\mathrm{d}u_1\mathrm{d}u_2.\end{aligned}$$

By (5.7), the latter term tends to 0. We also recall that

$$\begin{aligned}&\frac{1}{m_{\sigma_n}(u_1,t)}+\frac{1}{m_{\sigma_n}(u_2,t)}-\frac{2m_{\sigma_n}(u_1,u_2,t)}{m_{\sigma_n}(u_1,t)m_{\sigma_n}(u_2,t)} \\ &\qquad=\frac{\int_0^1 |\varphi_{\sigma_n}(y_{\sigma_n}(u_1,t_0)-y_{\sigma_n}(v,t_0))-\varphi_{\sigma_n}(y_{\sigma_n}(u_2,t_0)-y_{\sigma_n}(v,t_0))|^2\,\mathrm{d}v}{m_{\sigma_n}(u_1,t_0)m_{\sigma_n}(u_2,t_0)}\end{aligned}$$

is non-negative.

We define $f_{\sigma_n}(t,u_1,u_2):=\left(\frac{1}{m_{\sigma_n}(u_1,t)}+\frac{1}{m_{\sigma_n}(u_2,t)}-\frac{2m_{\sigma_n}(u_1,u_2,t)}{m_{\sigma_n}(u_1,t)m_{\sigma_n}(u_2,t)}\right)\mathbb{1}_{\{t\geqslant\tau_{u_1,u_2}\}}$. For every $\omega\in\Omega'$, $\int_0^T\int_0^1\int_0^1 f_{\sigma_n}(t,u_1,u_2)(\omega)\mathrm{d}u_1\mathrm{d}u_2\mathrm{d}t \underset{n\to\infty}{\longrightarrow} 0$. Therefore, for every $\varepsilon>0$, using

Markov's inequality as in (5.1), and since $f_{\sigma_n} \geqslant 0$:

$$\mathbb{P} \otimes \frac{1}{T} \operatorname{Leb}|_{[0,T]} \otimes \operatorname{Leb}|_{[0,1]} \otimes \operatorname{Leb}|_{[0,1]} \{(\omega, t, u_1, u_2) : f_{\sigma_n}(t, u_1, u_2)(\omega) \geqslant \varepsilon\}$$
$$\leqslant \mathbb{E}\left[1 \wedge \frac{1}{\varepsilon T} \int_0^T \int_0^1 \int_0^1 f_{\sigma_n}(t, u_1, u_2) \mathrm{d}u_1 \mathrm{d}u_2 \mathrm{d}t\right],$$

which tends to 0 when $n \to \infty$, whence we obtain a convergence in probability with respect to the probability space $\Omega \times [0,T] \times [0,1] \times [0,1]$. Up to extracting another subsequence (independent of the choice of ω), we deduce the existence of an almost sure event on which (f_{σ_n}) converges to 0.

Let Ω'', $\mathbb{P}[\Omega''] = 1$, be such that for every $\omega \in \Omega''$, we have $f_{\sigma_n}(t, u_1, u_2)(\omega) \to 0$ for almost every $(t, u_1, u_2) \in [0,T] \times [0,1] \times [0,1]$. Fix $\omega \in \Omega''$. Let us consider a Borel set $\mathcal{B} = \mathcal{B}(\omega)$ in $[0,T]$, $\operatorname{Leb}(\mathcal{B}) = T$, such that for every $t \in \mathcal{B}$, $f_{\sigma_n}(t, u_1, u_2) \to 0$ for almost every $(u_1, u_2) \in [0,1] \times [0,1]$.

Let $t_0 \in \mathcal{B}$. Let us consider a Borel set $\mathcal{A}$ (depending on ω and t_0) of measure 1 such that for all $u_1, u_2 \in \mathcal{A}$,

$$f_{\sigma_n}(t_0, u_1, u_2) \underset{n\to\infty}{\longrightarrow} 0. \tag{5.8}$$

Let $u \in \mathcal{A}$. We want to prove that $\liminf_{n\to\infty} m_{\sigma_n}(u, t_0) \geqslant m(u, t_0)$. Define $u_{\sup} = \sup\{v \in [0,1] : y(v, t_0) = y(u, t_0)\}$ and $u_{\inf}$ the infimum of that set. Since $v \mapsto y(v, t_0)$ is non-decreasing, $m(u, t_0) = u_{\sup} - u_{\inf}$. If $m(u, t_0) = 0$, then we clearly have:

$$\liminf_{n\to\infty} m_{\sigma_n}(u, t_0) \geqslant m(u, t_0).$$

Suppose now that $m(u, t_0) > 0$. Choose $\delta > 0$ such that $\delta < \frac{u_{\sup} - u_{\inf}}{6}$. Let $u_{\max} \in \mathcal{A} \cap (u_{\sup} - \delta, u_{\sup})$, $u_{\min} \in \mathcal{A} \cap (u_{\inf}, u_{\inf} + \delta)$ and $u_{\text{med}} \in \mathcal{A} \cap \left(\frac{u_{\min}+u_{\max}}{2} - \delta, \frac{u_{\min}+u_{\max}}{2} + \delta\right)$. We have: $u_{\max} - u_{\min} \geqslant u_{\sup} - u_{\inf} - 2\delta = m(u, t_0) - 2\delta$ and by definition of $u_{\sup}$ and $u_{\inf}$ and since $u_{\max}$, $u_{\min}$ and u_{med} belongs to $(u_{\inf}, u_{\sup})$, we have $t_0 \geqslant \tau_{u_1,u_2}$ for $(u_1, u_2) = (u, u_{\max})$, $(u, u_{\min})$, $(u_{\max}, u_{\min})$ and (u, u_{med}).

We deduce from (5.8) and the fact that $u, u_{\max}, u_{\min}, u_{\text{med}}$ belongs to $\mathcal{A}$ that there exists N such that for each $n \geqslant N$, $f_{\sigma_n}(t_0, u_1, u_2) \leqslant \delta$ for $(u_1, u_2) = (u, u_{\max})$, $(u, u_{\min})$, $(u_{\max}, u_{\min})$ and (u, u_{med}). It implies that for each $n \geqslant N$,

$$\frac{\int_0^1 |\varphi_{\sigma_n}(y_{\sigma_n}(u_1, t_0) - y_{\sigma_n}(v, t_0)) - \varphi_{\sigma_n}(y_{\sigma_n}(u_2, t_0) - y_{\sigma_n}(v, t_0))|^2 \,\mathrm{d}v}{m_{\sigma_n}(u_1, t_0) m_{\sigma_n}(u_2, t_0)} = f_{\sigma_n}(t_0, u_1, u_2) \leqslant \delta. \tag{5.9}$$

Since the mass m_{σ_n} is bounded by 1, we deduce in particular that for all $n \geqslant N$,

$$\int_0^1 |\varphi_{\sigma_n}(y_{\sigma_n}(u_1, t_0) - y_{\sigma_n}(v, t_0)) - \varphi_{\sigma_n}(y_{\sigma_n}(u_2, t_0) - y_{\sigma_n}(v, t_0))|^2 \,\mathrm{d}v \leqslant \delta. \tag{5.10}$$

Inequalities (5.9) and (5.10) are satisfied for $(u_1, u_2) = (u, u_{\max})$, $(u, u_{\min})$, $(u_{\max}, u_{\min})$ and (u, u_{med}).

Let $n \geqslant N$ and $d := y_{\sigma_n}(u_{\max}, t_0) - y_{\sigma_n}(u_{\min}, t_0) \geqslant 0$. We distinguish three cases:

- $d \geqslant \sigma_n$**:** Recall that φ_{σ_n} is equal to 0 on $[\frac{\sigma_n}{2}, +\infty)$. Thus for every $v \in [0,1]$, the terms $\varphi_{\sigma_n}(y_{\sigma_n}(u_{\max}, t_0) - y_{\sigma_n}(v, t_0))$ and $\varphi_{\sigma_n}(y_{\sigma_n}(u_{\min}, t_0) - y_{\sigma_n}(v, t_0))$ can not be simultaneously different from 0, because $d \geqslant \sigma_n$. Therefore, selecting $(u_1, u_2) = (u_{\max}, u_{\min})$, inequality (5.9) implies:

$$\frac{\int_0^1 \varphi_{\sigma_n}^2(y_{\sigma_n}(u_{\max}, t_0) - y_{\sigma_n}(v, t_0))\mathrm{d}v + \int_0^1 \varphi_{\sigma_n}^2(y_{\sigma_n}(u_{\min}, t_0) - y_{\sigma_n}(v, t_0))\mathrm{d}v}{m_{\sigma_n}(u_{\max}, t_0) m_{\sigma_n}(u_{\min}, t_0)} \leqslant \delta,$$

that is:

$$\frac{1}{m_{\sigma_n}(u_{\min}, t_0)} + \frac{1}{m_{\sigma_n}(u_{\max}, t_0)} \leqslant \delta.$$

Thus, we obtain $\delta \geqslant 2$, which is excluded by definition of δ.

- $d \leqslant \sigma_n - \eta$**:** Recall that η is chosen so that $\eta < \frac{\sigma_n}{3}$. Define the two following sets

$$V_{\max} = \{v \in [u_{\min}, u_{\max}] : y_{\sigma_n}(u_{\max}, t_0) - y_{\sigma_n}(v, t_0) \leqslant \tfrac{\sigma_n - \eta}{2}\},$$
$$V_{\min} = \{v \in [u_{\min}, u_{\max}] : y_{\sigma_n}(u_{\max}, t_0) - y_{\sigma_n}(v, t_0) > \tfrac{\sigma_n - \eta}{2}\}.$$

Clearly, we have: $\mathrm{Leb}(V_{\max}) + \mathrm{Leb}(V_{\min}) = u_{\max} - u_{\min} \geqslant m(u, t_0) - 2\delta$. Recall that φ_{σ_n} is equal to 1 on $[0, \frac{\sigma_n - \eta}{2}]$. Thus, for each $v \in V_{\max}$, $\varphi_{\sigma_n}(y_{\sigma_n}(u_{\max}, t_0) - y_{\sigma_n}(v, t_0)) = 1$, and for each $v \in V_{\min}$, using $d \leqslant \sigma_n - \eta$, $\varphi_{\sigma_n}(y_{\sigma_n}(u_{\min}, t_0) - y_{\sigma_n}(v, t_0)) = 1$. We have

$$m_{\sigma_n}(u, t_0) \geqslant \textstyle\int_{V_{\max}} \varphi^2_{\sigma_n}(y_{\sigma_n}(u, t_0) - y_{\sigma_n}(v, t_0))\mathrm{d}v + \int_{V_{\min}} \varphi^2_{\sigma_n}(y_{\sigma_n}(u, t_0) - y_{\sigma_n}(v, t_0))\mathrm{d}v. \tag{5.11}$$

We can deduce from inequality (5.10) applied to $(u_1, u_2) = (u, u_{\max})$ that:

$$\textstyle\int_{V_{\max}} |\varphi_{\sigma_n}(y_{\sigma_n}(u, t_0) - y_{\sigma_n}(v, t_0)) - \varphi_{\sigma_n}(y_{\sigma_n}(u_{\max}, t_0) - y_{\sigma_n}(v, t_0))|^2 \, \mathrm{d}v \leqslant \delta.$$

