

Local law for random Gram matrices

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Abstract

We prove a local law in the bulk of the spectrum for random Gram matrices XX^* , a generalization of sample covariance matrices, where X is a large matrix with independent, centered entries with arbitrary variances. The limiting eigenvalue density that generalizes the Marchenko-Pastur law is determined by solving a system of nonlinear equations. Our entrywise and averaged local laws are on the optimal scale with the optimal error bounds. They hold both in the square case (hard edge) and in the properly rectangular case (soft edge). In the latter case we also establish a macroscopic gap away from zero in the spectrum of XX^* .

Keywords: capacity of MIMO channels; Marchenko-Pastur law; hard edge; soft edge; general variance profile.

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1 Introduction

Random matrices were introduced in pioneering works by Wishart [43] and Wigner [42] for applications in mathematical statistics and nuclear physics, respectively. Wigner argued that the energy level statistics of large atomic nuclei could be described by the eigenvalues of a large *Wigner matrix*, i.e., a hermitian matrix $H = (h_{ij})_{i,j=1}^N$ with centered, identically distributed and independent entries (up to the symmetry constraint $H = H^*$). He proved that the empirical spectral measure (or density of states) converges to the *semicircle law* as the dimension of the matrix N goes to infinity. Moreover, he postulated that the statistics of the gaps between consecutive eigenvalues depend only on the symmetry type of the matrix and are independent of the distribution of the entries in the large N limit. The precise formulation of this phenomenon is called the *Wigner-Dyson-Mehta universality conjecture*, see [33].

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Historically, the second main class of random matrices is the one of *sample covariance matrices*. These are of the form XX^* where X is a $p \times n$ matrix with centered, identically distributed independent entries. In statistics context, its columns contain n samples of a p -dimensional data vector. In the regime of high dimensional data, i.e., in the limit when $n, p \rightarrow \infty$ in such a way that the ratio p/n converges to a constant, the empirical spectral measure of XX^* was explicitly identified by Marchenko and Pastur [32]. Random matrices of the form XX^* also appear in the theory of wireless communication; the spectral density of these matrices is used to compute the transmission capacity of a Multiple Input Multiple Output (MIMO) channel. This fundamental connection between random matrix theory and wireless communication was established by Telatar [39] and Foschini [23, 22] (see also [40] for a review). In this model, the element x_{ij} of the *channel matrix* X represents the transmission coefficient from the j -th transmitter to the i -th receiver antenna. The received signal is given by the linear relation $y = Xs + w$, where s is the input signal and w is a Gaussian noise with variance σ^2 . In case of i.i.d. Gaussian input signals, the channel capacity is given by

$$\text{Cap} = \frac{1}{p} \log \det \left(I + \sigma^{-2} XX^* \right). \quad (1.1)$$

The assumption in these models that the matrix elements of H or X have identical distribution is a simplification that does not hold in many applications. In Wigner's model, the matrix elements h_{ij} represent random quantum transition rates between physical states labelled by i and j and their distribution may depend on these states. Analogously, the transmission coefficients in X may have different distributions. This leads to the natural generalizations of both classes of random matrices by allowing for general variances, $s_{ij} := \mathbb{E}|h_{ij}|^2$ and $s_{ij} := \mathbb{E}|x_{ij}|^2$, respectively. We will still assume the independence of the matrix elements and their zero expectation. Under mild conditions on the variance matrix $S = (s_{ij})$, the limiting spectral measure depends only on the second moments, i.e., on S , and otherwise it is independent of the fine details of the distributions of the matrix elements. However, in general there is no explicit formula for the limiting spectral measure. In fact, the only known way to find it in the general case is to solve a system of nonlinear deterministic equations, known as the Dyson (or Schwinger-Dyson) equation in this context, see [8, 41, 24, 30].

For the generalization of Wigner's model, the Dyson equation is a system of equations of the form

$$-\frac{1}{m_i(z)} = z + \sum_{j=1}^N s_{ij} m_j(z), \quad \text{for } i = 1, \dots, N, \quad z \in \mathbb{H}, \quad (1.2)$$

where z is a complex parameter in the upper half plane $\mathbb{H} := \{z \in \mathbb{C} : \text{Im } z > 0\}$. The average $\langle m(z) \rangle = \frac{1}{N} \sum_i m_i(z)$ in the large N limit gives the Stieltjes transform of the limiting spectral density, which then can be computed by inverting the Stieltjes transform. In fact, $m_i(z)$ approximates individual diagonal matrix elements $G_{ii}(z)$ of the resolvent $G(z) = (H - z)^{-1}$, thus the solution of (1.2) gives much more information on H than merely the spectral density. In the case when S is a stochastic matrix, i.e., $\sum_j s_{ij} = 1$ for every i , the solution $m_i(z)$ to (1.2) is independent of i and the density is still the semicircle law. The corresponding generalized Wigner matrix was introduced in [18] and the optimal local law was proven in [19, 20]. For the general case, a detailed analysis of (1.2) and the shapes of the possible density profiles was given in [2, 3] with the optimal local law in [4].

Considering the XX^* model with a general variance matrix for X , we note that in statistical applications the entries of X within the same row still have the same variance, i.e., $s_{ik} = s_{il}$ for all i and all k, l . However, beyond statistics, for example modeling the

capacity of MIMO channels, applications require to analyze the spectrum of XX^* with a completely general variance profile for X [28, 13]. These are called *random Gram matrices*, see e.g. [24, 26]. The corresponding Dyson equation is (see [24, 13, 40] and references therein)

$$-\frac{1}{m_i(\zeta)} = \zeta - \sum_{k=1}^n s_{ik} \frac{1}{1 + \sum_{j=1}^n s_{jk} m_j(\zeta)}, \quad \text{for } i = 1, \dots, p, \quad \zeta \in \mathbb{H}. \quad (1.3)$$

We have $m_i(\zeta) \approx (XX^* - \zeta)_{ii}^{-1}$ and the average of $m_i(\zeta)$ yields the Stieltjes transform of the spectral density exactly as in case of the Wigner-type ensembles. In fact, there is a direct link between these two models: Girko's symmetrization trick reduces (1.3) to studying (1.2) on \mathbb{C}^N with $N = n + p$, where S and H are replaced by

$$\mathfrak{S} = \begin{pmatrix} 0 & S \\ S^t & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix}, \quad (1.4)$$

respectively, and $z^2 = \zeta$.

The limiting spectral density, also called the *global law*, is typically the first question one asks about random matrix ensembles. It can be strengthened by considering its *local* versions. In most cases, it is expected that the deterministic density computed via the Dyson equation accurately describes the eigenvalue density down to the smallest possible scale which is slightly above the typical eigenvalue spacing (we choose the standard normalization such that the spacing in the bulk spectrum is of order $1/N$). This requires to understand the trace of the resolvent $G(z)$ at a spectral parameter very close to the real axis, down to the scales $\text{Im } z \gg 1/N$. Additionally, *entry-wise local laws* and *isotropic local laws*, i.e., controlling individual matrix elements $G_{ij}(z)$ and bilinear forms $\langle v, G(z)w \rangle$, carry important information on eigenvectors and allow for perturbation theory. Moreover, effective error bounds on the speed of convergence as N goes to infinity are also of great interest.

Local laws have also played a crucial role in the recent proofs of the Wigner-Dyson-Mehta conjecture. The three-step approach, developed in a series of works by Erdős, Schlein, Yau and Yin [15, 16] (see [17] for a review), was based on establishing the local law as the first step. Similar input was necessary in the alternative approach by Tao and Vu in [37, 38].

In this paper, we establish the optimal local law for random Gram matrices with a general variance matrix S in the bulk spectrum; edge analysis and local spectral universality is deferred to a forthcoming work. We show that the empirical spectral measure of XX^* can be approximated by a deterministic measure ν on \mathbb{R} with a continuous density away from zero and possibly a point mass at zero. The convergence holds locally down to the smallest possible scale and with an optimal speed of order $1/N$. In the special case when X is a square matrix, $n = p$, the measure ν does not have a point mass but the density has an inverse square-root singularity at zero (called the *hard edge case*). In the *soft edge case*, $n \neq p$, the continuous part of ν is supported away from zero and it has a point mass of size $1 - n/p$ at zero if $p > n$. All these features are well-known for the classical Marchenko-Pastur setup, but in the general case we need to demonstrate them without any explicit formula.

We now summarize some previous related results on Gram matrices. If each entry of X has the same variance, local Marchenko-Pastur laws have first been proved in [16, 34] for the soft edge case; and in [12, 10] for the hard edge case. The isotropic local law was given in [9]. Relaxing the assumption of identical variances to a doubly stochastic variance matrix of X , the optimal local Marchenko-Pastur law has been established in [1] for the hard edge case. Sample correlation matrices in the soft edge case were considered in [5].

Motivated by the linear model in multivariate statistics and to depart from the identical distribution, random matrices of the form TZZ^*T^* have been extensively studied where T is a deterministic matrix and the entries of Z are independent, centered and have unit variance. If T is diagonal, then they are generalizations of sample covariance matrices as $TZZ^*T^* = XX^*$ and the elements of $X = TZ$ are also independent. With this definition, all entries within one row of X have the same variance since $s_{ij} = \mathbb{E}|x_{ij}|^2 = (TT^*)_{ii}$, i.e., it is a special case of our random Gram matrix. In this case the Dyson system of equations (1.3) can be reduced to a single equation for the average $\langle m(z) \rangle$, i.e., the limiting density can still be obtained from a *scalar self-consistent equation*. This is even true for matrices of the form XX^* with $X = TZ\tilde{T}$, where both T and \tilde{T} are deterministic, investigated for example in [14]. For general T the elements of $X = TZ$ are not independent, so general sample covariance matrices are typically not Gram matrices. The global law for TZZ^*T^* has been proven by Silverstein and Bai in [36]. Knowles and Yin showed optimal local laws for a general deterministic T in [31].

Finally, we review some existing results on random Gram matrices with general variance S , when (1.3) cannot be reduced to a simpler scalar equation. The global law, even with nonzero expectation of X , has been determined by Girko [24] via (1.3) who also established the existence and uniqueness of the solution to (1.3). More recently, motivated by the theory of wireless communication, Hachem, Loubaton and Najim initiated a rigorous study of the asymptotic behaviour of the channel capacity (1.1) with a general variance matrix S [27, 28], This required to establish the global law under more general conditions than Girko; see also [26] for a review from the point of view of applications. Hachem *et. al.* have also established Gaussian fluctuations of the channel capacity (1.1) around a deterministic limit in [29] for the centered case. For a nonzero expectation of X , a similar result was obtained in [25], where S was restricted to a product form. Very recently in [7], a special k -fold clustered matrix XX^* was considered, where the samples came from k different clusters with possibly different distributions. The Dyson equation in this case reduces to a system of k equations. In an information-plus-noise model of the form $(R + X)(R + X)^*$, the effect of adding a noise matrix to X with identically distributed entries was studied knowing the limiting density of RR^* [11].

In all previous works concerning general Gram matrices, the spectral parameter z was fixed, in particular $\text{Im } z$ had a positive lower bound independent of the dimension of the matrix. Technically, this positive imaginary part provided the necessary contraction factor in the fixed point argument that led to the existence, uniqueness and stability of the solution to the Dyson equation, (1.3). For local laws down to the optimal scales $\text{Im } z \gg 1/N$, the regularizing effect of $\text{Im } z$ is too weak. In the bulk spectrum $\text{Im } z$ is effectively replaced with the local density, i.e., with the average imaginary part $\text{Im } \langle m(z) \rangle$. The main difficulty with this heuristics is its apparent circularity: the yet unknown solution itself is necessary for regularizing the equation. This problem is present in all existing proofs of any local law. This circularity is broken by separating the analysis into three parts. First, we analyze the behavior of the solution $m(z)$ as $\text{Im } z \rightarrow 0$. Second, we show that the solution is stable under small perturbations of the equation and the stability is provided by $\text{Im } \langle m(E + i0) \rangle$ for any energy E in the bulk spectrum. Finally, we show that the diagonal elements of the resolvent of the random matrix satisfy a perturbed version of (1.3), where the perturbation is controlled by large deviation estimates. Stability then provides the local law.

While this program could be completed directly for the Gram matrix and its Dyson equation (1.3), the argument appears much shorter if we used Girko's linearization (1.4) to reduce the problem to a Wigner-type matrix and use the comprehensive analysis of (1.2) from [2, 3] and the local law from [4]. There are two major obstacles to this naive approach.

First, the results of [2, 3] are not applicable as \mathfrak{S} does not satisfy the uniform primitivity assumption imposed in these papers (recall that a matrix A is primitive if there is a positive integer L such that all entries of A^L are strictly positive). This property is crucial for many proofs in [2, 3] but \mathfrak{S} in (1.4) is a typical example of a nonprimitive matrix. It is not a mere technical subtlety, in fact in the current paper, the stability estimates of (1.2) require a completely different treatment, culminating in the key technical bound, the Rotation-Inversion lemma (see Lemma 3.6 later).

Second, Girko’s transformation is singular around $z \approx 0$ since it involves a $z^2 = \zeta$ change in the spectral parameter. This accounts for the singular behavior near zero in the limiting density for Gram matrices, while the corresponding Wigner-type matrix has no singularity at zero. Thus, we need to perform a more accurate analysis near zero. If $p \neq n$, the soft edge case, we derive and analyze two new equations for the first coefficients in the expansion of m around zero. Indeed, the solutions to these new equations describe the point mass at zero and provide information about the gap above zero in the support of the approximating measure. In the hard edge case, $n = p$, an additional symmetry allows us to exclude a point mass at zero.

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Notation For vectors $v, w \in \mathbb{C}^l$, the operations product and absolute value are defined componentwise, i.e., $vw = (v_i w_i)_{i=1}^l$ and $|v| = (|v_i|)_{i=1}^l$. Moreover, for $w \in (\mathbb{C} \setminus \{0\})^l$, we set $1/w := (1/w_i)_{i=1}^l$. For vectors $v, w \in \mathbb{C}^l$, we define $\langle w \rangle = l^{-1} \sum_{i=1}^l w_i$, $\langle v, w \rangle = l^{-1} \sum_{i=1}^l \bar{v}_i w_i$, $\|w\|_2^2 = l^{-1} \sum_{i=1}^l |w_i|^2$ and $\|w\|_\infty = \max_{i=1, \dots, l} |w_i|$, $\|v\|_1 := \langle |v| \rangle$. Note that $\langle w \rangle = \langle 1, w \rangle$ where we used the convention that 1 also denotes the vector $(1, \dots, 1) \in \mathbb{C}^l$. For a matrix $A \in \mathbb{C}^{l \times l}$, we use the short notations $\|A\|_\infty := \|A\|_{\infty \rightarrow \infty}$ and $\|A\|_2 := \|A\|_{2 \rightarrow 2}$ if the domain and the target are equipped with the same norm whereas we use $\|A\|_{2 \rightarrow \infty}$ to denote the matrix norm of A when it is understood as a map $(\mathbb{C}^l, \|\cdot\|_2) \rightarrow (\mathbb{C}^l, \|\cdot\|_\infty)$.

2 Main results

Let $X = (x_{ik})_{i,k}$ be a $p \times n$ matrix with independent, centered entries and variance matrix $S = (s_{ik})_{i,k}$, i.e.,

$$\mathbb{E}x_{ik} = 0, \quad s_{ik} := \mathbb{E}|x_{ik}|^2$$

for $i = 1, \dots, p$ and $k = 1, \dots, n$.

Assumptions:

(A) The variance matrix S is *flat*, i.e., there is $s_* > 0$ such that

$$s_{ik} \leq \frac{s_*}{p+n}$$

for all $i = 1, \dots, p$ and $k = 1, \dots, n$.

(B) There are $L_1, L_2 \in \mathbb{N}$ and $\psi_1, \psi_2 > 0$ such that

$$[(SS^t)^{L_1}]_{ij} \geq \frac{\psi_1}{p+n}, \quad [(S^t S)^{L_2}]_{kl} \geq \frac{\psi_2}{p+n}$$

for all $i, j = 1, \dots, p$ and $k, l = 1, \dots, n$.

(C) All entries of X have bounded moments in the sense that there are $\mu_m > 0$ for $m \in \mathbb{N}$ such that

$$\mathbb{E}|x_{ik}|^m \leq \mu_m s_{ik}^{m/2}$$

for all $i = 1, \dots, p$ and $k = 1, \dots, n$.

(D) The dimensions of X are comparable with each other, i.e., there are constants $r_1, r_2 > 0$ such that

$$r_1 \leq \frac{p}{n} \leq r_2.$$

In the following, we will assume that $s_*, L_1, L_2, \psi_1, \psi_2, r_1, r_2$ and the sequence $(\mu_m)_m$ are fixed constants which we will call, together with some constants introduced later, *model parameters*. The constants in all our estimates will depend on the model parameters without further notice. We will use the notation $f \lesssim g$ if there is a constant $c > 0$ that depends on the model parameter only such that $f \leq cg$ and their counterparts $f \gtrsim g$ if $g \lesssim f$ and $f \sim g$ if $f \lesssim g$ and $f \gtrsim g$. The model parameters will be kept fixed whereas the parameters p and n are large numbers which will eventually be sent to infinity.

We start with a theorem about the deterministic density.

Theorem 2.1. (i) *If (A) holds true, then there is a unique holomorphic function $m: \mathbb{H} \rightarrow \mathbb{C}^p$ satisfying*

$$-\frac{1}{m(\zeta)} = \zeta - S \frac{1}{1 + S^t m(\zeta)} \tag{2.1}$$

for all $\zeta \in \mathbb{H}$ such that $\text{Im } m(\zeta) > 0$ for all $\zeta \in \mathbb{H}$. Moreover, there is a probability measure ν on \mathbb{R} whose support is contained in $[0, 4s_*]$ such that

$$\langle m(\zeta) \rangle = \int_{\mathbb{R}} \frac{1}{\omega - \zeta} \nu(d\omega) \tag{2.2}$$

for all $\zeta \in \mathbb{H}$.

(ii) *Assume (A), (B) and (D). The measure ν is absolutely continuous wrt. the Lebesgue measure apart from a possible point mass at zero, i.e., there are a number $\pi_* \in [0, 1]$ and a locally Hölder-continuous function $\pi: (0, \infty) \rightarrow [0, \infty)$ such that $\nu(d\omega) = \pi_* \delta_0(d\omega) + \pi(\omega) \mathbf{1}(\omega > 0) d\omega$.*

Part (i) of this theorem has already been proved in [28] and we will see that it also follows directly from [2, 3]. We included this part only for completeness. Part (ii) is a new result.

For $\zeta \in \mathbb{C} \setminus \mathbb{R}$, we denote the resolvent of XX^* at ζ by

$$R(\zeta) := (XX^* - \zeta)^{-1}$$

and its entries by $R_{ij}(\zeta)$ for $i, j = 1, \dots, p$.

We state our main result, the local law, i.e., optimal estimates on the resolvent R , both in entrywise and in averaged form. In both cases, we provide different estimates when the real part of the spectral parameter ζ is in the bulk and when it is away from the spectrum. As there may be many zero eigenvalues, hence, a point mass at zero in the density ν , our analysis for spectral parameters ζ in the vicinity of zero requires a special treatment. We thus first prove the local law under the general assumptions (A) – (D) for ζ away from zero. Some additional assumptions in the following subsections will allow us to extend our arguments to all ζ .

All of our results are uniform in the spectral parameter ζ which is contained in some spectral domain

$$\mathbb{D}_\delta := \{\zeta \in \mathbb{H}: \delta \leq |\zeta| \leq 10s_*\} \tag{2.3}$$

for some $\delta \geq 0$. In the first result, we assume $\delta > 0$. In the next section, under additional assumptions on S , we will work on the bigger spectral domain \mathbb{D}_0 that also includes a neighbourhood of zero.

In the following theorem, we use the notation $[p] := \{1, \dots, p\}$ for $p \in \mathbb{N}$ and for $\gamma, \varepsilon > 0$, we set

$$\mathbb{S}_{\gamma, \varepsilon} := \{\zeta \in \mathbb{H} : \pi(\operatorname{Re} \zeta) \geq \varepsilon, \operatorname{Im} \zeta \geq p^{-1+\gamma}\}. \tag{2.4}$$

Theorem 2.2 (Local Law for Gram matrices). *Let $\delta, \varepsilon_* > 0$ and $\gamma \in (0, 1)$. If X is a random matrix satisfying (A) – (D) then for every $\varepsilon > 0$ and $D > 0$ there is a constant $C_{\varepsilon, D} > 0$ such that*

$$\mathbb{P}\left(\exists i, j \in [p], \zeta \in \mathbb{D}_\delta \cap \mathbb{S}_{\gamma, \varepsilon_*} : |R_{ij}(\zeta) - m_i(\zeta)\delta_{ij}| \geq \frac{p^\varepsilon}{\sqrt{p \operatorname{Im} \zeta}}\right) \leq \frac{C_{\varepsilon, D}}{p^D}, \tag{2.5a}$$

$$\mathbb{P}\left(\exists i, j \in [p], \zeta \in \mathbb{D}_\delta : \operatorname{dist}(\zeta, \operatorname{supp} \nu) \geq \varepsilon_*, |R_{ij}(\zeta) - m_i(\zeta)\delta_{ij}| \geq \frac{p^\varepsilon}{\sqrt{p}}\right) \leq \frac{C_{\varepsilon, D}}{p^D}, \tag{2.5b}$$

for all $p \in \mathbb{N}$. Furthermore, for any sequences of deterministic vectors $w \in \mathbb{C}^p$ satisfying $\|w\|_\infty \leq 1$, we have

$$\mathbb{P}\left(\exists \zeta \in \mathbb{D}_\delta \cap \mathbb{S}_{\gamma, \varepsilon_*} : \left|\frac{1}{p} \sum_{i=1}^p w_i [R_{ii}(\zeta) - m_i(\zeta)]\right| \geq \frac{p^\varepsilon}{p \operatorname{Im} \zeta}\right) \leq \frac{C_{\varepsilon, D}}{p^D}, \tag{2.6a}$$

$$\mathbb{P}\left(\exists \zeta \in \mathbb{D}_\delta : \operatorname{dist}(\zeta, \operatorname{supp} \nu) \geq \varepsilon_*, \left|\frac{1}{p} \sum_{i=1}^p w_i [R_{ii}(\zeta) - m_i(\zeta)]\right| \geq \frac{p^\varepsilon}{p}\right) \leq \frac{C_{\varepsilon, D}}{p^D}, \tag{2.6b}$$

for all $p \in \mathbb{N}$. In particular, choosing $w_i = 1$ for all $i = 1, \dots, p$ in (2.6) yields that $p^{-1} \operatorname{Tr} R(\zeta)$ is close to $\langle m(\zeta) \rangle$.

The constant $C_{\varepsilon, D}$ depends, in addition to ε and D , only on the model parameters and on γ, δ and ε_* .

