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Preferential attachment with fitness: unfolding the condensate

Steffen Dereich*

Abstract

Preferential attachment models with fitness are a substantial extension of the classical preferential attachment model, where vertices have an independent fitness that has a linear impact on its attractiveness in the network formation. As observed by Bianconi and Barabási [4] such network models show different phases. In the condensation phase a small number of exceptionally fit vertices collects a finite fraction of all new links and hence forms a condensate. In this article, we analyse the formation of the condensate for a variant of the model with deterministic normalisation. We consider the regime where the fitness distribution is bounded and has polynomial tail behaviour in its upper end. The central result is a law of large numbers for an appropriately scaled version of the condensate. It follows that a Γ-distributed shape is formed and, in particular, that the number of vertices contributing to the condensate rises to infinity with increasing network size, in probability.

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1 Introduction

Since the late 90s one can observe a strong research activity in complex networks driven by various fields of science and we mention the monographs [1, 12, 13, 15, 20]. A driving force are on one hand their abundant presence in real life, for instances as social networks, the world-wide-web. On the other hand they are themselves scientifically intriguing objects showing diverse behaviours and phase transitions.

Formally complex networks are modelled as sequences of random graphs that are built according to certain *plausible* rules. The *preferential attachment network* (PA model) is an archetypical example that had a strong impetus [2]. Here the network is formed dynamically. In each step, one vertex is added and connected randomly to a deterministic or random number of old vertices with a preference to connect to vertices with high degree. This building rule can be made precise in various ways and one may therefore rather speak of the preferential attachment paradigm. As observed by Barabási and Albert [2] a linear preference in the PA model (meaning that the conditional

*Westfälische Wilhelms-Universität Münster, Germany. E-mail: steffen.dereich@wwu.de

probability for a vertex in the network to establish a new link in one evolution step is proportional to its current degree) leads to empirical degree distributions following power laws as seen in real world networks.

In this article we will deal with an extension of the preferential attachment model. In the original preferential attachment model, vertices do not have an intrinsic attractivity. Only the degree of a vertex effects the proceeding network formation which, for instance, entails that vertices with high indices have typically no chance to overtake the ones that entered first. A natural extension is the assignment of a nonnegative *fitness* to each vertex which has a linear impact on its likeliness to establish new links. Formally this is modelled by a sequence of independent and identically distributed, say μ-distributed, random variables.

Such models were suggested and first analysed by [4]. A crucial quantity is the distribution induced by the fitness attached to a randomly chosen half-edge, the *degree-weighted fitness distribution*. It is believed that, in general, one can observe two regimes

- *Fit-get-richer phase.* The degree-weighted fitness distribution converges to a distribution that is absolutely continuous with respect to μ. The density is monotonically increasing in the fitness and it represents the relative strength of certain fitnesses.
- *Condensation or innovation-pays-off phase.* The degree-weighted fitness distribution converges to a distribution that is not absolutely continuous with respect to μ, but has a Dirac mass in the essential supremum of μ (possibly ∞). In the degree-weighted fitness distribution a constant fraction of the total mass shifts to the essential supremum of μ.

We mention that in the original work [4] a third regime is quoted which refers to the original preferential attachment model without fitness. In the condensation phase, one recovers the Bose statistic as limiting distribution when mapping fitness to energy in an appropriate way. This explains the term Bose-Einstein phase used in [4] for the condensation phase. A rigorous verification of the above classification was conducted in [7] for the preferential attachement model with fitness introduced in [4] to which we refer as the *classical variant* of preferential attachment with fitness. The crucial point in the proofs is that the preferential attachment model is a time-changed Crump-Mode-Jagers process and we mention [3] which contains a more direct proof based on a strong law of large numbers for Crump-Mode-Jagers processes of [19]. In line with physicists' intuition the condensation phenomenon is quite robust under the details of the model specification and a robust analysis which covers several variants of the model can be found in [11]. Beside complex networks, condensation phenomena appear at various other places and we refer the reader to Section 1.3, where we relate our findings to the ones made for related models.

Our research is focused on the dynamics of the degree-weighted fitness distribution close to the essential supremum of μ in the condensation regime. In this regime there is a small number of vertices which have exceptionally high degrees, to be called *condensate*, and when choosing an edge in a large network uniformly at random then this is attached with a certain probability strictly bigger than zero to a vertex of the condensate. The aim of this article is to understand which vertices belong to the condensate and to analyse the qualitative and quantitative behaviour of the condensate. One of the main findings is a weak limit theorem for the age and fitness of the vertices constituting the condensation phenomenon, see Theorem 1.4 below. In colloquial terms, we appropriately *unfold* the condensate and obtain a nontrivial limit. We will see that typically the number of vertices building the condensate tends to infinity (though very slowly) and that individual vertices typically have a negligible share. This is contradictory to the claim made in [4] that typically there is one vertex which gets a strictly positive fraction of all links. Unfortunately our research does not cover the classical variant of preferential attachment with fitness since we rely on a deterministic normalisation in the model specification. However we strongly believe that analogous results are true for the classical and other variants as we will explain in Section 1.3 below. We stress that the results proved in this article are new in the context of network models.

We proceed as follows. In Section 1.1 we introduce the main notation and, in particular, the network model. In Section 1.2, we give the main results and outline the structure of the remaining article. Section 1.3 contains a discussion of our finding in the context of other related models.

1.1 The model

In the following, μ denotes a probability measure on the Borel sets of $[0,\infty)$ with bounded support, the *fitness distribution*. Without loss of generality, we can and will assume that the essential supremum of μ is one. Our aim is to analyse a sequence of directed random (multi)graphs $(\mathcal{G}_n)_{n\in\mathbb{N}}$ whose formation is described in an informal way as follows: $\mathcal{G}_1$ consists of a single vertex, labelled 1, which is assigned a μ-distributed fitness $\mathcal{F}_1$. Given the network $\mathcal{G}_n$, the network $\mathcal{G}_{n+1}$ is formed by carrying out the following two steps independently:

- Insertion of a vertex, labelled $n+1$, with an independent μ-distributed fitness $\mathcal{F}_{n+1}$.
- Insertion of directed edges $n+1 \to m$ to old vertices $m \in \{1,\dots,n\}$ with *intensity* proportional to

$$\mathcal{F}_m \cdot \mathrm{imp}_{\mathcal{G}_n}(m), \tag{1.1}$$

where

$$\mathrm{imp}_{\mathcal{G}_n}(m) := \begin{cases} 1 + \text{ indegree of } m \text{ in } \mathcal{G}_n, & \text{if } m \leqslant n \\ 0, & \text{otherwise.} \end{cases}$$

Note that this is not a unique description of the network formation since, in particular, the term intensity is not specified. There are various ways to formalise the network formation and in the classical variant of the model each new vertex $n+1$ connects to precisely one old vertex, which is m with probability

$$\frac{\mathcal{F}_m \, \mathrm{imp}_{\mathcal{G}_n}(m)}{n \, \bar{\mathcal{F}}_n},$$

where $\bar{\mathcal{F}}_n = \frac{1}{n}\sum_{j=1}^{n} \mathcal{F}_j \, \mathrm{imp}_{\mathcal{G}_n}(j)$. The first rigorous verification of the condensation phenomenon was obtained by Borgs et al. [7] for this variant. In general, the phenomenon is quite robust and it prevails in various variants of the model, see [11]. Further, the limit distribution does only depend on the (average) outdegree of new vertices, but not on the specifics of the model specification.

The classical variants of preferential attachment with fitness feature random adaptive normalisations $\bar{\mathcal{F}}_n$. In general, adaptive normalisations induce severe complications in the analysis of complex networks and the most studied models have either deterministic normalisation (see, e.g., [6], [15]) or admit a representation as time changed Crump-Mode-Jagers process (see, e.g., [21], [23],[7]). To bypass this difficulties we work in a variant of the model with *deterministic* normalisation which is tuned in such a way that it mimics models with adaptive normalisation and, in particular, shows the same condensation phenomenon.

In the following, $(\bar{\mathcal{F}}_n)_{n\in\mathbb{N}}$ is a deterministic sequence and we denote by $(\mathcal{G}_n)_{n\in\mathbb{N}}$ the complex network model with fitness where in each step $n \to n+1$, the new vertex establishes to each old vertex $m \in \{1, \dots, n\}$ an independent Poisson-distributed number of edges with intensity

$$\frac{\mathcal{F}_m \, \mathrm{imp}_{\mathcal{G}_n}(m)}{n \, \bar{\mathcal{F}}_n}.$$

So $(\mathcal{G}_n)$ is a sequence of random *multigraphs*. Alternatively, one might consider models without multiple edges where each new vertex establishes an independent Bernoulli-distributed number of edges to each old vertex using the same parameter as before (provided that it is smaller or equal to one). Choosing a Poissonian number of edges has the advantage that we do not need to distinguish the two cases where the latter parameter is or is not less than or equal to one. Beyond that technical detail there is no difference between the two variants and one can easily carry over our analysis to the Bernoulli-model.

Clearly, the choice of $(\bar{\mathcal{F}}_n)$ has a severe influence on the behaviour of the network and we will need to impose appropriate assumptions to mimic the behaviour of the classical variant of the model. Here the main problem is to guarantee on one hand the existence of a condensate and on the other hand that the total number of edges increases proportionally in the number of vertices.

Exponential time scale

In the analysis of preferential attachment networks, the size of the network is often not a good representative for the age of the network. Instead one uses a concept of time, where the size $n \in \mathbb{N}$ of the network is associated with a time $t = \pi(n)$ growing logarithmically in n. In this case the network size grows exponentially in the time as is automatically the case for branching process representations. Typically under the new time concept, the evolution of the indegree of a vertex over time converges in distribution to a pure birth Markov chain in continuous time, see for instance [9]. This qualitative feature is also vital for our analysis.

We conceive the network model $(\mathcal{G}_n)_{n\in\mathbb{N}}$ as a dynamical process evolving in time. We assign the network $\mathcal{G}_n$ with $n \in \mathbb{N}$ vertices the *time*

$$\pi(n) := \sum_{j=1}^{n-1} \frac{1}{j}$$

and let $\mathbb{T} := \pi(\mathbb{N})$ denote the set of times at which new vertices are inserted. For $t \in [0, \infty)$, we denote by $N(t) \in \mathbb{N}$ the number of vertices in the network at time t, i.e., $N : [0, \infty) \to \mathbb{N}$ is the generalized right continuous inverse of π given by

$$N(t) = \max\{n \in \mathbb{N} : \pi(n) \le t\}.$$

In particular, one has $N(\pi(n)) = n$ for $n \in \mathbb{N}$. The number of vertices in the network increases exponentially in the time t: there exists a constant $C_N > 0$ such that

$$N(t) \sim C_N \, e^t,$$

see Lemma 5.1.

We consider all relevant quantities in the newly introduced concept of *time* and let for $s, t \geqslant 0$, $m = N(s)$ and $n = N(t)$:

$$G_t = \mathcal{G}_n, \ F_t = \mathcal{F}_n, \text{ and } \bar{F}_t = \bar{\mathcal{F}}_n.$$

The time intervals at which the network changes decrease exponentially fast in the time t and we let

$$\Delta t = \frac{1}{N(t)} = \frac{1}{n}$$

so that for $t \in \mathbb{T}$, $t + \Delta t$ is the successor of t in $\mathbb{T}$. Further, we denote by

$$Z[s,t] = \mathrm{imp}_{G_t}(m) = \mathrm{imp}_{\mathcal{G}_n}(m)$$

the impact of the vertex s (actually the vertex with label $N(s)$) at time t. We call for fixed $s \in \mathbb{T}$ the process

$$Z[s,\cdot] = (Z[s,t] : t \in \mathbb{T}, t \geqslant s)$$

impact evolution of s. By definition all impact evolutions are independent.

In general, we make use of the Landau symbols o and $\mathcal{O}$ and of asymptotic (in)equalities. For two functions f and g, we write

$$f \precsim g$$

if

$$\limsup \frac{f}{g} < \infty.$$

Further we write $f \approx g$, if $f \precsim g$ and $g \precsim f$. We also make use of strong equivalence and write

$$f \lesssim g \quad \text{and} \quad f \sim g,$$

if $\limsup \frac{f}{g} \leqslant 1$ and $\lim \frac{f}{g} = 1$, respectively. The asymptotic inequalities will be either used along a parameter send to infinity, likewise $t \to \infty$, or on countable sets of parameters (with the obvious meaning).

1.2 Main results

In this article we do a detailed analysis of the vertices that form the condensate. New vertices establish links to old vertices proportional to their impact and to understand the network formation we consider the *impact-weighted fitness distribution*: we consider for each $t \in \mathbb{T}$ the random measure

$$\Xi_t = \frac{1}{N(t)} \sum_{\substack{s \in \mathbb{T} \\ s \leqslant t}} Z[s,t]\, \delta_{F_s} = \frac{1}{N(t)} \sum_{m=1}^{N(t)} \mathrm{imp}_{\mathcal{G}_{N(t)}}(m)\, \delta_{\mathcal{F}_m},$$

which indicates how the overall impact is distributed over the fitnesses. In the classical model each new vertex connects to exactly one old vertex. In that case the impact-weighted fitness distribution equals the degree-weighted fitness distribution. However in our model the impact-weighted fitness distribution differs from the degree-weighted fitness distribution and we restrict attention to the former one since the impact is the crucial quantity governing the dynamics of the network formation. We will also refer to the impact-weighted fitness distribution as *fitness profile* or even shorter as *profile*. We state a consequence of [11, Proposition 4.1]:

Theorem 1.1. *If*

$$\lim_{t \to \infty} \bar{F}_t = 1, \tag{A0}$$

one has for $[a,b)$ *with* $0 \leqslant a < b < 1$

$$\lim_{t \to \infty} \Xi_t([a,b)) = \int_{[a,b)} \frac{1}{1-f}\, \mu(\mathrm{d}f), \text{ almost surely.}$$

Informally, we distinguish between vertices of the *condensate* and the *bulk*. Vertices of the condensate have exceptionally high impact with fitness very close to the essential supremum of μ. They are responsible for the condensation phenomenon. Vertices of the bulk have moderate impact and they rather contribute by their large numbers. We mention that many vertices are asymptotically negligible and we do not assign them to one of the two categories. Further whether vertices qualify for either of the categories depends on the reference time t at which one looks at the network and a clear classification is only possible in an asymptotic way. We stress that Theorem 1.1 is only a statement about the behaviour away from the essential supremum of μ so it is a statement about the contribution of the *bulk*. In particular, it implies that the contribution of the bulk coincides with that in models with adaptive normalisation as long as $\bar{\mathcal{F}}_n$ tends to 1. Somehow the behaviour of the condensate is more subtle and we need to impose stronger assumptions on the asymptotic behaviour of $\bar{\mathcal{F}}_n$ below.

In the condensation phase the impact profile accumulates mass in the essential supremum of μ which we assumed to be one. The aim is to find an appropriate scaling window which intuitively unfolds the contribution of the condensate in the sense that non-degenerate limit theorems can be proved. In particular, we would like to understand qualitative properties of the condensation phenomenon. Intuitively, one may expect that one of the following two scenarios can be observed:

A. *Travelling wave.* The number of vertices that belong to the condensate converges to infinity and the appropriately scaled impact profile Ξ_t converges to a deterministic measure as the time t tends to infinity. In this case we call the limit (asymptotic) wavefront.

B. *Winner-takes-all.* The number of vertices that belong to the condensate is of finite order meaning that individual vertices of the condensate contribute a positive fraction of the condensating mass in the fitness profile.

In [4] it is conjectured that in Bose-Einstein condensation configurations of the 'Winner-takes-all' type appear. As we show in our research the 'travelling wave' scenario typically prevails. However, the number of vertices that constitute the wave increases very slowly in the size of the network and it is not possible to see the wavefront in simulations of moderate sizes.

In this article we restrict attention to the case where μ has polynomial tails at one. We assume existence of $\alpha > 1$ and a slowly varying function ℓ in zero with

$$\mu([1-\delta, 1]) = \delta^{\alpha} \ell(\delta) \tag{A1}$$

for $\delta \in (0,1)$ and

$$\bar{\mathcal{F}}_n = 1 - \alpha(\log n)^{-1} + o((\log n)^{-1}) \text{ as } n \to \infty. \tag{A2}$$

In terms of the exponential time scale Assumption (A2) corresponds to

$$\bar{F}_t = 1 - \alpha t^{-1} + o(t^{-1}) \text{ as } n \to \infty. \tag{A2}$$

Here we used that $\pi(n) \sim \log n$ as n tends to infinity, see Lemma 5.1. We will see in the discussion below that Assumption (A2) is natural and, in particular, the parameter α appearing in both assumptions has to be the same in order to have the contributions of the condensate and the steady mass of the same order.

In order to state the main results, we use $t \geqslant 0$ as representative for the time in the network formation. It is associated to two parameters $\delta = \delta(t)$ and $T = T(t)$ via

$$\delta = \frac{1}{t} \wedge 1 \text{ and } \mu([1-\delta, 1)) = e^{-T}.$$

We keep the notation for δ and T in the whole article. In particular, δ and T always depend on t, although we mostly omit the identifier t, for ease of notation.

Further, we define, for $0 \leqslant s \leqslant t$,

$$\Upsilon[s,t] := \int_s^t (1 - \bar{F}_v)\,\mathrm{d}v.$$

In the rest of this section, we always assume Assumptions (A1) and (A2) without further mentioning. By Lemma 5.3, one has for $s,t \to \infty$ with $s \leqslant t$

$$T \sim \alpha \log t \;\text{ and }\; \Upsilon[s,t] \sim \alpha \log \frac{t}{s}.$$

In the original variant of the model each vertex establishes exactly one new edge and the adaptive normalisation $\bar{\mathcal{F}}_n$ has a stabilising effect. In our Poissonian model with deterministic normalisation the choice of the normalisation $\bar{\mathcal{F}}_n$ is quite subtle and nontrivial as the following theorem shows.

Theorem 1.2. *Assume that the limit*

$$w := \lim_{t\to\infty} T(t)\, e^{\Upsilon[T(t),t]-T(t)} \in [0,\infty) \tag{COEX}$$

exists. Then one has

$$\lim_{t\to\infty} \Xi_t = \Xi, \quad \textit{in the weak topology,}$$

in probability, where Ξ is the measure on the Borel sets of $[0,1]$ with

$$\Xi(\mathrm{d}f) = \frac{1}{1-f}\,\mu(\mathrm{d}f) + \tfrac{\alpha}{\alpha-1}\,\Gamma(\alpha)\,w\,\delta_1(\mathrm{d}f)$$

and $\Gamma(\alpha)$ denotes the Gamma function with argument α.

