Electronic Journal of Probability

Electron. J. Probab. **22** (2017), no. 13, 1–49.
ISSN: 1083-6489 DOI: 10.1214/17-EJP32

Intermediate disorder directed polymers and the multi-layer extension of the stochastic heat equation

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Abstract

We consider directed polymer models involving multiple non-intersecting random walks moving through a space-time disordered environment in one spatial dimension. For a single random walk, Alberts, Khanin and Quastel proved that under intermediate disorder scaling (in which time and space are scaled diffusively, and the strength of the environment is scaled to zero in a critical manner) the polymer partition function converges to the solution to the stochastic heat equation with multiplicative white noise. In this paper we prove the analogous result for multiple non-intersecting random walks started and ended grouped together. The limiting object now is the multi-layer extension of the stochastic heat equation introduced by O'Connell and Warren.

Keywords: directed polymers; stochastic heat equation; KPZ; scaling limits; partition function; non-intersecting processes.
AMS MSC 2010: 60F05; 82C05.
Submitted to EJP on January 18, 2017, final version accepted on January 29, 2017.
Supersedes arXiv:1603.08168.

1 Introduction

Last passage percolation (LPP) involves finding the maximal sum of iid weights along random walk trajectories. Based on rigorous results for a few solvable choices of weights (e.g. exponential weights [31]), it is widely conjectured that for generic weight distributions the fluctuations of this maximal sum grows like the cube-root of time, and has a limit under this scaling described by the GUE Tracy-Widom distribution. This distribution owes its name to the fact that it also arises as the fluctuations of the largest eigenvalue of an $N \times N$ Gaussian Unitary Ensemble (GUE) matrix, as N goes to infinity [53].

The asymptotics for the second, third, and so on eigenvalues also come up in LPP when one considers the maximal sum of weights along two, three, and so on non-intersecting trajectories which start and end clumped together [8]. Again, this has only been shown

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for the solvable LPP models. By varying the endpoints of the random walks suitably, one encounters the Airy line ensemble [49, 32, 11], which also arises from the asymptotics of Dyson's Brownian motion near its edge. Therefore, a strengthened version of the LPP conjecture would be that the Airy line ensemble arises in this manner for arbitrary choices of weights. This conjecture can be considered a form of the conjecture that LPP with general weights is in the Kardar-Parisi-Zhang universality class [10, 51]

In this paper we prove an *intermediate disorder* version of this conjecture. We consider the positive temperature version of LPP (called directed polymers) whereby one essentially transforms the $(\max, +)$ algebra to $(+, \times)$. In other words, one considers sums over non-intersecting random walks of the products of weights along their trajectories. These are often called polymer partition functions. At fixed positive temperature it is still conjectured that the Airy line ensemble arises in the exact same manner as for LPP. As before, this is only proved for certain solvable models and even then only in terms of the one-point marginal of the single path partition function – see e.g. [13, 7, 46, 14, 3]. (See also [43] for a two-point marginal formula which has not yet yielded to asymptotics, as well as [50, 17, 29] for non-rigorous physics work regarding multi-point Airy asymptotics.)

In this paper we scale time and space diffusively as well as simultaneously weaken the strength of the disorder (this is known as intermediate disorder, or weak noise scaling). Under this scaling we prove convergence to the KPZ line ensemble [12] for general choices of weight distributions. (In fact our result is stated in terms of the closely related O'Connell and Warren multi-layer extension to the solution of the stochastic heat equation [47].) This main result is stated as Theorem 1.5. In the case of a single random walk path this result was proved by Alberts, Khanin and Quastel [2] and the convergence was to the top curve of the O'Connell and Warren multi-layer extension which is simply the solution to stochastic heat equation (whose logarithm is the solution to KPZ equation).

It is quite intuitive to see why Theorem 1.5 should hold. Under diffusive scaling, non-intersecting random walks started and ended grouped together converge to non-intersecting Brownian bridges (sometimes called Brownian watermelons). Under the same space-time scaling, a field of iid random weights converges when their strength is simultaneously scaled to zero (in a critical manner) to space-time Gaussian white noise. Therefore one expects the limits in question should be given by the average with respect to the Brownian watermelon measure of the exponential of the integral of white noise along these trajectories. Such objects require some work to be made sense of, but that is essentially what O'Connell and Warren defined. See Remark 1.3 for more on this.

The result of Alberts, Khanin and Quastel for a single random walk polymer partition function relies on writing a discrete chaos series and then proving convergence of each term (with control over the tail of the series) to the corresponding Gaussian chaos series for the stochastic heat equation. Caravenna, Sun and Zygouras [9] provided a more general formulation of the approach of Alberts, Khanin and Quastel. In proving our main result, we appeal to this general formulation. Consequently, the proof of Theorem 1.5 boils down to proving convergence of the correlation functions for non-intersecting random walks to those of non-intersecting Brownian motions. This is the main technical result of this paper and is presented in Theorem 1.13. Pointwise versions of this convergence are present previously in the literature [33]. However, to apply the results of Caravenna, Sun and Zygouras we must show L^2 convergence (with respect to space and time). The fact that the starting and ending points are grouped together introduces some challenging technical impediments in proving this fact and requires us to use methods beyond those employed in the pointwise limits. See Remark 1.14 for more discussion on this.

One of our main motivations for the present investigation comes from the desire to better understand the properties of O'Connell and Warren's multi-layer extension to the solution of the stochastic heat equation. In [12], Corwin and Hammond considered a semi-discrete directed polymer model and showed that under the same intermediate disorder scaling considered here, the associated line ensemble is tight. They defined any subsequential limit as a KPZ line ensemble and conjectured that there is a unique such limit which can be identified with O'Connell and Warren's multi-layer extension. What was missing, their Conjecture 2.17, was exactly the analog of the convergence result which we provide herein in the case of discrete polymers. We expect that similar methods as developed here can be imported into that semi-discrete setting to prove the conjecture. (Essentially one must control the L^2 convergence of non-intersecting Poissonian walks rather than simple symmetric random walks.)

The KPZ line ensembles enjoy a certain resampling invariance called a Brownian Gibbs property. In [12] this was due to the existence of a corresponding property for the prelimiting semi-discrete directed polymer model which was shown to hold by O'Connell by utilizing a continuous version of the geometric lifting of the Robinson-Schensted correspondence [45]. On the discrete polymer side, Seppäläinen's log-gamma polymer [13] likewise enjoys a discrete Gibbs resampling property which essentially follows from the work of Corwin, O'Connell, Seppäläinen and Zygouras [13] by likewise utilizing the geometric lifting of the Robinson-Schensted-Knuth correspondence. Our results here also apply to the geometric RSK corresponence when the weights are critically scaled, which we present in Theorem 1.8. Corollary 1.11 shows how this applies exactly to the log-gamma polymer. This requires a bit of an argument simply because the log-gamma distributions do not come with an inverse temperature and instead one must identify an effective inverse temperature parameter.

There is also motivation for the study of directed polymers coming from various directions in physics. The general class was introduced to study domain walls of Ising type models with impurities [28, 41], and also applied to study vortices in superconductors [5], roughness of crack interfaces [26], Burgers turbulence [21], and interfaces in competing bacterial colonies [24] (see also the reviews [25] or [19] for more applications). Directed polymers with many non-intersecting paths was studied in [35] as a model for two-dimensional random interfaces subject to disorder. Recently there has also been interest [15] in studying the probability of non-cross for independent polymers in the same environment. This naturally leads to the type of generalized multi-path polymers considered herein.

1.1 Conventions

Let $\mathbb{N} = \{1, 2, \ldots\}$. We use $\triangleq$ for definitions. We use the letters $t \in (0, \infty)$, $z \in \mathbb{R}$ to denote continuous time and space and the letters $n \in \mathbb{N}$, $x \in \mathbb{Z}$ to denote discrete time and space. We use the vector symbol $\vec{\cdot}$ to denote vectors and use subscripts for their components, e.g. $\vec{v} = (v_1, \ldots, v_j)$. We often use $\vec{w}$ as a variable in k-fold space-time integrals:

$$\int\limits_A f(\vec{w})\mathrm{d}\vec{w} \triangleq \int \cdots \int_{\{(t_1,z_1),\ldots,(t_k,z_k)\}\in A} f\big((t_1, z_1), \ldots, (t_k, z_k)\big)\mathrm{d}t_1 \ldots \mathrm{d}t_k \mathrm{d}z_1 \ldots \mathrm{d}z_k$$

We use the superscript $\star$ to denote the endpoint of polymers; for example $(t^\star, z^\star)$ denotes the endpoint of non-intersecting Brownian bridges, and $(n^\star, x^\star)$ will denote the endpoint of non-intersecting random walk bridges.

For the rest of the article we always use $d \in \mathbb{N}$ to denote the number of random walks/Brownian motions in the non-intersecting ensembles we consider. We think of this as fixed throughout the paper.

1.2 Main results

The continuum partition function of d non-intersecting Brownian bridges in a space-time white noise environment was first introduced and studied by O'Connell and Warren in [47] in connection to the multi-layer extension of the stochastic heat equation.

Definition 1.1 (Continuum partition function)**.** *Fix any $t^\star > 0$ and $z^\star \in \mathbb{R}$. Let $\vec{D}^{(t^\star,z^\star)}(t) \in \mathbb{R}^d, t \in (0, t^\star)$ denote the stochastic process of d non-intersecting Brownian bridges which start at $\vec{D}^{(t^\star,z^\star)}(0) = (0, 0, \ldots, 0)$ and end at $\vec{D}^{(t^\star,z^\star)}(t^\star) = (z^\star, z^\star, \ldots, z^\star)$; see Figure 1 for an example of sample paths of $\vec{D}^{(t^\star,z^\star)}$ and Definition 2.2 for more details. We define the following Wiener chaos series:*

$$\mathcal{Z}_d^\beta(t^\star, z^\star) \triangleq \rho(t^\star, z^\star)^d \sum_{k=0}^{\infty} \beta^k \int\limits_{\Delta_k(0,t)} \int\limits_{\mathbb{R}^k} \psi_k^{(t^\star,z^\star)}\big((t_1, z_1), \ldots, (t_k, z_k)\big)\xi(dt_1, dz_1)\cdots\xi(dt_k, dz_k), \tag{1.1}$$

where ξ denotes space time white noise (see [30, 27, 2, 9] for the background on k-fold white noise integrals) and where

$$\begin{aligned} \Delta_k(s, s') &\triangleq \left\{\vec{t} \in (0, \infty)^k : s < t_1 < \cdots < t_k < s'\right\}, \\ \rho(t, z) &\triangleq (2\pi t)^{-\frac{1}{2}} \exp\left(-z^2/2t\right), \end{aligned}$$

and the functions ψ_k are the k-point correlation functions for $\vec{D}^{(t^\star,z^\star)}$; see Definition 2.2 for the precise definition of $\psi_k^{(t^\star,z^\star)}$. When $d = 1$, $\mathcal{Z}_1^\beta(t^\star, z^\star)$ is the continuum directed polymer studied in [1] and is shown to be a solution of the stochastic heat equation with multiplicative noise. For $d > 1$, $\mathcal{Z}_d^\beta(t^\star, z^\star)$ was first studied in [47]. (Note that in their original definition, they fixed $\beta = 1$)

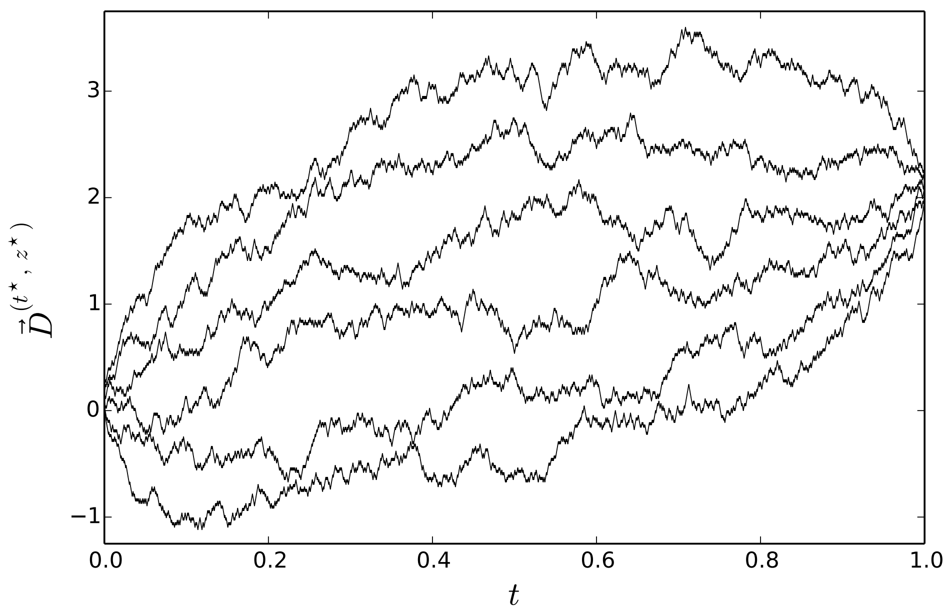

Figure 1: Sample path of the non-intersecting Brownian bridges $\vec{D}^{(t^\star,z^\star)}(t)$ with $d = 6$, $t^\star = 1.0$, $z^\star = 2.0$.

Proposition 1.2. *(Theorem 1.2. of [47]) The series in equation (1.1) that defines $\mathcal{Z}_d^\beta$ is convergent in $L^2(\xi)$.*

Remark 1.3. The chaos series defining $\mathcal{Z}_d^\beta$ has a continuum path integral interpretation

$$\mathcal{Z}_d^\beta(t^\star, z^\star) = \rho(t^\star, z^\star)^d \mathbb{E}\Big[: \exp : \Big\{ \beta \sum_{j=1}^{d} \int_0^{t^\star} \xi\big(t, D_j^{(t^\star, z^\star)}(t)\big) dt \Big\} \Big], \tag{1.2}$$

where $: \exp :$ denotes the Wick exponential, see [30, 27, 9, 1] for background. There is work necessary to fully understand this since Brownian motion is not smooth enough to make sense of the white noise integral in equation (1.2). When $d = 1$, it is shown in [4] that $\mathcal{Z}_1^\beta$ can be recovered by renormalization as a limit of integration against smoothed noise. The case $d > 1$ has not yet been treated.

In this article we prove that $\mathcal{Z}_d^\beta$ arises in the intermediate disorder scaling limit for the partition function for a particular ensemble of d non-intersecting random walks in a disordered environment. We also show that $\mathcal{Z}_d^\beta$ appears in a scaling limit for the d-th row of the geometric RSK correspondence when applied to suitably rescaled random weights. These are the statements of Theorems 1.5 and 1.8 which are both based on the main technical result of this article Theorem 1.13. Corollary 1.11 contains an application of Theorem 1.8 in the case of the exactly solvable log-gamma weights in [52, 13] (with parameters of the log-gamma weights scaled like $\gamma \sim \sqrt{N}$.)

Definition 1.4. *Fix any $n^\star \in \mathbb{N}$ and $x^\star \in \mathbb{Z}$ so that $n^\star + x^\star \equiv 0 (\text{mod } 2)$. Let $\vec{X}^{(n^\star, x^\star)}(n) \in \mathbb{Z}^d$, $n \in [0, n^\star] \cap \mathbb{N}$ denote an ensemble of d non-intersecting simple symmetric random walks conditioned to start at $\vec{X}^{(n^\star, x^\star)}(0) = (0, 2, \dots, 2d-2)$, end at $\vec{X}^{(n^\star, x^\star)}(n^\star) = (x^\star, x^\star + 2, \dots, x^\star + 2d - 2)$ and never intersect for all $n \in [0, n^\star] \cap \mathbb{N}$. See Figure 2 for an example of a sample path and Definition 2.6 for more details. Denote by $\mathbb{E}$ the expectation for this process.*

Let $\omega = \{\omega(n, x)\}_{n \in \mathbb{N}, x \in \mathbb{Z}}$ be an iid collection of random variables which we think of as a disordered environment. We will denote by $\mathcal{E}$ the expectation with respect to this disorder. Define the energy of the ensemble $\vec{X}^{(n^\star, x^\star)}$ in the environment ω by:

$$H^\omega\left(\vec{X}^{(n^\star, x^\star)}\right) = \sum_{j=1}^{d} \sum_{n=1}^{n^\star - 1} \omega\left(n, X_j^{(n^\star, x^\star)}(n)\right).$$

Define the partition function at inverse temperature $\beta > 0$ for the environment ω by:

$$Z_d^\beta(n^\star, x^\star) = \mathbb{E}\left[\exp\left(\beta H^\omega\big(\vec{X}^{(n^\star, x^\star)}\big)\right)\right].$$

We now state our main result.

Theorem 1.5. *Fix any $t^\star > 0$ and $z^\star \in \mathbb{R}$. Let $\omega = \{\omega(n, x)\}_{x \in \mathbb{Z}, n \in \mathbb{N}}$ be any iid environment of mean zero, unit variance random variables with finite exponential moments $\Lambda(\beta) \triangleq \log\left(\mathcal{E}\left(e^{\beta\omega(0,0)}\right)\right) < \infty$. For any $\beta > 0$, let $\beta_N \triangleq N^{-\frac{1}{4}}\beta$. We have the following convergence in distribution of the partition functions in the limit $N \to \infty$.*

$$Z_d^{\beta_N}\left(\big(Nt^\star, \sqrt{N} z^\star\big)_2\right) \exp\Big(-dNt^\star \Lambda\big(\beta_N\big)\Big) \Rightarrow \frac{\mathcal{Z}_d^{\sqrt{2}\beta}(t^\star, z^\star)}{\rho(t^\star, z^\star)^d}. \tag{1.3}$$

where the notation $(n^\star, x^\star) = (Nt^\star, \sqrt{N} z^\star)_2$ denotes the lattice point nearest $(Nt^\star, \sqrt{N} z^\star)$ which has $x^\star + n^\star \equiv 0 (\text{mod } 2)$ (see Definition 2.8 for more details on this notation).

Remark 1.6. When $d = 1$, equation (1.3) is the convergence of the partition function for a simple symmetric random walk to the Wiener chaos solution of the stochastic heat equation first proven in [2].

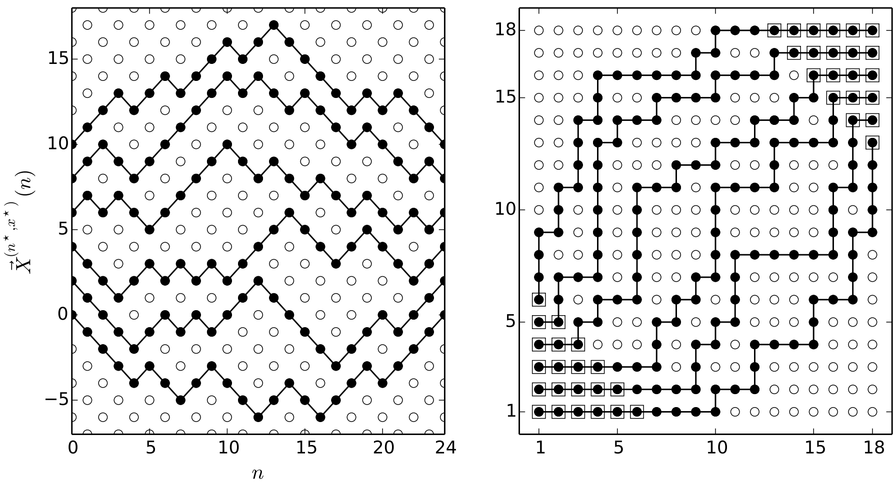

Figure 2: (Left) A sample path of the non-intersecting random walks $\vec{X}^{(n^\star,x^\star)}$ with $d=6$, $n^\star=24$, $x^\star=0$. Notice that only points with $x+n\equiv 0(\text{mod } 2)$ are accessible. (Right) An example of the non-intersecting lattice paths from $\Pi^d_{n,m}$ when $d=6$, $n=18$, $m=18$. The points that have a square around them are forced; they appear in *all* the paths in $\Pi^d_{n,m}$. The left and right plots are related by rotation by $45°$.

A related construction is the geometric RSK correspondence applied to a matrix of positive weights. We also have a limit theorem for this object.

Definition 1.7. *(From Section 2.2 in [13]) Fix any* $(n,m)\in\mathbb{N}^2$. *Let* $\Pi^d_{n,m}$ *denote the set of* d*-tuples* $\pi=(\pi_1,\dots,\pi_d)$ *of non-intersecting lattice paths in* $\{1,\dots,n\}\times\{1,\dots,m\}$ *such that for* $1\le r\le d$, π_r *is lattice path from* $(1,r)$ *to* $(n,m+r-d)$. *A "lattice path" is a path that takes unit steps in the coordinate directions (i.e. up or right) between nearest-neighbor lattice points of* $\mathbb{N}^2$; *non-intersecting means the paths never touch. See Figure 2 for an example.*

Let $g=\{g_{ij}\}_{(i,j)\in\mathbb{N}^2}$ *be an infinite matrix of positive real weights. The weight of a tuple* $\pi=(\pi_1,\dots,\pi_d)$ *of such paths in the environment* g *is defined by:*

$$wt(\pi)\triangleq\prod_{r=1}^{d}\prod_{(i,j)\in\pi_r}g_{ij}.$$

For $1\le d\le\min(n,m)$ *define:*

$$\tau_{m,d}(n)\triangleq\sum_{\pi\in\Pi^d_{n,m}}wt(\pi).$$

These are used to define the elements of the geometric RSK array $\{z_{m,j}\}_{m\in\mathbb{N},j\in\mathbb{N}}$ *by the prescription that* $z_{m,1}(n)\cdot z_{m,2}(n)\cdots z_{m,d}(n)=\tau_{m,d}(n)$.

Theorem 1.8. *Fix some* $\beta>0$. *For each* $N\in\mathbb{N}$, *suppose we are given* $g^{(N)}=\left\{g^{(N)}_{ij}\right\}_{(i,j)\in\mathbb{N}^2}$ *an iid collection of positive real weights with finite second moment. We use* $\mathcal{E}$ *and* $\mathcal{V}ar$ *to denote expectation and variance of these weights. Assume that:*

$$\lim_{N\to\infty}N^{\frac{1}{2}}\frac{\mathcal{V}ar\left(g^{(N)}_{1,1}\right)}{\mathcal{E}\left[g^{(N)}_{1,1}\right]^2}=\beta.$$

Assume also that the sequence of random variables $\left\{\left(g^{(N)}_{1,1}-\mathcal{E}(g^{(N)}_{1,1})\right)^2\right\}$ *is uniformly integrable. For* $(n,m)\in\mathbb{N}^2$*, let* $\tau^{(N)}_{m,d}(n)$ *be defined as in Definition 1.7 using the weights* $g^{(N)}$ *as input. Then we have convergence in distribution:*

$$\frac{\left(\mathcal{E}\left[g^{(N)}_{1,1}\right]\right)^{-d(2N+d)}}{2^{d(2N+d)}N^{-\frac{1}{2}d^2}\left(\prod_{j=0}^{d-1}j!\right)}\tau^{(N)}_{N+d,d}(N+d)\Rightarrow\mathcal{Z}^{\sqrt{2}\beta}_d(2,0). \tag{1.4}$$

Remark 1.9. The proof of Theorem 1.8 can be extended to give limit theorems for $\tau_{m(N),d}(n(N))$ when $m(N)\to\infty$, $n(N)\to\infty$ and $\frac{m(N)}{n(N)}=1+O(N^{-\frac{1}{2}})$ as $N\to\infty$.

Definition 1.10. *For* $\theta>0$*. A random variable* g *has the inverse-Gamma distribution with parameter* $\theta>0$ *if it is supported on the positive reals where it has the density:*

$$\mathbb{P}\left(g\in dx\right)=\frac{1}{\Gamma\left(\theta\right)}x^{-\theta-1}\exp\left(-\frac{1}{x}\right)dx.$$

We denote this by $g\sim\Gamma^{-1}(\theta)$*. An elementary calculation shows that for* $\theta>1$*,* $\mathcal{E}(g)=(\theta-1)^{-1}$ *and for* $\theta>2$*,* $\mathcal{V}ar(g)=(\theta-1)^{-2}(\theta-2)^{-1}$

Corollary 1.11. *If* $g^{(N)}\sim\Gamma^{-1}\left(\beta^{-1}N^{\frac{1}{2}}\right)$ *then Theorem 1.8 applies and we have:*

$$\frac{\beta^{-d(2N+d)}}{2^{d(2N+d)}N^{dN}\left(\prod_{j=0}^{d-1}j!\right)}\tau^{(N)}_{N+d,d}(N+d)\Rightarrow\mathcal{Z}^{\sqrt{2}\beta}_d(2,0).$$

Remark 1.12. The log-gamma weights are special in that they lead to an exactly solvable polymer model related to Whittaker measures which have been studied in [52, 6, 13]. Corollary 1.11 can be interpreted as a limit law for these Whittaker measures.

The main technical result needed for Theorems 1.5 and 1.8 is L^2 convergence of the correlation functions of the non-intersecting random walk bridges $\vec{X}^{(n^\star,x^\star)}$ to those of the non-intersecting Brownian bridges $\vec{D}^{(t^\star,z^\star)}$ under the diffusive scaling $(n^\star,x^\star)\approx\left(Nt^\star,\sqrt{N}z^\star\right)$.

Theorem 1.13. *Fix any* $t^\star>0$ *and* $z^\star\in\mathbb{R}$*. For all* $k\in\mathbb{N}$*, let* $\psi^{(N),(t^\star,z^\star)}_k:\left((0,t^\star)\times\mathbb{R}\right)^k\to\mathbb{R}$ *be the* k*-point correlation functions for* d *non-intersecting random walk bridges which are rescaled as in Definition 2.9. Let* $\psi^{(t^\star,z^\star)}_k:\left((0,t^\star)\times\mathbb{R}\right)^k\to\mathbb{R}$ *be the* k*-point correlation function for* d *non-intersecting Brownian bridges given in Definition 2.2. Then we have the following convergence as* $N\to\infty$*:*

$$\lim_{N\to\infty}\left\|\psi^{(N),(t^\star,z^\star)}_k-\psi^{(t^\star,z^\star)}_k\right\|_{L^2\left(\left((0,t^\star)\times\mathbb{R}\right)^k\right)}=0. \tag{1.5}$$

Remark 1.14. The *pointwise* convergence of these k-point correlation functions has been observed in the special case that $z^\star=0$ in Section 3.2 of [33]. This is proven by explicitly writing $\psi^{(N),(t^\star,z^\star)}_k$ as a $k\times k$ determinant of a kernel $K^{(N),(t^\star,z^\star)}$ and showing that this converges pointwise to a corresponding kernel for $\psi^{(t^\star,z^\star)}_k$ (This is explained in detail in Section 3.) The L^2 convergence is much harder because the conditioning on non-intersection is singular near the endpoints $t=0$ and $t=t^\star$ in a way that gives rise to singularities in the determinantal kernel which are not square integrable for $d\geq 2$. The fact that ψ_k is indeed in square integrable seems to arise due to cancellations in the determinant which cancel away these singularities.

The bulk of this article (Sections 4 and 5) is devoted to developing alternative techniques, which do not rely on the determinantal structure, to control any possible singularity of $\psi^{(N),(t^\star,z^\star)}$ at $t=0$ and $t=t^\star$. This is very similar in spirit and inspired by the analysis used to justify the convergence of the series defining $\mathcal{Z}_d^\beta$ which was carried out in [47]. The additional complexity in our analysis can be attributed to the fact that the generator for the non-intersecting random walk bridges $\vec{X}^{(n^\star,x^\star)}$ is more complicated than the generator for the non-intersecting Brownian bridges $\vec{D}^{(t^\star,z^\star)}$. (See Remark 4.17 for more discussion about this.)

1.3 Outline

Subsections 2.1 and 2.2 contain the precise definitions relating to the non-intersecting Brownian bridges and the non-intersecting random walks respectively. Subsection 2.3 contains the proof of Theorems 1.5 and 1.8 by showing how these fit into the general framework of polynomial chaos series, and then applying a result from [9] about the convergence of polynomial chaos series to Wiener chaos series. In Subsection 2.4, the main technical result, Theorem 1.13, is proven with the important estimates, Propositions 2.19, 2.20, 2.21 and 2.22, deferred to later sections.

Section 3 uses the theory of determinantal kernels and orthogonal polynomials to prove the pointwise convergence in Proposition 2.19 and the bound in Proposition 2.20. Section 4 introduces "overlap times" which are connected to the L^2 norm of $\psi_k^{(N),(t^\star,z^\star)}$. The main result of this section is Proposition 4.23 which gives a particular type of control on the moments of the overlap time. Section 5 uses Proposition 4.23 as a tool to prove the estimates in Propositions 2.21, 2.22 and 2.17.

2 Definitions and proof of main results

We now define the main objects of study in detail and give the proofs of the Theorems 1.5, 1.8 and 1.13, deferring technical estimate to Sections 3, 4 and 5.