These results are optimal up to the arbitrarily small tolerance exponents $\gamma > 0$ and $\varepsilon > 0$. We remark that under stronger (e.g. subexponential) moment conditions in (C), one may replace the p^γ and p^ε factors with high powers of $\log p$.

Owing to the symmetry of the assumptions (A) – (D) in X and X^* , we can exchange X and X^* in Theorem 2.2 and obtain a statement about X^*X instead of XX^* as well.

For the results in the up-coming subsections, we need the following notion of a sequence of high probability events.

Definition 2.3 (Overwhelming probability). *Let $N_0: (0, \infty) \rightarrow \mathbb{N}$ be a function that depends on the model parameters and the tolerance exponent γ only. For a sequence $A = (A^{(p)})_p$ of random events, we say that A holds true **asymptotically with overwhelming probability** (a.w.o.p.) if for all $D > 0$*

$$\mathbb{P}(A^{(p)}) \geq 1 - p^{-D}$$

for all $p \geq N_0(D)$.

We denote the eigenvalues of XX^* by $\lambda_1 \leq \dots \leq \lambda_p$ and define

$$i(\chi) := \left\lceil p \int_{-\infty}^\chi \nu(d\omega) \right\rceil, \quad \text{for } \chi \in \mathbb{R}. \tag{2.7}$$

For a spectral parameter $\chi \in \mathbb{R}$ in the bulk, the nonnegative integer $i(\chi)$ is the index of an eigenvalue expected to be close to χ .

Theorem 2.4. *Let $\delta, \varepsilon_* > 0$ and X be a random matrix satisfying (A) – (D).*

(i) (Bulk rigidity away from zero) *For every $\varepsilon > 0$ and $D > 0$, there exists a constant $C_{\varepsilon, D} > 0$ such that*

$$\mathbb{P}\left(\exists \tau \in (\delta, 10s_*] : \pi(\tau) \geq \varepsilon_*, |\lambda_{i(\tau)} - \tau| \geq \frac{p^\varepsilon}{p}\right) \leq \frac{C_{\varepsilon, D}}{p^D} \tag{2.8}$$

holds true for all $p \in \mathbb{N}$.

The constant $C_{\varepsilon,D}$ depends, in addition to ε and D , only on the model parameters as well as on δ and ε_* .

(ii) Away from zero, all eigenvalues lie in the vicinity of the support of ν , i.e., a.w.o.p.

$$\sigma(XX^*) \cap \{\tau; |\tau| \geq \delta, \text{dist}(\tau, \text{supp } \nu) \geq \varepsilon_*\} = \emptyset. \tag{2.9}$$

In the following two subsections, we distinguish between square Gram matrices, $n = p$, and properly rectangular Gram matrices, $|p/n - 1| \geq d_* > 0$, in order to extend the local law, Theorem 2.2, to include zero in the spectral domain \mathbb{D} . Since the density of states behaves differently around zero in these two cases, separate statements and proofs are necessary.

2.1 Square Gram matrices

The following concept is well-known in linear algebra. For understanding singularities of the density of states in random matrix theory, it was introduced in [2].

Definition 2.5 (Fully indecomposable matrix). A $K \times K$ matrix $T = (t_{ij})_{i,j=1}^K$ with nonnegative entries is called **fully indecomposable** if for any two subsets $I, J \subset \{1, \dots, K\}$ such that $\#I + \#J \geq K$, the submatrix $(t_{ij})_{i \in I, j \in J}$ contains a nonzero entry.

For square Gram matrices, we add the following assumptions.

(E1) The matrix X is square, i.e., $n = p$.

(F1) The matrix S is **block fully indecomposable**, i.e., there are constants $\varphi > 0$, $K \in \mathbb{N}$, a fully indecomposable matrix $Z = (z_{ij})_{i,j=1}^K$ with $z_{ij} \in \{0, 1\}$ and a partition $(I_i)_{i=1}^K$ of $\{1, \dots, p\}$ such that

$$\#I_i = \frac{p}{K}, \quad s_{xy} \geq \frac{\varphi}{p+n} z_{ij}, \quad x \in I_i \text{ and } y \in I_j$$

for all $i, j = 1, \dots, K$.

The constants φ and K in (F1) are considered model parameters as well.

Remark 2.6. Clearly, (E1) yields (D) with $r_1 = r_2 = 1$. Moreover, adapting the proof of Theorem 2.2.1 in [6], we see that (F1) implies (B) with L_1, L_2, ψ_1 and ψ_2 explicitly depending on φ and K .

For part (i) of the following theorem, we recall the definitions of $[p]$ and $\mathbb{S}_{\gamma,\varepsilon}$ from (2.4).

Theorem 2.7 (Local law for square Gram matrices). *If X satisfies (A), (C), (E1) and (F1), then*

(i) *The conclusions of Theorem 2.2 are valid with the following modifications: (2.5b) and (2.6) hold true for $\delta = 0$ (cf. (2.3)) while instead of (2.5a), we have*

$$\mathbb{P} \left(\exists i, j \in [p], \zeta \in \mathbb{D}_0 \cap \mathbb{S}_{\gamma,\varepsilon_*} : |R_{ij}(\zeta) - m_i(\zeta)\delta_{ij}| \geq p^\varepsilon \sqrt{\frac{\langle \text{Im } m(\zeta) \rangle}{p \text{Im } \zeta}} \right) \leq \frac{C_{\varepsilon,D}}{p^D}. \tag{2.10}$$

(ii) $\pi_* = 0$ and the limit $\lim_{\omega \downarrow 0} \pi(\omega)\sqrt{\omega}$ exists and lies in $(0, \infty)$.

(iii) (Bulk rigidity down to zero) For every $\varepsilon_* > 0$ and every $\varepsilon > 0$ and $D > 0$, there exists a constant $C_{\varepsilon,D} > 0$ such that

$$\mathbb{P} \left(\exists \tau \in (0, 10s_*) : \pi(\tau) \geq \varepsilon_*, |\lambda_{i(\tau)} - \tau| \geq \frac{p^\varepsilon}{p} \left(\sqrt{\tau} + \frac{1}{p} \right) \right) \leq \frac{C_{\varepsilon,D}}{p^D} \tag{2.11}$$

for all $p \in \mathbb{N}$. The constant $C_{\varepsilon,D}$ depends, in addition to ε and D , only on the model parameters and on ε_* .

- (iv) There are no eigenvalues away from the support of ν , i.e., (2.9) holds true with $\delta = 0$.

We remark that the bound of the individual resolvent entries (2.10) deteriorates as ζ gets close to zero since $\langle \text{Im } m(\zeta) \rangle \sim |\zeta|^{-1/2}$ in this regime while the averaged version (2.6), with $\delta = 0$, does not show this behaviour.

2.2 Properly rectangular Gram matrices

(E2) The matrix X is properly rectangular, i.e., there is $d_* > 0$ such that

$$\left| \frac{p}{n} - 1 \right| \geq d_*.$$

(F2) The matrix elements of S are bounded from below, i.e., there is a $\varphi > 0$ such that

$$s_{ik} \geq \frac{\varphi}{n+p}$$

for all $i = 1, \dots, p$ and $k = 1, \dots, n$.

The constants d_* and φ in (E2) and (F2), respectively, are also considered as model parameters. Note that (F2) is a simpler version of (F1). For properly rectangular Gram matrices we work under the stronger condition (F2) for simplicity but our analysis could be adjusted to some weaker condition as well.

Remark 2.8. Note that (F2) immediately implies condition (B) with $L = 1$.

We introduce the lower edge of the absolutely continuous part of the distribution ν for properly rectangular Gram matrices

$$\delta_\pi := \inf\{\omega > 0: \pi(\omega) > 0\}. \tag{2.12}$$

Theorem 2.9 (Local law for properly rectangular Gram matrices). *Let X be a random matrix satisfying (A), (C), (D), (E2) and (F2). We have*

- (i) The gap between zero and the lower edge is macroscopic $\delta_\pi \sim 1$.
- (ii) (Bulk rigidity down to zero) The estimate (2.8) holds true with $\delta = 0$.
- (iii) There are no eigenvalues away from the support of ν , i.e., (2.9) holds true with $\delta = 0$.
- (iv) If $p > n$, then $\pi_* = 1 - n/p$ and $\dim \ker(XX^*) = p - n$ a.w.o.p.
- (v) If $p < n$, then $\pi_* = 0$ and $\dim \ker(XX^*) = 0$ a.w.o.p.
- (vi) (Local law around zero) For every $\varepsilon_* \in (0, \delta_\pi)$, every $\varepsilon > 0$ and $D > 0$, there exists a constant $C_{\varepsilon,D} > 0$, such that

$$\mathbb{P} \left(\exists \zeta \in \mathbb{H}, i, j \in \{1, \dots, p\}: |\zeta| \leq \delta_\pi - \varepsilon_*, |R_{ij}(\zeta) - m_i(\zeta)\delta_{ij}| \geq \frac{p^\varepsilon}{|\zeta|\sqrt{p}} \right) \leq \frac{C_{\varepsilon,D}}{p^D}, \tag{2.13}$$

for all $p \in \mathbb{N}$ if $p > n$ and

$$\mathbb{P} \left(\exists \zeta \in \mathbb{H}, i, j \in \{1, \dots, p\}: |\zeta| \leq \delta_\pi - \varepsilon_*, |R_{ij}(\zeta) - m_i(\zeta)\delta_{ij}| \geq \frac{p^\varepsilon}{\sqrt{p}} \right) \leq \frac{C_{\varepsilon,D}}{p^D}, \tag{2.14}$$

for all $p \in \mathbb{N}$ if $p < n$. Moreover, in both cases

$$\mathbb{P} \left(\exists \zeta \in \mathbb{H}: |\zeta| \leq \delta_\pi - \varepsilon_*, \left| \frac{1}{p} \sum_{i=1}^p [R_{ii}(\zeta) - m_i(\zeta)] \right| \geq \frac{p^\varepsilon}{p} \right) \leq \frac{C_{\varepsilon,D}}{p^D}, \quad (2.15)$$

for all $p \in \mathbb{N}$.

The constant $C_{\varepsilon,D}$ depends, in addition to ε and D , only on the model parameters and on ε_* .

If $p > n$, then the Stieltjes transform of the empirical spectral measure of XX^* has a term proportional to $1/\zeta$ due to the macroscopic kernel of XX^* . This is the origin of the additional factor $1/|\zeta|$ in (2.13).

Remark 2.10. As a consequence of Theorem 2.7 and Theorem 2.9 and under the same conditions, the standard methods in [9] and [4] can be used to prove an anisotropic law and delocalization of eigenvectors in the bulk.

3 Quadratic vector equation

For the rest of the paper, without loss of generality we will assume that $s_* = 1$ in (A), which can be achieved by a simple rescaling of X . In the whole section, we will assume that the matrix S satisfies (A), (B) and (D) without further notice.

3.1 Self-consistent equation for resolvent entries

We introduce the random matrix H and the deterministic matrix \mathfrak{S} defined through

$$H = \begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix}, \quad \mathfrak{S} = \begin{pmatrix} 0 & S \\ S^t & 0 \end{pmatrix}. \quad (3.1)$$

Note that both matrices, H and \mathfrak{S} have dimensions $(p + n) \times (p + n)$. We denote their entries by $H = (h_{xy})_{x,y}$ and $\mathfrak{S} = (\sigma_{xy})_{x,y}$, respectively, where $\sigma_{xy} = \mathbb{E}|h_{xy}|^2$ with $x, y = 1, \dots, n + p$.

It is easy to see that condition (B) implies

(B') There are $L \in \mathbb{N}$ and $\psi > 0$ such that

$$\sum_{k=1}^L (\mathfrak{S}^k)_{xy} \geq \frac{\psi}{n + p} \quad (3.2)$$

for all $x, y = 1, \dots, n + p$.

In the following, a crucial part of the analysis will be devoted to understanding the resolvent of H at $z \in \mathbb{H}$, i.e., the matrix

$$G(z) := (H - z)^{-1} \quad (3.3)$$

whose entries are denoted by $G_{xy}(z)$ for $x, y = 1, \dots, n + p$. For $V \subset \{1, \dots, n + p\}$, we use the notation $G_{xy}^{(V)}$ to denote the entries of the resolvent $G^{(V)}(z) = (H^{(V)} - z)^{-1}$ of the matrix $H_{xy}^{(V)} = h_{xy} \mathbf{1}(x \notin V) \mathbf{1}(y \notin V)$ where $x, y = 1, \dots, n + p$.

The Schur complement formula and the resolvent identities applied to $G(z)$ yield the self-consistent equations

$$-\frac{1}{g_{1,i}(z)} = z + \sum_{k=1}^n s_{ik} g_{2,k}(z) + d_{1,i}(z), \quad (3.4a)$$

$$-\frac{1}{g_{2,k}(z)} = z + \sum_{i=1}^p s_{ik} g_{1,i}(z) + d_{2,k}(z), \quad (3.4b)$$

where $g_{1,i}(z) := G_{ii}(z)$ for $i = 1, \dots, p$ and $g_{2,k}(z) := G_{k+p,k+p}(z)$ for $k = 1, \dots, n$ with the error terms

$$d_{1,r} := \sum_{k,l=1, k \neq l}^n x_{rk} G_{kl}^{(r)} \bar{x}_{rl} + \sum_{k=1}^n (|x_{rk}|^2 - s_{rk}) G_{k+n,k+n}^{(r)} - \sum_{k=1}^n s_{rk} \frac{G_{k+n,r} G_{r,k+n}}{g_{1,r}},$$

$$d_{2,m} := \sum_{i,j=1, i \neq j}^p \bar{x}_{im} G_{ij}^{(m+p)} x_{jm} + \sum_{i=1}^p (|x_{im}|^2 - s_{im}) G_{ii}^{(m+p)} - \sum_{i=1}^p s_{im} \frac{G_{i,m+p} G_{m+p,i}}{g_{2,m}}$$

for $r = 1, \dots, p$ and $m = 1, \dots, n$.

We will prove a local law which states that $g_{1,i}(z)$ and $g_{2,k}(z)$ can be approximated by $M_{1,i}(z)$ and $M_{2,k}(z)$, respectively, where $M_1: \mathbb{H} \rightarrow \mathbb{C}^p$ and $M_2: \mathbb{H} \rightarrow \mathbb{C}^n$ are the unique solution of

$$-\frac{1}{M_1} = z + S M_2, \tag{3.5a}$$

$$-\frac{1}{M_2} = z + S^t M_1, \tag{3.5b}$$

which satisfy $\text{Im } M_1(z) > 0$ and $\text{Im } M_2(z) > 0$ for all $z \in \mathbb{H}$.

The system of self-consistent equations for g_1 and g_2 in (3.4) can be seen as a perturbation of the system (3.5). With the help of \mathfrak{S} , equations (3.5a) and (3.5b) can be combined to a vector equation for $\mathfrak{M} = (M_1, M_2)^t \in \mathbb{H}^{p+n}$, i.e.,

$$-\frac{1}{\mathfrak{M}} = z + \mathfrak{S} \mathfrak{M}. \tag{3.6}$$

Since \mathfrak{S} is symmetric, has nonnegative entries and fulfills (A) with $s_* = 1$, Theorem 2.1 in [2] is applicable to (3.6). Here, we take $a = 0$ in Theorem 2.1 of [2]. This theorem implies that (3.6) has a unique solution \mathfrak{M} with $\text{Im } \mathfrak{M}(z) > 0$ for any $z \in \mathbb{H}$. Moreover, by this theorem, \mathfrak{M}_x is the Stieltjes transform of a symmetric probability measure on \mathbb{R} whose support is contained in $[-2, 2]$ for all $x = 1, \dots, n + p$ and we have

$$\|\mathfrak{M}(z)\|_2 \leq \frac{2}{|z|} \tag{3.7}$$

for all $z \in \mathbb{H}$. The function $\langle \mathfrak{M} \rangle$ is the Stieltjes transform of a symmetric probability measure on \mathbb{R} which we denote by ρ , i.e.,

$$\langle \mathfrak{M}(z) \rangle = \int_{\mathbb{R}} \frac{1}{t - z} \rho(dt) \tag{3.8}$$

for $z \in \mathbb{H}$. Its support is contained in $[-2, 2]$.

We combine (3.4a) and (3.4b) to obtain

$$-\frac{1}{\mathfrak{g}} = z + \mathfrak{S} \mathfrak{g} + \mathfrak{d}, \tag{3.9}$$

where $\mathfrak{g} = (g_1, g_2)^t$ and $\mathfrak{d} = (d_1, d_2)^t$. We think of (3.9) as a perturbation of (3.6) and most of the subsequent subsection is devoted to the study of (3.9) for an arbitrary perturbation \mathfrak{d} .

Before we start studying (3.6) we want to indicate how m and R are related to \mathfrak{M} and G , respectively. The Stieltjes transforms as well as the resolvents are essentially related via the same transformation of the spectral parameter. If $\mathcal{G}_{11}(z)$ denotes the upper left $p \times p$ block of $G(z)$ then $R(z^2) = (X X^* - z^2)^{-1} = \mathcal{G}_{11}(z)/z$. In the proof of Theorem 2.1 in Subsection 3.4, we will see that m and M_1 are related via $m(\zeta) = M_1(\sqrt{\zeta})/\sqrt{\zeta}$. (We

always choose the branch of the square root satisfying $\text{Im} \sqrt{\zeta} > 0$ for $\text{Im} \zeta > 0$.) Assuming this relation and introducing $m_2(\zeta) := M_2(\sqrt{\zeta})/\sqrt{\zeta}$, we obtain

$$\begin{aligned} -\frac{1}{m(\zeta)} &= \zeta(1 + Sm_2(\zeta)), \\ -\frac{1}{m_2(\zeta)} &= \zeta(1 + S^t m(\zeta)) \end{aligned}$$

from (3.5). Solving the second equation for m_2 and plugging the result into the first one yields (2.1) immediately. In fact, m_2 is the analogue of m corresponding to X^*X , i.e, the Stieltjes transform of the deterministic measure approximating the eigenvalue density of X^*X .

3.2 Structure of the solution

We first notice that the inequality $s_{ik} \leq 1/(n + p)$ implies

$$\|S^t w\|_\infty = \max_{k=1, \dots, n} \sum_{i=1}^p s_{ik} |w_i| \leq \max_{k=1, \dots, n} \left(p \sum_{i=1}^p s_{ik}^2 \right)^{1/2} \left(\frac{1}{p} \sum_{i=1}^p |w_i|^2 \right)^{1/2} \leq \|w\|_2 \quad (3.10)$$

for all $w \in \mathbb{C}^p$, i.e., $\|S^t\|_{2 \rightarrow \infty} \leq 1$. Now, we establish some preliminary estimates on the solution of (3.6).

Lemma 3.1. *Let $z \in \mathbb{H}$ and $x \in \{1, \dots, n + p\}$. We have*

$$|\mathfrak{M}_x(z)| \leq \frac{1}{\text{dist}(z, \text{supp } \rho)}, \quad (3.11a)$$

$$\text{Im } \mathfrak{M}_x(z) \leq \frac{\text{Im } z}{\text{dist}(z, \text{supp } \rho)^2}. \quad (3.11b)$$

If $z \in \mathbb{H}$ and $|z| \leq 10$ then

$$|z| \lesssim |\mathfrak{M}_x(z)| \leq \|\mathfrak{M}(z)\|_\infty \lesssim \frac{|z|^{2-2L}}{\langle \text{Im } \mathfrak{M}(z) \rangle} \quad (3.12a)$$

$$|z|^{2L} \langle \text{Im } \mathfrak{M}(z) \rangle \lesssim \text{Im } \mathfrak{M}_x(z). \quad (3.12b)$$

In particular, the support of the measures representing \mathfrak{M}_x is independent of x away from zero.

The proof essentially follows the same line of arguments as the proof of Lemma 5.4 in [2]. However, instead of using the lower bound on the entries of S^L as in [2] we have to make use of the lower bound on the entries of $\sum_{k=1}^L \mathfrak{G}^k$.

To prove another auxiliary estimate on \mathfrak{G} , we define the vectors $\mathfrak{G}_x = (\sigma_{xy})_{y=1, \dots, n+p} \in \mathbb{R}^{n+p}$ for $x = 1, \dots, n + p$. Since (3.2) implies

$$\psi \leq \sum_{k=1}^L \sum_{y=1}^{n+p} (\mathfrak{G}^k)_{xy} \leq \sum_{k=1}^L \sum_{v=1}^{n+p} \sigma_{xv} \max_{t=1, \dots, n+p} \sum_{y=1}^{n+p} (\mathfrak{G}^{k-1})_{ty} \leq L \sum_{v=1}^{n+p} \sigma_{xv}$$

for any fixed $x = 1, \dots, n + p$, where we used $\|\mathfrak{G}^{k-1}\|_\infty \leq \|\mathfrak{G}\|_\infty^{k-1} \leq 1$ by (A), we obtain

$$\inf_{x=1, \dots, n+p} \|\mathfrak{G}_x\|_1 \geq \frac{\psi}{L}. \quad (3.13)$$

In particular, together with (A), this implies

$$\sum_{j=1}^p s_{jk} \sim 1, \quad \sum_{l=1}^n s_{il} \sim 1, \quad i = 1, \dots, p, \quad k = 1, \dots, n. \quad (3.14)$$

In the study of the stability of (3.6) when perturbed by a vector \mathfrak{d} , as in (3.9), the linear operator

$$\mathfrak{F}(z)v := |\mathfrak{M}(z)|\mathfrak{S}(|\mathfrak{M}(z)|v) \tag{3.15}$$

for $v \in \mathbb{C}^{n+p}$ plays an important role. Before we collect some properties of operators of this type in the next lemma, we first recall the definition of the gap of an operator from [2].

Definition 3.2. *Let T be a compact self-adjoint operator on a Hilbert space. The spectral gap $\text{Gap}(T) \geq 0$ is the difference between the two largest eigenvalues of $|T|$ (defined by spectral calculus). If the operator norm $\|T\|$ is a degenerate eigenvalue of $|T|$, then $\text{Gap}(T) = 0$.*

In the next lemma, we study matrices of the form $\widehat{\mathfrak{F}}(r)_{xy} := r_x \sigma_{xy} r_y$ where $r \in (0, \infty)^{n+p}$ and $x, y = 1, \dots, n + p$. If $\inf_x r_x > 0$ then (3.2) implies that all entries of $\sum_{k=1}^L \widehat{\mathfrak{F}}(r)^k$ are strictly positive. Therefore, by the Perron-Frobenius theorem, the eigenspace corresponding to the largest eigenvalue $\widehat{\lambda}(r)$ of $\widehat{\mathfrak{F}}(r)$ is one-dimensional and spanned by a unique non-negative vector $\widehat{\mathfrak{f}} = \widehat{\mathfrak{f}}(r)$ such that $\langle \widehat{\mathfrak{f}}, \widehat{\mathfrak{f}} \rangle = 1$.