We stress that the proof of this result differs significantly from the proofs of similar results for models with adaptive normalisation $\bar{\mathcal{F}}_n$. The main obstacle in the proof is the analysis of the total impact in the system which is an almost trivial problem in the case with adaptive normalisation.

Theorem 1.2 gives rise to the problem of constructing appropriate normalisations with nontrivial limits in (COEX). Unfortunately, it seems to be not feasible to give an appropriate normalisation $(\bar{F}_t)$ in closed form for general μ. However, the converse direction, i.e. to start with a normalisation and to construct an appropriate fitness distribution, is feasible as the following remark illustrates.

Remark 1.3. Let $\alpha > 1$ and $w \in (0,\infty)$. We start with a deterministic normalisation $(\bar{F}_t)$ such that we have a representation $\bar{F}_t = 1 - \alpha\frac{1}{t} + h(t)$ $(t \geqslant 1)$ with $h : [1,\infty) \to \mathbb{R}$ being integrable and of order $o(1/t)$. Then for $0 < s < t$

$$\Upsilon[s,t] = \log\Big(\frac{t}{s}\Big)^{\alpha} + o(1)$$

as $s,t \to \infty$, see Lemma 5.3, and we conclude that

$$Te^{\Upsilon[T,t]-T} \sim T\Big(\frac{t}{T}\Big)^{\alpha} e^{-T} = t^{\alpha}\,\mu([1-\delta,1))\big(\log \mu([1-\delta,1))^{-1}\big)^{1-\alpha}.$$

It is straight-forward to verify that for any fitness distributions μ with

$$\mu([1-\delta,1)) \sim w\delta^{\alpha}(\log \delta^{-\alpha})^{\alpha-1}, \quad \text{as } \delta \downarrow 0,$$

Assumption (COEX) is satisfied. In particular, $\delta \mapsto \mu([1-\delta,1))$ is regularly varying with index α and Assumption (A1) is satisfied.

The central point of this article is the identification of the vertices with significant contribution to the overall impact. We consider the *extended fitness profile* defined by

$$\bar{\Xi}_t = \frac{1}{N(t)} \sum_{\substack{s \in \mathbb{T} \\ s \leqslant t}} Z[s,t]\, \delta_{(s,F_s)}$$

for $t \in \mathbb{T}$. As t tends to infinity the mass of $\bar{\Xi}_t$ accumulates in two regions: The codensating mass at a late time t is carried by vertices s born around time $T \sim \alpha \log t$ with particularly high fitness, meaning that $F_s = 1 - \mathcal{O}(t^{-1})$. The mass in the bulk is carried by the numerous young vertices. Technically we will prove two weak limit theorems for appropriately scaled versions of $\bar{\Xi}_t$, one zooming into the region of the vertices constituting the condensate and the second zooming into the region of the bulk, see Figure 1 for an illustration of both regions. The mass outside these two regions will be asymptotically negligible.

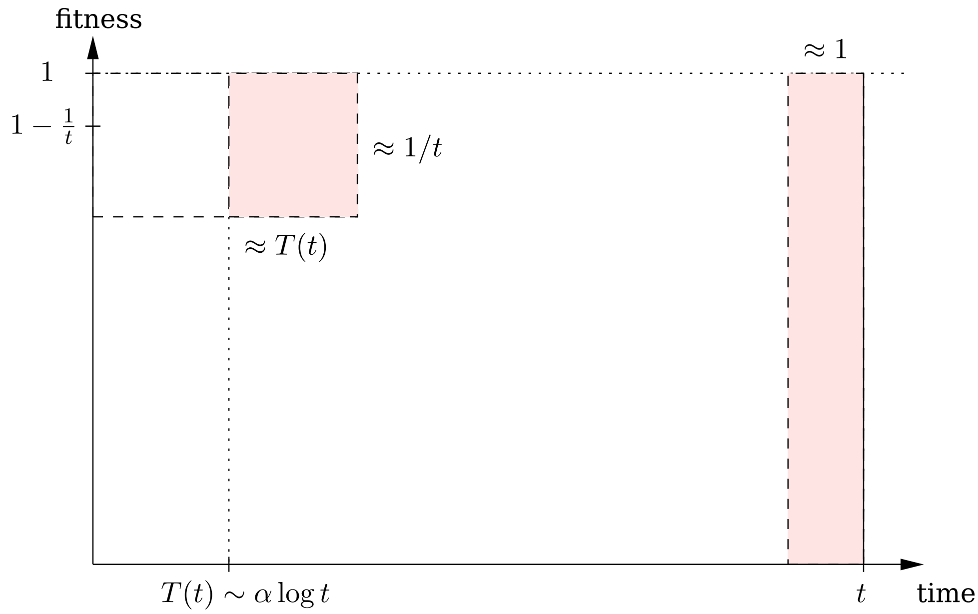

Figure 1: Illustration of the two windows that contribute to the overall impact.

The travelling wave

The condensate is mathematically represented by a family of finite measures $(\Gamma_t : t \in \mathbb{T})$ on $\mathbb{R}^2$, where the first marginal refers to the time of birth of the vertex scaled by $1/T$ and the second marginal represents the fitness relative to the essential supremum one of the fitness distribution μ scaled by t. We define the random measure Γ_t via

$$\Gamma_t = \frac{1}{N(t)} \sum_{\substack{s \in \mathbb{T} \\ s \leqslant t}} Z[s,t]\, \delta_{(s/T(t),(1-F_s)t)}$$

We call Γ_t the extended profile in the condensation window or briefly the *condensation profile*.

Theorem 1.4. *One has for any bounded and open interval* $K \subset \mathbb{R}$ *that*

$$\lim_{t\to\infty} \frac{1}{T(t)\, e^{\Upsilon[T(t),t]-T(t)}}\, \Gamma_t\big|_{\mathbb{R}\times K} = \gamma_\alpha\big|_{\mathbb{R}\times K}, \quad \textit{in the weak topology},$$

in probability, where γ_α is the measure on the Borel sets of $\mathbb{R}^2$ with Lebesgue density

$$\frac{\mathrm{d}\gamma_\alpha}{\mathrm{d}\ell^2}(s,f) = \alpha\, \mathbb{1}_{[1,\infty)\times[0,\infty)}(s,f)\, s^{-\alpha} f^{\alpha-1} e^{-f}.$$

We note that Theorem 1.4 does not require that the limit in (COEX) exists. If one projects the condensation profile onto the fitness component one retrieves a Γ-distribution with parameter α which is the same shape appearing in the Kingman model of selection and mutation, a corresponding mean field model, see [8].

Remark 1.5. The above theorem shows that the number of vertices in the condensate tends to infinity and that typically the "Travelling wave scenario" prevails. A natural question is whether the condensation profile also converges in the almost sure sense. In forthcoming work we will show that this is not the case and that at rare times exceptionally fit vertices are born that later will catch a significant amount of mass.

Next, we show that one has almost sure convergence when discarding vertices with atypically high fitness, so called *early birds*: let $\gamma \in (0,1)$ and set

$$\mathcal{E}_\gamma = \{\pi(n) : n(\log n)^\gamma \mu([\mathcal{F}_n, 1)) \leqslant 1\}.$$

That means $\mathcal{E}_\gamma$ contains the time of birth of every vertex that has a particular high fitness relative to its peers. If μ is a continuous distribution, then $\mu([\mathcal{F}_n, 1))$ is uniformly distributed on $(0,1)$ and by Borel-Cantelli, almost surely, the set $\mathcal{E}_\gamma$ contains infinitely many vertices since $\gamma < 1$.

We consider an analogous version of Γ_t where we exclude the early birds: for $t \geqslant 0$ let Γ_t^* denote the random measure on $\mathbb{R}^2$ with

$$\Gamma_t^* = \frac{1}{N(t)} \sum_{\substack{s\in\mathbb{T}\cap\mathcal{E}_\gamma^c \\ s\leqslant t}} Z[s,t]\, \delta_{(s/T,(1-F_s)t)}$$

Theorem 1.6. *One has for any bounded and open interval $K \subset \mathbb{R}$ that*

$$\lim_{t\to\infty} \frac{1}{T(t)\, e^{\Upsilon[T(t),t]-T(t)}}\, \Gamma_t^*\big|_{\mathbb{R}\times K} = \gamma_\alpha\big|_{\mathbb{R}\times K}, \quad \textit{in the weak topology},$$

almost surely.

The theorem shows that there is a tiny number of vertices that might be responsible for the nonconvergence of (Γ_t) in the almost sure sense. This is indeed the case and the detailed analysis of the early birds is the object of future research.

The steady mass/bulk

There is a second window where the steady mass/bulk concentrates: for $t \in \mathbb{T}$ consider the random measure

$$\Phi_t = \frac{1}{N(t)} \sum_{s\in\mathbb{T}} Z[s,t]\, \delta_{(t-s,F_s)}.$$

Theorem 1.7. *For every $\kappa \in (0,1)$, one has almost surely weak convergence*

$$\lim_{t\to\infty} \Phi_t\big|_{\mathbb{R}\times[0,\kappa)} = \phi\big|_{\mathbb{R}\times[0,\kappa)},$$

where ϕ denotes the measure on $[0,\infty)\times[0,1)$ with

$$\mathrm{d}\phi(s,f) = e^{-(1-f)s}\,\mathrm{d}s\,\mathrm{d}\mu(f).$$

Outline of the remaining article

The article is outlined as follows. In Section 2 we analyse individual and collections of impact evolutions. Section 3 introduces a binning argument (similar vertices are collected in bins). A strong law of large numbers is proved for the contribution of bins and estimates are provided for the mass of Ξ_t in various areas of $[0,t]\times[0,1]$. Finally, we combine in Section 4 the estimates from Section 3 to prove the main theorems. Technical estimates are given in the appendix.

1.3 Outlook and relations to other models

Condensation phenomena have been addressed recently by several research groups. The model which is closest to our model is the Kingman model of selection and mutation [17]. It can be interpreted as a deterministic analogue of our model. The model shows the same phase transition and when taking bounded fitness distributions with polynomial tails, a Γ-distributed condensation wave is formed [8]. Further, [22] observed a Gaussian condensation wave for certain unbounded fitness distributions. In the stochastic model analysed here, exceptionally strong vertices are not in the system right from the start. This causes a delay compared to the deterministic variant which is responsible for the particular structure that has to be assumed for the normalisation, see Assumption (COEX).

An extensive overview on condensation phenomena for conditioned Galton-Watson processes and random allocations can be found in [16]. We will discuss one model in detail and present [16, Thm. 7.1] in order to point out the similarities to our model. Given a weight sequence $(w_k)_{k\in\mathbb{N}_0}$ of non-negative reals with $w_0, w_k>0$ for a $k\geqslant 2$, we give a rooted finite tree T the weight

$$w(T)=\prod_{v\in T} w_{d_+(v)},$$

where $d_+(v)$ denotes the number of children of the vertex v in T. For $n\in\mathbb{N}$ denote by $\mathcal{I}_n$ the set of all ordered rooted trees (sometimes also referred to as rooted planar trees) with n vertices and denote by $\mathcal{T}_n$ an $\mathcal{I}_n$-valued random variables with

$$\mathbb{P}(\mathcal{T}_n=T)=\frac{w(T)}{Z_n},\ \ \text{for } T\in\mathcal{I}_n,$$

and Z_n denoting the right normalisation. These random trees converge in distribution for $n\to\infty$ and one distinguishes two regimes. To understand this, we consider the exponential family of distributions associated to (w_k): for $\tau\in[0,\infty)$ we denote by $\pi^\tau=(\pi^\tau_k)_{k\in\mathbb{N}_0}$ the distribution

$$\pi^\tau_k=\frac{\tau^k\, w_k}{\Pi_\tau}$$

provided that there exists an appropriate normalisation $\Pi_\rho\in(0,\infty)$. To understand the limit of the trees $\mathcal{T}_n$, one looks for τ for which π^τ induces a critical Galton-Watson process (meaning with expectation 1). If such a τ exists, the limit is a particular branching process that branches along a single spine with offspring distribution $\bar\pi^\tau:=(k\pi^\tau_k)_{k\in\mathbb{N}}$ with all other vertices branching with offspring distribution π^τ. The condensation phase is the case where such a τ does *not* exist. In this case one chooses τ as the maximal value for which π^τ is well-defined and notices that $\nu:=1-\sum_{k=1}^\infty k\pi^\tau_k>0$. The limit of $(\mathcal{T}_n)$ is a branching process with a spine of finite length branching with offspring distribution $\mathbb{1}_{\mathbb{N}}\,\bar\pi^\tau+\mathbb{1}_{\{\infty\}}\,\nu$ until it generates infinite offspring for the first time and with all other vertices branching with offspring distribution π^τ.

The preferential attachment model with fitness that is analysed in [7, 3] is a time changed Crump-Mode-Jagers process having as branching prototype the point process

$$\Pi = \sum_{k=1}^{\infty} \delta_{(T_1+\ldots+T_k)/\mathcal{F}},$$

where $\mathcal{F}, T_1, T_2, \ldots$ are independent random variables with $\mathcal{F}$, resp. T_k, being μ-distributed, resp. exponentially distributed with parameter k. To understand the two regimes one has to look for a Malthusian parameter, which is a solutions to

$$\mathbb{E}\Big[\int e^{-\lambda x}\,\mathrm{d}\Pi(x)\Big] = 1.$$

If such a solution exists one is in the fit-get-richer regime, otherwise one is in the condensation regime, in which case one chooses λ as essential supremum of μ. In that case the mass in the condensate is $\nu := 1-\mathbb{E}\Big[\int e^{-\lambda x}\,\mathrm{d}\Pi(x)\Big]$ in complete analogy to above. That means that in particular a late vertex n connects to a vertex in the condensate with probability ν and this vertex will have unbounded degree in the limit.

Random permutations with cycle weights appear in [14] as a very simplified model for Bose-Einstein condensation. Given positive weights $(w_k)_{k\in\mathbb{N}}$, we denote for $n \in \mathbb{N}$ by Σ_n a random permutation of n elements satisfying

$$\mathbb{P}(\Sigma_n = \sigma) = \frac{1}{Z_n} \prod_{\zeta \text{ cycle in } \sigma} w_{\ell(\zeta)},$$

where $\ell(\zeta)$ denotes the length of the cycle ζ. As observed in [14] the model has different phases and for particular choices of weights with polynomial growth one recovers again the Γ-distribution as limit shape. As observed in [10], the appearance of the Γ-distribution is robust in the sense that it appears whenever the weights are regularly varying.

We stress that the model analysed here is a random model that is built dynamically. This distinguishes it from the other models that are either dynamically built but deterministic or stochastic, but built completely afresh when increasing the system size.

2 Analysis of impact evolutions

By definition, the individual impact evolutions $Z[s,\cdot]$ are independent. Furthermore, given F_s, the process $Z[s,\cdot]$ is a Markov process on $\mathbb{T} \cap [s,\infty)$ with $\mathbb{T} = \pi(\mathbb{N})$ denoting the times when the network evolves. We associate a vertex with two further processes $A[s,\cdot] = (A[s,t])_{t\in\mathbb{T}, t\geqslant s}$ and $M[s,\cdot] = (M[s,t])_{t\in\mathbb{T}, t\geqslant s}$ defined via

$$A[s,s] = 1 \ \text{ and } \ A[s,t+\Delta t] = \Big(1 + \frac{F_s}{\bar{F}_t}\,\Delta t\Big) A[s,t] \ \text{ for } t \in \mathbb{T} \text{ with } t \geqslant s \tag{2.1}$$

and

$$Z[s,t] = A[s,t]\, M[s,t]. \tag{2.2}$$

This section is devoted to the analysis of $A[s,\cdot]$ and $M[s,\cdot]$ conditioned on the fitness F_s.

2.1 Single impact evolutions

In the first subsection, we analyse single impact evolutions conditionally on $\{F_s = f\}$ for $f \in [0,1)$. Fix $s \in \mathbb{T}$ and $f \in (0,1]$ and let $Z = Z^{f,s} = (Z_t^{f,s})_{t\in\mathbb{T}, t\geqslant s}$ be the Markov process whose distribution is specified by

$$\begin{cases} Z_s = 1 \\ \mathcal{L}(Z_{t+\Delta t}|Z_t) = Z_t + \mathrm{Pois}\big(\frac{f}{\bar{F}_t} Z_t\,\Delta t\big). \end{cases}$$

We use a martingale representation for (Z_t) which is a common tool in the analysis of preferential attachment networks, see [6] and [18]. We denote by $A = A^{f,s} = (A_t^{f,s})_{t\in\mathbb{T}}$ the deterministic process given by

$$A_t = A_t^{f,s} = \prod_{\substack{s \leqslant u<t \\ u\in\mathbb{T}}} \Big(1 + \frac{f}{\bar{F}_s}\Delta u\Big), \tag{2.3}$$

(with the convention that $A_t = 1$ if $t \leqslant s$) and the process $M = M^{f,s} = (Z_t/A_t)_{t\in\mathbb{T},t \geqslant s}$. Since A is deterministic, the process M is a (non-homogeneous) Markov process, too.

