Electronic Journal of Probability

Electron. J. Probab. **21** (2016), no. 12, 1–44.
ISSN: 1083-6489 DOI: 10.1214/16-EJP4079

Approximation of Markov semigroups in total variation distance *

Vlad Bally† Clément Rey‡

Abstract

In this paper, we consider Markov chains of the form $X^n_{(k+1)/n} = \psi_k(X^n_{k/n}, Z_{k+1}/\sqrt{n}, 1/n)$ where the innovation comes from the sequence $Z_k, k \in \mathbb{N}^*$ of independent centered random variables with arbitrary law. Then, we study the convergence $\mathbb{E}[f(X^n_t)] \to \mathbb{E}[f(X_t)]$ where $(X_t)_{t\geqslant 0}$ is a Markov process in continuous time. This may be considered as an invariance principle, which generalizes the classical Central Limit Theorem to Markov chains. Alternatively (and this is the main motivation of our paper), X^n may be an approximation scheme used in order to compute $\mathbb{E}[f(X_t)]$ by Monte Carlo methods. Estimates of the error are given for smooth test functions f as well as for measurable and bounded f. In order to prove convergence for measurable test functions we assume that Z_k satisfies Doeblin's condition and we use Malliavin calculus type integration by parts formulas based on the smooth part of the law of Z_k. As an application, we will give estimates of the error in total variation distance for the Ninomiya Victoir scheme.

Keywords: approximation schemes; Markov processes; total variation distance; invariance principles; Malliavin Calculus.
AMS MSC 2010: 60F17; 60H07; 65C40.
Submitted to EJP on January 27, 2015, final version accepted on December 27, 2015.

1 Introduction

In this paper, we consider a time grid $t^n_k = k/n, k \in \mathbb{N}$ with $n \in \mathbb{N}^*$ and a Markov chain

$$X^n_{t^n_{k+1}} = \psi_k(X^n_{t^n_k}, Z_{k+1}/\sqrt{n}, 1/n),$$

where $\psi_k : \mathbb{R}^d \times \mathbb{R}^N \times \mathbb{R}_+ \to \mathbb{R}^d$ is a smooth function and $Z_k, k \in \mathbb{N}^*$, is a sequence of independent centered random variables. We aim to study the convergence of the law of

*MathRisk ENPC-INRIA-UMLV Project, This research benefited from the support of the 'Chaire Risques Financiers', Fondation du Risque.
†Université de Marne-la-Vallée Cité Descartes. E-mail: vlad.bally@univ-mlv.fr
‡CERMICS-École des Ponts Paris Tech, France.
E-mail: clem6410@msn.com

X^n to the law of a Markov process X. More precisely, we will give estimates of the weak error

$$\varepsilon_n(f) = |\mathbb{E}[f(X^n_t)] - \mathbb{E}[f(X_t)]| .$$

This problem may be considered from two points of view. The first one is to look at this convergence result as to an invariance principle. We illustrate this approach with the Central Limit Theorem (CLT). Indeed, if $\psi_k(x,z,t) = x+z$ and $Z_k, k \in \mathbb{N}^*$, are independent and identically distributed with variance 1, we have $X^n_1 = n^{-1/2}\sum_{k=1}^n Z_k$. Using then the CLT, we know that $X^n_1 \overset{law}{\to} W_1$ where $(W_t)_{t\geqslant 0}$ is a standard Brownian motion and then $W_1 \sim \mathcal{N}(0,1)$ where $\mathcal{N}(0,1)$ is the standard Normal distribution. Since the law of $Z_k, k \in \mathbb{N}^*$ is arbitrary and the limit law of $(X^n_1)_{n\in\mathbb{N}}$ does not depend on this law, we say that it is an invariance principle. Keeping this in mind, we look at our Markov chain X^n as to a general Markovian scheme based on the sequence of random variables $Z_k, k \in \mathbb{N}^*$. Then, the convergence of X^n to a Markov process X which is universal (in the sense that it does not depend on the law of $Z_k, k \in \mathbb{N}^*$) represents an invariance principle. Our result can thus be seen as a direct generalization of the CLT. Notice that, when looking from this point of view, $\psi_k, k \in \mathbb{N}$ represents a scheme which naturally appears in a concrete modelization problem. A main interest is to approximate the law of X^n_1, which may be difficult to understand directly, by the law of X_1 which is simpler to study (as for W_1 above).

A second point of view comes from numerical probabilities: For instance, if X is a diffusion process and we want to compute $\mathbb{E}[f(X_t)]$, then we can use a discretization scheme X^n (for example the Euler scheme). Thereafter, we can obtain the approximation $\mathbb{E}[f(X^n_t)]$ using Monte Carlo methods. In this kind of approaches, we may choose the approximation scheme $(X^n_{t^n_k})_{k\in\mathbb{N}}$ as we want (in contrast with the previous situation when the Markov chain X^n was given by an external modelization).

Our initial motivation for the study of the error $\varepsilon_n(f)$ comes from the second point of view (numerical probabilities) but all the results of this paper are significant from both perspectives. Let us mention that the difficulty of the analysis and the interest of the result depend on the regularity of the test function f. It turns out that if f is a smooth function, then the analysis of the error is rather simple, using a Taylor type expansion in short time first, and a concatenation argument after. However, the study is much more subtle if f is simply a bounded and measurable test function - this is the so called convergence in total variation distance. A lot of work has been done in this direction in the case of the CLT. In particular, Bhattacharya and Rao [9] obtained the convergence when $f(x) = \mathbb{1}_A(x)$ where A is a measurable set that belongs to a large class (including convex sets). From that point, one would hope to get such results for every measurable set A and consequently for every measurable and bounded test function f. Eventually, the seminal result of Prokhorov [32] clarified this point: He proved that the convergence in total variation in the CLT may not be obtained without some regularity assumptions on the law of Z_k. Essentially, one has to assume that the law of Z_k has an absolute continuous component. In our framework this hypothesis has to be slightly strengthened. We assume that Z_k verifies the Doeblin's condition (see (1.8)). In this way, we extract some regular noise and use it in order to build some integration by parts formulas (inspired from Malliavin calculus). Then, we use those formulas to regularize the test function f and finally to achieve our error analysis.

Main results

Let us now present our results with more details. In order to do it, we have to introduce some notations. For fixed $T>0$ and $n \in \mathbb{N}^*$, we define the homogeneous time grid $\pi_{T,n} = \{t_k^n = kT/n, k \in \mathbb{N}\}$. We consider the d dimensional Markov chain

$$X^n_{t^n_{k+1}} = \psi_k(X^n_{t^n_k}, \frac{Z_{k+1}}{\sqrt{n}}, \delta^n_{k+1}), \qquad k \in \mathbb{N}, \tag{1.1}$$

where $\psi_k : \mathbb{R}^d \times \mathbb{R}^N \times \mathbb{R}_+ \to \mathbb{R}^d$ is a smooth function such that $\psi_k(x,0,0) = x$, and $Z_k \in \mathbb{R}^N$, $k \in \mathbb{N}^*$, is a sequence of independent and centered random variables and $\sup_{k\in\mathbb{N}^*} \delta^n_k \leqslant C/n$. The semigroup of the Markov chain $(X^n_t)_{t\in\pi_{T,n}}$ is denoted by $(Q^n_t)_{t\in\pi_{T,n}}$ and its transition probabilities are given by $\nu^n_{k+1}(x,dy) = \mathbb{P}(X^n_{t^n_{k+1}} \in dy | X^n_{t^n_k} = x)$, $k \in \mathbb{N}$. We recall that for $t \in \pi_{T,n}$, $Q^n_t f(x) = \mathbb{E}[f(X^n_t)|X^n_0 = x]$. We will also consider a Markov process in continuous time $(X_t)_{t\geqslant 0}$ with semigroup $(P_t)_{t\geqslant 0}$ and we define $\mu^n_{k+1}(x,dy) = \mathbb{P}(X_{t^n_{k+1}} \in dy | X_{t^n_k} = x)$.

Moreover, for $f \in \mathcal{C}^\infty(\mathbb{R}^d)$ and for a multi-index $\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{N}^d$ we denote $|\alpha| = \alpha_1 + ... + \alpha_d$ and $\partial_\alpha f = (\partial_1)^{\alpha_1} \ldots (\partial_d)^{\alpha_d} f = \partial^\alpha_x f(x) = \partial^{\alpha_1}_{x_1} \ldots \partial^{\alpha_d}_{x_d} f(x)$. We include the multi-index $\alpha = (0,...,0)$ and in this case $\partial_\alpha f = f$. We will use the norms

$$\|f\|_{q,\infty} = \sup_{x\in\mathbb{R}^d} \sum_{0\leqslant|\alpha|\leqslant q} |\partial_\alpha f(x)|, \qquad \|f\|_{q,1} = \sum_{0\leqslant|\alpha|\leqslant q} \int_{\mathbb{R}^d} |\partial_\alpha f(x)| dx.$$

In particular $\|f\|_{0,\infty} = \|f\|_\infty$ is the usual supremum norm and we will denote $\mathcal{C}^q_b(\mathbb{R}^d) = \{f \in \mathcal{C}^q(\mathbb{R}^d), \|f\|_{q,\infty} < \infty\}$ and $C^q_c(\mathbb{R}^d) \subset C^q(\mathbb{R}^d)$ the set of functions with compact support.

A first standard result is the following: Let us assume that there exists $h > 0$, $q \in \mathbb{N}$ such that for every $f \in \mathcal{C}^q(\mathbb{R}^d)$, $k \in \mathbb{N}^*$ and $x \in \mathbb{R}^d$,

$$\left|\mu^n_k f(x) - \nu^n_k f(x)\right| = \left|\int f(y)\mu^n_k(x,dy) - \int f(y)\nu^n_k(x,dy)\right| \leqslant C\|f\|_{q,\infty}/n^{1+h}. \tag{1.2}$$

Then, for all $T \geqslant 0$, there exists $C \geqslant 1$ such that we have

$$\sup_{t\in\pi_{T,n};t\leqslant T} \|P_t f - Q^n_t f\|_\infty \leqslant C\|f\|_{q,\infty}/n^h. \tag{1.3}$$

It means that $(X^n_t)_{t\in\pi_{T,n}}$ is an approximation scheme of weak order h for the Markov process $(X_t)_{t\geqslant 0}$. In the case of the Euler scheme for diffusion processes, this result, with $h = 1$, has initially been proved in the seminal papers of Milstein [27] and of Talay and Tubaro [34] (see also [18]). Similar results were obtained in various situations: Diffusion processes with jumps (see [33], [16]) or diffusion processes with boundary conditions (see [13], [12], [14]). An overview of this subject is given in [17]. More recently, approximation schemes of higher orders (*e.g.*, $h = 2$), based on cubature methods, have been introduced and studied by Kusuoka [22], Lyons [26], Ninomiya, Victoir [28] or Alfonsi [1]. The reader may also refer to the work of Kohatsu-Higa and Tankov [19] for a higher weak order scheme for jump processes.

Another result concerns convergence in total variation distance. We want to obtain (1.3) with $\|f\|_{q,\infty}$ replaced by $\|f\|_\infty$ when f is a measurable function. In the case of the Euler scheme for diffusion processes, a first result of this type has been obtained by Bally and Talay [6], [7] using the Malliavin calculus (see also Guyon [15]). Afterwards Konakov, Menozzi and Molchanov [20], [21] obtained similar results using a parametrix method. Recently Kusuoka [23] obtained estimates of the error in total variation distance for the Victoir Ninomiya scheme (which corresponds to the case $h = 2$). We will obtain a similar result using our approach. Moreover, we give estimates of the rate of convergence of the density function and its derivatives.