The block structure of \mathfrak{S} implies that there is a matrix $\widehat{F}(r) \in \mathbb{R}^{p \times n}$ such that

$$\widehat{\mathfrak{F}}(r) = \begin{pmatrix} 0 & \widehat{F}(r) \\ \widehat{F}(r)^t & 0 \end{pmatrix}. \tag{3.16}$$

However, for this kind of operator, we obtain $\sigma(\widehat{\mathfrak{F}}(r)) = -\sigma(\widehat{\mathfrak{F}}(r))$, i.e., $\text{Gap}(\widehat{\mathfrak{F}}(r)) = 0$ by above definition. Therefore, we will compute $\text{Gap}(\widehat{F}(r)\widehat{F}(r)^t)$, instead. We will apply these observations for $\mathfrak{F}(z)$ where the blocks $\widehat{F}(|\mathfrak{M}(z)|)$ will be denoted by $F(z)$.

Lemma 3.3. *For a vector $r \in (0, \infty)^{n+p}$ which is bounded by constants $r_+ \in (0, \infty)$ and $r_- \in (0, 1]$, i.e.,*

$$r_- \leq r_x \leq r_+$$

for all $x = 1, \dots, n + p$, we define the matrix $\widehat{\mathfrak{F}}(r)$ with entries $\widehat{\mathfrak{F}}(r)_{xy} := r_x \sigma_{xy} r_y$ for $x, y = 1, \dots, n + p$. Then the eigenspace corresponding to $\widehat{\lambda}(r) := \|\widehat{\mathfrak{F}}(r)\|_{2 \rightarrow 2}$ is one-dimensional and $\widehat{\lambda}(r)$ satisfies the estimates

$$r_-^2 \lesssim \widehat{\lambda}(r) \lesssim r_+^2. \tag{3.17}$$

There is a unique eigenvector $\widehat{\mathfrak{f}} = \widehat{\mathfrak{f}}(r)$ corresponding to $\widehat{\lambda}(r)$ satisfying $\widehat{\mathfrak{f}}_x \geq 0$ and $\|\widehat{\mathfrak{f}}\|_2 = 1$. Its components satisfy

$$\frac{r_-^{2L}}{r_+^4} \min \left\{ \widehat{\lambda}(r), \widehat{\lambda}(r)^{-L+2} \right\} \lesssim \widehat{\mathfrak{f}}_x \lesssim \frac{r_+^4}{\widehat{\lambda}(r)^2}, \tag{3.18}$$

for all $x = 1, \dots, n + p$. Moreover, $\widehat{F}(r)\widehat{F}(r)^t$ has a spectral gap

$$\text{Gap} \left(\widehat{F}(r)\widehat{F}(r)^t \right) \gtrsim \frac{r_-^{8L}}{r_+^{16}} \min \left\{ \widehat{\lambda}(r)^6, \widehat{\lambda}(r)^{-8L+10} \right\}. \tag{3.19}$$

The estimates in (3.17) and (3.18) can basically be proved following the proof of Lemma 5.6 in [2] where S^L is replaced by $\sum_{k=1}^L \mathfrak{S}^k$ and $(\widehat{F}/\widehat{\lambda})^L$ by $\sum_{k=1}^L (\widehat{\mathfrak{F}}/\widehat{\lambda})^k$. Therefore, we will only show (3.19) assuming the other estimates.

Proof. We write $\widehat{\mathfrak{f}} = (\widehat{f}_1, \widehat{f}_2)^t$ for $\widehat{f}_1 \in \mathbb{C}^p$ and $\widehat{f}_2 \in \mathbb{C}^n$ and define a linear operator on \mathbb{C}^p through

$$T := \sum_{k=1}^L \left(\frac{\widehat{F}\widehat{F}^t}{\widehat{\lambda}^2} \right)^k.$$

Thus, $\|T\|_2 = L$ as $T\hat{f}_1 = L\hat{f}_1$. Using (B') we first estimate the entries t_{ij} by

$$t_{ij} \geq \sum_{k=1}^L \frac{r^{4k}}{\widehat{\lambda}^{2k}} ((SS^t)^k)_{ij} \geq r_-^{4L} \min \left\{ \widehat{\lambda}^{-2}, \widehat{\lambda}^{-2L} \right\} \frac{\psi}{n+p}, \quad \text{for } i, j = 1, \dots, p.$$

Estimating $\|\hat{f}_1\|_2$ and $\|\hat{f}_1\|_\infty$ from (3.18) and applying Lemma 5.6 in [3] or Lemma 5.7 in [2] yields

$$\text{Gap}(T) \geq \frac{\|\hat{f}_1\|_2^2}{\|\hat{f}_1\|_\infty^2} p \inf_{i,j} t_{ij} \gtrsim \frac{r_-^{8L}}{r_+^{16}} \min \left\{ \widehat{\lambda}^4, \widehat{\lambda}^{-8L+8} \right\}.$$

Here we used (D) and note that the factor $\inf_{i,j} t_{ij}$ in Lemma 5.6 in [3] is replaced by $p \inf_{i,j} t_{ij}$ as t_{ij} are considered as the matrix entries of T and not as the kernel of an integral operator on $L^2(\{1, \dots, p\})$ where $\{1, \dots, p\}$ is equipped with the uniform probability measure. As $q(x) := x + x^2 + \dots + x^L$ is a monotonously increasing, differentiable function on $[0, 1]$ and $\sigma(\widehat{F}\widehat{F}^t/\widehat{\lambda}^2) \subset [0, 1]$ we obtain $\text{Gap}(T) \sim \text{Gap}(\widehat{F}\widehat{F}^t)/\widehat{\lambda}^2$ which concludes the proof. \square

Lemma 3.4. *The matrix $\mathfrak{F}(z)$ defined in (3.15) with entries $\mathfrak{F}_{xy}(z) = |\mathfrak{M}_x(z)|\sigma_{xy}|\mathfrak{M}_y(z)|$ has the norm*

$$\|\mathfrak{F}(z)\|_2 = 1 - \frac{\text{Im } z \langle \mathfrak{f}(z) | \mathfrak{M}(z) \rangle}{\langle \mathfrak{f}(z) | \text{Im } \mathfrak{M}(z) | \mathfrak{M}(z) |^{-1} \rangle}, \tag{3.20}$$

where $\mathfrak{f}(z)$ is the unique eigenvector of $\mathfrak{F}(z)$ associated to $\|\mathfrak{F}(z)\|_2$. In particular, we obtain

$$(1 - \|\mathfrak{F}(z)\|_2)^{-1} \lesssim \frac{1}{|z|} \min \left\{ \frac{1}{\text{Im } z}, \frac{1}{|z| \text{dist}(z, \text{supp } \rho)^2} \right\} \tag{3.21}$$

for $z \in \mathbb{H}$ satisfying $|z| \leq 10$.

Proof. The derivation of (3.20) follows the same steps as the proof of (4.4) in [3] (compare Lemma 5.5 in [2] as well). We take the imaginary part of (3.6), multiply the result by $|\mathfrak{M}|$ and take the scalar product with \mathfrak{f} . Thus, we obtain

$$\left\langle \mathfrak{f}, \frac{\text{Im } \mathfrak{M}}{|\mathfrak{M}|} \right\rangle = \text{Im } z \langle \mathfrak{f} | \mathfrak{M} \rangle + \|\mathfrak{F}\|_2 \left\langle \mathfrak{f}, \frac{\text{Im } \mathfrak{M}}{|\mathfrak{M}|} \right\rangle, \tag{3.22}$$

where we used the symmetry of \mathfrak{F} and $\mathfrak{F}\mathfrak{f} = \|\mathfrak{F}\|_2\mathfrak{f}$. Solving (3.22) for $\|\mathfrak{F}\|_2$ yields (3.20).

Now, (3.21) is a direct consequence of Lemma 3.1 and (3.20). \square

3.3 Stability away from the edges and continuity

All estimates of $\mathfrak{M} - \mathfrak{g}$, when \mathfrak{M} and \mathfrak{g} satisfy (3.6) and (3.9), respectively, are based on inverting the linear operator

$$\mathfrak{B}(z)v := \frac{|\mathfrak{M}(z)|^2}{\mathfrak{M}(z)^2} v - \mathfrak{F}(z)v$$

for $v \in \mathbb{C}^{n+p}$. The following lemma bounds $\mathfrak{B}^{-1}(z)$ in terms of $\langle \text{Im } \mathfrak{M}(z) \rangle$ if z is away from zero. For $\delta > 0$, we use the notation $f \lesssim_\delta g$ if and only if there is an $r > 0$ which is allowed to depend on model parameters such that $f \lesssim \delta^{-r}g$.

Lemma 3.5. *There is a universal constant $\kappa \in \mathbb{N}$ such that for all $\delta > 0$ we have*

$$\|\mathfrak{B}^{-1}(z)\|_2 \lesssim_\delta \min \left\{ \frac{1}{(\text{Re } z)^2 \langle \text{Im } \mathfrak{M}(z) \rangle^\kappa}, \frac{1}{\text{Im } z}, \frac{1}{\text{dist}(z, \text{supp } \rho)^2} \right\}, \tag{3.23}$$

$$\|\mathfrak{B}^{-1}(z)\|_\infty \lesssim_\delta \min \left\{ \frac{1}{(\text{Re } z)^2 \langle \text{Im } \mathfrak{M}(z) \rangle^{\kappa+2}}, \frac{1}{(\text{Im } z)^3}, \frac{1}{\text{dist}(z, \text{supp } \rho)^4} \right\} \tag{3.24}$$

for all $z \in \mathbb{H}$ satisfying $\delta \leq |z| \leq 10$.

For the proof of this result, we will need the two following lemmata. We recall that by the Perron-Frobenius theorem an irreducible matrix with nonnegative entries has a unique ℓ^2 -normalized eigenvector with positive entries corresponding to its largest eigenvalue. By the definition of the spectral gap, Definition 3.2, we observe that if AA^* is irreducible then $\text{Gap}(AA^*) = \|AA^*\|_2 - \max(\sigma(AA^*) \setminus \{\|AA^*\|_2\})$.

Lemma 3.6 (Rotation-Inversion Lemma). *There exists a positive constant C such that for all $n, p \in \mathbb{N}$, unitary matrices $U_1 \in \mathbb{C}^{p \times p}$, $U_2 \in \mathbb{C}^{n \times n}$ and $A \in \mathbb{R}^{p \times n}$ with nonnegative entries such that A^*A and AA^* are irreducible and $\|A^*A\|_2 \in (0, 1]$, the following bound holds:*

$$\left\| \begin{pmatrix} U_1 & A \\ A^* & U_2 \end{pmatrix}^{-1} \right\|_2 \leq \frac{C}{\text{Gap}(AA^*)|1 - \|A^*A\|_2 \langle v_1, U_1 v_1 \rangle \langle v_2, U_2 v_2 \rangle|}, \tag{3.25}$$

where $v_1 \in \mathbb{C}^p$ and $v_2 \in \mathbb{C}^n$ are the unique positive, normalized eigenvectors with $AA^*v_1 = \|A^*A\|_2 v_1$ and $A^*Av_2 = \|A^*A\|_2 v_2$. The norm on the left hand side of (3.25) is infinite if and only if the right hand side of (3.25) is infinite, i.e., in this case the inverse does not exist.

This lemma is proved in the appendix.

Lemma 3.7. *Let $R: \mathbb{C}^{n+p} \rightarrow \mathbb{C}^{n+p}$ be a linear operator and $D: \mathbb{C}^{n+p} \rightarrow \mathbb{C}^{n+p}$ a diagonal operator. If $R - D$ is invertible and $D_{xx} \neq 0$ for all $x = 1, \dots, n + p$ then*

$$\|(R - D)^{-1}\|_\infty \leq \left(\inf_{x=1}^{n+p} |D_{xx}| \right)^{-1} (1 + \|R\|_{2 \rightarrow \infty} \|(R - D)^{-1}\|_2). \tag{3.26}$$

The proof of (3.26) follows a similar way as the proof of (5.28) in [2].

Proof of Lemma 3.5. The bound on $\|\mathfrak{B}^{-1}(z)\|_\infty$, (3.24), follows from (3.23) by employing (3.26). We use (3.26) with $R = \mathfrak{F}(z)$ and $D = |\mathfrak{M}(z)|^2 / \mathfrak{M}(z)^2$ and observe that $\|\mathfrak{F}(z)\|_{2 \rightarrow \infty} \leq \|\mathfrak{M}\|_\infty^2 \|\mathfrak{G}\|_{2 \rightarrow \infty}$. Therefore, (3.24) follows from (3.23) due to the estimates

$$\begin{aligned} \|\mathfrak{M}\|_\infty &\lesssim \min\{(\text{Im } \mathfrak{M})^{-1}, (\text{Im } z)^{-1}, \text{dist}(z, \text{supp } \rho)^{-1}\}, \\ 1 &\lesssim_\delta \min\{(\text{Im } \mathfrak{M})^{-1}, (\text{Im } z)^{-1}, \text{dist}(z, \text{supp } \rho)^{-1}\}. \end{aligned}$$

Here, we used (3.12a) and $\delta \leq |z| \leq 10$.

Now we prove (3.23). Our first goal is the following estimate

$$\|\mathfrak{B}^{-1}(z)\|_2 \lesssim_\delta \frac{1}{\text{Gap}(F(z)F(z)^t)(\text{Re } z)^2(\text{Im } \mathfrak{M}(z))^\kappa} \tag{3.27}$$

for some universal $\kappa \in \mathbb{N}$ which will be a consequence of Lemma 3.6. We apply this lemma with

$$\begin{pmatrix} 0 & F(z) \\ F(z)^t & 0 \end{pmatrix} = \mathfrak{F}(z) := \widehat{\mathfrak{F}}(|\mathfrak{M}(z)|), \quad \mathfrak{U} := \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix} = \text{diag} \left(\frac{|\mathfrak{M}(z)|^2}{\mathfrak{M}(z)^2} \right)$$

and $v_1 := f_1 / \|f_1\|_2$ and $v_2 := f_2 / \|f_2\|_2$ where $\mathfrak{f} = (f_1, f_2) \in \mathbb{C}^{p+n}$. Note that $\lambda(z) := \widehat{\lambda}(|\mathfrak{M}(z)|) = \|\mathfrak{F}(z)\|_2$ in Lemma 3.3 and $F(z) = \widehat{F}(|\mathfrak{M}(z)|)$ in the notation of (3.16). In Lemma 3.3, we choose $r_- := \inf_x |\mathfrak{M}_x(z)|$ and $r_+ := \|\mathfrak{M}(z)\|_\infty$ and use the bounds $r_- \gtrsim |z|$ and $r_+ \lesssim |z|^{2-2L} / \langle \text{Im } \mathfrak{M}(z) \rangle$ by (3.12a). Moreover, we have

$$|z|^2 \lesssim \|\mathfrak{F}(z)\|_2 \leq 1 \tag{3.28}$$

by (3.12a), (3.17) and (3.20).

We write $\mathfrak{U} = \text{diag}(e^{-i2\psi})$, i.e., $e^{i\psi} = \mathfrak{M} / |\mathfrak{M}|$, to obtain

$$\langle v_1, U_1 v_1 \rangle = \langle v_1, (\cos \psi_1 - i \sin \psi_1)^2 v_1 \rangle = \langle v_1, (1 - 2(\sin \psi_1)^2 - 2i \cos \psi_1 \sin \psi_1) v_1 \rangle$$

and a similar relation holds for $\langle v_2, U_2 v_2 \rangle$. Thus, we compute

$$\begin{aligned} & \operatorname{Re} \left(1 - \|F(z)^t F(z)\|_2 \langle v_1, (1 - 2(\sin \psi_1)^2 - 2i \cos \psi_1 \sin \psi_1) v_1 \rangle \right. \\ & \quad \left. \times \langle v_2, (1 - 2(\sin \psi_2)^2 - 2i \cos \psi_2 \sin \psi_2) v_2 \rangle \right) \\ &= 1 - \|F(z)^t F(z)\|_2 \left(1 - 2\langle v_1, (\sin \psi_1)^2 v_1 \rangle - 2\langle v_2, (\sin \psi_2)^2 v_2 \rangle \right. \\ & \quad \left. + 4\langle v_1, (\sin \psi_1)^2 v_1 \rangle \langle v_2, (\sin \psi_2)^2 v_2 \rangle \right). \end{aligned}$$

Using $2a + 2b - 4ab \geq (a + b)(2 - a - b)$ for $a, b \in \mathbb{R}$, and estimating the absolute value by the real part yields

$$\begin{aligned} & \left| 1 - \|F(z)^t F(z)\|_2 \langle v_1, U_1 v_1 \rangle \langle v_2, U_2 v_2 \rangle \right| \\ & \geq 1 - \|F(z)^t F(z)\|_2 + \|F(z)^t F(z)\|_2 \left(\langle v_1, (\sin \psi_1)^2 v_1 \rangle + \langle v_2, (\sin \psi_2)^2 v_2 \rangle \right) \\ & \quad \times \left(\langle v_1, (\cos \psi_1)^2 v_1 \rangle + \langle v_2, (\cos \psi_2)^2 v_2 \rangle \right) \\ & \gtrsim |z|^4 \langle \mathfrak{f}, (\sin \psi)^2 \mathfrak{f} \rangle \langle \mathfrak{f}, (\cos \psi)^2 \mathfrak{f} \rangle \\ & \gtrsim_\delta \left(\inf_{x=1, \dots, n+p} \mathfrak{f}_x^4 \right) \left\langle \left(\frac{\operatorname{Im} \mathfrak{M}}{|\mathfrak{M}|} \right)^2 \right\rangle \left\langle \left(\frac{\operatorname{Re} \mathfrak{M}}{|\mathfrak{M}|} \right)^2 \right\rangle, \end{aligned} \tag{3.29}$$

where we used $1 \geq \|F(z)^t F(z)\|_2 = \|\mathfrak{F}\|_2^2 \gtrsim |z|^4$ by (3.28) and $\langle \mathfrak{f}, (\sin \psi)^2 \mathfrak{f} \rangle \langle \mathfrak{f}, (\cos \psi)^2 \mathfrak{f} \rangle \leq 1$ in the second step. In order to estimate the last expression, we use $r_- \gtrsim |z|$ and $\|\mathfrak{F}(z)\|_2 \leq 1$ by (3.28) as well as (3.12a), (3.17) and (3.18) to get for the first factor

$$\inf_{x=1, \dots, n+p} \mathfrak{f}_x^4 \gtrsim r_-^{8L+8} r_+^{-16} \gtrsim_\delta \langle \operatorname{Im} \mathfrak{M} \rangle^{16}. \tag{3.30}$$

To estimate the last factor in (3.29), we multiply the real part of (3.6) with $|\mathfrak{M}|$ and obtain

$$(1 + \mathfrak{F}) \frac{\operatorname{Re} \mathfrak{M}}{|\mathfrak{M}|} = -\tau |\mathfrak{M}|$$

if $z = \tau + i\eta$ for $\tau, \eta \in \mathbb{R}$. Estimating $\|\cdot\|_2$ of the last equation yields

$$|\tau| \|\mathfrak{M}\|_2 \leq 2 \left\| \frac{\operatorname{Re} \mathfrak{M}}{|\mathfrak{M}|} \right\|_2$$

by (3.28). As $\|\mathfrak{M}\|_2 \geq \|\operatorname{Im} \mathfrak{M}\|_2 \geq \langle \operatorname{Im} \mathfrak{M} \rangle$ we get

$$2 \left\| \frac{\operatorname{Re} \mathfrak{M}}{|\mathfrak{M}|} \right\|_2 \geq |\tau| \langle \operatorname{Im} \mathfrak{M} \rangle. \tag{3.31}$$

Finally, we use (3.30) for the first factor in (3.29) and (3.31) for the last factor and apply the last estimate in (3.12a) and Jensen's inequality, $\langle (\operatorname{Im} \mathfrak{M})^2 \rangle \geq \langle \operatorname{Im} \mathfrak{M} \rangle^2$, to estimate the second factor which yields

$$\left| 1 - \|F(z)^t F(z)\|_2 \langle v_1, U_1 v_1 \rangle \langle v_2, U_2 v_2 \rangle \right| \gtrsim_\delta |\tau|^2 \langle \operatorname{Im} \mathfrak{M} \rangle^\kappa. \tag{3.32}$$

This completes the proof of (3.27).

Next, we bound $\operatorname{Gap}(F(z)F(z)^t)$ from below by applying Lemma 3.3 with $r_- := \inf_x |\mathfrak{M}_x(z)|$ and $r_+ := \|\mathfrak{M}(z)\|_\infty$. As $F(z) = \widehat{F}(|\mathfrak{M}(z)|)$ we have

$$\operatorname{Gap}(F(z)F(z)^t) \gtrsim_\delta \langle \operatorname{Im} \mathfrak{M}(z) \rangle^{16},$$

where we used the estimates in (3.12a) and (3.28). Combining this estimate on $\operatorname{Gap}(F(z)F(z)^t)$ with (3.27) and (3.21) and increasing κ , we obtain

$$\|\mathfrak{B}^{-1}(z)\|_2 \lesssim_\delta \min \left\{ \frac{1}{(\operatorname{Re} z)^2 \langle \operatorname{Im} \mathfrak{M}(z) \rangle^\kappa}, \frac{1}{\operatorname{Im} z}, \frac{1}{\operatorname{dist}(\operatorname{Re} z, \operatorname{supp} \rho)^2} \right\}$$

as $\|\mathfrak{B}^{-1}(z)\|_2 \leq (1 - \|\mathfrak{F}(z)\|_2)^{-1}$ and $\delta \leq |z| \leq 10$. □

Lemma 3.8 (Continuity of the solution). *If \mathfrak{M} is the solution of the QVE (3.6) then $z \mapsto \langle \mathfrak{M}(z) \rangle$ can be extended to a locally Hölder-continuous function on $\overline{\mathbb{H}} \setminus \{0\}$. Moreover, for every $\delta > 0$ there is a constant c depending on δ and the model parameters such that*

$$|\langle \mathfrak{M}(z_1) \rangle - \langle \mathfrak{M}(z_2) \rangle| \leq c|z_1 - z_2|^{1/(\kappa+1)} \tag{3.33}$$

for all $z_1, z_2 \in \overline{\mathbb{H}} \setminus \{0\}$ such that $\delta \leq |z_1|, |z_2| \leq 10$ where κ is the universal constant of Lemma 3.5.