Lemma 2.1. *Under Assumption (A0) there exists a constant C such that the following hold. For $f \in (0, 1]$ and $s \in \mathbb{T}$, the process $M = M^{f,s}$ is a convergent L^2-martingale with*

$$\mathbb{E}\Big[\sup_{\substack{t\in\mathbb{T} \\ t \geqslant s}} M_t^2\Big] \leqslant C.$$

Proof. In the following, we will denote by $C_1, C_2, \ldots$ positive constants that do not depend on f, s and t. Note that for $t \in \mathbb{T}$

$$\mathbb{E}[Z_{t+\Delta t}|Z_t] = Z_t + \frac{f}{\bar{F}_t} Z_t\,\Delta t = \frac{A_{t+\Delta t}}{A_t}\, Z_t$$

and, hence, $M = Z/A$ is a martingale. Further one has for $t \in \mathbb{T}$ with $t \geqslant s$ that

$$\mathrm{var}(M_{t+\Delta t}|M_t) = \frac{1}{A_{t+\Delta t}^2} \frac{f}{\bar{F}_t} Z_t\,\Delta t.$$

As a consequence of Assumption (A0), there exist constants $C_1, C_2 > 0$ such that for all $u \in \mathbb{T}$, $C_1^{-1} \leqslant \bar{F}_u \leqslant C_2$. Hence, for all $t \geqslant s$

$$\mathrm{var}(M_{t+\Delta t}|M_t) \leqslant C_1\, \frac{f\, M_t}{A_{t+\Delta t}}\,\Delta t$$

By Lemma 5.2, there exists a constant C_3 such that for all t

$$A_t \geqslant C_3^{-1} \exp\{f a[s, t]\} \geqslant C_3^{-1} \exp(f(t - s)/C_2).$$

Since, further, $\mathbb{E}[M_t] = 1$, we conclude that

$$\mathbb{E}[(M_{t+\Delta t} - M_t)^2] \leqslant C_1\, C_2\, f\, e^{-f(t+\Delta t)/C_3}\Delta t.$$

The result follows since

$$\sum_{\substack{t\in\mathbb{T} \\ t \geqslant s}} \mathbb{E}[(M_{t+\Delta t} - M_t)^2] \leqslant C_1\, C_2\, f \int_s^\infty e^{-f(t-s)/C_3}\,\mathrm{d}t = C_1\, C_2\, C_3. \qquad \square$$

Remark 2.2. We assume Assumption (A0). For $f \in (0, 1]$ and $s \in \mathbb{T}$, one has

$$\log A_t^{f,s} = \sum_{\substack{s \leqslant u<t \\ u\in\mathbb{T}}} \log\big(1 + \frac{f}{\bar{F}_u}\,\Delta u\big) \sim ft,$$

as $t \to \infty$. Hence, by Lemma 2.1, one has that, almost surely, $\log Z^{f,s} \sim ft$ so that, in particular,

$$\lim_{t\to\infty} \frac{Z_t^{f,s}}{N(t)} = 0, \text{ almost surely.}$$

2.2 Collections of impact evolutions

In the following, we turn to the analysis of a process being a prototype for the *sum* of similar impact evolutions.

We fix $f,p \in (0,1]$ and $s,s' \in \mathbb{T}$ with $s < s'$. For each $u \in \mathbb{T}$, let $Z^{(u)} = (Z^{(u)}_t)_{t\in\mathbb{T}}$ be an independent Markov process whose distribution is specified by

$$\begin{cases} Z^{(u)}_v = 0, & \text{for } v < u \\ \mathcal{L}(Z^{(u)}_u) = \mathrm{Ber(p)}, & \\ \mathcal{L}(Z^{(u)}_{v+\Delta v} | Z^{(u)}_v) = Z^{(u)}_v + \mathrm{Pois}(\frac{f}{\bar F_v} Z^{(u)}_v \, \Delta v), & \text{for } v \geqslant u. \end{cases}$$

We consider the process $\bar Z = \bar Z^{f,p,s,s'} = (\bar Z^{f,p,s,s'}_t)_{t\in\mathbb{T}}$ defined by

$$\bar Z_t = \sum_{u \in \mathbb{T}\cap[s,s')} Z^{(u)}_t.$$

Further, we choose $A = A^{f,s}$ as before and set $\bar M := \bar M^{f,p,s,s'} := \bar Z / A$.

Proposition 2.3. *We assume Assumption (A2). The process $(\bar M_t)_{t\in\mathbb{T}}$ is a submartingale. Further, there exists a constant C only depending on μ and $(\bar F_t)$ such that for all $f,p \in (0,1]$ and $s,s' \in \mathbb{T}$ with $s < s'$, one has*

$$\mathbb{E}[\bar M_t] \leqslant C\, p\, e^s \int_0^{s'-s} \exp\{(1-f)\,u - f\,\Upsilon[s,s+u]\}\,\mathrm{d}u$$

and

$$\mathrm{var}(\bar M_t) \leqslant C\, p\, e^s \int_0^{s'-s} \exp\{(1-2f)\,u - 2f\Upsilon[s,s+u]\}\,\mathrm{d}u.$$

for all $t \in \mathbb{T}$.

Proof. As in the proof of Lemma 2.1 one easily checks that $\bar M$ is a submartingale.

1.) We start with proving the first moment estimate. It suffices to consider $t \geqslant s'$ since $\bar M$ is a submartingale. Conditionally on $\{Z^{(u)}_u = 1\}$ the process $Z^{(u)}$ has the same distribution as $Z^{f,u}$ from the last subsection. Hence,

$$\mathbb{E}[\bar M_t] = \frac{1}{A_t} \sum_{u\in\mathbb{T}\cap[s,s')} \mathbb{E}[\mathbb{1}_{\{Z^{(u)}_u=1\}} Z^{(u)}_t] = \frac{p}{A_t} \sum_{u\in\mathbb{T}\cap[s,s)} A^{f,u}_t = p \sum_{u\in\mathbb{T}\cap[s,s')} \frac{1}{A_u},$$

where we used that $A_t = A_u\, A^{f,u}_t$. By Lemmas 5.2 and 5.3, one has for $0 \leqslant s \leqslant u$

$$A_u \geqslant \frac{1}{C_1} \exp\{f\,(u-s) + f\,\Upsilon[s,u]\} \tag{2.4}$$

where C_1 is a constant not depending on s and u. We obtain that

$$\mathbb{E}[\bar M_t] \leqslant C_1\, p \sum_{u\in\mathbb{T}\cap[s,s')} \exp\{-f(u-s) - f\,\Upsilon[s,u]\}.$$

Since $1 \leqslant e^t \Delta t$ we arrive at

$$\mathbb{E}[\bar M_t] \leqslant C_1\, p \int_s^{s'} \exp\{(1-f)\,\iota(u) - f\,\Upsilon[s,\iota(u)]\}\,\mathrm{d}u,$$

where $\iota(u) := \sup \mathbb{T} \cap [0,u]$ for $u \geqslant 0$. By definition $\Upsilon[\iota(u),u] \leqslant 1$. Therefore, one has for all $t \geqslant s'$ that

$$\mathbb{E}[\bar M_t] \leqslant e\, C_1\, p \int_s^{s'} \exp\{(1-f)\,u - f\,\Upsilon[s,u]\}\,\mathrm{d}u.$$

2.) We denote by $C_2, C_3, \ldots$ constants that may change from line to line and that do not depend on the parameters f, p, s and s'. One has

$$\mathrm{var}(\bar{M}_t) = \frac{1}{A_t^2} \sum_{u \in \mathbb{T} \cap [s,s')} \mathrm{var}(Z_t^{(u)}) \leqslant \sum_{u \in \mathbb{T} \cap [s,s')} A_u^{-2}\, \mathbb{E}\Big[\mathbb{1}_{\{Z_u^{(u)}=1\}} \Big(\frac{Z_t^{(u)}}{A_t^{f,u}}\Big)^2\Big].$$

Conditionally on $\{Z_u^{(u)} = 1\}$, the process $\frac{Z_t^{(u)}}{A_t^{f,u}}$ has the same distribution as the martingale $M^{f,u}$ from before. By Lemma 2.1, there exists a constant C_2 such that

$$\mathbb{E}\Big[\mathbb{1}_{\{Z_u^{(u)}=1\}} \Big(\frac{Z_t^{(u)}}{A_t^{f,u}}\Big)^2\Big] \leqslant C_2\, p$$

for $u \in [s, s')$. Hence,

$$\mathrm{var}(\bar{M}_t) \leqslant C_2\, p \sum_{u \in \mathbb{T} \cap [s,s')} A_u^{-2}$$

We proceed as in part one. Using (2.4) and the inequality $1 \leqslant e^t \Delta t$, we get that

$$\mathrm{var}(\bar{M}_t) \leqslant C_3\, p \sum_{u \in \mathbb{T} \cap [s,s')} \exp\{-2f(u-s) + 2f\Upsilon[s,u]\}\, e^u\, \Delta u$$

for a constant C_3. Hence, we get as before that one has

$$\mathrm{var}(\bar{M}_t) \leqslant C_4\, p\, e^s \int_s^{s'} \exp\{(1-2f)\,(u-s) - 2f\Upsilon[s,u]\}\, \mathrm{d}u$$

for all $t \geqslant s'$. The estimate is also true for $t < s'$ since the variance is increasing in t. □

3 Riemann approximation/binning

The main tool of this section is a binning argument. We collect vertices of similar fitness and similar date of birth in individual bins. Further we give each bin a type (A), (B), (C) or (D). Bins of the same type will behave similarly in our forthcoming analysis. Mainly, there are three ways how bins contribute to the overall impact. All but finitely many of the bins of type (A) will be empty and their contribution is negligible. Bins of type (C) and (D) will typically comprise a large number of vertices and we will prove a strong law of large numbers for their contribution. Problematic are the bins of type (B). Most of these bins are empty, but there is an infinite number of vertices in these bins. We will prove that the probability that a vertex of type (B) effects the overall impact at a certain time vanishes as time tends to infinity.

The binning argument is based on two parameters $\zeta > 1$ and $\iota > 0$ that govern the size of the bins and that are fixed in the following discussion. Let

$$f_m := 1 - \zeta^{-m} \;\text{ and }\; S_{n,m} := \log \mu([f_m, f_{m+1}))^{-1} + \iota\, n,$$

for $m \in \mathbb{N}_0$ with $\mu([f_m, f_{m+1})) > 0$ and $n \in \mathbb{Z}$. We consider the (n,m)-bin

$$\mathbb{I}_{n,m} := \{s \in \mathbb{T} : S_{n,m} < s \leq S_{n+1,m}, F_s \in [f_m, f_{m+1})\}.$$

The relevant bins (n,m) satisfy $n \in \mathbb{Z}$ and $m \in \mathbb{N}_0$ with

$$\mu([f_m, f_{m+1})) > 0 \;\text{ and }\; S_{n+1,m} \geqslant 0$$

and we restrict attention to these choices of parameters!

We note that the index n refers to the date of birth and the index m to the fitness of the vertices of a bin and for fixed $n \in \mathbb{Z}$ one has as $m \to \infty$

$$\mathbb{E}[\#\mathbb{I}_{n,m}] = (N(S_{n+1,m}) - N(S_{n,m}))\mu([f_m, f_{m+1})) \sim C_N\, e^{\iota n}(e^{\iota} - 1).$$

Hence, the definition of the bins is achieved in such a way that n is closely related to the number of vertices in the respective bin.

Crucial to us will be the analysis of the contribution of an individual bin (n, m) to the overall impact at time $t \in \mathbb{T}$ and we set

$$\Sigma_{n,m}(t) := \sum_{s \in \mathbb{I}_{n,m}} Z[s,t] \qquad (t \in \mathbb{T}). \tag{3.1}$$

We distinguish different categories of bins. We set

$$n_0(m) = \inf \mathbb{N}_0 \cap [\zeta\iota^{-1} \log m, \infty) \;\text{ and }\; n_0'(m) = \inf \mathbb{N}_0 \cap [(\zeta\iota)^{-1} \log m, \infty)$$

and consider the following four categories:

(A) Bins (n, m) with $n < -n_0(m)$ are negligible since for all sufficiently large m these are empty.

(B) Bins (n, m) with with $-n_0(m) \leqslant n < -n_0'(m)$ typically are empty and do not contribute. However, infinitely many of them do contain vertices, the early birds.

(C) Bins (n, m) with $-n_0'(m) \leqslant n < n_0(m)$ contribute a negligible amount to the total edge weight.

(D) Bins (n, m) with $n \geqslant n_0(m)$ contribute to the total edge weight and we prove a law of large numbers.

We will keep the notation for the bins in the whole section and recall that their definition and classification depends on the fixed parameters $\zeta > 1$ and $\iota > 0$. Moreover, we call the set of all vertices with fitness in $[f_m, f_{m+1})$ the vertices of the mth stripe.

3.1 Analysis of the bins of type (A) and (B)

We start with showing that bins of type (A) are typically empty and that bins of type (B) contain only vertices that are in the random set $\mathcal{E}_\gamma$ as long as the fitness is sufficiently large.

Lemma 3.1. *Almost surely, the bins of category (A) contain only a finite number of vertices.*

Proof. Using Lemma 5.1, we conclude that

$$\mathbb{P}\big(\exists s \in \mathbb{T} \cap [0, S_{-n_0(m),m}] \text{ with } F_s \in [f_m, f_{m+1})\big) \leqslant N(S_{-n_0(m),m})\, \mu([f_m, f_{m+1})) \lesssim C_N m^{-\zeta}$$

which is summable. By the lemma of Borel-Cantelli all but finitely many stripes m do not contain vertices of type (A). This proves the statement since any stripe has only a finite number of bins and thus vertices of category (A) ☐

Recall the definition of the set of early birds. Depending on a parameter $\gamma \in (0, 1)$ it is the random set

$$\mathcal{E}_\gamma = \{\pi(n) : n(\log n)^\gamma \mu([\mathcal{F}_n, 1)) \leqslant 1\}.$$

Lemma 3.2. *Assume Assumption (A1). Provided that $\gamma < \zeta^{-1}$, almost surely, all but finitely many vertices in bins of type (B) belong to $\mathcal{E}_\gamma$.*

Proof. Let $m \in \mathbb{N}$ and suppose that $s \in \mathbb{T}$ is a vertex with $F_s \in [f_m, f_{m+1})$ that is of type (B). It suffices to show that s is in $\mathcal{E}_\gamma$ as long as m is sufficiently large since every stripe does only contain finitely many vertices of type (B).

We note that by Assumption (A1), $\mu([f_m, f_{m+1})) \approx \mu([F_s, 1))$ and we conclude that

$$s \leqslant \log \mu([f_m, f_{m+1}))^{-1} - \iota n_0'(m) = \log \mu([F_s, 1))^{-1} - \zeta^{-1} \log m + \mathcal{O}(1).$$

Hence there exists a constant C_1 such that as long as m is sufficiently large one has

$$e^s \leqslant C_1 \frac{\mu([F_s, 1))^{-1}}{m^{\zeta^{-1}}}.$$

Since $e^s \approx N(s)$ and $m \approx \log(1 - F_s)^{-1} \approx \log \mu([F_s, 1))^{-1}$ we obtain existence of a constant C_2 such that as long as m is large

$$N(s) \leqslant C_2 \, \mu([F_s, 1))^{-1} (\log \mu([F_s, 1))^{-1})^{-\zeta^{-1}}.$$

Applying the monotonically increasing function $[1, \infty) \ni x \mapsto x(\log x)^{\zeta^{-1}}$ implies existence of a constant C_3 such that for large m

$$N(s)(\log N(s))^{\zeta^{-1}} \leqslant C_3 \, \mu([F_s, 1))^{-1}.$$

and the statement follow by recalling that $\gamma < \zeta^{-1}$. □

3.2 Analysis of bins of type (D)

Next, we analyse the contribution of the bins of type (D). For each $s \in \mathbb{T}$, we represent its impact evolution as

$$Z[s, t] = M[s, t] \, A[s, t],$$

where

$$A[s, t] = \prod_{\substack{u \in \mathbb{T} \\ s \leqslant u < t}} \Big(1 + \frac{F_s}{\bar{F}_u} \Delta u\Big) \tag{3.2}$$

and $M[s, \cdot]$ is a convergent martingale (see Lemma 2.1). Further, we denote

$$a[s, t] = \sum_{\substack{u \in \mathbb{T} \\ s \leqslant u \leqslant t}} \frac{1}{\bar{F}_u} \Delta u \quad \text{and} \quad \Upsilon[s, t] = \int_s^t (1 - \bar{F}_v) \, \mathrm{d}v$$

for $0 \leqslant s \leqslant t$.

Lemma 3.3. *Assume Assumption (A0). For $\varepsilon > 0$, one has, almost surely, that for all but finitely many*

$$(n, m) \in \mathbb{S}_D := \{(n', m') \in \mathbb{N}_0^2 : n' \geqslant n_0(m'), \mu([f_{m'}, f_{m'+1})) > 0\} \tag{3.3}$$

that

$$\Sigma_{n,m}(t) \leqslant e^{\varepsilon} \, \mathbb{E}[\#\mathbb{I}_{n,m}] \exp\{f_{m+1} \, a[S_{n,m}, t]\}$$

for all $t \geqslant S_{n,m}$ and

$$\Sigma_{n,m}(t) \geqslant e^{-\varepsilon} \, \mathbb{E}[\#\mathbb{I}_{n,m}] \exp\{f_m \, a[S_{n+1,m}, t]\}$$

for all $t \geqslant S_{n+1,m}$, where $\Sigma_{n,m}$ is as in (3.1).

Proof. In the following, $C_1, C_2, \dots$ denote constants that may change from line to line and that do not depend on the variables n and m.