By Minkowski's inequality $|\|f_1\|_{L_2} - \|f_2\|_{L_2}| \leqslant \|f_1 - f_2\|_{L_2}$, we obtain:

$$\left| \left(\int_{V_{\max}} \varphi^2_{\sigma_n}(y_{\sigma_n}(u, t_0) - y_{\sigma_n}(v, t_0))\mathrm{d}v \right)^{1/2} - \mathrm{Leb}(V_{\max})^{1/2} \right| \leqslant \sqrt{\delta},$$

whence

$$\begin{aligned} \left| \int_{V_{\max}} \varphi^2_{\sigma_n}(y_{\sigma_n}(u, t_0) - y_{\sigma_n}(v, t_0))\mathrm{d}v - \mathrm{Leb}(V_{\max}) \right| &\leqslant (m^{1/2}_{\sigma_n}(u, t_0) + \mathrm{Leb}(V_{\max})^{1/2})\sqrt{\delta} \\ &\leqslant 2\sqrt{\delta}. \end{aligned}$$

Similarly, applying inequality (5.10) to $(u, u_{\min})$, we obtain:

$$\left| \textstyle\int_{V_{\min}} \varphi^2_{\sigma_n}(y_{\sigma_n}(u, t_0) - y_{\sigma_n}(v, t_0))\mathrm{d}v - \mathrm{Leb}(V_{\min}) \right| \leqslant 2\sqrt{\delta}.$$

Thus, by inequality (5.11), we conclude:

$$\begin{aligned} m_{\sigma_n}(u, t_0) &\geqslant \mathrm{Leb}(V_{\max}) + \mathrm{Leb}(V_{\min}) - 4\sqrt{\delta} \\ &\geqslant m(u, t_0) - 2\delta - 4\sqrt{\delta}. \end{aligned}$$

- $d \in (\sigma_n - \eta, \sigma_n)$**:** We now define three distinct sets

$$V_{\max} = \{v \in [u_{\min}, u_{\max}] : y_{\sigma_n}(u_{\max}, t_0) - y_{\sigma_n}(v, t_0) < \tfrac{\sigma_n - \eta}{2}\},$$
$$V_{\mathrm{med}} = \{v \in [u_{\min}, u_{\max}] : y_{\sigma_n}(u_{\max}, t_0) - y_{\sigma_n}(v, t_0) \in [\tfrac{\sigma_n - \eta}{2}, \tfrac{\sigma_n + \eta}{2}]\},$$
$$V_{\min} = \{v \in [u_{\min}, u_{\max}] : y_{\sigma_n}(u_{\max}, t_0) - y_{\sigma_n}(v, t_0) > \tfrac{\sigma_n + \eta}{2}\}.$$

By definition of those sets, and since $d \in (\sigma_n - \eta, \sigma_n)$, we have

$$\forall v \in V_{\max}, \;\; \varphi_{\sigma_n}(y_{\sigma_n}(u_{\max}, t_0) - y_{\sigma_n}(v, t_0)) = 1,$$
$$\forall v \in V_{\min}, \;\; \varphi_{\sigma_n}(y_{\sigma_n}(u_{\min}, t_0) - y_{\sigma_n}(v, t_0)) = 1.$$

Moreover, we have $y_{\sigma_n}(u_{\max}, t_0) - y_{\sigma_n}(u_{\mathrm{med}}, t_0) \in [\frac{\sigma_n - \eta}{2}, \frac{\sigma_n + \eta}{2}]$.

Indeed, if $y_{\sigma_n}(u_{\max}, t_0) - y_{\sigma_n}(u_{\text{med}}, t_0)$ was greater than $\frac{\sigma_n+\eta}{2}$, we would have, for all $v \in [u_{\min}, u_{\text{med}}]$, $\varphi_{\sigma_n}(y_{\sigma_n}(u_{\max}, t_0) - y_{\sigma_n}(v, t_0)) = 0$ and $\varphi_{\sigma_n}(y_{\sigma_n}(u_{\min}, t_0) - y_{\sigma_n}(v, t_0)) = 1$. By inequality (5.10) applied to $(u_1, u_2) = (u_{\max}, u_{\min})$, we would deduce that:

$$\begin{aligned}\delta &\geqslant \int_0^1 |\varphi_{\sigma_n}(y_{\sigma_n}(u_{\max}, t_0) - y_{\sigma_n}(v, t_0)) - \varphi_{\sigma_n}(y_{\sigma_n}(u_{\min}, t_0) - y_{\sigma_n}(v, t_0))|^2 \,\mathrm{d}v \\ &\geqslant \int_{u_{\min}}^{u_{\text{med}}} \mathrm{d}v = u_{\text{med}} - u_{\min} \geqslant \frac{u_{\max} - u_{\min}}{2} - \delta.\end{aligned}$$

However, since $\delta < \frac{u_{\sup} - u_{\inf}}{6}$ and $u_{\max} - u_{\min} \geqslant u_{\sup} - u_{\inf} - 2\delta$, we have $u_{\max} - u_{\min} > 4\delta$, which is in contradiction with the above inequality. Similarly, $y_{\sigma_n}(u_{\max}, t_0) - y_{\sigma_n}(u_{\text{med}}, t_0)$ can not be smaller than $\frac{\sigma_n - \eta}{2}$, otherwise $y_{\sigma_n}(u_{\text{med}}, t_0) - y_{\sigma_n}(u_{\min}, t_0)$ would be greater than $\frac{\sigma_n+\eta}{2}$ and we would obtain the same contradiction. Therefore, $y_{\sigma_n}(u_{\max}, t_0) - y_{\sigma_n}(u_{\text{med}}, t_0) \in [\frac{\sigma_n-\eta}{2}, \frac{\sigma_n+\eta}{2}]$, which implies that $u_{\text{med}} \in V_{\text{med}}$ and in particular that

$$\forall v \in V_{\text{med}},\ \varphi_{\sigma_n}(y_{\sigma_n}(u_{\text{med}}, t_0) - y_{\sigma_n}(v, t_0)) = 1.$$

As in the previous case, we deduce that

$$\begin{aligned}m_{\sigma_n}(u, t_0) &\geqslant \text{Leb}(V_{\max}) + \text{Leb}(V_{\text{med}}) + \text{Leb}(V_{\min}) - 6\sqrt{\delta} \\ &= u_{\max} - u_{\min} - 6\sqrt{\delta} \\ &\geqslant m(u, t_0) - 2\delta - 6\sqrt{\delta}.\end{aligned}$$

Actually, putting all the cases together, we have proved that for each $n \geqslant N$, $m_{\sigma_n}(u, t_0) \geqslant m(u, t_0) - 2\delta - 6\sqrt{\delta}$. Hence, for all $\delta < \frac{u_{\sup} - u_{\inf}}{6}$, we have:

$$\liminf_{n\to\infty} m_{\sigma_n}(u, t_0) \geqslant m(u, t_0) - 2\delta - 6\sqrt{\delta}.$$

By letting δ converge to 0, we have for every $t_0 \in \mathcal{B}$, $\liminf_{n\to\infty} m_{\sigma_n}(u, t_0) \geqslant m(u, t_0)$ for every $u \in \mathcal{A}$. Therefore, there exists a subsequence (σ_n) such that for almost every ω, for almost every $t \in [0, T]$ and almost every $u \in [0, 1]$, $m_{\sigma_n}(\omega, u, t) \to_{n\to\infty} m(\omega, u, t)$. □

It remains to give the proof of Lemma 5.10.

Proof (Lemma 5.10). The first step will be to prove that the sequence $(y_\sigma, C_\sigma)_{\sigma\in\mathbb{Q}_+}$ is tight in $L_2([0, 1], \mathcal{C}[0, T]) \times L_1([0, 1] \times [0, 1], \mathcal{C}[0, T])$. We have already proved that $(y_\sigma)_{\sigma\in\mathbb{Q}_+}$ is tight in $L_2([0, 1], \mathcal{C}[0, T])$. We will use a tightness criterion to prove that the sequence $(C_\sigma)_{\sigma\in\mathbb{Q}_+}$ is tight in $L_1([0, 1] \times [0, 1], \mathcal{C}[0, T])$. The space changed in comparison with $L_2([0, 1], \mathcal{C}[0, T])$, but the criterion remains very semilar to the one of Proposition 4.2.

We have, similarly to Proposition 4.2, three criteria to prove. We want to show the following criterion:

First criterion: Let $\delta > 0$. There is $M > 0$ such that for every σ in $\mathbb{Q}_+$, $\mathbb{P}[\|C_\sigma\| \geqslant M] \leqslant \delta$, where $\|C_\sigma\| := \int_0^1 \int_0^1 \sup_{t\leqslant T} |C_\sigma(u_1, u_2, t)| \mathrm{d}u_1 \mathrm{d}u_2$.

That statement follows from Markov's inequality and the existence of a constant C independent of σ such that:

$$\begin{aligned}\mathbb{E}\left[\int_0^1\int_0^1 \sup_{t\leqslant T} |C_\sigma(u_1, u_2, t)| \mathrm{d}u_1 \mathrm{d}u_2\right] &\leqslant 2\mathbb{E}\left[\int_0^1\int_0^T \frac{\mathrm{d}t\mathrm{d}u_1}{m_\sigma(u_1, t)}\right] + 2\mathbb{E}\left[\int_0^1\int_0^T \frac{\mathrm{d}t\mathrm{d}u_2}{m_\sigma(u_2, t)}\right] \\ &\leqslant C.\end{aligned}$$

The existence of C is a consequence of Lemma 4.15.

Then, we prove the following criterion:

Second criterion: Let $\delta > 0$. For each $k \geqslant 1$, there exists $\eta_k > 0$ such that for all σ in $\mathbb{Q}_+$,

$$\mathbb{P}\left[\int_0^1\int_0^1 \sup_{|t_2-t_1|<\eta_k} |C_\sigma(u_1,u_2,t_2) - C_\sigma(u_1,u_2,t_1)|\mathrm{d}u_1\mathrm{d}u_2 \geqslant \frac{1}{k}\right] \leqslant \frac{\delta}{2^k}.$$

The proof is very close to Proposition 4.10. We start by defining for every u_1, $u_2 \in (0,1)$: $K_1(u_1,u_2) := \mathbb{E}\left[\|C_\sigma(u_1,u_2,\cdot)\|_{\mathcal{C}[0,T]}\right]$ and $K_2(u_1) := \mathbb{E}\left[\int_0^T \frac{1}{m_\sigma^\beta(u_1,s)}\mathrm{d}s\right]$. Fix $\delta > 0$. There exists $C > 0$ such that $\int_0^1\int_0^1 \mathbb{1}_{\{K_1(u_1,u_2)\geqslant C\}}\mathrm{d}u_1\mathrm{d}u_2 \leqslant \delta$ and $\int_0^1 \mathbb{1}_{\{K_2(u)\geqslant C\}}\mathrm{d}u \leqslant \delta$. Define the following set $K := \{(u_1,u_2) : K_1(u_1,u_2) \leqslant C, K_2(u_1) \leqslant C, K_2(u_2) \leqslant C\}$.