2.1 Non-intersecting Brownian motions and bridges

Definition 2.1 (Non-intersecting Brownian motions)**.** *Let $\mathbb{W}^d = \left\{\vec{z} \in \mathbb{R}^d : \ z_1 \le \ldots \le z_d\right\}$ be the d-dimensional Weyl chamber. We denote by $\vec{D}(t) \in \mathbb{W}^d$, $t \in (0,\infty)$ an ensemble of d Brownian motions conditioned not to intersect for all time, and use $\mathbb{E}_{\vec{z}^0}[\cdot]$ to denote the expectation started from $\vec{D}(0) = \vec{z}^0$. This is also known as Dyson Brownian motion. $\vec{D}(t)$ is the Markov process which is obtained from d iid standard Brownian motions via a Doob h-transform by the Vandermonde determinant*

$$h_d(\vec{z}) \triangleq \prod_{1\le i<j\le d} (z_j - z_i).$$

(See Section 3 of [56] for more details.) More precisely this is the prescription that, for $\vec{z}^0 \in \mathbb{W}^d$ with $h_d(\vec{z}^0) > 0$ and for any continuous function $f : \mathbb{W}^d \to \mathbb{R}$, we have:

$$\begin{aligned} \mathbb{E}_{\vec{z}^0}\left[f\big(\vec{D}(t)\big)\right] &\triangleq \frac{1}{h_d\left(\vec{z}^0\right)}\mathbb{E}\left[f\big(\vec{B}(t)+\vec{z}^0\big)h_d\big(\vec{B}(t)+\vec{z}^0\big)\mathbf{1}\left\{\tau_{\vec{z}^0} > t\right\}\right] \\ \tau_{\vec{z}^0} &\triangleq \inf\left\{t > 0 : \vec{B}(t)+\vec{z}^0 \notin \mathbb{W}^d\right\}, \end{aligned}$$

where $\vec{B}(t)$ are d iid standard Brownian motions started from $\vec{B}(0) = \vec{0}$. This definition requires some interpretation for starting points $\vec{z}^0$ on the boundary of $\mathbb{W}^d$ (i.e. where $z_i^0 = z_j^0$ for some $i \neq j$) because $h_d(\vec{z}^0)$ vanishes here. When the starting point is $\vec{0} \triangleq (0,0,\ldots,0) \in \mathbb{W}^d$, this is carried out in Section 4 of [48] and the law of the GUE eigenvalues is obtained:

$$\mathbb{E}_{\vec{0}}\left[f(\vec{D}(t))\right] = \frac{1}{(2\pi)^{\frac{1}{2}d}\prod_{j=1}^{d-1} j!}\int\limits_{\vec{z}\in\mathbb{W}^d} f(z)\cdot t^{-\frac{1}{2}d^2}\exp\left(-\sum_{j=1}^{d}\frac{z_j^2}{2t}\right)h_d(\vec{z})^2 d\vec{z}. \tag{2.1}$$

Definition 2.2 (Non-intersecting Brownian bridges)**.** *Fix $t^\star > 0$ and $z^\star \in \mathbb{R}$. As in Definition 1.1, we will denote by $\vec{D}^{(t^\star,z^\star)}(t) \in \mathbb{W}^d$, $t \in [0,t^\star]$ an ensemble of d non-intersecting Brownian bridges that start at $\vec{D}^{(t^\star,z^\star)}(0) = (0,0\ldots,0)$ and end at the point $\vec{D}^{(t^\star,z^\star)}(t^\star) = (z^\star,z^\star,\ldots z^\star)$. This is sometimes called a Brownian watermelon; see Figure 1. $\vec{D}^{(t^\star,z^\star)}$ is defined by starting with the process $\vec{D}(t) \in \mathbb{W}^d$ from Definition 2.1 and applying the general Markovian construction of a bridge, see Proposition 1 of [20]. This construction is carried out in detail in Section 2 of [47], see in particular Lemma 2.1.*

Given any $k \in \mathbb{N}$, indices $\vec{j} = (j_1,\ldots,j_k) \in \{1,\ldots,d\}^k$ and space-time coordinates $\big((t_1,z_1),\ldots,(t_k,z_k)\big) \in \big((0,t^\star)\times\mathbb{R}\big)^k$, let $\rho_{j_1,j_2,\ldots,j_k}\big((t_1,z_1),\ldots,(t_k,z_k)\big)$ denote the probability density of the random vector $\Big(D^{(t^\star,z^\star)}_{j_1}(t_1),\ldots,D^{(t^\star,z^\star)}_{j_k}(t_k)\Big)$ with respect to Lebesgue measure on $\mathbb{R}^k$ evaluated at $(z_1,\ldots,z_k)$. The k-point correlation functions $\psi_k^{(t^\star,z^\star)}$ are defined for k-tuples $\big((t_1,z_1),\ldots,(t_k,z_k)\big) \in \big((0,t^\star)\times\mathbb{R}\big)^k$ where all the entries (t_i,z_i) are distinct by

$$\psi_k^{(t^\star,z^\star)}\big((t_1,z_1),\ldots,(t_k,z_k)\big) \triangleq \sum_{\vec{j}\in\{1,\ldots,d\}^k} \rho_{j_1,\ldots,j_k}\big((t_1,z_1),\ldots,(t_k,z_k)\big),$$

and declaring that $\psi_k^{(N),(t^\star,z^\star)} \triangleq 0$ if any of the space time coordinates are duplicated $(t_i,z_i) = (t_j,z_j)$ for $i \neq j$. Informally, this is the probability density of finding the position z_i occupied at time t_i by any *of the d Brownian bridges from the ensemble $\vec{D}^{(t^\star,z^\star)}$ for every $1 \leq i \leq k$ with no regard as to which curve of the ensemble is in which location. This definition agrees with the definition of $R_k^{(d)}$ from [47].*

Proposition 2.3. *(Lemma 4.1 and Proposition 4.2. of [47]) Fix any $z^\star \in \mathbb{R}, t^\star > 0$. For each $k \in \mathbb{N}$, we have that $\psi_k^{(t^\star,z^\star)} \in L^2\Big(\big((0,t^\star)\times\mathbb{R}\big)^k\Big)$ and moreover for any $\beta > 0$, the following series is absolutely convergent*

$$1 + \sum_{k=1}^{\infty} \frac{\beta^k}{k!} \left\| \psi_k^{(t^\star,z^\star)} \right\|^2_{L^2\left(\left((0,t^\star)\times\mathbb{R}\right)^k\right)} < \infty$$

Remark 2.4. This result is used to justify the convergence of the Wiener chaos series which defines $\mathcal{Z}_d^\beta(t^\star,z^\star)$ in equation (1.1).

2.2 Non-intersecting random walks and non-intersecting random walk bridges

Definition 2.5 (Non-intersecting random walks)**.** *Define the discrete period-2 Weyl chamber $\mathbb{W}_2^d = \mathbb{W}^d \cap \big\{x \in \mathbb{Z}^d : x_i + x_j \equiv 0 (\text{mod } 2)\, \forall\, 1 \leq i,j \leq d\big\}$. We denote by $\vec{X}(n) \in \mathbb{W}_2^d$, $n \in \mathbb{N}$ an ensemble of d non-intersecting simple symmetric random walks and use $\mathbb{E}_{\vec{x}^0}[\cdot]$ to denote the expectation from the initial condition $\vec{X}(0) = \vec{x}^0$. More precisely, this is the Markov process one gets by conditioning iid simple symmetric random walks not to intersect for all time via a Doob h-transform with the Vandermonde determinant h_d. (see [39] for more details) The transition probabilities are*

$$\mathbb{P}\left(\vec{X}(n') = \vec{x}' \middle| \vec{X}(n) = \vec{x}\right) \triangleq q_{n'-n}\left(\vec{x},\vec{x}'\right) \frac{h_d\left(\vec{x}'\right)}{h_d\left(\vec{x}\right)},$$

where $q_n(\vec{x},\vec{y})$ is defined to be the probability for d iid simple symmetric random walks to go from $\vec{x}$ to $\vec{y}$ in time n without intersections. By the Karlin-MacGregor/Lindstrom-Gessel-Viennot theorem (see e.g. [36, 22]), this is given by

$$q_n(\vec{x},\vec{y}) = \det\left[\frac{1}{2^n}\binom{n}{\frac{1}{2}(n+x_i-y_j)}\right]_{i,j=1}^d,$$

where we use the convention that $\binom{n}{x}$ is zero unless $x \in \mathbb{Z}$ and $0 \leq x \leq n$.

Definition 2.6 (Non-intersecting random walk bridges)**.** *Fix any $n^\star \in \mathbb{N}$ and $x^\star \in \mathbb{Z}$ so that $x^\star + n^\star \equiv 0(\text{mod } 2)$. For $x \in \mathbb{Z}$, define $\vec{\delta}_d(x) \triangleq (x, x+2, \ldots, x+2(d-1)) \in \mathbb{W}^d_2$. As in Definition 1.4, we will denote by $\vec{X}^{(n^\star,x^\star)}(n) \in \mathbb{W}^d_2, n \in [0,n^\star] \cap \mathbb{N}$ the ensemble of d non-intersecting simple symmetric random walks bridges that start at $\vec{X}^{(n^\star,x^\star)}(0) = \vec{\delta}_d(0)$ and end at $\vec{X}^{(n^\star,x^\star)}(n^\star) = \vec{\delta}_d(x^\star)$ after $n^\star$ steps. We take the uniform measure on $\vec{X}^{(n^\star,x^\star)}$ over the finite set of trajectories that satisfy these properties; equivalently one could run d i.i.d. simple symmetric walks started from $\vec{\delta}_d(0)$ and then conditioning on the positive-probability event that at time $n^\star$ the walks are exactly at the final position $\vec{\delta}_d(x^\star)$ and that there have been no collisions between the walks at any intermediate time $n \in [0,n^\star] \cap \mathbb{N}$. See Figure 2 for an example of a sample path $\vec{X}^{(n^\star,x^\star)}$. By the Karlin-MacGregor/Lindstrom-Gessel-Viennot theorem, one can explicitly write the transition probabilities for this Markov process as*

$$\mathbb{P}\left(\vec{X}^{(n^\star,x^\star)}(n') = \vec{x}' \middle| \vec{X}^{(n^\star,x^\star)}(n) = \vec{x}\right) = \frac{q_{n'-n}\big(\vec{x},\vec{x}'\big) q_{n^\star-n'}\big(\vec{x}',\vec{\delta}(x^\star)\big)}{q_{n^\star-n}\big(\vec{x},\vec{\delta}(x^\star)\big)} \;\; \forall\, n < n'.$$

By comparing with Definition 2.5, we see that this process is absolutely continuous with respect to the non-intersecting random walks $\vec{X}(n)$ started from $\vec{X}(0) = \vec{\delta}(0)$ on the interval $n \in [0,n^\star] \cap \mathbb{N}$ with Radon-Nikodym derivative given by

$$\frac{\mathbb{P}\big(\vec{X}^{(n^\star,x^\star)}(n) = \vec{x}\big)}{\mathbb{P}\big(\vec{X}(n) = \vec{x}\big)} = \frac{q_{n^\star-n}\big(\vec{x},\vec{\delta}_d(x^\star)\big)}{q_{n^\star}\big(\vec{\delta}_d(0),\vec{\delta}_d(x^\star)\big)} \frac{h_d\big(\vec{\delta}_d(0)\big)}{h_d\big(\vec{x}\big)}. \tag{2.2}$$

We may also think of $\vec{X}^{(n^\star,x^\star)}(n)$ as an unordered random subset of $\mathbb{Z}$ with d points by $\vec{X}^{(n^\star,x^\star)}(n) = \left\{X_1^{(n^\star,x^\star)}(n), \ldots, X_d^{(n^\star,x^\star)}(n)\right\}$. We will abuse notation in this way and write $\left\{x \in \vec{X}^{(n^\star,x^\star)}(n)\right\}$ to mean the event that $X_j^{(n^\star,x^\star)}(n) = x$ for some $1 \le j \le d$.

Definition 2.7. *(Following definitions in Section 2.3 of [9]) For some set $S \subset \mathbb{R}^2$, let $\mathcal{B}(S)$ denote the Borel subsets of S and let $\mathcal{L}eb\,(A) \in \mathbb{R}^+$ denote the Lebesgue measure for sets $A \in \mathcal{B}(S)$. Given a locally finite set $\mathbb{T} \subset S$ we call $\mathcal{C} : \mathbb{T} \to \mathcal{B}(S)$ a tessellation of S indexed by $\mathbb{T}$ if $\{\mathcal{C}(y)\}_{y\in\mathbb{T}}$ form a disjoint union of S and such that $y \in \mathcal{C}(y)$ for each $y \in \mathbb{T}$. We call $\mathcal{C}(y)$ the cell associated to $y \in \mathbb{T}$.*

Once a tessellation $\mathcal{C}$ is fixed, any function $f : \mathbb{T}^k \to \mathbb{R}$ can be extended to a function $f : S^k \to \mathbb{R}$ by declaring that f is constant on cells of the form $\mathcal{C}(y_1) \times \ldots \times \mathcal{C}(y_k)$ for every $(y_1, \ldots, y_k) \in \mathbb{T}^k$. Note that for such extensions we have:

$$\|f\|^2_{L^2(S^k)} = \sum_{\vec{y}\in\mathbb{T}^k} f(y_1,\ldots,y_k)^2 \prod_{j=1}^k \mathcal{L}eb\,(\mathcal{C}(y_j))\,. \tag{2.3}$$

Definition 2.8. *Define*

$$\mathbb{T}^{(N)} \;\; \triangleq \;\; \left\{(t,z) \; : \; t \in \frac{\mathbb{N}}{N}, z \in \frac{\mathbb{Z}}{\sqrt{N}}, \text{ and } z\sqrt{N} + tN \equiv 0(\text{mod } 2)\right\}.$$

This discrete set is the set of space-time points that are accessible to a diffusively rescaled simple symmetric random walk which takes steps of spatial size $\frac{1}{\sqrt{N}}$ and temporal size $\frac{1}{N}$. The map $\mathcal{C}^{(N)} : \mathbb{T}^{(N)} \to \mathcal{B}\big((0,t^\star) \times \mathbb{R}\big)$ given by $\mathcal{C}^{(N)}\left(\frac{n}{N}, \frac{x}{\sqrt{N}}\right) = \left[\frac{n}{N}, \frac{n}{N} + \frac{1}{N}\right) \times \left[\frac{x}{\sqrt{N}}, \frac{x}{\sqrt{N}} + \frac{2}{\sqrt{N}}\right)$ is a tesselation of $(0,t^\star) \times \mathbb{R}$ indexed by $\mathbb{T}^{(N)}$ in the sense of Definition 2.7. For a space-time point $(t,z) \in \mathbb{R} \times (0,\infty)$, we will sometimes want to access the closest lattice point in $\mathbb{T}^{(N)}$ to the point (t,z). Because of the periodicity *condition $x+n\equiv 0(\text{mod } 2)$, using simply $\lfloor z \rfloor, \lfloor t \rfloor$ will not give exactly what we want. We will instead use the following notation that takes into account this periodicity issue:*

$$(t,z)_2 \triangleq \Big(\lfloor t \rfloor, \max\left\{x\in\mathbb{Z} \,:\, x\le z,\ x+\lfloor t \rfloor \equiv 0(\text{mod } 2)\right\}\Big).$$

Definition 2.9. *Fix $t^\star > 0$ and $z^\star\in\mathbb{R}$. Let $X^{(N),(t^\star,z^\star)}(t)\in\mathbb{W}^d, t\in(0,t^\star)$ be the rescaled version of the non-intersecting random walk bridges $\vec{X}^{(n^\star,x^\star)}$, which is scaled by $N^{-\frac{1}{2}}$ in space and by N^{-1} in time:*

$$\vec{X}^{(N),(t^\star,z^\star)}(t) \triangleq \frac{1}{\sqrt{N}}\vec{X}^{\left(Nt^\star,\sqrt{N}z^\star\right)_2}\left(\lfloor Nt \rfloor\right).$$

Define the rescaled k-point correlation function $\psi_k^{(N),(t^\star,z^\star)} : \left(\mathbb{T}^{(N)}\right)^k \to \mathbb{R}$ by defining for k-tuples $\left((t_1,z_1),\dots,(t_k,z_k)\right)\in\left(\mathbb{T}^{(N)}\right)^k$ where all the entries (t_i,z_i) are distinct:

$$\begin{aligned}\psi_k^{(N),(t^\star,z^\star)}\left((t_1,z_1),\dots,(t_k,z_k)\right) &\triangleq \left(\frac{\sqrt{N}}{2}\right)^k \sum_{\vec{j}\in\{1,\dots,d\}^k} \mathbb{P}\left(\bigcap_{i=1}^k \left\{X_{j_i}^{(N),(t^\star,z^\star)}(t_i)=z_i\right\}\right)\\ &= \left(\frac{\sqrt{N}}{2}\right)^k \mathbb{P}\left(\bigcap_{i=1}^k\left\{z_i\in\vec{X}^{(N),(t^\star,z^\star)}(t_i)\right\}\right), \qquad (2.4)\end{aligned}$$

and declaring that $\psi_k^{(N),(t^\star,z^\star)}\triangleq 0$ if any of the space time coordinates are duplicated $(t_i,z_i)=(t_j,z_j)$ for $i\neq j$. We extend the domain of $\psi_k^{(N),(t^\star,z^\star)}$ to all of $\left((0,t^\star)\times\mathbb{R}\right)^k$ as in Definition 2.7 by declaring it to be constant on the cells $\mathcal{C}^{(N)}(t_1,z_1)\times\cdots\times\mathcal{C}^{(N)}(t_k,z_k)$. Notice that because $\psi_k^{(N),(t^\star,z^\star)}$ is constant on these cells, whenever we compute an integral of $\psi_k^{(N),(t^\star,z^\star)}$, which we will frequently do, such an integral is actually a sum over discrete cells as in equation (2.3).

Remark 2.10. The scaling by $N^{-\frac{1}{2}}$ in space and N^{-1} in time is to be expected as this is the scaling needed for convergence of a random walk to a Brownian motion. The factor of $\left(\sqrt{N}/2\right)^k$ in the definition of the k-point correlation function is the same factor that appears in the local central limit theorem for simple symmetric random walks (which have periodicity 2) to Brownian motion.

2.3 Polynomial chaos expansions – proof of Theorems 1.5 and 1.8

The proof of Theorems 1.5 and 1.8 follow by writing the discrete partition functions as polynomial chaos expansions and applying convergence results from [9] which show the convergence of polynomials chaos expansions to Wiener chaos expansions. Theorem 1.13 provides the key technical input in this application, and its proof is deferred to Section 2.4. We begin by reviewing some definitions about polynomial chaos expansions from Section 2 of [9].

Definition 2.11. *(From Section 2.2 of [9]) Let $\mathbb{T}$ be a finite or countable set. Define:*

$$\mathcal{P}^{fin}(\mathbb{T}) \triangleq \left\{ I\subset\mathbb{T} : |I|<\infty\right\},$$

be the collection of finite subsets of $\mathbb{T}$. Any function $\psi:\mathcal{P}^{fin}(\mathbb{T})\to\mathbb{R}$ determines a (formal if $|\mathbb{T}|=\infty$) multilinear polynomial Ψ by:

$$\Psi(x) = \sum_{I\in\mathcal{P}^{fin}(\mathbb{T})}\psi(I)x^I \ \text{ where } \ x^I\triangleq\prod_{i\in I}x_i$$

Let $\zeta=\{\zeta_i\}_{i\in\mathbb{T}}$ be a family of independent random variables. When $|\mathbb{T}|<\infty$, we say that a random variable X admits a polynomial chaos expansion with respect to ζ if it *can be expressed as* $X = \Psi(\zeta) = \Psi\left(\{\zeta_i\}_{i\in\mathbb{T}}\right)$. *When* $|\mathbb{T}| = \infty$, *we say that* X *admits a polynomial chaos expansion with respect to* ζ *if for any increasing sequence of subsets* $A_M \subset \mathbb{T}$ *with* $|A_M| < \infty$ *and* $\lim_{M\to\infty} A_M = \mathbb{T}$, *we have that*

$$X = \lim_{M\to\infty} \sum_{I\subset A_M} \psi(I)\zeta^I \text{ in probability}$$

When this is the case, the function ψ *is called a polynomial chaos kernel function with respect to* ζ.

Proposition 2.12. *(Theorem 2.3 from [9]) Fix a set* $S \subset \mathbb{R}^2$. *Assume that for each* $N \in \mathbb{N}$ *the following four ingredients are given:*

1. $\mathbb{T}^{(N)} \subset S$ *a locally finite subset of* S,
2. $\mathcal{C}^{(N)} = \left\{\mathcal{C}^{(N)}(x)\right\}_{x\in\mathbb{T}^{(N)}}$ *a tessellation of* S *indexed by* $\mathbb{T}^{(N)}$ *as in Definition 2.7 and so that every cell* $\mathcal{C}^{(N)}(x)$ *has the same volume* $v^{(N)} \triangleq \mathcal{L}eb(\mathcal{C}^{(N)}(x))$,
3. $\zeta^{(N)} = \left\{\zeta_x^{(N)}\right\}_{x\in\mathbb{T}^{(N)}}$ *an independent family of mean-zero random variables, i.e.* $\mathcal{E}\left[\zeta_x^{(N)}\right] = 0$, *all with the same variance* $\left(\sigma^{(N)}\right)^2 = \mathcal{V}ar\left(\zeta_x^{(N)}\right)$ *and such that the family* $\left\{\frac{1}{v^{(N)}}\left(\zeta_x^{(N)}\right)^2\right\}$, $N \in \mathbb{N}, x \in \mathbb{T}^{(N)}$ *is uniformly integrable,*
4. $\psi^{(N)} : \mathcal{P}^{fin}\left(\mathbb{T}^{(N)}\right) \to \mathbb{R}$ *a polynomial chaos kernel function.*

Let $\Psi^{(N)}(z)$ *be a formal multi-linear polynomial defined by the kernel* $\psi^{(N)} : \mathcal{P}^{fin}(\mathbb{T}^{(N)}) \to \mathbb{R}$ *as in Definition 2.11. Assume that* $v^{(N)} \to 0$ *as* $N \to \infty$ *and that the following conditions are satisfied:*
i) There exists $\sigma \in (0,\infty)$ *such that*

$$\lim_{N\to\infty} \frac{\left(\sigma^{(N)}\right)^2}{v^{(N)}} = \sigma^2.$$

ii) There exists $\psi : \mathcal{P}^{fin}(S) \to \mathbb{R}$ *so that for every* $k \in \mathbb{N}$, *the restriction of* ψ *to* k *elements subsets,* $\psi : S^k \to \mathbb{R}$ *has* $\|\psi\|_{L^2(S^k)} < \infty$ *and so that we have the convergence*

$$\lim_{N\to\infty} \left\|\psi^{(N)} - \psi\right\|_{L^2(S^k)} = 0.$$

iii) The following limit holds:

$$\lim_{\ell\to\infty} \limsup_{N\to\infty} \sum_{I\in\mathcal{P}^{fin}(\mathbb{T}^{(N)}),|I|>\ell} \left(\sigma^{(N)}\right)^{2|I|} \psi^{(N)}(I)^2 = 0.$$

Then the polynomial chaos expansion $\Psi^{(N)}\left(\zeta^{(N)}\right)$ *is well defined and converges in distribution as* $N \to \infty$ *to a random variable* Ψ *with explicit Wiener chaos expansion given in terms of a white noise* ξ *on* S:

$$\Psi^{(N)}\left(\zeta^{(N)}\right) \Rightarrow \Psi, \quad \Psi \triangleq \sum_{k=0}^{\infty} \frac{\sigma^k}{k!} \int_{S^k} \psi(y_{1,\dots,}y_k) \prod_{i=1}^{k} \xi(\mathrm{d}y_i).$$

Remark 2.13. In general, the white noise integral over the set S^k requires interpretation due to issues that arise on the set $E = \{\vec{y} : y_i = y_j, \text{ for some } i \neq j\}$; see Section 2.1 in [9] or the final Remark in Section 3.2 in [2]. When $S = [s,s'] \times A$, $A \subset \mathbb{R}$ and ψ is symmetric with $\psi(\vec{y}) = 0$ for $y \in E$, an equivalent way to write this integral which avoids this issue is:

$$\int_{([s,s']\times A)^k} \psi(y_1,\dots,y_k) \prod_{j=1}^{k} \xi(\mathrm{d}y_j) = k! \int_{\Delta_k[s,s']} \int_{A^k} \psi\big((t_1,z_1),\dots,(t_k,z_k)\big) \prod_{j=1}^{k} \xi(\mathrm{d}z_j, \mathrm{d}t_j)$$

Both Theorems 1.5 and 1.8 are proven using Proposition 2.12. The ingredients for the two theorems are very similar: only the collection of variables $\zeta^{(N)}$ differs. The other ingredients are defined in Definition 2.14 below. Lemma 2.15 will justify our choice for the variables $\zeta^{(N)}$. With these ingredients chosen, the verification of condition i) from Proposition 2.12 will be a straightforward calculation, ii) will follow by the the main technical Theorem 1.13 and iii) will follow by Proposition 2.17.

Definition 2.14. *Fix $t^\star > 0$ and $z^\star \in \mathbb{R}$. Set $S = (0, t^\star) \times \mathbb{R}$. For $N \in \mathbb{N}$, let $\mathbb{T}^{(N)} \subset (0, t^\star) \times \mathbb{R}$ and the tessellation $\mathcal{C}^{(N)}$ be as in Definition 2.8. Notice that $v^{(N)} = \frac{2}{N\sqrt{N}}$. Let $\psi^{(N)} : \mathcal{P}^{fin}\left(\mathbb{T}^{(N)}\right) \to \mathbb{R}$ be the polynomial chaos kernel which is defined by setting its action on sets of size $|I| = k$ to be the k-point correlation function for the non-intersecting random walks $\psi_k^{(N),(t^\star,z^\star)}$*

$$\psi^{(N)}\left(\{(t_1, z_1), \dots, (t_k, z_k)\} \right) \triangleq \psi_k^{(N),(t^\star,z^\star)}\big((t_1, z_1), \dots, (t_k, z_k)\big).$$

Define the target limiting kernel ψ in a similar way by defining its action on sets of size k to be the k-point correlation functions for the non-intersecting Brownian bridges $\psi_k^{(t^\star,z^\star)}$

$$\psi\left(\big\{(t_1, z_1), \dots, (t_k, z_k)\big\}\right) = \psi_k^{(t^\star,z^\star)}\big((t_1, z_1), \dots, (t_k, z_k)\big).$$

With this choice of ψ, by comparing Definition 1.1 and Ψ from Proposition 2.12, we observe

$$\Psi = \frac{\mathcal{Z}_d^\sigma(t^\star, z^\star)}{\rho(t^\star, z^\star)^d}.$$

Lemma 2.15. *Given an iid collection of random variables $\{\omega\,(n, x)\}_{x\in\mathbb{Z},n\in\mathbb{N}}$, define for every $x^\star \in \mathbb{Z}$, $n^\star \in \mathbb{N}$ with $x^\star + n^\star \equiv 0$(mod 2) the quantity*

$$Z_d^\omega\,(n^\star, x^\star) \triangleq \mathbb{E}\left[\prod_{j=1}^{d}\prod_{n=1}^{n^\star-1} \omega\left(n, X_j^{(n^\star,x^\star)}(n)\right)\right].$$

Let $\mathbb{T}^{(N)}$ be as in Definition 2.8. For each $N \in \mathbb{N}$ define a field of random variables $\zeta^{(N)} = \left\{\zeta^{(N)}\,(t, z)\right\}_{(t,z)\in\mathbb{T}^{(N)}}$ by

$$\zeta^{(N)}(t, z) \triangleq \frac{2}{\sqrt{N}}\left(\omega\left(Nt, \sqrt{N}z\right) - 1\right).$$

Then, $Z_d^\omega\left(\left(Nt^\star, \sqrt{N}z^\star\right)_2\right)$ is a polynomial chaos expansion in the variables $\zeta^{(N)}$ with the polynomial chaos kernel $\psi^{(N)}$ as in Definition 2.14:

$$Z_d^\omega\left(\left(Nt^\star, \sqrt{N}z^\star\right)_2\right) = \Psi^{(N)}\left(\zeta^{(N)}\right) = \sum_{I\subset\mathbb{T}^{(N)}} \psi^{(N)}(I)\left(\zeta^{(N)}\right)^I. \tag{2.5}$$

Remark 2.16. Even though $\mathbb{T}^{(N)}$ is an infinite space, there is no issue with the interpretation of the RHS of equation (2.5) since for each N, $\psi^{(N)}$ is nonzero on only finitely many cells of $\mathbb{T}^{(N)}$.