1.) In the first step we show that $(\#\mathbb{I}_{n,m})_{(n,m)\in\mathbb{S}_D}$ satisfies a strong law of large numbers. By Lemma 5.1, one has

$$N(S_{n+1,m}) - N(S_{n,m}) \sim C_N e^{S_{n,m}}(e^\iota - 1)$$

on $\mathbb{S}_D$, since $S_{n,m}$ tends to infinity on $\mathbb{S}_D$. Consequently, one has

$$\mathbb{E}[\#\mathbb{I}_{n,m}] = \mu([f_m, f_{m+1}))(N(S_{n+1,m}) - N(S_{n,m})) \sim C_N(e^\iota - 1)e^{\iota n}. \tag{3.4}$$

Note that $\mathrm{var}(\#\mathbb{I}_{n,m}) \leqslant \mathbb{E}[\#\mathbb{I}_{n,m}]$ so that by Chebyshev's inequality

$$\mathbb{P}(|\#\mathbb{I}_{n,m} - \mathbb{E}[\#\mathbb{I}_{n,m}]| > \varepsilon\mathbb{E}[\#\mathbb{I}_{n,m}]) \leq \varepsilon^{-2}\,\mathbb{E}[\#\mathbb{I}_{n,m}]^{-1}.$$

Consequently,

$$\begin{aligned}\sum_{n=n_0(m)}^{\infty} \mathbb{P}(|\#\mathbb{I}_{n,m} - \mathbb{E}[\#\mathbb{I}_{n,m}]| > \varepsilon\mathbb{E}[\#\mathbb{I}_{n,m}]) &\leqslant \varepsilon^{-2} \sum_{n=n_0(m)}^{\infty} \mathbb{E}[\#\mathbb{I}_{n,m}]^{-1} \\ &\leqslant C_1 \sum_{n=n_0(m)}^{\infty} e^{-\iota n} = C_1\,\frac{1}{1-e^{-\iota}}\,e^{-\iota n_0(m)}\end{aligned}$$

where $C_1 < \infty$ does not depend on $m \in \mathbb{N}$. Note that $e^{-\iota n_0(m)} \approx m^{-\zeta}$ as $m \to \infty$ and that $\zeta > 1$. Hence, one has by the lemma of Borel-Cantelli that, almost surely, for all but finitely $(n,m) \in \mathbb{S}_D$

$$(1-\varepsilon)\mathbb{E}[\#\mathbb{I}_{n,m}] \leq \#\mathbb{I}_{n,m} \leq (1+\varepsilon)\mathbb{E}[\#\mathbb{I}_{n,m}].$$

2.) Recall that the impact evolutions $Z[s,\cdot]$ $(s \in \mathbb{T})$ are all independent and each admits a representation

$$Z[s,\cdot] = M[s,\cdot]A[s,\cdot]$$

with $A[s,\cdot]$ as in (3.2) and $M[s,\cdot]$ being a martingale. For each $(n,m) \in \mathbb{S}_D$, we consider

$$\bar{M}_{n,m}(t) := \sum_{s\in\mathbb{I}_{n,m}} \mathbb{1}_{\{s\,\leqslant\,t\}}(M[s,t]-1) \qquad (t \in \mathbb{T}),$$

By Lemma 2.1, all processes appearing in the sum are uniformly square integrable martingales. Since the individual terms $M[s,\cdot]$ are independent one gets with Doob's maximal inequality that

$$\mathbb{E}\Big[\sup_{t\in\mathbb{T}} \bar{M}_{n,m}(t)^2\Big] \leq 4\,\mathbb{E}[\bar{M}_{n,m}(\infty)^2] = 4\,\mathbb{E}\Big[\sum_{s\in\mathbb{I}_{n,m}} \mathbb{E}[(M[s,\infty]-1)^2|F_s]\Big]$$

By Lemma 2.1 and (3.4), one has for $(n,m) \in \mathbb{S}_D$ that

$$\mathbb{E}\Big[\sup_{t\in\mathbb{T}} \bar{M}_{n,m}(t)^2\Big] \leq C_2\,\mathbb{E}[\#\mathbb{I}_{n,m}] \sim C_2\,C_N(e^\iota - 1)e^{\iota n},$$

where C_2 is as in the lemma. As before one verifies with Borel-Cantelli that, almost surely, for all but finitely many $(n,m) \in \mathbb{S}_D$

$$\sup_{t\in\mathbb{T}} |\bar{M}_{n,m}(t)| \leqslant \varepsilon\,\mathbb{E}[\#\mathbb{I}_{n,m}].$$

Altogether we get with Lemma 5.2 that, almost surely, for all but finitely many $(n, m) \in \mathbb{S}_D$ and for all $t \geqslant S_{n+1,m}$

$$\begin{aligned}\Sigma_{n,m}(t) &= \sum_{s \in \mathbb{I}_{n,m}} M[s,t]A[s,t] \geqslant \exp\{-4N(S_{n,m})^{-1}\} \exp\{f_m a[S_{n+1,m}, t]\} \sum_{s \in \mathbb{I}_{n,m}} M[s,t] \\ &= \exp\{-4N(S_{n,m})^{-1}\} \exp\{f_m a[S_{n+1,m}, t]\} (\#\mathbb{I}_{n,m} + \bar{M}_{n,m}(t)) \\ &\geqslant (1 - 2\varepsilon) \exp\{-4N(S_{n,m})^{-1}\} \exp\{f_m a[S_{n+1,m}, t]\} \, \mathbb{E}[\#\mathbb{I}_{n,m}].\end{aligned}$$

This immediately implies the second inequality of the statement since $N(S_{n,m})$ tends to infinity. The converse direction follows analogously. □

Remark 3.4. We remark that the same proof allows to verify that for $f, f' \in [0, 1]$ with $f \leqslant f'$, and

$$\bar{\mathbb{I}}_n := \{s \in \mathbb{T} : (n-1)\iota < s \leqslant n\iota, F_s \in [f, f')\}$$

one has for $\varepsilon > 0$ that, almost surely, for all but finitely many $n \in \mathbb{N}$,

$$\sum_{s \in \bar{\mathbb{I}}_n} Z[s,t] \geqslant e^{-\varepsilon} \, \mathbb{E}[\bar{\mathbb{I}}_{n,m}] \, \exp\{f \, a[n\iota, t]\}$$

for all $t \geqslant n\iota$.

We prove a modified version of Lemma 3.3 that expresses the contribution of the bins of type (D) in terms of Υ.

Proposition 3.5. *Assume Assumptions (A0) and (A2). For every $\varepsilon > 0$, one has that, almost surely, for all but finitely many $(n, m) \in \mathbb{S}_D$,*

$$\Sigma_{n,m}(t) \leqslant e^{\varepsilon + 2\iota} \, \mu([f_m, f_{m+1})) \, N(t) \int_{S_{n,m}}^{S_{n+1,m}} \exp\{-(1 - f_{m+1})(t - u) + \Upsilon[u,t]\} \, \mathrm{d}u,$$

for $t \geqslant 0$, and

$$\Sigma_{n,m}(t) \geqslant e^{-\varepsilon - 2\iota} \, \mu([f_m, f_{m+1})) \, N(t) \int_{S_{n,m}}^{S_{n+1,m}} \exp\{-(1 - f_m)(t - u) + f_m \Upsilon[u,t]\} \, \mathrm{d}u,$$

for $t \geqslant S_{n+1,m}$.

Proof. Let $\varepsilon > 0$. Note that by Lemma 5.3, one has for all but finitely many $(n, m) \in \mathbb{S}_D$ for all $t \geqslant S_{n,m}$ (we briefly say *eventually*) that

$$a[S_{n,m}, t] \leqslant t - S_{n,m} + \Upsilon[S_{n,m}, t] + \varepsilon$$

Further, by Lemma 5.1, eventually,

$$\mathbb{E}[\#\mathbb{I}_{n,m}] = \mu([f_m, f_{m+1})) \, (N(S_{n+1,m}) - N(S_{n,m})) \sim C_N \, \mu([f_m, f_{m+1})) \int_{S_{n,m}}^{S_{n+1,m}} e^u \, \mathrm{d}u$$

and for $u \in [S_{n,m}, S_{n+1,m}]$

$$(t - S_{n,m}) + \Upsilon[S_{n,m}, t] = \int_{S_{n,m}}^{t} (2 - \bar{F}_v) \, \mathrm{d}v \leqslant (t - u) + \Upsilon[u, t] + 2\iota.$$

Consequently, one has that, eventually,

$$\begin{aligned}&\mathbb{E}[\#\mathbb{I}_{n,m}] \exp\{f_{m+1} a[S_{n,m}, t]\} \\ &\quad \leqslant e^{2\varepsilon} C_N \, \mu([f_m, f_{m+1})) \int_{S_{n,m}}^{S_{n+1,m}} e^u \, \mathrm{d}u \; \exp\{f_{m+1}(t - S_{n,m}) + f_{m+1} \Upsilon[S_{n,m}, t]\} \\ &\quad \leqslant e^{2\varepsilon + 2\iota} C_N \, e^t \, \mu([f_m, f_{m+1})) \int_{S_{n,m}}^{S_{n+1,m}} \exp\{-(1 - f_{m+1})(t - u) + f_{m+1} \Upsilon[u,t]\} \, \mathrm{d}u\end{aligned}$$

Hence, by Lemma 3.3, one has that, almost surely, eventually,

$$\Sigma_{n,m}(t) \leqslant e^{3\varepsilon+2\iota}\,\mu([f_m,f_{m+1}))\,N(t)\int_{S_{n,m}}^{S_{n+1,m}} \exp\{-(1-f_{m+1})(t-u)+\Upsilon[u,t]\}\,\mathrm{d}u.$$

Conversely, it follows in complete analogy that for all but finitely many $(n,m)\in\mathbb{S}_D$ for $t\geqslant S_{n+1,m}$,

$$\begin{aligned}&\mathbb{E}[\#\mathbb{I}_{n,m}]\,\exp\{f_m a[S_{n+1,m},t]\}\\ &\quad\geqslant e^{-2\varepsilon}C_N\,\mu([f_m,f_{m+1}))\int_{S_{n,m}}^{S_{n+1,m}} e^u\,\mathrm{d}u\;\exp\{f_m(t-S_{n+1,m})+f_m\Upsilon[S_{n+1,m},t]\}\\ &\quad\geqslant e^{-2\varepsilon-2\iota}C_N\,e^t\,\mu([f_m,f_{m+1}))\int_{S_{n,m}}^{S_{n+1,m}}\exp\{-(1-f_m)(t-u)+f_m\Upsilon[u,t]\}\,\mathrm{d}u\end{aligned}$$

which implies that, almost surely, for all but finitely many $(n,m)\in\mathbb{S}_D$ for $t\geqslant S_{n+1,m}$

$$\Sigma_{n,m}(t)\geqslant e^{-3\varepsilon-2\iota}\,\mu([f_m,f_{m+1}))\,N(t)\int_{S_{n,m}}^{S_{n+1,m}}\exp\{-(1-f_m)(t-u)+f_m\Upsilon[u,t]\}\,\mathrm{d}u,$$

again by Lemma 3.3. □

3.3 Analysis of bins of type (C)

In this subsection we provide the main estimate to control the impact of the bins of category (C). For $m\in\mathbb{N}_0$ with $\mu([f_m,f_{m+1}))>0$ and $t\geqslant 0$, let

$$\Sigma_m^{(\mathrm{C})}(t):=\sum_{n=-n_0'(m)}^{n_0(m)-1}\sum_{s\in\mathbb{I}_{n,m}} Z[s,t].$$

Lemma 3.6. *Assume Assumptions (A1) and (A2). For every $\varepsilon>0$, one has that almost surely, for all but finitely many m and all $t\geqslant 0$*

$$\Sigma_m^{(\mathrm{C})}(t)\leqslant\varepsilon\,S_{0,m}\,e^{-(1-f_{m+1})t+\Upsilon[S_{0,m},t]}\,\mu([f_m,f_{m+1}))N(t),$$

where $\Upsilon[u,v]=0$ for $v<u$.

Proof. We use constants $C_1,C_2,\dots$ that may change from line to line and do not depend on m and t. We will apply Proposition 2.3. Fix $m\in\mathbb{N}$ with $\mu([f_m,f_{m+1}))>0$, let $f=f_{m+1}$, $p=\mu([f_m,f_{m+1}))$,

$$s=S_m^-:=\min\mathbb{T}\cap(S_{-n_0'(m),m},\infty)\text{ and } s'=S_m^+:=\min\mathbb{T}\cap(S_{n_0(m),m},\infty)$$

and consider the processes $\bar M=\bar M^{f,p,s,s'}$ and $A=A^{f,s}$ as introduced in Subsection 2.2. By definition of $\bar M$ and A, one can couple $\Sigma_m^{(\mathrm{C})}$ with $\bar M$ such that

$$\Sigma_m^{(\mathrm{C})}(t)\leqslant\bar M_t\,A_t \tag{3.5}$$

for all $t\in\mathbb{T}$. By Proposition 2.3, $\bar M$ is a non-negative submartingale and together with Doob's martingale inequality we get existence of a constant C_1 such that for sufficiently large m

$$\mathbb{E}\Big[\sup_{t\in\mathbb{T}}\bar M_t^2\Big]\leqslant C_1\,p^2\,e^{2S_m^-}(S_m^+-S_m^-)^2e^{2(1-f)(S_m^+-S_m^-)}+C_1\,p\,e^{S_m^-}\,(S_m^+-S_m^-).$$

We note that $1-f=\zeta^{-(m+1)}$ and $S_m^+ - S_m^- \leqslant 2\zeta\log m + 2 + 2\iota$ so that $e^{2(1-f)(S_m^+-S_m^-)} \to 1$ as $m\to\infty$. Hence,

$$\mathbb{E}\Big[\sup_{t\in\mathbb{T}} \bar{M}_t^2\Big] \leqslant 2\,C_1\,p^2\,e^{2S_m^-}(S_m^+-S_m^-)^2 + C_1\,p\,e^{S_m^-}\,(S_m^+-S_m^-)$$

for large m. Since $p=e^{-S_{0,m}}$ and $S_{0,m}-S_m^- \geqslant \zeta^{-1}\log m - 1$, we get with the above estimate for $S_m^+-S_m^-$ that

$$p\,e^{S_m^-}\,(S_m^+-S_m^-) \leqslant e\,m^{-\zeta^{-1}}(2\zeta\log m + 2 + 2\iota) \approx m^{-\zeta^{-1}}\log m.$$

Consequently, there exists a constant C_2 such that for sufficiently large $m\in\mathbb{N}$ one has

$$\mathbb{E}\Big[\sup_{t\in\mathbb{T}} \bar{M}_t^2\Big] \leqslant C_2\,m^{-\zeta^{-1}}\log m. \tag{3.6}$$

Let now $(\alpha_m)_{m\in\mathbb{N}}$ denote a sequence of positive numbers with

$$\sum_{m=1}^{\infty} \alpha_m^{-2} m^{-1/\zeta}\log m < \infty. \tag{3.7}$$

By the Markov inequality, one has

$$\mathbb{P}\Big(\sup_{t\in\mathbb{T}} \bar{M}_t^2 \leqslant \alpha_m^2\Big) \leqslant \frac{\mathbb{E}[\sup_{t\in\mathbb{T}_s}\bar{M}_t^2]}{\alpha_m^2}$$

and by Borel-Cantelli (see (3.6) and (3.7)) it follows that, almost surely,

$$\sup_{t\in\mathbb{T}} \bar{M}_t \leqslant \alpha_m$$

for all but finitely many $m\in\mathbb{N}$. Together with (3.5) and Lemma 5.2, we get that almost surely, for all sufficiently large m and $t\geqslant 0$ (we briefly say eventually),

$$\Sigma_m^{(\mathrm{C})}(t) \leqslant \alpha_m \exp\{f_{m+1}a[S_m^-,t]\}. \tag{3.8}$$

We distinguish two cases. First we restrict attention to $t\geqslant S_{0,m}$. Since $S_{0,m}\sim S_m^-$, we get with Lemma 5.3 existence of a constant C_3 with

$$a[S_m^-,S_{0,m}] \leqslant S_{0,m}-S_m^- + C_3 \leqslant \log m^{1/\zeta} + C_3 + \iota \tag{3.9}$$

for large m. Since $S_{0,m}\sim \alpha m\log\zeta$ as $m\to\infty$, we get that, eventually,

$$\Sigma_m^{(\mathrm{C})}(t) \leqslant C_4\alpha_m\,m^{\zeta^{-1}-1}S_{0,m}\,\exp\{f_{m+1}a[S_{0,m},t]\},$$

where the constant C_4 does not depend on the choice of (α_m). By Lemma 5.3, one has

$$f_{m+1}\,a[S_{0,m},t] = f_{m+1}(t-S_{0,m}+\Upsilon[S_{0,m},t]+o(1)) \leqslant f_{m+1}(t-S_{0,m}) + \Upsilon[S_{0,m},t] + o(1).$$

Consequently, we obtain with $\mu([f_m,f_{m+1}))=e^{-S_{0,m}}$ and $N(t)\sim C_N e^t$ that, eventually,

$$\Sigma_m^{(\mathrm{C})}(t) \leqslant C_5\alpha_m\,m^{\zeta^{-1}-1}S_{0,m}\,e^{-(1-f_{m+1})(t-S_{0,m})}e^{\Upsilon[S_{0,m},t]}\mu([f_m,f_{m+1}))\,N(t),$$

where C_5 is a constant not depending on (α_m). The choice $\alpha_m := \varepsilon\,(2C_5)^{-1}m^{1-\zeta^{-1}}$ is admissible in the sense that (3.7) is valid and we conclude that, eventually, for $t\geqslant S_{0,m}$

$$\Sigma_m^{(C)} \leqslant \frac{\varepsilon}{2}\,S_{0,m}\,e^{-(1-f_{m+1})(t-S_{0,m})+\Upsilon[S_{0,m},t]}\,\mu([f_m,f_{m+1}))N(t).$$

Next,we consider the case $t \in [S_m^-, S_{0,m})$ We use that by (3.8), almost surely, for sufficiently large m

$$\Sigma_m^{(C)}(t) \leqslant \alpha_m \exp\{f_{m+1}\, a[S_m^-, t]\}.$$

Estimate (3.9) remains true when replacing $S_{0,m}$ by $t \in [S_m^-, S_{0,m})$ with the same argument. Hence, there exists a constant C_6 such that, eventually,

$$\Sigma_m^{(C)}(t) \leqslant C_6\, \alpha_m\, e^{t - S_m^-}$$

The rest of the proof is in line with case one: using that $e^{-S_{0,m}} = \mu([f_m, f_{m+1}))$, $N(t) \approx e^{-t}$, $S_{0,m} - S_m^- = \log m^{1/\zeta} + \mathcal{O}(1)$ and $S_{0,m} \approx m$ we conclude existence of a constant C_7 such that, eventually,

$$\Sigma_m^{(C)}(t) \leqslant C_7\, \alpha_m\, m^{\zeta^{-1}-1}\, S_{0,m}\, \mu([f_m, f_{m+1}))\, N(t).$$

Choosing (α_m) as before gives that, eventually, for $t \in [S_m^-, S_{0,m})$

$$\Sigma_m^{(C)} \leqslant \frac{\varepsilon}{2}\, S_{0,m}\, \mu([f_m, f_{m+1}))N(t).$$

Noting that $S_{0,m} = \log \mu([f_m, f_m+1))^{-1} = \log(1 - f_{m+1})^{-\alpha + o(1)} = o((1-f_{m+1})^{-1})$ we get the result. ☐

3.4 The stripes in the condensate without early birds

In this subsection we control the contribution of vertices of type (C) and (D) in stripes contributing to the condensate. From now on we always assume validity of Assumptions (A1) and (A2) without further mentioning!

Proposition 3.7. *Assume Assumptions (A1) and (A2). Let $\kappa, \varepsilon > 0$. Almost surely, for all but finitely many m for all $t \in [0, \kappa/(1-f_m)]$ one has*

$$\Sigma_m(t) = \sum_{\substack{n \in \mathbb{Z} \\ n \geqslant -n_0'(m)}} \Sigma_{n,m} \leqslant \frac{e^{2\iota + \varepsilon}}{\alpha - 1}\, S_{0,m} e^{-(1-f_{m+1})t + \Upsilon[S_{0,m}, t]}\, \mu([f_m, f_{m+1}))\, N(t).$$

Proof. First we analyse the contribution of all boxes of category (D). We denote by m the index of the stripe which is sent to infinity. Most of the following equations are meant to hold almost surely, for all but finitely many m and all $t \in [S_{0,m}, \kappa/(1-f_m)]$. We briefly say the statement holds *eventually*.