Proof. In a first step, we prove that $z \mapsto \langle \text{Im } \mathfrak{M}(z) \rangle$ is locally Hölder-continuous. Taking the derivative of (3.6) with respect to $z \in \mathbb{H}$ yields

$$(1 - \mathfrak{M}^2(z)\mathfrak{S})\partial_z \mathfrak{M}(z) = \mathfrak{M}(z)^2.$$

By using that $\partial_z \phi = i2\partial_z \text{Im } \phi$ for every analytic function ϕ and taking the average, we get

$$i2\partial_z \langle \text{Im } \mathfrak{M} \rangle = \langle |\mathfrak{M}|, \mathfrak{B}^{-1}|\mathfrak{M}| \rangle.$$

Here, we suppressed the z -dependence of \mathfrak{B}^{-1} . We apply Cauchy-Schwarz inequality and use (3.7), (3.23) and (3.12a) to obtain

$$|\partial_z \langle \text{Im } \mathfrak{M} \rangle| \leq \|\mathfrak{M}\|_2 \|\mathfrak{B}^{-1}\|_{2 \rightarrow 2} \|\mathfrak{M}\|_2 \lesssim_\delta \min\{(\text{Re } z)^{-2} \langle \text{Im } \mathfrak{M} \rangle^{-\kappa}, (\text{Im } z)^{-1}\} \lesssim_\delta \langle \text{Im } \mathfrak{M} \rangle^{-\kappa}$$

for all $z \in \mathbb{H}$ satisfying $\delta \leq |z| \leq 10$. This implies that $z \mapsto \langle \text{Im } \mathfrak{M}(z) \rangle$ is Hölder-continuous with Hölder-exponent $1/(\kappa + 1)$ on $z \in \mathbb{H}$ satisfying $\delta \leq |z| \leq 10$. Moreover, it has a unique continuous extension to $I_\delta := \{\tau \in \mathbb{R}; \delta/3 \leq |\tau| \leq 10\}$. Multiplying this continuous function on I_δ by π^{-1} yields a Lebesgue-density of the measure ρ (cf. (3.8)) restricted to I_δ .

We conclude that the Stieltjes transform $\langle \mathfrak{M} \rangle$ has the same regularity by decomposing ρ into a measure supported around zero and a measure supported away from zero and using Lemma A.7 in [2]. □

For estimating the difference between the solution \mathfrak{M} of the QVE and a solution \mathfrak{g} of the perturbed QVE (3.9), we introduce the deterministic control parameter

$$\vartheta(z) := \langle \text{Im } \mathfrak{M}(z) \rangle + \text{dist}(z, \text{supp } \rho), \quad z \in \mathbb{H}.$$

Lemma 3.9 (Stability of the QVE). *Let $\delta \gtrsim 1$. Suppose there are some functions $\mathfrak{d}: \mathbb{H} \rightarrow \mathbb{C}^{p+n}$ and $\mathfrak{g}: \mathbb{H} \rightarrow (\mathbb{C} \setminus \{0\})^{n+p}$ satisfying (3.9). Then there exist universal constants $\kappa_1, \kappa_2 \in \mathbb{N}$ and a function $\lambda_*: \mathbb{H} \rightarrow (0, \infty)$, independent of n and p , such that $\lambda_*(10i) \geq 1/5$, $\lambda_*(z) \gtrsim_\delta \vartheta(z)^{\kappa_1}$ and*

$$\|\mathfrak{g}(z) - \mathfrak{M}(z)\|_\infty \mathbf{1}(\|\mathfrak{g}(z) - \mathfrak{M}(z)\|_\infty \leq \lambda_*(z)) \lesssim_\delta \vartheta(z)^{-\kappa_2} \|\mathfrak{d}(z)\|_\infty \tag{3.34}$$

for all $z \in \mathbb{H}$ satisfying $\delta \leq |z| \leq 10$. Moreover, there are a universal constant $\kappa_3 \in \mathbb{N}$ and a matrix-valued function $T: \mathbb{H} \rightarrow \mathbb{C}^{(p+n) \times (p+n)}$, depending only on S and satisfying $\|T(z)\|_{\infty \rightarrow \infty} \lesssim 1$, such that

$$|\langle w, \mathfrak{g}(z) - \mathfrak{M}(z) \rangle| \mathbf{1}(\|\mathfrak{g}(z) - \mathfrak{M}(z)\|_\infty \leq \lambda_*(z)) \lesssim_\delta \vartheta(z)^{-\kappa_3} (\|w\|_\infty \|\mathfrak{d}(z)\|_\infty^2 + |\langle T(z)w, \mathfrak{d}(z) \rangle|) \tag{3.35}$$

for all $w \in \mathbb{C}^{p+n}$ and $z \in \mathbb{H}$ satisfying $\delta \leq |z| \leq 10$.

Proof. We set $\Phi(z) := \max\{1, \|\mathfrak{M}(z)\|_\infty\}$, $\Psi(z) := \max\{1, \|\mathfrak{B}^{-1}(z)\|_\infty\}$ and $\lambda_*(z) := (2\Phi\Psi)^{-1}$. As $\Phi(z) \leq \max\{1, (\text{Im } z)^{-1}\}$ and $\|\mathfrak{B}^{-1}(z)\|_\infty \leq (1 - \|\mathfrak{F}(z)\|_\infty)^{-1} \leq (1 - (\text{Im } z)^{-2})^{-1}$ due to $\|\mathfrak{M}(z)\|_\infty \leq (\text{Im } z)^{-1}$ we obtain $\lambda_*(10i) \geq 1/5$. We obtain $\langle \text{Im } \mathfrak{M}(z) \rangle^{-1} \gtrsim_\delta 1$ by

(3.12a) and $\delta \leq |z|$. Thus, for $z \in \mathbb{H}$ satisfying $\delta \leq |z| \leq 10$ the first estimate in (3.11a), the last estimate in (3.12a) and (3.24) yield

$$\Phi \lesssim_{\delta} \vartheta^{-1}, \quad \Psi \lesssim_{\delta} \vartheta^{-\kappa-2},$$

where κ is the universal constant from Lemma 3.5. Therefore, $\lambda_*(z) \gtrsim_{\delta} \vartheta(z)^{\kappa+3}$ and Lemma 5.11 in [2] yield the assertion as $\|w\|_1 = (p+n)^{-1} \sum_i |w_i| \leq \|w\|_{\infty}$. \square

3.4 Proof of Theorem 2.1

Proof of Theorem 2.1. We start by proving the existence of the solution m of (2.1). Let $\mathfrak{M} = (M_1, M_2)^t$ be the solution of (3.6) satisfying $\text{Im } \mathfrak{M}(z) > 0$ for $z \in \mathbb{H}$. For $\zeta \in \mathbb{H}$, we set $m(\zeta) := M_1(\sqrt{\zeta})/\sqrt{\zeta}$. Then it is straightforward to check that m satisfies (2.1) by solving (3.5b) for M_2 and plugging the result into (3.5a). Note that $\text{Im } m(\zeta) > 0$ for all $\zeta \in \mathbb{H}$ since $M_{1,i}$ is the Stieltjes transform of a symmetric measure on \mathbb{R} (cf. the explanation before (3.7) for the symmetry of this measure).

Next, we show the uniqueness of the solution m of (2.1) with $\text{Im } m(\zeta) > 0$ for $\zeta \in \mathbb{H}$ which is a consequence of the uniqueness of the solution of (3.6). Therefore, we set $m_1(\zeta) := m(\zeta)$, $m_2(\zeta) := -1/(\zeta(1+S^t m_1(\zeta)))$ and $\mathfrak{m}(\zeta) := (m_1(\zeta), m_2(\zeta))^t$ for $\zeta \in \mathbb{H}$. From (2.1), we see that

$$|m_1| = \frac{1}{\left| \zeta - S \frac{1}{1+S^t m_1} \right|} \leq \frac{1}{\text{Im } \zeta + S \frac{1}{|1+S^t m_1|} S^t \text{Im } m_1} \leq \frac{1}{\text{Im } \zeta} \tag{3.36}$$

for all $\zeta \in \mathbb{H}$. Since m_2 satisfies

$$-\frac{1}{m_2(\zeta)} = \zeta + S^t \frac{1}{1+S m_2}(\zeta) \tag{3.37}$$

for $\zeta \in \mathbb{H}$, a similar argument yields $|m_2| \leq (\text{Im } \zeta)^{-1}$. Combining these two estimates, we obtain $|\mathfrak{m}(\zeta)| \leq (\text{Im } \zeta)^{-1}$ for all $\zeta \in \mathbb{H}$. Therefore, multiplying (2.1) and (3.37) with m_1 and m_2 , respectively, yields

$$|1 + i\xi \mathfrak{m}_x(i\xi)| \leq \|\mathfrak{m}(i\xi)\|_{\infty} \frac{1}{1 - \|\mathfrak{m}(i\xi)\|_{\infty}} \leq \frac{1}{\xi - 1} \rightarrow 0$$

for $\xi \rightarrow \infty$ and $x = 1, \dots, n+p$ where we used $|\mathfrak{m}(\zeta)| \leq (\text{Im } \zeta)^{-1}$ in the last but one step. Thus, \mathfrak{m}_x is the Stieltjes transform of a probability measure ν_x on \mathbb{R} for all $x = 1, \dots, n+p$. Multiplying (2.1) by m_1 , taking the imaginary part and averaging at $\zeta = \chi + i\xi$, for $\chi \in \mathbb{R}$ and $\xi > 0$, yields

$$\begin{aligned} \chi \langle \text{Im } m_1 \rangle + \xi \langle \text{Re } m_1 \rangle &= - \left\langle \text{Re } m_1, S \frac{1}{|1+S^t m_1|^2} S^t \text{Im } m_1 \right\rangle \\ &\quad + \left\langle \text{Im } m_1, S \frac{1}{|1+S^t m_1|^2} (1+S^t \text{Re } m_1) \right\rangle \\ &= \left\langle \text{Im } m_1, S \frac{1}{|1+S^t m_1|^2} \right\rangle \geq 0, \end{aligned} \tag{3.38}$$

where we used the definition of the transposed matrix and the symmetry of the scalar product in the last step. On the other hand, we have

$$\chi \langle \text{Im } m_1 \rangle + \xi \langle \text{Re } m_1 \rangle = \int_{\mathbb{R}} \frac{\xi t}{(t-\chi)^2 + \xi^2} \nu(dt).$$

Assuming that there is a $\chi < 0$ such that $\chi \in \text{supp } \nu$ we obtain that $\chi \langle \text{Im } m_1 \rangle + \xi \langle \text{Re } m_1 \rangle < 0$ for $\xi \downarrow 0$ which contradicts (3.38). Therefore $\text{supp } \nu_x \subset [0, \infty)$ for $x = 1, \dots, p$.

Together with a similar argument for m_2 , we get that $\text{supp } \nu_x \subset [0, \infty)$ for all $x = 1, \dots, n + p$. In particular, we can assume that \mathfrak{m} is defined on $\mathbb{C} \setminus [0, \infty)$. We set $M_1(z) := zm_1(z^2)$, $M_2(z) := zm_2(z^2)$ and $\mathfrak{M}(z) := (M_1(z), M_2(z))^t$ for all $z \in \mathbb{H}$. Hence, we get

$$\text{Im } \mathfrak{M}_x(\tau + i\eta) = \eta \int_{[0, \infty)} \frac{t + \tau^2 + \eta^2}{(t - \tau^2 + \eta^2)^2 + 4\eta^2\tau^2} \nu_x(dt)$$

as $\text{supp } \nu_x \subset [0, \infty)$. This implies $\text{Im } \mathfrak{M}(z) > 0$ for $z \in \mathbb{H}$ and thus the uniqueness of solutions of (3.6) with positive imaginary part implies the uniqueness of m_1 .

Finally, we verify the claim about the structure of the probability measure representing $\langle m \rangle$. By Lemma 3.8 and the statements following (3.6), $\langle M_1 \rangle$ is the Stieltjes transform of $\pi_* \delta_0 + \rho_1(\omega) d\omega$ for some $\pi_* \in [0, 1]$ and some symmetric Hölder-continuous function $\rho_1: \mathbb{R} \setminus \{0\} \rightarrow [0, \infty)$ whose support is contained in $[-2, 2]$. Therefore, m is the Stieltjes transform of $\nu(d\omega) := \pi_* \delta_0(d\omega) + \pi(\omega) \mathbf{1}(\omega > 0) d\omega$ where $\pi(\omega) = \omega^{-1/2} \rho_1(\omega^{1/2})$ for $\omega > 0$. Thus, the support of ν is contained in $[0, 4]$. \square

3.5 Square Gram matrices

In this subsection, we study the stability of (3.6) for $n = p$. Here, we assume (A), (E1) and (F1). These assumptions are strictly stronger than (A), (B) and (D) (cf. Remark 2.6).

For the following arguments, it is important that \mathfrak{M} is purely imaginary for $\text{Re } z = 0$ as $\mathfrak{M}(-\bar{z}) = -\overline{\mathfrak{M}(z)}$ for all $z \in \mathbb{H}$. If we set

$$\mathfrak{v}(z) = \text{Im } \mathfrak{M}(z) \tag{3.39}$$

for $z \in \mathbb{H}$, then \mathfrak{v} fulfills

$$\frac{1}{\mathfrak{v}(i\eta)} = \eta + \mathfrak{S}\mathfrak{v}(i\eta) \tag{3.40}$$

for all $\eta \in (0, \infty)$ due to (3.6). The study of this equation will imply the stability of the QVE at $z = 0$. The following proposition is the main result of this subsection.

Proposition 3.10. *Let $n = p$, i.e., (E1) holds true, and S satisfies (A) as well as (F1).*

- (i) *There exists a $\widehat{\delta} \sim 1$ such that $|\mathfrak{M}(z)| \sim 1$ uniformly for all $z \in \mathbb{H}$ satisfying $|z| \leq 10$ and $\text{Re } z \in [-\widehat{\delta}, \widehat{\delta}]$. Moreover, $\langle \text{Im } \mathfrak{M}(z) \rangle \gtrsim 1$ for all $z \in \mathbb{H}$ satisfying $|z| \leq 10$ and $\text{Re } z \in [-\widehat{\delta}, \widehat{\delta}]$ and there is a $\mathfrak{v}(0) = (v_1(0), v_2(0))^t \in \mathbb{R}^p \oplus \mathbb{R}^p$ such that $\mathfrak{v}(0) \sim 1$ and*

$$i\mathfrak{v}(0) = \lim_{\eta \downarrow 0} \mathfrak{M}(i\eta).$$

- (ii) *(Stability of the QVE at $z = 0$) Suppose that some functions $\mathfrak{d} = (d_1, d_2)^t: \mathbb{H} \rightarrow \mathbb{C}^{p+p}$ and $\mathfrak{g} = (g_1, g_2)^t: \mathbb{H} \rightarrow (\mathbb{C} \setminus \{0\})^{p+p}$ satisfy (3.9) and*

$$\langle g_1(z) \rangle = \langle g_2(z) \rangle \tag{3.41}$$

for all $z \in \mathbb{H}$. There are numbers $\lambda_*, \widehat{\delta} \gtrsim 1$, depending only on S , such that

$$\|\mathfrak{g}(z) - \mathfrak{M}(z)\|_\infty \mathbf{1}(\|\mathfrak{g}(z) - \mathfrak{M}(z)\|_\infty \leq \lambda_*) \lesssim \|\mathfrak{d}(z)\|_\infty \tag{3.42}$$

for all $z \in \mathbb{H}$ satisfying $|z| \leq 10$ and $\text{Re } z \in [-\widehat{\delta}, \widehat{\delta}]$. Moreover, there is a matrix-valued function $T: \mathbb{H} \rightarrow \mathbb{C}^{2p \times 2p}$, depending only on S and satisfying $\|T(z)\|_\infty \lesssim 1$, such that

$$|\langle w, \mathfrak{g}(z) - \mathfrak{M}(z) \rangle| \cdot \mathbf{1}(\|\mathfrak{g}(z) - \mathfrak{M}(z)\|_\infty \leq \lambda_*) \lesssim \|w\|_\infty \|\mathfrak{d}(z)\|_\infty^2 + |\langle T(z)w, \mathfrak{d}(z) \rangle| \tag{3.43}$$

for all $w \in \mathbb{C}^{2p}$ and $z \in \mathbb{H}$ satisfying $|z| \leq 10$ and $\text{Re } z \in [-\widehat{\delta}, \widehat{\delta}]$.

The remainder of this subsection will be devoted to the proof of this proposition. Therefore, we will always assume that (A), (E1) and (F1) are satisfied.

Lemma 3.11. *The function $\mathbf{v}: i(0, \infty) \rightarrow \mathbb{R}^{2p}$ defined in (3.39) satisfies*

$$1 \lesssim \inf_{\eta \in (0, 10]} \mathbf{v}(i\eta) \leq \sup_{\eta > 0} \|\mathbf{v}(i\eta)\|_\infty \lesssim 1. \tag{3.44}$$

If we write $\mathbf{v} = (v_1, v_2)^t$ for $v_1, v_2: i(0, \infty) \rightarrow \mathbb{R}^p$, then

$$\langle v_1(i\eta) \rangle = \langle v_2(i\eta) \rangle \tag{3.45}$$

for all $\eta \in (0, \infty)$.

The estimate in (3.44), with some minor modifications which we will explain next, is shown as in the proof of (6.30) of [2].

Proof. From (3.40) and the definition of \mathfrak{S} , we obtain $\eta \langle v_1 \rangle - \eta \langle v_2 \rangle = \langle v_1, S v_2 \rangle - \langle v_2, S^t v_1 \rangle = 0$ for all $\eta \in (0, \infty)$ which proves (3.45). Differing from [2], the discrete functional \tilde{J} is defined as follows:

$$\tilde{J}(u) = \frac{\varphi}{2K} \sum_{i,j=1}^{2K} u(i) \mathcal{Z}_{ij} u(j) - \sum_{i=1}^{2K} \log u(i) \tag{3.46}$$

for $u \in (0, \infty)^{2K}$ (we used the notation $u(i)$ to denote the i -th entry of u) where \mathcal{Z} is the $2K \times 2K$ matrix with entries in $\{0, 1\}$ defined by

$$\mathcal{Z} = \begin{pmatrix} 0 & Z \\ Z^t & 0 \end{pmatrix}. \tag{3.47}$$

Decomposing $u = (u_1, u_2)^t$ for $u_1, u_2 \in (0, \infty)^K$ and writing $u_1(i) = u(i)$ and $u_2(j) = u(K + j)$ for their entries we obtain

$$\tilde{J}(u) = \frac{\varphi}{K} \sum_{i,j=1}^K u_1(i) Z_{ij} u_2(j) - \sum_{i=1}^K (\log u_1(i) + \log u_2(i)). \tag{3.48}$$

Lemma 3.12. *If $\Psi < \infty$ is a constant such that $u = (u_1, u_2)^t \in (0, \infty)^K \times (0, \infty)^K$ satisfies*

$$\tilde{J}(u) \leq \Psi,$$

where \tilde{J} is defined in (3.46), and $\langle u_1 \rangle = \langle u_2 \rangle$, then there is a constant $\Phi < \infty$ depending only on (Ψ, φ, K) such that

$$\max_{k=1}^{2K} u(k) \leq \Phi.$$

Proof. We define $\tilde{Z}_{ij} := Z_{i\sigma(j)}$ where σ is a permutation of $\{1, \dots, K\}$ such that $\tilde{Z}_{ii} = 1$ for all $i = 1, \dots, K$ where we use the FID property of Z . Moreover, we set $M_{ij} := u_1(i) \tilde{Z}_{ij} u_2(\sigma(j))$ and follow the proof of Lemma 6.10 in [2] to obtain

$$u_1(i) u_2(\sigma(j)) \lesssim (M^{K-1})_{ij} \lesssim 1$$

for all $i, j = 1, \dots, K$. Averaging over i and j yields

$$\langle u_1 \rangle^2 = \langle u_2 \rangle^2 \lesssim 1$$

where we used $\langle u_1 \rangle = \langle u_2 \rangle$. This concludes the proof of Lemma 3.12. □

Recalling the function \mathbf{v} in Lemma 3.11, we set $u = (\langle \mathbf{v} \rangle_1, \dots, \langle \mathbf{v} \rangle_{2K})$ with $\langle \mathbf{v} \rangle_i = Kp^{-1} \sum_{x \in I_i} v_x$, where $I_i := p + I_{i-K}$ for $i \geq K + 1$. Then we have $\langle u_1 \rangle = \langle u_2 \rangle$ by (3.45) and since I_1, \dots, I_{2K} is an equally sized partition of $\{1, \dots, 2p\}$. Therefore, the assumptions of Lemma 3.12 are met which implies (3.44) of Lemma 3.11 as in [2]. \square

We recall from Lemma 3.4 that $\mathbf{f} = (f_1, f_2)$ is the unique nonnegative, normalized eigenvector of \mathfrak{F} corresponding to the eigenvalue $\|\mathfrak{F}\|_2$. Moreover, we define $\mathbf{f}_- := (f_1, -f_2)$ which clearly satisfies

$$\mathfrak{F}\mathbf{f}_- = -\|\mathfrak{F}\|_2\mathbf{f}_-. \tag{3.49}$$

Since the spectrum of \mathfrak{F} is symmetric, $\sigma(\mathfrak{F}) = -\sigma(\mathfrak{F})$ with multiplicities, and $\|\mathfrak{F}\|_2$ is a simple eigenvalue of \mathfrak{F} , the same is true for the eigenvalue $-\|\mathfrak{F}\|_2$ of \mathfrak{F} and \mathbf{f}_- spans its associated eigenspace. We introduce

$$\mathbf{e}_- := \begin{pmatrix} 1 \\ -1 \end{pmatrix} \in \mathbb{C}^p \oplus \mathbb{C}^p. \tag{3.50}$$

Lemma 3.13. *For $\eta \in (0, \infty)$, the derivative of \mathfrak{M} satisfies*

$$\mathfrak{M}'(i\eta) = \frac{d}{dz}\mathfrak{M}(i\eta) = -\mathbf{v}(i\eta)(1 + \mathfrak{F}(i\eta))^{-1}\mathbf{v}(i\eta). \tag{3.51}$$

Moreover, $|\mathfrak{M}'(i\eta)| \lesssim 1$ uniformly for $\eta \in (0, 10]$.

Proof. In the whole proof, the quantities \mathbf{v} , \mathbf{f} , \mathbf{f}_- and \mathfrak{F} are evaluated at $z = i\eta$ for $\eta > 0$. Therefore, we will mostly suppress the z -dependence of all quantities. Differentiating (3.6) with respect to z and using (3.39) yields

$$-(1 + \mathfrak{F})\frac{\mathfrak{M}'}{\mathbf{v}} = \mathbf{v}.$$

As $\|\mathfrak{F}\|_2 < 1$ by (3.20), the matrix $(1 + \mathfrak{F})$ is invertible which yields (3.51) for all $\eta \in (0, \infty)$.