By Aldous' tightness criterion, the collection $(C_\sigma(u_1,u_2,\cdot))_{\sigma\in\mathbb{Q}_+,(u_1,u_2)\in K}$ is tight in $\mathcal{C}[0,T]$. This fact relies on the following inequality, where $\eta > 0$ and τ is a stopping time for $C_\sigma(u_1,u_2,\cdot)$:

$$\begin{aligned}\mathbb{E}\left[|C_\sigma(u_1,u_2,\tau+\eta) - C_\sigma(u_1,u_2,\tau)|\right]&\\ &= \mathbb{E}\left[\left|\int_\tau^{\tau+\eta}\left(\frac{1}{m_\sigma(u_1,s)} + \frac{1}{m_\sigma(u_2,s)} - \frac{2m_\sigma(u_1,u_2,s)}{m_\sigma(u_1,s)m_\sigma(u_2,s)}\right)\mathrm{d}s\right|\right]\\ &\leqslant 2\mathbb{E}\left[\int_\tau^{\tau+\eta}\left(\frac{1}{m_\sigma(u_1,s)} + \frac{1}{m_\sigma(u_2,s)}\right)\mathrm{d}s\right],\end{aligned}$$

and the rest of the proof is an adaptation of the proof of Proposition 4.10.

Finally we show the third criterion:

Third criterion: Let $\delta > 0$. For each $k \geqslant 1$, there is $H > 0$ such that for all σ in $\mathbb{Q}_+$,

$$\begin{aligned}\mathbb{P}\Bigg[&\forall h = (h_1,h_2), 0 < h_1 < H, 0 < h_2 < H,\\ &\int_0^{1-h_1}\int_0^{1-h_2} \sup_{t\leqslant T}|C_\sigma(u_1+h_1,u_2+h_2,t) - C_\sigma(u_1,u_2,t)|\mathrm{d}u_1\mathrm{d}u_2 \leqslant \frac{1}{k}\Bigg] \geqslant 1 - \frac{\delta}{2^k}. \quad (5.12)\end{aligned}$$

Let $h_1 > 0$ and begin by estimating

$$E_\sigma := \mathbb{E}\left[\int_0^{1-h_1}\int_0^1 \sup_{t\leqslant T}|C_\sigma(u_1+h_1,u_2,t) - C_\sigma(u_1,u_2,t)|\mathrm{d}u_1\mathrm{d}u_2\right].$$

We compute (for the sake of simplicity, we will write from now on $y_\sigma(u)$ instead of $y_\sigma(u,\cdot)$ if there is no possibility of confusion):

$$\begin{aligned}C_\sigma(u_1+h_1,u_2,t) - C_\sigma(u_1,u_2,t) &= \langle y_\sigma(u_1+h_1) - y_\sigma(u_2), y_\sigma(u_1+h_1) - y_\sigma(u_2)\rangle_t\\ &\quad - \langle y_\sigma(u_1) - y_\sigma(u_2), y_\sigma(u_1) - y_\sigma(u_2)\rangle_t\\ &= \langle y_\sigma(u_1+h_1) - y_\sigma(u_1), y_\sigma(u_1+h_1) - y_\sigma(u_2)\rangle_t\\ &\quad + \langle y_\sigma(u_1) - y_\sigma(u_2), y_\sigma(u_1+h_1) - y_\sigma(u_1)\rangle_t.\end{aligned}$$

Therefore,

$$\begin{aligned}\sup_{t\leqslant T}|C_\sigma(u_1+h_1,u_2,t) - C_\sigma(u_1,u_2,t)|&\\ &\leqslant \sup_{t\leqslant T}|\langle y_\sigma(u_1+h_1) - y_\sigma(u_1), y_\sigma(u_1+h_1) - y_\sigma(u_2)\rangle_t|\\ &\quad + \sup_{t\leqslant T}|\langle y_\sigma(u_1) - y_\sigma(u_2), y_\sigma(u_1+h_1) - y_\sigma(u_1)\rangle_t|. \quad (5.13)\end{aligned}$$

Then, we use Kunita-Watanabe's inequality on the first term of the right hand side:

$$\begin{aligned}
&|\langle y_\sigma(u_1+h_1)-y_\sigma(u_1), y_\sigma(u_1+h_1)-y_\sigma(u_2)\rangle_t| \\
&\qquad \leqslant |\langle y_\sigma(u_1+h_1)-y_\sigma(u_1), y_\sigma(u_1+h_1)-y_\sigma(u_1)\rangle_t|^{\frac{1}{2}} \\
&\qquad\qquad |\langle y_\sigma(u_1+h_1)-y_\sigma(u_2), y_\sigma(u_1+h_1)-y_\sigma(u_2)\rangle_t|^{\frac{1}{2}} \\
&\qquad \leqslant |\langle y_\sigma(u_1+h_1)-y_\sigma(u_1), y_\sigma(u_1+h_1)-y_\sigma(u_1)\rangle_T|^{\frac{1}{2}} \\
&\qquad\qquad |\langle y_\sigma(u_1+h_1)-y_\sigma(u_2), y_\sigma(u_1+h_1)-y_\sigma(u_2)\rangle_T|^{\frac{1}{2}}.
\end{aligned}$$

By doing the same computation on the second term of the right hand side of (5.13), by Cauchy-Schwarz inequality and by the substitution of u_1+h_1 by u_1, we obtain:

$$\begin{aligned}
E_\sigma &\leqslant 2\mathbb{E}\left[\int_0^{1-h_1}\int_0^1 \langle y_\sigma(u_1+h_1)-y_\sigma(u_1), y_\sigma(u_1+h_1)-y_\sigma(u_1)\rangle_T \mathrm{d}u_1\mathrm{d}u_2\right]^{1/2} \\
&\quad \times \mathbb{E}\left[\int_0^1\int_0^1 \langle y_\sigma(u_1)-y_\sigma(u_2), y_\sigma(u_1)-y_\sigma(u_2)\rangle_T \mathrm{d}u_1\mathrm{d}u_2\right]^{1/2} \\
&\leqslant 2\mathbb{E}\left[\int_0^{1-h_1} \langle y_\sigma(u_1+h_1)-y_\sigma(u_1), y_\sigma(u_1+h_1)-y_\sigma(u_1)\rangle_T \mathrm{d}u_1\right]^{1/2} C^{1/2},
\end{aligned}$$

where C is the same constant as the one in the first criterion. By Fubini's Theorem:

$$\begin{aligned}
E_\sigma &\leqslant 2C^{1/2}\mathbb{E}\left[\int_0^{1-h_1} (y_\sigma(u_1+h_1,T)-y_\sigma(u_1,T)+g(u_1)-g(u_1+h_1))^2 \mathrm{d}u_1\right]^{1/2} \\
&\leqslant 2C^{1/2}\mathbb{E}\left[\int_0^{1-h_1} (y_\sigma(u_1+h_1,T)-y_\sigma(u_1,T))^2 \mathrm{d}u_1\right]^{1/2} \\
&\quad + 2C^{1/2}\mathbb{E}\left[\int_0^{1-h_1} (g(u_1+h_1)-g(u_1))^2 \mathrm{d}u_1\right]^{1/2}.
\end{aligned}$$

We recall inequalities (4.18) and (4.19). Therefore, there are $\alpha>0$ and $C>0$ such that for each $\sigma\in\mathbb{Q}_+$ and each $h_1>0$,

$$E_\sigma \leqslant C h_1^\alpha.$$

We deduce that for each $n\in\mathbb{N}$, by Markov's inequality,

$$\begin{aligned}
p_n := \mathbb{P}\left[\int_0^{1-\frac{1}{2^n}}\int_0^1 \sup_{t\leqslant T}|C_\sigma(u_1+\tfrac{1}{2^n},u_2,t)-C_\sigma(u_1,u_2,t)|\mathrm{d}u_1\mathrm{d}u_2 \geqslant \tfrac{1}{2^{\frac{n\alpha}{2}}}\right] &\leqslant 2^{\frac{n\alpha}{2}} C\left(\frac{1}{2^n}\right)^\alpha \\
&= \frac{C}{2^{\frac{n\alpha}{2}}}.
\end{aligned}$$

Since $\alpha>0$, $\sum_{n\geqslant 0} p_n$ converges. By Borel-Cantelli's Lemma, for each $k\geqslant 1$, there is $n_0\geqslant 0$ such that, with probability greater than $1-\frac{\delta}{2^k}$, for all $n\geqslant n_0$,

$$\int_0^{1-\frac{1}{2^n}}\int_0^1 \sup_{t\leqslant T}|C_\sigma(u_1+\tfrac{1}{2^n},u_2,t)-C_\sigma(u_1,u_2,t)|\mathrm{d}u_1\mathrm{d}u_2 \leqslant \frac{1}{2^{\frac{n\alpha}{2}}}.$$

Furthermore, up to choosing a greater n_0, we can suppose that for all $n\geqslant n_0$, we also have:

$$\int_0^1\int_0^{1-\frac{1}{2^n}} \sup_{t\leqslant T}|C_\sigma(u_1,u_2+\tfrac{1}{2^n},t)-C_\sigma(u_1,u_2,t)|\mathrm{d}u_1\mathrm{d}u_2 \leqslant \frac{1}{2^{\frac{n\alpha}{2}}}.$$

We will now extend these estimations to more general perturbations. Let $h = (h_1, h_2)$ be such that $0 < h_1 < \frac{1}{2^{n_0}}$, $0 < h_2 < \frac{1}{2^{n_0}}$. We decompose:

$$\begin{aligned} \int_0^{1-h_1}\int_0^{1-h_2} & \sup_{t\leqslant T} |C_\sigma(u_1+h_1,u_2+h_2,t) - C_\sigma(u_1,u_2,t)| \mathrm{d}u_1\mathrm{d}u_2 \\ & \leqslant \int_0^{1-h_1}\int_0^{1} \sup_{t\leqslant T} |C_\sigma(u_1+h_1,u_2,t) - C_\sigma(u_1,u_2,t)| \mathrm{d}u_1\mathrm{d}u_2 \\ & \quad + \int_0^{1}\int_0^{1-h_2} \sup_{t\leqslant T} |C_\sigma(u_1,u_2+h_2,t) - C_\sigma(u_1,u_2,t)| \mathrm{d}u_1\mathrm{d}u_2. \end{aligned} \tag{5.14}$$

Suppose $h_1 \geqslant 0$. Since $h_1 < \frac{1}{2^{n_0}}$, there exists a sequence $(\varepsilon_n)_{n>n_0}$ with values in $\{0,1\}$ such that $h_1 = \sum_{n\geqslant n_0+1} \frac{\varepsilon_n}{2^n}$. Moreover, we have for every $q \geqslant 1$:

$$\begin{aligned} & \int_0^{1-h_1}\int_0^{1} \sup_{t\leqslant T} |C_\sigma(u_1+h_1,u_2,t) - C_\sigma(u_1+\textstyle\sum_{n\geqslant n_0+q} \frac{\varepsilon_n}{2^n},u_2,t)| \mathrm{d}u_1\mathrm{d}u_2 \\ & \leqslant \sum_{k=1}^{q-1} \int_0^{1-h_1}\int_0^{1} \sup_{t\leqslant T} |C_\sigma(u_1+\textstyle\sum_{n\geqslant n_0+k} \frac{\varepsilon_n}{2^n},u_2,t) - C_\sigma(u_1+\textstyle\sum_{n\geqslant n_0+k+1} \frac{\varepsilon_n}{2^n},u_2,t)| \mathrm{d}u_1\mathrm{d}u_2 \\ & \leqslant \sum_{k=1}^{q-1} \int_0^{1-\frac{1}{2^{n_0+k}}}\int_0^{1} \sup_{t\leqslant T} |C_\sigma(u_1+\tfrac{1}{2^{n_0+k}},u_2,t) - C_\sigma(u_1,u_2,t)| \mathrm{d}u_1\mathrm{d}u_2 \leqslant \sum_{k=1}^{q-1} \frac{1}{2^{(n_0+k)\frac{\alpha}{2}}}. \end{aligned} \tag{5.15}$$