Regularization properties. We first remark that the crucial property which is used in order to replace $\|f\|_{q,\infty}$ by $\|f\|_\infty$ in (1.3), is the regularization property of the semigroup. Let us be more precise: Let $\eta:\mathbb{R}_+\to\mathbb{R}_+$ an increasing function, $q\in\mathbb{N}$ be fixed. Given the time grid $\pi_{T,n}=\{t_k^n=kT/n,k\in\mathbb{N}\}$, we say that a semigroup $(P_t^n)_{t\in\pi_{T,n}}$ satisfies $R_{q,\eta}$, if

$$R_{q,\eta} \qquad \forall t\in\pi_{T,n}, t>0, \quad \|P_t^n f\|_{q,\infty}\leqslant \frac{C}{t^{\eta(q)}}\|f\|_\infty. \tag{1.4}$$

We also introduce a dual regularization property: We consider the dual semigroup $P_t^{n,*}$ (*i.e.* $\left\langle P_t^{n,*}g,f\right\rangle=\langle g,P_t^n f\rangle$ with the scalar product in $\mathrm{L}^2(\mathbb{R}^d)$) and we assume that

$$R_{q,\eta}^* \qquad \forall t\in\pi_{T,n}, t>0, \quad \|P_t^{n,*} f\|_{q,1}\leqslant \frac{C}{t^{\eta(q)}}\|f\|_1. \tag{1.5}$$

Finally, we consider the following stronger regularization property: For every multi-index α,β with $|\alpha|+|\beta|=q$,

$$\overline{R}_{q,\eta} \qquad \forall t\in\pi_{T,n}, t>0, \quad \|\partial_\alpha P_t^n \partial_\beta f\|_{\infty}\leqslant \frac{C}{t^{\eta(q)}}\|f\|_\infty. \tag{1.6}$$

We notice that $\overline{R}_{q,\eta}$ implies both $R_{q,\eta}$ and $R^*_{q,\eta}$ and that a semigroup satisfying $\overline{R}_{q,\eta}$ is absolutely continuous with respect to the Lebesgue measure.

In addition to (1.2), we will also suppose that the following dual estimate of the error in short time holds:

$$|\left\langle g,(\mu_k^n-\nu_k^n)f\right\rangle|\leqslant C\|g\|_{q,1}\|f\|_\infty/n^{1+h}. \tag{1.7}$$

Using those hypothesis, we can obtain a first result.

Theorem 1.1. *We recall that $T>0$ and $n\in\mathbb{N}^*$. We fix $h>0$, $q\in\mathbb{N}$ and we assume that the short time estimates (1.2) and (1.7) hold (with this q and h). Moreover, we assume that (1.4) holds for $(P_t)_{t\in\pi_{T,n}}$ and that (1.5) holds for $(Q_t^n)_{t\in\pi_{T,n}}$. Then, for every $S\in[T/n,T/2)$,*

$$\forall t\in\pi_{T,n}, t>2S, \quad \|P_t f-Q_t^n f\|_\infty\leqslant\frac{C}{S^{\eta(q)}}\|f\|_\infty/n^h.$$

Integration by parts formulas. Once we have this abstract result, the following step is to give sufficient conditions in order to obtain $R_{q,\eta}$, $R^*_{q,\eta}$ and $\overline{R}_{q,\eta}$. The method we adopt in this paper is to use Malliavin type integration by parts formulas based on the noise $Z_k\in\mathbb{R}^N$, $k\in\mathbb{N}^*$. Then we will have to bound the weights that appear in those formulas and the regularization properties will follow.

In order to obtain those estimates, we assume that the law of each Z_k is locally lower bounded by the Lebesgue measure: There exists some $z_{*,k}\in\mathbb{R}^N$ and $r_*,\varepsilon_*>0$ such that for every measurable set $A\subset B_{r_*}(z_{*,k})$ one has

$$\mathbb{P}(Z_k\in A)\geqslant\varepsilon_*\lambda(A) \tag{1.8}$$

where λ is the Lebesgue measure. If this property holds then a “splitting method” can be used in order to represent Z_k as

$$\frac{Z_k}{\sqrt{n}}=\chi_k U_k+(1-\chi_k)V_k,$$

where χ_k,U_k,V_k are independent random variables, χ_k is a Bernoulli random variable and $\sqrt{n}U_k\sim\varphi_{r_*}(u)du$ with $\varphi_{r_*}\in\mathcal{C}^\infty(\mathbb{R}^N)$. Then we use the abstract Malliavin calculus based on U_k, developed in [5] and [3], in order to obtain integration by parts formulas. The crucial point is that the density φ_{r_*} of $\sqrt{n}U_k$ is smooth and we control its logarithmic derivatives. Using this property, we build integration by parts formulas and we obtain relevant estimates for the weights which appear in these formulas. It is worth mentioning that, a variant of the Malliavin calculus based on a similar splitting method has already been used by Nourdin and Poly [30] (see also [29] and [24]). They use the so called Γ calculus introduced by Bakry, Gentil and Ledoux [2]. Roughly speaking, the difference between our approach and the one in [2] is the following: Our construction is similar to the "simple functionals" approach in Malliavin calculus and has the derivative operator as basic object. In contrast, in the Γ calculus, the basic object is the Ornstein Uhlenbeck operator.

In order to state the main result of our paper, we introduce some additional assumptions:

$$\forall p \in \mathbb{N}, \quad \sup_{k\in\mathbb{N}^*} \mathbb{E}[|Z_k|^p] < \infty, \tag{1.9}$$

$$\forall r \in \mathbb{N}^*, \quad \sup_{k\in\mathbb{N}} \|\psi_k\|_{1,r,\infty} = \sup_{k\in\mathbb{N}} \sum_{|\alpha|=0}^{r} \sum_{|\beta|+|\gamma|=1}^{r-|\alpha|} \|\partial_x^\alpha \partial_z^\beta \partial_t^\gamma \psi_k\|_\infty < \infty \tag{1.10}$$

$$\exists \lambda_* > 0,\ \forall k \in \mathbb{N}, \quad \inf_{x\in\mathbb{R}^d} \inf_{|\eta|=1} \sum_{i=1}^{N} \langle \partial_{z_i}\psi_k(x,0,0), \eta\rangle^2 \geqslant \lambda_*. \tag{1.11}$$

Moreover, we introduce the following regularized version of the approximation scheme $(X_t^n)_{t\in\pi_{T,n}}$:

$$\forall t \in \pi_{T,n}, \quad X_t^{n,\theta}(x) = \frac{1}{n^\theta} G + X_t^n(x),$$

with G a standard normal random variable independent from $X^n_{t^n_k}$ and $\theta > h+1$. Here $X_t^n(x)$ is the Markov chain which starts from x: $X_0^n(x) = x$. We denote

$$Q_t^{n,\theta}(x,dy) = \mathbb{P}(X_t^{n,\theta}(x) \in dy) = p_t^{n,\theta}(x,y)dy.$$

Theorem 1.2. *We recall that $T > 0$ and $n \in \mathbb{N}^*$. We fix $h > 0$, $q \in \mathbb{N}$ and and we consider a Markov semigroup $(P_t)_{t\geqslant 0}$ and the discrete Markov chain $(Q_t^n)_{t\in\pi_{T,n}}$ defined in (1.1). We assume that the short time estimates (1.2) and (1.7) hold (with this q and h). Moreover, we assume (1.8), (1.9), (1.10) and (1.11).*

- ***A.*** *For every $S \in [T/n, T/2)$, we have*

$$\forall t \in \pi_{T,n}, t \in (2S,T], \quad \|P_t f - Q_t^n f\|_\infty \leqslant \frac{C}{(\lambda_* S)^{\eta(q)}} \|f\|_\infty / n^h. \tag{1.12}$$

- ***B.*** *For every $t > 0$, $P_t(x,dy) = p_t(x,y)dy$ with $(x,y) \mapsto p_t(x,y)$ belonging to $\mathcal{C}^\infty(\mathbb{R}^d \times \mathbb{R}^d)$.*

- ***C.*** *For every $x_0, y_0, R > 0, \varepsilon \in (0,1)$ and every multi-index α, β, we have*

$$\forall t \in \pi_{T,n}, t \in (2S,T], \quad \sup_{\overline{B}_R(x_0,y_0)} |\partial_x^\alpha \partial_y^\beta p_t(x,y) - \partial_x^\alpha \partial_y^\beta p_t^{n,\theta}(x,y)| \leqslant C_\varepsilon / n^{h(1-\varepsilon)}, \tag{1.13}$$

with a constant C_ε which depends on $R, x_0, y_0, S, \lambda_*, T, \varepsilon, \eta$ and on $|\alpha| + |\beta|$ (and may go to infinity as ε tends to 0). Moreover we denote $\overline{B}_R(x_0,y_0) = \{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d, |(x,y) - (x_0,y_0)| \leqslant R\}$.

We notice that (1.12) gives the total variation convergence between the semigroups $(P_t)_{t\geqslant 0}$ and $(Q_t^n)_{t\in\pi_{T,n}}$. Once the appropriate regularization properties are obtained (using the abstract Malliavin calculus), the proof of (1.12) is rather elementary. In contrast, the estimate (1.13) is based on a non trivial interpolation result recently obtained in [8]. Notice, however, that the estimate (1.13) is sub-optimal (because of $\varepsilon > 0$). We will illustrate (1.12) by taking X^n to be the Ninomiya Victoir scheme of a diffusion process. This is a variant of the result already obtained by Kusuoka [23] in the case where Z_k has a Gaussian distribution (and so the standard Malliavin calculus is available). As we have mentioned in the beginning of this paper, the random variables Z_k, $k \in \mathbb{N}^*$ have an arbitrary distribution (except the property (1.8)) and our result can be seen as an invariance principle as well.

The paper is organized as follows. In Section 2, we prove Theorem 1.1. In Section 3, we settle the abstract Malliavin calculus based on the splitting method. We use it in Section 4 in order to prove the regularization properties for the approximation scheme X^n (in fact for the regularization $X^{n,\theta}$) and we obtain Theorem 1.2. Finally, in Section 5, we use the previous results in order to give estimates of the total variation distance for the Ninomiya Victoir approximation scheme. In order to enlighten the presentation of our results, Section 6 is devoted to the proof of Theorem 4.2 on Sobolev norms of X^n which is presented in Section 4

2 The distance between two Markov semigroups

Throughout this section the following notations will prevail. We fix $T > 0$ and we denote $n \in \mathbb{N}^*$, the number of time step between 0 and T. Then, for $k \in \mathbb{N}$ we define $t_k^n = kT/n$ and we introduce the homogeneous time grid $\pi_{T,n} = \{t_k^n = kT/n, k \in \mathbb{N}\}$ and its bounded version $\pi_{T,n}^{\tilde{T}} = \{t \in \pi_{T,n}, t \leqslant \tilde{T}\}$ for $\tilde{T} \geqslant 0$. Finally, for $S \in [0, \tilde{T})$ we will denote $\pi_{T,n}^{S,\tilde{T}} = \{t \in \pi_{T,n}^{\tilde{T}}, t > S\}$. Notice that, all the results from this paper remain true with non homogeneous time step but, for sake of simplicity, we will not consider this case. First, we state some results for smooth test functions.

2.1 Regular test functions

We consider a sequence of finite transition measures $\mu_k^n(x, dy),\ k \in \mathbb{N}^*$ from $\mathbb{R}^d$ to $\mathbb{R}^d$. This means that for each fixed x and k, $\mu_k^n(x, dy)$ is a finite measure on $\mathbb{R}^d$ with the borelian σ field and for each bounded measurable function $f : \mathbb{R}^d \to \mathbb{R}$, the application

$$x \mapsto \mu_k^n f(x) := \int_{\mathbb{R}^d} f(y)\mu_k^n(x, dy)$$

is measurable. We also denote

$$|\mu_k^n| := \sup_{x\in\mathbb{R}^d} \sup_{\|f\|_\infty \leqslant 1} \Big| \int_{\mathbb{R}^d} f(y)\mu_k^n(x, dy)\Big|,$$

and, we assume that all the sequences of measures we consider in this paper satisfy:

$$\sup_{k\in\mathbb{N}^*} |\mu_k^n| < \infty. \tag{2.1}$$

Although the main application concerns the case where $\mu_k^n(x, dy)$ is a probability measure, we do not assume this here. Indeed, $\mu_k^n(x, dy)$ is only supposed be a signed measure of finite (but arbitrary) total mass. This is because one may use the results from this section not only in order to estimate the distance between two semigroups but also in order to obtain an expansion of the error.

Now we associate the sequence of measures μ^n to the time grid $\pi_{T,n}$.