Proof. Using $X^{(N),(t^\star,z^\star)}(t)$ from Definition 2.9, the definition of $\zeta^{(N)}$, and the definition of $Z_d^\omega\,(n^\star, x^\star)$ we have

$$\begin{aligned} Z_d^\omega\left(\left(Nt^\star, \sqrt{N}z^\star\right)_2\right) &= \mathbb{E}\left[\prod_{j=1}^{d}\prod_{t\in(0,t^\star)\cap\frac{\mathbb{N}}{N}} \omega\left(\left(Nt, \sqrt{N}X_j^{(N),(t^\star,z^\star)}(t)\right)_2\right)\right] \\ &= \mathbb{E}\left[\prod_{j=1}^{d}\prod_{t\in(0,t^\star)\cap\frac{\mathbb{N}}{N}} \left(1 + \frac{\sqrt{N}}{2}\cdot\zeta^{(N)}\left(t, X_j^{(N),(t^\star,z^\star)}(t)\right)\right)\right]. \end{aligned}$$

From here, we expand the product completely into a finite sum of monomials and then bring the expectation into each monomial individually. Focus attention for the moment only on those monomials which are a product of exactly $k \in \mathbb{N}$ random variables. Since the walks $\vec{X}^{(N),(t^\star,z^\star)}$ is independent of the disorder, the expectation of each monomial is a weighted sum over all the possible positions $\vec{X}^{(N),(t^\star,z^\star)}$ could take. For the monomial corresponding to indices $\vec{j} = (j_1,\dots,j_k) \in \{1,\dots,d\}^k$ and times $\vec{t} = (t_1,\dots,t_k) \in \left((0,t^\star)\cap\frac{\mathbb{N}}{N}\right)^k$ this is:

$$\begin{aligned}&\mathbb{E}\left[\prod_{\ell=1}^{k}\frac{N^{\frac{1}{2}}}{2}\zeta^{(N)}\left(t_\ell, X_{j_\ell}^{(N),(t^\star,z^\star)}(t_\ell)\right)\right]\\ =&\frac{N^{\frac{k}{2}}}{2^k}\sum_{\vec{z}\in\frac{\mathbb{Z}^k}{\sqrt{N}}}\mathbb{P}\left(\bigcap_{\ell=1}^{k}\left\{X_{j_\ell}^{(N),(t^\star,z^\star)}(t_\ell)=z_\ell\right\}\right)\prod_{\ell=1}^{k}\zeta^{(N)}\left(t_\ell,z_\ell\right).\end{aligned}$$

We now recognize by comparing with the definition of $\psi_k^{(N),(t^\star,z^\star)}$ from equation (2.4) that when summing over all indices $\vec{j}$ and all times $\vec{t}$ this yields exactly the contribution from sets of size $|I| = k$ in the sum from the RHS of equation (2.5). Doing this for every $k \in \mathbb{N}$ gives exactly the desired result. ☐

Proposition 2.17. *For any $\gamma > 0$, we have that*

$$\lim_{\ell\to\infty}\sup_{N\in\mathbb{N}}\sum_{k=\ell}^{\infty}\frac{\gamma^k}{k!}\int\limits_{\left((0,t^\star)\times\mathbb{R}\right)^k}\left|\psi_k^{(N),(t^\star,z^\star)}(\vec{w})\right|^2 d\vec{w} = 0. \tag{2.6}$$

The proof of Proposition 2.17 is deferred until Section 5, where it is proven using tools which are developed on the way to the proof of Theorem 1.13.

Proof. (Of Theorem 1.5) The proof will be an application of Proposition 2.12. Take the ingredients $S = (0,t^\star)\times\mathbb{R}$, $\mathbb{T}^{(N)}$, $\mathcal{C}^{(N)}$, $\psi^{(N)}$ and ψ as in Definition 2.14. We define the variables $\zeta^{(N)} = \left(\zeta^{(N)}(t,z)\right)_{(t,z)\in\mathbb{T}^{(N)}}$ by:

$$\zeta^{(N)}(t,z) \triangleq \frac{2}{\sqrt{N}}\left(\frac{\exp\left(\beta_N\omega\big(Nt,\sqrt{N}z\big)_2\right)}{\exp\left(\Lambda(\beta_N)\right)} - 1\right).$$

The collection $\left\{\frac{1}{v^{(N)}}\big(\zeta_y^{(N)}\big)^2\right\}_{N\in\mathbb{N},y\in\mathbb{T}^{(N)}}$ can be verified to be uniformly integrable by finding a uniform bound on the second moment: this computation is carried out in equation (6.7) in [9]. Comparing Definition 1.4 to the result of Lemma 2.15, we see that $Z_d^{\beta_N}$ can be expressed as a polynomial chaos series in these variables:

$$Z_d^{\beta_N}\left(\big(Nt^\star,\sqrt{N}z^\star\big)_2\right)\cdot\exp\big(-d\lfloor Nt^\star\rfloor\Lambda(\beta_N)\big) = \Psi^{(N)}\big(\zeta^{(N)}\big),$$

and so the desired convergence to $\mathcal{Z}_d^{\sqrt{2}\beta}$ would be the conclusion of Proposition 2.12 with $\sigma = \sqrt{2}\beta$, provided we verify conditions i), ii) and iii) from Proposition 2.12. This is carried out below:

i) Recall the definition $\beta_N = N^{-\frac{1}{4}}\beta$. By the definition of $\left(\sigma^{(N)}\right)^2 = \mathcal{V}ar\left(\zeta^{(N)}(t,z)\right)$ we have as $N \to \infty$:

$$\begin{aligned}\frac{\left(\sigma^{(N)}\right)^2}{v^{(N)}} &= \left(\frac{N\sqrt{N}}{2}\right)\left(\frac{2}{\sqrt{N}}\right)^2\left(\frac{\mathcal{V}ar\left[\exp\left(\beta_N\omega\right)\right]}{\mathcal{E}\left[\exp\left(\beta_N\omega\right)\right]^2}\right)\\ &= 2\sqrt{N}\left(\frac{\exp\left(\Lambda\left(2\beta_N\right)\right)}{\exp\left(2\Lambda(\beta_N)\right)}-1\right)\\ &= 2\sqrt{N}\Big(\beta_N^2+o\big(N^{-\frac{1}{2}}\big)\Big)=2\beta^2+o(1).\end{aligned}$$

The above limit follows from the Taylor series expansion $\exp(\Lambda(\gamma)) = \mathcal{E}[\exp(\gamma\omega)] = 1+\frac{1}{2}\gamma^2+o(\gamma^2)$ as $\gamma \to 0$ since the weights ω is assumed to be mean zero and unit variance.

ii) By the definition $\psi^{(N)}$ and ψ from Definition 2.14, the condition to be verified is that for each $k \in \mathbb{N}$, we have

$$\lim_{N\to\infty}\left\|\psi_k^{(N),(t^\star,z^\star)}-\psi_k^{(t^\star,z^\star)}\right\|_{L^2\left(\left((0,t^\star)\times\mathbb{R}\right)^k\right)}=0.$$

This is exactly the conclusion of Theorem 1.13.

iii) Since all the terms of the sum are non-negative, we can rearrange the sum into sets of size $|I| = k$. This gives:

$$\begin{aligned}&\lim_{\ell\to\infty}\limsup_{N\to\infty}\sum_{k=\ell+1}^{\infty}\sum_{\substack{I\subset\mathbb{T}^{(N)}\\|I|=k}}\left(\sigma^{(N)}\right)^{2k}\left|\psi_k^{(N),(t^\star,z^\star)}(I)\right|^2\\ =\;&\lim_{\ell\to\infty}\limsup_{N\to\infty}\sum_{k=\ell+1}^{\infty}\left(\frac{\left(\sigma^{(N)}\right)^2}{v^{(N)}}\right)^k\left(v^{(N)}\right)^k\sum_{\substack{I\subset\mathbb{T}^{(N)}\\|I|=k}}\left|\psi_k^{(N),(t^\star,z^\star)}(I)\right|^2\\ =\;&\lim_{\ell\to\infty}\sup_{N\to\infty}\sum_{k=\ell+1}^{\infty}\left(\frac{\left(\sigma^{(N)}\right)^2}{v^{(N)}}\right)^k\frac{1}{k!}\int\limits_{\left((0,t^\star)\times\mathbb{R}\right)^k}\left|\psi_k^{(N),(t^\star,z^\star)}(\vec{w})\right|^2\mathrm{d}\vec{w},\end{aligned}\tag{2.7}$$

where we have recognized the sum over cells times the volume of the cell as the integral of $\psi_k^{(N),(t^\star,z^\star)}$ as in equation (2.3). Finally, since $\lim_{N\to\infty}\frac{\left(\sigma^{(N)}\right)^2}{v^{(N)}} = 2\beta^2$, we have for N sufficiently large, this ratio is less than $2\beta^2+1$. With this bound in place, we see that the limit on the RHS of equation (2.7) is 0 by application of Proposition 2.17. □

Proof. (Of Theorem 1.8) We first notice that any tuple of paths $\pi = (\pi_1,\ldots,\pi_d)$ in $\Pi^d_{N+d,N+d}$ is forced to pass through a triangle of $\frac{1}{2}d(d+1)$ points near $(1,1)$ and a triangle of $\frac{1}{2}d(d+1)$ points near $(N+d,N+d)$. These points are indicated in Figure 2 with a square around them. Specifically, for index $1 \le \ell \le d$ the lattice path π_ℓ from $(1,\ell)$ to $(N+d,N+\ell)$ is forced to pass through the points $\{(i,\ell)\}_{i=1}^{d-\ell+1}$ and the points $\{(N+i,N+\ell)\}_{i=d-\ell+1}^{d}$. Let us denote by A this set of $d(d+1)$ lattice points and by $\tilde{\Pi}^d_{N+d,N+d}$ the "free" part of the lattice paths, that is d non-intersecting lattice paths from the diagonal $\{(d+1-\ell,\ell)\}_{\ell=1}^d$ to the opposite diagonal $\{(N+d+1-\ell,N+\ell)\}_{\ell=1}^d$. We can thus factor $\tau^{(N)}_{N+d,d}(N+d)$ as:

$$\mathcal{E}\left[g_{1,1}^{(N)}\right]^{-d(2N+d)}\tau^{(N)}_{N+d,d}(N+d)=\left(\prod_{(i,j)\in A}\frac{g_{ij}^{(N)}}{\mathcal{E}\left[g_{1,1}^{(N)}\right]}\right)\left(\sum_{\pi\in\tilde{\Pi}^d_{N+d,N+d}}\prod_{\ell=1}^{d}\prod_{(i,j)\in\pi_\ell}\frac{g_{ij}^{(N)}}{\mathcal{E}\left[g_{1,1}^{(N)}\right]}\right).\tag{2.8}$$

The factor $\left(\prod_{(i,j)\in A} g_{ij}^{(N)}\mathcal{E}\left[g_{1,1}^{(N)}\right]^{-1}\right) \to 1$ as $N\to\infty$ since the $g_{ij}^{(N)}$ are independent and since we are given that $\mathcal{V}ar\left(g_{ij}^{(N)}\right)/\mathcal{E}\left[g_{ij}^{(N)}\right]^2 = O\left(N^{-\frac{1}{2}}\right)$. Thus this prefactor coming from weights in A is asymptotically negligible. Notice now that rotating by $45°$ sends every tuple of lattice path in $\tilde{\Pi}^d_{N+d,N+d}$ to an ensemble of non-intersecting random walks $\vec{X}^{(2N,0)}$ as defined in Definition 1.4. This is illustrated in Figure 2 by comparing the relationship between the left and right subfigures. More precisely the map $\lambda : \{1,\dots,N+d\}^2\setminus A \to \mathbb{Z}\times[0,2N]$ by $\lambda(n,m) = (n+m-d-1, n-m+d-1)$ gives a bijection from the set of lattice paths $\tilde{\Pi}^d_{N+d,N+d}$ to the set of non-intersecting random walks $\vec{X}^{(2N,0)}$. Writing $\Omega^{(2N,0)}$ as the set of all such paths, we have:

$$\begin{aligned}\text{LHS (2.8)} &= (1+o(1))\left(\sum_{\vec{X}^{(2N,0)}\in\Omega^{(2N,0)}}\prod_{\ell=1}^{d}\prod_{n=1}^{2N-1}\frac{g^{(N)}\left(\lambda^{-1}\left(n,\vec{X}^{(2N,0)}(n)\right)\right)}{\mathcal{E}\left[g_{1,1}^{(N)}\right]}\right)\\ &= (1+o(1))\left|\Omega^{(2N,0)}\right|\mathbb{E}\left[\prod_{\ell=1}^{d}\prod_{n=1}^{2N-1}\frac{g^{(N)}\left(\lambda^{-1}\left(n,\vec{X}^{(2N,0)}(n)\right)\right)}{\mathcal{E}\left[g_{1,1}^{(N)}\right]}\right]. \qquad (2.9)\end{aligned}$$

where $\mathbb{E}$ is the uniform measure on $\Omega^{(2N,0)}$, as in Definition 2.6. There is an exact formula, MacMahon's formula, for $\left|\Omega^{(2N,0)}\right|$. We use the form of MacMahon's formula given in equation (3.34) of [33] to obtain

$$\left|\Omega^{(2N,0)}\right| = \prod_{i=0}^{d-1}\binom{2N+2i}{N+i}\binom{2N+2i}{i}^{-1}.$$

By Stirling's formula, we have that

$$\left|\Omega^{(2N,0)}\right| = 2^{d(2N+d-1)}N^{-\frac{1}{2}d^2}\pi^{-\frac{1}{2}d}\left(\prod_{j=0}^{d-1} j!\right)(1+o(1)),$$

as $N\to\infty$. Along with $\rho(2,0)^d = (4\pi)^{-d/2}$, this accounts for the prefactor that appears on the LHS of equation (1.4), and it remains only to show that:

$$\mathbb{E}\left[\prod_{\ell=1}^{d}\prod_{n=1}^{2N-1}\frac{g^{(N)}\left(\lambda^{-1}\left(n,\vec{X}^{(2N,0)}(n)\right)\right)}{\mathcal{E}\left[g_{1,1}^{(N)}\right]}\right] \Rightarrow \frac{\mathcal{Z}_d^{\sqrt{2}\beta}(2,0)}{\rho(2,0)^d}.$$

This is very similar to Theorem 1.5. We take the ingredients $S=(0,t^\star)\times\mathbb{R}$, $\mathbb{T}^{(N)}$, $\mathcal{C}^{(N)}$, $\psi^{(N)}$ and ψ as in Definition 2.14. Set the variables $\zeta^{(N)} = \left(\zeta^{(N)}(t,z)\right)_{(t,z)\in\mathbb{T}^{(N)}}$ by:

$$\zeta^{(N)}(t,z) \triangleq \frac{2}{\sqrt{N}}\left(\frac{g^{(N)}\left(\lambda^{-1}\left(Nt,\sqrt{N}z\right)\right)}{\mathcal{E}\left[g_{1,1}^{(N)}\right]}-1\right).$$

With this definition, we recognize by Lemma 2.15 the expectation on the RHS of equation (2.9) as a polynomial chaos series in $\zeta^{(N)}$, so the desired result follows by application of Proposition 2.12 with $\sigma=\sqrt{2}\beta$. The justification of condition ii) from Proposition 2.12 proceed exactly as in proof of Theorem 1.5 with no changes. Condition iii) follows by the uniform integrability assumption and the same variance calculation as in Theorem 1.5. Condition i) follows since:

$$\frac{\left(\sigma^{(N)}\right)^2}{v^{(N)}} = \left(\frac{N\sqrt{N}}{2}\right)\left(\frac{2}{\sqrt{N}}\right)^2\left(\frac{\mathcal{V}ar\left[g_{1,1}^{(N)}\right]}{\mathcal{E}[g_{1,1}^{(N)}]^2}\right) \to 2\beta,$$

by the hypothesis on the weights $g^{(N)}$. □

2.4 L^2 convergence – proof of Theorem 1.13

The main technical result of this article is the L^2 convergence in Theorem 1.13. The proof of Theorem 1.13 goes by dividing the space $\big((0,t^\star)\times\mathbb{R}\big)^k$ into four parts and analyzing contribution to the integral on each one.

Definition 2.18. *Instead of working with $\big((0,t^\star)\times\mathbb{R}\big)^k$, it will be more convenient to work with k-tuples where the times are ordered so that $t_1 < t_2 < \ldots < t_k$. To this end, define $\mathcal{S}_k(0,t^\star)\subset\big((0,t^\star)\times\mathbb{R}\big)^k$ by:*

$$\mathcal{S}_k(0,t^\star) \triangleq \left\{\big((t_1,z_1);\ldots,(t_k,z_k)\big):\ z_i\in\mathbb{R},\ 0<t_1<\ldots<t_k<t^\star\right\}.$$

Since the set of points omitted is a set of measure 0 and since there are $k!$ ways to permute the points in $\mathcal{S}_k(0,t^\star)$, we have for any integrable f which is symmetric with respect to permutations of its entries that:

$$\int\limits_{\mathcal{S}_k(0,t^\star)} f(\vec{w})d\vec{w} = \frac{1}{k!}\int\limits_{\big((0,t^\star)\times\mathbb{R}\big)^k} f(\vec{w})d\vec{w}. \tag{2.10}$$

For any parameters $\delta,\eta,M>0$, define the sets:

$$\begin{aligned}
D_1(\delta,\eta,M) &\triangleq \mathcal{S}_k(0,t^\star)\cap\bigcap_{i=1}^{k}\{|z_i|\le M,\ t_i\in[\delta,t^\star-\delta]\}\cap\bigcap_{i=1}^{k-1}\{t_{i+1}-t_i>\eta\},\\
D_2(\delta,\eta,M) &\triangleq \mathcal{S}_k(0,t^\star)\cap\bigcap_{i=1}^{k}\{|z_i|\le M,\ t_i\in[\delta,t^\star-\delta]\}\cap\bigcup_{i=1}^{k-1}\{t_{i+1}-t_i\le\eta\},\\
D_3(\delta) &\triangleq \mathcal{S}_k(0,t^\star)\cap\bigcup_{i=1}^{k}\{t_i\in(0,\delta)\cup(t^\star-\delta,t^\star)\},\\
D_4(M) &\triangleq \mathcal{S}_k(0,t^\star)\cap\bigcup_{i=1}^{k}\{|z_i|>M\}.
\end{aligned}$$

These sets, of course, depend on $t^\star$ and k as well but we suppress this from our notation for simplicity. Notice that for any choice of parameters these four sets subdivide $\mathcal{S}_k(0,t^\star)$,

$$\mathcal{S}_k(0,t^\star) = D_1(\delta,\eta,M)\cup D_2(\delta,\eta,M)\cup D_3(\delta)\cup D_4(M).$$

The set $D_1(\delta,\eta,M)$, for small $\delta,\eta>0$ and large $M>0$ can be thought of as covering the typical part of the space $\mathcal{S}_k(0,t^\star)$ and the sets $D_2(\delta,\eta,M)$, $D_3(\delta)$ and $D_4(M)$ can be thought of as exceptional sets. This subdivision is chosen to make $D_1(\delta,\eta,M)$ a bounded set on which the function $\psi_k^{(N),(t^\star,z^\star)}$ develops no singularities as $N\to\infty$; this makes the L^2 convergence in equation (1.5) more straightforward on $D_1(\delta,\eta,M)$. All of the singularities/non-boundedness issues occur on the exceptional sets $D_2(\delta,\eta,M)$, $D_3(\delta)$ and $D_4(M)$ where we will separately argue that they have a negligible contribution to the integral in equation (1.5). With this strategy, the substance of the proof of Theorem 1.13 is divided into Propositions 2.19, 2.20, 2.21 and 2.22, each handling one of these four sets.

Proposition 2.19. *Fix $t^\star>0$ and $z^\star\in\mathbb{R}$. For any $\delta,\eta,M>0$, we have pointwise convergence uniformly over all $\vec{w}\in D_1(\delta,\eta,M)$:*

$$\lim_{N\to\infty}\psi_k^{(N),(t^\star,z^\star)}(\vec{w}) = \psi_k^{(t^\star,z^\star)}(\vec{w}).$$

In addition, there is a constant $C_{D_1}=C_{D_1}(\delta,\eta,M)$ so that that for all $\vec{w}\in D_1(\delta,\eta,M)$ we have the bounds:

$$\sup_N \psi_k^{(N),(t^\star,z^\star)}(\vec{w})\le C_{D_1},\quad \psi_k^{(t^\star,z^\star)}(\vec{w})\le C_{D_1}.$$

Proposition 2.20. *Fix $t^\star > 0$ and $z^\star \in \mathbb{R}$. For any $\delta, M > 0$, and any given $\epsilon > 0$, there exists $\eta > 0$ small enough so that:*

$$\limsup_{N\to\infty} \int\limits_{D_2(\delta,\eta,M)} \left|\psi_k^{(N),(t^\star,z^\star)}(\vec{w})\right|^2 d\vec{w} < \epsilon.$$

Proposition 2.21. *Fix $t^\star > 0$ and $z^\star \in \mathbb{R}$. For any $\epsilon > 0$, there exists $\delta > 0$ small enough so that:*

$$\limsup_{N\to\infty} \int\limits_{D_3(\delta)} \left|\psi^{(N),(t^\star,z^\star)}(\vec{w})\right|^2 d\vec{w} \quad < \quad \epsilon.$$

Proposition 2.22. *Fix $t^\star > 0$ and $z^\star \in \mathbb{R}$. For any $\epsilon > 0$, there exists $M > 0$ large enough so that:*

$$\limsup_{N\to\infty} \int\limits_{D_4(M)} \left|\psi_k^{(N),(t^\star,z^\star)}(\vec{w})\right|^2 d\vec{w} < \epsilon. \tag{2.11}$$

We defer the proof of these estimates to later sections. Proposition 2.19 and Proposition 2.20 are proven using tools from determinantal point processes and orthogonal polynomials in Section 3. Proposition 2.21 and Proposition 2.22 are proven in Section 5 by a connection to the overlap time of two independent non-intersecting random walk bridges which is analyzed in Section 4.

Proof. (Of Theorem 1.13) By the relationship in equation (2.10), it suffices to show L^2 convergence on $\mathcal{S}_k(0,t^\star)$. Fix any $\epsilon > 0$. The strategy will be to first choose $\delta, \eta, M > 0$ so that the contribution on the exceptional sets $D_2(\delta,\eta,M)$, $D_3(\delta)$ and $D_4(M)$ are less than ϵ, and once δ, η, M are fixed, we will argue that on the typical set $D_1(\delta,\eta,M)$ we have L^2 convergence.

By Proposition 2.21 and Proposition 2.22 and a union bound, we can find $\delta > 0$ small enough and $M > 0$ large enough so that:

$$\limsup_{N\to\infty} \int\limits_{D_3(\delta)\bigcup D_4(M)} \left|\psi^{(N),(t^\star,z^\star)}(\vec{w})\right|^2 \mathrm{d}\vec{w} < \epsilon.$$

With this δ, M chosen in this way, by Proposition 2.20 we can now find $\eta > 0$ so small so that

$$\limsup_{N\to\infty} \int\limits_{D_2(\delta,\eta,M)} \left|\psi_k^{(N),(t^\star,z^\star)}(\vec{w})\right|^2 \mathrm{d}\vec{w} < \epsilon.$$

Now, since $\int_{\mathcal{S}_k(0,t^\star)} \left|\psi_k^{(t^\star,z^\star)}\right|^2 < \infty$ by Proposition 2.3, and since $\mathbf{1}\{D_1(\delta,\eta,M)\} \to one\{\mathcal{S}_k(0,t^\star)\}$ as $\delta \to 0, \eta \to 0, M \to \infty$, we know by the dominated convergence theorem that it is possible to further shrink δ, η and enlarge M so that on $D_2(\delta,\eta,M) \cup D_3(\delta) \cup D_4(M) = \mathcal{S}_k(0,t^\star) \setminus D_1(\delta,\eta,M)$ we have

$$\int\limits_{D_2(\delta,\eta,M)\cup D_3(\delta)\cup D_4(M)} \left|\psi_k^{(t^\star,z^\star)}(\vec{w})\right|^2 \mathrm{d}\vec{w} < \epsilon.$$

On the remaining set $D_1(\delta,\eta,M)$, by Proposition 2.19, we have pointwise convergence $\left|\psi_k^{(N),(t^\star,z^\star)}(\vec{w}) - \psi_k^{(t^\star,z^\star)}(\vec{w})\right|^2 \to 0$ as $N \to \infty$. Since $\left|\psi_k^{(N),(t^\star,z^\star)}(\vec{w}) - \psi_k^{(t^\star,z^\star)}(\vec{w})\right|^2 \leq$

$4C_{D_1}^2$ is bounded by Proposition 2.19, and since $D_1(\delta,\eta,M)$ is a compact set hence by an application of the bounded convergence theorem we have

$$\lim_{N\to\infty} \int\limits_{D_1(\delta,\eta,M)} \left|\psi_k^{(N),(t^\star,z^\star)}(\vec{w}) - \psi_k^{(t^\star,z^\star)}(\vec{w})\right|^2 = 0.$$

Finally, we use $\left|\psi_k^{(N),(t^\star,z^\star)}(\vec{w}) - \psi_k^{(t^\star,z^\star)}(\vec{w})\right|^2 \leq \left|\psi_k^{(N),(t^\star,z^\star)}(\vec{w})\right|^2 + \left|\psi_k^{(t^\star,z^\star)}(\vec{w})\right|^2$ (both are non-negative) and a union bound to arrive at:

$$\begin{aligned}\int\limits_{\mathcal{S}_k(0,t^\star)} \left|\psi_k^{(N),(t^\star,z^\star)}(\vec{w}) - \psi_k^{(t^\star,z^\star)}(\vec{w})\right|^2 \mathrm{d}\vec{w} \leq & \int\limits_{D_1(\delta,\eta,M)} \left|\psi_k^{(N),(t^\star,z^\star)}(\vec{w}) - \psi_k^{(t^\star,z^\star)}(\vec{w})\right|^2 \mathrm{d}\vec{w} \\ &+ \int\limits_{D_2(\delta,\eta,M)\cup D_3(\delta)\cup D_4(M)} \left|\psi_k^{(N),(t^\star,z^\star)}(\vec{w})\right|^2 \mathrm{d}\vec{w} \\ &+ \int\limits_{D_2(\delta,\eta,M)\cup D_3(\delta)\cup D_4(M)} \left|\psi_k^{(t^\star,z^\star)}(\vec{w})\right|^2 \mathrm{d}\vec{w}\end{aligned} \tag{2.12}$$

Taking the limit $N\to\infty$ of equation (2.12), we see by choice of δ,η,M that the RHS is less than 3ϵ. Since ϵ arbitrary, the limit $N\to\infty$ of the LHS of equation (2.12) is 0, as desired. ☐

3 Determinantal kernels and orthogonal polynomials

In this section, we will prove the pointwise convergence in Proposition 2.19 and the bound in Proposition 2.20. To do this, we will exploit the fact the fact that k-point correlation functions $\psi_k^{(t^\star,z^\star)}$, $\psi_k^{(N),(t^\star,z^\star)}$ can be written as $k\times k$ determinants of kernel functions $K^{(t^\star,z^\star)}, K^{(N),(t^\star,z^\star)}$, namely:

$$\begin{aligned}\psi_k^{(t^\star,z^\star)}\big((t_1,z_1),\dots,(t_k,z_k)\big) &= \det\Big[K^{(t^\star,z^\star)}\big((t_i,z_i);(t_j,z_j)\big)\Big]_{i,j=1}^k \\ \psi_k^{(N),(t^\star,z^\star)}\big((t_1,z_1),\dots,(t_k,z_k)\big) &= \det\Big[K^{(N),(t^\star,z^\star)}\big((t_i,z_i);(t_j,z_j)\big)\Big]_{i,j=1}^k.\end{aligned}$$

Our analysis will proceed by writing the determinantal kernels $K^{(N),(t^\star,z^\star)}$ and $K^{(t^\star,z^\star)}$ explicitly in terms of orthogonal polynomials.

Remark 3.1. Given any determinantal kernel $K\big((t,z);(t',z')\big)$, one can construct an equivalent kernel by choosing any function $g(t,z)\neq 0$ and then setting $\tilde{K}\big((t,z);(t',z')\big) \triangleq \frac{g(t',z')}{g(t,z)}K\big((t,z);(t',z')\big)$. The resulting kernel is equivalent in the sense that the $k\times k$ determinants are the same, $\det\big[K\big((t_i,z_i);(t_j,z_j)\big)\big]_{i,j=1}^k = \det\Big[\tilde{K}\big((t_i,z_i);(t_j,z_j)\big)\Big]_{i,j=1}^k$. This is because the factors of $g(t,z)$ and $g(t,z)^{-1}$ can be factored out of the rows and columns to cancel each other. For this reason, the choice of kernel is not unique and it is often helpful to choose a function $g(t,z)$ in order to get a kernel that is more convenient to work with.