Consider

$$\Sigma_m^{(D)}(t) := \sum_{n = n_0(m)}^{\infty} \Sigma_{n,m}(t)$$

for $t \geqslant 0$. By Proposition 3.5, one has that, eventually,

$$\begin{aligned}\Sigma_m^{(D)}(t) &\leqslant e^{\varepsilon + 2\iota}\, \mu([f_m, f_{m+1}))\, N(t) \int_{S_{0,m}}^{t+\iota} \exp\{-(1-f_{m+1})(t-u) + \Upsilon[u,t]\}\, \mathrm{d}u \\ &= e^{\varepsilon + 2\iota}\, \mu([f_m, f_{m+1}))\, e^{-(1-f_{m+1})t + \Upsilon[S_{0,m}, t]} \\ &\qquad \times N(t) \int_{S_{0,m}}^{t+\iota} \exp\{(1-f_{m+1})u - \Upsilon[S_{0,m}, u]\}\, \mathrm{d}u. \end{aligned} \tag{3.10}$$

Let $\alpha' \in (1, \alpha)$ arbitrary and note that, by Lemma 5.3, one has $\Upsilon[S_{0,m}, u] \geqslant \log(u/S_{0,m})^{\alpha'}$ for $u \geqslant S_{0,m}$, as long as m is sufficiently large. Consequently, one has that, eventually,

$$\begin{aligned}\int_{S_{0,m}}^{t+\iota} \exp\{(1-f_{m+1})u - \Upsilon[S_{0,m}, u]\}\, \mathrm{d}u &\leqslant \int_{S_{0,m}}^{t+\iota} \exp\{(1-f_{m+1})u\} \Big(\frac{S_{0,m}}{u}\Big)^{\alpha'}\, \mathrm{d}u \\ &\leqslant S_{0,m} \int_{\mathbb{R}} \mathbb{1}_{[1, (\frac{\kappa}{1-f_m} + \iota)/S_{0,m}]}(v)\, \exp\{(1-f_{m+1})S_{0,m} v\}\, v^{-\alpha'}\, \mathrm{d}v, \end{aligned} \tag{3.11}$$

where we applied the substitution $v = u/S_{0,m}$ and the inequality $t \leqslant \kappa/(1-f_m)$ in the second transformation. We note that the function

$$v \mapsto \mathbb{1}_{[1,\infty)}(v)\, e^{\kappa+\iota}\, v^{-\alpha'}$$

is an integrable majorant for the latter integral. Furthermore, one has $\lim_{m\to\infty}(1-f_{m+1})S_{0,m} = 0$ so that the integrand converges pointwise to $\mathbb{1}_{[1,\infty)}(v)v^{-\alpha'}$. Consequently, one has by dominated convergence that for sufficiently large m

$$\int_{\mathbb{R}} \mathbb{1}_{[1,(\frac{\kappa}{1-f_m}+\iota)/S_{0,m}]}(v)\, \exp\{(1-f_{m+1})S_{0,m}v\}\, v^{-\alpha'}\, \mathrm{d}v \leqslant e^{\varepsilon}\int_1^\infty v^{-\alpha'}\,\mathrm{d}v = \frac{e^{\varepsilon}}{\alpha'-1}.$$

In combination with (3.10) and (3.11), we conclude that, eventually,

$$\Sigma_m^{(\mathrm{D})}(t) \leqslant \frac{e^{2\varepsilon+2\iota}}{\alpha'-1}\,\mu([f_m,f_{m+1}))\,S_{0,m}\,e^{-(1-f_{m+1})t+\Upsilon[S_{0,m},t]}\,N(t). \tag{3.12}$$

Clearly, the estimate is true for all $m \in \mathbb{N}$ for all $t \in [0, S_{0,m})$ since for these parameters the left hand side is zero. Note that $\varepsilon > 0$, resp. $\alpha' \in (1,\alpha)$ can be chosen arbitrarily close to zero or α so that the statement of the proposition follows with Lemma 3.6. □

3.5 The contribution outside the two main windows

First we derive an estimate which allows us to control the contribution of vertices born before time $t/2$ that are no early birds and have fitness smaller than $1-\kappa/t$ with κ being a large constant.

Proposition 3.8. *For every $\varepsilon > 0$, there exists $\kappa > 1$ such that almost surely for sufficiently large $t \geqslant 0$, one has*

$$\sum_{\substack{s\in\mathcal{E}_\gamma^c\cap[0,t/2]\\ F_s\,\leqslant\,1-\kappa\delta(t)}} Z[s,t] \leqslant \varepsilon\, T\, e^{\Upsilon[T,t]-T}\,N(t).$$

Proof. We fix $\zeta > 1$ with $\gamma < \zeta^{-1}$ and recall that by Lemmas 3.1 and 3.2, almost surely, all but finitely many vertices in $\mathcal{E}_\gamma^c$ are of type (C) or (D). We start with controlling the contribution of vertices of type (D) in stripes with large index m.

In the following $C_1, C_2, \ldots$ denote constants that do not depend on the parameters κ, n, m and t. By Proposition 3.5, one has that, almost surely, for all but finitely many $(n,m) \in \mathbb{S}^{(\mathrm{D})}$ for $t \geqslant 0$

$$\Sigma_{n,m}(t) \leqslant 2\,\mu([f_m,f_{m+1}))\,N(t)\,e^{\Upsilon[S_{0,m},t]}\int_{S_{n,m}}^{S_{n+1,m}} \exp\{-(1-f_{m+1})(t-u)-\Upsilon[S_{0,m},u]\}\,\mathrm{d}u, \tag{3.13}$$

and, hence almost surely, for sufficiently large $m \in \mathbb{N}$ and all $t \geqslant 0$ (we will briefly say eventually)

$$\begin{aligned}\Sigma_m^{(\mathrm{D})}(t) :=& \sum_{\substack{n\,\geqslant\, n_0(m)\\ \text{with } S_{n,m}\,\leqslant\, t/2}} \Sigma_{n,m}(t)\\ &\leqslant 2\,\mu([f_m,f_{m+1}))\,N(t)\,e^{\Upsilon[S_{0,m},t]}e^{-(1-f_{m+1})t/3}\int_{S_{0,m}}^{2t/3} e^{-\Upsilon[S_{0,m},u]}\,\mathrm{d}u.\end{aligned}$$

Fix $\alpha' \in (1,\alpha)$. By Lemma 5.3, one has that, eventually,

$$\Sigma_m^{(\mathrm{D})}(t) \leqslant 3\mu([f_m,f_{m+1}))\,N(t)\,e^{\Upsilon[S_{0,m},t]-(1-f_{m+1})t/3}\int_{S_{0,m}}^\infty \Big(\frac{u}{S_{0,m}}\Big)^{-\alpha'}\,\mathrm{d}u.$$

Hence, there exists a constant C_1 depending only on α' such that, eventually,

$$\Sigma_m^{(\mathrm{D})}(t) \leqslant C_1\, S_{0,m}\, \mu([f_m, f_{m+1}))\, N(t)\, e^{\Upsilon[S_{0,m},t]-(1-f_{m+1})t/3}$$

By Lemma 3.6, the same estimate is true for the bins of type (C) and we conclude that, eventually,

$$\Sigma_m(t) := \sum_{\substack{n \geqslant -n_0'(m) \\ \text{with } S_{n,m} \leqslant t/2}} \Sigma_{n,m}(t) \leqslant C_2\, S_{0,m}\, \mu([f_m, f_{m+1}))\, N(t)\, e^{\Upsilon[S_{0,m},t]-(1-f_{m+1})t/3}. \quad (3.14)$$

Let m_0 denote a random integer such that the latter estimate is true for all $m \geqslant m_0$ and $t \in \mathbb{T}$. For large t we now analyse the contribution of the bins in the stripes $m_0, m_0+1, \dots$ of type (C) and (D) with $1 - f_m \geqslant \kappa\delta$ for $\kappa > 1$ with $\kappa^\alpha(1-\zeta^{-\alpha}) > 1$: let

$$\Sigma(t) := \sum_{\substack{m\colon m \geqslant m_0 \\ \text{with } 1-f_m \geqslant \kappa\delta}} \Sigma_m(t).$$

First we show that, $S_{0,m} \leqslant T$ for all m with $1 - f_m \geqslant \kappa\delta$ as long as t is sufficiently large. As $m \to \infty$, one has for $t \geqslant 0$ with $1 - f_m \geqslant \kappa/t$

$$\mu([f_m, f_{m+1})) \sim (1-\zeta^{-\alpha})\, \mu([f_m, 1)) \geqslant (1-\zeta^{-\alpha})\, \mu([1-\kappa\delta, 1)) \sim \kappa^\alpha(1-\zeta^{-\alpha})\, \mu([1-\delta, 1)).$$

Consequently, $\mu([f_m, f_{m+1})) \geqslant \mu([1-\delta, 1))$ and, equivalently, $S_{m,0} \leqslant T$ as long as m is sufficiently large. For the finitely many remaining indices m the inequality is trivially true for large t. Conversely, by the Potter bound (Lemma 5.5), there exists $\alpha'' > \alpha$ and a constant C_3 such that for m with $1 - f_m \geqslant \delta$

$$\mu([f_m, f_{m+1})) \leqslant \mu([f_m, 1)) \leqslant C_3 \Big(\frac{1-f_m}{\delta}\Big)^{\alpha''} \underbrace{\mu([1-\delta, 1))}_{=e^{-T}}. \quad (3.15)$$

In combination with $v^{\alpha''} = \frac{\alpha''}{1-\zeta^{-\alpha''}} \int_{v/\zeta}^{v} u^{\alpha''-1}\, \mathrm{d}u$ for $v = (1-f_m)/\delta$ we deduce with (3.14) that for sufficiently large t

$$\begin{aligned} \Sigma(t) &\leqslant C_4 T e^{\Upsilon[T,t]-T}\, N(t) \sum_{\substack{m\colon m \geqslant m_0 \\ \text{with } 1-f_m \geqslant \kappa\delta}} \Big(\frac{1-f_m}{\delta}\Big)^{\alpha''} e^{-\frac{1-f_{m+1}}{3\delta}+\Upsilon[S_{0,m},T]} \\ &\leqslant C_5 T e^{\Upsilon[T,t]-T} N(t) \sum_{\substack{m\colon m \geqslant m_0 \\ \text{with } 1-f_m \geqslant \kappa\delta}} e^{\Upsilon[S_{0,m},T]} \int_{\frac{1-f_{m+1}}{\delta}}^{\frac{1-f_m}{\delta}} u^{\alpha''-1} e^{-\frac{u}{3\zeta}}\, \mathrm{d}u. \end{aligned} \quad (3.16)$$

Note that $e^{\Upsilon[T/2,T]} \to 2^\alpha$ and $e^{\Upsilon[0,T]} = T^{\alpha+o(1)}$ by Lemma 5.3. Therefore, one has for sufficiently large T that

$$e^{\Upsilon[s,T]} \leqslant \begin{cases} 2^{\alpha+1} & \text{if } s \in [T/2, T] \\ T^{\alpha+1} & \text{if } s \in [0, T]. \end{cases}$$

Next, we show that the contribution of bins with $S_{0,m} < T/2$ is asymptotically negligible. By (3.15),

$$S_{0,m} = -\log \mu([f_m, f_{m+1})) \geqslant T - \log C_3 \Big(\frac{1-f_m}{\delta}\Big)^{\alpha''}$$

so that $S_{0,m} < T/2$ implies that $\frac{1-f_m}{\delta} > (e^{T/(2\alpha)}/C_3)^{1/\alpha''}$. Hence,

$$\sum_{\substack{m\colon m \geqslant m_0 \\ \text{with } S_{0,m} < T/2}} e^{\Upsilon[S_{0,m},T]} \int_{\frac{1-f_{m+1}}{\delta}}^{\frac{1-f_m}{\delta}} u^{\alpha''-1} e^{-\frac{u}{3\zeta}}\, \mathrm{d}u \leqslant T^{\alpha+1} \int_{(e^{T/(2\alpha)}/C_3)^{1/\alpha''}/\zeta}^{\infty} u^{\alpha''} e^{-\frac{u}{3\zeta}}\, \mathrm{d}u$$

which tends to zero as $t \to \infty$. It follows with (3.16) that there exists a constant C_6 not depending on κ such that, eventually,

$$\Sigma(t) \leqslant C_6 T e^{\Upsilon[T,t]-T} N(t) \int_{\kappa/\zeta}^{\infty} u^{\alpha''} e^{-\frac{u}{2\zeta}} \, \mathrm{d}u.$$

It remains to consider the stripes $m < m_0$. Note that (3.13) is true for all but finitely many bins of type (D) and we can proceed as above to control the contribution of these "nice" bins. Altogether, one ends with an estimate

$$C_7 T e^{\Upsilon[T,t]-T} N(t) \int_{\kappa/\zeta}^{\infty} u^{\alpha''} e^{-\frac{u}{2\zeta}} \, \mathrm{d}u,$$

for the contribution of all bins for which the law of large numbers applies. Note that choosing κ sufficiently large finishes the proof of the statement for the contribution of these vertices.

The remaining bins contain only finitely many vertices and each impact evolution satisfies $\log Z[s,t] \sim F_s t$ (see Remark 2.2). Conversely, one has $\log T e^{\Upsilon[T,t]-T} N(t) \sim t$ as $t \to \infty$, which shows that in general finitely many vertices are asymptotically negligible. □

Next, we control the contribution of vertices that are born after time $t/2$ with fitness smaller than $1 - \kappa/t$, again with κ denoting a large constant.

Proposition 3.9. *Suppose that Assumptions (A1) and (A2) are true. Let $\kappa > 9\alpha$. One has almost surely, that for all sufficiently large t*

$$\frac{1}{N(t)} \sum_{\substack{s \in \mathbb{T} \cap [t/2,t] \\ F_s \leqslant 1-\kappa\delta}} Z[s,t] \leqslant \frac{1}{1 - 9\alpha/\kappa} \int \frac{1}{1-f} \, \mu(\mathrm{d}f).$$

Proof. Let $\mathbb{S}(t)$ denote the indices of the bins (n,m) that contribute (at least partially) to

$$\Sigma(t) := \sum_{\substack{s \in \mathbb{T} \cap [t/2,t] \\ F_s \leqslant 1-\kappa\delta}} Z[s,t]$$

For sufficiently large t these are all of type (D) and, moreover, one has that, almost surely, for sufficiently large t the estimates of Proposition 3.5 are valid for all bins in $\mathbb{S}(t)$. Consequently, one has almost surely, for all sufficiently large t and all bins $(n,m) \in \mathbb{S}(t)$ (again we say briefly eventually) that

$$\Sigma_{n,m}(t) \leqslant e^{\varepsilon+2\iota} \mu([f_m, f_{m+1})) N(t) \int_{S_{n,m}}^{S_{n+1,m}} \exp\{-(1-f_{m+1})(t-u) + \Upsilon[u,t]\} \, \mathrm{d}u.$$

By assumption, $f_m \leqslant 1 - \kappa\delta$ for $(n,m) \in \mathbb{S}(t)$ which implies that $1 - f_{m+1} = (1 - f_m)/\zeta \geqslant \kappa\delta/\zeta$. Further, $\bar{F}_u \geqslant 1 - 4\alpha\delta$ for $u \in [t/3, t]$ as long as t is sufficiently large. Hence,

$$\Upsilon[u,t] \leqslant 4\alpha\delta(t-u) \leqslant \tfrac{4\alpha\zeta}{\kappa}(1 - f_{m+1})(t-u)$$

for $u \in [t/3, t]$ as long as t is large. Suppose that $\zeta \in (1,2)$. Then

$$\Sigma_{n,m}(t) \leqslant e^{\varepsilon+2\iota} \mu([f_m, f_{m+1})) N(t) \int_{S_{n,m}}^{S_{n+1,m}} \exp\{-(1 - \tfrac{8\alpha}{\kappa})(1 - f_{m+1})(t-u)\} \, \mathrm{d}u,$$

eventually. Since $1 - f_{m+1} \geqslant \zeta^{-1}(1-f)$ for $f \in [f_m, f_{m+1})$, we conclude that, eventually,

$$\Sigma_{n,m}(t) \leqslant e^{\varepsilon+2\iota} N(t) \int_{[f_m, f_{m+1})} \int_{S_{n,m}}^{S_{n+1,m}} \exp\{-(1 - \tfrac{8\alpha}{\kappa})\zeta^{-1}(1-f)(t-u)\} \, \mathrm{d}u \, \mu(\mathrm{d}f).$$

Each bin corresponds to an integration over the set $[f_m, f_{m+1}) \times [S_{n,m}, S_{n+1,m})$. The sets are pairwise disjoint and the union of all relevant sets is contained in the set $[0, 1-\kappa/\zeta) \times [0, t+\iota]$ so that, eventually,

$$\begin{aligned}\Sigma(t) &\leqslant e^{\varepsilon+2\iota} N(t) \int_{[0,1-\kappa/\zeta)} \int_0^{t+\iota} \exp\{-(1-\tfrac{8\alpha}{\kappa})\zeta^{-1}(1-f)(t-u)\}\,\mathrm{d}u\,\mu(\mathrm{d}f) \\ &\leqslant e^{\varepsilon+3\iota} \frac{\zeta}{1-8\alpha\kappa^{-1}}\, N(t) \int_{[0,1)} \frac{1}{1-f}\,\mu(\mathrm{d}f).\end{aligned}$$

The statement follows since $\varepsilon, \iota > 0$ and $\zeta > 1$ can be chosen arbitrarily small. □

3.6 Negligibility of early birds

The former two propositions allow to control the contribution outside the condensation window in an almost sure sense when excluding early birds. It remains to show that, with high probability, early birds, i.e., vertices in $\mathcal{E}_\gamma$, do typically not contribute.

Again we denote by $t \geqslant 0$ the time in the network formation. Depending on a parameter $\kappa > 0$, we analyse separately the contribution of vertices $s \in \mathbb{T} \cap [0,T]$ with

(I) $1 - \frac{T-s-\kappa}{t} \leqslant F_s$

(II) $F_s \leqslant 1 - \frac{(T-s-\kappa)\vee 1}{t}$

First we provide an estimate for the number of vertices of type (I).