We want to let q tend to $+\infty$ in (5.15). To do that, we prove that:

$$\int_0^{1-h_1}\int_0^{1} \sup_{t\leqslant T} |C_\sigma(u_1+\textstyle\sum_{n\geqslant n_0+q} \frac{\varepsilon_n}{2^n},u_2,t) - C_\sigma(u_1,u_2,t)| \mathrm{d}u_1\mathrm{d}u_2 \underset{q\to+\infty}{\longrightarrow} 0. \tag{5.16}$$

By definition of C_σ,

$$\begin{aligned} & \int_0^{1-h_1}\int_0^{1} \sup_{t\leqslant T} |C_\sigma(u_1+\textstyle\sum_{n\geqslant n_0+q} \frac{\varepsilon_n}{2^n},u_2,t) - C_\sigma(u_1,u_2,t)| \mathrm{d}u_1\mathrm{d}u_2 \\ & \leqslant \int_0^{1-h_1}\int_0^{T} \left| \frac{1}{m_\sigma(u_1+\sum_{n\geqslant n_0+q} \frac{\varepsilon_n}{2^n},s)} - \frac{1}{m_\sigma(u_1,s)} \right| \mathrm{d}s\mathrm{d}u_1 \\ & + \int_0^{1-h_1}\int_0^{1}\int_0^{T} \frac{2}{m_\sigma(u_2,s)} \left| \frac{m_\sigma(u_1+\sum_{n\geqslant n_0+q} \frac{\varepsilon_n}{2^n},u_2,s)}{m_\sigma(u_1+\sum_{n\geqslant n_0+q} \frac{\varepsilon_n}{2^n},s)} - \frac{m_\sigma(u_1,u_2,s)}{m_\sigma(u_1,s)} \right| \mathrm{d}s\mathrm{d}u_1\mathrm{d}u_2. \end{aligned} \tag{5.17}$$

For each $s \in [0,T]$, $m_\sigma(\cdot,s)$ is right-continuous. Therefore, $m_\sigma(u_1 + \sum_{n\geqslant n_0+q} \frac{\varepsilon_n}{2^n}, s)$ converges to $m_\sigma(u_1,s)$ when $q \to +\infty$. Furthermore, there is $\beta > 1$ such that almost surely,

$$\sup_{u\in\left[0,\frac{1}{2^{n_0-1}}\right]} \int_0^{1-u}\int_0^{T} \left| \frac{1}{m_\sigma(u_1+u,s)} - \frac{1}{m_\sigma(u_1,s)} \right|^\beta \mathrm{d}s\mathrm{d}u_1 < +\infty.$$

Indeed,

$$\begin{aligned} \mathbb{E}\left[\sup_{u\in\left[0,\frac{1}{2^{n_0-1}}\right]} \int_0^{1-u}\int_0^{T} \left| \frac{1}{m_\sigma(u_1+u,s)} - \frac{1}{m_\sigma(u_1,s)} \right|^\beta \mathrm{d}s\mathrm{d}u_1 \right] & \\ \leqslant C_\beta \mathbb{E}\left[\int_0^1\int_0^T \frac{1}{m_\sigma(u_1,s)^\beta} \mathrm{d}s\mathrm{d}u_1 \right] & < +\infty, \end{aligned}$$

by Lemma 4.6. Therefore, since $\sum_{n\geqslant n_0+q}\frac{\varepsilon_n}{2^n}\leqslant h_1<\frac{1}{2^{n_0-1}}$ for every $q\geqslant 1$,

$$\begin{aligned}\int_0^{1-h_1}\int_0^T\left|\frac{1}{m_\sigma(u_1+\sum_{n\geqslant n_0+q}\frac{\varepsilon_n}{2^n},s)}-\frac{1}{m_\sigma(u_1,s)}\right|^\beta \mathrm{d}s\mathrm{d}u_1 \\ \leqslant \int_0^{1-\sum_{n\geqslant n_0+q}\frac{\varepsilon_n}{2^n}}\int_0^T\left|\frac{1}{m_\sigma(u_1+\sum_{n\geqslant n_0+q}\frac{\varepsilon_n}{2^n},s)}-\frac{1}{m_\sigma(u_1,s)}\right|^\beta \mathrm{d}s\mathrm{d}u_1 \\ \leqslant \sup_{u\in\left[0,\frac{1}{2^{n_0-1}}\right]}\int_0^{1-u}\int_0^T\left|\frac{1}{m_\sigma(u_1+u,s)}-\frac{1}{m_\sigma(u_1,s)}\right|^\beta \mathrm{d}s\mathrm{d}u_1,\end{aligned}$$

which is almost surely finite. Thus the first term of the right hand side of (5.17) tends almost surely to 0 for every $h_1<\frac{1}{2^{n_0}}$. A similar argument shows that the second term of the right hand side of (5.17) also converges to 0. Hence we have justified convergence (5.16).

When $q\to\infty$ in inequality (5.15), we obtain:

$$\int_0^{1-h_1}\int_0^1\sup_{t\leqslant T}|C_\sigma(u_1+h_1,u_2,t)-C_\sigma(u_1,u_2,t)|\mathrm{d}u_1\mathrm{d}u_2\leqslant\sum_{k=1}^{+\infty}\frac{1}{2^{(n_0+k)\frac{\alpha}{2}}}\leqslant\frac{C_\alpha}{2^{\frac{n_0\alpha}{2}}}.$$

Then, we proceed similarly for the second term of the right hand side of (5.14) and we finally obtain, for each $h=(h_1,h_2)$ such that $0<h_1<\frac{1}{2^{n_0}}$ and $0<h_2<\frac{1}{2^{n_0}}$,

$$\int_0^{1-h_1}\int_0^{1-h_2}\sup_{t\leqslant T}|C_\sigma(u_1+h_1,u_2+h_2,t)-C_\sigma(u_1,u_2,t)|\mathrm{d}u_1\mathrm{d}u_2\leqslant\frac{C}{2^{\frac{n_0\alpha}{2}}}.$$

Choosing $H=\frac{1}{2^{n_0}}$ such that $CH^{\alpha/2}\leqslant\frac{1}{k}$, we get (5.12) for each σ in $\mathbb{Q}_+$.

Conclusion of the proof. By Simon's tightness criterion on $L_1([0,1]\times[0,1],\mathcal{C}[0,T])$, the collection of laws of $(C_\sigma)_{\sigma\in\mathbb{Q}_+}$ is relatively compact in $\mathcal{P}(L_1([0,1]\times[0,1],\mathcal{C}[0,T]))$. Thus the collection of laws of $(y_\sigma,C_\sigma)_{\sigma\in\mathbb{Q}_+}$ is also relatively compact in $\mathcal{P}(L_2([0,1],\mathcal{C}[0,T])\times L_1([0,1]\times[0,1],\mathcal{C}[0,T]))$. Thus there is a subsequence, $(y_{\sigma_n},C_{\sigma_n})_{n\geqslant 1}$ converges in distribution in $L_2([0,1],\mathcal{C}[0,T])\times L_1([0,1]\times[0,1],\mathcal{C}[0,T])$. We denote by (y,C) the limit. We want to prove that for almost every $u_1,u_2\in[0,1]$, $C(u_1,u_2,\cdot)$ is the quadratic variation of $y(u_1,\cdot)-y(u_2,\cdot)$ relatively to the filtration generated by Y and C.

Let $l\geqslant 1$, $0\leqslant s_1\leqslant s_2\leqslant\ldots\leqslant s_l\leqslant s\leqslant t$ and $f_l:(L_2(0,1))^l\times L_1([0,1]\times[0,1])^l\to\mathbb{R}$ be a bounded and continuous function. For every non-negative $\psi_1,\psi_2\in L_\infty(0,1)$, we have for every $n\geqslant 1$:

$$\begin{aligned}\mathbb{E}\Bigg[\int_0^1\int_0^1\psi_1(u_1)\psi_2(u_2)\Big((y_{\sigma_n}(u_1,t)-y_{\sigma_n}(u_2,t)-g(u_1)+g(u_2))^2 \\ -(y_{\sigma_n}(u_1,s)-y_{\sigma_n}(u_2,s)-g(u_1)+g(u_2))^2-C_{\sigma_n}(u_1,u_2,t)+C_{\sigma_n}(u_1,u_2,s)\Big)\mathrm{d}u_1\mathrm{d}u_2 \\ f_l(Y_{\sigma_n}(s_1),\ldots,Y_{\sigma_n}(s_l),C_{\sigma_n}(s_1),\ldots,C_{\sigma_n}(s_l))\Bigg]=0,\end{aligned}$$

since the process $(C_{\sigma_n}(t))_{t\in[0,T]}:=(C_{\sigma_n}(\cdot,\cdot,t))_{t\in[0,T]}$ is $(\mathcal{F}_t^{\sigma_n})_{t\in[0,T]}$-adapted. By the convergence in distribution, we obtain when n goes to ∞:

$$\mathbb{E}\Bigg[\int_0^1\int_0^1 \psi_1(u_1)\psi_2(u_2)\Big((y(u_1,t)-y(u_2,t)-g(u_1)+g(u_2))^2 - (y(u_1,s)-y(u_2,s)-g(u_1)+g(u_2))^2 - C(u_1,u_2,t)+C(u_1,u_2,s)\Big)\mathrm{d}u_1\mathrm{d}u_2 \; f_l(Y(s_1),\dots,Y(s_l),C(s_1),\dots,C(s_l))\Bigg]=0.$$

By Fubini's Theorem, we obtain that for almost every $u_1,u_2 \in (0,1)$, for all rational numbers $(s_1,\dots,s_l,s,t)$ such that $0\leqslant s_1\leqslant s_2\leqslant\ldots\leqslant s_l\leqslant s\leqslant t$:

$$\mathbb{E}\Bigg[\Big((y(u_1,t)-y(u_2,t)-g(u_1)+g(u_2))^2-(y(u_1,s)-y(u_2,s)-g(u_1)+g(u_2))^2 - C(u_1,u_2,t)+C(u_1,u_2,s)\Big)f_l(Y(s_1),\dots,Y(s_l),C(s_1),\dots,C(s_l))\Bigg]=0.$$

By continuity in time, the latter equality remains true for every $0\leqslant s_1\leqslant s_2\leqslant\ldots\leqslant s_l\leqslant s\leqslant t$. Furthermore, for almost every u_1,u_2, $(C_{\sigma_n}(u_1,u_2,t))_{t\in[0,T]}$ is a non-decreasing bounded variation process. This remains true for the limit $(C(u_1,u_2,t))_{t\in[0,T]}$. Therefore, we deduce that

$$C(u_1,u_2,t)=\langle y(u_1)-y(u_2),y(u_1)-y(u_2)\rangle_t,$$

for almost every $u_1,u_2\in(0,1)$, with respect to the filtration generated by (Y,C). □