3.1 Determinantal kernel for non-intersecting Brownian bridges

Definition 3.2. *Fix any $t^\star>0$. For $t\in(0,t^\star)$, define the shorthand $\alpha_t \triangleq \sqrt{\frac{t^\star}{2t(t^\star-t)}}$. For $z,z'\in\mathbb{R}$ and $t,t'\in(0,t^\star)$, define the kernel $K^{(0,t^\star)}\big((t,z);(t',z')\big)$ by:*

$$K^{(t^\star,0)}\big((t,z);(t',z')\big) \tag{3.1}$$
$$\triangleq -\frac{1}{\sqrt{2\pi\,(t'-t)}}\exp\left(-\frac{(z'-z)^2}{2\,(t'-t)}\right)\mathbf{1}\{t<t'\}$$
$$+\left(\frac{t^\star}{2t'(t^\star-t)}\right)^{\frac{1}{2}}\sum_{j=0}^{d-1}\left(\frac{t(t^\star-t')}{(t^\star-t)t'}\right)^{j/2}p_j(z\alpha_t)\exp\left(-\frac{z^2}{2(t^\star-t)}\right)p_j(z'\alpha_{t'})\exp\left(-\frac{z'^2}{2t'}\right).$$

where $p_j(y)$, $j\in\mathbb{N}$, $y\in\mathbb{R}$ are the normalized Hermite polynomials:

$$p_j(y) \triangleq \frac{H_j(y)}{\sqrt{\sqrt{\pi}\cdot j!\cdot 2^j}}, \tag{3.2}$$
$$H_j(y) \triangleq (-1)^j e^{y^2}\frac{\mathrm{d}^j}{\mathrm{d}y^j}e^{-y^2}.$$

Finally, for any $z^\star\in\mathbb{R}$, we will define:

$$K^{(t^\star,z^\star)}\big((t,z);(t',z')\big)\triangleq K^{(t^\star,0)}\left(\Big(t,z-z^\star\frac{t}{t^\star}\Big);\Big(t',z'-z^\star\frac{t'}{t^\star}\Big)\right)\frac{\exp\left(\frac{z^\star}{t^\star}\big(z'-z^\star\frac{t'}{t^\star}\big)\right)}{\exp\left(\frac{z^\star}{t^\star}\big(z-z^\star\frac{t}{t^\star}\big)\right)}. \tag{3.3}$$

Lemma 3.3. *Fix any $t^\star>0$ and $z^\star\in\mathbb{R}$. Recall from Definition 2.2 the k-point correlation functions $\psi_k^{(t^\star,z^\star)}$ for non-intersecting Brownian bridges $\vec{D}^{(t^\star,z^\star)}$. We have that $\psi_k^{(t^\star,z^\star)}$ is determinantal with kernel $K^{(t^\star,z^\star)}$:*

$$\psi_k^{(t^\star,z^\star)}\big((t_1,z_1);\ldots,;(t_k,z_k)\big)=\det\Big[K^{(t^\star,z^\star)}\big((t_i,z_i);(t_j,z_j)\big)\Big]_{i,j=1}^k. \tag{3.4}$$

Proof. When $z^\star=0$, the fact that $K^{(t^\star,0)}$ is the determinantal kernel for $\vec{D}^{(t^\star,0)}$ is exactly equation (2.21) in [33]. For $z^\star\neq 0$, we notice that the shift of coordinates $(t,z)\to\big(t,z-z^\star\frac{t}{t^\star}\big)$ maps the non-intersecting Brownian bridges $\vec{D}^{(t^\star,z^\star)}$ to the bridges $\vec{D}^{(t^\star,0)}$ in a measure preserving way. This shows that $K^{(t^\star,0)}\left(\big(t,z-z^\star\frac{t}{t^\star}\big);\big(t',z'-z^\star\frac{t'}{t^\star}\big)\right)$ is a determinantal kernel for $\vec{D}^{(t^\star,z^\star)}$. The multiplication by the factor $\exp\left(\frac{z^\star}{t^\star}\big(z'-z^\star\frac{t'}{t^\star}\big)\right)\exp\left(\frac{z^\star}{t^\star}\big(z-z^\star\frac{t}{t^\star}\big)\right)^{-1}$ yields an equivalent kernel, as explained in Remark 3.1. □

Lemma 3.4. *Fix any $t^\star>0$ and $z^\star\in\mathbb{R}$. For any $\delta,M>0$, there exist constants $C_K^{<}=C_K^{<}(\delta,M)$, $C_K^{\geq}=C_K^{\geq}(\delta,M)$ so that for all pairs $\big((t,z);(t',z')\big)$ that satisfy $t,t'\in(\delta,t^\star-\delta)$ and $z,z'\in(-M,M)$ we have:*

$$\begin{aligned}\Big|K^{(t^\star,z^\star)}\big((t,z);(t',z')\big)\Big| &\leq C_K^{<}\,(t'-t)^{-1/2} && \textit{if } t<t',\\ \Big|K^{(t^\star,z^\star)}\big((t,z);(t',z')\big)\Big| &\leq C_K^{\geq} && \textit{if } t\geq t'.\end{aligned}$$

Proof. First notice that the multiplicative factor that appears in equation (3.3) is bounded here by a constant we denote by $C_\times=C_\times(M)$:

$$\frac{\exp\left(\frac{z^\star}{t^\star}\left(z'-z^\star\frac{t'}{t^\star}\right)\right)}{\exp\left(\frac{z^\star}{t^\star}\left(z-z^\star\frac{t}{t^\star}\right)\right)}\leq\exp\left(2\frac{z^\star}{t^\star}\,(M+|z^\star|)\right)\triangleq C_\times.$$

Now, if $t\geq t'$, then inspecting Definition 3.2 reveals that $K^{(t^\star,z^\star)}$ consists of a sum of d terms. By the hypothesis on t,t' we have $\alpha_t\leq\sqrt{\frac{t^\star}{2\delta^2}}$ and $\alpha_{t'}\leq\sqrt{\frac{t^\star}{2\delta^2}}$. From this we notice that the domain $\left\{\big(z-z^\star\frac{t}{t^\star}\big)\,\alpha_t:\ (t,z)\in(-M,M)\times(\delta,t^\star-\delta)\right\}\subset\mathbb{R}$ is a bounded set, so we have for each $0\leq j\leq d-1$ a constant $C_j=C_j(\delta,M)<\infty$ defined by:

$$C_j\triangleq\sup_{(t,z)\in(\delta,t^\star-\delta)\times(-M,M)}\left|p_j\left(\left(z-z^\star\frac{t}{t^\star}\right)\alpha_t\right)\right|.$$

Thus, using the bounds on t, t' from the hypothesis and the definition $K^{(t^\star,z^\star)}$ we have for $t \geq t'$:

$$\left|K^{(t^\star,z^\star)}\big((t,z);(t',z')\big)\right| \leq C_\times \cdot \left(\frac{t^\star}{2\delta^2}\right)^{\frac{1}{2}} \sum_{j=0}^{d-1} \left(\frac{t^\star}{\delta}\right)^j C_j^2.$$

This is some constant depending on δ, M as desired. When $t < t'$, there is an additional term in $K^{(t^\star,z^\star)}$. Using the bound $\sqrt{t'-t} < \sqrt{t^\star}$ and the triangle inequality now gives

$$\left|K^{(t^\star,z^\star)}\big((t,z);(t',z')\big)\right| \leq \frac{C_\times}{\sqrt{t'-t}} \left(\frac{1}{\sqrt{2\pi}} + \sqrt{t^\star}\left(\frac{t^\star}{2\delta^2}\right)^{\frac{1}{2}} \sum_{j=0}^{d-1} \left(\frac{t^\star}{\delta}\right)^j C_j^2\right).$$

This is some other constant that depends on δ, M as desired. □

Corollary 3.5. *Fix any $t^\star > 0$ and $z^\star \in \mathbb{R}$. For any $\delta, \eta, M > 0$ there exists a constant $C_{D_1,K} = C_{D_1,K}(\delta,\eta,M)$ such that $\left|K^{(t^\star,z^\star)}\big((t,z);(t',z')\big)\right| \leq C_{D_1,K}$ for all pairs $\big((t,z);(t',z')\big)$ that satisfy $t,t' \in (\delta, t^\star - \delta)$, $|t'-t| > \eta$ and $z, z' \in (-M, M)$.*

Proof. For such $(t,z);(t',z')$, the bound $\sqrt{t'-t} > \sqrt{\eta}$ and the result from Lemma 3.4 show together that the constant $C_{D_1,K} = \max\left(C_K^{\geq}, \frac{1}{\sqrt{\eta}} C_K^{<}\right)$ works for the stated inequality. □

3.2 Determinantal kernel for non-intersecting random walk bridges

Definition 3.6. *The Hahn polynomials are a family of orthogonal polynomials depending on three parameters α, β, N and given explicitly in terms of the hypergeometric function ${}_3F_2$ by:*

$$Q_j(x,\alpha,\beta,M) = {}_3F_2\left(\begin{matrix}-j, -j+\alpha+\beta+1, -x\\ \alpha+1, -M\end{matrix}\right)(1). \tag{3.5}$$

See [37] for extensive details of the Hahn polynomials. Fix $n^\star \in \mathbb{N}$ and $x^\star \in \mathbb{Z}$ with $n^\star + x^\star \equiv 0 (\text{mod } 2)$. For any $x \in \mathbb{Z}, n \in \mathbb{N}$ with $x + n \equiv 0 (\text{mod } 2)$ define now $P_j^{(n^\star,x^\star)}(n,x)$ and $\tilde{P}_j^{(n^\star,x^\star)}(n,x)$ in terms of Hahn polynomials with parameters depending on $n, x, x^\star, n^\star$:

$$\begin{aligned}
P_j^{(n^\star,x^\star)}(n,x) &\triangleq Q_j\left(\frac{n+x}{2}, -\frac{n^\star+x^\star}{2} - d, -\frac{n^\star-x^\star}{2} - d, n^\star + d - 1\right),\\
\tilde{P}_j^{(n^\star,x^\star)}(n,x) &\triangleq Q_j\left(\frac{n^\star - n + x - x^\star}{2}, -\frac{n^\star-x^\star}{2} - d, -\frac{n^\star+x^\star}{2} - d, n^\star - n + d - 1\right).
\end{aligned}$$

Define for each $0 \leq j \leq d-1$:

$$F_j^{(n^\star,x^\star)} \triangleq \frac{(n^\star + 2d - 2j - 1)\,(n^\star + 2d - j)_{(j-1)}}{j!} \binom{n^\star + 2d - 2}{\frac{1}{2}n^\star + \frac{1}{2}x^\star + d - 1}^{-1},$$

where we use the notation for the Pochhammer symbol $(x)_m = x(x+1)\cdots(x+m-1)$. For $x, x' \in \mathbb{Z}$, $n, n' \in \mathbb{N}$, that have $x + n \equiv 0 (\text{mod } 2)$, $x' + n' \equiv 0 (\text{mod } 2)$ define the kernel:

$$\begin{aligned}
&K_{RW}^{(n^\star,x^\star)}\left((n,x);(n',x')\right) \qquad (3.6)\\
&\triangleq 2^{n-n'} \binom{n'-n}{\frac{1}{2}(n'-n) + \frac{1}{2}(x'-x)} \mathbf{1}\{n < n'\}\\
&\quad + 2^{n-n'} \sum_{j=0}^{d-1} F_j^{(n^\star,x^\star)} P_j^{(n^\star,x^\star)}(n',x') \binom{n'+d-1}{\frac{1}{2}n' + \frac{1}{2}x'} \tilde{P}_j^{(n^\star,x^\star)}(n,x) \binom{n^\star - n + d - 1}{\frac{1}{2}(n^\star - n) + \frac{1}{2}(x - x^\star)}.
\end{aligned}$$

Finally, for any $N \in \mathbb{N}$, $z^\star \in \mathbb{R}$, $t^\star > 0$, and any pair of coordinates $\big((t,z);(t',z')\big) \in \big((0,t^\star)\times\mathbb{R}\big)^2$ define the rescaled kernel $K^{(N),(t^\star,z^\star)}\big(\,(t,z)\,;(t',z')\,\big)$ by (recall the notation $(t,z)_2$ from Definition 2.8)

$$K^{(N),(t^\star,z^\star)}\Big(\big(t,z\big);\big(t',z'\big)\Big) \triangleq \frac{\sqrt{N}}{2} K_{RW}^{(Nt^\star,\sqrt{N}z^\star)_2}\left(\big(Nt,\sqrt{N}z\big)_2;\big(Nt',\sqrt{N}z'\big)_2\right).$$

Lemma 3.7. *Fix any $z^\star \in \mathbb{R}$,$t^\star > 0$ and $k \in \mathbb{N}$. We have that $\psi_k^{(N),(t^\star,z^\star)}$ is determinantal with kernel $K^{(N),(t^\star,z^\star)}$, namely:*

$$\psi_k^{(N),(t^\star,z^\star)}\big((t_1,z_1),\ldots,(t_k,z_k)\big) = \det\Big[K^{(N),(t^\star,z^\star)}\big((t_i,z_i);(t_j,z_j)\big)\Big]_{i,j=1}^k. \tag{3.7}$$

Proof. It suffices to show that for $x^\star \in \mathbb{Z}, n^\star \in \mathbb{N}$ that $K_{RW}^{(x^\star,n^\star)}\big(\,(n,x)\,;(n',x')\big)$ is the determinantal kernel for d non-intersecting simple symmetric random walk bridges that start at $\vec{X}(0) = \vec{\delta}(0)$ and end at $\vec{X}(n^\star) = \vec{\delta}(x^\star)$; the result then follows by the scaling in equation (2.4). $K_{RW}^{(n^\star,x^\star)}$ is a rewriting of the determinantal kernel for non-intersecting simple symmetric random walks that appears in [33] with the identification of the three main parameters a,b,c from [33] to the parameters $d,x^\star,t^\star$ in our setting by $a = d$, $b = \frac{1}{2}n^\star - \frac{1}{2}x^\star, c = \frac{1}{2}n^\star + \frac{1}{2}x^\star$.

Specifically, comparing the definition of $F_j^{(n^\star,x^\star)}$ and the definitions of $P_j^{(n^\star,x^\star)}(n,x)$, $\tilde{P}_j^{(n^\star,x^\star)}(n,x)$ from Definition 3.6, to equation (3.21), equation (3.22) and equation (3.29) in [33], we have the following identification to the quantities $C_j(a,b,c)$, $\phi_{0,n}\left(j,x\right)$ and $\phi_{n,b+c}\left(x,j\right)$ that appear in that paper:

$$\begin{aligned}
\frac{1}{d!}P_j^{(n^\star,x^\star)}(n,x)\cdot\binom{n+d-1}{\frac{1}{2}n+\frac{1}{2}x} &= \phi_{0,n}\left(j,x\right),\\
\frac{1}{d!}\tilde{P}_j^{(n^\star,x^\star)}(n,x)\cdot\binom{n^\star-n+d-1}{\frac{1}{2}\left(n^\star-n\right)+\frac{1}{2}\left(x-x^\star\right)} &= \phi_{n,b+c}\left(x,j\right),\\
\left(d!\right)^2 F_j^{(n^\star,x^\star)} &= C_j(a,b,c).
\end{aligned}$$

With this identification, the fact that $2^{n'-n}K_{RW}^{(n^\star,x^\star)}\big((n,x);(n',x')\big)$ is the determinantal kernel for simple symmetric random walks is equation (3.24) in [33]. Finally, we have multiplied by the factor $2^{-\left(n'-n\right)}$ which yields an equivalent kernel, see Remark 3.1. □

Remark 3.8. Another description of the kernel for $\psi^{(N),(t^\star,z^\star)}$ also appears in [23]. The parameters chosen in the Hahn polynomials $P_j^{(n^\star,x^\star)}$ and $\tilde{P}_j^{(n^\star,x^\star)}$ are motivated by the choice of parameters from this paper. By using hypergeometric identities, it is possible to show that the kernel there is equivalent to the kernel $K_{RW}^{(N),(n^\star,x^\star)}$.

3.3 Pointwise convergence on $D_1(\delta,\eta,M)$ – proof of Proposition 2.19

Lemma 3.9. *Fix $0 < p < 1$, $c \in \mathbb{R}$, $\gamma \in \mathbb{R}\backslash\{0\}$ such that $1+\gamma^{-1} > 0$, and a compact set $E \subset \mathbb{R}$. Let $p_M = p + cM^{-\frac{1}{2}}$ and let $\tilde{y}_M, \alpha_M, \beta_M$ be such that $\tilde{y}_M = p_M M + y\sqrt{2p(1-p)M(1+\gamma^{-1})} + O(1)$, $\alpha_M = \gamma p_M M + O(1)$ and $\beta_M = \gamma(1-p_M)M + O(1)$ as $M \to \infty$. Define the polynomials $G_j^{(M)}(y)$ in terms of the Hahn polynomials $Q_j(x,\alpha,\beta,M)$ by:*

$$G_j^{(M)}(y) \quad \triangleq \quad (-1)^j\sqrt{\binom{M}{j}2^j j!\left(\frac{p}{1-p}\right)^j\left(\frac{\gamma}{1+\gamma}\right)^j} Q_j\left(\tilde{y}_M,\alpha_M,\beta_M,M\right).$$

Then we have pointwise convergence of these polynomials to the Hermite polynomials H_j as $M \to \infty$, uniformly over all $y \in E$. In fact:

$$G_j^{(M)}(y) = H_j(y) + O\big(M^{-\frac{1}{2}}\big).$$

Proof. This is a very slight extension of Theorem A.1. from [34] where we allow the $O(1)$ corrections in the parameters $\tilde{y}_M, \alpha_M, \beta_M$ and the $O\big(M^{-\frac{1}{2}}\big)$ correction on the factor $p_M = p + cM^{-\frac{1}{2}}$. An inspection of the proof there shows that these corrections give the same asymptotic three term recurrence for the polynomials $G_j^{(M)}(y)$, namely:

$$G_{j+1}^{(M)}(y) = \Big(2y + O\big(M^{-\frac{1}{2}}\big)\Big) G_j^{(M)}(y) + \Big(2j + O\big(M^{-\frac{1}{2}}\big)\Big) G_{j-1}^{(M)}(y).$$

With this observation, the convergence follows exactly by the same induction argument as in Theorem A.1 in [34] . ☐

Corollary 3.10. *Fix any $t^\star > 0$ and $z^\star \in \mathbb{R}$. Recall the definition of the polynomials P_j, $\tilde{P}_j$ from Definition 3.6 and the factor α_t from Definition 3.2. For any $\delta, M > 0$, we have the following limit as $N \to \infty$, uniformly over the set $(t,z) \in (\delta, t^\star - \delta) \times (-M, M)$:*

$$\sqrt{N}^j P_j^{(Nt^\star, \sqrt{N}z^\star)_2}\Big(\big(Nt, \sqrt{N}z\big)_2\Big) = \left(\frac{-1}{\sqrt{2}}\right)^j \left(\frac{t^\star - t}{t^\star t}\right)^{j/2} H_j\left(\Big(z - z^\star \frac{t}{t^\star}\Big)\alpha_t\right) + O\big(N^{-\frac{1}{2}}\big),$$

$$\sqrt{N}^j \tilde{P}_j^{(Nt^\star, \sqrt{N}z^\star)_2}\Big(\big(Nt, \sqrt{N}z\big)_2\Big) = \left(\frac{-1}{\sqrt{2}}\right)^j \left(\frac{t^\star t}{t^\star - t}\right)^{j/2} H_j\left(\Big(z - z^\star \frac{t}{t^\star}\Big)\alpha_t\right) + O\big(N^{-\frac{1}{2}}\big).$$

Proof. This follows by the definition P_j and $\tilde{P}_j$ in terms of Hahn polynomials from Definition 3.6 and the asymptotics from Lemma 3.9. For P_j the parameters from Lemma 3.9 are fixed as $p = \frac{1}{2}$, $c = \frac{1}{2}\sqrt{t}\frac{z^\star}{t^\star}$, $M = \lfloor tN \rfloor + d - 1$, $\gamma = -\frac{t^\star}{t}$ and $y = \big(z - z^\star \frac{t}{t^\star}\big)\alpha_t$. For $\tilde{P}_j$ the parameters are fixed as $p = \frac{1}{2}, c = -\frac{1}{2}\sqrt{t^\star - t}\frac{z^\star}{t^\star}, M = \lfloor (t^\star - t)N \rfloor + d - 1$, $\gamma = -\frac{t^\star}{t^\star - t}$ and $y = \big(z - z^\star \frac{t}{t^\star}\big)\alpha_t$. ☐

Lemma 3.11. *Fix $t^\star > 0$ and $z^\star \in \mathbb{R}$. For any $\delta, \eta, M > 0$, we have the following pointwise convergence uniformly over all pairs $(t,z);(t',z')$ that satisfy $z, z' \in (-M, M)$, $t, t' \in (\delta, t^\star - \delta)$ and $|t - t'| > \eta$:*

$$\lim_{N\to\infty} K^{(N),(t^\star,z^\star)}\left((t,z);(t',z')\right) = K^{(t^\star,z^\star)}\big((t,z);(t',z')\big).$$

Proof. Define for convenience the variables (which depend on N), $n, n', n^\star \in \mathbb{N}$ and $x, x', x^\star \in \mathbb{Z}$ by $\big(n', x'\big) \triangleq \big(Nt', \sqrt{N}z'\big)_2$, $\big(n, x\big) \triangleq \big(Nt, \sqrt{N}z\big)_2$, $\big(n^\star, x^\star\big) \triangleq \big(Nt^\star, \sqrt{N}z^\star\big)_2$ (recall the notation $(t,z)_2$ from Definition 2.8)). By inspecting equation (3.1) and equation (3.6), we see that both kernels consist of a sum of $d+1$ terms. We will show convergence of each term separately. In the convergence of each term, we will use the local central limit theorem for binomial coefficients (see e.g. Theorem 3.5.2 in [18]) that:

$$\lim_{M\to\infty} \sup_{\ell\in\mathbb{Z}} \left| \sqrt{M} 2^{-M} \binom{M}{\ell} - \sqrt{\frac{2}{\pi}} \exp\left(-\frac{(2\ell - M)^2}{2M}\right) \right| = 0.$$

The convergence of the first term in equation (3.1) and equation (3.6) is a direct application of this result. Notice that uniformly over all t', t with $|t' - t| > \eta$ that we have $n' - n > N\eta$. By application of the local central limit theorem we have uniformly over all such t', t and any choice of z, z' that

$$\lim_{N\to\infty} \frac{\sqrt{N}}{2} 2^{n-n'} \binom{n' - n}{\frac{1}{2}(n' - n) + \frac{1}{2}(x' - x)} = \frac{1}{\sqrt{2\pi\,(t' - t)}} \exp\left(-\frac{(z' - z)^2}{2\,(t' - t)}\right),$$

and it is clear that $\mathbf{1}\{n < n'\} = \mathbf{1}\{t < t'\}$. It remains to see convergence of the remaining d terms in $K^{(t^\star,z^\star)}$ and $K^{(N),(t^\star,z^\star)}$. Focusing attention on the j-th term of the sum in the definition of $K^{(t^\star,z^\star)}$ equation (3.6), we observe that since $t > \delta$ we have $n' > \delta N$ and since $t^\star - t' > \delta$ we have $n^\star - n' > \delta N$. By application of the local central limit theorem, we have uniformly over all such t', t and any choice of $z, z' \in \mathbb{R}$ that:

$$\begin{aligned}\lim_{N\to\infty} \frac{\sqrt{N}}{2} 2^{-(n'+d-1)} \binom{n'+d-1}{\frac{1}{2}n' + \frac{1}{2}x'} &= \frac{1}{\sqrt{2\pi t'}} \exp\left(-\frac{z'^2}{2t'}\right), \\ \lim_{N\to\infty} \frac{\sqrt{N}}{2} 2^{-(n^\star-n+d-1)} \binom{n^\star - n + d - 1}{\frac{1}{2}(n^\star - n) + \frac{1}{2}(x - x^\star)} &= \frac{1}{\sqrt{2\pi\,(t^\star - t)}} \exp\left(-\frac{z^2}{2\,(t^\star - t)}\right).\end{aligned}$$

Since $n^\star > t^\star N \to \infty$ we also have

$$\lim_{N\to\infty} \frac{\sqrt{N}}{2} 2^{-(n^\star+2d-2)} \binom{n^\star + 2d - 2}{\frac{1}{2}n^\star + \frac{1}{2}x^\star + d - 1} = \frac{1}{\sqrt{2\pi t^\star}} \exp\left(-\frac{z^{\star 2}}{2t^\star}\right).$$

By the definition of $F_j^{(n^\star,x^\star)}$ from Definition 3.6 then

$$\lim_{N\to\infty} \frac{2}{\sqrt{N}} \frac{2^{(n^\star+2d-2)} F_j^{(n^\star,x^\star)}}{N^j} = \frac{1}{j!}\sqrt{2\pi}(t^\star)^{j+\frac{1}{2}} \exp\left(\frac{z^{\star 2}}{2t^\star}\right).$$

Combining these asymptotics with the asymptotics for P_j and $\tilde{P}_j$ from Corollary 3.10 we have the following limit for the j-th term in the sum from equation (3.6):

$$\begin{aligned}&\lim_{N\to\infty} \frac{\sqrt{N}}{2} 2^{n-n'} F_j^{(n^\star,x^\star)} P_j^{(n^\star,x^\star)}(n',x') \binom{n'+d-1}{\frac{1}{2}n' + \frac{1}{2}x'} \tilde{P}_j^{(n^\star,x^\star)}(n,x) \binom{n^\star - n + d - 1}{\frac{1}{2}(n^\star - n) + \frac{1}{2}(x - x^\star)} \\ =&\lim_{N\to\infty}\left(\frac{2}{\sqrt{N}} 2^{n^\star+2d-2} N^{-j} F_j^{(n^\star,x^\star)}\right) \left(\sqrt{N}^j P_j^{(n^\star,x^\star)}(n',x')\right) \left(\frac{\sqrt{N}}{2} 2^{-(n'+d+1)} \binom{n'+d-1}{\frac{1}{2}n' + \frac{1}{2}x'}\right) \\ &\times \left(\sqrt{N}^j \tilde{P}_j^{(n^\star,x^\star)}\right) \left(\frac{\sqrt{N}}{2} 2^{-(n^\star-n+d+1)} \binom{n^\star - n + d - 1}{\frac{1}{2}(n^\star - n) + \frac{1}{2}(x - x^\star)}\right)\Bigg) \\ =&\frac{1}{\sqrt{2\pi}} \frac{1}{j!} \left(\frac{t^\star}{t'(t^\star - t)}\right)^{\frac{1}{2}} \left(\frac{t(t^\star - t')}{(t^\star - t)t'}\right)^{j/2} \left(-\frac{1}{\sqrt{2}}\right)^j H_j\left(\left(z' - z^\star\frac{t'}{t^\star}\right)\alpha_{t'}\right) \exp\left(-\frac{z'^2}{2t'}\right) \\ &\times \left(-\frac{1}{\sqrt{2}}\right)^j H_j\left(\left(z - z^\star\frac{t}{t^\star}\right)\alpha_t\right) \exp\left(-\frac{(z - z^\star)^2}{2(t^\star - t)}\right).\end{aligned}$$

Keeping in mind the normalization of the Hermite polynomials from equation (3.2), we see that this is exactly the corresponding j-th term in equation (3.1), as desired. ☐

Remark 3.12. When $z^\star = 0$, Lemma 3.11 confirms equation (3.36) from [33].

Corollary 3.13. *Fix $t^\star > 0$ and $z^\star \in \mathbb{R}$. For any choice of parameters $\delta, M > 0$, there exist constants $C_K^< = C_K^<(\delta, M)$, and $C_K^\geq = C_K^\geq(\delta, M)$ so that for pairs $(t,z);(t',z')$ with $t,t' \in (\delta, T-\delta)$ and $z,z' \in (-M,M)$ we have*

$$\begin{aligned}\sup_N \left|K^{(N),(t^\star,z^\star)}\big((t,z);(t',z')\big)\right| &\leq C_K^<\,(t'-t)^{-\frac{1}{2}} \quad \textit{if } t < t', \\ \sup_N \left|K^{(N),(t^\star,z^\star)}\big((t,z);(t',z')\big)\right| &\leq C_K^\geq \quad \textit{if } t \geq t'.\end{aligned}$$

Proof. When $t \geq t'$, the first term in the definition of $K^{(N),(t^\star,z^\star)}$ and $K^{(t^\star,z^\star)}$ vanishes, and the proof of Lemma 3.11 shows that that regardless of η, $K^{(N),(t^\star,z^\star)}$ converges uniformly to $K^{(t^\star,z^\star)}$ on the set $t,t' \in (\delta, t^\star - \delta)$ and $z,z' \in (-M,M)$. Thus when $t \geq t'$, since $K^{(t^\star,z^\star)}$ is bounded by $C_K^\geq$ here by Lemma 3.4, and since the convergence in Lemma 3.11 is uniform, hence $K^{(N),(t^\star,z^\star)}$ is also bounded with a possibly larger constant. By enlarging if necessary, we denote by $C_K^\geq$ a constant large enough to bound both of them.