Proposition 3.10. *Under Assumption (A1), there exists a constant $C > 0$ such that for $\kappa > 0$*

$$\limsup_{t\to\infty} \mathbb{E}\Big[\#\big\{s \in \mathbb{T} : 1 - \frac{T-s-\kappa}{t} \leqslant F_s\big\}\Big] \leqslant Ce^{-\kappa}.$$

Proof. Let $\rho(s) = \rho^{(t)}(s) := \frac{T-s-\kappa}{t}$ and denote by $N = N^{(t)}$ the random number of vertices

$$N = N^{(t)} := \#\big\{s \in \mathbb{T} : 1 - \rho(s) \leqslant F_s\big\}$$

One has

$$\mathbb{E}[N] = \sum_{\substack{s\in\mathbb{T}\\ s\,\leqslant\, s_0}} \mu\Big(\Big[1 - \frac{T-s-\kappa}{t}, 1\Big]\Big),$$

where $s_0 = s_0^{(t)} := T - \kappa$. By Lemma 5.1, one has $1 \leqslant e^s\,\Delta s$ for $s \in \mathbb{T}$ so that

$$\begin{aligned}\mathbb{E}[N] &\leqslant \mu([1-\rho(0),1)) + \sum_{\substack{s\in\mathbb{T}\\ 0<s\,\leqslant\, s_0}} e^s\,\mu([1-\rho(s),1))\,\Delta s \\ &\leqslant \mu([1-\rho(0),1)) + e\int_0^{s_0} e^s\,\mu([1-\rho(s),1))\,\mathrm{d}s.\end{aligned}$$

For all $s \in [0, s_0 - 1]$, one has $\rho(s) \geqslant \delta$ and by Potter's bound (Lemma 5.5) there is a constant $C > 1$ depending only on μ with

$$\mu([1-\rho(s),1)) \leqslant C\mu([1-\delta,1))\,(T-s-\kappa)^{\alpha+1},$$

for $s \in [0, s_0 - 1]$. Consequently,

$$\mathbb{E}[N] \leqslant \mu([1-\rho(0),1)) + C\,e\,\mu([1-\delta,1)) \int_0^{s_0} e^s\,((T-s-\kappa)\vee 1)^{\alpha+1}\,\mathrm{d}s.$$

We apply the substitution $u = T - s - \kappa$ and recall that $e^{-T} = \mu([1-\delta,1))$ to deduce that

$$\mathbb{E}[N] \leqslant \mu([1-\rho(0),1)) + C\,e\,e^{-\kappa} \int_0^\infty e^{-u}(u \vee 1)^{\alpha+1}\,\mathrm{d}u.$$

The statement follows since $\mu([1-\rho(0),1)) = \mu([1-\frac{T-\kappa}{t},1)) \to 0$ as $t \to \infty$. □

Next, we use a first moment method to bound the impact of vertices of type (II).

Proposition 3.11. *We assume Assumptions (A1) and (A2). For* $\kappa > 0$*, there exists a constant* C *such that for all sufficiently large* t*, one has*

$$\sum_{\substack{s\in\mathbb{T}\\0\leqslant s\leqslant T(t)}} \mathbb{E}\big[Z[s,t]\,\mathbb{1}_{\{F_s\,\leqslant\,1-\rho(s)\}}\big] \leqslant C\,e^{t-T(t)+\Upsilon[T(t),t]},$$

where $\rho(s) := \rho^{(t)}(s) := \frac{(T(t)-s-\kappa)\vee 1}{t}$ $(s\in[0,\infty))$.

Proof. In the following $C_1, C_2, \dots$ denote constants that do not depend on t. Further the following estimates are only valid for sufficiently large $t\in\mathbb{T}$. We analyse

$$\Sigma(t) := \sum_{s\in\mathbb{T}\cap[0,T(t)]} \mathbb{E}\big[Z[s,t]\,\mathbb{1}_{\{F_s<1-\rho(s)\}}\big]$$

By Lemmas 5.2 and 5.3, there is a constant C_1 such that one has for $s,t\in\mathbb{T}$ with $s\leqslant t$

$$\mathbb{E}[Z[s,t]|\mathcal{F}_s] = A[s,t] \leqslant C_1\,\exp\{F_s(t-s)+\Upsilon[s,t]\}.$$

By Lemma 5.6, there exists a constant C_2 such that for $s\leqslant t$ with $s\leqslant T=T(t)$

$$\mathbb{E}\big[Z[s,t]\,\mathbb{1}_{\{F_s\,\leqslant\,1-\rho(s)\}}\big] \leqslant 1 + C_2 e^{\Upsilon[s,t]}\exp\{(1-\rho(s))(t-s)\}\,\mu([1-\rho(s),1)).$$

Consequently, using that $1\leqslant e^s\Delta s$ we get

$$\begin{aligned}\Sigma(t) &= \sum_{\substack{s\in\mathbb{T}\\0\leqslant s\leqslant T}} \mathbb{E}\big[Z[s,t]\,\mathbb{1}_{\{F_s\,\leqslant\,1-\rho(s)\}}\big]\\ &\leqslant N(T) + C_2\,e^{t+\Upsilon[T,t]} \sum_{\substack{s\in\mathbb{T}\\0\leqslant s\leqslant T}} e^{-\rho(s)(t-s)+\Upsilon[s,T]}\,\mu([1-\rho(s),1))\,\Delta s\\ &=: \Sigma_1(t)+\Sigma_2(t).\end{aligned}$$

Note that $\Sigma_1(t)\sim C_N e^{T(t)}\sim C_N\,t^\alpha/\ell(t^{-1})$ is of negligible order and we restrict attention to $\Sigma_2(t)$ in the following. The map $s\mapsto\rho(s)(t-s)$ is decreasing on $[0,t]$ and one has for $0\leqslant s\leqslant s'\leqslant s+1$, $\rho(s)\leqslant 2\rho(s')$ so that by the Potter bound (Lemma 5.5)

$$\mu([1-\rho(s),1)) \leqslant C_3\,\mu([1-\rho(s'),1))$$

for a constant C_3. Using further that $\Upsilon[s,s']\leqslant 1$ we conclude that

$$\Sigma_2(t) \leqslant C_4\,e^{t+\Upsilon[T,t]} \int_0^{T+1} e^{-\rho(s)(t-s)+\Upsilon[s,T+1]}\,\mu([1-\rho(s),1))\,\mathrm{d}s.$$

For sufficiently large t, one has $t\geqslant 2(T+1)$ and it follows that, for $s\in[0,T+1]$, $\rho(s)(t-s)\geqslant\frac{1}{2}[(T-s-\kappa)\vee 1]$. Further, by Lemma 5.3, there exists a constant $\alpha'>\alpha$ not depending on t such that $e^{\Upsilon[s,T+1]}\leqslant\big(\frac{T+2}{s+1}\big)^{\alpha'}$ for all $s\leqslant T$. Hence, for large t

$$\Sigma_2(t) \leqslant C_4\,e^{t+\Upsilon[T,t]} \int_0^{T+1} e^{-\frac{1}{2}[(T-s-\kappa)\vee 1]}\Big(\frac{T+2}{s+1}\Big)^{\alpha'}\mu([1-\rho(s),1))\,\mathrm{d}s$$

and the Potter bound implies that for a further constant C_5

$$\Sigma_2(t) \leqslant C_5\,e^{t+\Upsilon[T,t]}\,\mu([1-\delta,1)) \int_0^{T+1} e^{-\frac{1}{2}[(T-s-\kappa)\vee 1]}\Big(\frac{T+2}{s+1}\Big)^{\alpha'}((T-s-\kappa)\vee 1)^{\alpha'}\,\mathrm{d}s.$$

If T is sufficiently large to ensure that $\frac{T+2}{T-\kappa-1+1} \leqslant 2$ one gets that

$$\frac{T+2}{s+1} \leqslant 2[(T-s-\kappa)\vee 1] \text{ for all } s \geqslant 0.$$

Hence, there exists a constant C_6 such that for large t

$$\begin{aligned}\Sigma_2(t) &\leqslant C_6\, e^{t+\Upsilon[T,t]}\,\mu([1-\delta,1))\int_0^{T+1} e^{-\frac{1}{2}[(T-s-\kappa)\vee 1]}((T-s-\kappa)\vee 1)^{2\alpha'}\,\mathrm{d}s\\ &\leqslant C_6\, e^{t+\Upsilon[T,t]}\,\mu([1-\delta,1))\int_{-1}^{\infty} e^{-\frac{1}{2}[(u-\kappa)\vee 1]}((u-\kappa)\vee 1)^{2\alpha'}\,\mathrm{d}u,\end{aligned}$$

where we applied the substitution $u=T-s$ in the latter transformation. Clearly, the latter integral is finite and the statement follows since $\mu([1-\delta,1))=e^{-T}$, by definition. □

The combination of the above propositions allows us to conclude that for large times, typically, early birds have no impact:

Proposition 3.12. *For any $\varepsilon>0$, one has*

$$\lim_{t\to\infty}\frac{1}{T(t)\,e^{t+\Upsilon[T(t),t]-T(t)}}\sum_{\substack{s\in\mathbb{T}\cap[0,T(t)]:\\ F_s\,\leqslant\,1-1/t}} Z[s,t]=0, \quad \textit{in probability.}$$

Proof. We fix $\kappa>0$ and note that any vertex $s\in[0,T]\cap\mathbb{T}$ with $F_s\leqslant 1-\delta$ satisfies property (I) or (II) above. Hence, there exists a constants C_1 not depending on κ and a further constant C_2 such that for every $\varepsilon>0$ and t large

$$\begin{aligned}\mathbb{P}\Big(\sum_{\substack{s\in\mathbb{T}\cap[0,T(t)]:\\ F_s\,\leqslant\,1-1/t}} Z[s,t]\geqslant \varepsilon T(t)\,e^{t+\Upsilon[T(t),t]-T(t)}\Big)&\\ \leqslant \mathbb{P}\Big(\Big\{\exists s\in\mathbb{T}\cap[0,T] \text{ with } 1-\frac{T-s-\kappa}{t}\leqslant F_s\Big\}\Big)&\\ +\mathbb{P}\Big(\sum_{\substack{s\in\mathbb{T}\cap[0,T]\\ F_s\,\leqslant\,1-\rho(s)}} Z[s,t]\geqslant \varepsilon T(t)\,e^{t+\Upsilon[T(t),t]-T(t)}\Big)&\\ \leqslant C_1e^{-\kappa}+C_2\frac{1}{\varepsilon\,T(t)},&\end{aligned}$$

Here we used Proposition 3.10 to bound the first term and Proposition 3.11 together with the Markov inequality to bound the second term. The statement follows since the second term tends to 0 and the first one can be made arbitrarily small by choosing κ large. □

4 Proof of the main theorems

We start with proving Theorem 1.6, that is we prove that for $\gamma\in(0,1)$, almost surely, the scaled impact distributions with discarded early birds

$$\Gamma_t^* = \frac{1}{N(t)}\sum_{\substack{s\in\mathbb{T}\\ s\,\leqslant\,t}} \mathbb{1}_{\mathcal{E}_\gamma^c}(s)\,\delta_{(s/T,(1-F_s)t)}$$

converge, almost surely, in an appropriate topology to γ_α. Recall that

$$\mathcal{E}_\gamma=\{s\in\mathbb{T}:(\log(1-F_s)^{-1})^\gamma\,N(s)\leqslant\mu([F_s,1))^{-1}\}.$$

Proof of Theorem 1.6. Step 1: We first prove that for arbitrary bounded intervals $A = (a_0, a_1] \subset \mathbb{R}_+$ and $B = (b_0, b_1] \subset \mathbb{R}_+$

$$\liminf_{t\to\infty} \frac{1}{T(t)\, e^{\Upsilon[T(t),t]-T}}\, \Gamma_t^*(A\times B) \geqslant \gamma_\alpha(A\times B), \text{ almost surely.} \tag{4.1}$$

Since γ_α is supported on $(1,\infty)\times(0,\infty)$, we can and will assume without loss of generality that $1 < a_0 < a_1$ and $0 < b_0 < b_1$.

We use the concepts of bins introduced in Section 3 with fixed $\zeta \in (1, \gamma^{-1})$ and $\iota > 0$. Associate each bin $\mathbb{I}_{n,m}$ with a cube $C_{n,m} := [f_m, f_{m+1}) \times (S_{n,m}, S_{n+1,m}]$ and its t-scaled analogue

$$C_{n,m;t} := T^{-1}(S_{n,m}, S_{n+1,m}] \times t(1-f_{m+1}, 1-f_m].$$

We denote by $\mathbb{S}(t)$ the set of bins (n,m) with

$$C_{n,m} \subset T\,A \times (1 - B/t) \text{ or, equivalently, } C_{n,m;t} \subset A\times B.$$

These bins contribute to $\Gamma_t^*(A\times B)$. First we need to convince ourselves that the vertices of these bins are for sufficiently large t of type (D) which then allows us to apply the estimates of Proposition 3.5. One has for $(n,m)\in\mathbb{S}(t)$

$$1 - f_m \approx 1/t \text{ and } S_{n,m} \approx T \approx \log t \tag{4.2}$$

as $t\to\infty$, so that on $\mathbb{S}(t)$

$$\log \mu([f_m, f_{m+1}))^{-1} + \iota\, n_0(m) \sim \log\mu([f_m,1])^{-1} \sim \log\mu([1-t^{-1},1])^{-1} = T(t) \tag{4.3}$$

as $t\to\infty$. For $(n,m)\in\mathbb{S}(t)$ one has $S_{n,m} \geqslant a_0 T$ with $a_0 > 1$ so that (4.3) implies that $n \geqslant n_0(m)$ for sufficiently large m.

By Proposition 3.5 one has, almost surely, for sufficiently large $t\in\mathbb{T}$ for all $(n,m)\in\mathbb{S}(t)$ (we again say eventually) that

$$\Sigma_{n,m}(t) \geqslant e^{-\varepsilon-2\iota}\mu([f_m, f_{m+1}))\, N(t) \int_{S_{n,m}}^{S_{n+1,m}} \exp\{-(1-f_m)(t-u) + f_m\,\Upsilon[u,t]\}\,\mathrm{d}u.$$

Recalling that on $\mathbb{S}(t)$, $1 - f_m \approx 1/t$, we conclude with Lemma 5.3 that $(1-f_m)\Upsilon[u,t]\to 0$ uniformly for all bins in $\mathbb{S}(t)$ and u in the respective domains of integration. Hence, one has eventually that

$$\Sigma_{n,m}(t) \geqslant e^{-2\varepsilon-2\iota}\mu([f_m, f_{m+1}))\, N(t)\, e^{-(1-f_m)t} \int_{S_{n,m}}^{S_{n+1,m}} e^{\Upsilon[u,t]}\,\mathrm{d}u.$$

The domain of integration is for sufficiently large t to the right of T so that we have $\Upsilon[u,t] = \Upsilon[T,t] - \Upsilon[T,u]$. Further the domain is of order T (uniformly on $\mathbb{S}(t)$) so that $e^{\Upsilon[T,u]} \leqslant e^{\varepsilon}\,(u/T)^\alpha$ by Lemma 5.3. Hence, eventually

$$\Sigma_{n,m}(t) \geqslant e^{-3\varepsilon-2\iota}\mu([f_m, f_{m+1}))\, N(t)\, e^{\Upsilon[T,t]}\, e^{-(1-f_m)t} \int_{S_{n,m}}^{S_{n+1,m}} (u/T)^\alpha\,\mathrm{d}u.$$

Using that

$$\mu([f_m, f_{m+1})) = \mu([f_m,1)) - \mu([f_{m+1},1)) \sim \int_{f_m}^{f_{m+1}} \alpha(1-f)^{\alpha-1}\,\mathrm{d}f\ \ell(t^{-1}), \tag{4.4}$$

$(1-f_m)t \leqslant b_1$ and $1-f_m = \zeta(1-f_{m+1})$ we get that

$$\begin{aligned}
&\Sigma_{n,m}(t) \\
&\quad \geqslant \alpha\, e^{-4\varepsilon-2\iota-(\zeta-1)b_1}\, N(t)\, e^{\Upsilon[T,t]} \int_{f_m}^{f_{m+1}} e^{-(1-f)t}\,(1-f)^{\alpha-1}\,\mathrm{d}f\;\ell(t^{-1}) \int_{S_{n,m}}^{S_{n+1,m}} (u/T)^\alpha\,\mathrm{d}u \\
&\quad = \alpha\, e^{-4\varepsilon-2\iota-(\zeta-1)b_1}\, N(t)\, e^{\Upsilon[T,t]} \int_{C_{n,m}} e^{-(1-f)t}\,(1-f)^{\alpha-1}\;\ell(t^{-1})(u/T)^\alpha\,\mathrm{d}(u,f) \\
&\quad = e^{-4\varepsilon-2\iota-(\zeta-1)b_1}\, N(t)\, e^{\Upsilon[T,t]}\, \underbrace{t^{-\alpha}\ell(t^{-1})}_{e^{-T}}\, T \int_{C_{n,m;t}} \alpha e^{-g}\, g^{\alpha-1}\, v^\alpha\,\mathrm{d}(v,g),
\end{aligned}$$

where we substituted (u,f) by (v,g) with $v=u/T$ and $g=(1-f)t$ in the latter step. Consequently, eventually,

$$\Gamma_t^*(A\times B) \geqslant \frac{1}{N(t)} \sum_{(n,m)\in\mathbb{S}(t)} \Sigma_{n,m}(t) \geqslant e^{-4\varepsilon-2\iota-(\zeta-1)b_1}\, T\, e^{\Upsilon[T,t]-T}\, \gamma_\alpha\Big(\bigcup_{(n,m)\in\mathbb{S}(t)} C_{n,m;t}\Big).$$

Note that for large t, the cubes $(C_{n,m;t} : (n,m)\in\mathbb{S}(t))$ cover

$$[a_0+\varepsilon, a_1-\varepsilon]\times[b_0\zeta, b_1/\zeta].$$

Since the measure γ_α has Lebesgue density and since $\varepsilon>0$ and $\zeta>1$ can be chosen arbitrarily small it follows validity of (4.1).