We conclude this Paragraph by using Fatou's Lemma to extend the statement of Lemma 4.15 to the limit process:

Proposition 5.11. *Let $g\in L_p(0,1)$. For all $\beta\in(0,\frac{3}{2}-\frac{1}{p})$, there is a constant $C>0$ depending only on β and $\|g\|_{L_p}$ such that for all $0\leqslant s<t\leqslant T$, we have the following inequality:*

$$\mathbb{E}\left[\int_s^t\int_0^1\frac{1}{m(u,r)^\beta}\mathrm{d}u\mathrm{d}r\right]\leqslant C\sqrt{t-s}.$$

By Burkholder-Davis-Gundy inequality, we deduce the following estimation:

Corollary 5.12. *For each $\beta\in(0,\frac{3}{2}-\frac{1}{p})$, $\sup_{t\leqslant T}\mathbb{E}\left[\int_0^1(y(u,t)-g(u))^{2\beta}\mathrm{d}u\right]<+\infty$.*

5.3 Covariation of $y(u,\cdot)$ and $y(u',\cdot)$

In this Paragraph, we want to complete the proof of property $(C5)$ of Theorem 1.4. It remains to prove the following Proposition:

Proposition 5.13. *Let y be the version in $\mathcal{D}((0,1),\mathcal{C}[0,T])$ of the limit process given by Proposition 5.5. For every $u,u'\in(0,1)$,*

$$\langle y(u,\cdot),y(u',\cdot)\rangle_{t\wedge\tau_{u,u'}}=0, \tag{5.18}$$

where $\tau_{u,u'}=\inf\{t\geqslant 0: y(u,t)=y(u',t)\}\wedge T$.

As in the previous Paragraph, we will need to prove the convergence of the joint law of y_σ and a quadratic covariation. More precisely, define:

$$K_\sigma(u,u',t):=\int_0^t\frac{m_\sigma(u,u',s)}{m_\sigma(u,s)m_\sigma(u',s)}\mathrm{d}s.$$

We state the following result:

Lemma 5.14. *For every sequence $(\sigma_n)_n$ of rational numbers tending to 0, we can extract a subsequence $(\widetilde{\sigma}_n)_n$ such that the sequence $(y_{\widetilde{\sigma}_n}, K_{\widetilde{\sigma}_n})_{n\to\infty}$ converges in distribution to (y,K) in $L_2([0,1],\mathcal{C}[0,T]) \times L_1([0,1]\times[0,1],\mathcal{C}[0,T])$, where*

$$K(u,u',t) := \langle y(u,\cdot), y(u',\cdot)\rangle_t.$$

Proof (Lemma 5.14). We follow the same structure as in the proof of Lemma 5.10. First, we define $K_{\sigma,\varepsilon} = \langle y_{\sigma,\varepsilon}(u,\cdot), y_{\sigma,\varepsilon}(u',\cdot)\rangle_t = \int_0^t \frac{m_{\sigma,\varepsilon}(u,u',s)}{(\varepsilon+m_{\sigma,\varepsilon}(u,s))(\varepsilon+m_{\sigma,\varepsilon}(u',s))}\mathrm{d}s$. We show that $K_{\sigma,\varepsilon}$ satisfies the three criteria of tightness in $L_1([0,1]\times[0,1],\mathcal{C}[0,T])$. For the first criterion, we want to bound

$$\mathbb{E}\left[\int_0^1\int_0^1 \sup_{t\leqslant T}|K_{\sigma,\varepsilon}(u,u',t)|\mathrm{d}u\mathrm{d}u'\right]$$

uniformly for $\sigma,\varepsilon\in\mathbb{Q}_+$. This follows from Kunita-Watanabe's inequality:

$$\begin{aligned}|K_{\sigma,\varepsilon}(u,u',t)| = |\langle y_{\sigma,\varepsilon}(u), y_{\sigma,\varepsilon}(u')\rangle_t| &\leqslant \langle y_{\sigma,\varepsilon}(u), y_{\sigma,\varepsilon}(u)\rangle_t^{1/2}\langle y_{\sigma,\varepsilon}(u'), y_{\sigma,\varepsilon}(u')\rangle_t^{1/2}\\ &\leqslant \langle y_{\sigma,\varepsilon}(u), y_{\sigma,\varepsilon}(u)\rangle_T^{1/2}\langle y_{\sigma,\varepsilon}(u'), y_{\sigma,\varepsilon}(u')\rangle_T^{1/2}\end{aligned}$$

and from Cauchy-Schwarz inequality:

$$\begin{aligned}\mathbb{E}\left[\int_0^1\int_0^1 \sup_{t\leqslant T}|K_{\sigma,\varepsilon}(u,u',t)|\mathrm{d}u\mathrm{d}u'\right] &\leqslant \mathbb{E}\left[\int_0^1 \langle y_{\sigma,\varepsilon}(u), y_{\sigma,\varepsilon}(u)\rangle_T\mathrm{d}u\right]\\ &= \mathbb{E}\left[\int_0^1 (y_{\sigma,\varepsilon}(u,T)-g(u))^2\mathrm{d}u\right],\end{aligned}$$

which is bounded uniformly for $\sigma,\varepsilon\in\mathbb{Q}_+$ by Corollary 4.8.

We refer to the proof of Lemma 5.10 for the second and the third criteria of tightness, and for the rest of the proof, which follows in the same way. It remains to explain why $(K(u,u',t))_{t\in[0,T]}$ is a bounded variation process for almost every $u,u'\in(0,1)$. It follows from Kunita-Watanabe's inequality that:

$$\begin{aligned}\sum_{k=0}^{p-1}|K_{\sigma,\varepsilon}(u,u',t_{k+1})-K_{\sigma,\varepsilon}(u,u',t_k)| &= \sum_{k=0}^{p-1}|\langle y_{\sigma,\varepsilon}(u), y_{\sigma,\varepsilon}(u')\rangle_{t_{k+1}} - \langle y_{\sigma,\varepsilon}(u), y_{\sigma,\varepsilon}(u')\rangle_{t_k}|\\ &\leqslant \sum_{k=0}^{p-1}\left(\int_{t_k}^{t_{k+1}}\mathrm{d}\langle y_{\sigma,\varepsilon}(u), y_{\sigma,\varepsilon}(u)\rangle_s\right)^{\frac12}\left(\int_{t_k}^{t_{k+1}}\mathrm{d}\langle y_{\sigma,\varepsilon}(u'), y_{\sigma,\varepsilon}(u')\rangle_s\right)^{\frac12}\\ &\leqslant \frac12\int_{t_0}^{t_p}\mathrm{d}\langle y_{\sigma,\varepsilon}(u), y_{\sigma,\varepsilon}(u)\rangle_s + \frac12\int_{t_0}^{t_p}\mathrm{d}\langle y_{\sigma,\varepsilon}(u'), y_{\sigma,\varepsilon}(u')\rangle_s\\ &= \frac12\int_{t_0}^{t_p}\frac{\mathrm{d}s}{M_{\sigma,\varepsilon}(u,s)} + \frac12\int_{t_0}^{t_p}\frac{\mathrm{d}s}{M_{\sigma,\varepsilon}(u',s)},\end{aligned}$$

Therefore, for every $p\geqslant 1$ and $0\leqslant t_0\leqslant t_1\leqslant\ldots\leqslant t_p$, $\sum_{k=0}^{p-1}|K(u,u',t_{k+1})-K(u,u',t_k)|\leqslant \frac12\int_{t_0}^{t_p}\frac{\mathrm{d}s}{m(u,s)}+\frac12\int_{t_0}^{t_p}\frac{\mathrm{d}s}{m(u',s)}$. By Proposition 5.11, we know that almost surely and for almost every $u\in(0,1)$, $\int_0^T\frac{\mathrm{d}s}{m(u,s)}$ is finite. Thus for almost every u and u' in $(0,1)$, $K(u,u',\cdot)$ is a bounded variation process. This concludes the proof of the Lemma. ☐

We use the latter Lemma to prove Proposition 5.13.

Proof (Proposition 5.13). By Lemma 5.14 and Skorohod's representation Theorem, we may suppose that $(y_\sigma,K_\sigma)_{\sigma\in\mathbb{Q}_+}$ converges almost surely in $L_2([0,1],\mathcal{C}[0,T])\times L_1([0,1]\times$

$[0,1], \mathcal{C}[0,T])$ to (y,K). As previously, up to extracting a subsequence, we deduce that for almost every $(\omega,u,u') \in \Omega \times [0,1] \times [0,1]$,

$$\sup_{t \leqslant T} |y_\sigma(u,t) - y(u,t)|(\omega) \underset{\sigma \to 0}{\longrightarrow} 0, \tag{5.19}$$

and

$$\sup_{t \leqslant T} |K_\sigma(u,u',t) - K(u,u',t)|(\omega) \underset{\sigma \to 0}{\longrightarrow} 0. \tag{5.20}$$

Therefore, there exists a (non-random) subset $\mathcal{A}$ of $[0,1]$, such that for every $u,u' \in \mathcal{A}$, (5.19) and (5.20) holds almost surely.

Let $u,u' \in \mathcal{A}$. If $g(u) = g(u')$ then $\tau_{u,u'} = 0$ almost surely, thus (5.18) is clear. Up to exchanging u and u', assume that $g(u) < g(u')$. Let $\delta < 2(g(u') - g(u))$. Almost surely, by (5.19), there exists σ_0 such that for all $\sigma \in (0,\sigma_0) \cap \mathbb{Q}_+$,

$$\sup_{t \leqslant T} |y_\sigma(u,t) - y(u,t)| \leqslant \frac{\delta}{4},$$
$$\sup_{t \leqslant T} |y_\sigma(u',t) - y(u',t)| \leqslant \frac{\delta}{4}.$$