In order to prove $|\mathfrak{M}'(i\eta)| \lesssim 1$ uniformly for $\eta \in (0, \infty)$, we first prove that

$$|\langle \mathbf{f}_-(i\eta)\mathbf{v}(i\eta) \rangle| \leq O(\eta). \tag{3.52}$$

We define the auxiliary operator $A := \|\mathfrak{F}\|_2 + \mathfrak{F} = 1 + \mathfrak{F} - \eta\frac{\langle \mathbf{f}\mathbf{v} \rangle}{\langle \mathbf{f} \rangle}$ where we used (3.20) and (3.39). Note that

$$A\mathbf{f}_- = 0, \quad A\mathbf{e}_- = \mathbf{e}_- + \mathfrak{F}\mathbf{e}_- - \eta\frac{\langle \mathbf{f}\mathbf{v} \rangle}{\langle \mathbf{f} \rangle}\mathbf{e}_- = O(\eta), \tag{3.53}$$

where we used $\mathfrak{F}\mathbf{e}_- = -\mathbf{e}_- + \eta\begin{pmatrix} v_1 \\ -v_2 \end{pmatrix}$ which follows from (3.6) and the definition of \mathfrak{F} .

Defining $Qu := u - \langle \mathbf{f}_-u \rangle\mathbf{f}_-$ for $u \in \mathbb{C}^{2p}$ and decomposing

$$\mathbf{e}_- = \langle \mathbf{f}_-\mathbf{e}_- \rangle\mathbf{f}_- + Q\mathbf{e}_-$$

yield $AQ\mathbf{e}_- = O(\eta)$ because of (3.53). As $|\mathfrak{M}(i\eta)| \sim 1$ by (3.44) for $\eta \in (0, 10]$ the bound (3.19) in Lemma 3.3 implies that there is an $\varepsilon \sim 1$ such that for all $\eta \in (0, 10]$ we have

$$\sigma(\mathfrak{F}) \subset \{-\|\mathfrak{F}\|_2\} \cup [-\|\mathfrak{F}\|_2 + \varepsilon, \|\mathfrak{F}\|_2 - \varepsilon] \cup \{\|\mathfrak{F}\|_2\}. \tag{3.54}$$

Since $-\|\mathfrak{F}\|_2$ is a simple eigenvalue of \mathfrak{F} and (3.49) the symmetric matrix $A = \|\mathfrak{F}\|_2 + \mathfrak{F}$ is invertible on \mathbf{f}_-^\perp and $\|(A|_{\mathbf{f}_-^\perp})^{-1}\|_2 = \varepsilon^{-1} \sim 1$. As $\mathbf{f}_- \perp Q\mathbf{e}_-$ we conclude $Q\mathbf{e}_- = O(\eta)$ and hence

$$(1 - \langle \mathbf{f} \rangle)(1 + \langle \mathbf{f} \rangle) = 1 - \langle \mathbf{f} \rangle^2 = 1 - \langle \mathbf{f}_-\mathbf{e}_- \rangle^2 = \|Q\mathbf{e}_-\|_2^2 = O(\eta^2). \tag{3.55}$$

Thus, using (3.45) and (3.55), this implies

$$|\langle f_{-}(i\eta)\mathbf{v}(i\eta)\rangle| = |\langle \mathbf{v}\mathbf{e}_{-}\rangle + \langle \mathbf{v}[f_{-} - \mathbf{e}_{-}]\rangle| \lesssim \|f_{-} - \mathbf{e}_{-}\|_2 = \sqrt{2(1 - \langle f\rangle)} = O(\eta),$$

which concludes the proof of (3.52).

In (3.51), we decompose $\mathbf{v} = \langle f_{-}\mathbf{v}\rangle f_{-} + Q\mathbf{v}$ and, using $\mathfrak{F}f_{-} = -\|\mathfrak{F}\|_2 f_{-}$ and (3.20), we obtain

$$\mathfrak{M}' = -\mathbf{v} \frac{\langle f_{-}\mathbf{v}\rangle}{\eta} \frac{\langle f\rangle}{\langle f\mathbf{v}\rangle} f_{-} - \mathbf{v}(1 + \mathfrak{F})^{-1}Q\mathbf{v}.$$

Using (3.54), we see that $\|(1 + \mathfrak{F})^{-1}Q\mathbf{v}\|_2 \sim 1$ uniformly for $\eta \in (0, 10]$. Together with $\langle f_{-}(i\eta)\mathbf{v}(i\eta)\rangle = O(\eta)$ by (3.52), this yields $|\mathfrak{M}'(i\eta)| \lesssim 1$ uniformly for $\eta \in (0, 10]$. \square

The previous lemma, (3.40) and Lemma 3.11 imply that $\mathbf{v}(0) := \lim_{\eta \downarrow 0} \mathbf{v}(i\eta)$ exists and satisfies

$$\mathbf{v}(0) \sim 1, \quad 1 = \mathbf{v}(0)\mathfrak{S}\mathbf{v}(0) = \mathfrak{F}(0)1, \quad \langle v_1(0)\rangle = \langle v_2(0)\rangle, \tag{3.56}$$

where $\mathbf{v}(0) = (v_1(0), v_2(0))^t$.

In the next lemma, we establish an expansion of $\mathfrak{M}(z)$ on the upper half-plane around $z = 0$. The proof of this result and later the stability estimates on $\mathfrak{g} - \mathfrak{M}$ will be a consequence of the equation

$$\mathfrak{B}u = e^{-i\psi}u\mathfrak{F}u + e^{-i\psi}\mathfrak{g}\mathfrak{d} \tag{3.57}$$

where $u = (\mathfrak{g} - \mathfrak{M})/|\mathfrak{M}|$ and $e^{i\psi} = \mathfrak{M}/|\mathfrak{M}|$. This quadratic equation in u was derived in Lemma 5.8 in [2].

Lemma 3.14. *For $z \in \mathbb{H}$, we have*

$$\mathfrak{M}(z) = i\mathbf{v}(0) - z\mathbf{v}(0)(1 + \mathfrak{F}(0))^{-1}\mathbf{v}(0) + O(|z|^2), \tag{3.58a}$$

$$\frac{\mathfrak{M}(z)}{|\mathfrak{M}(z)|} = i - (\operatorname{Re} z)(1 + \mathfrak{F}(0))^{-1}\mathbf{v}(0) + O(|z|^2). \tag{3.58b}$$

In particular, there is a $\hat{\delta} \sim 1$ such that $|\mathfrak{M}(z)| \sim 1$ uniformly for $z \in \mathbb{H}$ satisfying $\operatorname{Re} z \in [-\hat{\delta}, \hat{\delta}]$ and $|z| \leq 10$. Moreover,

$$\|f(z) - 1\|_{\infty} = O(|z|), \quad \|f_{-}(z) - \mathbf{e}_{-}\|_{\infty} = O(|z|). \tag{3.59}$$

Proof. In order to prove (3.58a), we consider (3.6) at z as a perturbation of (3.6) at $z = 0$ perturbed by $\mathfrak{d} = z$ in the notation of (3.9). The solution of the unperturbed equation is $\mathfrak{M} = i\mathbf{v}(0)$. Following the notation of (3.9), we find that (3.57) holds with $\mathfrak{g} = \mathfrak{M}(z)$ and $u(z) = (\mathfrak{M}(z) - i\mathbf{v}(0))/\mathbf{v}(0)$. We write $u(z) = \theta(z)\mathbf{e}_{-} + w(z)$ with $w \perp \mathbf{e}_{-}$. (We will suppress the z -dependence in our notation.) Plugging this into (3.57) and projecting onto \mathbf{e}_{-} yields

$$\theta \langle \mathbf{v}(0)\rangle = -\langle \mathbf{e}_{-}\mathbf{v}(0)w\rangle, \tag{3.60}$$

where we used that $\mathfrak{F}(0)1 = 1$, i.e., $\langle \mathfrak{F}(0)w\rangle = \langle w\rangle$, $\langle \mathbf{e}_{-}w\mathfrak{F}(0)w\rangle = 0$ and $\langle v_1(0)\rangle = \langle v_2(0)\rangle$. Thus, we have $\theta = O(\|w\|_{\infty})$ because of (3.56), so that we conclude $-(1 + \mathfrak{F}(0))w = z\mathbf{v}(0) + O(\|w\|_{\infty}^2 + |z|\|w\|_{\infty})$. As w , $(1 + \mathfrak{F}(0))w$ and $\mathbf{v}(0)$ are orthogonal to \mathbf{e}_{-} , the error term is also orthogonal to it which implies

$$w = -z(1 + \mathfrak{F}(0))^{-1}\mathbf{v}(0) + O(|z|^2) \tag{3.61}$$

using that $(1 + \mathfrak{F}(0))^{-1}$ is bounded on \mathbf{e}_{-}^{\perp} .

Observing that $\langle M_1(z)\rangle = \langle M_2(z)\rangle$ for $z \in \mathbb{H}$ by (3.6) and differentiating this relation yields $\langle \mathfrak{M}'(i\eta)\mathbf{e}_{-}\rangle = 0$ for all $\eta \in (0, \infty)$. Hence,

$$\langle \mathbf{e}_{-}\mathbf{v}(0)(1 + \mathfrak{F}(0))^{-1}\mathbf{v}(0)\rangle = -\lim_{\eta \downarrow 0} \langle \mathbf{e}_{-}\mathfrak{M}'(i\eta)\rangle = 0 \tag{3.62}$$

by Lemma 3.13.

Plugging (3.61) into (3.60), we obtain

$$\theta \langle \mathbf{v}(0) \rangle = \langle \mathbf{e}_- \mathbf{v}(0) (1 + \mathfrak{F}(0))^{-1} \mathbf{v}(0) \rangle + O(|z|^2) = O(|z|^2),$$

where we used (3.62). Hence, $\mathfrak{M}(z) = \mathbf{v}(0)(u + i\mathbf{v}(0))$ concludes the proof of (3.58a) which immediately implies (3.58b).

Using the expansion of \mathfrak{M} in (3.58a) in a similar argument as in the proof of $\|\mathfrak{f}_-(i\eta) - \mathbf{e}_-\|_2 = O(\eta)$ in Lemma 3.13 yields

$$\|\mathfrak{f}(z) - 1\|_2 = \|\mathfrak{f}_-(z) - \mathbf{e}_-\|_2 = O(|z|).$$

Similarly, using (3.26), we obtain (3.59). □

By a standard argument from perturbation theory and possibly reducing $\widehat{\delta} \sim 1$, we can assume that $\mathfrak{B}(z)$ has a unique eigenvalue $\beta(z)$ of smallest modulus for $z \in \mathbb{H}$ satisfying $|\operatorname{Re} z| \leq \widehat{\delta}$ and $|z| \leq 10$ such that $|\beta'| - |\beta| \gtrsim 1$ for $\beta' \in \sigma(\mathfrak{B}(z))$ and $\beta' \neq \beta$. This follows from $|\mathfrak{M}| \sim 1$ and thus $\operatorname{Gap}(F(z)F(z)^t) \gtrsim 1$ by Lemma 3.3. For $z \in \mathbb{H}$ satisfying $|\operatorname{Re} z| \leq \widehat{\delta}$ and $|z| \leq 10$, we therefore find a unique (unnormalized) vector $b(z) \in \mathbb{C}^{2p}$ such that $\mathfrak{B}(z)b(z) = \beta(z)b(z)$ and $\langle \mathfrak{f}_-, b(z) \rangle = 1$.

We introduce the spectral projection P onto the spectral subspace of $\mathfrak{B}(z)$ associated to $\beta(z)$ which fulfills the relation

$$P = \frac{\langle \bar{b}, \cdot \rangle}{\langle b, b \rangle} b.$$

Note that P is not an orthogonal projection in general. Let $Q := 1 - P$ denote the complementary projection onto the spectral subspace of $\mathfrak{B}(z)$ not containing $\beta(z)$ (this Q is different from the one in the proof of Lemma 3.13). Since $\mathfrak{B}(z) = -1 - \mathfrak{F}(z) + O(|z|)$ we obtain

$$\|b(z) - \mathbf{e}_-\|_\infty = \|\overline{b(z)} - \mathbf{e}_-\|_\infty = O(|z|) \tag{3.63}$$

for $z \in \mathbb{H}$ satisfying $|\operatorname{Re} z| \leq \widehat{\delta}$ and $|z| \leq 10$.

Lemma 3.15. *By possibly reducing $\widehat{\delta}$ from Lemma 3.14, but still $\widehat{\delta} \gtrsim 1$, we have*

$$\|\mathfrak{B}^{-1}(z)\|_\infty \lesssim \frac{1}{|z|}, \quad \|\mathfrak{B}^{-1}(z)Q\|_\infty + \|(\mathfrak{B}^{-1}(z)Q)^*\|_\infty \lesssim 1 \tag{3.64}$$

for $z \in \mathbb{H}$ satisfying $|\operatorname{Re} z| \leq \widehat{\delta}$ and $|z| \leq 10$.

Proof. Due to $|\mathfrak{M}(z)| \sim 1$ and using (3.26) with $R = \mathfrak{F}(z)$ and $D = |\mathfrak{M}(z)|^2/\mathfrak{M}(z)^2$, it is enough to prove the estimates in (3.64) with $\|\cdot\|_\infty$ replaced by $\|\cdot\|_2$. We first remark that $|\mathfrak{M}(z)| \sim 1$ and arguing similarly as in the proof of Lemma 3.4 imply $\|\mathfrak{B}^{-1}(z)\|_2 \lesssim (\operatorname{Im} z)^{-1}$.

Now we prove $\|\mathfrak{B}^{-1}(z)\|_2 \lesssim (\operatorname{Re} z)^{-1}$. We apply Lemma 3.6 and recall $U_1 = |M_1|^2/M_1^2$ and $U_2 = |M_2|^2/M_2^2$ to get

$$\begin{aligned} \operatorname{Im} \left(1 - \|F(z)^t F(z)\|_2 \left\langle \frac{f_1}{\|f_1\|_2}, U_1 \frac{f_1}{\|f_1\|_2} \right\rangle \left\langle \frac{f_2}{\|f_2\|_2}, U_2 \frac{f_2}{\|f_2\|_2} \right\rangle \right) \\ = \frac{\|F(z)^t F(z)\|_2}{\|f_1\|_2 \|f_2\|_2} \langle \mathbf{v}(0) \rangle \operatorname{Re} z + O(|z|^2), \end{aligned} \tag{3.65}$$

where we used (3.58b), (3.59) and $\|f_1\|_2, \|f_2\|_2, \|F(z)^t F(z)\|_2 \sim 1$. Since $\mathbf{v}(0) \sim 1$ and $\operatorname{Gap}(F(z)F(z)^t) \gtrsim 1$ by Lemma 3.3 and $|\mathfrak{M}(z)| \sim 1$, (3.65) and Lemma 3.6 yield $\|\mathfrak{B}^{-1}(z)\|_2 \lesssim (\operatorname{Re} z)^{-1}$ and hence $\|\mathfrak{B}^{-1}(z)\|_2 \lesssim \min\{(\operatorname{Im} z)^{-1}, (\operatorname{Re} z)^{-1}\} \lesssim |z|^{-1}$.

The estimate $\|\mathfrak{B}^{-1}(z)Q\|_\infty \lesssim 1$ in (3.64) follows from $\text{Gap}(F(z)F(z)^t) \gtrsim 1$ by Lemma 3.3, $|\mathfrak{M}(z)| \sim 1$ and a standard argument from perturbation theory as presented in Lemma 8.1 of [2]. Here, it might be necessary to reduce $\widehat{\delta}$. We remark that $\mathfrak{B}^* = |\mathfrak{M}|^2/\overline{\mathfrak{M}}^2 - \mathfrak{F}$ and similarly $P^* = \langle b, \cdot \rangle / \langle \overline{b}^2 \rangle \overline{b}$, i.e., \mathfrak{B}^* and P^* emerge by the same constructions where \mathfrak{M} is replaced by $\overline{\mathfrak{M}}$. Therefore, we obtain $\|(\mathfrak{B}^{-1}(z)Q)^*\|_\infty \lesssim 1$. \square

Proof of Proposition 3.10. The part (i) follows from the previous lemmata.

The part (ii) has already been proved for $|z| \geq \delta$ in Lemma 3.9 and for any $\delta \gtrsim 1$. Therefore, we restrict ourselves to $|z| \leq \delta$ for a sufficiently small $\delta \gtrsim 1$. We recall $e^{i\psi} = \mathfrak{M}/|\mathfrak{M}|$.

Owing to Lemma 3.14 and (3.63), there are positive constants $\delta, \Phi, \widehat{\Phi} \sim 1$ which only depend on the model parameters such that

$$\|\mathfrak{M}(z)\|_\infty \leq \Phi, \quad \|b(z) - \mathbf{e}_-\|_2 \|b\|_\infty + \|e^{-i\psi} + i\|_\infty \|b\|_\infty^2 \leq \widehat{\Phi} |\langle b^2 \rangle| |z| \tag{3.66}$$

for all $z \in \mathbb{H}$ satisfying $|z| \leq \delta$. Here, we used $\|w\|_2 \leq \|w\|_\infty$ for all $w \in \mathbb{C}^{2p}$. Note that we employed (3.63) for estimating $\|b - \mathbf{e}_-\|_2$ as well as to obtain $\|b\|_\infty \sim 1$ and $|\langle b^2 \rangle| \sim 1$ for all $z \in \mathbb{H}$ satisfying $|z| \leq \delta$ if $\delta \gtrsim 1$ is small enough.

Lemma 3.15 implies the existence of $\Psi, \widehat{\Psi} \sim 1$ such that

$$\|\mathfrak{B}^{-1}(z)\|_\infty \leq \Psi |z|^{-1}, \quad \|\mathfrak{B}^{-1}(z)Q\|_\infty \leq \widehat{\Psi} \tag{3.67}$$

for all $z \in \mathbb{H}$ satisfying $|z| \leq \delta$ if $1 \lesssim \delta \leq \widehat{\delta}$ is sufficiently small. With these definitions, we set

$$\lambda_* := \frac{1}{2\Phi(\Psi\widehat{\Phi} + \widehat{\Psi})}. \tag{3.68}$$

The estimate on $h := \mathfrak{g}(z) - \mathfrak{M}(z) = u|\mathfrak{M}|$ will be obtained from inverting \mathfrak{B} in (3.57). In order to control the right-hand side of (3.57), we decompose it, according to $1 = P + Q$, as

$$e^{-i\psi} u \mathfrak{F} u = \frac{\langle b e^{-i\psi} u \mathfrak{F} u \rangle}{\langle b^2 \rangle} b + Q e^{-i\psi} u \mathfrak{F} u, \quad e^{-i\psi} \mathfrak{g} \mathfrak{d} = \frac{\langle e^{-i\psi} \mathfrak{g} \mathfrak{d} b \rangle}{\langle b^2 \rangle} b + Q e^{-i\psi} \mathfrak{g} \mathfrak{d}.$$

Clearly, as $\|\mathfrak{G}\|_\infty \leq 1$ we have $\|(\mathfrak{B}^{-1}Q)(e^{-i\psi} u \mathfrak{F} u)\|_\infty \leq \widehat{\Psi} \|h\|_\infty^2$ and $\|(\mathfrak{B}^{-1}Q)(e^{-i\psi} \mathfrak{g} \mathfrak{d})\|_\infty \leq \widehat{\Psi} \|\mathfrak{g}\|_\infty \|\mathfrak{d}\|_\infty$ due to (3.67). Using $\langle \mathbf{e}_- h \mathfrak{G} h \rangle = 0$ and (3.66), we obtain

$$\begin{aligned} \left\| \langle b e^{-i\psi} u \mathfrak{F} u \rangle \frac{b}{\langle b^2 \rangle} \right\|_\infty &\leq (|-i \langle h \mathfrak{G} h \mathbf{e}_- \rangle| + |-i \langle (b - \mathbf{e}_-) h \mathfrak{G} h \rangle| + |\langle (e^{-i\psi} + i) b h \mathfrak{G} h \rangle|) \frac{\|b\|_\infty}{|\langle b^2 \rangle|} \\ &\leq \widehat{\Phi} |z| \|h\|_\infty^2. \end{aligned}$$

Similarly, due to (3.66) and $\langle \mathfrak{g} \mathfrak{d} \mathbf{e}_- \rangle = \langle g_1(z) d_1(z) \rangle - \langle g_2(z) d_2(z) \rangle = 0$ by the perturbed QVE (3.9), we get

$$\begin{aligned} \left\| \langle e^{-i\psi} \mathfrak{g} \mathfrak{d} b \rangle \frac{b}{\langle b^2 \rangle} \right\|_\infty &\leq (|\langle \mathfrak{g} \mathfrak{d} \mathbf{e}_- \rangle| + |\langle (b - \mathbf{e}_-) \mathfrak{g} \mathfrak{d} \rangle| + |\langle (e^{-i\psi} + i) b \mathfrak{g} \mathfrak{d} \rangle|) \frac{\|b\|_\infty}{|\langle b^2 \rangle|} \\ &\leq \widehat{\Phi} |z| \|\mathfrak{g}\|_\infty \|\mathfrak{d}\|_\infty. \end{aligned}$$

Thus, inverting \mathfrak{B} in (3.57), multiplying the result with $|\mathfrak{M}|$, taking its norm and using (3.67) yield

$$\|h\|_\infty \leq \Phi(\Psi\widehat{\Phi} + \widehat{\Psi}) \|h\|_\infty^2 + \Phi(\Psi\widehat{\Phi} + \widehat{\Psi}) \|\mathfrak{g}\|_\infty \|\mathfrak{d}\|_\infty,$$

which implies

$$\|h\|_\infty \mathbf{1}(\|h\|_\infty \leq \lambda_*) \leq \Phi(1 + 2\Phi(\Psi\widehat{\Phi} + \widehat{\Psi})) \|\mathfrak{d}\|_\infty$$

by the definition of λ_* in (3.68). This concludes the proof of (3.42).

For the proof of (3.43), inverting \mathfrak{B} in (3.57) and taking the scalar product with w yield

$$\begin{aligned} \langle w, h \rangle &= \langle w, \mathfrak{B}^{-1}(e^{-i\psi} h \mathfrak{S} h) \rangle + \frac{\langle w, |\mathfrak{M}| \mathfrak{B}^{-1} b \rangle}{\langle b^2 \rangle} \langle h \mathfrak{d} [(e^{-i\psi} + i)b - i(b - \epsilon_-)] \rangle \\ &\quad + \langle (\mathfrak{B}^{-1} Q)^*(|\mathfrak{M}|w), e^{-i\psi} h \mathfrak{d} \rangle + \langle Tw, \mathfrak{d} \rangle, \end{aligned} \tag{3.69}$$

where we used $\langle \epsilon_- \mathfrak{g} \mathfrak{d} \rangle = 0$ and set $Tw := \langle b^2 \rangle^{-1} \langle |\mathfrak{M}| \mathfrak{B}^{-1} b, w \rangle \overline{\mathfrak{M}} [(e^{i\psi} - i)\bar{b} + i(\bar{b} - \epsilon_-)] + e^{i\psi} \overline{\mathfrak{M}} (\mathfrak{B}^{-1} Q)^*(|\mathfrak{M}|w)$.