To see the bound when $t < t'$, it remains to study the effect of the first term. To do this we use the bound on binomial coefficients which follows from Stirling's formula:

$$\sup_{M\in\mathbb{N}} \sup_{\ell} \sqrt{M} 2^{-M} \binom{M}{\ell} \leq \sup_{M\in\mathbb{N}} \sqrt{M} 2^{-M} \binom{M}{\lfloor \frac{1}{2} M \rfloor} \leq C_{Binom},$$

Applying this bound to the first term in $K^{(N),(t^\star,z^\star)}$, along with the triangle inequality and the bound $\sqrt{t'-t} < \sqrt{t^\star}$ we conclude that:

$$\sup_{N} \sqrt{t'-t} \left| K^{(N),(t^\star,z^\star)}\big((t,z);(t',z')\big) \right| \leq \left(C_{Binom} + \sqrt{t^\star} C_K^{\geq} \right).$$

This is some constant depending on δ, M as desired. By enlarging the already defined constant $C_K^{<}$ if necessary, we may denote by $C_K^{<}$ a constant large enough to bound both. □

Corollary 3.14. *Fix $t^\star > 0$ and $z^\star \in \mathbb{R}$. For any choice of $\delta, \eta, M > 0$, there exists a constant $C_{D_1,K} = C_{D_1,K}(\delta,\eta,M,t^\star,z^\star)$ such that $\left|K^{(N),(t^\star,z^\star)}\big((t,z);(t',z')\big)\right| \leq C_{D_1,K}$ for all pairs $(t,z);(t',z')$ that satisfy $t,t' \in (\delta, t^\star - \delta)$, $|t'-t| > \eta$ and $z,z' \in (-M,M)$.*

Proof. Use the bound $\sqrt{t'-t} > \sqrt{\eta}$ and the result from Corollary 3.13 to see that the constant $C_{D_1,K} = \max\left(C_K^{\geq}, \frac{1}{\sqrt{\eta}} C_K^{<} \right)$ will do. □

Proof. (Of Proposition 2.19) By Lemma 3.3 and Lemma 3.7, $\psi_k^{(t^\star,z^\star)}$ and $\psi_k^{(N),(t^\star,z^\star)}$ are given by $k \times k$ determinants of the kernels $K^{(t^\star,z^\star)}$ and $K^{(N)(t^\star,z^\star)}$ respectively. The stated bounds by $C_{D_1}(\delta,\eta)$ follows by the bound for $\left|K^{(t^\star,z^\star)}(\cdot)\right| \leq C_{D_1,K}$ in Corollary 3.5 and the bound for $\left|K^{(N),(t^\star,z^\star)}(\cdot)\right| < C_{D_1,K}$ in Corollary 3.14. Finally, by Lemma 3.11, we have uniform convergence $K^{(N),(t^\star,z^\star)}\big((t_i,z_i);(t_j,z_j)\big) \to K^{(t^\star,z^\star)}\big((t_i,z_i);(t_j,z_j)\big)$ for any pairs (t_i,z_i) and (t_j,z_j) chosen from the k-tuple $\big((t_1,z_1),\ldots,(t_k,z_k)\big) \in D_1(\delta,\eta,M)$. Since determinants are polynomials of the entries, and the entries are always bounded by $C_{D_1,K}$, the uniform convergence of the entries implies uniform convergence of the whole determinant, yielding the desired result. □

3.4 Bounds on $D_2(\delta,\eta,M)$ – proof of Proposition 2.20

Lemma 3.15. *Fix $t^\star > 0$. Also fix subsets $I_z \subset \mathbb{R}$ and $I_t \subset (0,t^\star)$. Suppose that $K\big((t,z);(t',z')\big)$, $z,z' \in I_z$, $t,t' \in I_t$ is any determinantal kernel for which there are constants C_1 and C_2 so that we have the following bounds*

$$\begin{array}{lcll} \left|K\big((t,z);(t',z')\big)\right| & \leq & C_1 (t'-t)^{-\frac{1}{2}} & \text{if } t < t', \\ \left|K\big((t,z);(t',z')\big)\right| & \leq & C_2 & \text{if } t \geq t'. \end{array}$$

Then, for any $k \in \mathbb{N}$, there exists a constant $C_{sq} = C_{sq}(k,C_1,C_2)$ so that for all $\big((t_1,z_1);\ldots;(t_k,z_k)\big) \in (I_t \times I_z)^k$ with $0 < t_1 < \ldots < t_k < t^\star$ we have the bound

$$\det\Big[K\big((t_i,z_i);(t_j,z_j)\big)\Big]_{i,j=1}^k \leq \frac{C_{sq}}{\sqrt{t_2-t_1}\sqrt{t_3-t_2}\cdots\sqrt{t_k-t_{k-1}}}.$$

Proof. Consider $\sqrt{t_2-t_1}\sqrt{t_3-t_2}\cdots\sqrt{t_k-t_{k-1}} \det\Big[K\big((t_i,z_i);(t_j,z_j)\big)\Big]_{i,j=1}^k$. By pulling the factors into the rows of the determinant, this is:

$$\sqrt{t_2-t_1}\sqrt{t_3-t_2}\cdots\sqrt{t_k-t_{k-1}}\det\Big[K\big((t_i,z_i);(t_j,z_j)\big)\Big]_{i,j=1}^{k}$$

$$=\det\begin{bmatrix} \sqrt{t_2-t_1}K\big((t_1,z_1);(z_1,t_1)\big) & .. & \sqrt{t_2-t_1}K\big((t_1,z_1);(t_k,z_k)\big) \\ \sqrt{t_3-t_2}K\big((t_2,z_2);(z_1,t_1)\big) & .. & \sqrt{t_3-t_2}K\big((t_2,z_2);(t_k,z_k)\big) \\ \vdots & .. & \vdots \\ \sqrt{t_k-t_{k-1}}K\big((t_{k-1},z_{k-1});(t_1,z_1)\big) & .. & \sqrt{t_k-t_{k-1}}K\big((t_{k-1},z_{k-1});(t_k,z_k)\big) \\ K\big((t_k,z_k);(t_1,z_1)\big) & .. & K\big((t_k,z_k);(t_k,z_k)\big) \end{bmatrix}.$$

We now consider the entries above the diagonal and the entries on-or-below the diagonal of this matrix separately. Above the diagonal, the (i,j)-th entry with $j>i$ is bounded by:

$$\sqrt{t_{i+1}-t_i}\,\big|K\big((t_i,z_i);(t_j,z_j)\big)\big|\le\sqrt{t_j-t_i}\,\big|K\big((t_i,z_i);(t_j,z_j)\big)\big|\le C_1,$$

by hypothesis. On-or-below the diagonal, we use the bound $\sqrt{t_{i+1}-t_i}\le\sqrt{t^\star}$ and $\big|K\big((t_i,z_i);(t_j,z_j)\big)\big|\le C_2$ to conclude that these entries are bounded by $\sqrt{t^\star}C_2$. Thus all the entries of the matrix are bounded in absolute value by $\max\{C_2,\sqrt{t^\star}C_2\}$. Since all the entries are bounded in this way, and determinants are polynomials of the entries, hence the determinant is bounded by some constant which depends on k, C_1, C_2 and $t^\star$ as desired. □

Corollary 3.16. *Fix $t^\star>0$ and $z^\star\in\mathbb{R}$. For any $\delta, M>0$, there exists a constant $C_{D_2}=C_{D_2}(\delta,M)$ such that for all $\big((t_1,z_1);\ldots;(t_k,z_k)\big)\in D_2(\delta,\eta,M)$ we have:*

$$\sup_N \psi_k^{(N),(t^\star,z^\star)}\big((t_1,z_1);\ldots;(t_k,z_k)\big)\le\frac{C_{D_2}}{\sqrt{t_2-t_1}\sqrt{t_3-t_2}\cdots\sqrt{t_k-t_{k-1}}}. \tag{3.8}$$

Proof. This follows by applying Lemma 3.15 to the bounds on $K^{(N),(t^\star,z^\star)}$ from Corollary 3.13 and then finally using the fact that $K^{(N),(t^\star,z^\star)}$ is the determinantal kernel for $\psi_k^{(N),(t^\star,z^\star)}$ from Lemma 3.7. □

Proof. (Of Proposition 2.20) Recall from Definition 2.9 that $\psi^{(N),(t^\star,z^\star)}$ is constant on the cells of $\mathbb{T}^{(N)}$. Thus, as in equation (2.3), we may rewrite the integral as a sum over the discrete set of points in $\big(\mathbb{T}^{(N)}\big)^k\cap D_2(\delta,\eta)$. For convenience, we will define the set

$$E(\delta,\eta)\triangleq\Big\{\vec{t}\in\big(\mathbb{R}^+\big)^k:\delta<t_1<\ldots<t_k<t^\star-\delta, t_{i+1}-t_i<\eta\text{ for some }1\le i\le k\Big\}.$$

With this notation in hand, we now apply the bound from Corollary 3.16 on $\psi_k^{(N),(t^\star,z^\star)}(\vec{w})$ to get:

$$\int\limits_{D_2(\delta,\eta)}\Big|\psi_k^{(N),(t^\star,z^\star)}(\vec{w})\Big|^2\mathrm{d}\vec{w} \tag{3.9}$$

$$=\left(\frac{2}{N\sqrt{N}}\right)^k\sum_{\vec{w}\in\left(\mathbb{T}^{(N)}\right)^k\cap D_2(\delta,\eta)}\Big|\psi_k^{(N),(t^\star,z^\star)}(\vec{w})\Big|^2$$

$$\le\frac{1}{N^k}\sum_{\vec{t}\in E(\delta,\eta)\cap\frac{\mathbb{N}^k}{N}}\frac{C_{D_2}}{\sqrt{t_2-t_1}\cdots\sqrt{t_k-t_{k-1}}}\sum_{\vec{z}\in\frac{\mathbb{Z}^k}{\sqrt{N}}}\frac{2^k}{N^{\frac{k}{2}}}\psi_k^{(N),(t^\star,z^\star)}\big((t_1,z_1);\ldots;(t_k,z_k)\big).$$

We notice now from Definition 2.9 that $\frac{2^k}{N^{\frac{k}{2}}}\psi_k^{(N),(t^\star,z^\star)}(\vec{y})=\mathbb{P}\left(\bigcap_{j=1}^k\{z_j\in\vec{X}^{(N),(t^\star,z^\star)}(t_j)\}\right)$, namely the probability of finding a particle occupying each position $z_1,\ldots,z_k$ at the times $t_1, \ldots, t_k$ respectively. With $\vec{t}$ fixed, summing these probabilities simply counts the d particles:

$$\sum_{\vec{z} \in \frac{\mathbb{Z}^k}{\sqrt{N}}} \mathbb{P}\left(\bigcap_{j=1}^{k} \left\{ z_j \in \vec{X}^{(N),(t^\star,z^\star)}(t_j) \right\}\right) = \mathbb{E}\left[\prod_{j=1}^{k}\left(\sum_{z_j \in \frac{\mathbb{Z}}{\sqrt{N}}} \mathbf{1}\left\{ z_j \in \vec{X}^{(N),(t^\star,z^\star)}(t_j) \right\}\right)\right] = \mathbb{E}\left[d^k\right] = d^k.$$

Thus the sum over $\vec{z} \in \mathbb{Z}^k/\sqrt{N}$ in equation (3.9) gives d^k. We finally recognize the remaining piece as a Riemann sum approximation to an integral, thus concluding that:

$$\begin{aligned} \int_{D_2(\delta,\eta)} \left|\psi_k^{(N),(t^\star,z^\star)}(\vec{w})\right|^2 \mathrm{d}\vec{w} &\leq d^k C_{D_2} \frac{1}{N^k} \sum_{\vec{t} \in E(\delta,\eta) \cap \frac{\mathbb{N}^k}{N}} \frac{1}{\sqrt{t_2 - t_1}\sqrt{t_3 - t_2} \cdots \sqrt{t_k - t_{k-1}}} \\ &\leq d^k C_{D_2} \int_{\vec{t} \in E(\delta,\eta)} \frac{\mathrm{d}t_1 \mathrm{d}t_2 \ldots \mathrm{d}t_k}{\sqrt{t_2 - t_1}\sqrt{t_3 - t_2} \cdots \sqrt{t_k - t_{k-1}}}. \end{aligned} \tag{3.10}$$

Since $(t_{i+1} - t_i)^{-\frac{1}{2}}$ is integrable around the singularity at $t_{i+1} - t_i = 0$, the integrand in equation (3.10) is integrable over the range of times $\left\{\vec{t} \in \mathbb{R}^k : \delta < t_1 < \ldots < t_k < t^\star - \delta\right\}$ with finite total integral. Since $\lim_{\eta \to 0} \mathbf{1}\left\{E(\delta,\eta)\right\} = 0$ $a.s$, we have by the dominated convergence theorem that the RHS of equation (3.10) tends to 0 as $\eta \to 0$. Hence, given any $\epsilon > 0$, we can find η so small so that of equation (3.10) is less than ϵ, as desired. □

4 Overlap times and exponential moment control

The main object of study in this section are the "overlap times" introduced in Definition 4.1. These can be thought of as discrete version of the local times studied in Section 4 of [47]. The moments of this object are naturally related to the L^2 norm of $\psi_k^{(N),(t^\star,z^\star)}$ (see Corollary 5.2). The main result of this section is Proposition 4.23 which gives a particular type of control, which we call exponential moment control, on the overlap time. This result is the key ingredient in Section 5 to prove Proposition 2.21, Proposition 2.22 and Proposition 2.17.

Definition 4.1. *Recall from Definition 2.5 that $\vec{X}(n)$, $n \in \mathbb{N}$ denotes d non-intersecting random walks started from $\vec{X}(0) = \vec{\delta}_d(0)$. Let $\vec{X}'(n)$, $n \in \mathbb{N}$ be an independent copy of the same ensemble. For indices $1 \leq k, \ell \leq d$ and times $a, b \in \mathbb{N}$ with $a < b$, define the overlap time on $[a,b]$ between the k-th walk of $\vec{X}$ and the ℓ-th walk of $\vec{X}'$ by:*

$$O_{k,\ell}[a,b] \triangleq \sum_{n=a}^{b} \mathbf{1}\left\{X_k(n) = X'_\ell(n)\right\}. \tag{4.1}$$

For times $a, b \in \mathbb{N}$ with $a < b$, define the total overlap time on the interval $[a,b]$ of these processes by:

$$O[a,b] \triangleq \sum_{1 \leq k,\ell \leq d} O_{k,\ell}[a,b] = \sum_{n=a}^{b} \left|\left\{\vec{X}(n) \cap \vec{X}'(n)\right\}\right|,$$

where we think of $\vec{X}(n)$ and $\vec{X}'(n)$ as sets and $\left|\left\{\vec{X}(n) \cap \vec{X}'(n)\right\}\right|$ is the number of elements in their intersection.

Definition 4.2. *Fix any $x^\star \in \mathbb{Z}$ and $n^\star \in \mathbb{N}$ with $x^\star + n^\star \equiv 0 (\mathrm{mod}\, 2)$. Recall from Definition 2.6 that we have denoted by $\vec{X}^{(n^\star,x^\star)}(n)$, $n \in [0,n^\star] \cap \mathbb{N}$ the ensemble of d non-intersecting random walk bridges started from $\vec{X}^{(n^\star,x^\star)}(0) = \vec{\delta}_d(0)$ and ended at*

$\vec{X}^{(n^\star,x^\star)}(n^\star) = \vec{\delta}_d(x^\star)$. *Let* $\vec{X}'^{(n^\star,x^\star)}(n)$, $n \in [0,n^\star] \cap \mathbb{N}$ *be an independent copy of the same ensemble. For times* $a, b \in \mathbb{N}$ *with* $a < b$, *define the total overlap time on the interval* $[a,b] \subset [0,n^\star]$ *of these processes by:*

$$O^{(n^\star,x^\star)}[a,b] \triangleq \sum_{n=a}^{b} \left| \left\{ \vec{X}^{(n^\star,x^\star)}(n) \cap \vec{X}'^{(n^\star,x^\star)}(n) \right\} \right|.$$

For any fixed $t^\star > 0$ *and* $z^\star \in \mathbb{R}$, *and any* $0 < s < s' < t^\star$ *define the rescaled version of this by:*

$$O^{(N),(t^\star,z^\star)}[s,s'] \triangleq \frac{1}{\sqrt{N}} O^{\left(Nt^\star,\sqrt{N}z^\star\right)_2} \left[\lfloor Ns \rfloor , \lfloor Ns' \rfloor \right].$$

4.1 Exponential moment control – definition and properties

Definition 4.3. *We say that a collection of non-negative valued processes*

$$\left\{ Z^{(N)}(t) \ : \ t \in [0,t^\star] \right\}_{N \in \mathbb{N}},$$

is "exponential moment controlled as $t \to 0$*" if the following conditions are all met:*
i) For any fixed $t \in [0,t^\star]$, $\gamma > 0$*:*

$$\sup_{N\in\mathbb{N}} \mathbb{E}\left[\exp\left(\gamma Z^{(N)}(t) \right) \right] < \infty.$$

ii) For any fixed $\gamma > 0$*:*

$$\lim_{t\to 0} \sup_{N\in\mathbb{N}} \mathbb{E}\left[\exp\left(\gamma Z^{(N)}(t) \right) \right] = 1.$$

iii) For any fixed $t \in [0,t^\star]$ *and* $\gamma > 0$*:*

$$\lim_{\ell\to\infty} \sup_{N\in\mathbb{N}} \mathbb{E}\left[\sum_{k=\ell}^{\infty} \frac{1}{k!} \gamma^k \left(Z^{(N)}(t) \right)^k \right] = 0.$$

When there is no risk for ambiguity, we will call this "exponential moment controlled" and omit the "as $t \to 0$*".*

Lemma 4.4. *If* $\left\{ Z^{(N)}(t) \ : \ t \in [0,t^\star] \right\}_{N\in\mathbb{N}}$ *is a collection of non-negative valued process which are exponential moment controlled, then for any exponent* $m \in \mathbb{N}$ *we have that:*

$$\lim_{\ell\to\infty} \sup_{N\in\mathbb{N}} \mathbb{E}\left[\left(\sum_{k=\ell}^{\infty} \frac{1}{k!} \gamma^k \left(Z^{(N)}(t) \right)^k \right)^m \right] = 0.$$

Proof. Since each $Z^{(N)}(t)$ is non-negative, there is no harm in rearranging the order of the terms in the infinite sum. For any $m \in \mathbb{N}$, $t \in [0,t^\star]$ and $\gamma > 0$ we have:

$$\begin{aligned} \left(\sum_{k=\ell}^{\infty} \frac{1}{k!} \gamma^k \left(Z^{(N)}(t) \right)^k \right)^m &= \sum_{k_1,\dots,k_m=\ell}^{\infty} \frac{1}{k_1! \dots k_m!} \gamma^{k_1+\dots+k_m} \left(Z^{(N)}(t) \right)^{k_1+\dots+k_m} \\ &\le \sum_{k\ge m\ell} \left(\sum_{k_1+\ldots+k_m=k} \binom{k}{k_1,\dots,k_m} \frac{1}{k!} \gamma^k Z^{(N)}(t)^k \right) \\ &= \sum_{k\ge m\ell} (m\gamma)^k \frac{1}{k!} Z^{(N)}(t)^k, \end{aligned}$$

so the desired result holds by property iii) from Definition 4.3 of exponential moment control with parameter chosen to be $m\gamma$. □

Lemma 4.5. *Suppose* $\left\{Z^{(N)}(t) \,:\, t\in[0,t^\star]\right\}_{N\in\mathbb{N}}$ *and* $\left\{Y^{(N)}(t) \,:\, t\in[0,t^\star]\right\}_{N\in\mathbb{N}}$ *are both exponential moment controlled as* $t\to 0$. *If* $\left\{W^{(N)}(t) \,:\, t\in[0,t^\star]\right\}_{N\in\mathbb{N}}$ *is a collection of non-negative valued processes so that for all* $t\in[0,t^\star]$ *and all* $N\in\mathbb{N}$ *we have*

$$W^{(N)}(t)\le Z^{(N)}(t)+Y^{(N)}(t),$$

then $\left\{W^{(N)}(t) \,:\, t\in[0,t^\star]\right\}$ *is exponential moment controlled as* $t\to 0$.

Proof. We verify properties i), ii) and iii) from Definition 4.3. For any $t\in[0,t^\star]$ and $\gamma>0$, we have by the Cauchy Shwarz inequality:

$$\mathbb{E}\left[\exp\left(\gamma W^{(N)}(t)\right)\right]\le\sqrt{\mathbb{E}\left[\exp\left(2\gamma Z^{(N)}(t)\right)\right]\cdot\mathbb{E}\left[\exp\left(2\gamma Z^{(N)}(t)\right)\right]}.$$

From this inequality, properties i) and ii) for $W^{(N)}$ follow by the hypothesis that $Z^{(N)}(t)$ and $Y^{(N)}(t)$ satisfy properties i) and ii). To see property iii) for $W^{(N)}(t)$, consider that for any $t\in[0,t^\star]$ and $\gamma>0$,

$$\begin{aligned}
\mathbb{E}\left[\sum_{k=2\ell}^{\infty}\frac{1}{k!}\gamma^k\left(W^{(N)}(t)\right)^k\right] \le&\mathbb{E}\left[\sum_{k=2\ell}^{\infty}\frac{1}{k!}\gamma^k\left(Z^{(N)}(t)+Y^{(N)}(t)\right)^k\right]\\
=&\mathbb{E}\left[\sum_{k=2\ell}^{\infty}\sum_{\substack{a\ge 0,b\ge 0\\ a+b=k}}\left(\frac{1}{a!}\gamma^a\left(Z^{(N)}(t)\right)^a\right)\left(\frac{1}{b!}\gamma^b\left(Y^{(N)}(t)\right)^b\right)\right]\\
\le&\mathbb{E}\left[\left(\sum_{a=0}^{\infty}\left(\frac{1}{a!}\gamma^a\left(Z^{(N)}(t)\right)^a\right)\right)\left(\sum_{b=\ell}^{\infty}\left(\frac{1}{b!}\gamma^b\left(Y^{(N)}(t)\right)^b\right)\right)\right.\\
&\left.+\left(\sum_{a=\ell}^{\infty}\left(\frac{1}{a!}\gamma^a\left(Z^{(N)}(t)\right)^a\right)\right)\left(\sum_{b=0}^{\infty}\left(\frac{1}{b!}\gamma^b\left(Y^{(N)}(t)\right)^b\right)\right)\right]\\
\le&\sqrt{\mathbb{E}\left[\exp\left(2\gamma Z^{(N)}(t)\right)\right]}\sqrt{\mathbb{E}\left[\left(\sum_{b=\ell}^{\infty}\left(\frac{1}{b!}\gamma^b\left(Y^{(N)}(t)\right)^b\right)\right)^2\right]}\\
&+\sqrt{\mathbb{E}\left[\exp\left(2\gamma Y^{(N)}(t)\right)\right]}\sqrt{\mathbb{E}\left[\left(\sum_{a=\ell}^{\infty}\left(\frac{1}{a!}\gamma^a\left(Z^{(N)}(t)\right)^a\right)\right)^2\right]},
\end{aligned}\tag{4.2}$$

where we have applied the Cauchy-Schwarz inequality in the last line. Since we have that $\mathbb{E}\left[\exp\left(2\gamma Z^{(N)}(t)\right)\right]$ and $\mathbb{E}\left[\exp\left(2\gamma Y^{(N)}(t)\right)\right]$ are bounded over all $N\in\mathbb{N}$ by hypothesis i) of the exponential moment control, the desired limit as $\ell\to\infty$ of equation (4.2) follows by application of Lemma 4.4. □

Lemma 4.6. *Suppose that* $\{a_j\}_{j=1}^{\infty}$, $a_j\ge 0$, *are coefficients such that the power series* $f(x)=\sum_{j=1}^{\infty}a_jx^j$ *has an infinite radius of convergence. If* $\left\{Z^{(N)}(t) \,:\, t\in[0,t^\star]\right\}_{N\in\mathbb{N}}$ *is a collection of non-negative valued processes and there is a constant* $c>0$ *and an exponent* $\alpha>0$ *so that for all* $k\in\mathbb{N}$

$$\sup_{N\in\mathbb{N}}\frac{1}{k!}\mathbb{E}\left[Z^{(N)}(t)^k\right]\le a_k(ct^\alpha)^k,$$

then $\left\{Z^{(N)}(t) \,:\, t\in[0,t^\star]\right\}_{N\in\mathbb{N}}$ *is exponential moment controlled as* $t\to 0$.

Proof. To verify property i) of Definition 4.3, since all the terms are non-negative, we have by application of the monotone convergence theorem that for any $\gamma > 0$, $t \in [0, t^\star]$:

$$\begin{aligned}\sup_{N\in\mathbb{N}} \mathbb{E}\left[\exp\left(\gamma Z^{(N)}(t)\right)\right] &\leq 1 + \sum_{k=1}^{\infty} \frac{\gamma^k}{k!} \sup_{N\in\mathbb{N}} \mathbb{E}\left[Z^{(N)}(t)^k\right] \\ &\leq 1 + \sum_{k=1}^{\infty} a_k \left(ct^\alpha\gamma\right)^k = 1 + f(ct^\alpha\gamma).\end{aligned}$$

Since f has an infinite radius of convergence, this is finite as desired. Property ii) also follows from this display since we notice that $f(ct^\alpha\gamma) \to 0$ as $t \to 0$. Finally to see iii), notice that for fixed $t \in [0, t^\star]$ and $\gamma > 0$ we have in the same way that for any $\ell \in \mathbb{N}$, $\sup_{N\in\mathbb{N}} \mathbb{E}\left[\sum_{k=\ell}^{\infty} \frac{1}{k!}\gamma^k \left(Z^{(N)}(t)\right)^k\right] \leq \sum_{k=\ell}^{\infty} a_k \left(ct^\alpha\gamma\right)^k$. This tends to zero as $\ell \to \infty$ since $f(ct^\alpha\gamma) = \sum_{k=1}^{\infty} a_k \left(ct^\alpha\gamma\right)^k$ is a convergent series. □

Lemma 4.7. *If $\left\{Z^{(N)}(t) \ : \ t \in [0, t^\star]\right\}_{N\in\mathbb{N}}$ is a collection non-negative valued processes, for which there exist constants C and c so that:*

$$\sup_{N\in\mathbb{N}} \mathbb{P}\left(Z^{(N)}(t) > \alpha\right) \leq C\exp\left(-c\frac{\alpha^2}{t}\right) \ \forall\alpha > 0,$$

then $\left\{Z^{(N)}(t) \ : \ t \in [0, t^\star]\right\}_{N\in\mathbb{N}}$ is exponential moment controlled as $t \to 0$.

Proof. The k-th moments are bounded as follows:

$$\begin{aligned}\mathbb{E}\left[\left(Z^{(N)}(t)\right)^k\right] &= k\int_0^\infty x^{k-1}\mathbb{P}\left(Z^{(N)}(t) > x\right)\mathrm{d}x \\ &\leq k\int_0^\infty C\exp\left(-c\frac{x^2}{t}\right)x^{k-1}\mathrm{d}x = k\frac{C}{2}\left(\frac{t}{c}\right)^{\frac{1}{2}k}\Gamma\left(\frac{k}{2}\right),\end{aligned}$$

and the result follows by Lemma 4.6 because the power series $f(x) = \sum_{k=1}^{\infty} k\frac{1}{k!}\Gamma\left(\frac{k}{2}\right)x^k$ has infinite radius of convergence. □

4.2 Positions of non-intersecting random walks

In this subsection we prove exponential moment control for the rescaled position of the random walks. This is used as an ingredient in Subsection 4.5 to prove that the total overlap time is exponential moment controlled.

Lemma 4.8. *Recall from Definition 2.5 that $\vec{X}(n)$, $n \in \mathbb{N}$ denotes an ensemble of d non-intersecting random walks started from $\vec{X}(0) = \vec{\delta}_d(0)$. For any fixed $t^\star$, the absolute value of the rescaled top line process*

$$\left\{\frac{1}{\sqrt{N}}\left|X_d\left(\lfloor tN\rfloor\right)\right|, \ t \in [0, t^\star]\right\}_{N\in\mathbb{N}}$$

is exponential moment controlled as $t \to 0$.

Proof. By Lemma 4.7, it suffices to show that there are constants c, C so that for all N, $\mathbb{P}\left(|X_d(\lfloor tN\rfloor)| > \sqrt{N}\alpha\right) \leq C\exp\left(-c\frac{\alpha^2}{t}\right)$. We will prove the stronger statement that $\mathbb{P}\left(\sup_{0<n<tN}|X_d\left(n\right)| > \sqrt{N}\alpha\right) \leq C\exp\left(-c\frac{\alpha^2}{t}\right)$ by induction on d, using the reflected construction of d non-intersecting random walks from Section 2 of [44] (this is the only part of the paper where d is un-fixed). The case $d = 1$ is clear since in this case $\frac{1}{\sqrt{N}}X_1(n)$

is a rescaled simple symmetric random walk and the estimate above is standard. Now suppose the result holds for $d-1$. The reflected construction in [44] is a coupling of the process $\vec{X}$ of d non-intersecting walks started from $\vec{\delta}_d(0)$ and the process $\vec{Y}$ of $d-1$ non-intersecting walks started from $\vec{\delta}_{d-1}(0)$. In this coupling, the process $\vec{Y}$ is first constructed, and then the top line X_d is realized as a simple symmetric random walk which is reflected upward upon collisions with the top line Y_{d-1}, namely:

$$X_d(n+1) - X_d(n) = \beta(n) + 2 \cdot \mathbf{1}\big\{X_d(n) + \beta(n) = Y_{d-1}(n+1)\big\},$$

where $\beta(t)$ are iid $\{-1,+1\}$ fair coinflips, independent of the process $\vec{Y}$. Define the range of a simple random walk up to time M by $R(\beta)[0,M] \triangleq \sup_{0<s<M} \sum_{i=1}^{s} \beta(i) - \inf_{0<s<M} \sum_{i=1}^{s} \beta(i)$. From this construction, we notice that for any α:

$$\left\{ \sup_{0<n<tN} X_d(n) > \alpha\sqrt{N} \right\} \subset \left\{ \sup_{0<n<tN} Y_{d-1}(n) > \frac{\alpha}{2}\sqrt{N} \right\} \cup \left\{ R(\beta)[0,\lfloor tN \rfloor] > \frac{\alpha}{2}\sqrt{N} \right\}.$$

This is because if $\sup_{0<n<tN} Y_{d-1}(n) \leq \frac{\alpha}{2}\sqrt{N}$, then in order for X_d to advance from position $\frac{\alpha}{2}\sqrt{N}$ to $\alpha\sqrt{N}$, the process X_d will need a boost of at least $\frac{\alpha}{2}\sqrt{N}$ from the coinflip sequence β.