Define $\tau^\delta_{u,u'} := \inf\{t \geqslant 0 : |y(u,t) - y(u',t)| \leqslant \delta\} \wedge T$. Therefore, for all $t < \tau^\delta_{u,u'}$ and for all $\sigma < \sigma_0$, $|y_\sigma(u,t) - y_\sigma(u',t)| \geqslant \frac{\delta}{2}$. Let $\sigma < \min(\sigma_0, \frac{\delta}{2})$. For all $t < \tau^\delta_{u,u'}$, we have $|y_\sigma(u,t) - y_\sigma(u',t)| \geqslant \sigma$ and thus $m_\sigma(u,u',t) = 0$, hence $K_\sigma(u,u',t) = \int_0^t \frac{m_\sigma(u,u',s)}{m_\sigma(u,s) m_\sigma(u',s)} \mathrm{d}s = 0$ for $t \leqslant \tau^\delta_{u,u'}$. By (5.20), we obtain

$$\sup_{t \leqslant \tau^\delta_{u,u'}} |K(u,u',t)| = 0.$$

Thus for every $\delta > 0$, for every $u,u' \in \mathcal{A}$ and $t \leqslant \tau^\delta_{u,u'}$, $\langle y(u), y(u') \rangle_t = 0$. Since $\tau^\delta_{u,u'} \to \tau_{u,u'}$ when $\delta \to 0$, we have for each $u,u' \in \mathcal{A}$:

$$\langle y(u), y(u') \rangle_{t \wedge \tau_{u,u'}} = 0. \tag{5.21}$$

It remains to show that (5.21) holds for every $(u,u') \in (0,1)^2$. Let $(u,u') \in (0,1)^2$. As previously, we may assume that $g(u) < g(u')$. By continuity of the processes $(y(u,t))_{t \in [0,T]}$ and $(y(u',t))_{t \in [0,T]}$, the first time of coalescence $\tau_{u,u'}$ is almost surely positive. Fix $l \geqslant 1$, $0 \leqslant s_1 \leqslant s_2 \leqslant \ldots \leqslant s_l \leqslant s \leqslant t$ and a bounded and continuous function $f_l : (L_2(0,1))^l \to \mathbb{R}$. Suppose that $s > 0$. We want to prove that:

$$\mathbb{E}\left[(y(u,t \wedge \tau_{u,u'}) y(u',t \wedge \tau_{u,u'}) - y(u,s \wedge \tau_{u,u'}) y(u',s \wedge \tau_{u,u'})) f_l(Y(s_1),\ldots,Y(s_l))\right] = 0. \tag{5.22}$$

Let $\varepsilon > 0$. For each $v \in (u,u+\varepsilon) \cap \mathcal{A}$ and $v' \in (u',u'+\varepsilon) \cap \mathcal{A}$ (since $\mathcal{A}$ is of plain measure in $(0,1)$, both sets are non-empty), since we have equality (5.21),

$$0 = \mathbb{E}\big[(y(v,t \wedge \tau_{v,v'}) y(v',t \wedge \tau_{v,v'}) - y(v,s \wedge \tau_{v,v'}) y(v',s \wedge \tau_{v,v'})) f_l(Y(s_1),\ldots,Y(s_l))\big]. \tag{5.23}$$

Let $t_0 \in (0,s)$. We define

$$\eta := \sup\{h \geqslant 0 : y(u+h,t_0) = y(u,t_0) \text{ and } y(u'+h,t_0) = y(u',t_0)\}.$$

By the coalescence property given by Proposition 5.5, under the event $\{\tau_{u,u'} > t_0\}$, we know that for every $r \geqslant t_0$, for each $v \in (u,u+\eta)$ and $v' \in (u',u'+\eta)$, $y(v,r) = y(u,r)$

and $y(v',r)=y(u',r)$, whence $\tau_{v,v'}=\tau_{u,u'}$. Thus, by equality (5.23), we deduce that for each $v\in(u,u+\varepsilon)\cap\mathcal{A}$ and $v'\in(u',u'+\varepsilon)\cap\mathcal{A}$,

$$\begin{aligned}0=\mathbb{E}\big[&\mathbb{1}_{\{\eta>\varepsilon\}}\mathbb{1}_{\{\tau_{u,u'}>t_0\}}(y(u,t\wedge\tau_{u,u'})y(u',t\wedge\tau_{u,u'})\\&\qquad-y(u,s\wedge\tau_{u,u'})y(u',s\wedge\tau_{u,u'}))f_l(Y(s_1),\dots,Y(s_l))\big]\\&+\mathbb{E}\big[\mathbb{1}_{\{\eta\leqslant\varepsilon\}\cup\{\tau_{u,u'}\leqslant t_0\}}(y(v,t\wedge\tau_{v,v'})y(v',t\wedge\tau_{v,v'})-y(v,s\wedge\tau_{v,v'})y(v',s\wedge\tau_{v,v'}))\\&\qquad f_l(Y(s_1),\dots,Y(s_l))\big].\end{aligned}\tag{5.24}$$

Let $h>0$ be such that $(u,u+\varepsilon)$ and $(u',u'+\varepsilon)$ are contained in $(h,1-h)$. Thus for every $v\in(u,u+\varepsilon)\cap\mathcal{A}$, for every $r\in[0,T]$, by inequality (5.5) and by Doob's inequality, we deduce that:

$$\mathbb{E}\left[\sup_{r\leqslant T}y(v,r)^{2\beta}\right]\leqslant\frac{2}{h}\mathbb{E}\left[\int_0^1\sup_{r\leqslant T}y(x,r)^{2\beta}\mathrm{d}x\right]\leqslant\frac{C_\beta}{h}\mathbb{E}\left[\int_0^1 y(x,T)^{2\beta}\mathrm{d}x\right]\leqslant\frac{\widetilde{C}_\beta}{h},$$

for a β arbitrarily chosen in $(1,\frac{3}{2}-\frac{1}{p})$ (by Corollary 5.12). Thus, there exists $\beta>1$ such that $\mathbb{E}\left[(y(v,t\wedge\tau_{v,v'})y(v',t\wedge\tau_{v,v'}))^\beta\right]$ is uniformly bounded for $v\in(u,u+\varepsilon)$ and $v'\in(u',u'+\varepsilon)$. Let $\alpha=1-\frac{1}{\beta}$. Therefore, we deduce from (5.24) that there is a constant C depending only on u, u' and α such that:

$$\begin{aligned}\mathbb{E}\big[&\mathbb{1}_{\{\eta>\varepsilon\}}\mathbb{1}_{\{\tau_{u,u'}>t_0\}}(y(u,t\wedge\tau_{u,u'})y(u',t\wedge\tau_{u,u'})-y(u,s\wedge\tau_{u,u'})y(u',s\wedge\tau_{u,u'}))\\&f_l(Y(s_1),\dots,Y(s_l))\big]\leqslant C\left(\mathbb{P}\left[\eta\leqslant\varepsilon\right]^\alpha+\mathbb{P}\left[\tau_{u,u'}\leqslant t_0\right]^\alpha\right).\end{aligned}\tag{5.25}$$

We divide the left hand side of inequality (5.25) into two parts by writing

$$\mathbb{1}_{\{\eta>\varepsilon\}}\mathbb{1}_{\{\tau_{u,u'}>t_0\}}=1-\mathbb{1}_{\{\eta\leqslant\varepsilon\}\cup\{\tau_{u,u'}\leqslant t_0\}}$$

and we estimate the second term in the same way as above. We deduce that there is a constant C' such that:

$$\begin{aligned}\mathbb{E}&\left[(y(u,t\wedge\tau_{u,u'})y(u',t\wedge\tau_{u,u'})-y(u,s\wedge\tau_{u,u'})y(u',s\wedge\tau_{u,u'}))f_l(Y(s_1),\dots,Y(s_l))\right]\\&\leqslant C'\left(\mathbb{P}\left[\eta\leqslant\varepsilon\right]^\alpha+\mathbb{P}\left[\tau_{u,u'}\leqslant t_0\right]^\alpha\right).\end{aligned}$$

Let $\delta>0$. Since $\tau_{u,u'}>0$ almost surely, we choose $t_0\in(0,s)$ such that $\mathbb{P}\left[\tau_{u,u'}\leqslant t_0\right]^\alpha\leqslant\delta$. Since $t_0>0$, we know by Proposition 5.1 that $y(\cdot,t_0)$ is almost surely a step function, so $\eta>0$ almost surely. Therefore, we can choose $\varepsilon>0$ so that $\mathbb{P}\left[\eta\leqslant\varepsilon\right]^\alpha\leqslant\delta$. This concludes the proof of equality (5.22).

Recall that we suppose that $t\geqslant s>0$. By continuity of time of $y(u,\cdot)$ and $y(u',\cdot)$, equality (5.22) also holds for $s=0$. Therefore, $y(u,t\wedge\tau_{u,u'})y(u',t\wedge\tau_{u,u'})$ is a $(\mathcal{F}_t)_{t\in[0,T]}$-martingale and $\langle y(u),y(u')\rangle_{t\wedge\tau_{u,u'}}=0$. This concludes the proof of Proposition 5.13. □

A Appendix: Itô's formula for the Wasserstein diffusion

Let $g\in\mathcal{L}^{\uparrow}_{2+}[0,1]$. We assume, to simplify the notations, that $g(1)$ is finite, but the proof can be easily adapted to functions g with $g(u)\underset{u\to1}{\longrightarrow}+\infty$. Let y be a process in $\mathcal{D}([0,1],\mathcal{C}[0,T])$ satisfying $(i)-(iv)$ (see Introduction).

Recall that the process $y(\cdot,t)_{t\in[0,T]}$ can be considered as the quantile function of $(\mu_t)_{t\in[0,T]}$, by setting $\mu_t = \mathrm{Leb}\,|_{[0,1]} \circ y(\cdot,t)^{-1}$. The latter process has every feature of a Wasserstein diffusion. We describe in this Paragraph the dynamics of the process $(\mu_t)_{t\in[0,T]}$, after having introduced a differential calculus on $\mathcal{P}_2(\mathbb{R})$ due to Lions (see [15, 5]). We prove that, for a smooth function $U : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$, the process $(U(\mu_t))_{t\in[0,T]}$ is a semi-martingale with quadratic variation proportional to the square of the gradient of U (see Theorem A.3). This result is a generalization of the formula given by Konarovskyi and von Renesse in [13]. We compare it to a similar result obtained by von Renesse and Sturm [22] for the Wasserstein diffusion on $[0,1]$ (see Remark A.4).

In order to describe the dynamics of $(\mu_t)_{t\in[0,T]}$, we begin by a discretization in space and by writing the classical Itô formula for that discretized process. Let introduce $\widetilde{\mu}^n_t := \frac{1}{n}\sum_{k\in[n]} \delta_{y(\frac{k}{n},t)}$, where $[n]$ denotes the set $\{1,\dots,n\}$. Fix $U : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ a continuous function, with respect to the Wasserstein distance W_2 on $\mathcal{P}_2(\mathbb{R})$. Let define $U^n(x_1,\dots,x_n) := U(\frac{1}{n}\sum_{j\in[n]}\delta_{x_j})$. Remark that $U(\widetilde{\mu}^n_t) = U^n\left(y(\frac{1}{n},t), y(\frac{2}{n},t),\dots,y(1,t)\right)$. Assuming that U^n belongs to $C^2(\mathbb{R}^n)$, and using that $y(\frac{k}{n},\cdot)$ is a square integrable continuous martingale on $[0,T]$, we have (recall that $g(1)$ is finite):

$$\begin{aligned} U(\widetilde{\mu}^n_t) =& U^n(g(\tfrac{1}{n}),\dots,g(1)) + \sum_{k\in[n]} \int_0^t \partial_k U^n(y(\tfrac{1}{n},s),\dots,y(1,s))\,\mathrm{d}y(\tfrac{k}{n},s) \\ &+ \frac{1}{2}\sum_{k,l\in[n]} \int_0^t \partial^2_{k,l} U^n(y(\tfrac{1}{n},s),\dots,y(1,s))\,\mathrm{d}\langle y(\tfrac{k}{n},\cdot), y(\tfrac{l}{n},\cdot)\rangle_s. \end{aligned} \tag{A.1}$$

In order to write the derivatives of U^n in terms of derivatives of U, we should introduce a differential calculus on $\mathcal{P}_2(\mathbb{R})$, well-adapted to the differentiation of empirical measures. P.L. Lions introduces in his lectures at Collège de France (see Section 6.1 of Cardaliaguet's notes [5]) a differential calculus on $\mathcal{P}_2(\mathbb{R})$ by using the Hilbertian structure of $L_2(\Omega)$. We set $\widetilde{U}(X) := U(\mathrm{Law}(X))$ for all $X \in L_2(\Omega)$.