Definition 2.1. *We define the discrete semigroup P^n in the following way.*

$$P_0^n f(x) = f(x), \qquad P^n_{t^n_{k+1}} f(x) = P^n_{t^n_k} \mu^n_{k+1} f(x) = P^n_{t^n_k} \int_{\mathbb{R}^d} f(y) \mu^n_{k+1}(x, dy).$$

More generally, we define $(P_{t,s})_{t,s\in\pi_{T,n};t\leqslant s}$ by

$$P^n_{t^n_k,t^n_k} f(x) = f(x), \qquad \forall k, r \in \mathbb{N}^*, k \leqslant r,\ P^n_{t^n_k,t^n_{r+1}} f(x) = P^n_{t^n_k,t^n_r} \mu^n_{r+1} f(x).$$

We notice that for $t, s, u \in \pi_{T,n}$, $t \leqslant s \leqslant u$, we have the semigroup property $P^n_{t,u} f = P^n_{t,s} P^n_{s,u} f$. We will consider the following hypothesis: Let $q \in \mathbb{N}$ and $t \leqslant s \in \pi_{T,n}$. If $f \in \mathcal{C}^q(\mathbb{R}^d)$ then $P_{t,s} f \in \mathcal{C}^q(\mathbb{R}^d)$ and

$$\sup_{t,s\in\pi_{T,n};t\leqslant s} \|P^n_{t,s} f\|_{q,\infty} \leqslant C\|f\|_{q,\infty}. \tag{2.2}$$

Notice that (2.1) implies that (2.2) holds for $q = 0$.

We consider now a second sequence of finite transition measures $\nu^n_k(x, dy), k \in \mathbb{N}^*$. Moreover, we introduce the corresponding semigroup Q^n defined in a similar way as P^n with μ^n replaced by ν^n which also satisfies (2.1) and (2.2).

We aim to estimate the distance between $P^n f$ and $Q^n f$ in terms of the distance between the transition measures $\mu^n_k(x, dy)$ and $\nu^n_k(x, dy)$, so we denote

$$\Delta^n_k = \mu^n_k - \nu^n_k.$$

$(P^n_t)_{t\in\pi_{T,n}}$ can be seen as a semigroup in continuous time, $(P_t)_{t\geqslant 0}$, considered on the time grid $\pi_{T,n}$, while $(Q_t)_{t\in\pi_{T,n}}$ would be its approximation discrete semigroup. Let $q \in \mathbb{N},\ h \geqslant 0$ be fixed. We introduce a short time error approximation assumption: There exists a constant $C > 0$ (depending on q only) such that for every $k \in \mathbb{N}^*$, we have

$$E_n(h, q) \qquad \|\Delta^n_k f\|_\infty \leqslant C\|f\|_{q,\infty}/n^{h+1}. \tag{2.3}$$

Proposition 2.2. *Let $q \in \mathbb{N}$ be fixed. Suppose that ν^n satisfies (2.2) for this q and that we have $E_n(h, q)$ (see (2.3)). Then for every $f \in \mathcal{C}^q(\mathbb{R}^d)$,*

$$\sup_{t\in\pi^T_{T,n}} \|P^n_t f - Q^n_t f\|_\infty \leqslant C\|f\|_{q,\infty}/n^h. \tag{2.4}$$

Proof. Let $m \in \mathbb{N}^*$, $m \leqslant n$. We have

$$\begin{aligned} \|P^n_{t^n_m} f - Q^n_{t^n_m} f\|_\infty &\leqslant \sum_{k=0}^{m-1} \|P^n_{t^n_k} P^n_{t^n_k,t^n_{k+1}} Q^n_{t^n_{k+1},t^n_m} f - P^n_{t^n_k} Q^n_{t^n_k,t^n_{k+1}} Q^n_{t^n_{k+1},t^n_m} f\|_\infty \\ &= \sum_{k=0}^{m-1} \|P^n_{t^n_k} \Delta^n_{k+1} Q^n_{t^n_{k+1},t^n_m} f\|_\infty. \end{aligned} \tag{2.5}$$

Using (2.1) for μ^n, (2.3) and then (2.2) for ν^n, we obtain

$$\|P^n_{t^n_{k+1},t^n_m} \Delta^n_{k+1} Q^n_{t^n_k} f\|_\infty \leqslant C\|\Delta^n_{k+1} Q^n_{t^n_k} f\|_\infty \leqslant C\|Q^n_{t^n_k} f\|_{q,\infty}/n^{h+1} \leqslant C\|f\|_{q,\infty}/n^{1+h}.$$

Summing over $k = 0, ..., m-1$, we conclude. □

2.2 Measurable test functions (convergence in total variation distance)

The estimate (2.4) requires a lot of regularity for the test function f. We aim to show that, if the semigroups at work have a regularization property, then we may obtain estimates of the error for measurable and bounded test functions. In order to state this result we have to give some hypothesis on the adjoint semigroup. Let $q \in \mathbb{N}$. We assume that there exists a constant $C \geqslant 1$ such that for every measurable and bounded function f and any $g \in \mathcal{C}^q(\mathbb{R}^d)$

$$E_n^*(h,q) \qquad |\langle g, \Delta_k^n f\rangle| \leqslant C\|g\|_{q,1}\|f\|_\infty/n^{1+h}. \tag{2.6}$$

where $\langle g,f\rangle = \int g(x)f(x)dx$ is the scalar product in $\mathrm{L}^2(\mathbb{R}^d)$.

Our regularization hypothesis is the following. Let $q \in \mathbb{N},\ S > 0$ and $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ an increasing function be given. We assume that there exists a constant $C \geqslant 1$ such that

$$R_{q,\eta}(S) \qquad \forall t,s \in \pi_{T,n},\ \text{with } S \leqslant s-t, \quad \|P_{t,s}^n f\|_{q,\infty} \leqslant \frac{C}{S^{\eta(q)}}\|f\|_\infty. \tag{2.7}$$

We also consider the "adjoint regularization hypothesis". We assume that there exists an adjoint semigroup $P_{t,s}^{n,*}$, that is

$$\left\langle P_{t,s}^{n,*}g, f\right\rangle = \left\langle g, P_{t,s}^n f\right\rangle$$

for every measurable and bounded function f and every function $g \in \mathcal{C}_c^\infty(\mathbb{R}^d)$. We assume that $P_{t,s}^{n,*}$ satisfies

$$R_{q,\eta}^*(S) \qquad \forall t,s \in \pi_{T,n},\ \text{with } S \leqslant s-t, \quad \|P_{t,s}^{n,*} f\|_{q,1} \leqslant \frac{C}{S^{\eta(q)}}\|f\|_1. \tag{2.8}$$

Notice that a sufficient condition in order that $R_{q,\eta}^*(S)$ holds is the following: For every multi index α with $|\alpha| \leqslant q$

$$\forall t,s \in \pi_{T,n},\ \text{with } S \leqslant s-t, \quad \|P_{t,s}^n \partial_\alpha f\|_\infty \leqslant \frac{C}{S^{\eta(q)}}\|f\|_\infty. \tag{2.9}$$

Indeed:

$$\begin{aligned}
\|\partial_\alpha P_{t,s}^{n,*} f\|_1 &\leqslant \sup_{\|g\|_\infty \leqslant 1} |\left\langle \partial_\alpha P_{t,s}^{n,*} f, g\right\rangle| = \sup_{\|g\|_\infty \leqslant 1} |\left\langle f, P_{t,s}^n(\partial_\alpha g)\right\rangle| \\
&\leqslant \|f\|_1 \sup_{\|g\|_\infty \leqslant 1} \|P_{t,s}^n(\partial_\alpha g)\|_\infty \leqslant \frac{C}{S^{\eta(q)}}\|f\|_1.
\end{aligned}$$

Proposition 2.3. *Let $q \in \mathbb{N}, \quad h \geqslant 0,\ S \in [T/n, T/2)$ and $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ an increasing function be fixed. We assume that $E_n(h,q)$ (see (2.3)) and $E_n^*(h,q)$ (see (2.6)) hold for P^n and Q^n. We also suppose that P^n satisfies $R_{q,\eta}(S)$ (see (2.7)) and Q^n satisfies $R_{q,\eta}^*(S)$ (see (2.8)) and that (2.2) hold with $q = 0$ for both of them. Then,*

$$\sup_{t \in \pi_{T,n}^{2S,T}} \|P_t^n f - Q_t^n f\|_\infty \leqslant \frac{C}{S^{\eta(q)}}\|f\|_\infty/n^h.$$

Proof. Using a density argument we may assume that $f \in \mathcal{C}(\mathbb{R}^d)$. Moreover, by (2.5), it is sufficient to prove that

$$\|Q_{t_k^n}^n \Delta_{k+1}^n P_{t_{k+1}^n, t_m^n}^n f\|_\infty \leqslant \frac{C}{S^{\eta(q)}}\|f\|_\infty/n^{1+h},$$

for $m \in \{2,\ldots,n\}$. Since $t_m^n > 2S$ we have $t_k^n \geqslant S$ or $t_m^n - t_{k+1}^n \geqslant S$. Suppose first that $t_m^n - t_{k+1}^n \geqslant S$. Using (2.1) for Q^n, (2.3) and (2.7) for P^n,

$$\begin{aligned}
\|Q_{t_k^n}^n \Delta_{k+1}^n P_{t_{k+1}^n, t_m^n}^n f\|_\infty &\leqslant C\|\Delta_{k+1}^n P_{t_{k+1}^n, t_m^n}^n f\|_\infty \\
&\leqslant C\|P_{t_{k+1}^n, t_m^n}^n f\|_{q,\infty}/n^{1+h} \leqslant CS^{-\eta(q)}\|f\|_\infty/n^{1+h}.
\end{aligned}$$

Suppose now that $t_k^n \geqslant S$. We take $\phi_\varepsilon(x) = \varepsilon^{-d}\phi(\varepsilon^{-1}x)$ with $\phi \in \mathcal{C}_c(\mathbb{R}^d),\ \phi \geqslant 0$. Then, for a fixed x_0, we define $\phi_{\varepsilon,x_0}(x) = \phi_\varepsilon(x - x_0)$. Since we have (2.2), $Q^n_{t_k^n}\Delta^n_{k+1}P^n_{t^n_{k+1},t^n_m} f$ is continuous. Then

$$|Q^n_{t_k^n}\Delta^n_{k+1}P^n_{t^n_{k+1},t^n_m} f(x_0)| = \lim_{\varepsilon\to 0} |\langle \phi_{\varepsilon,x_0}, Q^n_{t_k^n}\Delta^n_{k+1}P^n_{t^n_{k+1},t^n_m} f\rangle|.$$

Using (2.6), (2.8) for Q^n and then (2.1) for P^n, we obtain

$$\begin{aligned}|\langle \phi_{\varepsilon,x_0}, Q^n_{t_k^n}\Delta^n_{k+1}P^n_{t^n_{k+1},t^n_m} f\rangle| &= |\langle Q^{n,*}_{t_k^n}\phi_{\varepsilon,x_0}, \Delta^n_{k+1}P^n_{t^n_{k+1},t^n_m} f\rangle| \\ &\leqslant C\|Q^{n,*}_{t_k^n}\phi_{\varepsilon,x_0}\|_{q,1}\|P^n_{t^n_{k+1},t^n_m} f\|_\infty / n^{1+h} \\ &\leqslant CS^{-\eta(q)}\|\phi_{\varepsilon,x_0}\|_1\|f\|_\infty/n^{1+h}\end{aligned}$$

and since $\|\phi_{\varepsilon,x_0}\|_1 = \|\phi\|_1 \leqslant C$, the proof is completed. □

In concrete applications the following slightly more general variant of the above proposition will be useful.