By the inductive hypothesis, $\mathbb{P}\left(\sup_{0<s<tN} Y_{d-1}(s) > \frac{\alpha}{2}\sqrt{N}\right) \leq C_{d-1} \exp\left(-c_{d-1}\frac{\alpha^2}{4t}\right)$ for some constants C_{d-1}, c_{d-1} (which depend on $d-1$). On the other hand the range of the walk, $R(\beta)[0,\lfloor tN \rfloor]$ has been classically studied see e.g. [55], and is known to have subguassian tails $\mathbb{P}\left(R(\beta)[0,\lfloor tN \rfloor] > \frac{\alpha}{2}\sqrt{N}\right) \leq C_{RW} \exp\left(-c_{RW}\frac{\alpha^2}{4t}\right)$. A union bound then completes the bound on $\mathbb{P}\left(\sup_{0<s<t} X_d(s) > \sqrt{N}\alpha\right)$. The bound on $\mathbb{P}\left(\sup_{0<s<tT} -X_d(t) > \alpha\sqrt{N}\right)$ is even easier since in this coupling above we have $X_d(n) \geq \sum_{i=1}^{n} \beta(i)$, and the result follows by a standard bound for the simple symmetric random walk. □

Corollary 4.9. *For any $t^\star > 0$ and any $1 \leq k \leq d$, the rescaled k-th line process:*

$$\left\{ \frac{1}{\sqrt{N}} \big| X_k\left(\lfloor tN \rfloor\right) \big|,\ t \in [0,t^\star] \right\}_{N\in\mathbb{N}},$$

is exponential moment controlled as $t \to 0$.

Proof. The case $k=d$ is exactly Lemma 4.8. The case $k=1$ (the bottom line) is immediate by the invariance of the random walk under flipping the process vertically, namely: $X_1(t) \stackrel{d}{=} 2d - X_d(t)$. Finally then notice that for $1<k<d$, because the walks are always ordered so that $X_1(t) < X_k(t) < X_d(t)$, we have:

$$\frac{1}{\sqrt{N}}\left|X_k(\lfloor tN \rfloor)\right| \leq \frac{1}{\sqrt{N}}\left|X_d(\lfloor tN \rfloor)\right| + \frac{1}{\sqrt{N}}\left|X_1(\lfloor tN \rfloor)\right|,$$

and the exponential moment control follows by application of Lemma 4.5 using the cases $k=1$, $k=d$ already proven. □

4.3 Inverse gaps of non-intersecting random walks

In this subsection we study some bounds involving the inverse gaps between walks in the ensemble of d non-intersecting random walks: these are quantities involving $\left|X_b(n) - X_a(n)\right|^{-1}$ for $1 \leq a,b \leq d$.

Definition 4.10. *For fixed $n \in \mathbb{N}$, $\epsilon > 0$, define $\mathbb{S}_{n,\epsilon} \subset \mathbb{Z}^d$ by*

$$\mathbb{S}_{n,\epsilon} \triangleq \left\{ x \in \mathbb{Z}^d : |x_j - x_i| > n^{\frac{1}{2}-\epsilon}\ \forall 1 \leq i,j \leq d, i \neq j \right\}.$$

Lemma 4.11. *Recall from Definition 2.5 that $\vec{X}(n) \in \mathbb{W}_2^d$, $n \in \mathbb{N}$ is an ensemble of d non-intersecting walks and $\mathbb{E}_{\vec{x}^0}[\cdot]$ denotes the expected value of this ensemble started from $\vec{X}(0) = \vec{x}^0 \in \mathbb{W}_2^d$. Recall also from Definition 2.2 that $\vec{D}(t) \in \mathbb{W}^d$, $t \in (0,\infty)$ denotes d non-intersecting Brownian motions and $\mathbb{E}_{\vec{0}}[\cdot]$ is the expectation of these walks started from $\vec{D}(0) = (0,0,\dots,0)$. For any $\epsilon > 0$, there exists a constant C_ϵ so that for any indices $1 \le a < b \le d$ and any $n \in \mathbb{N}$ we have the bound*

$$\sup_{n\in\mathbb{N}} \sup_{\vec{x}^0 \in \mathbb{S}_{n,\epsilon}\cap\mathbb{W}_2^d} \mathbb{E}_{\vec{x}^0}\left[\frac{1}{\frac{1}{\sqrt{n}}\big(X_b(n) - X_a(n)\big)}\right] \quad \le \quad 3^{\binom{d}{2}} \mathbb{E}_{\vec{0}}\left[\frac{1}{D_b(1) - D_a(1)}\right] + C_\epsilon. \tag{4.3}$$

Remark 4.12. This argument is based on ideas from [16] which goes by coupling random walks with Brownian motions and using the Doob h-transform to evaluate the expectations. Using these ideas, it is possible to show that for smooth functions f and for fixed $\vec{x}^0$ that $\mathbb{E}_{\vec{x}^0}\left[f\left(\frac{1}{\sqrt{n}}\vec{X}(n)\right)\right] = (1+o(1))\mathbb{E}_{\vec{0}}\left[f\left(\vec{D}(1)\right)\right]$ (see Lemma 18 in [16]). We need a bound which holds uniformly over starting positions $\vec{x}^0$ which is why our bound is relaxed by the factor $3^{\binom{d}{2}}$ and constant C_ϵ. Note also that the expectation on the RHS of equation (4.3) is finite since the possible singularity when $D_b(1) - D_a(1) = 0$ is canceled away by the Vandermonde determinant in the density from equation (2.1).

Proof. Let $\vec{S}(n) = (S_1(n),\dots,S_d(n))$ be d iid simple symmetric walks started from $\vec{S}(0) = (0,0,\dots 0)$, and denote their expectation simply by $\mathbb{E}$. By the Definition 2.5 for $\vec{X}(n)$ as a Doob h-transform using the Vandermonde determinant h_d, the expectation on the LHS of equation (4.3) can be written as

$$\begin{aligned}\mathbb{E}_{\vec{x}^0}\left[\frac{1}{\frac{1}{\sqrt{n}}\left(X_b(n) - X_a(n)\right)}\right] &= \frac{1}{h_d(\vec{x}^0)}\mathbb{E}\left[\frac{h_d\left(\vec{S}(n)+\vec{x}^0\right)}{\frac{1}{\sqrt{n}}\left(S_b(n)+x_b^0 - S_a(n) - x_a^0\right)}\mathbf{1}\left\{\tau^S_{\vec{x}^0} > n\right\}\right] \\ &= \frac{\sqrt{n}}{h_d(\vec{x}^0)}\mathbb{E}\left[\prod_{\substack{i<j \\ (i,j)\neq(a,b)}} \left(S_j(n)+x_j^0 - S_i(n) - x_i^0\right)\mathbf{1}\left\{\tau^S_{\vec{x}^0} > n\right\}\right],\end{aligned} \tag{4.4}$$

$$\tau^S_{\vec{x}^0} \triangleq \inf_{m\in\mathbb{N}}\left\{S_i(m)+x_i^0 = S_j(m)+x_j^0 \text{ for } 1\le i<j\le d\right\}.$$

By the KMT coupling [38], we couple the symmetric random walks $\vec{S}(i)$ with d iid Brownian motions, $\vec{B}(t) = (B_1(t),\dots,B_d(t))$ started from $\vec{B}(0) = (0,0,\dots,0)$ so that for absolute constants $K_1, K_2, K_3 > 0$ we have

$$\mathbb{P}\left(\sup_{1\le j\le d}\sup_{1\le m\le n}|S_j(m) - B_j(m)| > K_3\log n + x\right) \le K_1\exp\left(-K_2 x\right),$$

for all $n\in\mathbb{N}, x\in\mathbb{R}$. For our purposes, we do not need the full power of this $O(\log n)$ coupling, so we will put $x = \frac{1}{2}n^{\frac{1}{2}-2\epsilon}$ and enlarge the constants if necessary to get the weaker inequality

$$\mathbb{P}\left(\sup_{1\le j\le d}\sup_{1\le m\le n}|S_j(m) - B_j(m)| \ge \frac{1}{2}n^{\frac{1}{2}-2\epsilon}\right) \le K_1\exp\left(-\frac{1}{2}K_2 n^{\frac{1}{2}-2\epsilon}\right). \tag{4.5}$$

Now define the event $A_{\epsilon,n} = \left\{\sup_{1\le j\le d}\sup_{t\in[0,n]}|S_j(\lfloor t\rfloor) - B_j(t)| < n^{\frac{1}{2}-2\epsilon}\right\}$. We have by a union bound that:

$$A^c_{\epsilon,n} \subset \left\{\sup_{\substack{1\le j\le d\\1\le m\le n}}|S_j(m) - B_j(m)| \ge \frac{n^{\frac{1}{2}-2\epsilon}}{2}\right\}\bigcup_{\substack{1\le j\le d\\1\le m\le n}}\left\{\sup_{0\le t\le 1}|B_j(m+t) - B_j(m)| \ge \frac{n^{\frac{1}{2}-2\epsilon}}{2}\right\}.$$

Thus:

$$\begin{aligned}\mathbb{P}\left(A^c_{\epsilon,n}\right) &\leq K_1\exp\left(-K_2\left(\frac{1}{2}n^{1-2\epsilon}\right)\right)+(2nd)\mathbb{P}\left(\sup_{0\leq t\leq 1}B(t)\geq\frac{1}{2}n^{\frac{1}{2}-2\epsilon}\right)\\ &\leq K_1\exp\left(-K_2\left(\frac{1}{2}n^{1-2\epsilon}\right)\right)+\frac{4d}{\sqrt{2\pi}}n^{\frac{1}{2}+2\epsilon}\exp\left(-\frac{1}{8}n^{1-4\epsilon}\right). \qquad (4.6)\end{aligned}$$

In the above, we have used equation (4.5) along with the reflection principle and the Mill's ratio estimate $\mathbb{P}\left(\sup_{0\leq t\leq 1}B(t)\geq x\right)=2\mathbb{P}\left(B(1)\geq x\right)\leq\frac{2}{\sqrt{2\pi}}\frac{1}{x}\exp\left(-x^2/2\right)$.

We now analyze the expectation in equation (4.4) by separately examining the contribution on $A_{\epsilon,n}$ and $A^c_{\epsilon,n}$. On the event $A^c_{\epsilon,n}$ we use the bound $|S_i(n)|\leq n$ and then expand the Vandermonde determinant to see that:

$$\begin{aligned}&\frac{\sqrt{n}}{h_d(\vec{x}^0)}\mathbb{E}\left[\prod_{i<j,(i,j)\neq(a,b)}\left(S_j(n)+x^0_j-S_i(n)-x^0_i\right)\mathbf{1}\left\{\tau^S_{\vec{x}^0}>n\right\}\mathbf{1}\left\{A^c_{\epsilon,n}\right\}\right] \qquad (4.7)\\ \leq\ &\frac{\sqrt{n}}{x^0_b-x^0_a}\prod_{i<j,(i,j)\neq(a,b)}\left(\frac{2n}{x^0_j-x^0_i}+1\right)\mathbb{E}\left[\mathbf{1}\left\{\tau^S_{\vec{x}^0}>n\right\}\mathbf{1}\left\{A^c_{\epsilon,n}\right\}\right]\\ \leq\ &\sqrt{n}(n+1)^{\binom{d}{2}-1}\mathbb{P}\left(A^c_{\epsilon,n}\right),\end{aligned}$$

where the last equality follows since $x^0_j-x^0_i\geq 2$ for $1\leq i<j\leq d$. By equation (4.6), $\mathbb{P}\left(A^c_{\epsilon,n}\right)$ is *exponentially* small as $n\to\infty$, and thus the LHS of equation (4.7) converges to 0 as $n\to\infty$. In particular then, it is bounded for all n by some constant C_ϵ.

To analyze the contribution to equation (4.4) on $A_{\epsilon,n}$, we first define $\vec{x}^+$ to be a slightly dilated version of the initial position $\vec{x}^0$ by setting $x^+_i\triangleq x^0_i+2i\left\lfloor n^{\frac{1}{2}-2\epsilon}\right\rfloor$, i.e. by expanding the initial gaps between adjacent walks by $2n^{\frac{1}{2}-2\epsilon}$. On event $A_{\epsilon,n}$ we take advantage of the following inequality which holds for any $j>i$ and at all times $t\in[0,n]$:

$$\begin{aligned}S_j(\lfloor t\rfloor)+x^0_j-S_i(\lfloor t\rfloor)-x^0_i &\leq B_j(t)+x^0_j-B_i(t)-x^0_i+2n^{\frac{1}{2}-2\epsilon}\\ &\leq B_j(t)+x^0_j-B_i(t)-x^0_i+2(j-i)n^{\frac{1}{2}-2\epsilon}\\ &\leq B_j(t)+x^+_j-B_i(t)-x^+_i.\end{aligned}$$

In other words, on the event $A_{\epsilon,n}$, we have for all times $t\in[0,n]$, the gaps between the Brownian motions $\vec{B}(t)+\vec{x}^+$ are all strictly greater than the the gaps between the walks $\vec{S}(t)+\vec{x}^0$. As a consequence of this, the first intersection for the Brownian motions happens strictly after the first intersection time for the random walks and therefore $\mathbf{1}\left\{\tau^S_{\vec{x}^0}>n\right\}\leq\mathbf{1}\left\{\tau^B_{\vec{x}^+}>n\right\}$, where $\tau^B_{\vec{x}^+}\triangleq\inf_{t\in[0,n]}\left\{B_i(t)+x^+_i=B_j(t)+x^+_j\text{ for }i\neq j\right\}$. Thus we have the following bound on the contribution on $A_{\epsilon,n}$ to the expectation in equation (4.4):

$$\begin{aligned}&\frac{\sqrt{n}}{h_d(\vec{x}^0)}\mathbb{E}\left[\prod_{i<j,(i,j)\neq(a,b)}\left(S_j(n)+x^0_j-S_i(n)-x^0_i\right)\mathbf{1}\left\{\tau^S_{\vec{x}^0}>n\right\}\mathbf{1}\left\{A_{\epsilon,n}\right\}\right] \qquad (4.8)\\ \leq\ &\frac{\sqrt{n}}{h_d(\vec{x}^0)}\mathbb{E}\left[\prod_{i<j,(i,j)\neq(a,b)}\left(B_j(n)+x^+_j-B_i(n)-x^+_i\right)\mathbf{1}\left\{\tau^B_{\vec{x}^+}>n\right\}\mathbf{1}\left\{A_{\epsilon,n}\right\}\right]\\ \leq\ &\frac{h_d(\vec{x}^+)}{h_d(\vec{x}^0)}\frac{1}{h_d(\vec{x}^+)}\mathbb{E}\left[\frac{h_d\left(\vec{B}(n)+\vec{x}^+\right)}{\frac{1}{\sqrt{n}}\left(B_b(n)+x^+_b-B_a(n)-x^+_a\right)}\mathbf{1}\left\{\tau^B_{\vec{x}^+}>n\right\}\right]\\ =\ &\frac{h_d(\vec{x}^+)}{h_d(\vec{x}^0)}\mathbb{E}_{\vec{x}^+}\left[\frac{1}{\frac{1}{\sqrt{n}}\left(D_b(n)-D_a(n)\right)}\right].\end{aligned}$$

where we have recognized the Doob h-transform definition of the non-intersecting Brownian motions $\vec{D}$ from Definition 2.2. We now use a coupling result for non-intersecting Brownian motions that will allow us to compare this to a non-intersecting Brownian motion started from $\vec{D}(0)=\vec{0}$. By Lemma 3.7. of [54], if two initial positions $\vec{x}^{(1)}$ and $\vec{x}^{(2)}$ have $x^{(1)}_j - x^{(1)}_i > x^{(2)}_j - x^{(2)}_i$ for all pairs $1\leq i<j\leq d$, then there exists a coupling of two ensembles non-intersecting Brownian motions with $\vec{D}^{(1)}(0)=\vec{x}^{(1)}$ and $\vec{D}^{(2)}(0)=\vec{x}^{(2)}$ so that the gaps are always ordered, $D^{(1)}_j(t)-D^{(1)}_i(t) > D^{(2)}_j(t)-D^{(2)}_i(t)$ for $1\leq i<j\leq d$, $t\in(0,\infty)$. By this coupling, we can make the comparison

$$\frac{h_d(\vec{x}^{+})}{h_d(\vec{x}^{0})}\mathbb{E}_{\vec{x}^{+}}\left[\frac{1}{\frac{1}{\sqrt{n}}\left(D_b(n)-D_a(n)\right)}\right] \leq \frac{h_d(\vec{x}^{+})}{h_d(\vec{x}^{0})}\mathbb{E}_{\vec{0}}\left[\frac{1}{\frac{1}{\sqrt{n}}\left(D_b(n)-D_a(n)\right)}\right].$$

Finally, since $\vec{x}^0\in\mathbb{S}_{n,\epsilon}\cap\mathbb{W}^d_2$ has gaps $x^0_j-x^0_i > (j-i)n^{\frac{1}{2}-\epsilon}$ for $1\leq i<j\leq d$, we observe the inequality $0<x^+_j-x^+_i = x^0_j-x^0_i+2(j-i)\lfloor n^{\frac{1}{2}-2\epsilon}\rfloor \leq 3(x^0_j-x^0_i)$. Since a Vandermonde determinant is the product of $\binom{d}{2}$ such gaps, we have the inequality:

$$\begin{aligned}\frac{h_d(\vec{x}^{+})}{h_d(\vec{x}^{0})}\mathbb{E}_{\vec{0}}\left[\frac{1}{\frac{1}{\sqrt{n}}\left(D_b(n)-D_a(n)\right)}\right] &\leq 3^{\binom{d}{2}}\mathbb{E}_{\vec{0}}\left[\frac{1}{\frac{1}{\sqrt{n}}\left(D_b(n)-D_a(n)\right)}\right] \\ &= 3^{\binom{d}{2}}\mathbb{E}_{\vec{0}}\left[\frac{1}{D_b(1)-D_a(1)}\right],\end{aligned}$$

where the last equality follows by Brownian scaling. The conclusion of the lemma follows by combining the estimates for the contribution on $A^c_{\epsilon,n}$ from equation (4.7) and for the contribution on $A_{\epsilon,n}$ from equation (4.8). □

Lemma 4.13. *Fix any indices $1\leq a<b\leq d$. There is a universal constant $C^g_{b,a}$ that bounds the expected inverse gap size uniformly over all initial conditions $\vec{x}^0\in\mathbb{W}^d_2$ and all times $n\in\mathbb{N}$. Namely:*

$$\sup_{n\in\mathbb{N}}\sup_{\vec{x}^0\in\mathbb{W}^d_2}\mathbb{E}_{\vec{x}^0}\left[\frac{1}{\frac{1}{\sqrt{n}}\left(X_b(n)-X_a(n)\right)}\right]\leq C^g_{b,a}.$$

Proof. Fix some $0<\epsilon<\frac{1}{4}$ ($\epsilon=\frac{1}{10}$ will do). Let $\vec{S}(n)=(S_1(n),\dots,S_d(n))$ be d iid simple symmetric walks started from $\vec{S}(0)=(0,0,\dots 0)$, and denote their expectation simply by $\mathbb{E}$. Let $\nu_{n,\epsilon}=\min\left\{t\geq 1:\vec{x}^0+\vec{S}(t)\in\mathbb{S}_{n,\epsilon}\right\}$ be the first time these walk shifted by $\vec{x}^0$ enters $\mathbb{S}_{n,\epsilon}$. (Note that $\nu_{n,\epsilon}$ depends on $\vec{x}^0$ as well, but we suppress this for notational convenience.) By Lemma 8 from [16], we have some constants c_1,c_2 so that the following bounds holds:

$$\mathbb{E}\left[\left|h_d\big(\vec{x}^{\,0}+\vec{S}(n)\big)\right|\cdot\mathbf{1}\left\{\nu_{n,\epsilon}>n^{1-\epsilon}\right\}\right]\leq c_1 h_d(\vec{x}^{\,0})\exp\left(-c_2n^{\epsilon}\right). \tag{4.9}$$

(Actually, Lemma 8 in [16] includes a stronger statement that

$$\mathbb{E}\left[\left|h_d\big(\vec{x}^0+\vec{S}(n)\big)\right|\mathbf{1}\left\{\nu_{n,\epsilon}>n^{1-\epsilon}\right\}\right]\leq c_1\prod_{1\leq i<j\leq d}\left(1+\left|x^0_i-x^0_j\right|\right)\exp\left(-c_2n^{\epsilon}\right),$$

but since $1\leq\left|x^0_i-x^0_j\right|$, we can easily reduce to the above by enlarging the constant c_1 by a factor $3^{\binom{d}{2}}$.)

Define now the first intersection time $\tau_{\vec{x}^0} \triangleq \inf\left\{t : x_i^0 + S_i^0(t) = x_j^0 + S_j^0(t), i \neq j\right\}$ and using the Doob h-transform definition of $\vec{X}$ as in Definition 2.5, we find that:

$$\begin{aligned}
&\mathbb{E}_{\vec{x}^0}\left[\frac{1}{\frac{1}{\sqrt{n}}\left(X_b(n) - X_a(n)\right)}\right] = \frac{1}{h_d(\vec{x}^0)}\mathbb{E}\left[\frac{h_d\left(\vec{S}(n) + \vec{x}^0\right)}{\frac{1}{\sqrt{n}}\left(S_b(n) + x_b^0 - S_a(n) - x_a^0\right)}\mathbf{1}_{\left\{\tau_{\vec{x}^0} > n\right\}}\right] \\
&= \frac{1}{h_d(\vec{x}^0)}\mathbb{E}\left[\frac{h_d\left(\vec{S}(n) + \vec{x}^0\right)}{\frac{1}{\sqrt{n}}\left(S_b(n) + x_b^0 - S_a(n) - x_a^0\right)}\mathbf{1}_{\left\{\tau_{\vec{x}^0} > n\right\}}\mathbf{1}_{\{\nu_{n,\epsilon} > n^{1-\epsilon}\}}\right] \\
&+ \frac{1}{h_d(\vec{x}^0)}\mathbb{E}\left[\frac{h_d\left(\vec{S}(n) + \vec{x}^0\right)}{\frac{1}{\sqrt{n}}\left(S_b(n) + x_b^0 - S_a(n) - x_a^0\right)}\mathbf{1}_{\left\{\tau_{\vec{x}^0} > n\right\}}\mathbf{1}_{\{\nu_{n,\epsilon} \leq n^{1-\epsilon}\}}\right] \\
&\leq c_1\sqrt{n}\exp\left(-c_2 n^{\epsilon}\right) + \frac{1}{h_d(\vec{x}^0)}\mathbb{E}\left[\frac{h_d\left(\vec{S}(n) + \vec{x}^0\right)}{\frac{1}{\sqrt{n}}\left(S_b(n) + x_b^0 - S_a(n) - x_a^0\right)}\mathbf{1}_{\left\{\tau_{\vec{x}^0} > n\right\}}\mathbf{1}_{\{\nu_{n,\epsilon} \leq n^{1-\epsilon}\}}\right],
\end{aligned} \tag{4.10}$$

where we have applied the bound from equation (4.9) and also the simple bound $S_b(n) + x_b^0 - S_a(n) - x_a^0) \geq 1$ on the event $\{\tau_{\vec{x}^0} > n\}$. In the second term of the RHS of equation (4.10), since we are on the event $\left\{\nu_{n,\epsilon} \leq n^{1-\epsilon}\right\} \cap \{\tau_{\vec{x}^0} > n\}$, we know that there is no self intersection up to time $\nu_{n,\epsilon}$, i.e. $\vec{S}(\nu_{n,\epsilon}) + \vec{x}^0 \in \cap\mathbb{W}_2^d$ is still in the Weyl chamber. We now use the strong Markov property to think of $\vec{S}(\nu_{n,\epsilon}) + \vec{x}^0 \in \mathbb{S}_{n,\epsilon} \cap \mathbb{W}_2^d$ as the initial position of the walks, which we run for the remaining time $n - \nu_{n,\epsilon}$. Since $\vec{S}(\nu_{n,\epsilon}) + \vec{x}^0 \in \mathbb{S}_{n,\epsilon} \cap \mathbb{W}_2^d$ here, we are in a position to apply the bound from Lemma 4.11 for this initial position. Integrating over all possible times for $\nu_{n,\epsilon}$ and all possible positions for $\vec{S}(\nu_{n,\epsilon})$ gives:

$$\begin{aligned}
&\mathbb{E}\left[\frac{h_d\left(\vec{S}(n) + \vec{x}^0\right)}{\frac{1}{\sqrt{n}}\left(S_b(n) + x_b^0 - S_a(n) - x_a^0\right)}\mathbf{1}\left\{\tau_{\vec{x}^0} > n\right\}\mathbf{1}\left\{\nu_{n,\epsilon} \leq n^{1-\epsilon}\right\}\right] \qquad (4.11)\\
&= \sum_{k=1}^{n^{1-\epsilon}}\sum_{\vec{y}\in\mathbb{S}_{n,\epsilon}\cap\mathbb{W}_2^d}\mathbb{P}\left(\nu_{n,\epsilon} = k, \vec{S}(k) + \vec{x}^0 = \vec{y}, \tau_{\vec{x}^0} > k\right) \\
&\quad\times\mathbb{E}\left[\frac{h_d\left(\vec{S}(n) + \vec{x}^0\right)}{\frac{1}{\sqrt{n}}\left(S_b(n) + x_b^0 - S_a(n) - x_a^0\right)}\mathbf{1}\left\{\tau_{\vec{x}^0} > n\right\}\middle|\nu_{n,\epsilon} = k, \vec{S}(k) + \vec{x}^0 = \vec{y}, \tau_{\vec{x}^0} > k\right] \\
&= \sum_{k=1}^{n^{1-\epsilon}}\frac{\sqrt{n}}{\sqrt{n-k}}\sum_{\vec{y}\in\mathbb{S}_{n,\epsilon}\cap\mathbb{W}_2^d}h_d(\vec{y})\mathbb{P}\left(\nu_{n,\epsilon} = k, \vec{S}(k) + \vec{x}^0 = \vec{y}, \tau_{\vec{x}^0} > k\right) \\
&\quad\times\left(\frac{1}{h_d(\vec{y})}\mathbb{E}\left[\frac{h_d\left(\vec{S}(n-k) + \vec{y}\right)}{\frac{1}{\sqrt{n-k}}\left(S_b(n-k) + y_b - S_a(n-k) - y_a\right)}\mathbf{1}\left\{\tau_{\vec{x}^0} > n-k\right\}\middle|\vec{S}(0) = \vec{0}\right]\right) \\
&= \sum_{k=1}^{n^{1-\epsilon}}\frac{\sqrt{n}}{\sqrt{n-k}}\sum_{\vec{y}\in\mathbb{S}_{n,\epsilon}\cap\mathbb{W}_2^d}h_d(\vec{y})\mathbb{P}\left(\nu_{n,\epsilon} = k, \vec{S}(k) + \vec{x}^0 = \vec{y}, \tau_{\vec{x}^0} > k\right) \\
&\quad\times\left(\mathbb{E}_{\vec{y}}\left[\frac{1}{\frac{1}{\sqrt{n-k}}\left(X_b(n-k) - X_a(n-k)\right)}\right]\right) \\
&\leq \frac{\mathbb{E}\left[h_d\big(\vec{S}(\nu_{n,\epsilon}) + \vec{x}^0\big)\mathbf{1}\{\nu_{n,\epsilon} \leq n^{1-\epsilon}\}\mathbf{1}\{\tau_{\vec{x}^0} > \nu_{n,\epsilon}\}\right]}{\sqrt{1 - n^{-\epsilon}}}\left(3^{\binom{d}{2}}\mathbb{E}_{\vec{0}}\left[\frac{1}{D_b(1) - D_a(1)}\right] + C_\epsilon\right).
\end{aligned}$$

where we have applied Lemma 4.11 and recognized the remaing sum as an expectation in the last line of equation (4.11). We now claim that:

$$\mathbb{E}\left[h_d\left(\vec{S}(\nu_{n,\epsilon})+\vec{x}^0\right)\mathbf{1}\left\{\nu_{n,\epsilon}\leq n^{1-\epsilon}\right\}\mathbf{1}\left\{\tau_{\vec{x}^0}>\nu_{n,\epsilon}\right\}\right]\leq h_d(\vec{x}^0)\left(1+c_1\exp\left(-c_2 n^{\epsilon}\right)\right) \tag{4.12}$$

Indeed, one verifies that (with the notation $x\wedge y=\min(x,y)$)

$$\mathbf{1}\left\{\nu_{n,\epsilon}\leq n^{1-\epsilon}\right\}\mathbf{1}\left\{\tau_{\vec{x}^0}>\nu_{n,\epsilon}\right\}=1-\mathbf{1}\left\{\tau_{\vec{x}^0}\leq\nu_{n,\epsilon}\wedge n^{1-\epsilon}\right\}-\mathbf{1}\left\{\tau_{\vec{x}^0}>n^{1-\epsilon}\right\}\mathbf{1}\left\{\nu_{n,\epsilon}>n^{1-\epsilon}\right\},$$

and then we have:

$$\begin{aligned}\text{LHS (4.12)} =&\mathbb{E}\left[h_d\left(\vec{S}(\nu_{n,\epsilon})+\vec{x}^0\right)\right]-\mathbb{E}\left[h_d\left(\vec{S}(\nu_{n,\epsilon})+\vec{x}^0\right)\mathbf{1}\left\{\tau_{\vec{x}^0}\leq\nu_{n,\epsilon}\wedge n^{1-\epsilon}\right\}\right]\\ &-\mathbb{E}\left[h_d\left(\vec{S}(\nu_{n,\epsilon})+\vec{x}^0\right)\mathbf{1}\left\{\tau_{\vec{x}^0}>n^{1-\epsilon}\right\}\mathbf{1}\left\{\nu_{n,\epsilon}>n^{1-\epsilon}\right\}\right]\\ \leq& h_d(\vec{x}^0)-0+c_1 h_d(\vec{x}^0)\exp\left(-c_2 n^{\epsilon}\right),\end{aligned}$$

where we have used the bound from equation (4.9) and the fact that $h_d\left(\vec{S}(\cdot)+\vec{x}^0\right)$ is a martingale (see e.g. [39]), so $\mathbb{E}\left[h_d\left(\vec{S}(\nu_{n,\epsilon})+\vec{x}^0\right)\right]=h_d(\vec{x}^0)$. The middle term is zero since $h_d\left(\vec{S}(\cdot)+\vec{x}^0\right)$ is a martingale and reaches zero at the earlier time $\tau_{\vec{x}^0}\leq\nu_{n,\epsilon}$. Finally, combining equations (4.10), (4.11) and (4.12) we have

$$\mathbb{E}_{\vec{x}^0}\left[\frac{1}{\frac{1}{\sqrt{n}}\left(X_b(n)-X_a(n)\right)}\right]\leq\frac{3^{\binom{d}{2}}\mathbb{E}_{\vec{0}}\left[\frac{1}{D_b(1)-D_a(1)}\right]+C_\epsilon}{\sqrt{1-n^{-\epsilon}}}\left(1+(1+\sqrt{n})c_1\exp\left(-c_2 n^{\epsilon}\right)\right).$$

This upper bound does not depend on $\vec{x}^0$ and has a finite limit as $n\to\infty$, and is hence bounded above by some constant, as desired. □

Lemma 4.14. *Let $\vec{X}(n)$, $n\in\mathbb{N}$ denote d non-intersecting random walks started from any $\vec{x}^0\in\mathbb{W}_2^d$ as in Definition 2.6. For any $t^\star>0$ and any indices $1\leq a<b\leq d$, the collection*

$$\left\{\frac{1}{\sqrt{N}}\sum_{i=1}^{\lfloor tN\rfloor}\frac{1}{X_b(i)-X_a(i)}:t\in[0,t^\star]\right\}_{N\in\mathbb{N}},$$

is exponential moment controlled as $t\to 0$.