Step 2: In the next step we show that for $K=[0,b]$ with $b>0$ one has that, almost surely,

$$\limsup_{t\to\infty} \frac{1}{T\, e^{\Upsilon[T,t]-T}}\, \Gamma_t^*(\mathbb{R}\times K) \leqslant \gamma_\alpha(\mathbb{R}\times K).$$

By Lemma 3.2, the vertices that contribute to $\Gamma_t^*(\mathbb{R}\times K)$ are of type (C) and (D) provided that t is sufficiently large. We analyse

$$\Sigma_m(t) := \sum_{n=-n_0'(m)}^{\infty} \sum_{s\in\mathbb{I}_{n,m}} Z[s,t]$$

for the t-dependent set of indices

$$\mathbb{S}(t) := \{m\in\mathbb{N} : (1-f_{m+1})t \leqslant b\}.$$

By Proposition 3.7, one has that, almost surely, for sufficiently large t, for all $m\in\mathbb{S}(t)$, (we briefly say eventually)

$$\Sigma_m(t) \leqslant \frac{e^{2\iota+\varepsilon}}{\alpha-1}\mu([f_m,f_{m+1}))S_{0,m}e^{-(1-f_{m+1})t+\Upsilon[S_{0,m},t]}N(t). \tag{4.5}$$

As $t\to\infty$ the terms T and $S_{0,m}$ with $m\in\mathbb{S}(t)$ tend to infinity so that by Lemma 5.3

$$S_{0,m}\, e^{\Upsilon[S_{0,m},t]} = \Big(\frac{T}{S_{0,m}}\Big)^{\alpha-1+o(1)} T\, e^{\Upsilon[T,t]}.$$

Since for $m\in\mathbb{S}(t)$, $e^{-S_{0,m}} = \mu([f_m,f_{m+1})) \leqslant \mu([1-(\zeta b)/t,1)) \approx e^{-T}$, there exists a constant C such that for sufficiently large t, $S_{0,m}\geqslant T-C$ on $\mathbb{S}(t)$, and we conclude that, eventually,

$$S_{0,m}\, e^{\Upsilon[S_{0,m},t]} \leqslant e^{\varepsilon}\, T\, e^{\Upsilon[T,t]}. \tag{4.6}$$

Now we distinguish two cases. First we consider the contribution of $m \in \mathbb{S}(t)$ with $f_m \geqslant 1 - t^{-1}$ in which case $S_{0,m} \geqslant T$. We apply the Potter bound and note that for arbitrarily fixed $\alpha' \in (0, \alpha)$ one has for sufficiently large t that

$$\mu([f_m, f_{m+1})) \leqslant e^{\varepsilon} \mu([f_m, 1))(1 - \zeta^{-\alpha}) \leq e^{2\varepsilon} \mu([1 - t^{-1}, 1))((1 - f_m)t)^{\alpha'}(1 - \zeta^{-\alpha})$$
$$= e^{2\varepsilon} \frac{1 - \zeta^{-\alpha}}{1 - \zeta^{-\alpha'}}\, e^{-T}\, \alpha' \int_{(1-f_{m+1})t}^{(1-f_m)t} f^{\alpha'-1}\, \mathrm{d}f.$$

Combining this estimates with (4.5) and (4.6), we conclude that, eventually,

$$\Sigma_m(t) \leqslant \frac{e^{2\iota+5\varepsilon}}{\alpha - 1} \frac{1 - \zeta^{-\alpha}}{1 - \zeta^{-\alpha'}}\, T\, e^{\Upsilon[T,t]-T}\, \alpha' \int_{(1-f_{m+1})t}^{(1-f_m)t} f^{\alpha'-1}\, e^{-\zeta^{-1} f}\, \mathrm{d}f\, N(t).$$

Next, we consider the contribution of $m \in \mathbb{S}(t)$ with $f_m < 1 - t^{-1}$. In this case we can use as in step one of the proof, see (4.4), that $\mu([f_m, f_{m+1})) \leqslant e^{\varepsilon} \int_{f_m}^{f_{m+1}} \alpha(1 - f)^{\alpha-1}\, \mathrm{d}f\, \ell(t^{-1})$ to deduce that, eventually,

$$\Sigma_m(t) \leqslant \frac{e^{2\iota+4\varepsilon}}{\alpha - 1}\, T\, e^{\Upsilon[T,t]-T}\, \alpha \int_{(1-f_{m+1})t}^{(1-f_m)t} f^{\alpha-1}\, e^{-\zeta^{-1} f}\, \mathrm{d}f\, N(t).$$

Altogether we get that, eventually,

$$\Sigma_m(t) \leqslant \frac{e^{2\iota+5\varepsilon}}{\alpha - 1} \frac{1 - \zeta^{-\alpha}}{1 - \zeta^{-\alpha'}}\, T\, e^{\Upsilon[T,t]-T}\, \alpha \int_{(1-f_{m+1})t}^{(1-f_m)t} (f^{\alpha-1} \vee f^{\alpha'-1})\, e^{-\zeta^{-1} f}\, \mathrm{d}f\, N(t).$$

irrespective of the case. Consequently, it follows that almost surely for sufficiently large t

$$\Gamma_t^*(\mathbb{R} \times K) \leqslant \frac{\sum_{m \in \mathbb{S}(t)} \Sigma_m(t)}{N(t)} \leqslant \frac{e^{2\iota+5\varepsilon}}{\alpha - 1} \frac{1 - \zeta^{-\alpha}}{1 - \zeta^{-\alpha'}}\, T\, e^{\Upsilon[T,t]-T}\, \alpha \int_0^{\zeta b} (f^{\alpha-1} \vee f^{\alpha'-1})\, e^{-\zeta^{-1} f}\, \mathrm{d}f.$$

By choosing $\varepsilon, \iota > 0$ and $\zeta > 1$ small and $\alpha' \in (0, \alpha)$ large, we get that, almost surely,

$$\limsup_{t \to \infty} \frac{1}{T\, e^{\Upsilon[T,t]-T}} \Gamma_t^*(\mathbb{R}_+ \times K) \leq \frac{\alpha}{\alpha - 1} \int_0^b f^{\alpha-1}\, e^{-f}\, \mathrm{d}f = \gamma_\alpha(\mathbb{R} \times K).$$

Step 3: Let $K \subset \mathbb{R}_+$ be a bounded open interval. We prove that, almost surely,

$$\lim_{t \to \infty} \frac{1}{T\, e^{\Upsilon[T,t]-T}}\, \Gamma_t^*\big|_{\mathbb{R} \times K} = \gamma_\alpha\big|_{\mathbb{R} \times K},$$

in the weak topology. Without loss of generality, we can assume that $K = (0, b)$ with $b \in \mathbb{Q} \cap (0, \infty)$, since we make the statement stronger by enlarging the set K and since neither of the measures puts mass on $\mathbb{R} \times (-\infty, 0]$. (Indeed, for every open interval K the boundary of $\mathbb{R} \times K$ is a γ_α-zero set and hence restrictions to such sets preserve weak convergence.) By step 1, we have that, almost surely,

$$\liminf_{t \to \infty} \frac{\Gamma_t^*((a_0, a_1] \times (b_0, b_1])}{T\, e^{\Upsilon[T,t]-T}} \geqslant \gamma_\alpha((a_0, a_1] \times (b_0, b_1]), \tag{4.7}$$

for all $a_0, a_1, b_0, b_1 \in \mathbb{Q}$ with $0 < a_0 < a_1$ and $0 < b_0 < b_1$. Say this property is true on the almost sure set Ω_0. We denote by $\mathcal{K}$ the collection of all sets that can be represented as finite disjoint unions of such cubes. Then clearly the statement (4.7) remains true on Ω_0 for all sets from $\mathcal{K}$. Let now $U \subset \mathbb{R}^2$ be an arbitrary open set. It is straight-forward to construct an increasing sequence $(U_n)_{n \in \mathbb{N}}$ of sets from $\mathcal{K}$ with $\bigcup_{n \in \mathbb{N}} U_n = U \cap (0, \infty)^2$.

Since on one hand, by monotone convergence $\gamma_\alpha(U_n) \to \gamma_\alpha(U \cap (0,\infty)^2) = \gamma_\alpha(U)$ and, on the other hand, $\liminf \Gamma_t^*(U) \geqslant \liminf \Gamma_t^*(U_n) \geqslant \gamma_\alpha(U_n)$ on Ω_0, we conclude that on Ω_0

$$\liminf_{t\to\infty} \frac{\Gamma_t^*(U)}{T\, e^{\Upsilon[T,t]-T}} \geqslant \gamma_\alpha(U).$$

In order to apply Portmanteau's thorem, we still need to verify that

$$\lim_{t\to\infty} \frac{\Gamma_t^*(\mathbb{R}\times K)}{T\, e^{\Upsilon[T,t]-T}} = \gamma_\alpha(\mathbb{R}\times K)$$

and that the right hand side is finite. The finiteness is clear and the lower bound follows immediately by choosing $U = \mathbb{R}\times(0,b)$ above. The converse estimate is an immediate consequence of step 2. □

Proof of Theorem 1.4. In view of Theorem 1.6, it suffices to show that for $a > 0$

$$\lim_{t\to\infty} \frac{1}{T\, e^{\Upsilon[T,t]-T}\, N(t)} \sum_{s\in\mathbb{T}} \mathbb{1}_{\mathcal{E}_\gamma}(s)\, \mathbb{1}_{[1-\delta a,1)}(F_s)\, Z[s,t] = 0, \text{ in probability.} \tag{4.8}$$

Again we make use of the binning techniques introduced in Section 3. We choose as parameters $\iota > 0$ and $\zeta \in (1,\gamma^{-1})$ and let $\varepsilon > 0$ denote an arbitrarily small number. We distinguish three cases.

First we consider the contribution of stripes m with index in

$$\mathbb{S}(t) = \{m \in \mathbb{N}_0 : f_{m+1} \geqslant 1 - a\delta(t), f_m < 1 - \varepsilon\delta(t)\}.$$

By construction the number of indices in $\mathbb{S}_1(t)$ is uniformly bounded by $2 + \frac{\log(a/\varepsilon)}{\log\zeta}$. Further, by Lemma 3.2, one has for sufficiently large t that

$$\begin{aligned}&\big\{\exists s \in \mathcal{E}_\gamma \text{ that belongs to a stripe in } \mathbb{S}(t)\big\}\\ &\qquad\qquad \subset \big\{\exists \text{ stripe in } \mathbb{S}(t) \text{ containing a vertex of type (A) or (B)}\big\}\end{aligned}$$

and for $m \in \mathbb{N}_0$ large

$$\begin{aligned}&\mathbb{P}(\text{stripe } m \text{ contains a vertex of type (A) or (B)}) \leqslant N(S_{m,-n_0'(m)})\, \mu([f_m, f_{m+1}))\\ &\quad \sim C_N \exp\{-\iota n_0'(m)\} \to 0.\end{aligned}$$

Hence with probability tending to one all stripes in $\mathbb{S}(t)$ do not contain vertices from $\mathcal{E}_\gamma$.

Second, we consider the contribution of vertices $s \in \mathbb{T}$ with fitness $F_s \geqslant 1 - \varepsilon t^{-1}$ that are born before T. Using the regular variation of the tails of μ we get that

$$\begin{aligned}\mathbb{P}(\exists s \in \mathbb{T}\cap[0,T] \text{ with } F_s \geqslant 1-\varepsilon t^{-1}) &\leqslant N(T)\, \mu([1-\varepsilon\delta,1))\\ &\sim C_N\, \frac{\mu([1-\varepsilon\delta,1))}{\mu([1-\delta,1))} \to C_N\, \varepsilon^{-\alpha}.\end{aligned}$$

Third we control the contribution of all vertices with fitness $F_s \geqslant 1 - \varepsilon t^{-1}$ born after time T with the help of Proposition 2.3. Set

$$\Sigma_\varepsilon(t) := \sum_{s\in\mathbb{T}\cap[T,t)} \mathbb{1}_{\{F_s \,\geqslant\, 1-\varepsilon t^{-1}\}} Z[s,t].$$

The process $\Sigma_\varepsilon(t)$ is stochastically dominated by $A_t^{1,T^*} \bar{M}_t^{1,p,T^*,t}$ (see Section 2.2 for its definition), where

$$p = \mu([1-\varepsilon t^{-1}, 1)) \text{ and } T^* = \min \mathbb{T}\cap[T,\infty).$$

Now Proposition 2.3 implies that for a constant C not depending on $\varepsilon > 0$

$$\mathbb{E}[\Sigma_\varepsilon(t)] \leqslant C\, p\, A_t^{1,T^*} e^{T^*} \int_{T^*}^t \exp\{-\Upsilon[T^*,u]\}\,\mathrm{d}u.$$

As a consequence of Lemma 5.3, one obtains that

$$\int_{T^*}^t \exp\{-\Upsilon[T^*,u]\}\,\mathrm{d}u \sim \int_T^\infty \big(\frac{u}{T}\big)^{-\alpha}\,\mathrm{d}u = \frac{1}{\alpha-1}\,T.$$

Further, by Lemma 5.3, $A_t^{1,T^*} \sim \exp\{a[T,t]\} \sim \exp\{t - T + \Upsilon[T,t]\}$ and $e^{T^*} \sim e^T$. Consequently,

$$\mathbb{E}[\Sigma_\varepsilon(t)] \lesssim \frac{C}{\alpha-1}\,T\,\exp\{t+\Upsilon[T,t]\}\,\mu([1-\varepsilon t^{-1},1)) \sim \frac{C}{C_N(\alpha-1)}\,\varepsilon^\alpha\,T\,e^{\Upsilon[T,t]-T}N(t),$$

where we used that $\mu([1-\varepsilon\delta,1)) \sim \varepsilon^\alpha\,\mu([1-\delta,1)) = \varepsilon^\alpha e^{-T}$ in the last step. Since $\varepsilon > 0$ can be chosen arbitrarily small it is now straight-forward to combine the estimates of the three cases and deduce the statement of the theorem. ☐

Proof of Theorem 1.7. Similarly as in the proof of Theorem 1.6, one can deduce the statement of Theorem 1.7 from the following two properties:

1. For every rectangle $A = [s_1,s_2)\times[f_1,f_2)$ with $0 \leqslant s_1 < s_2$ and $0 \leqslant f_1 < f_2 < 1$, one has almost surely

$$\liminf_{n\to\infty} \Phi_t(A) \geqslant e^{-(1-f_1)s_2}\,(s_2-s_1)\,\mu([f_1,f_2))$$

2. For all $0 \leqslant f_1 < f_2 < 1$, one has almost surely that

$$\limsup_{t\to\infty} \Phi_t(\mathbb{R}\times[f_1,f_2)) \leqslant \frac{1}{1-f_2}\,\mu([f_1,f_2)).$$

We start with proving the first statement. It suffices to consider the case with $\mu([f_1,f_2)) > 0$. Fix $\varepsilon > 0$ and $\iota \in (0,(s_2-s_1)/2)$ and denote for $n \in \mathbb{N}$

$$\mathbb{I}_n := \{s \in \mathbb{T}\cap((n-1)\iota, n\iota] : F_s \in [f_1,f_2)\} \;\text{ and }\; \Sigma_n(t) = \sum_{s\in\mathbb{I}_n} Z[s,t].$$

In complete analogy to the binning analysed in Section 3, see Remark 3.4, one notices that almost surely for all but finitely many $n \in \mathbb{N}$ for all $t \geqslant n\iota$ (we briefly say eventually)

$$\Sigma_n(t) \geqslant e^{-\varepsilon}\,\mu([f_1,f_2))\,\mathbb{E}[\#\mathbb{I}_n]\,\exp\{f_1 a[n\iota,t]\}.$$

By Lemma 5.1, $\mathbb{E}[\#\mathbb{I}_n] \sim C_N e^{\iota n}(1-e^{-\iota})$ and we assume that $\iota > 0$ is chosen sufficiently small to ensure that $(1-e^{-\iota}) > e^{-\varepsilon}\iota$. Since $a[n\iota,t] \geqslant e^{-\varepsilon}(t-n\iota)$ as long as n is sufficiently large we get that eventually

$$\Sigma_n(t) \geqslant e^{-2\varepsilon} C_N\,\iota\,e^{\iota n}\,\mu([f_1,f_2))\,\exp\{e^{-\varepsilon} f_1(t-n\iota)\}.$$

For given $t \in \mathbb{T}$ the vertices in bins $\mathbb{I}_n$ with

$$((n-1)\iota, n\iota] \subset (t-s_2, t-s_1]$$

and using that $N(t) \leqslant e^\varepsilon C_N e^t$ for sufficiently large t we get that almost surely

$$\Phi_t(A) \geqslant \sum_{\substack{n\in\mathbb{N}\\ ((n-1)\iota,n\iota]\subset(t-s_2,t-s_1]}} \frac{\Sigma_n(t)}{N(t)} \geqslant e^{-3\varepsilon}\,e^{-(1-e^{-\varepsilon}f_1)s_2}\,(s_2-s_1-2\iota)\,\mu([f_1,f_2))$$

for large t. Since ι and ε can be chosen arbitrarily small this proves the first statement.