Proposition 2.4. *Let $q \in \mathbb{N},\ h \geqslant 0,\ S \in [T/n, T/2)$ and $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ an increasing function be fixed. We assume that $E_n(h,q)$ (see (2.3)) and $E^*_n(h,q)$ (see (2.6)) hold for P^n and Q^n. Moreover, we assume that there exists some kernels $(\overline{P}^n_{t,s})_{t,s\in\pi_{T,n};t\leqslant s}$ which satisfies $R_{q,\eta}(S)$ (see(2.7)) and $(\overline{Q}^n_{t,s})_{t,s\in\pi_{T,n};t\leqslant s}$ which satisfies $R^*_{q,\eta}(S)$ (see (2.8)) and that (2.2) hold with $q = 0$ for both of them. We also assume that for every $t, s \in \pi_{T,n}$ with $s - t \geqslant S$,*

$$\|Q^n_{t,s}f - \overline{Q}^n_{t,s}f\|_\infty + \|P^n_{t,s}f - \overline{P}^n_{t,s}f\|_\infty \leqslant CS^{-\eta(q)}\|f\|_\infty/n^{h+1}. \tag{2.10}$$

Then,

$$\sup_{t\in\pi^{2S,T}_{T,n}} \|P^n_t f - Q^n_t f\|_\infty \leqslant C\sup_{k\leqslant n}(|\mu^n_k| + |\nu^n_k|)S^{-\eta(q)}\|f\|_\infty/n^h.$$

Remark 2.5. *Notice that $\overline{P}^n$ and $\overline{Q}^n$ are not supposed to satisfy the semigroup property and are not directly related to μ^n and ν^n.*

Proof. The proof follows the same line as the one of the previous proposition. Suppose first that $t^n_m - t^n_k \geqslant S$. Then, (2.1) implies

$$\begin{aligned}\|Q^n_{t_k^n}\Delta^n_{k+1}P^n_{t^n_{k+1},t^n_m} f\|_\infty &\leqslant \|Q^n_{t_k^n}\Delta^n_{k+1}\overline{P}^n_{t^n_{k+1},t^n_m} f\|_\infty + \|Q^n_{t_k^n}\Delta^n_{k+1}(P^n_{t^n_{k+1},t^n_m} - \overline{P}^n_{t^n_{k+1},t^n_m})f\|_\infty \\ &\leqslant \|\Delta^n_{k+1}\overline{P}_{t^n_{k+1},t^n_m} f\|_\infty + \|\Delta^n_{k+1}(P^n_{t^n_{k+1},t^n_m} - \overline{P}^n_{t^n_{k+1},t^n_m})f\|_\infty.\end{aligned}$$

Since $\overline{P}^n$ verifies $R_{q,\eta}(S)$, we deduce from (2.3) that

$$\|\Delta^n_{k+1}\overline{P}^n_{t_k^n} f\|_\infty \leqslant C\|\overline{P}^n_{t_k^n} f\|_{q,\infty}/n^{h+1} \leqslant CS^{-\eta(q)}\|f\|_\infty/n^{h+1}.$$

Using (2.10), it follows

$$\begin{aligned}|\Delta^n_{k+1}(P^n_{t^n_{k+1},t^n_m} - \overline{P}^n_{t^n_{k+1},t^n_m})f(x)| &\leqslant |\int (P^n_{t^n_{k+1},t^n_m} - \overline{P}^n_{t^n_{k+1},t^n_m})f(y)\nu_{k+1}(x,dy)| \\ &\quad + |\int (P^n_{t^n_{k+1},t^n_m} - \overline{P}^n_{t^n_{k+1},t^n_m})f(y)\mu_{k+1}(x,dy)| \\ &\leqslant (|\nu^n_{k+1}| + |\mu^n_{k+1}|)\|(P^n_{t^n_{k+1},t^n_m} - \overline{P}^n_{t^n_{k+1},t^n_m})f\|_\infty \\ &\leqslant C(|\nu^n_{k+1}| + |\mu^n_{k+1}|)S^{-\eta(q)}\|f\|_\infty/n^{h+1}.\end{aligned}$$

Suppose now that $t^n_k \geqslant S$. We write

$$\|Q^n_{t_k^n}\Delta^n_{k+1}P^n_{t^n_{k+1},t^n_m} f\|_\infty \leqslant \|\overline{Q}^n_{t_k^n}\Delta^n_{k+1}P^n_{t^n_{k+1},t^n_m} f\|_\infty + \|(Q^n_{t_k^n} - \overline{Q}^n_{t_k^n})\Delta^n_{k+1}P^n_{t^n_k,t^n_m} f\|_\infty.$$

In order to bound $\|\overline{Q}^n_{t_k^n}\Delta^n_{k+1}P^n_{t^n_{k+1},t^n_m} f\|_\infty$ we use the same reasoning as in the proof of the previous proposition. And the second term is bounded using (2.10). □

2.3 Convergence of the density functions

In this section we will consider a Markov semigroup $(P_t)_{t\geqslant 0}$ and we will give an approximation result and a regularity criterion for it. The regularization property that we assume for the approximation processes is stronger than the one considered in the previous section and, instead of Proposition 2.3 we will use a general approximation result based on an interpolation inequality, proved in [8]. We recall that we have fixed $T>0$, and for $n\in\mathbb{N}^*$ we denote $t^n_k=kT/n$. For $k\in\mathbb{N}^*$, we consider $\mu^n_k(x,dy)=\mu^n(x,dy)=P_{T/n}(x,dy)$, for all $k\in\mathbb{N}$, the homogeneous sequence of finite transition measures which satisfy (2.2). To this sequence of measures, we associate the discrete version $(P^n_t)_{t\in\pi_{T,n}}$ of P such that for all $t,s\in\pi_{T,n}$, $t\leqslant s$, $P^n_{t,s}f(x)=P_{s-t}f(x)$. Moreover we introduce a sequence of transition probability measures $\nu^n_k(x,dy),\ k\in\mathbb{N}^*$, and the corresponding discrete semigroups $Q^n(x,dy)$ defined by $Q^n_{t,t}=Id$ and $Q^n_{t^n_k,t^n_{r+1}}=Q^n_{t^n_k,t^n_r}\nu^n_{r+1}$. We recall that for all $t\in\pi_{T,n}$ then $Q^n_tf=Q^n_{0,t}f$. We assume that for $f\in\mathcal{C}^q(\mathbb{R}^d)$, we have $Q^n_{t,s}f\in\mathcal{C}^q(\mathbb{R}^d)$ for all $t,s\in\pi_{T,n},t\leqslant s$, and it verifies (2.2):

$$\sup_{t,s\in\pi_{T,n};t\leqslant s}\|Q^n_{t,s}f\|_{q,\infty}\leqslant C\|f\|_{q,\infty}.$$

For $h>0$ and $q\in\mathbb{N}$, we assume that we have (2.3) and (2.6):

$$E_n(h,q)\qquad \|(\mu^n-\nu^n_k)f\|_\infty\leqslant C\|f\|_{q,\infty}/n^{1+h}.$$

and,

$$E^*_n(h,q)\qquad |\langle g,(\mu^n-\nu^n_k)f\rangle|\leqslant C\|g\|_{q,1}\|f\|_\infty/n^{1+h}.$$

In concrete applications, it may be cumbersome to prove the regularization properties of the underlying semigroups P^n and Q^n. In order to treat this problem, we introduce now $(\overline{Q}^n_t)_{t\in\pi_{T,n}}$, a modification of $(Q^n_t)_{t\in\pi_{T,n}}$ in the sense that for every measurable and bounded function $f:\mathbb{R}^d\to\mathbb{R}$, we have

$$\forall t,s\in\pi_{T,n},\ \text{with } S\leqslant s-t,\quad \|Q^n_{t,s}f-\overline{Q}^n_{t,s}f\|_\infty\leqslant CS^{-\eta(q)}\|f\|_\infty/n^{h+1}. \tag{2.11}$$

We assume that $(\overline{Q}^n_t)_{t\in\pi_{T,n}}$ satisfies the following strong regularization property. We fix $q\in\mathbb{N}$ $S,\eta>0$, and we assume that for every multi-index α,β with $|\alpha|+|\beta|\leqslant q$ and $f\in\mathcal{C}^q(\mathbb{R}^d)$ one has

$$\overline{R}_{q,\eta}(S)\qquad \forall t,s\in\pi_{T,n},\ \text{with } S\leqslant s-t,\quad \|\partial_\alpha\overline{Q}^n_{t,s}\partial_\beta f\|_\infty\leqslant CS^{-\eta(q)}\|f\|_\infty. \tag{2.12}$$

Notice that if $\overline{R}_{q+2d,\eta}(S)$ holds, then for all $t\in\pi_{T,n}$, there exists $\overline{p}^n_t\in\mathcal{C}^q(\mathbb{R}^d\times\mathbb{R}^d)$ such that $\overline{Q}^n_t(x,dy)=\overline{p}^n_t(x,y)dy$. Moreover, if $t\geqslant S$, then for every $|\alpha|+|\beta|\leqslant q$, we have

$$\sup_{(x,y)\in\mathbb{R}^d\times\mathbb{R}^d}|\partial^\alpha_x\partial^\beta_y\overline{p}^n_t(x,y)|\leqslant CS^{-\eta(q+2d)}. \tag{2.13}$$

Indeed, let $f_\zeta:\mathbb{R}^d\to\mathbb{C},x\mapsto e^{-i\langle\zeta,x\rangle}$. Using the Fourier representation of the density function, we have

$$\overline{p}^n_t(x,y)=\int_{\mathbb{R}^d}e^{i\langle\zeta,y\rangle}\overline{Q}^n_tf_\zeta(x)d\zeta$$

Now we notice that $\partial^\beta_yf_\zeta(y)=f_\zeta(y)(-i)^{|\beta|}\prod_{i=1}^{|\beta|}\zeta_{\beta_i}$ and it follows that for all $x,y,\in\mathbb{R}^d$,

$$\partial^\alpha_x\partial^\beta_y\overline{p}^n_t(x,y)=\int_{\mathbb{R}^d}i^{|\beta|}\big(\prod_{i=1}^{|\beta|}\zeta_{\beta_i}\big)e^{i\langle\zeta,y\rangle}\partial^\alpha_x(\overline{Q}^n_tf_\zeta)(x)d\zeta$$

$$
\begin{aligned}
&= \int_{[-1,1]^d} i^{|\beta|} \big(\prod_{i=1}^{|\beta|} \zeta_{\beta_i} \big) e^{i\langle \zeta, y\rangle} \partial_x^\alpha (\overline{Q}_t^n f_\zeta)(x) d\zeta \\
&+ \int_{\mathbb{R}^d \setminus [-1,1]^d} i^{|\beta|} \big(\prod_{i=1}^{|\beta|} \zeta_{\beta_i} \big) e^{i\langle \zeta, y\rangle} \partial_x^\alpha (\overline{Q}_t^n f_\zeta)(x) d\zeta \\
&=: I + J
\end{aligned}
$$

Since $\|f_\zeta\|_\infty = 1$, we use (2.12) and we obtain: $|I| \leqslant CS^{-\eta(|\alpha|)} \leqslant CS^{-\eta(q)}$. Moreover, for any multi-index β', we have

$$
J = (-1)^{|\beta|} i^{|\beta'|} \int_{\mathbb{R}^d \setminus [-1,1]^d} \frac{e^{i\langle \zeta, y\rangle}}{\prod_{i=1}^{|\beta'|} \zeta_{\beta'_i}} \partial_x^\alpha (\overline{Q}_t^n \partial_{\beta'} \partial_\beta f_\zeta)(x) d\zeta.
$$

We take $\beta' = (2, \ldots, 2)$ and we obtain similarly $|J| \leqslant CS^{-\eta(q+2d)}$. We gather all the terms together and we obtain (2.13). Finally, we recall that the regularization properties $R_{q,\eta}(S)$ and $R^*_{q,\eta}(S)$ hold when $\overline{R}_{q,\eta}(S)$ is satisfied.

Theorem 2.6. *We recall that $T > 0$ and $n \in \mathbb{N}^*$. We have the following properties.*

A. *We fix $q \in \mathbb{N}$, h, $S \in [T/n, T/2)$ and $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ an increasing function. We assume that for every $m \in \mathbb{N}$, $m \geqslant n$, there exists some modifications $(\overline{Q}_t^m)_{t \in \pi_{T,m}}$ of $(Q_t^m)_{t \in \pi_{T,m}}$ such that (2.11) and (2.12) hold for these q, h, η and S. Moreover we assume that $E_m(h, q)$ (see (2.3)) and $E^*_m(h, q)$ (see (2.6)) hold between $(P_t^m)_{t \in \pi_{T,m}} = (P_t)_{t \in \pi_{T,m}}$ and $(Q_t^m)_{t \in \pi_{T,m}}$ and that (2.2) hold for Q^m. Then, we have*

$$
\sup_{t \in \pi_{T,n}^{2S,T}} \|P_t f - Q_t^n f\|_\infty \leqslant CS^{-\eta(q)} \|f\|_\infty / n^h. \tag{2.14}
$$

B. *Moreover, we suppose that the modifications $\overline{Q}$ of Q satisfy also $\overline{R}_{\overline{q},\eta}(S)$ (see (2.12)) for every $\overline{q} \in \mathbb{N}$. Then, for every $t > 0$, $P_t(x, dy) = p_t(x, y) dy$ with $(x, y) \mapsto p_t(x, y)$ belonging to $\mathcal{C}^\infty(\mathbb{R}^d \times \mathbb{R}^d)$.*

C. *For every $R > 0, \varepsilon \in (0, 1)$ and every multi-index α, β with $|\alpha| + |\beta| = u$, we also have*

$$
\sup_{t \in \pi_{T,n}^{2S,T}} \sup_{(x,y) \in \overline{B}_R(x_0, y_0)} |\partial_x^\alpha \partial_y^\beta p_t(x, y) - \partial_x^\alpha \partial_y^\beta \overline{p}_t^n(x, y)| \leqslant CS^{-\eta(p_{u,\varepsilon} \vee q)} / n^{h(1-\varepsilon)} \tag{2.15}
$$

with a constant C which depends on R, x_0, y_0, T and on $|\alpha| + |\beta|$ and $p_{u,\varepsilon} = (u + 2d + 1 + 2\lceil (1-\varepsilon)(u+d)/(2\varepsilon) \rceil)$.