Using (3.66) and (3.67) as well as a similar argument as in the proof of (3.42) for the first term in the definition of T and $\|(\mathfrak{B}^{-1} Q)^*\|_\infty \lesssim 1$ by (3.64) for the second term, we obtain $\|T\|_\infty \lesssim 1$. Moreover, as above we see that the first term on the right-hand side of (3.69) is $\lesssim \|w\|_\infty \|h\|_\infty^2$. The estimates (3.66) and (3.67) imply that the second term on the right-hand side of (3.69) is $\lesssim \|w\|_\infty \|h\|_\infty \|\mathfrak{d}\|_\infty$. Applying (3.42) to these bounds yields (3.43). \square

3.6 Properly rectangular Gram matrices

In this subsection, we study the behaviour of M_1 and M_2 for z close to zero for p/n different from one. We establish that the density of the limiting distribution is zero around zero – a well-known feature of the Marchenko-Pastur distribution for p/n different from one.

We suppose that the assumptions (A), (C) and (D) are fulfilled and we will study the case $p > n$. More precisely, we assume that

$$\frac{p}{n} \geq 1 + d_* \tag{3.70}$$

for some $d_* > 0$ which will imply that each component of M_1 diverges at $z = 0$ whereas each component of M_2 stays bounded at $z = 0$. Later, we will see that these properties carry over to m_1 and m_2 . We use the notation $D_\delta(w) := \{z \in \mathbb{C} : |z - w| < \delta\}$ for $\delta > 0$ and $w \in \mathbb{C}$.

Proposition 3.16 (Solution of the QVE close to zero). *If (F2) and (3.70) are satisfied then there exist a vector $u \in \mathbb{C}^p$, a constant $\delta_* \gtrsim 1$ and analytic functions $a: D_{\delta_*}(0) \rightarrow \mathbb{C}^p$, $b: D_{\delta_*}(0) \rightarrow \mathbb{C}^n$ such that the unique solution $\mathfrak{M} = (m_1, m_2)^t$ of (3.6) with $\text{Im } \mathfrak{M} > 0$ fulfills*

$$M_1(z) = za(z) - \frac{u}{z}, \quad M_2(z) = zb(z) \tag{3.71}$$

for all $z \in D_{\delta_*}(0) \cap \mathbb{H}$. Moreover, we have

- (i) $\sum_{i=1}^p u_i = p - n$ and $1 \lesssim u_i \leq 1$ for all $i = 1, \dots, p$,
- (ii) $b(0) = 1/S^t u \sim 1$,
- (iii) $\|a(z)\|_\infty + \|b(z)\|_\infty \lesssim 1$ uniformly for all $z \in D_{\delta_*}(0)$,
- (iv) $\lim_{\eta \downarrow 0} \text{Im } M_1(\tau + i\eta) = 0$ and $\lim_{\eta \downarrow 0} \text{Im } M_2(\tau + i\eta) = 0$ locally uniformly for all $\tau \in (-\delta_*, \delta_*) \setminus \{0\}$.

The ansatz (3.71) is motivated by the following heuristics. Considering H as an operator $\mathbb{C}^p \oplus \mathbb{C}^n \rightarrow \mathbb{C}^n \oplus \mathbb{C}^p$, we expect that the first component described by $X^*: \mathbb{C}^p \rightarrow \mathbb{C}^n$ has a nontrivial kernel for dimensional reasons whereas the second component has a trivial kernel. Since the nonzero eigenvalues of H^2 correspond to the nonzero eigenvalues of XX^* and X^*X , the Marchenko-Pastur distribution indicates that there is a constant $\delta_* \gtrsim 1$ such that H has no nonzero eigenvalue in $(-\delta_*, \delta_*)$. As the first

component M_1 of \mathfrak{M} corresponds to the “first component” of H , the term $-u/z$ in (3.71) implements the expected kernel. For dimensional reasons, the kernel should be $p - n$ dimensional which agrees with part (i) of Proposition 3.16. The factor z in the terms $za(z)$ and $zb(z)$ in (3.71) realizes the expected gap in the eigenvalue distribution around zero.

Proof of Proposition 3.16. We start with the defining equations for u and b . We assume that $u \in (0, 1]^p$ fulfills

$$\frac{1}{u} = 1 + S \frac{1}{S^t u} \tag{3.72}$$

and $b: D_{\delta_*}(0) \rightarrow \mathbb{C}^p$ fulfills

$$-\frac{1}{b(z)} = z^2 - S^t \frac{1}{1 + Sb(z)} \tag{3.73}$$

for some $\delta_* > 0$. We then define $a: D_{\delta_*}(0) \rightarrow \mathbb{C}^p$ through

$$z^2 a(z) = u - \frac{1}{1 + Sb(z)} \tag{3.74}$$

and set $\widehat{M}_1(z) := za(z) - u/z$ and $\widehat{M}_2(z) := zb(z)$ for $z \in D_{\delta_*}(0)$. Thus, for $z \in D_{\delta_*}(0)$, we obtain

$$z + S^t \widehat{M}_1(z) = z - S^t \frac{1}{1 + Sb(z)} = -\frac{1}{zb(z)} = -\frac{1}{\widehat{M}_2(z)},$$

where we used (3.74) in the first step and (3.73) in the second step. Similarly, solving (3.74) for $Sb(z)$ yields

$$z + S \widehat{M}_2(z) = z + z \left(\frac{1}{u - z^2 a(z)} - 1 \right) = -\frac{1}{\widehat{M}_1(z)}, \quad z \in D_{\delta_*}(0). \tag{3.75}$$

Thus, $(\widehat{M}_1, \widehat{M}_2)$ satisfy (3.6), the defining equation for $\mathfrak{M} = (M_1, M_2)$ and we will be able to conclude that $\widehat{M}_1 = M_1$ and $\widehat{M}_2 = M_2$.

For the rigorous argument, we first establish the existence and uniqueness of u and b that follow from the next two lemmata whose proofs are given later.

Lemma 3.17. *If (F2) and (3.70) are satisfied then there is a unique solution of (3.72) in the set $u \in (0, 1]^p$. Moreover,*

$$1 > u_i \gtrsim 1, \quad (S^t u)_k \gtrsim 1 \tag{3.76}$$

for all $i = 1, \dots, p$ and $k = 1, \dots, n$ and $\sum_{i=1}^p u_i = p - n$.

Lemma 3.18. *If (F2) and (3.70) are satisfied, then there are a $\delta_* \sim 1$ and a unique holomorphic function $b: D_{\delta_*}(0) \rightarrow \mathbb{C}^n$ satisfying (3.73) with $b(0) = 1/(S^t u)$, where u is the solution of (3.72). Moreover, we have $\|b(z)\|_\infty \lesssim 1$ and $\|(1 + Sb(z))^{-1}\|_\infty \leq 1/2$ for all $z \in D_{\delta_*}(0)$, $b(0) \sim 1$, $b'(0) = 0$, $\text{Im}(zb(z)) > 0$ for all $z \in D_{\delta_*}(0)$ with $\text{Im} z > 0$ and $\text{Im}(zb(z)) = 0$ for $z \in (-\delta_*, \delta_*)$.*

Given u and $b(z)$, the formula (3.74) defines $a(z)$ for $z \neq 0$. To extend its definition to $z = 0$, we observe that the right-hand side of (3.74) is a holomorphic function for all $z \in D_{\delta_*}(0)$ by Lemma 3.18. Since $b(0) = 1/(S^t u)$ and the derivative of the right-hand side of (3.74) vanishes as $b'(0) = 0$, the first two coefficients of the Taylor series of the right-hand side on $D_{\delta_*}(0)$ are zero by (3.72). Thus, (3.74) defines a holomorphic function $a: D_{\delta_*}(0) \rightarrow \mathbb{C}^p$.

Furthermore, $\text{Im} \widehat{M}_2(z) > 0$ for $\text{Im} z > 0$ by Lemma 3.18. Taking the imaginary part of (3.75) yields

$$\frac{\text{Im} \widehat{M}_1(z)}{|\widehat{M}_1(z)|^2} = \text{Im} z + S \text{Im} \widehat{M}_2(z), \tag{3.77}$$

which implies $\text{Im } \widehat{M}_1(z) > 0$ for $\text{Im } z > 0$ as $\text{Im } \widehat{M}_2(z) > 0$ for $z \in \mathbb{H} \cap D_{\delta_*}(0)$. Since the solution \mathfrak{M} of (3.6) with $\text{Im } \mathfrak{M}(z) > 0$ for $\text{Im } z > 0$ is unique by Theorem 2.1 in [2], we have $\mathfrak{M}(z) = \widehat{\mathfrak{M}}(z) := (\widehat{M}_1(z), \widehat{M}_2(z))^t$ for all $z \in \mathbb{H} \cap D_{\delta_*}(0)$. The statements in (i), (ii) and (iii) follow from Lemma 3.17, Lemma 3.18 and (3.74).

For the proof of (iv), we note that $\lim_{\eta \downarrow 0} \text{Im } M_2(\tau + i\eta) = 0$ for all $\tau \in (-\delta_*, \delta_*)$ locally uniformly by Lemma 3.18. Because of (3.77) and the locally uniform convergence of $M_1(\tau + i\eta)$ to $\tau a(\tau) - u/\tau$ for $\eta \downarrow 0$ and $\tau \in (-\delta_*, \delta_*) \setminus \{0\}$, we have $\lim_{\eta \downarrow 0} \text{Im } M_1(\tau + i\eta) = 0$ locally uniformly for all $\tau \in (-\delta_*, \delta_*) \setminus \{0\}$ as well, which concludes the proof of (iv). \square

We conclude this subsection with the proofs of Lemma 3.17 and Lemma 3.18.

Proof of Lemma 3.17. We will show that the functional

$$J: (0, 1]^p \rightarrow \mathbb{R}, \quad u \mapsto \frac{1}{p} \sum_{j=1}^n \log \left(\sum_{i=1}^p s_{ij} u_i \right) + \frac{1}{p} \sum_{i=1}^p (u_i - \log u_i)$$

has a unique minimizer u with $u_i > 0$ for all $i = 1, \dots, p$ which solves (3.72). Note that

$$J(1, \dots, 1) = \frac{1}{p} \sum_{j=1}^n \log \left(\sum_{i=1}^p s_{ij} \right) + \frac{p}{p} \leq 1. \tag{3.78}$$

We start with an auxiliary bound on the components of u . Using (F2) and Jensen’s inequality, we get

$$\begin{aligned} J(u) &\geq \frac{1}{p} \sum_{k=1}^n \log \left(\sum_{i=1}^p \frac{\varphi}{n+p} u_i \right) + \frac{1}{p} \sum_{i=1}^p (u_i - \log u_i) \\ &\geq \frac{1}{p} \left(\sum_{i=1}^p \frac{n}{p} \log \left(\frac{\varphi}{2} u_i \right) - \sum_{i=1}^p \log u_i \right) \\ &\geq -\frac{1}{p} \frac{d_*}{1+d_*} \sum_{i=1}^p \log u_i + \frac{n}{p} \log \left(\frac{\varphi}{2} \right), \end{aligned} \tag{3.79}$$

where we used (3.70) in the last step. For any $u \in (0, 1]^p$ with $J(u) \leq J(1, \dots, 1)$, using (3.78), we obtain

$$\begin{aligned} 1 \geq J(1, \dots, 1) &\geq J(u) \geq -\frac{d_*}{p(1+d_*)} \sum_{i=1}^p \log u_i + \frac{n}{p} \log \left(\frac{\varphi}{2} \right) \\ &\geq -\frac{d_*}{p(1+d_*)} \log u_i + \frac{1}{r_1} \log \left(\frac{\varphi}{2} \right), \end{aligned}$$

for any $i = 1, \dots, p$, i.e., $u_i \geq \exp(-p(1+d_*)(1-r_1^{-1} \log(\varphi/2))/d_*) > 0$.

Therefore, taking a minimizing sequence, using a compactness argument and the continuity of J , we obtain the existence of $u^* \in (0, 1]^p$ such that $J(u^*) = \inf_{u \in (0, 1]^p} J(u)$ and

$$u_i^* \geq \exp \left(-p \frac{1+d_*}{d_*} \left(1 - \frac{1}{r_1} \log \left(\frac{\varphi}{2} \right) \right) \right), \quad i = 1, \dots, p. \tag{3.80}$$

Next, we show that $u_i^* < 1$ for all $i = 1, \dots, p$. Assume that $u_i^* = 1$ for some $i \in \{1, \dots, p\}$. Consider a vector \widehat{u} that agrees with u^* except that u_i^* is replaced by $\lambda \in (0, 1)$. An elementary calculation then shows that $J(\widehat{u}) \geq J(u^*)$ implies $s_{ik} = 0$ for all $k = 1, \dots, n$ which contradicts (3.14).

Therefore, evaluating the derivative $J(u^* + \tau h)$ for $h \in \mathbb{R}^p$ at $\tau = 0$, which vanishes since $u^* \in (0, 1)^p$ is a minimizer, we see that u^* satisfies (3.72).

To see the uniqueness of the solution of (3.72), we suppose that $u^*, v^* \in (0, 1]^p$ satisfy (3.72), i.e., $u^* = f(u^*)$ and $v^* = f(v^*)$ where the function $f: (0, 1]^p \rightarrow (0, 1]^p$ is defined as $f(u) = (1 + S((S^t u)^{-1}))^{-1}$. On $(0, 1]^p$ we define the distance function

$$D(u, v) := \sup_{i=1, \dots, p} d(u_i, v_i) \tag{3.81}$$

where $d(a, b) = (a - b)^2 / (ab)$ for $a, b > 0$. This function d defined on $(0, \infty)^2$ is the analogue of D defined in (A.6) of [3] on \mathbb{H}^2 . Therefore, we can apply Lemma A.2 in [3] with the natural substitutions which yields

$$\begin{aligned} D(u^*, v^*) &= D(f(u^*), f(v^*)) = \left(1 + \frac{1}{S(S^t u^*)^{-1}}\right)^{-1} \left(1 + \frac{1}{S(S^t v^*)^{-1}}\right)^{-1} D(u^*, v^*) \\ &\leq cD(u^*, v^*). \end{aligned}$$

for some number c . Here we used 1. and 2. of Lemma A.2 in [3] in the second step and 3. of Lemma A.2 in [3] in the last step. Since we can choose $c < 1$ by (3.80), we conclude $u^* = v^*$. This argument applies particularly to minimizers of J on $(0, 1]^p$.

In the following, we will denote the unique minimizer of J by u . To compute the sum of the components of u we multiply (3.72) by u and sum over $i = 1, \dots, p$ and obtain

$$p = \sum_{i=1}^p u_i + \sum_{i=1}^p u_i \left(S \frac{1}{S^t u} \right)_i = \sum_{i=1}^p u_i + \sum_{j=1}^n (S^t u)_j \frac{1}{(S^t u)_j} = \sum_{i=1}^p u_i + n,$$

i.e., $\sum_{i=1}^p u_i = p - n$.

Finally, we show that the components of the minimizer u are bounded from below by a positive constant which only depends on the model parameters. For $k \in \{1, \dots, n\}$, we obtain

$$(S^t u)_k \geq \frac{\varphi}{n + p} \sum_{i=1}^p u_i \geq \frac{\varphi}{2} \langle u \rangle = \frac{\varphi}{2} \left(1 - \frac{n}{p}\right) \geq \frac{\varphi d_*}{2(1 + d_*)}, \tag{3.82}$$

where we used (F2) in the first step, $n \leq p$ in the second step, $\sum_{i=1}^p u_i = p - n$ in the third step and (3.70) in the last step. This implies the third bound in (3.76).

Therefore, we obtain for all $i = 1, \dots, p$ from (3.72)

$$\frac{1}{u_i} = 1 + \sum_{k=1}^n s_{ik} \frac{1}{(S^t u)_k} \leq 1 + \frac{2(1 + d_*)}{\varphi d_*},$$

where we used (A) with $s_* = 1$ in the last step. This shows that u_i is bounded from below by a positive constant which only depends on the model parameters, i.e., the second bound in (3.76). □

Proof of Lemma 3.18. Instead of solving (3.73) directly, we solve a differential equation with the correctly chosen initial condition in order to obtain b . Note that $b_0 := 1/(S^t u)$ fulfills (3.73) for $z = 0$ and $b_0 \sim 1$ by (3.76) and (3.14).

For any $b \in \mathbb{C}^n$ satisfying $(Sb)_i \neq -1$ for $i = 1, \dots, p$, we define the linear operator

$$L(b): \mathbb{C}^n \rightarrow \mathbb{C}^n, \quad v \mapsto L(b)v := bS^t \frac{1}{(1 + Sb)^2} S(bv),$$

where bv is understood as componentwise multiplication. Using the definition of $L(b)$, $b_0 = 1/(S^t u)$ and (3.72), we get

$$L(b_0)1 = \frac{1}{S^t u} S^t u^2 S \frac{1}{S^t u} = \frac{1}{S^t u} (S^t u - S^t u^2) = 1 - \frac{S^t u^2}{S^t u} \leq 1 - \kappa \tag{3.83}$$

for some $\kappa \sim 1$. Here we used (3.14), $u^2 \gtrsim 1$ and (3.76) in the last step. As

$$L(b_0) = \frac{1}{S^t u} S^t u^2 S \left(\frac{1}{S^t u} \cdot \right)$$

is symmetric and positivity-preserving, Lemma 4.6 in [2] implies $\|L(b_0)\|_{2 \rightarrow 2} \leq 1 - \kappa$ because of (3.83). Therefore, $(1 - L(b_0))$ is invertible and $\|(1 - L(b_0))^{-1}\|_{2 \rightarrow 2} \leq \kappa^{-1}$. Moreover, $\|(1 - L(b_0))^{-1}\|_{\infty \rightarrow \infty} \leq 1 + \|L(b_0)\|_{2 \rightarrow \infty} \kappa^{-1}$ by (3.26) with $R = L(b_0)$ and $D = 1$. The estimate (3.10) and the submultiplicativity of the operator norm $\|\cdot\|_2$ yield $\|L(b_0)\|_{2 \rightarrow \infty} \lesssim 1$. Thus, we obtain

$$\|(1 - L(b_0))^{-1}\|_{\infty} \lesssim 1.$$

We introduce the notation $U_{\delta'} := \{b \in \mathbb{C}^n; \|b - b_0\|_{\infty} < \delta'\}$. If we choose $\delta' \leq (2\|S\|_{\infty \rightarrow \infty})^{-1}$ then

$$|(1 + Sb)_i| = |u_i^{-1} + (S(b - b_0))_i| \geq |u_i^{-1}| - \|S\|_{\infty \rightarrow \infty} \|b - b_0\|_{\infty} \geq 1/2$$

for all $i = 1, \dots, p$, where we used the definition of b_0 , (3.72) and $u_i \leq 1$. Therefore, $\|(1 + Sb)^{-1}\|_{\infty} \leq 1/2$ for all $b \in U_{\delta'}$, i.e., $U_{\delta'} \rightarrow \mathbb{C}^{n \times n}$, $b \mapsto L(b)$ will be a holomorphic map. In particular,

$$\|L(b) - L(b_0)\|_{\infty} \lesssim \|b - b_0\|_{\infty}. \tag{3.84}$$

If $D := L(b) - L(b_0)$ and $\|(1 - L(b_0))^{-1}D\|_{\infty \rightarrow \infty} \leq 1/2$ then $(1 - L(b))$ will be invertible and

$$(1 - L(b))^{-1} = (1 - (1 - L(b_0))^{-1}D)^{-1} (1 - L(b_0))^{-1},$$

as well as $\|(1 - L(b))^{-1}\|_{\infty \rightarrow \infty} \leq 2\|(1 - L(b_0))^{-1}\|_{\infty \rightarrow \infty}$. Therefore, (3.84) implies the existence of $\delta' \sim 1$ such that $(1 - L(b))$ is invertible and $\|(1 - L(b))^{-1}\|_{\infty} \lesssim 1$ for all $b \in U_{\delta'}$.

Hence, the right-hand side of the differential equation

$$b' := \frac{\partial}{\partial z} b = 2zb(1 - L(b))^{-1}b =: f(z, b) \tag{3.85}$$

is holomorphic on $D_{\delta'}(0) \times U_{\delta'}$. As $\delta' \sim 1$ and $\sup\{\|f(z, w)\|_{\infty}; z \in D_{\delta'}(0), w \in U_{\delta'}\} \lesssim 1$, the standard theory of holomorphic differential equations yields the existence of $\delta_* \gtrsim 1$ and a holomorphic function $b: D_{\delta_*}(0) \rightarrow \mathbb{C}^n$ which is the unique solution of (3.85) on $D_{\delta_*}(0)$ satisfying $b(0) = b_0$.

The solution of the differential equation (3.85) is a solution of (3.73) since dividing by b , multiplying by $(1 - L(b))$ and dividing by b in (3.85) yields

$$\frac{b'}{b^2} = 2z + \frac{1}{b}L(b)\frac{b'}{b}.$$

This is the derivative of (3.73). Since $b(0) = b_0$ fulfils (3.73) for $z = 0$ the unique solution of (3.85) with this initial condition is a solution of (3.73) for $z \in D_{\delta_*}(0)$. There is only one holomorphic solution of (3.73) due to the uniqueness of the solution of (3.85). This proves the existence and uniqueness of $b(z)$ in Lemma 3.18.

Since b is a holomorphic function on $D_{\delta_*}(0)$ such that $|b(z)| \lesssim 1$ on $D_{\delta_*}(0)$ and $\delta_* \sim 1$ there is a holomorphic function $b_1: D_{\delta_*}(0) \rightarrow \mathbb{C}^n$ such that

$$b(z) = b_0 + b_1(z)z$$

and $|b_1(z)| \lesssim 1$. Thus, we can assume that $\delta_* \gtrsim 1$ is small enough such that $\text{Im } zb(z) \geq (b_0 - |z|b_1(z))\text{Im } z > 0$ for all $z \in D_{\delta_*}(0) \cap \mathbb{H}$.