The proof of the second statement is similar to the proof of the first one. Using the same bins as in part one we get with analogous reasoning that almost surely for sufficiently large $n \in \mathbb{N}$ and all $t \geqslant (n-1)\iota$ (we briefly say eventually)

$$\Sigma_n(t) \leqslant e^{\varepsilon}\, \mu([f_1, f_2))\, \mathbb{E}[\#\mathbb{I}_n]\, \exp\{f_2\, a[(n-1)\iota, t]\}.$$

For sufficiently large n, we have $\mathbb{E}[\#\mathbb{I}_n] \leqslant e^{\varepsilon} C_N (1-e^{-\iota}) e^{\iota n} \leqslant e^{\varepsilon} C_N \iota e^{\iota n}$ and $a[(n-1)\iota, t] \leqslant e^{\varepsilon}(t-(n-1)\iota)$ so that eventually

$$\Sigma_n(t) \leqslant e^{2\varepsilon}\, C_N\, \mu([f_1, f_2))\, e^{e^{\varepsilon} f_2 (t+\iota)}\, \iota\, e^{\iota n (1-e^{\varepsilon} f_2)}. \tag{4.9}$$

We assume that ε s sufficiently small to ensure that $e^{\varepsilon} f_2 < 1$ and use that

$$\iota\, e^{\iota n(1-e^{\varepsilon} f_2)} \leqslant e^{\iota} \int_{(n-1)\iota}^{n\iota} e^{(1-e^{\varepsilon} f_2) u}\, \mathrm{d}u$$

We denote by n_0 a random index for which (4.9) is true for all $n \geqslant n_0$ and conclude that

$$\sum_{n=n_0}^{\lceil t/\iota \rceil} \Sigma_n(t) \leqslant e^{2\varepsilon + 2\iota}\, C_N\, \mu([f_1, f_2))\, e^{e^{\varepsilon} f_2 t} \int_{-\infty}^{t+\iota} e^{(1-e^{\varepsilon} f_2) u}\, \mathrm{d}u.$$

Evaluating the integral and using that $N(t) \geqslant e^{-\varepsilon} C_N e^{t}$ for sufficiently large t, we get that eventually

$$\sum_{n=n_0}^{\lceil t/\iota \rceil} \Sigma_n(t) \leqslant e^{2\varepsilon+3\iota} \frac{1}{1-e^{\varepsilon} f_2}\, \mu([f_1, f_2))\, N(t).$$

Since the vertices born before time $n_0 \iota$ have an asymptotically negligible influence, see Remark 2.2, the second statement follows by noticing that ε and ι can be chosen arbitrarily small. □

Proof of Theorem 1.2. In view of Theorems 1.1 and 1.4 it suffices to show that for every $\varepsilon > 0$

$$\lim_{t\to\infty} \mathbb{P}\Big(\Xi_t([0,1)) \leqslant \int \frac{1}{1-f}\, \mathrm{d}\mu + \tfrac{\alpha}{\alpha-1}\, \Gamma(\alpha)\, w + \varepsilon\Big) = 1$$

By Propositions 3.8 and 3.12 together with Assumption COEX, we conclude that for sufficiently large $\kappa > 0$,

$$\mathbb{P}\Big(\sum_{\substack{s \in \mathbb{T} \cap [0, t/2] \\ F_s \,\leqslant\, \kappa/t}} Z[s,t] \leqslant \frac{\varepsilon}{3} N(t) \Big) \to 1.$$

Further assuming that κ is sufficiently large we conclude with Proposition 3.9 that, almost surely, for large $t \in \mathbb{T}$

$$\sum_{\substack{s \in \mathbb{T} \cap [t/2, t] \\ F_s \,\leqslant\, \kappa/t}} Z[s,t] \leqslant \Big(\int \frac{1}{1-f}\, \mathrm{d}\mu(f) + \frac{\varepsilon}{3} \Big)\, N(t).$$

The result follows by recalling that by Theorem 1.4, almost surely for large $t \in \mathbb{T}$

$$\sum_{\substack{s \in \mathbb{T} \\ F_s \,\geqslant\, 1-\kappa/t}} Z[s,t] \leqslant (w + \frac{\varepsilon}{3})\, N(t).$$

When disregarding the early birds, the estimate is also true in the almost sure sense. □

5 Appendix

Lemma 5.1 (Technical estimates)**.** *1. One has, for* $n \in \mathbb{N}$*,*

$$\ln n \leqslant \pi(n) \leqslant \ln(2n-1)$$

and for $t \geqslant 0$*,*

$$e^{t-1} < N(t) \leqslant e^t$$

2. There exists a positive constant C_N *such that*

$$N(t) \sim C_N e^t$$

Proof. 1.) Using the convexity of $x \mapsto 1/x$ on $(0,\infty)$, we get that

$$\pi(n) \leqslant \int_{\frac{1}{2}}^{n-\frac{1}{2}} \frac{1}{x}\,\mathrm{d}x = \ln(2n-1)$$

for $n \in \mathbb{N}$. Further, $\pi(n) \geqslant \int_1^n \frac{1}{x}\,\mathrm{d}x = \ln(n)$. For $t \geqslant 0$, one has $\pi(N(t)) \leqslant t$, which implies with the lower bound for $\pi(n)$ that $N(t) \leqslant e^t$. The converse estimate is an immediate consequence of the estimate

$$\pi(n) \leqslant 1 + \int_1^{n-1} \frac{1}{x}\,\mathrm{d}x \leqslant 1 + \ln(n-1) \qquad (n \geqslant 2)$$

and $\pi(N(t)+1) > t$.

2.) Since

$$|\pi(n+1) - \pi(n) - (\log(n+1) - \log n)| \leqslant \frac{1}{n^2}$$

is summable, there exists a constant C such that

$$\log n = \pi(n) + C + o(1).$$

As t tends to infinity, $N(t)$ tends to infinity and we get with $\pi(N(t)) = t + o(1)$ that

$$\log N(t) = t + C + o(1)$$

so that applying an exponential yields

$$N(t) \sim e^C e^t. \qquad \square$$

Often we consider expressions of the form

$$A_t = A_t^{f,s} = \prod_{\substack{u \in \mathbb{T} \\ s \leqslant u < t}} \Big(1 + \frac{f}{\bar{F}_u} \Delta u\Big)$$

for $t \in \mathbb{T}$ with $t \geqslant s$, where $s \in \mathbb{T}$ and $f \in (0,1]$ are parameters.

Lemma 5.2. *Let* $f \in (0,1]$ *and* $s \in \mathbb{T}$ *with* $\bar{F}_u \geqslant \frac{1}{2}$ *for all* $u \geqslant s$*. Then one has for* $t \in \mathbb{T} \cap [s,\infty)$

$$e^{-4\Delta s} \exp\{f\, a[s,t]\} \leqslant A_t^{f,s} \leqslant \exp\{f\, a[s,t]\}$$

where

$$a[s,t] = \sum_{\substack{u \in \mathbb{T} \\ s \leqslant u < t}} \frac{1}{\bar{F}_u} \Delta u.$$

Proof. Using the concavity of the logarithm one gets

$$\log A_t = \sum_{u\in[s,t)\cap\mathbb{T}} \log\Big(1+\frac{f}{\bar F_u}\Delta u\Big) \le f a[s,t].$$

Conversely, since $\log(1+u) \ge u - \frac{1}{2}u^2$ for $u \geqslant 0$, one has

$$\log A_t \ge f\, a[s,t] - \frac{1}{2}\sum_{u\in[s,t)\cap\mathbb{T}} \frac{f^2}{\bar F_u^2}(\Delta u)^2 \ge f a[s,t] - 2\sum_{u\in[s,t)\cap\mathbb{T}} (\Delta u)^2 \ge f\, a[s,t] - 4\Delta s. \quad \square$$

Let Υ be defined via

$$\Upsilon[s,t] = \int_s^t (1-\bar F_u)\,\mathrm{d}u.$$

Lemma 5.3. *Under Assumption (A0), one has for* $s \leqslant t$

$$a[s,t] = t - s + (1+o(1))\Upsilon[s,t]$$

as $s,t\to\infty$*. If, additionally, Assumption (A1) is satisfied, one has for* $s\leqslant t$

$$a[s,t] = t-s+\Upsilon[s,t] + o(1)$$

and

$$\Upsilon[s,t] = (1+o(1))\log\Big(\frac{t}{s}\Big)^\alpha$$

as $s,t\to\infty$*. Further, there exists a finite constant* C *such that for all* $0\leqslant s\leqslant t$

$$a[s,t] \leqslant t-s+\Upsilon[s,t]+C \ \text{ and } \ \Upsilon[s,t]\leqslant C\log\frac{t+1}{s+1}$$

Proof. The first assertion follows since

$$a[s,t]-(t-s) = \int_s^t \frac{1-\bar F_u}{\bar F_u}\,\mathrm{d}u.$$

The second statement is a direct consequence of the estimate

$$\Big|\frac{1}{x} - (1+1-x)\Big| \leqslant 8(1-x)^2 \text{ for } x\in(\frac{1}{2},\infty)$$

which follows itself by the Taylor theorem. Indeed, it implies that for sufficiently large s, one has

$$|a[s,t] - \big(t-s+\Upsilon[s,t])| = \Big|\int_s^t \frac{1}{\bar F_u} - \big(1+1-\bar F_u)\,\mathrm{d}u\Big| \leqslant 8\int_s^\infty (1-\bar F_u)^2\,\mathrm{d}u$$

which becomes small when s is large. It is straight-forward to verify the remaining statements. $\square$

Lemma 5.4. *Suppose that Assumption (A1) is satisfied, i.e.* $\mu([1-\delta,1]) = \delta^\beta\ell(\varepsilon)$ *for* ℓ *being slowly varying at zero and* $\beta>0$*. Then for* $0<f<g$*, one has for* $f^*(\delta) = 1-\delta f$ *and* $g^*(\delta) = 1-\delta g$

$$\mu([g^*(\delta), f^*(\delta))) \sim \ell(\delta)\delta^\beta(g^\beta - f^\beta) = \beta\ell(\delta)\delta^\beta\int_f^g y^{\beta-1}\,\mathrm{d}y.$$

Suppose that $v:(0,\infty)\to[0,\infty)$ *with* $\lim_{t\to\infty} tv_t = w\in[0,\infty]$*. Then*

$$\lim_{t\to\infty}\frac{t^\beta}{\ell(1/t)}\int_{[1-v_t,1)} \exp\{(1-y)t\}\,\mathrm{d}\mu(y) = \int_0^w e^{-u}\,\mathrm{d}u^\beta.$$

Proof. First suppose that $w \in (0,\infty)$. One has

$$\begin{aligned}\int_{[1-v_t,1)} \exp\{(1-y)t\}\,\mathrm{d}\mu(y) &= \int_{(0,1]} \mu\Big(\Big[1-(\frac{1}{t}\log\frac{1}{s})\wedge v_t, 1\Big)\Big)\,\mathrm{d}s \\ &= \int_0^\infty \mu([1-\frac{u}{t}\wedge v_t,1))e^{-u}\,\mathrm{d}u \\ &= t^{-\beta}\int_0^\infty \big(u\wedge(tv_t)\big)^\beta \ell(\frac{u}{t}\wedge v_t)e^{-u}\,\mathrm{d}u\end{aligned}$$

where we carried out the substitution $s = e^{-u}$. Standard reasoning for regularly varying functions now gives that

$$\int_{[1-v_t,1)} \exp\{(1-y)t\}\,\mathrm{d}\mu(y) \sim t^{-\beta}\ell(\frac{1}{t})\int_0^\infty \big(u\wedge(tv_t)\big)^\beta e^{-u}\,\mathrm{d}u \sim t^{-\beta}\ell(\frac{1}{t})\int_0^w e^{-u}\,\mathrm{d}u^\beta,$$

where we used partial integration in the last step.

The case where $tv_t \to 0$ follows imediately by domination. Furthermore, it suffices to consider the case $v_t = 1$ to achieve the result in the general case $tv_t \to \infty$. This is done in analogy to the above reasoning. One has

$$\int_{[0,1)} \exp\{(1-y)t\}\,\mathrm{d}\mu(y) = t^{-\beta}\int_0^\infty u^\beta \ell(\frac{u}{t})e^{-u}\,\mathrm{d}u$$

and a standard argument for regularly varying functions implies the assertion. □

Lemma 5.5 (Potter bounds)**.** *Under Assumption (A1) there exists for every $\eta > 0$ a constant C such that for all $\delta,\delta' \in (0,1]$ with $\delta \leqslant \delta'$*

$$\frac{1}{C}\big(\frac{\delta'}{\delta}\big)^{\alpha-\eta}\mu([1-\delta,1]) \leqslant \mu([1-\delta',1]) \leqslant C\big(\frac{\delta'}{\delta}\big)^{\alpha+\eta}\mu([1-\delta,1])$$

Moreover, for any fixed $C > 1$ the estimate is valid for all sufficiently small δ'.

Under Assumption (A2), for every $\eta > 0$ and $C > 1$ that exists $s_0 > 0$ such that for all $s_2 \geqslant s_1 \geqslant s_0$

$$\frac{1}{C}\big(\frac{s_2}{s_1}\big)^{\alpha-\eta} \leqslant \exp\Upsilon[s_1,s_2] \leqslant C\big(\frac{s_2}{s_1}\big)^{\alpha+\eta}.$$

Proof. By Theorem 1.5.6 of [5] the first statement is valid for all sufficiently small δ,δ' say for $\delta,\delta' \leqslant \delta_0$. The general statement follows straight-forwardly from the fact that

$$\frac{\mu([1-\delta',1])}{\mu([1-\delta,1])}$$

is uniformly bounded for $\delta,\delta' \in [\delta_0,1]$. The second statement is again a consequence of the classical Potter bound: Use that $[0,\infty) \ni u \mapsto \exp\Upsilon[0,u]$ is regularly varying at infinity with index α combined with the representation

$$\exp\Upsilon[s_1,s_2] = \frac{\exp\Upsilon[0,s_2]}{\exp\Upsilon[0,s_1]}.$$

□

Lemma 5.6. *There is a constant C such that for all $v \in (0,1]$ and $\theta > 0$ with $v\theta \geqslant 1$, one has*

$$\int_{[0,1-v]} e^{\theta y}\,\mathrm{d}\mu(y) \leqslant 1 + C\,e^{(1-v)\theta}\,\mu([1-v,1)).$$

Proof. One has

$$I := \int_{[0,1-v]} e^{\theta y}\,\mathrm{d}\mu(y) \leqslant 1 + \int_1^\infty \mu([\theta^{-1}\log s, 1-v])\,\mathrm{d}s$$

We apply two substitutions $s = e^u$ and $w = \theta - u$ and obtain

$$I \leqslant 1 + \int_0^{(1-v)\theta} e^u\,\mu([\theta^{-1}u, 1-v])\,\mathrm{d}u = 1 + e^\theta \int_{v\theta}^{\theta} e^{-w}\,\mu([1-w/\theta, 1-v])\,\mathrm{d}w.$$

The Potter bound implies that for a constant C_1 only depending on μ, one has

$$I \leqslant 1 + C_1 e^\theta\,\mu([1-v,1))\int_{v\theta}^\infty e^{-w}\,\big(\frac{w}{v\theta}\big)^{\alpha+1}\,\mathrm{d}w.$$

By the rule of de l'Hôpital, the function

$$z \mapsto \int_z^\infty e^{-w} w^{\alpha+1} \Big/ \Big(e^{-z} z^{\alpha+1}\Big)$$

is bounded on $[1,\infty)$ so that there exists a constant C with

$$I \leqslant 1 + C\,e^{(1-v)\theta}\,\mu([1-v,1)).$$ □

References

[1] Réka Albert and Albert-László Barabási, *Statistical mechanics of complex networks*, Rev. Modern Phys. **74** (2002), no. 1, 47–97. MR-1895096

[2] Albert-László Barabási and Réka Albert, *Emergence of scaling in random networks*, Science **286** (1999), no. 5439, 509–512. MR-2091634

[3] Shankar Bhamidi, *Universal techniques to analyze preferential attachment trees: Global and local analysis*, Available from `http://www.unc.edu/~bhamidi/preferent.pdf`, 2007.

[4] Ginestra Bianconi and Albert-László Barabási, *Bose-Einstein condensation in complex networks*, Phys. Rev. Lett. **86** (2001), 5632–35.

[5] N. H. Bingham, C. M. Goldie, and J. L. Teugels, *Regular variation*, Encyclopedia of Mathematics and its Applications, vol. 27, Cambridge University Press, Cambridge, 1987. MR-898871

[6] Béla Bollobás, Oliver Riordan, Joel Spencer, and Gábor Tusnády, *The degree sequence of a scale-free random graph process*, Random Structures Algorithms **18** (2001), no. 3, 279–290. MR-1824277

[7] Christian Borgs, Jennifer Chayes, Constantinos Daskalakis, and Sebastien Roch, *First to market is not everything: an analysis of preferential attachment with fitness*, STOC'07—Proceedings of the 39th Annual ACM Symposium on Theory of Computing, ACM, New York, 2007, pp. 135–144. MR-2402437

[8] Steffen Dereich and Peter Mörters, *Emergence of condensation in Kingman's model of selection and mutation*, Acta Appl. Math. **127** (2013), 17–26. MR-3101142

[9] Steffen Dereich and Peter Mörters, *Random networks with sublinear preferential attachment: the giant component*, Ann. Probab. **41** (2013), no. 1, 329–384. MR-3059201

[10] Steffen Dereich and Peter Mörters, *Cycle length distributions in random permutations with diverging cycle weights*, Random Structures Algorithms **46** (2015), no. 4, 635–650. MR-3346460

[11] Steffen Dereich and Marcel Ortgiese, *Robust analysis of preferential attachment models with fitness*, Combin. Probab. Comput. **23** (2014), no. 3, 386–411. MR-3189418

[12] Sergey N. Dorogovtsev and José F. F. Mendes, *Evolution of networks*, Oxford University Press, Oxford, 2003, From biological nets to the Internet and WWW. MR-1993912

[13] Rick Durrett, *Random graph dynamics*, Cambridge Series in Statistical and Probabilistic Mathematics, Cambridge University Press, Cambridge, 2010. MR-2656427

[14] Nicholas M. Ercolani and Daniel Ueltschi, *Cycle structure of random permutations with cycle weights*, Random Structures Algorithms **44** (2014), no. 1, 109–133. MR-3143592

[15] Remco van der Hofstad, *Random graphs and complex networks*, Eindhoven, 2014, Lecture Notes.

[16] Svante Janson, *Simply generated trees, conditioned Galton-Watson trees, random allocations and condensation*, Probab. Surv. **9** (2012), 103–252. MR-2908619

[17] John F. C. Kingman, *A simple model for the balance between selection and mutation*, J. Appl. Probability **15** (1978), no. 1, 1–12. MR-0465272

[18] Tamás F. Móri, *The maximum degree of the Barabási-Albert random tree*, Combin. Probab. Comput. **14** (2005), no. 3, 339–348. MR-2138118

[19] Olle Nerman, *On the convergence of supercritical general (C-M-J) branching processes*, Z. Wahrsch. Verw. Gebiete **57** (1981), no. 3, 365–395. MR-629532

[20] Mark E. J. Newman, *The structure and function of complex networks*, SIAM Rev. **45** (2003), no. 2, 167–256 (electronic). MR-2010377

[21] Roberto Oliveira and Joel Spencer, *Connectivity transitions in networks with super-linear preferential attachment*, Internet Math. **2** (2005), no. 2, 121–163. MR-2193157

[22] Su-Chan Park and Joachim Krug, *Evolution in random fitness landscapes: the infinite sites model*, J. Stat. Mech. (2008), P04014.

[23] Anna Rudas, Bálint Tóth, and Benedek Valkó, *Random trees and general branching processes*, Random Structures Algorithms **31** (2007), no. 2, 186–202. MR-2343718