Remark 2.7. *The inequality (2.14) is essentially a consequence of Proposition 2.4. However, we may not use directly this result, because we do not assume that the semigroup $(P_t)_{t \geqslant 0}$ has the regularization property (2.7) or even the less restrictive hypothesis (2.2). It simply satisfies (2.1). This is a result of main interest since we have to check the regularization properties for the approximation scheme Q^n only (more precisely for every $Q^m, m \geqslant n$). Indeed, in concrete applications, it can be cumbersome to study the regularization property for P. Using this result, it is not necessary anymore. Consequently in this paper, we will only study the regularization properties of the approximation Markov chain (1.1) and we will give sufficient conditions in order to obtain those properties.*

Remark 2.8. *The estimate (2.15) is sub-optimal because of $\varepsilon > 0$. One may wonder if optimal estimates (with n^h instead of $n^{h(1-\varepsilon)}$) may be obtained - as it was the case in* *the paper of Bally and Talay [6] concerning the Euler scheme. Notice that, in the above paper, specific properties related to the dynamics of the diffusion process which gives the semigoup are used, and in particular properties of the tangent flow. For example, if $X_t(x)$ denotes the diffusion process starting from x then we have $\mathbb{E}[f'(X_t(x))] = \partial_x \mathbb{E}[f(X_t(x))(\partial_x X_t(x))^{-1}] - \mathbb{E}[f(X_t(x))\partial_x(\partial_x X_t(x))^{-1})]$. Such properties are crucial in the above paper - but are difficult to express in terms of general semigroups.*

Proof. We prove **A** first. We fix $n \in \mathbb{N}^*$. Now we introduce the sequence of discrete semigroups $((Q_t^{n,m})_{t\in\pi_{T,n}})_{m\in\mathbb{N}^*}$ defined in the following way: For all $t \in \pi_{T,n}$ we have $Q_t^{n,m} f(x) = Q_t^{nm} f(x)$. Let $m' \geqslant m$, then

$$\begin{aligned}\|Q^{n,m}_{t^n_k,t^n_{k+1}} f - Q^{n,m'}_{t^n_k,t^n_{k+1}} f\|_\infty &= \|Q^{n,m}_{t^{nm}_{mk},t^{nm}_{m(k+1)}} f - Q^{n,m'}_{t^{nm'}_{m'k},t^{nm'}_{m'(k+1)}} f\|_\infty \\ &\leqslant \|Q^{nm}_{t^{nm}_{mk},t^{nm}_{m(k+1)}} f - P^{nm}_{t^{nm}_{mk},t^{nm}_{m(k+1)}} f\|_\infty \\ &\quad + \|P^{nm'}_{t^{nm'}_{m'k},t^{nm'}_{m'(k+1)}} f - Q^{nm'}_{t^{nm'}_{m'k},t^{nm'}_{m'(k+1)}} f\|_\infty\end{aligned}$$

Since Q^{nm} and $Q^{nm'}$ verify respectively $E_{nm}(h,q)$ and $E_{nm'}(h,q)$ and both Q^{nm} and $Q^{nm'}$ satisfy (2.2), we use the Lindeberg decomposition (2.5) in order to obtain: $\|Q^{n,m}_{t^n_k,t^n_{k+1}} f - Q^{n,m'}_{t^n_k,t^n_{k+1}} f\|_\infty \leqslant C\|f\|_{q,\infty}/(n^{h+1}m^h)$. In the same way we obtain $|\langle g, Q^{n,m}_{t^n_k,t^n_{k+1}} f - Q^{n,m'}_{t^n_k,t^n_{k+1}} f\rangle| \leqslant C\|g\|_{1,q}\|f\|_\infty/(n^{h+1}m^h)$. Now, since both Q^{nm} and $Q^{nm'}$ have modifications which satisfy (2.11) and (2.12), we use the same reasoning as in the proof of Proposition 2.4 and it follows that: $\forall t \in \pi^{2S,T}_{T,n}, \|Q^{n,m}_t f - Q^{n,m'}_t f\|_\infty \leqslant CS^{-\eta(q)}\|f\|_\infty/(n^h m^h)$. The sequence $((Q_t^{n,m})_{t\in\pi_{T,n}})_{m\in\mathbb{N}^*}$ is thus Cauchy and it converges toward $(P^n_t)_{t\in\pi_{T,n}}$ for smooth test functions using Proposition 2.2. In particular, taking $m=1$ and letting m' tend to infinity in the previous inequality we have

$$\forall t \in \pi^{2S,T}_{T,n}, \quad \|Q^{n,1}_t f - P^n_t f\|_\infty \leqslant CS^{-\eta(q)}\|f\|_\infty/n^h,$$

which is (2.14). Let us prove **C**. We are going to use a result from [8]. First, we introduce some notations. For $q \in \mathbb{N}$, we introduce the distance d_q defined by

$$d_q(\mu,\nu) = \sup\left\{|\textstyle\int f d\mu - \int f d\nu\| : \|f\|_{q,\infty} \leqslant 1\right\}.$$

For $q,\ l \in \mathbb{N}$, $r > 1$ and $f \in \mathcal{C}^q(\mathbb{R}^d \times \mathbb{R}^d)$, we denote

$$\|f\|_{q,l,r} = \sum_{0\leqslant|\alpha|\leqslant q} \left(\textstyle\int\int (1+|x|^l+|y|^l)|\partial_\alpha f(x,y)|^r dxdy\right)^{1/r}.$$

Since we want to show how the constant depends from S in the right hand side of (2.15), we will use a variant of Theorem 2.11 from [8].

Proposition 2.9. *Let $p,\tilde{p} \in \mathbb{N}$, $m \in \mathbb{N}^*$ and $r > 1$ be given and let r^* be the conjugate of r. We consider some measures $\mu(dx,dy)$ and $\mu_{g_n}(dx,dy) = g_n(x,y)dxdy$ with $g_n \in \mathcal{C}^{p+2m}(\mathbb{R}^d \times \mathbb{R}^d)$ and we assume that there exists $K_\mu, K_{g,p,m} \geqslant 1$, $h \in \mathbb{N}^*$, such that*

$$d_{\tilde{p}}(\mu,\mu_{g_n}) \leqslant K_\mu/n^h, \quad \|g_n\|_{p+2m,2m,r} \leqslant K_{g,p,m}, \qquad \forall n \in \mathbb{N}. \tag{2.16}$$

Then $\mu(dx,dy) = g(x,y)dxdy$ where g belongs to the Sobolev space $W^{p,r}(\mathbb{R}^d)$ and for all $\zeta > (p+\tilde{p}+d/r^*)/m$, there exists a universal constant $C \geqslant 1$ such that

$$\|g-g_n\|_{W^{p,r}(\mathbb{R}^d)} \leqslant C\mathfrak{C}_{h,m\zeta,p+\tilde{p}+d/r^*}(K_{g,p,m}n^{-2h/\zeta} + K_\mu n^{-h+h(p+\tilde{p}+d/r^*)/(\zeta m)}). \tag{2.17}$$

with $\mathfrak{C}_{h,\xi,u} = 2^{h+u}(1-2^{-\xi+u})^{-1}$.

Proof. For $k,n\in\mathbb{N}$, we introduce

$$n_k=\min\{n;n^h\geqslant 2^{\zeta km}\},\quad \text{and}\quad k_n=\min\{k\in\mathbb{N};n_k\geqslant n\}$$

First, we notice that $n_{k_n-1}<n\leqslant n_{k_n}$. Moreover, if we define $C_2=2^{\zeta m},C_1=2^{-h}$, we have

$$C_1n^h\leqslant 2^{\zeta k_n m}\leqslant C_2n^h. \tag{2.18}$$

Indeed $n^h\geqslant n^h_{k_n-1}$ which gives C_2. In order to obtain C_1, we notice that $n_{k_n}\leqslant 1+2^{\zeta k_n m/h}$. Now, we fix $n\in\mathbb{N}$ and for $k\in\mathbb{N}^*$, we define

$$\tilde g_k=0 \text{ if } k<k_n \text{ and } \tilde g_k=g_{n_k}-g_n,\ \text{ if } k\geqslant k_n$$

and $\nu(dx)=\mu(dx)-g_n(x)dx,\nu_k(dx)=\tilde g_k(x)dx$. Using Proposition 2.5 and Theorem 2.6 in [8], it follows that

$$\|g-g_n\|_{W^{p,r}(\mathbb{R}^d)}\leqslant\sum_{k=1}^\infty 2^{k(p+\tilde p+d/r^*)}d_{\tilde p}(\nu,\nu_k)+\sum_{k=1}^\infty 2^{-2mk}\|\tilde g_k\|_{p+2m,2m,\tilde p}=:T_1+T_2$$

First, we estimate T_1. If $k<k_n$, we have $\nu_k=0$ so that $d_{\tilde p}(\nu,\nu_k)=d_{\tilde p}(\nu,0)=d_{\tilde p}(\mu,\mu_{g_n})\leqslant K_\mu/n^h$. On the other and, if $k\geqslant k_n$, we have $d_{\tilde p}(\nu,\nu_k)=d_{\tilde p}(\mu,\mu_{g_{n_k}})\leqslant K_\mu n_k^{-h}\leqslant K_\mu 2^{-km\zeta}$. Using (2.18) together with all $\zeta>(p+\tilde p+d/r^*)/m$, it follows that

$$\begin{aligned}T_1\leqslant&K_\mu 2^{k_n(p+\tilde p+d/r^*)}n^{-h}+(1-2^{-m\zeta+p+\tilde p+d/r^*})^{-1}K_\mu 2^{-k_n(m\zeta-p-\tilde p-d/r^*)}\\ \leqslant&2(1-2^{-m\zeta+p+\tilde p+d/r^*})^{-1}C_2^{(p+\tilde p+d/r^*)/(\zeta m)}C_1^{-1}K_\mu/n^{h(1-(p+\tilde p+d/r^*)/(\zeta m))}.\end{aligned}$$

Now, we estimate T_2. Using (2.16) and (2.18) again, we have

$$T_2\leqslant 2K_{g,p,m}\sum_{k=k_n}^\infty 2^{-2mk}\leqslant 2(1-2^{-2m})^{-1}C_1^{-1}K_{g,p,m}n^{-2h/\zeta},$$

and since $m\geqslant 1$, the proof is completed. □

We come back to our framework. We fix $R>0$, $t\in\pi^{2S,T}_{T,n}$. We consider a function $\Phi_R\in\mathcal{C}^\infty_b(\mathbb{R}^d\times\mathbb{R}^d)$ such that $\mathbb{1}_{\overline{B}_R(x_0,y_0)}(x,y)\leqslant\Phi_R(x,y)\leqslant\mathbb{1}_{B_{R+1}(x_0,y_0)}$ and we denote

$$g^{n,R}_t(x,y)=\Phi_R(x,y)\overline{p}^n_t(x,y).$$

We use the result above for the sequence $g_n:=g^{n,R}_t,\ n\in\mathbb{N}$ and $\mu(dx,dy)=\Phi_R(x,y)\times P_t(x,dy)dx$. In our specific case (2.11) and (2.14) give $d_0(\mu,\mu_{g_n})\leqslant CS^{-\eta(q)}n^{-h}$. Since we have also (2.13), it follows that (2.16) hold with $K_\mu=CS^{-\eta(q)}$ and $K_{g,p,m}=CS^{-\eta(p+2m+2d)}$. We deduce from Proposition 2.9 that $\Phi_R(x,y)P_t(x,dy)dx=\mu(dx,dy)=g(x,y)dxdy$ with $g\in W^{p,r}(\mathbb{R}^d)$. Moreover, using Sobolev's embedding theorem, for $\zeta>(p+d/r^*)/m$ and $u\leqslant p-d/r$ we have

$$\begin{aligned}\|g-g_n\|_{u,\infty}&\leqslant C\|g-g_n\|_{W^{p,r}(\mathbb{R}^d)}\\&\leqslant C\mathfrak{C}_{h,m\zeta,p+d/r^*}(S^{-\eta(p+2m+2d)}n^{-2h/\zeta}+S^{-\eta(q)}n^{-h+h(p+d/r^*)/(\zeta m)}).\end{aligned}$$