Proof. We assume without loss that the constant from Lemma 4.13 has $C^g_{b,a}>1$. We will show that the k-th moment obeys the bound

$$\frac{1}{k!}\mathbb{E}_{\vec{x}^0}\left[\left(\frac{1}{\sqrt{N}}\sum_{i=1}^{\lfloor tN\rfloor}\frac{1}{X_b(i)-X_a(i)}\right)^k\right]\leq\left(C^g_{b,a}\sqrt{t}\right)^k\frac{\Gamma(\frac{1}{2})^k}{\Gamma\left(\frac{1}{2}k+1\right)}, \tag{4.13}$$

From here the exponential moment control follows from Lemma 4.6 since the power series with coefficents given by the RHS of equation (4.13) is easily verified to have infinite radius of convergence. To simplify notation, we will use the shorthands $G(i)\triangleq\left(X_b(i)-X_a(i)\right)^{-1}$ and $E_k(s)\triangleq\left\{\vec{t}\in\mathbb{R}^k:0\leq t_1\leq\ldots\leq t_k\leq s\right\}$ and $E^{\mathbb{N}}_k(s)\triangleq E_k(s)\cap\mathbb{N}^k$. We will show the slightly stronger statement that for any $k\in\mathbb{N}$, $k\geq 1$, and for any non-negative non-increasing function $f:\mathbb{R}^{\geq 0}\to\mathbb{R}^{\geq 0}$ we have:

$$\mathbb{E}_{\vec{x}^0}\left[\sum_{\vec{i}\in E^{\mathbb{N}}_k(\lfloor tN\rfloor)}\left(\prod_{\ell=1}^k G(i_\ell)\right)f(i_k)\right]\leq\left(C^g_{b,a}\right)^k\int\limits_{\vec{t}\in E_k(\lfloor tN\rfloor)}\frac{1}{\sqrt{t_1}}\frac{1}{\sqrt{t_2-t_1}}\cdots\frac{1}{\sqrt{t_k-t_{k-1}}}f(t_k)\mathrm{d}\vec{t}. \tag{4.14}$$

Once this is established equation (4.13) follows by setting $f(x) \equiv 1$, using the inequality $\mathbb{E}_{\vec{x}^0}\left[\left(\frac{1}{\sqrt{N}}\sum_{i=1}^{\lfloor tN \rfloor}\frac{1}{X_b(i)-X_a(i)}\right)^k\right] \leq \frac{k!}{\sqrt{N}^k}\mathbb{E}_{\vec{x}^0}\left[\sum_{\vec{i}\in E_k^{\mathbb{N}}(\lfloor tN \rfloor)}\left(\prod_{\ell=1}^k G(i_\ell)\right)\right]$ and evaluating the integral that appears on the RHS of equation (4.14). (The integral appears when computing the normalizing constant for the Dirichlet distribution, see e.g. Section 3.4 in [2])

The proof of equation (4.14) goes by induction on k. The base case $k = 1$ follows by direct application of Proposition 4.13, the bound $0 < G(i) < 1$ and the integral comparison test for non-increasing functions:

$$\mathbb{E}_{\vec{x}^0}\left[\sum_{i=1}^{\lfloor tN \rfloor} G(i)f(i)\right] = \sum_{i=1}^{\lfloor tN \rfloor}\mathbb{E}_{\vec{x}^0}\left[\frac{1}{X_b(i)-X_a(i)}\right]f(i) \leq \sum_{i=1}^{\lfloor tN \rfloor}\frac{C_{b,a}^g}{\sqrt{i}}f(i) \leq C_{b,a}^g\int_0^{\lfloor tN \rfloor}\frac{f(s)}{\sqrt{s}}\mathrm{d}s.$$

Now suppose that the statement holds for $k-1$. Let $\mathcal{F}_i = \sigma\left(\vec{X}(1),\dots,\vec{X}(i)\right)$ be the filtration generated by the first i steps of the ensemble, and then consider by the law of total expectation that:

$$\begin{aligned} &\mathbb{E}_{\vec{x}^0}\left[\sum_{\vec{i}\in E_k^{\mathbb{N}}(\lfloor tN \rfloor)}\left(\prod_{\ell=1}^k G(i_\ell)\right)f(i_k)\right] \qquad (4.15)\\ \leq\ &\mathbb{E}_{\vec{x}^0}\left[\sum_{\vec{i}\in E_{k-1}^{\mathbb{N}}(\lfloor tN \rfloor)}\left(\prod_{\ell=1}^{k-1} G(i_\ell)\right)\left(\sum_{i_k=i_{k-1}+2}^{\lfloor tN \rfloor}\mathbb{E}\left[G(i_k)\middle|\mathcal{F}_{i_{k-1}}\right]f(i_k)+2f(i_{k-1})\right)\right]\\ \leq\ &\mathbb{E}_{\vec{x}^0}\left[\sum_{\vec{i}\in E_{k-1}^{\mathbb{N}}(\lfloor tN \rfloor)}\left(\prod_{\ell=1}^{k-1} G(i_\ell)\right)\left(\sum_{i_k=i_{k-1}+2}^{\lfloor tN \rfloor}\frac{C_{b,a}^g}{\sqrt{i_k-i_{k-1}}}f(i_k)+2f(i_{k-1})\right)\right], \end{aligned}$$

where we have used $0 \leq G(i) \leq 1$ and the fact that $\vec{X}(\cdot)$ is a Markov process, so $\mathbb{E}_{\vec{x}^0}\left[G(i_k)\middle|\mathcal{F}_{i_{k-1}}\right] = \mathbb{E}_{\vec{X}(i_{k-1})}\left[G(i_k-i_{k-1})\right] \leq C_{b,a}^g\left(i_k-i_{k-1}\right)^{-\frac{1}{2}}$ by an application of the inequality from Lemma 4.13. We now bound the sum over i_k that appears by the integral comparison test for non-increasing functions and use the fact that $\int_0^1\frac{1}{\sqrt{x}}\mathrm{d}x = 2$ to see that:

$$C_{b,a}^g\sum_{i_k=i_{k-1}+2}^{\lfloor tN \rfloor}\frac{f(i_k)}{\sqrt{i_k-i_{k-1}}}+2f(i_{k-1}) \leq C_{b,a}^g\int_{i_{k-1}}^{\lfloor tN \rfloor}\frac{f(s)}{\sqrt{s-i_{k-1}}}\mathrm{d}s.$$

The result then follows by applying the inductive hypothesis to the RHS of equation (4.15) applied with the non-increasing function $\tilde{f}(x_{k-1}) = C_{b,a}^g\int_{x_{k-1}}^{\lfloor tN \rfloor}\frac{f(s)}{\sqrt{s-x_{k-1}}}\mathrm{d}s$ ($\tilde{f}$ can be verified to be non-increasing by doing a change of variable $u = s - x_{k-1}$). □

4.4 Conditional drift of non-intersecting random walks

In this subsection, we study the conditional drift of the k-th walk at time n, namely $\mathbb{E}\left[X_k(n+1)-X_k(n)\middle|\vec{X}(n)=\vec{x}\right]$.

Lemma 4.15. *Fix any $m \in \mathbb{N}$. Then for any collection of real numbers $\alpha_1,\dots,\alpha_m$ with $|\alpha_i| \leq 1\ \forall i$, we have:*

$$\left|\prod_{i=1}^m(1+\alpha_i)-\left(1+\sum_{i=1}^m\alpha_i\right)\right| \leq (2^m-1)\sum_{i=1}^m|\alpha_i|.$$

Proof. The proof is a straightforward proof by induction on m. □

Lemma 4.16. *For any* $1 \leq k \leq d$*, denote by* $\Delta X_k(n) \triangleq X_k(n+1) - X_k(n)$ *the* n*-th increment of the* k*-th non-intersecting random walk in the ensemble. We have the following bound on the conditional expectation:*

$$\left|\mathbb{E}\left[\Delta X_k(n)\middle|\vec{X}(n) = \vec{x}\right]\right| \leq 2^d \sum_{\substack{i=1 \\ i\neq k}}^{d} \frac{1}{|x_k - x_i|}.$$

Remark 4.17. A more careful version of the proof of Lemma 4.16 shows actually that

$$\mathbb{E}\left[\Delta X_k(n)\middle|\vec{X}(n) = \vec{x}\right] = \sum_{\substack{i=1 \\ i\neq k}}^{d} \frac{1}{x_k - x_i} + \epsilon(\vec{x}),$$

where $\epsilon(\vec{x})$ involves only terms that have the product of at least two different gaps in the denominator:

$$\epsilon(\vec{x}) \sim \sum \frac{1}{(x_{i_1} - x_{j_1})(x_{i_2} - x_{j_2}) \cdots}.$$

In the limit $N \to \infty$, these terms $\epsilon(\vec{x})$ are expected to be typically negligible compared to the first term $\sum_{i=1,i\neq k}^{d} (x_k - x_i)^{-1}$. This would recover the generator for the non-intersecting Brownian motion $\vec{D}$, whose k-th walker has drift exactly equal to $\sum_{i=1,i\neq k}^{d} (z_k - z_i)^{-1}$.

Part of the additional difficulty in studying the non-intersecting random walks is these additional terms $\epsilon(\vec{x})$ give much more complicated interactions between the walks. In $\vec{D}$, the i-th and j-th Brownian motion interact in a symmetric way by both pushing on each other with the same "force" equal to $(D_i - D_j)^{-1}$; in $\vec{X}$ the exact interaction between the i-th and j-th walk will depend on the positions of *all* the other elements of $\vec{X}$ too. This symmetry for $\vec{D}$ was used in [47] as part of the analysis to control singularities of the k-point correlation function $\psi_k^{(t^\star,z^\star)}$, so this complication can be thought of as one of the reasons why our analysis of $\psi_k^{(N),(t^\star,z^\star)}$ is more involved.

Proof. (Of Lemma 4.16) The proof goes by expanding the Vandermonde determinant h_d that appears in the generator for $\vec{X}$ in terms of a smaller Vandermonde determinant h_{d-1} by factoring out the k-th term. We will use the following notation with the absentee hat " $\hat{\cdot}$ ", to denote the vector $\vec{x}$ with the k-th component removed

$$\vec{x}^{\hat{k}} \triangleq (x_1, \ldots, \hat{x}_k, \ldots x_d) \in \mathbb{Z}^{d-1}.$$

Notice with this notation we can decompose the Vandermonde determinant as

$$h_d(\vec{x}) = \prod_{i=1}^{k-1} (x_k - x_i) \cdot \prod_{i=k+1}^{d} (x_i - x_k) \cdot h_{d-1}\left(\vec{x}^{\hat{k}}\right).$$

We will also write $\vec{\delta}^{\hat{k}} \in \{-1,+1\}^{d-1}$ to mean the set of $\vec{\delta}^{\hat{k}} = \left(\delta_1, \ldots, \hat{\delta}_k, \ldots \delta_d\right)$ that have $\delta_i \in \{-1,+1\}$ for all $1 \leq i \leq k-1$ and $k+1 \leq i \leq d$ (i.e the labelling is shifted to not include the index k).

From the definition of the non-intersecting random walks in Definition 2.5 as a Doob h-transform of a simple symmetric random walk, we have:

$$
\begin{aligned}
&\mathbb{P}\left[\Delta X_k(n)=+1\middle|\vec{X}(n)=\vec{x}\right]\\
&=\sum_{\vec{\delta}\in\{-1,+1\}^d,\delta_k=+1}\frac{1}{2^d}\frac{h_d(\vec{x}+\vec{\delta})}{h_d(\vec{x})}\\
&=\sum_{\vec{\delta}^{\hat{k}}\in\{-1,+1\}^{d-1}}\frac{1}{2}\prod_{i=1}^{k-1}\left(1+\frac{1-\delta_i}{x_k-x_i}\right)\cdot\prod_{i=k+1}^{d}\left(1+\frac{\delta_i-1}{x_i-x_k}\right)\cdot\frac{1}{2^{d-1}}\frac{h_{d-1}\left(\vec{x}^{\hat{k}}+\vec{\delta}^{\hat{k}}\right)}{h_{d-1}\left(\vec{x}^{\hat{k}}\right)}\\
&=\sum_{\vec{\delta}^{\hat{k}}\in\{-1,+1\}^{d-1}}\frac{1}{2}\prod_{\substack{i\neq k\\ i:\delta_i=-1}}\left(1+\frac{2}{x_k-x_i}\right)\frac{1}{2^{d-1}}\frac{h_{d-1}\left(\vec{x}^{\hat{k}}+\vec{\delta}^{\hat{k}}\right)}{h_{d-1}\left(\vec{x}^{\hat{k}}\right)},
\end{aligned}
$$

and similarly,

$$
\mathbb{P}\left[\Delta X_k(n)=-1\middle|\vec{X}(n)=\vec{x}\right]=\sum_{\vec{\delta}^{\hat{k}}\in\{-1,+1\}^{d-1}}\frac{1}{2}\prod_{\substack{i\neq k\\ i:\delta_i=+1}}\left(1-\frac{2}{x_k-x_i}\right)\frac{1}{2^{d-1}}\frac{h_{d-1}(\vec{x}^{\hat{k}}+\delta^{\hat{k}})}{h_{d-1}(\vec{x}^{\hat{k}})}.
$$

Subtracting these from each other, we have that $\mathbb{E}\left[\Delta X_k(n)\middle|X(n)=\vec{x}\right]$ is given by:

$$
\begin{aligned}
&\mathbb{P}\left[\Delta X_k(n)=+1\middle|X(n)=\vec{x}\right]-\mathbb{P}\left[\Delta X_k(n)=-1\middle|X(n)=\vec{x}\right] \qquad (4.16)\\
=&\frac{1}{2}\sum_{\vec{\delta}^{\hat{k}}\in\{-1,+1\}^{d-1}}\left(\prod_{\substack{i\neq k\\ i:\delta_i=-1}}\left(1+\frac{2}{x_k-x_i}\right)-\prod_{\substack{i\neq k\\ i:\delta_i=+1}}\left(1-\frac{2}{x_k-x_i}\right)\right)\frac{1}{2^{d-1}}\frac{h_{d-1}(\vec{x}^{\hat{k}}+\delta^{\hat{k}})}{h_{d-1}(\vec{x}^{\hat{k}})}
\end{aligned}
$$

To approximate these products by sums, we now define error terms $\epsilon^{-}_{\delta,k}$, $\epsilon^{+}_{\delta,k}$ by:

$$
\begin{aligned}
\epsilon^{-}_{\delta,k}(\vec{x}) &\triangleq \prod_{\substack{i\neq k\\ i:\delta_i=-1}}\left(1+\frac{2}{x_k-x_i}\right)-\left(1+\sum_{\substack{i\neq k\\ i:\delta_i=-1}}\frac{2}{x_k-x_i}\right), \qquad (4.17)\\
\epsilon^{+}_{\delta,k}(\vec{x}) &\triangleq \prod_{\substack{i\neq k\\ i:\delta_i=+1}}\left(1-\frac{2}{x_k-x_i}\right)-\left(1-\sum_{\substack{i\neq k\\ i:\delta_i=+1}}\frac{2}{x_k-x_i}\right),
\end{aligned}
$$

where, since $\left|\frac{2}{x_k-x_i}\right|\leq 1$ for all $i\neq k$, the errors are bounded by application of Lemma 4.15,

$$
\left|\epsilon^{-}_{\delta,k}(\vec{x})\right|\leq\left(2^d-1\right)\sum_{\substack{i\neq k\\ i:\delta_i=-1}}\frac{2}{|x_k-x_i|},\quad\left|\epsilon^{+}_{\delta,k}(\vec{x})\right|\leq\left(2^d-1\right)\sum_{\substack{i\neq k\\ i:\delta_i=+1}}\frac{2}{|x_k-x_i|}. \qquad (4.18)
$$

Plugging equation (4.17) into equation (4.16) now yields

$$
\begin{aligned}
&\mathbb{E}\left[\Delta X_k(n)\middle|X(n)=\vec{x}\right]\\
=&\frac{1}{2}\sum_{\vec{\delta}^{\hat{k}}\in\{-1,+1\}^{d-1}}\Big(\sum_{\substack{i\neq k\\ i:\delta_i=-1}}\frac{2}{x_k-x_i}+\epsilon^{-}_{\delta,k}(\vec{x})+\sum_{\substack{i\neq k\\ i:\delta_i=+1}}\frac{2}{x_k-x_i}-\epsilon^{+}_{\delta,k}(\vec{x})\Big)\frac{1}{2^{d-1}}\frac{h_{d-1}(\vec{x}^{\hat{k}}+\delta^{\hat{k}})}{h_{d-1}(\vec{x}^{\hat{k}})}\\
=&\sum_{\vec{\delta}^{\hat{k}}\in\{-1,+1\}^{d-1}}\Big(\sum_{i\neq k}\frac{1}{x_k-x_i}+\frac{\epsilon^{-}_{\delta,k}(\vec{x})-\epsilon^{+}_{\delta,k}(\vec{x})}{2}\Big)\frac{1}{2^{d-1}}\frac{h_{d-1}(\vec{x}^{\hat{k}}+\delta^{\hat{k}})}{h_{d-1}(\vec{x}^{\hat{k}})}\\
=&\sum_{i\neq k}^{d}\frac{1}{x_k-x_i}+\sum_{\delta^{\hat{k}}\in\{-1,+1\}^{d-1}}\frac{\epsilon^{-}_{\delta,k}(\vec{x})-\epsilon^{+}_{\delta,k}(\vec{x})}{2}\frac{1}{2^{d-1}}\frac{h_{d-1}(\vec{x}^{\hat{k}}+\delta^{\hat{k}})}{h_{d-1}(\vec{x}^{\hat{k}})},
\end{aligned}
$$

where we have pulled out the term that does not depend on $\vec{\delta}^{\hat{k}}$ and used the fact that h_{d-1} is harmonic for the simple symmetric random walk in dimension $d-1$ (see e.g. [39]) to evaluate the sum over $\vec{\delta}^{\hat{k}}$. In the error term, we use the triangle inequality to bound $\left|\epsilon^{-}_{\delta,k}(\vec{x}) - \epsilon^{+}_{\delta,k}(\vec{x})\right| \leq \left|\epsilon^{-}_{\delta,k}(\vec{x})\right| + \left|\epsilon^{+}_{\delta,k}(\vec{x})\right| \leq \left(2^d-1\right)\sum_{i\neq k}\frac{2}{|x_k-x_i|}$ by equation (4.18) and use the fact that h_{d-1} is harmonic again to finally arrive at

$$\left|\mathbb{E}\left[\Delta X_k(n)\middle|\vec{X}(n)=\vec{x}\right]\right| \leq \left|\sum_{i=1,i\neq k}^{d}\frac{1}{x_k-x_i}\right| + \left(2^d-1\right)\sum_{i=1,i\neq k}^{d}\frac{1}{|x_k-x_i|} \leq \sum_{i=1,i\neq k}^{d}\frac{2^d}{|x_k-x_i|},$$

as desired. □

Corollary 4.18. *Fix* $t^\star > 0$. *For any* $1 \leq k \leq d$, *the collection*

$$\left\{\frac{1}{\sqrt{N}}\sum_{n=1}^{\lfloor tN\rfloor}\left|\mathbb{E}\left[X_k(n+1)-X_k(n)\middle|\vec{X}(n)\right]\right|, t\in[0,t^\star]\right\}_{N\in\mathbb{N}},$$

is exponential moment controlled as $t\to 0$.

Proof. By Lemma 4.16, we have for any $N\in\mathbb{N}$ and $t\in[0,t^\star]$:

$$\frac{1}{\sqrt{N}}\sum_{n=1}^{\lfloor tN\rfloor}\left|\mathbb{E}\left[X_k(n+1)-X_k(n)\middle|\vec{X}(i)\right]\right| \quad\leq\quad 2^d\sum_{i=1,i\neq k}^{d}\left(\frac{1}{\sqrt{N}}\sum_{n=1}^{\lfloor tN\rfloor}\frac{1}{|X_k(n)-X_i(n)|}\right).$$

Each term $\frac{1}{\sqrt{N}}\sum_{n=1}^{\lfloor tN\rfloor}\left|X_k(n)-X_i(n)\right|^{-1}$ is exponential moment controlled by application of Lemma 4.14. Thus, this whole quantity is a sum of $d-1$ random variables each of which are exponential moment controlled. Since finite sums of exponentially moment controlled variables are still exponential moment controlled by Lemma 4.5, the result follows. □

4.5 Overlap times of non-intersecting random walks

In this subsection we establish the exponential moment control for overlap times by using a discrete version of Tanaka's formula to write the overlap time as a finite sum of quantities we can control.

Lemma 4.19 (Discrete Tanaka formula)**.** *Let* $\alpha(i),\beta(i)$ *be any sequences with* $\alpha(i),\beta(i)\in\{-1,+1\}$. *For any* $A(0),B(0)\in\mathbb{Z}$ *with* $B(0)+A(0)\equiv 0(\text{mod } 2)$, *let* $A(n)\triangleq\sum_{i=1}^{n}\alpha(i)+A(0)$ *and* $B(n)\triangleq\sum_{i=1}^{n}\beta(i)+B(0)$. *Then:*

$$\sum_{i=0}^{n}\mathbf{1}\left\{A(i)=B(i)\right\} = |A(n+1)-B(n+1)| - |A(0)-B(0)| - \sum_{i=0}^{n}\operatorname{sgn}\big(A(i)-B(i)\big)\alpha(i+1) + \sum_{i=0}^{n}\operatorname{sgn}\big(A(i+1)-B(i)\big)\beta(i+1),$$

where we use the convention on the sign function that $\operatorname{sgn}(0)=0$.

Proof. Consider:

$$\begin{aligned}
&|A(i)-B(i)| - |A(i-1)-B(i-1)| \\
=&|A(i)-B(i)| - |A(i)-B(i-1)| + |A(i)-B(i-1)| - |A(i-1)-B(i-1)| \\
=&\big(|B(i)-A(i)| - |B(i-1)-A(i)|\big) + \big(|A(i)-B(i-1)| - |A(i-1)-B(i-1)|\big) \\
=&\operatorname{sgn}\big(B(i-1)-A(i)\big)\beta(i) + \mathbf{1}\{B(i-1)=A(i)\} \\
&+\operatorname{sgn}\big(A(i-1)-B(i-1)\big)\alpha(i) + \mathbf{1}\{A(i-1)=B(i-1)\}
\end{aligned}$$

Summing from $i = 1$ to $n+1$ and rearranging then gives the result, taking into account that $\mathbf{1}\{B(i-1) = A(i)\}$ is always zero since $A(i)$ and $B(i-1)$ always have opposite parity. ☐

Lemma 4.20. *For any indices $1 \le k, \ell \le d$, recall the definition of the overlap time $O_{k,\ell}[a,b]$ from Definition 4.1. For any fixed $t^\star > 0$, the collection of processes*

$$\left\{ \frac{1}{\sqrt{N}} O_{k,\ell}[0, \lfloor tN \rfloor] : t \in [0, t^\star] \right\}_{N \in \mathbb{N}}$$

is exponential moment controlled as $t \to 0$.

Proof. As in Definition 4.1, let $\left\{ \vec{X}(n) : n \in \mathbb{N} \right\}$ and $\left\{ \vec{X}'(n) : n \in \mathbb{N} \right\}$ denote two independent copies of the non-intersecting walks started from $\vec{\delta}(0)$. For notational convenience, we use the shorthand $\Delta X_k(i) \triangleq X_k(i+1) - X_k(i)$ and $\Delta X'_\ell(i) \triangleq X'_\ell(i+1) - X'_\ell(i)$. By the discrete version of Tanaka's formula, Lemma 4.19, applied to the definition of $O_{k,\ell}[0, \lfloor tN \rfloor]$ in equation (4.1) we have:

$$\begin{aligned} O_{k,\ell}[0, \lfloor tN \rfloor] &= \left| X_k(\lfloor tN \rfloor) - X'_\ell(\lfloor tN \rfloor) \right| - \left| 2k - 2\ell \right| - S(\lfloor tN \rfloor) + S'\left(\lfloor tN \rfloor\right) \quad (4.19) \\ &\le \left| X_k(\lfloor tN \rfloor) \right| + \left| X'_\ell(\lfloor tN \rfloor) \right| + \left| S(\lfloor tN \rfloor) \right| + \left| S'\left(\lfloor tN \rfloor\right) \right|, \end{aligned}$$

where we define

$$S(n) \triangleq \sum_{i=0}^{n} \operatorname{sgn}\left(X_k(i) - X'_\ell(i) \right) \Delta X_k(i), \;\; S'(n) \triangleq \sum_{i=0}^{n} \operatorname{sgn}\left(X_k(i+1) - X'_\ell(i) \right) \Delta X'_k(i)$$

By Lemma 4.5, to see the exponential moment control for $N^{-\frac{1}{2}} O_{k,\ell}[0, \lfloor tN \rfloor]$, it suffices to show that the four terms that appear on the RHS of equation (4.19) are each exponential moment controlled when scaled by $N^{-\frac{1}{2}}$. The first two terms on the RHS of equation (4.19) are exponential moment controlled by Corollary 4.9. We show the remaining two terms are exponential moment controlled below.

To see that $\left\{ \frac{1}{\sqrt{N}} \left| S\left(\lfloor tN \rfloor\right) \right| : t \in [0, t^\star] \right\}_{N \in \mathbb{N}}$ is exponential moment controlled as $t \to 0$, notice by triangle inequality we have that

$$\frac{1}{\sqrt{N}} \left| S(\lfloor tN \rfloor) \right| \le \frac{1}{\sqrt{N}} \left| M(\lfloor tN \rfloor) \right| + \frac{1}{\sqrt{N}} \sum_{i=1}^{\lfloor tN \rfloor} \left| \mathbb{E}\left[\Delta X_k(i) \middle| \vec{X}(i) \right] \right|, \tag{4.20}$$

where we define

$$M(n) \triangleq \sum_{i=0}^{n} \operatorname{sgn}\left(X_k(i) - X'_\ell(i) \right) \left(\Delta X_k(i) - \mathbb{E}\left[\Delta X_k(i) \middle| \vec{X}(i) \right] \right).$$

By Lemma 4.5, it suffices to check that both terms that appear on the RHS of equation (4.20) are exponential moment controlled. The second term in equation (4.20) is exponential moment controlled by Corollary 4.18. To see the exponential moment control for the first term, we observe that $\{M(n)\}_{n \in \mathbb{N}}$ is a martingale with respect to the the filtration $\mathcal{F}_n \triangleq \sigma\left(\vec{X}(1), \vec{X}'(1), \ldots, \vec{X}(n+1), \vec{X}'(n+1) \right)$. Indeed, its increments are

$$M(n) - M(n-1) = \operatorname{sgn}\left(X_k(n) - X'_\ell(n) \right) \left(\Delta X_k(n) - \mathbb{E}\left[\Delta X_k(n) \middle| \vec{X}(n) \right] \right), \tag{4.21}$$

which have $\mathbb{E}\left[M(n) - M(n-1) \middle| \mathcal{F}_{n-1} \right] = 0$ since $\operatorname{sgn}\left(X_k(n) - X'_\ell(n) \right)$ is $\mathcal{F}_{n-1}$ measurable and since $\vec{X}(\cdot)$ is a Markov process. Moreover, since $\Delta X_k(n) \in \{-1, +1\}$, we also notice from equation (4.21) that $|M(n)-M(n-1)| \leq 2$. We are thus in a position to apply Azuma's inequality for martingales with bounded differences (see e.g. Lemma 4.1 of [42]). This gives for any $N \in \mathbb{N}$ that

$$\mathbb{P}\left(\frac{1}{\sqrt{N}}\left|M(\lfloor tN\rfloor)\right| > \alpha\right) \leq 2\exp\left(-\frac{\alpha^2}{8t}\right).$$

By Lemma 4.7, this bound shows that $\left\{\frac{1}{\sqrt{N}}\left|M\left(\lfloor tN\rfloor\right)\right| : t\in[0,t^\star]\right\}_{N\in\mathbb{N}}$ is exponential moment controlled as desired.