A function $U : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ is said to be *L-differentiable* (or differentiable in the sense of Lions) at a point $\mu_0 \in \mathcal{P}_2(\mathbb{R})$ if there is a random variable X_0 with law μ_0 such that $\widetilde{U}$ is Fréchet-differentiable at X_0. The definition does not depend on the choice of the representative X_0 of the law μ_0, and if X_0 and X_1 have the same law, then the laws of $D\widetilde{U}(X_0)$ and $D\widetilde{U}(X_1)$ are equal (see e.g. [5]). Furthermore, if $D\widetilde{U} : L_2(\Omega) \to L_2(\Omega)$ is a continuous function, then for all $\mu_0 \in \mathcal{P}_2(\mathbb{R})$, there exists a measurable function $\mathbb{R} \to \mathbb{R}$, denoted by $\partial_\mu U(\mu_0)$, such that for each $X \in L_2(\Omega)$ with law μ_0, we have $D\widetilde{U}(X) = \partial_\mu U(\mu_0)(X)$ almost surely (see [5]).

In [6], Carmona and Delarue prove that the L-differentiability of $U : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ implies the differentiability of U^n on $\mathbb{R}^n$, and that we have for each $k \in [n]$:

$$\partial_k U^n(x_1,\dots,x_n) = \frac{1}{n}\partial_\mu U(\tfrac{1}{n}\textstyle\sum_{j\in[n]}\delta_{x_j})(x_k).$$

Furthermore, assume that U is L-differentiable and that $(\mu,v) \in \mathcal{P}_2(\mathbb{R}) \times \mathbb{R} \mapsto \partial_\mu U(\mu)(v) \in \mathbb{R}$ is continuous. Moreover, we assume that for every $\mu \in \mathcal{P}_2(\mathbb{R})$, the map $v \in \mathbb{R} \mapsto \partial_\mu U(\mu)(v) \in \mathbb{R}$ is differentiable on $\mathbb{R}$ in the classical sense and that its derivative is given by a jointly continuous function $(\mu,v) \mapsto \partial_v\partial_\mu U(\mu)(v)$. We also assume that for every $v \in \mathbb{R}$, the map $\mu \mapsto \partial_\mu U(\mu)(v)$ is L-differentiable and its derivative is denoted by $(\mu,v,v') \mapsto \partial^2_\mu U(\mu)(v,v')$. Then, U^n is $\mathcal{C}^2$ on $\mathbb{R}^n$ and for all $k,l \in [n]$:

$$\partial^2_{k,l} U^n(x_1,\dots,x_n) = \frac{1}{n}\partial_v\partial_\mu U(\tfrac{1}{n}\textstyle\sum_{j\in[n]}\delta_{x_j})(x_k)\mathbb{1}_{\{k=l\}} + \displaystyle\frac{1}{n^2}\partial^2_\mu U(\tfrac{1}{n}\textstyle\sum_{j\in[n]}\delta_{x_j})(x_k,x_l).$$

Therefore, we obtain from equation (A.1):

$$\begin{aligned} U(\widetilde{\mu}^n_t) = U(\widetilde{\mu}^n_0) &+ \frac{1}{n}\sum_{k\in[n]}\int_0^t \partial_\mu U(\widetilde{\mu}^n_s)(y(\tfrac{k}{n},s))\mathrm{d}y(\tfrac{k}{n},s) \\ &+ \frac{1}{2n}\sum_{k\in[n]}\int_0^t \partial_v\partial_\mu U(\widetilde{\mu}^n_s)(y(\tfrac{k}{n},s))\frac{\mathrm{d}s}{m(\frac{k}{n},s)} \\ &+ \frac{1}{2n^2}\sum_{k,l\in[n]}\int_0^t \partial^2_\mu U(\widetilde{\mu}^n_s)(y(\tfrac{k}{n},s),y(\tfrac{l}{n},s))\frac{\mathbb{1}_{\{\tau_{\frac{k}{n},\frac{l}{n}}\leqslant s\}}}{m(\frac{k}{n},s)}\mathrm{d}s. \end{aligned} \tag{A.2}$$

By property of coalescence, if $\tau_{\frac{k}{n},\frac{l}{n}} \leqslant s$, we have $y(\frac{k}{n},s) = y(\frac{l}{n},s)$, so that the last term in the latter equation is equal to:

$$\frac{1}{2n}\sum_{k\in[n]}\int_0^t \partial^2_\mu U(\widetilde{\mu}^n_s)(y(\tfrac{k}{n},s),y(\tfrac{k}{n},s))\frac{\frac{1}{n}\sum_{l\in[n]}\mathbb{1}_{\{\tau_{\frac{k}{n},\frac{l}{n}}\leqslant s\}}}{m(\frac{k}{n},s)}\mathrm{d}s.$$

Observe that the difference between $\frac{1}{n}\sum_{l\in[n]}\mathbb{1}_{\{\tau_{\frac{k}{n},\frac{l}{n}}\leqslant s\}}$ and $m(\frac{k}{n},s) = \int_0^1 \mathbb{1}_{\{\tau_{\frac{k}{n},u}\leqslant s\}}\mathrm{d}u$ is bounded by $\frac{2}{n}$, since the set $\{u : \tau_{\frac{k}{n},u} \leqslant s\}$ is an interval.

We want to let n tend to $+\infty$ in order to obtain an Itô formula for the limit process. We start by proving the convergence of a subsequence of $((\widetilde{\mu}^n_t)_{t\in[0,T]})_{n\geqslant 1}$ to $(\mu_t)_{t\in[0,T]}$ with respect to the L_2-Wasserstein distance.

Proposition A.1. *There exists a subsequence* $((\widetilde{\mu}^{\varphi(n)}_t)_{t\in[0,T]})_{n\geqslant 1}$ *of* $((\widetilde{\mu}^n_t)_{t\in[0,T]})_{n\geqslant 1}$ *such that, for almost every* $t \in [0,T]$*, the sequence* $(\widetilde{\mu}^{\varphi(n)}_t)_{n\geqslant 1}$ *converges almost surely to* μ_t *with respect to the Wasserstein distance* W_2*.*

Remark A.2. We point out that the extraction function φ does not depend on $t \in [0,T]$.

Proof. To obtain the statement of the Proposition, it is sufficient to prove that:

$$\mathbb{E}\left[\int_0^T W_2(\widetilde{\mu}^n_t,\mu_t)^2\mathrm{d}t\right] \to 0.$$

Let V be a uniform random variable on $[0,1]$, defined on a probability space $(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{\mathbb{P}})$. Therefore, μ_t is the law of $y(V,t)$ and $\widetilde{\mu}^n_t$ the law of $\sum_{k\in[n]}\mathbb{1}_{\{\frac{k-1}{n}<V\leqslant\frac{k}{n}\}}y(\frac{k}{n},t)$. Hence we have:

$$\begin{aligned} W_2(\widetilde{\mu}^n_t,\mu_t)^2 &\leqslant \widetilde{\mathbb{E}}\left[\left(\sum_{k\in[n]}\mathbb{1}_{\{\frac{k-1}{n}<V\leqslant\frac{k}{n}\}}y(\tfrac{k}{n},t) - y(V,t)\right)^2\right] \\ &= \int_0^1\sum_{k\in[n]}\mathbb{1}_{\{\frac{k-1}{n}<u\leqslant\frac{k}{n}\}}|y(\tfrac{k}{n},t)-y(u,t)|^2\mathrm{d}u. \end{aligned}$$

Therefore, it is sufficient to show that:

$$\mathbb{E}\left[\int_0^T\int_0^1\sum_{k\in[n]}\mathbb{1}_{\{\frac{k-1}{n}<u\leqslant\frac{k}{n}\}}|y(\tfrac{k}{n},t)-y(u,t)|^2\mathrm{d}u\mathrm{d}t\right] \underset{n\to+\infty}{\longrightarrow} 0. \tag{A.3}$$

Fixing $u \in (0,1), t\in(0,T)$, $\sum_{k\in[n]}\mathbb{1}_{\{\frac{k-1}{n}<u\leqslant\frac{k}{n}\}}|y(\frac{k}{n},t)-y(u,t)|^2$ converges almost surely to 0 by the right-continuity of $y(\cdot,t)$ at point u. To prove (A.3), we have to show a uniform

integrability property, *i.e.* that for a certain $\beta > 1$,

$$\sup_{n\geqslant 1} \mathbb{E}\left[\left(\int_0^T \int_0^1 \sum_{k\in[n]} \mathbb{1}_{\{\frac{k-1}{n}<u\leqslant\frac{k}{n}\}}|y(\tfrac{k}{n},t)-y(u,t)|^2\mathrm{d}u\mathrm{d}t\right)^\beta\right] < +\infty.$$

We compute:

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^T\int_0^1 \sum_{k\in[n]} \mathbb{1}_{\{\frac{k-1}{n}<u\leqslant\frac{k}{n}\}}|y(\tfrac{k}{n},t)-y(u,t)|^2\mathrm{d}u\mathrm{d}t\right)^\beta\right]^{1/(2\beta)}\\
&\quad\leqslant T^{\frac{\beta-1}{2\beta}}\,\mathbb{E}\left[\int_0^T\int_0^1 \sum_{k\in[n]} \mathbb{1}_{\{\frac{k-1}{n}<u\leqslant\frac{k}{n}\}}|y(\tfrac{k}{n},t)-y(u,t)|^{2\beta}\mathrm{d}u\mathrm{d}t\right]^{1/(2\beta)}\\
&\qquad\leqslant T^{\frac{\beta-1}{2\beta}}\,\mathbb{E}\left[\int_0^T\int_0^1 \sum_{k\in[n]} \mathbb{1}_{\{\frac{k-1}{n}<u\leqslant\frac{k}{n}\}}|M_0|^{2\beta}\mathrm{d}u\mathrm{d}t\right]^{1/(2\beta)}\\
&\qquad\quad+T^{\frac{\beta-1}{2\beta}}\,\mathbb{E}\left[\int_0^T\int_0^1 \sum_{k\in[n]} \mathbb{1}_{\{\frac{k-1}{n}<u\leqslant\frac{k}{n}\}}|M_t-M_0|^{2\beta}\mathrm{d}u\mathrm{d}t\right]^{1/(2\beta)},
\end{aligned}$$

where $M_t = y(\frac{k}{n},t)-y(u,t)$. Recall that by property (i) of the process y, $M_0 = g(\frac{k}{n})-g(u)$. We deduce that:

$$\mathbb{E}\left[\int_0^T\int_0^1 \sum_{k\in[n]} \mathbb{1}_{\{\frac{k-1}{n}<u\leqslant\frac{k}{n}\}}|g(\tfrac{k}{n})-g(u)|^{2\beta}\mathrm{d}u\mathrm{d}t\right] \leqslant TC_\beta\mathbb{E}\left[\int_0^1 g(u)^{2\beta}\mathrm{d}u\right].$$