We take $u=|\alpha|+|\beta|$, $r=d$, $p=u+1$ and $m=\lceil(1-\varepsilon)(u+d)/(2\varepsilon)\rceil$ and put $\zeta=2/(1-\epsilon)$. In this case $\zeta\geqslant(p+d/r^*)/m+2$ and we obtain

$$\begin{aligned}&\|g-g_n\|_{|\alpha|+|\beta|,\infty}\\&\quad\leqslant C2^{h+u+d}(S^{-\eta(u+2d+1+2\lceil(1-\varepsilon)(u+d)/(2\varepsilon)\rceil)}n^{-h(1-\varepsilon)}+S^{-\eta(q)}n^{-h(1-\varepsilon)}).\end{aligned}$$ □

3 Integration by parts using a splitting method

In this section, we aim to build some integration by part formulas in order to prove the regularization properties. This kind of formulas is widely studied in Malliavin calculus for the Gaussian framework. However, since we are interested in random variables with form (1.1), where the random variables laws of $Z_k, k \in \mathbb{N}^*$ are arbitrary (and thus not Gaussian) the standard Malliavin calculus is not adapted anymore. Therefore, we whether develop a finite dimensional differential calculus which happens to be well suited to our framework as soon as Z_k involves a regular part.

Concretely, we consider a sequence of independent random variables $Z_k = (Z_k^1, \dots, Z_k^N) \in \mathbb{R}^N$, $k \in \{1, \dots, n\}$ and we denote $Z = (Z_1, \dots, Z_n)$. The number n is fixed throughout this section (so there is no asymptotic procedure going on even if n is large in concrete applications since we are interested in estimating the error as $n \to \infty$). We aim to build integration by parts formulas based on the random vectors Z. The basic required assumption to obtain those formulas is the following: There exists $z_* = (z_{*,k})_{k \in \mathbb{N}^*}$ taking its values in $\mathbb{R}^N$ and $\varepsilon_*, r_* > 0$ such that for every Borel set $A \subset \mathbb{R}^N$ and every $k \in \{1, \dots, n\}$

$$L_{z_*}(\varepsilon_*, r_*) \qquad \mathbb{P}(Z_k \in A) \geqslant \varepsilon_* \lambda(A \cap B_{r_*}(z_{*,k})) \tag{3.1}$$

where λ is the Lebesgue measure on $\mathbb{R}^N$. We also define

$$M_p(Z) := 1 \vee \sup_{k \in \{1, \dots, n\}} \mathbb{E}[|Z_k|^p] \tag{3.2}$$

and assume that $M_p(Z) < \infty$ for every $p \geqslant 1$.

It is easy to check that (3.1) holds if and only if there exists some non negative measures μ_k with total mass $\mu_k(\mathbb{R}^N) < 1$ and a lower semi-continuous function $\varphi \geqslant 0$ such that $\mathbb{P}(Z_k \in dz) = \mu_k(dz) + \varphi(z - z_{*,k})dz$. Notice that the random variables $Z_1, \dots, Z_n$ are not assumed to be identically distributed. However, the fact that $r_* > 0$ and $\varepsilon_* > 0$ are the same for all k represents a mild substitute of this property. In order to construct φ we have to introduce the following function: For $v > 0$, set $\varphi_v : \mathbb{R}^N \to \mathbb{R}$ defined by

$$\varphi_v(z) = \mathbb{1}_{|z| \leqslant v} + \exp\left(1 - \frac{v^2}{v^2 - (|z| - v)^2}\right) \mathbb{1}_{v < |z| < 2v}. \tag{3.3}$$

Then $\varphi_v \in \mathcal{C}_b^\infty(\mathbb{R}^N)$, $0 \leqslant \varphi_v \leqslant 1$ and we have the following crucial property: For every $p, k \in \mathbb{N}$ there exists a universal constant $C_{q,p}$ such that for every $z \in \mathbb{R}^N$, $q \in \mathbb{N}$ and $i_1, \dots, i_q \in \{1, \dots, N\}$, we have

$$\varphi_v(z) \left| \frac{\partial^q}{\partial z^{i_1} \cdot \partial z^{i_q}} (\ln \varphi_v)(z) \right|^p \leqslant \frac{C_{q,p}}{v^{pq}}, \tag{3.4}$$

with the convention $\ln \varphi_v(z) = 0$ for $|z| \geqslant 2v$. As an immediate consequence of (3.1), for every non negative function $f : \mathbb{R}^N \to \mathbb{R}_+$

$$\mathbb{E}[f(Z_k)] \geqslant \varepsilon_* \int_{\mathbb{R}^N} \varphi_{r_*/2}(\ z - z_{*,k}\ ) f(z) dz.$$

By a change of variable

$$\mathbb{E}[f(\frac{1}{\sqrt{n}} Z_k)] \geqslant \varepsilon_* \int_{\mathbb{R}^N} n^{N/2} \varphi_{r_*/2}\big(\sqrt{n}(z - \frac{z_{*,k}}{\sqrt{n}})\big) f(z) dz. \tag{3.5}$$

We denote

$$m_* = \varepsilon_* \int_{\mathbb{R}^N} \varphi_{r_*/2}(z) dz = \varepsilon_* \int_{\mathbb{R}^N} \varphi_{r_*/2}(z - z_{*,k}) dz$$

and

$$\phi_n(z) = n^{N/2}\varphi_{r_*/2}(\sqrt{n}z)$$

and we notice that $\int \phi_n(z)dz = m_*\varepsilon_*^{-1}$.

We consider a sequence of independent random variables $\chi_k \in \{0,1\},\ U_k, V_k \in \mathbb{R}^N$, $k \in \{1,\ldots,n\}$ with laws given by

$$\begin{aligned}
\mathbb{P}(\chi_k &= 1) = m_*, \qquad \mathbb{P}(\chi_k = 0) = 1-m_*, \\
\mathbb{P}(U_k &\in dz) = \frac{\varepsilon_*}{m_*}\phi_n(z-\frac{z_{*,k}}{\sqrt{n}})dz, \\
\mathbb{P}(V_k &\in dz) = \frac{1}{1-m_*}(\mathbb{P}(\frac{1}{\sqrt{n}}Z_k \in dz) - \varepsilon_*\phi_n(z-\frac{z_{*,k}}{\sqrt{n}})dz).
\end{aligned} \tag{3.6}$$

Notice that (3.5) guarantees that $\mathbb{P}(V_k \in dz) \geqslant 0$. Then a direct computation shows that

$$\mathbb{P}(\chi_k U_k + (1-\chi_k)V_k \in dz) = \mathbb{P}(\frac{1}{\sqrt{n}}Z_k \in dz). \tag{3.7}$$

This is the splitting procedure for $\frac{1}{\sqrt{n}}Z_k$. Now on we will work with this representation of the law of $\frac{1}{\sqrt{n}}Z_k$. So, we always take

$$\frac{1}{\sqrt{n}}Z_k = \chi_k U_k + (1-\chi_k)V_k.$$

Remark 3.1. *The above splitting procedure has already been widely used in the litterature: In [31] and [25], it is used in order to prove convergence to equilibrium of Markov processes. In [10], [11] and [35], it is used to study the Central Limit Theorem. Last but not least, in [30], the above splitting method (with $\mathbb{1}_{B_{r_*}(z_{*,k})}$ instead of $\phi_n(z-\frac{z_{*,k}}{\sqrt{n}})$) is used in a framework which is similar to the one in this paper.*

In the following, we will denote $\chi = (\chi_1,\ldots,\chi_n)$, $U = (U_1,\ldots,U_n)$ and $V = (V_1,\ldots,V_n)$ and we will consider the class of random variables:

$$\mathcal{S} = \{F = f(\chi,U,V) : f \text{ is measurable and } u \to f(\chi,u,v) \in \mathcal{C}_b^\infty(\mathbb{R}^n\times\mathbb{R}^N), \forall \chi, v\}. \tag{3.8}$$

We will also denote $\mathcal{S}^d$ the space of d-dimensional vectors with components that belong to $\mathcal{S}$. For a multi index $\alpha = (\alpha_1,\ldots,\alpha_q)$ with $\alpha_j = (k_j,i_j)$, $k_j \in \{1,\ldots,n\},\ i_j \in \{1,\ldots,N\}$, we denote $|\alpha| = q$ the length of α and

$$\partial_u^\alpha f(\chi,u,v) = \frac{\partial^q}{\partial u_{k_1}^{i_1}\cdots\partial u_{k_q}^{i_q}} f(\chi,u,v).$$

We construct now a differential calculus based on the laws of the random variables $U_k,\ k=1,\ldots,n$ which mimics the Malliavin calculus, following the ideas from [5], [3] and [4]. In order to be self contained, we shortly present the results that we need. For $F = f(\chi,U,V) \in \mathcal{S}$ we define the Malliavin derivatives

$$D_{(k,i)}F = \chi_k\frac{1}{\sqrt{n}}\frac{\partial F}{\partial U_k^i} = \chi_k\frac{1}{\sqrt{n}}\frac{\partial f}{\partial u_k^i}(\chi,U,V), \qquad k=1,\ldots,n,\ \ i=1,\ldots,N. \tag{3.9}$$

We denote by $\langle\cdot,\cdot\rangle$ the usual scalar product on $\mathbb{R}^N\times\mathbb{R}^n$. The Malliavin covariance matrix for a multi dimensional functional $F = (F^1,\ldots,F^d)$ is defined as

$$\sigma_F^{i,j} = \left\langle DF^i, DF^j\right\rangle = \sum_{k=1}^n\sum_{r=1}^N D_{(k,r)}F^i \times D_{(k,r)}F^j, \qquad i,j=1,\ldots,d. \tag{3.10}$$

The higher order derivatives are defined by iterating D:

$$D_\alpha F = D_{\alpha_1}\cdots D_{\alpha_m}F.$$

Now we define the Ornstein Uhlenbeck operator $L:\mathcal{S}\to\mathcal{S}$. We denote

$$\Gamma_k = \ln\phi_n(U_k - \frac{z_{*,k}}{\sqrt{n}}) \in \mathcal{S}$$

and we notice that

$$\begin{aligned}
D_{(k,i)}\Gamma_k &= \frac{1}{\sqrt{n}}\chi_k\partial_{u_k^i}\ln\phi_n(U_k - \frac{z_{*,k}}{\sqrt{n}}) = \frac{1}{\sqrt{n}}\chi_k\partial_{u_k^i}\ln\phi_n(u_k - \frac{z_{*,k}}{\sqrt{n}})|_{u_k=U_k} \\
&= \chi_k\partial_{z^i}\ln\varphi_{r_*/2}(z)|_{z=\sqrt{n}(U_k-\frac{z_{*,k}}{\sqrt{n}})}.
\end{aligned}$$

Finally, we define

$$-LF = \sum_{k=1}^n\sum_{i=1}^N D_{(k,i)}D_{(k,i)}F + \sum_{k=1}^n\sum_{i=1}^N D_{(k,i)}F\times D_{(k,i)}\Gamma_k.$$

Remark 3.2. *The basic random variables in our calculus are $Z_k,\ k=1,\ldots,n$ so we precise the way in which the differential operators act on them. Since $Z_k=\sqrt{n}\chi_kU_k+\sqrt{n}(1-\chi_k)V_k$, it follows that*

$$D_{(m,j)}Z_k^i = \chi_k\delta_{m,k}\delta_{i,j}, \tag{3.11}$$
$$LZ_k^i = -\chi_k\partial_{z^i}\ln\varphi_{r_*/2}(z)|_{z=\sqrt{n}(U_k-\frac{z_{*,k}}{\sqrt{n}})}. \tag{3.12}$$

where $\delta_{i,j}=1$ if $i=j$ and 0 if $i\neq j$, stands for the Kroenecker symbol.