Taking the imaginary part of (3.73) for $\tau \in \mathbb{R}$, we get

$$\frac{\text{Im } b(\tau)}{|b(\tau)|^2} = S^t \frac{1}{|1 + Sb(\tau)|^2} S \text{Im } b(\tau)$$

or equivalently, introducing

$$\tilde{L}(z): \mathbb{C}^n \rightarrow \mathbb{C}^n, \quad v \mapsto \tilde{L}(z)v := |b(z)|S^t|1 + Sb(z)|^{-2}S(|b(z)|v)$$

for $z \in D_{\delta_*}(0)$, we have

$$\left(1 - \tilde{L}(\tau)\right) \frac{\text{Im } b(\tau)}{|b(\tau)|} = 0. \tag{3.86}$$

As $\|(1 + Sb(z))^{-1}\|_\infty \leq 1/2$ for all $z \in D_{\delta_*}(0)$, the linear operator $\tilde{L}(z)$ is well-defined for all $z \in D_{\delta_*}(0)$. Because $\tilde{L}(0) = L(b_0)$ and $\|\tilde{L}(b) - \tilde{L}(b_0)\|_\infty \lesssim \|b - b_0\|_\infty$ we can assume that $\delta_* \gtrsim 1$ is small enough such that $(1 - \tilde{L}(z))$ is invertible for all $z \in D_{\delta_*}(0)$. Thus, (3.86) implies that $\text{Im } b(\tau) = 0$ for all $\tau \in (-\delta_*, \delta_*)$ and consequently, $\text{Im } \tau b(\tau) = 0$ for all $\tau \in (-\delta_*, \delta_*)$. \square

4 Local laws

4.1 Local law for H

In this section, we will follow the approach used in [4] to prove a local law for the Wigner-type matrix H . We will not give all details but refer the reader to [4]. Therefore, we consider (3.4) as a perturbed QVE of the form (3.9) with $\mathbf{g} := (g_1, g_2)^t: \mathbb{H} \rightarrow \mathbb{C}^{p+n}$ and $\mathfrak{d} := (d_1, d_2)^t: \mathbb{H} \rightarrow \mathbb{C}^{p+n}$, in particular $\mathbf{g}(z) = (G_{xx}(z))_{x=1, \dots, n+p}$ where G_{xx} are the diagonal entries of the resolvent of H defined in (3.3). We recall that ρ is the probability measure on \mathbb{R} whose Stieltjes transform is $\langle \mathfrak{M} \rangle$, cf. (3.8), where \mathfrak{M} is the solution of (3.6) satisfying $\text{Im } \mathfrak{M}(z) > 0$ for $z \in \mathbb{H}$.

Definition 4.1 (Stochastic domination). *Let $P_0: (0, \infty)^2 \rightarrow \mathbb{N}$ be a given function which depends only on the model parameters and the tolerance exponent γ . If $\varphi = (\varphi^{(p)})_p$ and $\psi = (\psi^{(p)})_p$ are two sequences of nonnegative random variables then we will say that φ is **stochastically dominated** by ψ , $\varphi \prec \psi$, if for all $\varepsilon > 0$ and $D > 0$ we have*

$$\mathbb{P}\left(\varphi^{(p)} \geq p^\varepsilon \psi^{(p)}\right) \leq p^{-D}$$

for all $p \geq P_0(\varepsilon, D)$.

In the following, we will use the convention that $\tau := \text{Re } z$ and $\eta := \text{Im } z$ for $z \in \mathbb{C}$.

Theorem 4.2 (Local law for H away from the edges). *Fix any $\delta, \varepsilon_* > 0$ and $\gamma \in (0, 1)$ independent of p . If the random matrix X satisfies (A) – (D) then the resolvent entries $G_{xy}(z)$ of H defined in (3.3) and (3.1), respectively, fulfill*

$$\max_{x, y=1, \dots, n+p} |G_{xy}(z) - \mathfrak{M}_x(z)\delta_{xy}| \prec \frac{1}{\sqrt{p\eta}}, \quad \text{if } \text{Im } z \geq p^{-1+\gamma} \text{ and } \langle \text{Im } \mathfrak{M}(z) \rangle \geq \varepsilon_*, \tag{4.1a}$$

$$\max_{x, y=1, \dots, n+p} |G_{xy}(z) - \mathfrak{M}_x(z)\delta_{xy}| \prec \frac{1}{\sqrt{p}}, \quad \text{if } \text{dist}(z, \text{supp } \rho) \geq \varepsilon_*, \tag{4.1b}$$

uniformly for $z \in \mathbb{H}$ satisfying $\delta \leq |z| \leq 10$. For any sequence of deterministic vectors $w \in \mathbb{C}^{n+p}$ satisfying $\|w\|_\infty \leq 1$, we have

$$|\langle w, \mathbf{g}(z) - \mathfrak{M}(z) \rangle| \prec \frac{1}{p\eta}, \quad \text{if } \text{Im } z \geq p^{-1+\gamma} \text{ and } \langle \text{Im } \mathfrak{M}(z) \rangle \geq \varepsilon_*, \tag{4.2a}$$

$$|\langle w, \mathbf{g}(z) - \mathfrak{M}(z) \rangle| \prec \frac{1}{p}, \quad \text{if } \text{dist}(z, \text{supp } \rho) \geq \varepsilon_*, \tag{4.2b}$$

uniformly for $z \in \mathbb{H}$ satisfying $\delta \leq |z| \leq 10$. Here, the threshold function P_0 in the definition of the relation \prec depends on the model parameters as well as δ, ε_* and γ .

Remark 4.3. The proof of Theorem 4.2 actually shows an explicit dependence of the estimates (4.1) and (4.2) on ε_* . More precisely, if the right-hand sides of (4.1) and (4.2) are multiplied by a universal inverse power of ε_* and the right-hand side of the condition $\text{Im } z \geq p^{-1+\gamma}$ is multiplied by the same inverse power of ε_* then Theorem 4.2 holds true where the relation \prec does not depend on ε_* any more.

Let $\mu_1 \leq \dots \leq \mu_{n+p}$ be the eigenvalues of H . We define

$$I(\tau) := \left[(n+p) \int_{-\infty}^{\tau} \rho(d\omega) \right], \quad \tau \in \mathbb{R}. \tag{4.3}$$

Thus, $I(\tau)$ denotes the index of an eigenvalue expected to be close to the spectral parameter $\tau \in \mathbb{R}$.

Corollary 4.4 (Bulk rigidity, Absence of eigenvalues outside of $\text{supp } \rho$). *Let $\delta, \varepsilon_* > 0$.*

(i) *Uniformly for all $\tau \in [-10, -\delta] \cup [\delta, 10]$ satisfying $\rho(\tau) \geq \varepsilon_*$ or $\text{dist}(\tau, \text{supp } \rho) \geq \varepsilon_*$, we have*

$$\left| \#\{j; \mu_j \leq \tau\} - (n+p) \int_{-\infty}^{\tau} \rho(d\omega) \right| \prec 1. \tag{4.4}$$

(ii) *Uniformly for all $\tau \in [-10, -\delta] \cup [\delta, 10]$ satisfying $\rho(\tau) \geq \varepsilon_*$, we have*

$$|\mu_{I(\tau)} - \tau| \prec \frac{1}{n+p}. \tag{4.5}$$

(iii) *Asymptotically with overwhelming probability, we have*

$$\#\left(\sigma(H) \cap \{\tau \in [-10, -\delta] \cup [\delta, 10]; \text{dist}(\tau, \text{supp } \rho) \geq \varepsilon_*\}\right) = 0. \tag{4.6}$$

The estimates (4.2a) and (4.2b) in Theorem 4.2 imply Corollary 4.4 in the same way as the corresponding results, Corollary 1.10 and Corollary 1.11, in [4] were proved. In fact, inspecting the proofs in [4], rigidity at a particular point τ_0 in the bulk requires only (i) the local law, (4.2a), around $\tau_0 = \text{Re } z$, (ii) the local law somewhere outside of the support of ρ , (4.2b), and (iii) a uniform global law with optimal convergence rate, (4.2b), for any z away from $\text{supp } \rho$.

Proof of Theorem 4.2. In the proof, we will use the following shorter notation. We introduce the spectral domain

$$\mathbb{D}_H := \{z \in \mathbb{H}: \delta \leq |z| \leq 10, \text{Im } z \geq p^{-1+\gamma}, \langle \text{Im } \mathfrak{M}(z) \rangle \geq \varepsilon_* \text{ or } \text{dist}(z, \text{supp } \rho) \geq \varepsilon_*\}$$

for the parameters $\gamma > 0, \varepsilon_* > 0$ and $\delta > 0$. Moreover, we define the random control parameters

$$\Lambda_d(z) := \|\mathfrak{g}(z) - \mathfrak{M}(z)\|_{\infty}, \quad \Lambda_o(z) := \max_{\substack{x,y=1,\dots,n+p \\ x \neq y}} |G_{xy}(z)|, \quad \Lambda(z) := \max\{\Lambda_d(z), \Lambda_o(z)\}.$$

Before proving (4.1) and (4.2), we establish the auxiliary estimates: Uniformly for all $z \in \mathbb{D}_H$, we have

$$\Lambda_d(z) + \|\mathfrak{d}(z)\|_{\infty} \prec \sqrt{\frac{\langle \text{Im } \mathfrak{M}(z) \rangle}{(n+p)\eta}} + \frac{1}{(n+p)\eta} + \frac{1}{\sqrt{n+p}}, \tag{4.7a}$$

$$\Lambda_o(z) \prec \sqrt{\frac{\langle \text{Im } \mathfrak{M}(z) \rangle}{(n+p)\eta}} + \frac{1}{(n+p)\eta} + \frac{1}{\sqrt{n+p}}. \tag{4.7b}$$

Moreover, for every sequence of vectors $w \in \mathbb{C}^{p+n}$ satisfying $\|w\|_\infty \leq 1$,

$$|\langle w, \mathfrak{g}(z) - \mathfrak{M}(z) \rangle| \prec \frac{\langle \text{Im } \mathfrak{M}(z) \rangle}{(n+p)\eta} + \frac{1}{(n+p)^2\eta^2} + \frac{1}{n+p} \tag{4.8}$$

uniformly for $z \in \mathbb{D}_H$.

Now, we show that (4.8) follows from (4.7a) and (4.7b). To that end, we use the following lemma which is proved as Theorem 3.5 in [4].

Lemma 4.5 (Fluctuation Averaging). *For any $z \in \mathbb{D}_H$ and any sequence of deterministic vectors $w \in \mathbb{C}^{n+p}$ with the uniform bound, $\|w\|_\infty \leq 1$ the following holds true: If $\Lambda_o(z) \prec \Phi$ for some deterministic (n and p -dependent) control parameter Φ with $\Phi \leq (n+p)^{-\gamma/3}$ and $\Lambda(z) \prec (n+p)^{-\gamma/3}$ a.w.o.p., then*

$$|\langle w, \mathfrak{d}(z) \rangle| \prec \Phi^2 + \frac{1}{n+p}. \tag{4.9}$$

By (4.7a), the indicator function in (3.35) is nonzero a.w.o.p. Moreover, (4.7b) ensures the applicability of the fluctuation averaging, Lemma 4.5, which implies that the last term in (3.35) is stochastically dominated by the right-hand side in (4.8). Using (4.7a) again, we conclude that the first term of the right-hand side of (3.35) is dominated by the right-hand side of (4.8).

In order to show (4.7a) and (4.7b) we use the following lemma whose proof we omit, since it follows exactly the same steps as the proof of Lemma 2.1 in [4].

Lemma 4.6. *Let $\lambda_* : \mathbb{H} \rightarrow (0, \infty)$ be the function from Lemma 3.9. We have*

$$\|\mathfrak{d}(z)\|_\infty \mathbf{1}(\Lambda(z) \leq \lambda_*(z)) \prec \sqrt{\frac{\langle \text{Im } \mathfrak{g}(z) \rangle}{(n+p)\eta}} + \frac{1}{\sqrt{n+p}}, \tag{4.10a}$$

$$\Lambda_o(z) \mathbf{1}(\Lambda(z) \leq \lambda_*(z)) \prec \sqrt{\frac{\langle \text{Im } \mathfrak{g}(z) \rangle}{(n+p)\eta}} + \frac{1}{\sqrt{n+p}} \tag{4.10b}$$

uniformly for all $z \in \mathbb{D}_H$.

By (3.34) and (4.10a), we obtain

$$(\Lambda_d(z) + \|\mathfrak{d}(z)\|_\infty) \mathbf{1}(\Lambda_d(z) \leq \lambda_*(z)) \prec \sqrt{\frac{\langle \text{Im } \mathfrak{M} \rangle}{(n+p)\eta}} + (n+p)^{-\varepsilon} \Lambda_d + \frac{(n+p)^\varepsilon}{(n+p)\eta} + \frac{1}{\sqrt{n+p}}$$

for any $\varepsilon \in (0, \gamma)$. Here we used $\text{Im } \mathfrak{g} = \text{Im } \mathfrak{M} + \mathcal{O}(\Lambda_d)$. We absorb $(n+p)^{-\varepsilon} \Lambda_d$ into the left-hand side and get

$$(\Lambda_d(z) + \|\mathfrak{d}(z)\|_\infty) \mathbf{1}(\Lambda_d(z) \leq \lambda_*(z)) \prec \sqrt{\frac{\langle \text{Im } \mathfrak{M} \rangle}{(n+p)\eta}} + \frac{1}{(n+p)\eta} + \frac{1}{\sqrt{n+p}} \tag{4.11}$$

as $\varepsilon \in (0, \gamma)$ is arbitrary. From (4.10b), we conclude

$$\Lambda_o(z) \mathbf{1}(\Lambda(z) \leq \lambda_*(z)) \prec \sqrt{\frac{\langle \text{Im } \mathfrak{M} \rangle}{(n+p)\eta}} + \frac{1}{(n+p)\eta} + \frac{1}{\sqrt{n+p}}, \tag{4.12}$$

where we used $\text{Im } \mathfrak{g} = \text{Im } \mathfrak{M} + \mathcal{O}(\Lambda_d)$ and (4.11) and the fact that $\Lambda_d \leq \Lambda$.

We will conclude the proof by establishing that $\mathbf{1}(\Lambda(z) \leq \lambda_*(z)) = 1$ a.w.o.p. due to an application of Lemma A.1 in [4]. Combining (4.11) and (4.12) and using $\langle \text{Im } \mathfrak{M}(z) \rangle \lesssim (\text{Im } z)^{-1}$, we obtain

$$\Lambda(z) \mathbf{1}(\Lambda(z) \leq \lambda_*(z)) \prec (n+p)^{-\gamma/2} \tag{4.13}$$

for $z \in \mathbb{D}_H$ by the definition of \mathbb{D}_H . We define the function $\Phi(z) := (n+p)^{-\gamma/3}$ and note that $\Lambda(z) = \|\mathfrak{g}(z) - \mathfrak{M}(z)\|_\infty$ is Hölder-continuous since \mathfrak{g} and \mathfrak{M} are Hölder-continuous by

$$\max_{x,y=1,\dots,n+p} |G_{xy}(z_1) - G_{xy}(z_2)| \leq \frac{|z_1 - z_2|}{(\operatorname{Im} z_1)(\operatorname{Im} z_2)} \leq (n+p)^2 |z_1 - z_2| \tag{4.14}$$

for $z_1, z_2 \in \mathbb{D}_H$ and Lemma 3.8, respectively. We set $z_0 := 10i$. Since $|G_{xy}(z)| \leq (\operatorname{Im} z)^{-1}$ and $|\mathfrak{M}_x(z)| \leq (\operatorname{Im} z)^{-1}$ we get $\Lambda(10i) \leq 1$ and hence $\mathbf{1}(\Lambda(10i) \leq \lambda_*(10i)) = 1$ by Lemma 3.9. Therefore, we conclude $\Lambda(z_0) \leq (n+p)^{-\gamma/2} \leq \Phi(z_0)$ from (4.13). Moreover, (4.13) implies $\Lambda \cdot \mathbf{1}(\Lambda \in [\Phi - (n+p)^{-1}, \Phi]) < \Phi - (n+p)^{-1}$ a.w.o.p. uniformly on \mathbb{D}_H . Thus, we get $\Lambda(z) \leq (n+p)^{-\gamma/3}$ a.w.o.p. for all $z \in \mathbb{D}_H$ by applying Lemma A.1 in [4] to Λ and Φ on the connected domain \mathbb{D}_H , i.e., $\mathbf{1}(\Lambda(z) \leq \lambda_*(z)) = 1$ a.w.o.p. Therefore, (4.11) and (4.12) yield (4.7a) and (4.7b), respectively. As remarked above this also implies (4.1a).

For the proof of (4.1b) and (4.2b), we first notice that

$$G_{xx}(z) = \sum_{a=1}^{n+p} \frac{|u_a(x)|^2}{\mu_a - z}$$

for all $x = 1, \dots, n+p$, where $u_a(x)$ denotes the x -component of a $\|\cdot\|_2$ normalized eigenvector u_a corresponding to the eigenvalue μ_a of H . Therefore, we conclude

$$\operatorname{Im} G_{xx}(z) = \eta \sum_{a=1}^{n+p} \frac{|u_a(x)|^2}{(\mu_a - \tau)^2 + \eta^2} < \eta \sum_{a=1}^{n+p} \mathbf{1}(A_a) \frac{|u_a(x)|^2}{(\mu_a - \tau)^2 + \eta^2} < \eta$$

for all $z \in \mathbb{H}$ satisfying $\delta \leq |z| \leq 10$ and $\operatorname{dist}(z, \operatorname{supp} \rho) \geq \varepsilon_*$. Here we used that $A_a := \{\operatorname{dist}(\mu_a, \operatorname{supp} \rho) \leq \varepsilon_*/2\}$ occurs a.w.o.p by (4.6) and thus $1 - \mathbf{1}(A_a) < 0$. In particular, we have $\langle \operatorname{Im} \mathfrak{g} \rangle < \eta$. Now, (4.10a) and (4.10b) yield

$$\|\mathfrak{d}(z)\|_\infty \mathbf{1}(\Lambda(z) \leq \lambda_*(z)) < \frac{1}{\sqrt{n+p}}, \tag{4.15a}$$

$$\Lambda_o(z) \mathbf{1}(\Lambda(z) \leq \lambda_*(z)) < \frac{1}{\sqrt{n+p}}. \tag{4.15b}$$

Following the previous argument but using (4.15a) and (4.15b) instead of (4.10a) and (4.10b), we obtain (4.1b) and (4.2b) and this completes the proof of Theorem 4.2. \square

4.2 Local law for Gram matrices

Proofs of Theorem 2.2 and Theorem 2.4. Splitting the resolvent of H at $z \in \mathbb{C} \setminus \mathbb{R}$ into blocks

$$G(z) = \begin{pmatrix} \mathcal{G}_{11}(z) & \mathcal{G}_{12}(z) \\ \mathcal{G}_{21}(z) & \mathcal{G}_{22}(z) \end{pmatrix}$$

and computing the product $G(z)(H - z)$ blockwise, we obtain that $(XX^* - z^2)^{-1} = \mathcal{G}_{11}(z)/z$ and $(X^*X - z^2)^{-1} = \mathcal{G}_{22}(z)/z$ for $z \in \mathbb{C} \setminus \mathbb{R}$. Therefore, (2.5) follows from (4.1) as well as $|z| \geq \delta$ and $m(\zeta) = M_1(\sqrt{\zeta})/\sqrt{\zeta}$ for $\zeta \in \mathbb{H}$.

As $p \sim n$ we obtain

$$|\langle w, \operatorname{diag}(XX^* - \zeta)^{-1} - m(\zeta) \rangle| \lesssim \left| \left\langle (w, 0)^t, \frac{1}{\sqrt{\zeta}} \left(\mathfrak{g}(\sqrt{\zeta}) - \mathfrak{M}(\sqrt{\zeta}) \right) \right\rangle \right|$$

for $w \in \mathbb{C}^p$. Using $p \sim n$, this implies (2.6) by (4.2). This concludes the proof of Theorem 2.2.

Theorem 2.4 is a consequence of the corresponding result for H , namely Corollary 4.4. \square

Proof of Theorem 2.7. As $m(\zeta) = M_1(\sqrt{\zeta})/\sqrt{\zeta}$ for $\zeta \in \mathbb{H}$, Proposition 3.10 implies $|m(\zeta)| \lesssim |\zeta|^{-1/2}$. Thus, $\pi_* = 0$. Recalling $\pi(\omega) = \omega^{-1/2}\rho_1(\omega^{1/2})\mathbf{1}(\omega > 0)$, where ρ_1 is the bounded density representing $\langle M_1 \rangle$, yields

$$\lim_{\omega \downarrow 0} \pi(\omega)\sqrt{\omega} = \frac{1}{\pi} \langle v_1(0) \rangle \in (0, \infty)$$

by (3.58a) which proves part (ii) of Theorem 2.7.

Since $n = p$, in this case we have $\sigma(XX^*) = \sigma(X^*X)$. Thus, $\langle g_1 \rangle = \langle g_2 \rangle$, i.e., (3.41) is fulfilled and Proposition 3.10 is applicable.

Using Proposition 3.10 instead of Lemma 3.9 and following the argument in Subsection 4.1, we obtain the same result as Theorem 4.2 without the restriction $|z| \geq \delta$. As in the proof of Theorem 2.2, we obtain

$$|R_{ij}(\zeta) - \delta_{ij}m_i(\zeta)| \prec \frac{\sqrt{\operatorname{Re} \sqrt{\zeta}}}{|\sqrt{\zeta}|\sqrt{p\operatorname{Im} \zeta}} \lesssim \sqrt{\frac{\langle \operatorname{Im} m(\zeta) \rangle}{p\operatorname{Im} \zeta}}.$$

Here, we deviated from the proof of Theorem 2.2 since $|z|$ can be arbitrarily small for $z \in \mathbb{D}_0$ and used part (ii) of Theorem 2.7 in the last step. This concludes the proof of part (i) of Theorem 2.7.

Consequently, a version of Corollary 4.4 for $\delta = 0$ holds true. Then, part (iii) and (iv) of the theorem follow immediately. \square

4.3 Proof of Theorem 2.9

In this subsection, we will assume that (A), (C), (D) and (F2) as well as

$$\frac{p}{n} \geq 1 + d_* \tag{4.16}$$

for some $d_* > 0$ hold true.

Theorem 4.7 (Local law for H around $z = 0$). *If (A), (C), (D), (F2) and (4.16) hold true, then*

(i) *The kernel of H and the kernel of H^2 have dimension $p - n$ a.w.o.p.*

(ii) *There is a $\gamma_* \gtrsim 1$ such that*

$$|\mu| \geq \gamma_* \tag{4.17}$$

a.w.o.p. for all $\mu \in \sigma(H)$ such that $\mu \neq 0$.

(iii) *For every $\varepsilon_* > 0$, we have*

$$\max_{x,y=1,\dots,n+p} |G_{xy}(z) - \mathfrak{M}_x(z)\delta_{xy}| \prec \frac{1}{|z|\sqrt{n+p}}, \tag{4.18a}$$

$$|\langle \mathfrak{g} \rangle - \langle \mathfrak{M} \rangle| \prec \frac{|z|}{n+p}. \tag{4.18b}$$

uniformly for $z \in \mathbb{H}$ satisfying $|z| \leq \sqrt{\delta_\pi} - \varepsilon_$.*

We will prove that the kernel of H^2 has dimension $p - n$ by using a result about the smallest nonzero eigenvalue of XX^* from [21]. Since this result requires the entries of X to have the same variance and a symmetric distribution, in order to cover the general case, we employ a continuity argument which replaces x_{ik} , for definiteness, by centered Gaussians with variance $(n+p)^{-1}$. This will immediately imply Theorem 4.7 and consequently Theorem 2.9.

We recall the definition of δ_π from (2.12) and choose δ_* as in Proposition 3.16 for the whole section. Note that $\delta_*^2 \leq \delta_\pi$.