The proof that $\left\{\frac{1}{\sqrt{N}}\left|S'\left(\lfloor tN\rfloor\right)\right| : t\in[0,t^\star]\right\}_{N\in\mathbb{N}}$ is exponential moment controlled is similar to the above argument, this time using

$$M'(n) \triangleq \sum_{i=0}^{n} \operatorname{sgn}\left(X_k(i+1)-X'_\ell(i)\right)\left(\Delta X'_k(i) - \mathbb{E}\left[\Delta X'_k(i)\middle|\vec{X}'(i)\right]\right),$$

which is a martingale on $\mathcal{F}'_n \triangleq \sigma\left(\vec{X}(1),\vec{X}'(1),\ldots,\vec{X}(n+1),\vec{X}'(n+1),\vec{X}(n+2)\right)$. ☐

Corollary 4.21. *For any fixed $t^\star$, the collection of total overlap time*

$$\left\{\frac{1}{\sqrt{N}}O[0,\lfloor tN\rfloor] : t\in[0,t^\star]\right\}_{N\in\mathbb{N}},$$

is exponential moment controlled as $t\to 0$.

Proof. This follows from Lemma 4.20 since $O[0,\lfloor tN\rfloor] = \sum_{1\leq k,\ell\leq d} O_{k,\ell}[0,\lfloor tN\rfloor]$ and because exponential moment control is maintained under finite sums by Lemma 4.5. ☐

4.6 Overlap times of non-intersecting random walk bridges

In this subsection we will use the exponential moment control for overlap times of non-intersecting random walks proved in Corollary 4.21 to obtain the exponential moment control for non-intersecting random walk bridges in Proposition 4.23.

Lemma 4.22. *Fix any $t^\star > 0$ and $z^\star \in \mathbb{R}$. Recall from Definitions 2.5, 2.6, and 2.9 the definition of the non-intersecting random walks, the non-intersecting random walk bridges and its rescaled version. There is a constant $C_R^{(t^\star,z^\star)} < \infty$ so that the Radon-Nikodym derivative of the rescaled non-intersecting random walk bridges $\vec{X}^{(N),(t^\star,z^\star)}(t)$ with respect to the rescaled non-intersecting walks $\frac{1}{\sqrt{N}}\vec{X}(\lfloor tN\rfloor)$ is uniformly bounded by $C_R^{(t^\star,z^\star)}$ over all possible positions and at all times t with $t<\frac{2}{3}t^\star$:*

$$\sup_{N\in\mathbb{N}}\sup_{t<\frac{2}{3}t^\star}\sup_{\vec{z}\in\left(\frac{\mathbb{Z}}{\sqrt{N}}\right)^d}\frac{\mathbb{P}\left(\vec{X}^{(N),(t^\star,z^\star)}(t)=\vec{z}\right)}{\mathbb{P}\left(\frac{1}{\sqrt{N}}\vec{X}(\lfloor tN\rfloor)=\vec{z}\right)} \leq C_R^{(t^\star,z^\star)}.$$

Proof. Define the variables (which depend on N), $(n^\star,x^\star) \triangleq \left(Nt^\star,\sqrt{N}z^\star\right)_2$. From Definition 2.6, $\vec{X}^{(n^\star,x^\star)}$ is absolutely continuous with respect to the non-intersecting walks $\vec{X}$ with Radon-Nikodym derivative explicitly given in equation (2.2) in terms of the non-intersection probability $q_n(\vec{x},\vec{y})$ given in Definition 2.5. There is an exact formula for for $q_n\left(\vec{\delta}_d(0),\vec{x}\right)$ from Theorem 1 in [40]:

$$q_n(\vec{\delta}_d(0),\vec{x}) = 2^{-nd}2^{-\binom{d}{2}}\prod_{i=1}^{d}\binom{n+d-1}{\frac{n+x_i}{2}}\frac{1}{(n+d-i+1)_{i-1}}h_d(\vec{x}), \tag{4.22}$$

where $(x)_n = x(x+1)\cdots(x+n-1)$. By shifting and time reversal, this gives an explicit formula for $q_n\big(\vec{x}, \vec{\delta}_d(x^\star)\big)$, namely $q_n\big(\vec{x}, \vec{\delta}_d(x^\star)\big) = q_n\big(\vec{x} - x^\star\vec{1}, \vec{\delta}_d(0)\big) = q_n\big(\vec{\delta}_d(0), \vec{x} - x^\star\vec{1}\big)$ where $\vec{1}$ denotes $\vec{1} \triangleq (1,1,\ldots 1)$. Using this explicit formula twice in equation (2.2) gives:

$$\text{LHS (2.2)} = \prod_{i=1}^{d} 2^n \binom{n^\star - n + d - 1}{\frac{n^\star - n}{2} + \frac{x_i - x^\star}{2}} \binom{n^\star + d - 1}{\frac{n^\star + x^\star}{2} + i - 1}^{-1} \frac{(n^\star + d - i + 1)_{i-1}}{(n^\star - n + d - i + 1)_{i-1}}. \quad (4.23)$$

We now use the local limit theorem

$$\lim_{M\to\infty} \sup_{\ell\in\mathbb{Z}} \left| \sqrt{M} 2^{-M} \binom{M}{\ell} - \sqrt{\frac{2}{\pi}} \exp\left(-\frac{(2\ell - M)^2}{2M} \right) \right| = 0.$$

Using this, since $n^\star - n > \frac{1}{3}t^\star N \to \infty$ for $n = Nt$ with $t < \frac{2}{3}t^\star$ as $N \to \infty$, we have:

$$\begin{aligned} \limsup_{N\to\infty} \sqrt{N} 2^{-(n^\star - n)} \binom{n^\star - n + d - 1}{\frac{n^\star - n}{2} + \frac{x_i - x^\star}{2}} \leq \quad & \limsup_{N\to\infty} \sqrt{N} 2^{-(n^\star - n)} \binom{n^\star - n + d - 1}{\frac{n^\star - n}{2}} \\ = \quad & \frac{1}{\sqrt{t^\star - t}} \sqrt{\frac{2}{\pi}} 2^{d-1}. \end{aligned}$$

Similarly, since $n^\star + d - 1 > t^\star N \to \infty$ as $N \to \infty$, we have:

$$\limsup_{N\to\infty} \frac{1}{\sqrt{N}} 2^{n^\star} \binom{n^\star + d - 1}{\frac{1}{2}n^\star + \frac{1}{2}x^\star + i - 1}^{-1} \leq \sqrt{\frac{\pi}{2}} \sqrt{t^\star} \exp\left(\frac{z^{\star 2}}{2t^\star} \right) 2^{-(d-1)}.$$

Combining these and also using $\frac{(n^\star + d - i + 1)_{i-1}}{(n^\star - n + d - i + 1)_{i-1}} \leq \left(\frac{t^\star}{t^\star - t} \right)^{i-1}$ gives:

$$\begin{aligned} \limsup_{N\to\infty} (\text{LHS of equation (4.23)}) \quad & \leq \quad \prod_{i=1}^{d} \left(\frac{t^\star}{t^\star - t} \right)^{i - \frac{1}{2}} \exp\left(\frac{z^{\star 2}}{2t^\star} \right) \\ & \leq \quad \prod_{i=1}^{d} 3^{i - \frac{1}{2}} \exp\left(\frac{z^{\star 2}}{2t^\star} \right), \end{aligned}$$

where we have used finally that $t^\star - t \geq \frac{1}{3}t^\star$. Since this is finite, there is some constant $C_R^{(t^\star, z^\star)}$ that acts as an upper bound for all $N \in \mathbb{N}$ as desired. ☐

Proposition 4.23. *Recall the definition of the rescaled overlap time $O^{(N),(t^\star,z^\star)}[0,t]$ from Definition 4.1. For any $t^\star > 0$ and $z^\star \in \mathbb{R}$, the collection of rescaled overlap times*

$$\left\{ O^{(N),(t^\star,z^\star)}[0,t], t \in [0, t^\star] \right\}_{N\in\mathbb{N}}$$

is exponential moment controlled as $t \to 0$.

Proof. We first show the result holds for $t \in [0, \frac{1}{2}t^\star]$, i.e. $\left\{ O^{(N),(t^\star,z^\star)}[0,t], t \in [0, \frac{1}{2}t^\star] \right\}_{N\in\mathbb{N}}$ is exponential moment controlled. Indeed, for any $t < \frac{1}{2}t^\star$, the Radon-Nikodym bound from Lemma 4.22 shows that for any $k \in \mathbb{N}$ we have the bound:

$$\mathbb{E}\left[\left(O^{(N),(t^\star,z^\star)}[0,t] \right)^k \right] \leq C_R^{(t^\star,z^\star)} \mathbb{E}\left[\left(\frac{1}{\sqrt{N}} O[0, \lfloor tN \rfloor] \right)^k \right].$$

The exponential moment control follows from this bound by inspecting the conditions for exponential moment control in Definition 4.3 and using the exponential moment of $\left\{ N^{-\frac{1}{2}} O[0, tN], t \in [0, \frac{1}{2}t^\star] \right\}$ from Corollary 4.21.

To extend from exponential moment control for $t \in [0, \frac{1}{2}t^\star]$ to exponential moment control on all $t \in [0, t^\star]$, since $O^{(N),(t^\star,z^\star)}[0,t]$ is monotone non-deceasing in t, it suffices to check only the point $t = t^\star$. This is verified by doing the following subdivision:

$$O^{(N),(t^\star,z^\star)}[0,t^\star] \le O^{(N),(t^\star,z^\star)}\left[0, \frac{1}{2}t^\star\right] + O^{(N),(t^\star,z^\star)}\left[\frac{1}{2}t^\star, t^\star\right]. \tag{4.24}$$

By the symmetry $t \leftrightarrow t^\star - t$ of the non-intersecting random walk bridges, we have equality in distribution $O^{(N),(t^\star,z^\star)}\left[\frac{1}{2}t^\star, t^\star\right] \overset{d}{=} O^{(N),(t^\star,z^\star)}\left[0, \frac{1}{2}t^\star\right]$. By the exponential moment control of $\left\{O^{(N),(t^\star,z^\star)}[0,t], t \in [0, \frac{1}{2}t^\star]\right\}_{n\in\mathbb{N}}$, we see then that both terms on the RHS of equation (4.24) are exponential moment controlled and the desired result holds by Lemma 4.5. □

5 L^2 bounds from overlap times

The purpose of this section is to make the connection between $\left\|\psi_k^{(N),(t^\star,z^\star)}\right\|^2_{L^2(\mathcal{S}_k(0,t^\star))}$ and the moments of the overlap times $O^{(N),(t^\star,z^\star)}$ introduced and studied in Section 4. This connection, along with Lemma 5.3 which relates exponential moment control to bounds on moments, will allow us to prove the bounds in Propositions 2.21, 2.22 and 2.17 as consequences of the exponential moment control of $O^{(N),(t^\star,z^\star)}$ proven earlier in Proposition 4.23.

Lemma 5.1. *For $k \in \mathbb{N}$, recall the definition of the simplex $\Delta_k(s,s') \subset (0,t^\star)^k$ from Definition 1.1. Define also $\Delta_k^{(N)}(s,s') \triangleq \Delta_k(s,s') \cap \left(\frac{\mathbb{N}}{N}\right)^k$. We have the inequality:*

$$\begin{aligned} &\frac{1}{k!}\left(O^{(N),(t^\star,z^\star)}[s,s']\right)^k \\ \ge &\frac{1}{N^{\frac{k}{2}}} \sum_{\vec{t}\in\Delta_k^{(N)}(s,s')} \sum_{\vec{z}\in\frac{\mathbb{Z}^k}{\sqrt{N}}} \mathbf{1}\left\{\bigcap_{i=1}^k \left\{z_i \in \vec{X}^{(N),(t^\star,z^\star)}(t_i)\right\}\right\} \mathbf{1}\left\{\bigcap_{i=1}^k \left\{z_i \in \vec{X}'^{(N),(t^\star,z^\star)}(t_i)\right\}\right\}. \end{aligned} \tag{5.1}$$

Proof. Recall from Definition 4.1 that $O^{(N),(t^\star,z^\star)}$ is defined in terms of two independent copies of the bridges $\vec{X}^{(N),(t^\star,z^\star)}$, $\vec{X}'^{(N),(t^\star,z^\star)}$. Expanding this definition gives:

$$\begin{aligned} \frac{1}{k!}\left(O^{(N)(t^\star,z^\star)}[s,s']\right)^k &= \frac{1}{k!N^{\frac{k}{2}}}\left(\sum_{t\in(s,s')\cap\frac{\mathbb{N}}{N}} \left|\left\{\vec{X}^{(N),(t^\star,z^\star)}(t) \cap \vec{X}'^{(N),(t^\star,z^\star)}(t)\right\}\right|\right)^k \\ &= \frac{1}{k!N^{\frac{k}{2}}}\left(\sum_{t\in(s,s')\cap\frac{\mathbb{N}}{N}} \sum_{z\in\frac{\mathbb{Z}}{\sqrt{N}}} \mathbf{1}\left\{z \in \vec{X}^{(N),(t^\star,z^\star)}(t)\right\}\mathbf{1}\left\{z \in \vec{X}'^{(N),(t^\star,z^\star)}(t)\right\}\right)^k \end{aligned} \tag{5.2}$$

The desired inequality follows by expanding the RHS of equation (5.2) as a k-fold sum, and discarding the contribution from the indices in the sum $\vec{t} = (t_1,\dots,t_k)$ that have $t_i = t_\ell$ for some $i \ne \ell$. In the remaining sum, we can switch from an un-ordered k-fold sum to an ordered sum $\vec{t} \in \Delta_k^{(N)}(s,s')$ at the cost of the factor $k!$, which gives the desired result. □

Corollary 5.2. *Have for $0 < s < s' < t^\star$ that:*

$$\int_{\mathbb{R}^k}\int_{\Delta_k(s,s')} \left|\psi_k^{(N),(t^\star,z^\star)}\big((t_1,z_1),\dots,(t_k,z_k)\big)\right|^2 d\vec{t}d\vec{z} \le \mathbb{E}\left[\frac{1}{2^k k!}\left(O^{(N)(t^\star,z^\star)}[s,s']\right)^k\right].$$

Proof. Taking $\mathbb{E}$ of the RHS of equation (5.1) and keeping in mind that $\vec{X}^{(N),(t^\star,z^\star)}$ and $\vec{X}'^{(N),(t^\star,z^\star)}$ are independent copies, we have:

$$\mathbb{E}\left[\text{RHS of (5.1)}\right] = 2^k \left(\frac{2}{N\sqrt{N}}\right)^k \sum_{\vec{t}\in\Delta_k^{(N)}(s,s')} \sum_{\vec{z}\in\frac{\mathbb{Z}^k}{\sqrt{N}}} \left(\frac{N^{\frac{k}{2}}}{2^k}\mathbb{P}\left(\bigcap_{i=1}^k \left\{z_i \in \vec{X}^{(N),(t^\star,z^\star)}(t_i)\right\}\right)\right)^2$$

$$= 2^k \int\limits_{\mathbb{R}^k}\int\limits_{\Delta_k(s,s')} \left|\psi_k^{(N),(t^\star,z^\star)}\left((t_1,z_1),\dots,(t_k,z_k)\right)\right|^2 \mathrm{d}\vec{t}\mathrm{d}\vec{z}.$$

The last line follows since we recall from Definition 2.9 that $\psi_k^{(N),(t^\star,z^\star)}$ is constant on the cells $\prod_{i=1}^k \left[\frac{n_i}{N}, \frac{n_i}{N}+\frac{1}{N}\right) \times \left[\frac{x_i}{\sqrt{N}}, \frac{x_i}{\sqrt{N}}+\frac{2}{\sqrt{N}}\right)$ of volume $\left(\frac{2}{N\sqrt{N}}\right)^k$, so the integral is a sum over discrete cells in the same manner as explained in equation (2.3). Dividing by 2^k on both sides gives the desired result. ☐

Lemma 5.3. *If $\left\{Z^{(N)}(t) \,:\, t\in[0,t^\star]\right\}_{N\in\mathbb{N}}$ is a collection of non-negative valued processes that is exponential moment controlled as $t\to 0$, then for each $t\in[0,t^\star]$:*

$$\forall k\in\mathbb{N}, \forall t>0 \ \sup_{N\in\mathbb{N}} \mathbb{E}\left[\left(Z^{(N)}(t)\right)^k\right] < \infty.$$

Moreover, for any fixed $k\in\mathbb{N}$, the k-th moment can be made arbitrarily small by taking t small enough:

$$\forall k\in\mathbb{N},\ \lim_{t\to 0}\sup_{N\in\mathbb{N}} \mathbb{E}\left[\left(Z^{(N)}(t)\right)^k\right] = 0.$$

Proof. For any $t\in[0,t^\star]$ and any $\gamma>0$, we use the bound $x^k \le \frac{k!}{\gamma^k}e^{\gamma x}$ for $x\ge 0$ to see

$$\sup_{N\in\mathbb{N}} \mathbb{E}\left[\left(Z^{(N)}(t)\right)^k\right] \le \frac{k!}{\gamma^k}\sup_{N\in\mathbb{N}} \mathbb{E}\left[e^{\gamma Z^{(N)}(t)}\right], \tag{5.3}$$

which is finite by property i) of the exponential moment control from Definition 4.3. Moreover, we can take the limit $t\to 0$ and apply property ii) of the same definition to see that:

$$0 \le \lim_{t\to 0}\sup_{N\in\mathbb{N}} \mathbb{E}\left[\left(Z^{(N)}(t)\right)^k\right] \le \frac{k!}{\gamma^k}\lim_{t\to 0}\sup_{N\in\mathbb{N}} \mathbb{E}\left[e^{\gamma Z^{(N)}(t)}\right] = \frac{k!}{\gamma^k}.$$

By taking γ arbitrarily large, we conclude that the limit as $t\to 0$ is actually 0 as desired. ☐

5.1 Bounds on $D_3(\delta)$ – proof of Proposition 2.21

Proof. (Of Proposition 2.21) Let $D_3^0(\delta) = \mathcal{S}_k(0,t^\star)\cap\{t_1\le\delta\}$ and let $D_3^{t^\star}(\delta) = \mathcal{S}_k(0,t^\star)\cap\{t_k \ge t^\star-\delta\}$ so that $D_3(\delta) = D_3^0(\delta)\cup D_3^{t^\star}(\delta)$. It suffices to show that for that for all $\epsilon>0$, there exists $\delta>0$ so that:

$$\sup_{N\in\mathbb{N}} \int\limits_{D_3^0(\delta)} \left|\psi^{(N),(t^\star,z^\star)}(\vec{w})\right|^2 \mathrm{d}\vec{w} < \epsilon, \quad \sup_{N\in\mathbb{N}} \int\limits_{D_3^{t^\star}(\delta)} \left|\psi^{(N),(t^\star,z^\star)}(\vec{w})\right|^2 \mathrm{d}\vec{w} < \epsilon, \tag{5.4}$$

since once this is proven we can use a union bound to complete the result. It suffices to prove the inequality in equation (5.4) for D_3^0 only, as the result for $D_3^{t^\star}$ follows by the symmetry $t\leftrightarrow t^\star-t$. We first claim:

$$\int\limits_{D_3^0(\delta)} \left|\psi_k^{(N),(t^\star,z^\star)}(\vec{w})\right|^2 \mathrm{d}\vec{w} \le \mathbb{E}\left[\frac{1}{2}\left(O^{(N),(t^\star,z^\star)}[0,\delta]\right)\frac{1}{2^{k-1}(k-1)!}\left(O^{(N),(t^\star,z^\star)}[\delta,t^\star]\right)^{k-1}\right]. \tag{5.5}$$

Equation (5.5) is verified in the same way as the proof of Corollary 5.2: first using Lemma 5.1 to bound the RHS of equation (5.5) as a sum over indicator function, and then recognizing by the definition of $D_3^0(\delta)$ that this sum coincides with $\int_{D_3^0(\delta)} \left|\psi_k^{(N),(t^\star,z^\star)}(\vec{w})\right|^2 \mathrm{d}\vec{w}$. With equation (5.5) established, we use the fact that $O^{(N),(t^\star,z^\star)}[0,t^\star] \geq O^{(N),(t^\star,z^\star)}[\delta,t^\star]$ and the Cauchy-Schwarz inequality to see that

$$\int\limits_{D_3^0(\delta)} \left|\psi_k^{(N),(t^\star,z^\star)}(\vec{w})\right|^2 \mathrm{d}\vec{w} \leq \frac{2^{-k}}{(k-1)!}\sqrt{\mathbb{E}\left[\left(O^{(N),(t^\star,z^\star)}[0,\delta]\right)^2\right]\mathbb{E}\left[\left(O^{(N),(t^\star,z^\star)}[0,t^\star]\right)^{2(k-1)}\right]}.$$

We now use the exponential moment control of $\left\{O^{(N),(t^\star,z^\star)}[0,t] : t \in (0,t^\star)\right\}_{N\in\mathbb{N}}$ from Proposition 4.23. By Lemma 5.3, this implies that $\sup_{N\in\mathbb{N}} \mathbb{E}\left[\left(O^{(N),(t^\star,z^\star)}[0,t^\star]\right)^{2(k-1)}\right] < \infty$ is some finite constant. Also by Lemma 5.3 we have that $\sup_N \mathbb{E}\left[\left(O^{(N),(t^\star,z^\star)}[0,\delta]\right)^2\right] \to 0$ as $\delta \to 0$. Thus the limit as $\delta \to 0$ of the LHS of equation (5.4) is 0, and we can thus choose $\delta > 0$ small enough to bound above by ϵ, as desired. □

5.2 Bounds on $D_4(M)$ – proof of Proposition 2.22

Proof. (Of Proposition 2.22) Define $W^{(N),(t^\star,z^\star)} \triangleq \max_{i\in\{1,\ldots,d\}} \sup_{t\in[0,t^\star]} \left|X_i^{(N),(t^\star,z^\star)}(t)\right|$ to be largest absolute value achieved by the ensemble at any time $t \in [0,t^\star]$, and let $W'^{(N),(t^\star,z^\star)}$ be the same for the independent copy $\vec{X}'^{(N),(t^\star,z^\star)}$. Writing the integral over $D_4(M)$ in equation (2.11) as the expectation of a sum of indicator functions in the same way as was done in Corollary 5.2, by the definition of $D_4(M)$ we have:

$$\begin{aligned}
&\text{LHS equation (2.11)}\\
=&\frac{1}{2^k N^{\frac{k}{2}}}\mathbb{E}\Bigg[\sum_{\substack{(\vec{z},\vec{t})\in\left(\mathbb{T}^{(N)}\right)^k,\\ \bigcup_{i=1}^k\{|z_i|>M\}}} \mathbf{1}\Big\{\bigcap_{i=1}^k \left\{z_i \in \vec{X}^{(N),(t^\star,z^\star)}(t_i)\right\}\Big\}\mathbf{1}\Big\{\bigcap_{i=1}^k \left\{z_i \in \vec{X}'^{(N),(t^\star,z^\star)}(t_i)\right\}\Big\}\Bigg]\\
\leq&\frac{1}{2^k N^{\frac{k}{2}}}\mathbb{E}\Bigg[\mathbf{1}\left\{W^{(N),(t^\star,z^\star)} > M\right\}\mathbf{1}\left\{W'^{(N),(t^\star,z^\star)} > M\right\}\\
&\quad\times \sum_{(\vec{z},\vec{t})\in\left(\mathbb{T}^{(N)}\right)^k} \mathbf{1}\Big\{\bigcap_{i=1}^k \left\{z_i \in \vec{X}^{(N),(t^\star,z^\star)}(t_i)\right\}\Big\}\mathbf{1}\Big\{\bigcap_{i=1}^k \left\{z_i \in \vec{X}'^{(N),(t^\star,z^\star)}(t_i)\right\}\Big\}\Bigg].
\end{aligned}$$

This inequality follows by inclusion since if $|z_i| > M$ and $z_i \in \vec{X}^{(N),(t^\star,z^\star)}(t_i)$, then certainly the maximum has $W^{(N),(t^\star,z^\star)} > M$. By application of Lemma 5.1 and Cauchy-Schwarz we have:

$$\begin{aligned}
\text{LHS (2.11)} &\leq \mathbb{E}\left[\mathbf{1}\left\{W^{(N),(t^\star,z^\star)} > M\right\}\mathbf{1}\left\{W'^{(N),(t^\star,z^\star)} > M\right\}\frac{1}{2^k k!}\left(O^{(N)(t^\star,z^\star)}[0,t^\star]\right)^k\right]\\
&\leq \frac{1}{2^k k!}\mathbb{P}\left(W^{(N),(t^\star,z^\star)} > M\right)\sqrt{\mathbb{E}\left[\left(O^{(N)(t^\star,z^\star)}[0,t^\star]\right)^{2k}\right]}. \qquad (5.6)
\end{aligned}$$

Finally, by the exponential moment control from Proposition 4.23 and Lemma 5.3 we have $\sup_{N\in\mathbb{N}} \mathbb{E}\left[\left(O^{(N)(t^\star,z^\star)}[0,t^\star]\right)^{2k}\right] < \infty$. By this bound and $\sup_{N\in\mathbb{N}} \mathbb{P}\left(W^{(N),(t^\star,z^\star)} > M\right)$ goes to 0 as $M \to \infty$ (this is justified for example by the argument in Lemma 4.8) it is possible to choose an M so large so that the RHS of equation (5.6) is less than ϵ, as desired. □

5.3 Proof of Proposition 2.17

Proof. (Of Proposition 2.17) As explained in equation (2.10), the integral over $\mathcal{S}_k(0,t^\star)$ and the integral over $((0,t^\star)\times\mathbb{R})^k$ differ by a factor of $k!$; we will find it more convenient to work with $\mathcal{S}_k(0,t^\star)$ for this proof. Noticing that $\Delta_k(0,t^\star)\times\mathbb{R}^k$ is in natural bijection with $\mathcal{S}_k(0,t^\star)$, we have by Corollary 5.2 applied to each term of the sum on the LHS of equation (2.6) that

$$\sup_{N\in\mathbb{N}}\sum_{k=\ell}^{\infty}\gamma^k\int\limits_{\mathcal{S}_k(0,t^\star)}\left|\psi_k^{(N),(t^\star,z^\star)}(\vec{w})\right|^2\mathrm{d}\vec{w}\le\sup_{N\in\mathbb{N}}\sum_{k=\ell}^{\infty}\gamma^k\frac{1}{2^k k!}\mathbb{E}\left[\left(O^{(N)(t^\star,z^\star)}\left[0,t^\star\right]\right)^k\right]$$
$$=\sup_{N\in\mathbb{N}}\mathbb{E}\left[\sum_{k=\ell}^{\infty}\frac{1}{k!}\left(\frac{\gamma}{2}\right)^k\left(O^{(N)(t^\star,z^\star)}\left[0,t^\star\right]\right)^k\right]. \tag{5.7}$$

The interchange of expectation with the infinite sum is justified by the monotone convergence theorem since $O^{(N),(t^\star,z^\star)}[0,t^\star]$ is non-negative. Finally, if we take the limit $\ell\to\infty$, then the RHS of equation (5.7) goes to 0 by property iii) of exponential moment control from Definition 4.3 since $\left\{O^{(N)(t^\star,z^\star)}\left[0,t\right]:t\in[0,t^\star]\right\}_{N\in\mathbb{N}}$ is exponential moment controlled by Proposition 4.23. □

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Acknowledgments. IC thanks Jeremy Quastel for helpful discussions. MN thanks his advisor Gérard Ben Arous for his constant support and also Vadim Gorin, Wolfgang König, Neil O'Connell and Mykhaylo Shkolnikov for their friendly email responses to queries related to this project. Thanks also to an anonymous reviewer for a very careful reading of the paper which led to many improvements throughout. IC was partially supported by the NSF through grant DMS-1208998 and grant PHY11-25915 as well as by the Clay Mathematics Institute through the Clay Research Fellowship, by the Institute Henri Poincare through the Poincare Chair, and by the Packard Foundation through a Packard Fellowship in Science and Engineering. MN was partially supported by the NSF through grant DMS-1209165 as well as by the MacCracken Fellowship from New York University.