In our framework, the duality formula in Malliavin calculus reads as follows: For each $F,G\in\mathcal{S}$

$$\mathbb{E}[FLG] = \mathbb{E}[\langle DF, DG\rangle] = \mathbb{E}[GLF]. \tag{3.13}$$

This follows immediately using the independence structure and standard integration by parts on $\mathbb{R}^N$: Indeed, if $f,g\in\mathcal{C}_b^1(\mathbb{R}^N)$ and $k\in\{1,\ldots,n\}$, then

$$\begin{aligned}
&\sum_{i=1}^N\mathbb{E}[\partial_{u_k^i}f(U_k)\partial_{u_k^i}g(U_k)] \\
&= \frac{\varepsilon_*}{m_*}\sum_{i=1}^N\int_{\mathbb{R}^N}\partial_{u_k^i}f(u)\partial_{u_k^i}g(u)\phi_n(u-\frac{z_{*,k}}{\sqrt{n}})du \\
&= -\frac{\varepsilon_*}{m_*}\sum_{i=1}^N\int_{\mathbb{R}^N}f(u)(\partial^2_{u_k^i}g(u)+\partial_{u_k^i}g(u)\frac{\partial_{u_k^i}\phi_n(u-\frac{z_{*,k}}{\sqrt{n}})}{\phi_n(u-\frac{z_{*,k}}{\sqrt{n}})})\phi_n(u-\frac{z_{*,k}}{\sqrt{n}})du \\
&= -\mathbb{E}\Big[f(U_k)\sum_{i=1}^N\partial^2_{u_k^i}g(U_k)+\partial_{u_k^i}g(U_k)\partial_{u_k^i}\ln\phi_n(U_k-\frac{z_{*,k}}{\sqrt{n}})\Big].
\end{aligned}$$

It follows that

$$\begin{aligned}
&\sum_{k=1}^n\sum_{i=1}^N\mathbb{E}[D_{(k,i)}F\times D_{(k,i)}G] \\
&= \frac{1}{n}\sum_{k=1}^n\sum_{i=1}^N\mathbb{E}[\chi_k\partial_{u_k^i}f(\chi,U,V)\times\partial_{u_k^i}g(\chi,U,V)]
\end{aligned}$$

$$
\begin{aligned}
&= -\mathbb{E}\Big[f(\chi,U,V)\sum_{k=1}^{n}\chi_k\sum_{i=1}^{N}\frac{1}{n}\partial^2_{u^i_k}g(\chi,U,V)+\frac{1}{\sqrt{n}}\partial_{u^i_k}g(\chi,U,V)\frac{1}{\sqrt{n}}\partial_{u^i_k}\ln\phi_n(U_k-\frac{z_{*,k}}{\sqrt{n}})\Big] \\
&= -\mathbb{E}\Big[f(\chi,U,V)\sum_{k=1}^{n}\sum_{i=1}^{N}D_{(k,i)}D_{(k,i)}G+D_{(k,i)}GD_{(k,i)}\Gamma_k\Big] \\
&= \mathbb{E}[FLG],
\end{aligned}
$$

which is exactly (3.13). We have the following standard chain rule: For $\phi\in\mathcal{C}^1(\mathbb{R}^d)$ and $F\in\mathcal{S}^d$

$$
D\phi(F)=\sum_{j=1}^{d}\partial_j\phi(F)DF^j. \tag{3.14}
$$

Moreover, one may prove, using (3.14) and the duality relation (or direct computation), that

$$
L\phi(F)=\sum_{j=1}^{d}\partial_j\phi(F)LF^j+\sum_{i,j=1}^{d}\partial_i\partial_j\phi(F)\left\langle DF^i,DF^j\right\rangle. \tag{3.15}
$$

In particular for $F,G\in\mathcal{S}$,

$$
L(FG)=FLG+GLF+2\left\langle DF,DG\right\rangle. \tag{3.16}
$$

We are now able to give the Malliavin integration by parts formula:

Theorem 3.3. *Let $F\in\mathcal{S}^d$ and $G\in\mathcal{S}$ be such that $\mathbb{E}[(\det\sigma_F)^{-p}]<\infty$ for every $p\geqslant 1$. We denote $\gamma_F=\sigma_F^{-1}$. Then for every $\phi\in\mathcal{C}_c^\infty(\mathbb{R}^d)$ and every $i=1,\ldots,d$*

$$
\mathbb{E}[\partial_i\phi(F)G]=\mathbb{E}[\phi(F)H_i(F,G)] \tag{3.17}
$$

with

$$
-H(F,G)=G\gamma_F LF+\langle D(G\gamma_F),DF\rangle \tag{3.18}
$$

and

$$
H_i(F,G)=-\sum_{j=1}^{d}G\gamma_F^{i,j}LF^j+\langle D(G\gamma_F^{i,j}),DF^j\rangle.
$$

Moreover, for every multi index $\alpha=(\alpha_1,\ldots,\alpha_m)\in\{1,\ldots,d\}^m$

$$
\mathbb{E}[\partial_\alpha\phi(F)G]=\mathbb{E}[\phi(F)H_\alpha(F,G)] \tag{3.19}
$$

with $H_\alpha(F,G)$ defined by the recurrence relation $H_{(\alpha_1,\cdots,\alpha_m)}(F,G)=H_{\alpha_m}(F,H_{(\alpha_1,\ldots,\alpha_{m-1})}(F,G))$.

Proof. Using the chain rule $D\phi(F)=\nabla\phi(F)DF$ we have

$$
\langle D\phi(F),DF\rangle=\nabla\phi(F)\left\langle DF,DF\right\rangle=\nabla\phi(F)\sigma_F.
$$

It follows that $\nabla\phi(F)=\gamma_F\left\langle D\phi(F),DF\right\rangle$. Then, using (3.16) and the duality formula (3.13),

$$
\begin{aligned}
\mathbb{E}[G\nabla\phi(F)] &= \mathbb{E}[G\gamma_F\left\langle D\phi(F),DF\right\rangle]=\frac{1}{2}\mathbb{E}[G\gamma_F(L(\phi(F)F)-\phi(F)LF-FL\phi(F))] \\
&= \frac{1}{2}\mathbb{E}[\phi(F)(FL(G\gamma_F)-G\gamma_F LF-L(G\gamma_F F))].
\end{aligned}
$$

We use once again (3.16) in order to obtain $H(F,G)$ in (3.18). □

We give now estimates of the weights $H_\alpha(F,G)$ which appear in the above integration by parts formulas. We will work with the norms:

$$|F|_{1,q}^2 = \sum_{1\leqslant|\alpha|\leqslant q} |D_\alpha F|^2, \qquad |F|_q^2 = |F|^2 + |F|_{1,q}^2, \tag{3.20}$$

and

$$\begin{aligned} \|F\|_{1,q,p} &= \big\||F|_{1,q}\big\|_p = \mathbb{E}[|F|_{1,q}^p]^{1/p} \\ \|F\|_{q,p} &= \|F\|_p + \big\||F|_{1,q}\big\|_p. \end{aligned} \tag{3.21}$$

Proposition 3.4. *For each $m,q\in\mathbb{N}$, there exists a universal constant $C\geqslant 1$ (depending on d,m,q only) such that for every multi index α with $|\alpha|\leqslant q$ and every $F\in\mathcal{S}^d$ and $G\in\mathcal{S}$ on has*

$$|H_\alpha(F,G)|_m \leqslant C(1\vee(\det\sigma_F)^{-1})^{q(q+m+1)}(1+|F|_{1,m+q+1}^{2dq(q+m+2)}+|LF|_{m+q-1}^{2q})|G|_{m+q}. \tag{3.22}$$

The proof is long but straightforward so we skip it. The reader may find the detailed proof in [5] and in [3], Theorem 3.4.

We end this section with an estimate of $\|LZ_k^i\|_{q,p}$:

Lemma 3.5. *We have the following properties.*

A. *For every $k=1,\dots,n$ and $i=1,\dots,N$, we have*

$$\mathbb{E}[LZ_k^i]=0. \tag{3.23}$$

B. *For every $q\in\mathbb{N}$ and $p\geqslant 2$ there exists a constant C depending on $q,\ p$ only*

$$\|LZ_k^i\|_{q,p} \leqslant \frac{Cm_*^{1/p}}{r_*}(1+r_*^{-q}) \tag{3.24}$$

Proof. **A**. Using the duality relation we have $\mathbb{E}[1\times LZ_k^i]=\mathbb{E}[\langle D1, DZ_k^i\rangle]=0$. In order to prove **B** we recall (see (3.12)) that

$$LZ_k^i = -\chi_k\partial_i(\ln\varphi_{r_*/2})\big(\sqrt{n}(U_k-\frac{z_{*,k}}{\sqrt{n}})\big).$$

Let $\Lambda_{k,q}$ be the set of the multi-index $\alpha=(\alpha_1,\dots,\alpha_q)$ such that $\alpha_j=(k,i_j)$. Notice that for a multi-index α of length q, such that $\alpha\notin\Lambda_{k,q}$, we have $D_\alpha LZ_k^i=0$. Suppose now that $\alpha\in\Lambda_{k,q}$ and let $\overline{\alpha}=(i_1,\dots,i_q,i)$. It follows

$$D_\alpha LZ_k^i = -\chi_k\partial_{\overline{\alpha}}(\ln\varphi_{r_*/2})\big(\sqrt{n}(U_k-\frac{z_{*,k}}{\sqrt{n}})\big).$$

Since the function $\varphi_{r_*/2}$ is constant on $B_{r_*/2}(0)$ and on $\mathbb{R}^d\setminus\overline{B}_{r_*}(0)$, using (3.4), we obtain

$$\begin{aligned} \|D_\alpha LZ_k^i\|_p^p &= \frac{\varepsilon_*\|\chi_k\|_p^p}{m_*}\int_{\mathbb{R}^N} n^{N/2}\big|\partial_{\overline{\alpha}}(\ln\varphi_{r_*/2})\big(\sqrt{n}(u-\frac{z_{*,k}}{\sqrt{n}})\big)\big|^p\varphi_{r_*/2}\big(\sqrt{n}(u-\frac{z_{*,k}}{\sqrt{n}})\big)du \\ &= \frac{\varepsilon_*\|\chi_k\|_p^p}{m_*}\int_{r_*/2\leqslant|u|\leqslant r_*}\big|\partial_{\overline{\alpha}}(\ln\varphi_{r_*/2})(u)\big|^p\varphi_{r_*/2}(u)du \\ &\leqslant \frac{C_{q+1,p}m_*}{r_*^{p(q+1)}}. \end{aligned}$$

and then

$$\|LZ_k^i\|_{q,p}\leqslant C\sup_{l\leqslant q}\sup_{\alpha\in\Lambda_{k,l}}\|D_\alpha LZ_k^i\|_p \leqslant \frac{Cm_*^{1/p}}{r_*}(1+r_*^{-q}). \qquad \square$$

3.1 Localization

We have seen in Proposition 3.4 that we can bound the Sobolev norms of the weight which appear in the integration by part formula (3.19). In order to obtain the regularization properties, we will have to bound the moments of those Sobolev norms or more particularly, the moments of the terms which appear in the right hand side of (3.22). However, in many cases it is cumbersome to estimate $\mathbb{E}[(\det \sigma_F)^{-p}]$, $p \in \mathbb{N}$. The method adopted in this paper comes down to localize the calculus when $\det \sigma_F$ does not belong to a neighborhood of zero. Then, we will prove a similar property as (2.11) and we will obtain the convergence in total variation distance. More specifically, when $F = X^n$, we will have to localize the random variables Z_k and χ_k which appear in (1.1) with the decomposition (3.7). We introduce a suited framework to treat this problem.