Since g belongs to $\mathcal{L}_{2+}^{\uparrow}[0,1]$, there exists $p>2$ such that $g\in L_p(0,1)$. Therefore, we can choose $\beta>1$ such that $2\beta\leqslant p$. By Burkholder-Davis-Gundy inequality and the martingale property of M, we have:

$$\mathbb{E}\left[(M_t-M_0)^{2\beta}\right] \leqslant C_\beta\mathbb{E}\left[\langle M,M\rangle_t^\beta\right].$$

By property (iv),

$$\begin{aligned}
\langle M,M\rangle_t &= \int_0^t \frac{\mathrm{d}s}{m(\frac{k}{n},s)} + \int_0^t \frac{\mathrm{d}s}{m(u,s)} - 2\int_0^t \frac{\mathbb{1}_{\{\tau_{\frac{k}{n},u}\leqslant s\}}}{m(\frac{k}{n},s)^{1/2}m(u,s)^{1/2}}\mathrm{d}s\\
&\leqslant \int_0^t \frac{\mathrm{d}s}{m(\frac{k}{n},s)} + \int_0^t \frac{\mathrm{d}s}{m(u,s)},
\end{aligned}$$

so that there is a constant C_β satisfying:

$$\mathbb{E}\left[\langle M,M\rangle_t^\beta\right] \leqslant C_\beta t^{\beta-1}\mathbb{E}\left[\int_0^t \frac{\mathrm{d}s}{m(\frac{k}{n},s)^\beta} + \int_0^t \frac{\mathrm{d}s}{m(u,s)^\beta}\right].$$

To conclude, we use the following statement: provided $\beta < \frac{3}{2}-\frac{1}{p}$, there is a constant C_β such that for each t and u:

$$\mathbb{E}\left[\int_0^1\int_0^t \frac{\mathrm{d}s}{m(u,s)^\beta}\mathrm{d}u\right] \leqslant C_\beta\sqrt{t}. \tag{A.4}$$

This statement is Proposition 5.11 for the limit process that we constructed in this paper, or in [11, Prop. 4.3] for the process constructed by Konarovskyi. This completes the proof. □

By similar arguments of convergence, equation (A.2) leads to the following Itô formula for $(\mu_t)_{t\in[0,T]}$, by letting n tend to ∞. The estimation (A.4) is the key of the proof of those convergences.

Theorem A.3. *Let $U : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ be smooth enough so that U and its derivatives $\partial_\mu U$, $\partial_v \partial_\mu U$ and $\partial_\mu^2 U$ exist, are uniformly continuous and bounded. Almost surely, for each $t \in [0,T]$, we have:*

$$U(\mu_t) = U(\mu_0) + \int_0^1 \int_0^t \partial_\mu U(\mu_s)(y(u,s))\mathrm{d}y(u,s)\mathrm{d}u + \frac{1}{2}\int_0^1 \int_0^t \frac{\partial_v \partial_\mu U(\mu_s)(y(u,s))}{m(u,s)}\mathrm{d}s\mathrm{d}u + \frac{1}{2}\int_0^1 \int_0^t \partial_\mu^2 U(\mu_s)(y(u,s),y(u,s))\mathrm{d}s\mathrm{d}u,$$

where $\int_0^1 \int_0^t \partial_\mu U(\mu_s)(y(u,s))\mathrm{d}y(u,s)\mathrm{d}u$ is a square integrable continuous martingale with a quadratic variation process equal to $t \mapsto \int_0^1 \int_0^t (\partial_\mu U(\mu_s))^2 (y(u,s))\mathrm{d}s\mathrm{d}u$.

Remark A.4. Choose in particular $U : \mu \mapsto V\left(\int_\mathbb{R} \alpha_1 \mathrm{d}\mu, \dots, \int_\mathbb{R} \alpha_m \mathrm{d}\mu\right) =: V(\int \overrightarrow{\alpha}\mathrm{d}\mu)$, where $V \in \mathcal{C}^2(\mathbb{R}^m)$ and $\alpha_1, \dots, \alpha_m$ are bounded $\mathcal{C}^2(\mathbb{R})$-functions, with bounded first and second-order derivatives. In this case, $\partial_\mu U(\mu)(v) = \sum_{i=1}^m \partial_i V\left(\int \overrightarrow{\alpha}\mathrm{d}\mu\right)\alpha_i'(v)$ for all $\mu \in \mathcal{P}_2(\mathbb{R})$ and $v \in \mathbb{R}$. Computing the second-order derivatives, we show that

$$U(\mu_t) - U(\mu_0) - \frac{1}{2}\int_0^t \mathcal{L}_1 U(\mu_s)\mathrm{d}s - \frac{1}{2}\int_0^t \mathcal{L}_2 U(\mu_s)\mathrm{d}s$$

is a martingale with quadratic variation process

$$t \mapsto \int_0^t \int_0^1 \left(\sum_{i=1}^m \partial_i V\left(\int \overrightarrow{\alpha}\mathrm{d}\mu_s\right)\alpha_i'(y(u,s))\right)^2 \mathrm{d}u\mathrm{d}s$$

and an operator $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$ of the form $\mathcal{L}_1 U(\mu_s) := \sum_{i=1}^m \partial_i V\left(\int \overrightarrow{\alpha}\mathrm{d}\mu_s\right)\int_0^1 \frac{\alpha_i''(y(u,s))}{m(u,s)}\mathrm{d}u$ and $\mathcal{L}_2 U(\mu_s) := \sum_{i,j=1}^m \partial_{i,j}^2 V\left(\int \overrightarrow{\alpha}\mathrm{d}\mu_s\right)\int_0^1 \alpha_i'(y(u,s))\alpha_j'(y(u,s))\mathrm{d}u$.

Remark that we have some restrictions on the domain of the generator $\mathcal{L}_1$. We know that for measures with finite support, $\int_0^1 \frac{\mathrm{d}u}{m(u,s)}$ is finite and is equal to the cardinality of the support (see the Paragraph preceding Corollary 5.3). The fact that the generator of the martingale problem is not defined on the whole Wasserstein space is related to the fact that the process $(\mu_t)_{t\in[0,T]}$ takes values, for every positive time t, on the space of measures with finite support.

We compare this result to Theorem 7.17 in [22]. The generator of the martingale in the case of von Renesse and Sturm's Wasserstein diffusion is $\mathbb{L} = \mathbb{L}_1 + \mathbb{L}_2 + \beta\mathbb{L}_3$, with $\mathbb{L}_1 = \mathcal{L}_2$ and $\mathbb{L}_3$ similar to $\mathcal{L}_1$ up to the lack of the mass function, whereas $\mathbb{L}_2$, which is the part of the generator considering the gaps of the measure μ, does not appear in our model.

References

[1] Luigi Ambrosio, Nicola Gigli, and Giuseppe Savaré, *Gradient flows in metric spaces and in the space of probability measures*, second ed., Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 2008. MR-2401600

[2] Sebastian Andres and Max-K. von Renesse, *Particle approximation of the Wasserstein diffusion*, J. Funct. Anal. **258** (2010), no. 11, 3879–3905. MR-2606878

[3] Richard Alejandro Arratia, *Coalescing brownian motions on the line*, ProQuest LLC, Ann Arbor, MI, 1979, Thesis (Ph.D.)–The University of Wisconsin - Madison. MR-2630231

[4] Patrick Billingsley, *Convergence of probability measures*, second ed., Wiley Series in Probability and Statistics: Probability and Statistics, John Wiley & Sons, Inc., New York, 1999, A Wiley-Interscience Publication. MR-1700749

[5] Pierre Cardaliaguet, *Notes on Mean Field Games (from P.-L. Lions' lectures at collège de France)*, www.ceremade.dauphine.fr/~cardalia/, 2013.

[6] René Carmona and François Delarue, *Probabilistic theory of mean field games with applications. I*, Probability Theory and Stochastic Modelling, vol. 83, Springer, Cham, 2018, Mean field FBSDEs, control, and games. MR-3752669

[7] Andrey A. Dorogovtsev, *One Brownian stochastic flow*, Theory Stoch. Process. **10** (2004), no. 3-4, 21–25. MR-2329772

[8] Masatoshi Fukushima, Yoichi Oshima, and Masayoshi Takeda, *Dirichlet forms and symmetric Markov processes*, extended ed., De Gruyter Studies in Mathematics, vol. 19, Walter de Gruyter & Co., Berlin, 2011. MR-2778606

[9] Leszek Gawarecki and Vidyadhar Mandrekar, *Stochastic differential equations in infinite dimensions with applications to stochastic partial differential equations*, Probability and its Applications (New York), Springer, Heidelberg, 2011. MR-2560625

[10] V. V. Konarovskii, *On an infinite system of diffusing particles with coalescing*, Theory Probab. Appl. **55** (2011), no. 1, 134–144. MR-2768524

[11] Vitalii Konarovskyi, *On asymptotic behavior of the modified Arratia flow*, Electron. J. Probab. **22** (2017), Paper No. 19, 31. MR-3622889

[12] Vitalii Konarovskyi, *A system of coalescing heavy diffusion particles on the real line*, Ann. Probab. **45** (2017), no. 5, 3293–3335. MR-3706744

[13] Vitalii Konarovskyi and Max-K von Renesse, *Modified Massive Arratia Flow and Wasserstein Diffusion*, Communications on Pure and Applied Mathematics (2018), doi:10.1002/cpa.21758.

[14] Yves Le Jan and Olivier Raimond, *Flows, coalescence and noise*, Ann. Probab. **32** (2004), no. 2, 1247–1315. MR-2060298

[15] Pierre-Louis Lions, *Cours au Collège de France*, www.college-de-france.fr.

[16] James Norris and Amanda Turner, *Weak convergence of the localized disturbance flow to the coalescing Brownian flow*, Ann. Probab. **43** (2015), no. 3, 935–970. MR-3342655

[17] Vladimir V. Piterbarg, *Expansions and contractions of isotropic stochastic flows of homeomorphisms*, Ann. Probab. **26** (1998), no. 2, 479–499. MR-1626162

[18] Jacques Simon, *Compact sets in the space $L^p(0,T;B)$*, Ann. Mat. Pura Appl. (4) **146** (1987), 65–96. MR-916688

[19] Karl-Theodor Sturm, *A monotone approximation to the Wasserstein diffusion*, Singular phenomena and scaling in mathematical models, Springer, Cham, 2014, pp. 25–48. MR-3205035

[20] Cédric Villani, *Topics in optimal transportation*, Graduate Studies in Mathematics, vol. 58, American Mathematical Society, Providence, RI, 2003. MR-1964483

[21] Cédric Villani, *Optimal transport*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 338, Springer-Verlag, Berlin, 2009, Old and new. MR-2459454

[22] Max-K. von Renesse and Karl-Theodor Sturm, *Entropic measure and Wasserstein diffusion*, Ann. Probab. **37** (2009), no. 3, 1114–1191. MR-2537551

[23] John B. Walsh, *An introduction to stochastic partial differential equations*, École d'été de probabilités de Saint-Flour, XIV—1984, Lecture Notes in Math., vol. 1180, Springer, Berlin, 1986, pp. 265–439. MR-876085

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