Lemma 4.8. *If (4.16) holds true then for all $\delta_1, \delta_2 > 0$ such that $\delta_1 < \delta_2 < \delta_*^2/2$, the matrix H^2 has no eigenvalues in $[\delta_1, \delta_2]$ a.w.o.p.*

Proof. Part (iii) of Corollary 4.4 with $\delta = \delta_1$ and $\varepsilon_* = \min\{\delta_1, \delta_\pi - \delta_2\}$ implies

$$\# \left(\sigma(H) \cap [\sqrt{\delta_1}, \sqrt{\delta_2}] \right) = 0$$

a.w.o.p. because there is a gap in the support of ρ by part (iii) of Proposition 3.16. Since $\sigma(H^2) = \sigma(H)^2$ this concludes the proof. \square

For the remainder of the section, let $\widehat{X} = (\widehat{x}_{ik})_{i=1, \dots, p}^{k=1, \dots, n}$ consist of independent centered Gaussians with $\mathbb{E}|\widehat{x}_{ik}|^2 = (n+p)^{-1}$. We set

$$\widehat{H} := \begin{pmatrix} 0 & \widehat{X} \\ \widehat{X}^* & 0 \end{pmatrix}.$$

Lemma 4.9. *If (4.16) holds true then the kernel of $\widehat{X}\widehat{X}^*$ has dimension $p-n$ a.w.o.p., $\ker(\widehat{X}^*\widehat{X}) = \{0\}$ a.w.o.p. and there is a $\widehat{\gamma} \sim 1$ such that*

$$\widehat{\lambda} \geq \widehat{\gamma} \tag{4.19}$$

for all $\widehat{\lambda} \in \sigma(\widehat{X}^*\widehat{X})$.

Proof. Let $\widehat{\lambda}_1 \leq \dots \leq \widehat{\lambda}_p$ be the eigenvalues of $\widehat{X}\widehat{X}^*$. The assertion will follow once we have established that $\widehat{\lambda}_{p-n+1} \gtrsim 1$ a.w.o.p. since $\widehat{X}\widehat{X}^*$ and $\widehat{X}^*\widehat{X}$ have the same nonzero eigenvalues and $\dim \ker \widehat{X}\widehat{X}^* \geq p-n$ for dimensional reasons. Corollary V.2.1 in [21] implies that $\widehat{\lambda}_{p-n+1} \geq \gamma_- - p^{-2/3+\varepsilon}$ a.w.o.p. for each $\varepsilon > 0$ where $\gamma_- := 1 - 2\sqrt{pn}/(n+p) \gtrsim 1$, thus $\widehat{\lambda}_{p-n+1} \gtrsim 1$ a.w.o.p. In fact, our proof only requires that $\widehat{\lambda}_{p-n+1} \geq \gamma_- - \varepsilon$ for any $\varepsilon > 0$ a.w.o.p., which already follows from the argument in [35]. \square

Proof of Theorem 4.7. We define $H_t := \sqrt{1-t}H + \sqrt{t}\widehat{H}$ for $t \in [0, 1]$ and set $\gamma_* := \min\{\delta_*/2, \sqrt{\widehat{\gamma}}\}$, where $\widehat{\gamma}$ is chosen as in (4.19). By Lemma 4.8 with $\delta_2 := \gamma_*^2$ and $\delta_1 := \gamma_*^2/2$, H_t^2 has no eigenvalues in $[\delta_1, \delta_2]$ a.w.o.p. for every $t \in [0, 1]$. Clearly, the eigenvalues of H_t^2 depend continuously on t . Therefore, $\#(\sigma(H^2) \cap [0, \delta_1]) = \#(\sigma(\widehat{H}^2) \cap [0, \delta_1])$. Thus, we get the chain of inequalities

$$\begin{aligned} p-n &\leq \dim \ker H = \dim \ker H^2 \\ &\leq \#(\sigma(H^2) \cap [0, \delta_1]) = \#(\sigma(\widehat{H}^2) \cap [0, \delta_1]) = \dim \ker \widehat{H}^2 = p-n. \end{aligned}$$

Here we used Lemma 4.9 in the last step. As the left and the right-hand-side are equal all of the inequalities are equalities which concludes the proof of part (i) and part (ii).

We will omit the proof of part (iii) of Theorem 4.7 as it is very similar to the proof of part (vi) of Theorem 2.9 below which will be independent of part (iii) of Theorem 4.7. \square

Proof of Theorem 2.9. Since δ_* is chosen as in Proposition 3.16 we conclude $\delta_\pi \geq \delta_*^2 \gtrsim 1$ from part (iv) of this proposition. Part (ii) and (iii) of the theorem follow immediately from (4.17) in Theorem 4.7.

If $p > n$, then $\dim \ker XX^* = p-n$ a.w.o.p. as $p-n \leq \dim \ker XX^* \leq \dim \ker H^2 = p-n$ a.w.o.p. by part (i) of Theorem 4.7. By Proposition 3.16, we obtain $\pi_* = \langle u \rangle = 1 - n/p$, where u is defined as in this proposition. This proves part (iv). If $p < n$, then part (v) follows from interchanging the roles of X and X^* and following the same steps as in the proof of part (iv).

For the proof of part (vi), we first assume $p > n$. By Proposition 3.16 we can uniquely extend $\zeta m(\zeta) = \sqrt{\zeta}M_1(\sqrt{\zeta})$ to a holomorphic function on $D_{\delta_*^2}(0)$. We fix γ_* as in (4.17).

On the event $\{\lambda_i \geq \gamma_*^2$ for all $i = p - n + 1, \dots, p\}$, which holds true a.w.o.p. by (4.17), the function $\zeta R(\zeta)$ can be uniquely extended to a holomorphic function on $D_{\gamma_*^2}(0)$. We set $\delta := \min\{\gamma_*^2/2, \delta_*^2\}$ and assume without loss of generality that $\delta \leq \delta_\pi - \varepsilon_*$. For $\zeta \in \mathbb{H}$ satisfying $\delta \leq |\zeta| \leq \delta_\pi - \varepsilon_*$, (2.13) is immediate from (2.5b). We apply (2.5b) to obtain $\max_{i,j} |R_{ij}(\zeta) - m_i(\zeta)\delta_{ij}| < 1/p$ for $\zeta \in \mathbb{H}$ satisfying $|\zeta| = \delta$. By the symmetry of $R(\zeta)$ and $m(\zeta)$ this estimate holds true for all $\zeta \in \mathbb{C}$ satisfying $|\zeta| = \delta$. Thus, the maximum principle implies that $\max_{i,j} |\zeta R_{ij}(\zeta) - \zeta m_i(\zeta)\delta_{ij}| < 1/p$ which proves (2.13) since $\{\lambda_i \geq 2\delta$ for all $i = p - n + 1, \dots, p\}$ which holds true a.w.o.p. by $2\delta \leq \gamma_*^2$ and (4.17). If $p < n$ then XX^* does not have a kernel a.w.o.p. by (v). Therefore, a similar argument yields (2.14).

For the proof of (2.15), we observe that $\dim \ker(XX^*) = p\pi_*$ a.w.o.p. in both cases by (iv) and (v). Thus,

$$\frac{1}{p} \sum_{i=1}^p [R_{ii}(\zeta) - m_i(\zeta)] = \frac{1}{p} \left(\sum_{j: \lambda_j \geq \gamma_*^2} \frac{1}{\lambda_j - \zeta} - \sum_{i=1}^p a_i(\sqrt{\zeta}) \right)$$

a.w.o.p. for $\zeta \in D_\delta(0)$, δ chosen as above, by (4.17), where a is the holomorphic function on $D_{\delta_*}(0)$ defined in Proposition 3.16. The right-hand side of the previous equation can therefore be uniquely extended to a holomorphic function on $D_{\delta_*}(0)$. As before, the estimate (2.5b) can be extended to $\zeta \in \mathbb{H}$ with $|\zeta| \leq \delta$ by the maximum principle. \square

The local law for ζ around zero needed an extra argument, Theorem 2.9, due to the possible singularity at $\zeta = 0$. We note that this separate treatment is necessary even if $p < n$, in which case XX^* does not have a kernel and $R(\zeta)$ is regular at $\zeta = 0$, since we study XX^* and X^*X simultaneously. Our main stability results are formulated and proved in terms of H , as defined in (3.1). Therefore, these results are not sensitive to whether p or n is bigger which means whether XX^* has a kernel or X^*X .

A Appendix: Proof of the Rotation-Inversion lemma

In this appendix, we prove the Rotation-Inversion lemma, Lemma 3.6.

Proof of Lemma 3.6. In this proof, we will write $\|A\|$ to denote $\|A\|_2$. Moreover, we introduce a few short hand notations,

$$\mathcal{U} := \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix}, \quad \mathcal{A} := \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}, \quad a_\pm := \frac{1}{\sqrt{2}} \begin{pmatrix} v_1 \\ \pm v_2 \end{pmatrix}, \quad \rho := \|A^*A\|^{1/2}.$$

In particular, we have $Av_2 = \rho e^{i\psi} v_1$ and $A^*v_1 = \rho e^{-i\psi} v_2$ for some $\psi \in \mathbb{R}$. By redefining v_1 to be $e^{i\psi} v_1$ we may assume that $\psi = 0$ and get $Aa_\pm = \pm \rho a_\pm$ as well.

Let us check that indeed $\mathcal{U} + \mathcal{A}$ is not invertible if the right hand side of (3.25) is infinite, i.e., if

$$\|A^*A\| \langle v_1, U_1 v_1 \rangle \langle v_2, U_2 v_2 \rangle = 1.$$

In this case we find $\|A^*A\| = 1$, $\langle v_1, U_1 v_1 \rangle = e^{i\varphi}$ and $\langle v_2, U_2 v_2 \rangle = e^{-i\varphi}$ for some $\varphi \in \mathbb{R}$. Thus, v_1 and v_2 are eigenvectors of U_1 and U_2 , respectively. Therefore, both \mathcal{U} and \mathcal{A} leave the 2-dimensional subspace spanned by $(v_1, 0)$ and $(0, v_2)$ invariant and in this basis the restriction of $\mathcal{U} + \mathcal{A}$ is represented by the 2×2 -matrix

$$\begin{pmatrix} e^{i\varphi} & 1 \\ 1 & e^{-i\varphi} \end{pmatrix},$$

which is not invertible.

We will now show that in every other case $\mathcal{U} + \mathcal{A}$ is invertible and its inverse satisfies (3.25). To this end we will derive a lower bound on $\|(\mathcal{U} + \mathcal{A})w\|$ for an arbitrary normalized vector $w \in \mathbb{C}^{n+p}$. Any such vector admits a decomposition,

$$w = \alpha_+ a_+ + \alpha_- a_- + \beta b,$$

where $\alpha_{\pm} \in \mathbb{C}$, $\beta \geq 0$ and b is a normalized vector in the orthogonal complement of the 2-dimensional space spanned by a_+ and a_- . The normalization of w implies

$$|\alpha_+|^2 + |\alpha_-|^2 + \beta^2 = 1. \tag{A.1}$$

The case $\beta = 1$ is trivial because the spectral gap of A^*A implies a spectral gap of \mathcal{A} in the sense that

$$\sigma(\mathcal{A}/\rho) \subseteq \{-1\} \cup [-1 + \rho^{-2} \text{Gap}(AA^*), 1 - \rho^{-2} \text{Gap}(AA^*)] \cup \{1\}. \tag{A.2}$$

Thus, we will from now on assume $\beta < 1$.

We will use the notations \mathcal{P}_{\parallel} and \mathcal{P}_{\perp} for the orthogonal projection onto the 2-dimensional subspace spanned by a_{\pm} and its orthogonal complement, respectively. We also introduce

$$\begin{aligned} \lambda &:= \frac{1}{2} \frac{|\alpha_+ + \alpha_-|^2}{|\alpha_+|^2 + |\alpha_-|^2} \in [0, 1], \\ \kappa &:= (|\alpha_+|^2 + |\alpha_-|^2)^{-1/2} \|\mathcal{P}_{\parallel}(1 + \mathcal{U}^* \mathcal{A})(\alpha_+ a_+ + \alpha_- a_-)\|. \end{aligned} \tag{A.3}$$

With this notation we will now prove

$$\|(\mathcal{U} + \mathcal{A})w\| \geq c_1 \text{Gap}(AA^*) \kappa, \tag{A.4}$$

for some positive numerical constant c_1 . The analysis is split into the following regimes:

Regime 1: $\kappa^{1/2} < 10\beta$,

Regime 2: $\kappa^{1/2} \geq 10\beta$ and $\lambda < 1/10$,

Regime 3: $\kappa^{1/2} \geq 10\beta$ and $\lambda > 9/10$,

Regime 4: $\kappa^{1/2} \geq 10\beta$ and $1/10 \leq \lambda \leq 9/10$ and $|\langle v_1, U_1 v_1 \rangle|^2 + |\langle v_2, U_2 v_2 \rangle|^2 \leq 2 - \kappa/2$,

Regime 5: $\kappa^{1/2} \geq 10\beta$ and $1/10 \leq \lambda \leq 9/10$ and $|\langle v_1, U_1 v_1 \rangle|^2 + |\langle v_2, U_2 v_2 \rangle|^2 > 2 - \kappa/2$.

These regimes can be chosen more carefully in order to optimize the constant c_1 in (A.4), but we will not do that here.

Regime 1: In this regime we make use of the spectral gap of A^*A by simply using the triangle inequality,

$$\|(\mathcal{U} + \mathcal{A})w\| \geq \|w\| - \|\mathcal{A}w\| = 1 - \sqrt{\rho^2 |\alpha_+|^2 + \rho^2 |\alpha_-|^2 + \beta^2 \|\mathcal{A}b\|^2}.$$

We use the inequality $1 - \sqrt{1 - \tau} \geq \tau/2$ for $\tau \in [0, 1]$ as well as the normalization (A.1) and find

$$2\|(\mathcal{U} + \mathcal{A})w\| \geq 1 - \rho^2 + \rho^2 \beta^2 - \beta^2 \|\mathcal{A}b\|^2 \geq \rho \beta^2 (\rho - \|\mathcal{A}b\|) \geq \beta^2 \text{Gap}(AA^*).$$

The last inequality follows from (A.2) and because b is orthogonal to a_{\pm} . Since $\beta^2 \geq \kappa/100$, we conclude that in the first regime (A.4) is satisfied.

Regime 2: In this regime we project on the second component of $(\mathcal{U} + \mathcal{A})w$.

$$\begin{aligned} \sqrt{2}\|(\mathcal{U} + \mathcal{A})w\| &\geq \|(\alpha_+ - \alpha_-)U_2v_2 + \sqrt{2}\beta U_2b_2 - (\alpha_+ + \alpha_-)A^*v_1 - \sqrt{2}\beta A^*b_1\| \\ &\geq |\alpha_+ - \alpha_-|\|U_2v_2\| - \sqrt{2}\beta\|U_2b_2\| - \rho|\alpha_+ + \alpha_-|\|v_2\| - \sqrt{2}\beta\|A^*b_1\| \\ &\geq \sqrt{2}\sqrt{|\alpha_+|^2 + |\alpha_-|^2}(\sqrt{1-\lambda} - \sqrt{\lambda}) - 2\sqrt{2}\beta. \end{aligned}$$

Here we used the notation $b = (b_1, b_2)$ for the components of b . The last inequality holds by the normalization of v_2 and b , by $\rho \leq 1$ and by the definition of λ from (A.3), which also implies

$$|\alpha_+ - \alpha_-|^2 = 2(1 - \lambda)(|\alpha_+|^2 + |\alpha_-|^2).$$

Since $\lambda < 1/4$ in this regime and $\kappa \leq 2$ by the definition of κ in (A.3) we find $\beta \leq \kappa^{1/2}/10 \leq 1/5$ and infer

$$\|(\mathcal{U} + \mathcal{A})w\| \geq \sqrt{1 - \beta^2}(\sqrt{1 - \lambda} - \sqrt{\lambda}) - 2\beta \geq 1/10 \geq \kappa/20.$$

Regime 3: By the symmetry in a_+ and a_- and therefore in λ and $1 - \lambda$ this regime is treated in the same way as Regime 2 by estimating the norm of the first component of $(\mathcal{U} + \mathcal{A})w$ from below.

Regime 4: Here we project onto the orthogonal complement of the subspace spanned by a_+ and a_- ,

$$\|(\mathcal{U} + \mathcal{A})w\| \geq \|\mathcal{P}_\perp(\mathcal{U} + \mathcal{A})w\| \geq \|\mathcal{P}_\perp\mathcal{U}(\alpha_+a_+ + \alpha_-a_-)\| - \beta\|\mathcal{P}_\perp(\mathcal{U} + \mathcal{A})b\|. \quad (\text{A.5})$$

We compute the first term in this last expression more explicitly,

$$\begin{aligned} \|\mathcal{P}_\perp\mathcal{U}(\alpha_+a_+ + \alpha_-a_-)\|^2 &= \|\alpha_+a_+ + \alpha_-a_-\|^2 - \|\mathcal{P}_\parallel\mathcal{U}(\alpha_+a_+ + \alpha_-a_-)\|^2 \\ &= |\alpha_+|^2 + |\alpha_-|^2 - \frac{1}{2}|\alpha_+ + \alpha_-|^2|\langle v_1, U_1v_1 \rangle|^2 \\ &\quad - \frac{1}{2}|\alpha_+ - \alpha_-|^2|\langle v_2, U_2v_2 \rangle|^2 \\ &= (1 - \beta^2)(1 - \lambda|\langle v_1, U_1v_1 \rangle|^2 - (1 - \lambda)|\langle v_2, U_2v_2 \rangle|^2). \end{aligned} \quad (\text{A.6})$$

For the second equality we used that

$$\|\mathcal{P}_\parallel u\|^2 = |\langle v_1, u_1 \rangle|^2 + |\langle v_2, u_2 \rangle|^2, \quad u = (u_1, u_2) \in \mathbb{C}^{p+n}.$$

With the choice of variables

$$\xi := |\langle v_1, U_1v_1 \rangle|^2, \quad \eta := |\langle v_2, U_2v_2 \rangle|^2,$$

we are minimizing the last line in (A.6) under the restrictions that are satisfied in this regime,

$$\min\{1 - \lambda\xi - (1 - \lambda)\eta : \xi, \eta \in [0, 1], 2\xi + 2\eta \leq 4 - \kappa\} \geq \frac{1}{2}\kappa \min\{1 - \lambda, \lambda\}.$$

We use the resulting estimate in (A.5) and in this way we arrive at

$$\|(\mathcal{U} + \mathcal{A})w\| \geq \frac{1}{\sqrt{2}}\kappa^{1/2}\sqrt{1 - \beta^2} \min\{1 - \lambda, \lambda\}^{1/2} - 2\beta \geq \frac{\kappa^{1/2}}{100} \geq \frac{\kappa}{200}.$$

In the second to last inequality we used $\beta \leq 1/5$ which was already established in the consideration of Regime 2 and in the last inequality we used $\kappa \leq 2$.

Regime 5: In this regime we project onto the span of a_+ and a_- ,

$$\begin{aligned} \|(\mathcal{U} + \mathcal{A})w\| &= \|(1 + \mathcal{U}^* \mathcal{A})w\| \\ &\geq \|\mathcal{P}_{\parallel}(1 + \mathcal{U}^* \mathcal{A})(\alpha_+ a_+ + \alpha_- a_-)\| - \beta \|\mathcal{P}_{\parallel}(1 + \mathcal{U}^* \mathcal{A})b\| \\ &= \sqrt{|\alpha_+|^2 + |\alpha_-|^2} \kappa - \beta \|\mathcal{P}_{\parallel} \mathcal{U}^* \mathcal{A} b\|. \end{aligned} \tag{A.7}$$

The second term in the last line is estimated by using

$$\|\mathcal{P}_{\parallel} \mathcal{U}^* \mathcal{A} b\|^2 \leq \|\mathcal{A} b\| \sup_{h \parallel a_{\pm}} \sup_{u \perp a_{\pm}} |\langle h, \mathcal{U}^* u \rangle|^2,$$

where the suprema are taken over normalized vectors h and u in the 2-dimensional subspace spanned by a_{\pm} and its orthogonal complement, respectively. First we perform the supremum over h and get

$$\begin{aligned} \|\mathcal{P}_{\parallel} \mathcal{U}^* \mathcal{A} b\|^2 &\leq \sup_{u \perp a_{\pm}} (|\langle v_1, U_1^* u_1 \rangle|^2 + |\langle v_2, U_2^* u_2 \rangle|^2) \\ &\leq \sup_{u_1 \perp v_1} |\langle v_1, U_1^* u_1 \rangle|^2 + \sup_{u_2 \perp v_2} |\langle v_2, U_2^* u_2 \rangle|^2, \end{aligned} \tag{A.8}$$

where the vectors $u_1 \in \mathbb{C}^p$ and $u_2 \in \mathbb{C}^n$ are normalized. Computing

$$\sup_{u_1 \perp v_1} |\langle v_1, U_1^* u_1 \rangle|^2 = 1 - |\langle v_1, U_1 v_1 \rangle|^2, \quad \sup_{u_2 \perp v_2} |\langle v_2, U_2^* u_2 \rangle|^2 = 1 - |\langle v_2, U_2 v_2 \rangle|^2,$$

we get

$$\|\mathcal{P}_{\parallel} \mathcal{U}^* \mathcal{A} b\|^2 \leq 2 - |\langle v_1, U_1 v_1 \rangle|^2 - |\langle v_2, U_2 v_2 \rangle|^2 \leq \kappa/2,$$

where we used the choice of Regime 5 in the last step. Plugging this bound into (A.7) and using $\beta \leq \kappa^{1/2}/10$ as well as $\beta \leq 1/5$ yields

$$\|(\mathcal{U} + \mathcal{A})w\| \geq \sqrt{1 - \beta^2} \kappa - \beta \kappa^{1/2} \geq \kappa/2.$$

This finishes the proof of (A.4). In order to show (3.25), and thus the lemma, we notice that

$$\kappa \geq \inf_{u \parallel a_{\pm}} \|\mathcal{P}_{\parallel}(1 + \mathcal{U}^* \mathcal{A})u\|,$$

where the infimum is taken over normalized vectors u in the span of a_+ and a_- . Thus, it suffices to estimate the norm of the inverse of $\mathcal{P}_{\parallel}(1 + \mathcal{U}^* \mathcal{A})\mathcal{P}_{\parallel}$, restricted to the 2-dimensional subspace with orthonormal basis $(v_1, 0)$ and $(0, v_2)$. In this basis this linear operator takes the form of the simple 2×2 -matrix,

$$\begin{pmatrix} 1 & \rho \langle v_1, U_1 v_1 \rangle \\ \rho \langle v_2, U_2 v_2 \rangle & 1 \end{pmatrix}.$$

Its inverse is bounded by the right hand side of (3.25), up to the factor $\text{Gap}(AA^*)$ that we encountered in (A.4), and the lemma is proven. \square

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