In the following, we will not work under $\mathbb{P}$, but under a localized probability measure which we define now. We fix $S > 0$ such that $S \leqslant T$ and we consider the set

$$\Lambda_S = \{\frac{1}{\lfloor Sn/T \rfloor} \sum_{k=1}^{\lfloor Sn/T \rfloor} \chi_k \geqslant \frac{m_*}{2}\}. \tag{3.25}$$

Using Hoeffding's inequality and the fact that $\mathbb{E}[\chi_k] = m_*$, it can be checked that

$$\mathbb{P}(\Lambda_S^c) \leqslant \exp(-m_*^2 \lfloor Sn/T \rfloor /2). \tag{3.26}$$

We consider also the localization function $\varphi_{n^{1/4}/2}$, defined in (3.3), and we construct the random variable

$$\Theta = \Theta_{S,n} = \mathbb{1}_{\Lambda_S} \times \prod_{k=1}^{n} \varphi_{n^{1/4}/2}(Z_k). \tag{3.27}$$

Since Z_k has finite moments of any order, the following inequality can be shown: For every $l \in \mathbb{N}$ there exists C such that

$$\mathbb{P}(\Theta_{S,n} = 0) \leqslant \mathbb{P}(\Lambda_M^c) + \sum_{k=1}^{n} \mathbb{P}(|Z_k| \geqslant n^{1/4}) \leqslant \exp(-m_*^2 \lfloor Sn/T \rfloor /2) + \frac{M_{4(l+1)}(Z)}{n^l}. \tag{3.28}$$

We define the probability measure

$$d\mathbb{P}_\Theta = \frac{1}{\mathbb{E}[\Theta]} \Theta d\mathbb{P}. \tag{3.29}$$

Corollary 3.6. *Let $F \in \mathcal{S}^d$ and $G \in \mathcal{S}$ be such that $\mathbb{E}_\Theta[(\det \sigma_F)^{-p}] < \infty$ for every $p \geqslant 1$. We denote $\gamma_F = \sigma_F^{-1}$. Then, for every $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ and every $i = 1, \ldots, d$*

$$\mathbb{E}_\Theta[\partial_i \phi(F) G] = \mathbb{E}_\Theta[\phi(F) H_i^\Theta(F, G)] \tag{3.30}$$

with

$$-H^\Theta(F,G) = G\gamma_F LF + \langle D(G\gamma_F), DF \rangle + G\gamma_F \langle D \ln \Theta, DF \rangle$$

and

$$H_i^\Theta(F,G) = -\sum_{j=1}^{d} G\gamma_F^{i,j} LF^j + \langle D(G\gamma_F^{i,j}), DF^j \rangle + G\gamma_F^{i,j} \langle D \ln \Theta, DF^j \rangle.$$

And for every multi index $\alpha = (\alpha_1, \ldots, \alpha_m) \in \{1, \ldots, d\}^m$,

$$\mathbb{E}_\Theta[\partial_\alpha \phi(F) G] = \mathbb{E}_\Theta[\phi(F) H_\alpha^\Theta(F, G)], \tag{3.31}$$

with $H^{\Theta}_{\alpha}(F,G)$ *defined by the recurrence relation* $H^{\Theta}_{(\alpha_1,\cdots,\alpha_m)}(F,G) = H^{\Theta}_{\alpha_m}(F, H^{\Theta}_{(\alpha_1,\cdots,\alpha_{m-1})}(F,G))$, *and the convention* $\ln(\Theta)=0$ *for* $\Theta=0$. *Moreover there exists an universal constant* C *such that for every multi index* α *with* $|\alpha|=q$

$$\mathbb{E}_\Theta[|H^{\Theta}_{\alpha}(F,G)|^p_m] \leqslant CC_{q,\Theta}(F,G), \tag{3.32}$$

with

$$\begin{aligned} C_{q,\Theta}(F,G) &= \mathbb{E}_\Theta[(1\vee(\det\sigma_F)^{-1})^{2pq(q+m+1)}]^{1/2} \\ &\quad\times(1+\mathbb{E}_\Theta[|F|^{8pqd(q+m+2)}_{1,m+q+1}]^{1/4}+\mathbb{E}_\Theta[|LF|^{8pq}_{m+q-1}]^{1/4})\mathbb{E}_\Theta[|G|^{4p}_{m+q}]^{1/4}. \end{aligned} \tag{3.33}$$

Proof. Using (3.18) with G replaced by $G\Theta$ we obtain $\mathbb{E}[\partial_i\phi(F)G\Theta]=\mathbb{E}[\phi(F)\overline{H}_i]$ with

$$\overline{H} = -\Theta G\gamma_F LF - \langle D(\Theta G\gamma_F), DF\rangle = \Theta H(F,G) - G\gamma_F\langle D\Theta, DF\rangle.$$

It follows that

$$\begin{aligned} \mathbb{E}_\Theta[\partial_i\phi(F)G] &= \frac{1}{\mathbb{E}[\Theta]}\mathbb{E}[\partial_i\phi(F)G\Theta] = \frac{1}{\mathbb{E}[\Theta]}\mathbb{E}[\phi(F)(\Theta H_i(F,G) - G\sum_{j=1}^d \gamma_F^{i,j}\left\langle D\Theta, DF^j\right\rangle] \\ &= \mathbb{E}_\Theta[\phi(F)(H_i(F,G) - G\sum_{j=1}^d\gamma_F^{i,j}\left\langle D\ln\Theta, DF^j\right\rangle]. \end{aligned}$$

So (3.30) is proved and (3.31) follows by recurrence. Moreover

$$\begin{aligned} &\mathbb{E}_\Theta[|G\sum_{j=1}^d\gamma_F^{i,j}\left\langle D\ln\Theta, DF^j\right\rangle|^p] \\ &\leqslant C\mathbb{E}_\Theta[|D\ln(\Theta)^{4p}]^{1/4}\mathbb{E}_\Theta[|\gamma_F|^{4p}]^{1/4}\mathbb{E}_\Theta[|DF|^{4p}]^{1/4}\mathbb{E}_\Theta[|G|^{4p}]^{1/4}. \end{aligned}$$

Notice that by (3.4) we have

$$\mathbb{E}_\Theta[|D\ln\Theta|^{4p}]^{1/4p} \leqslant C/n^{1/4}.$$

Then (3.32) follows from (3.22). □

4 Convergence results for a class of Markov chain

Now we have introduced the integration by parts formulas which are adapted to our study, we are in a position to prove the regularization properties. In order to do it, we have to bound the miscellaneous terms which appear in the right hand side of (3.33). This section is devoted to the estimation of those terms. We will treat separately the estimation of the norm of the inverse of the covariance matrix from the other terms. Indeed, this study requires localization techniques which are not necessary in order to bound the Sobolev norms of the others terms. Then we will give the regularization properties and the total variation convergence results that follow from those estimates.

Throughout this section, $n\in\mathbb{N}^*$ will still be fixed and will be the number of time step between 0 and T and also the number of increments that we consider in our abstract Malliavin calculus. We consider two sequences of independent random variables $Z_{k+1}\in\mathbb{R}^N, \kappa_k\in\mathbb{R},\ k\in\mathbb{N}$ and we assume that Z_k, $k\in\mathbb{N}^*$, are centered and verify (3.1) and (3.2).

We suppose that, there exists $C\geqslant 1$ such that $\sup_{k\in\mathbb{N}^*}\delta_k^n\leqslant C/n$ and we construct the $\mathbb{R}^d$ valued Markov chain $(X^n_t)_{t\in\pi_{T,n}}$ in the following way:

$$X^n_{t^n_{k+1}} = \psi(\kappa_k, X^n_{t^n_k}, \frac{Z_{k+1}}{\sqrt{n}}, \delta^n_{k+1}), \qquad k\in\mathbb{N} \tag{4.1}$$

where

$$\psi \in \mathcal{C}^{\infty}(\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^N \times \mathbb{R}_+; \mathbb{R}^d) \quad \text{and} \quad \psi(\kappa, x, 0, 0) = x. \tag{4.2}$$

We introduce the norm

$$\|\psi\|_{1,r,\infty} = 1 \vee \sum_{|\alpha|=0}^{r} \sum_{|\beta|+|\gamma|=1}^{r-|\alpha|} \|\partial_x^{\alpha} \partial_z^{\beta} \partial_t^{\gamma} \psi\|_{\infty}. \tag{4.3}$$

Remark 4.1. *Notice that the random variables κ_k can be useful in concrete applications. Indeed, in the Ninomiya Victoir scheme, at each time step k, one throws a coin $\kappa_k \in \{1, -1\}$ and uses different forms for the function ψ according to the fact that κ_k is equal to 1 or to -1.*

Since the function ψ only needs to be measurable with respect to κ and that all our estimates will be done in terms of $\|\psi\|_{1,r,\infty}$, then without loss of generality, we can simplify the notations and denote

$$\psi_k(x, z, t) = \psi(\kappa_k, x, z, t).$$

Then, we slightly modify the definition (4.3) and instead, in the sequel, we will consider the norm

$$\|\psi\|_{1,r,\infty} = \sup_{k \in \mathbb{N}} \|\psi_k\|_{1,r,\infty} = 1 \vee \sup_{k \in \mathbb{N}} \sum_{|\alpha|=0}^{r} \sum_{|\beta|+|\gamma|=1}^{r-|\alpha|} \|\partial_x^{\alpha} \partial_z^{\beta} \partial_t^{\gamma} \psi_k\|_{\infty}, \tag{4.4}$$

with $(\psi_k)_{k \in \mathbb{N}}$ a sequence of functions that belong to $\mathcal{C}^r(\mathbb{R}^d \times \mathbb{R}^N \times \mathbb{R}_+; \mathbb{R}^d)$. It is worth noticing that all our results remain true if we replace the supremum over $k \in \mathbb{N}$ by the supremum over $k \in \mathbb{N}$ with $t_k^n < T$. However, for the sake of clarity, we will work with (4.4). Finally for $r \in \mathbb{N}^*$, we denote

$$\mathfrak{K}_r(\psi) = (1 + \|\psi\|_{1,r,\infty}) \exp(\|\psi\|_{1,3,\infty}^2). \tag{4.5}$$

We aim to give sufficient conditions under which the above Markov chain has the regularization property (2.12). In order to do it, we consider the following new representation of X^n. Let us introduce some notations. We denote

$$H_k = \frac{Z_k}{\sqrt{n}} = \chi_k U_k + (1 - \chi_k) V_k.$$

Using a Taylor expansion of order one, we write

$$\begin{aligned} X_{t_{k+1}^n}^n &= X_{t_k^n}^n + \sum_{i=1}^{N} \partial_{z_i} \psi_k(X_{t_k^n}^n, 0, 0) H_{k+1}^i + \delta_{k+1}^n \int_0^1 \partial_t \psi_k(X_{t_k^n}^n, H_{k+1}, \lambda \delta_{k+1}^n) d\lambda \\ &+ \frac{1}{2} \sum_{i,j=1}^{N} H_{k+1}^i H_{k+1}^j \int_0^1 (1 - \lambda) \partial_{z_i} \partial_{z_j} \psi_k(X_{t_k^n}^n, \lambda H_{k+1}, 0) d\lambda. \end{aligned}$$

We denote

$$\begin{aligned} a_k^i = \partial_{z_i} \psi_k(X_{t_k^n}^n, 0, 0),\ b_k^{i,j} &= \int_0^1 (1 - \lambda) \partial_{z_i} \partial_{z_j} \psi_k(X_{t_k^n}^n, \lambda H_{k+1}, 0) d\lambda,\ \tilde{b}_k \\ &= \int_0^1 \partial_t \psi_k(X_{t_k^n}^n, H_{k+1}, \lambda \delta_{k+1}^n) d\lambda, \end{aligned}$$

and then, we write

$$X^n_{t^n_m} = x + \sum_{i=1}^{N}\sum_{k=0}^{m-1} a^i_k H^i_{k+1} + \frac{1}{2}\sum_{i,j=1}^{N}\sum_{k=0}^{m-1} b^{i,j}_k H^i_{k+1} H^j_{k+1} + \sum_{k=0}^{m-1} \tilde{b}_k \delta^n_{k+1}. \tag{4.6}$$

Moreover we denote by $X^n(x)$ the Markov chain which starts from x (i.e. $X^n_0(x) = x$) and we denote by $\partial^\alpha X^n$ the derivative with respect to the starting point x. We will use the results from the previous section for X^n. In order to do it we have to estimate the Sobolev norms of X^n:

Theorem 4.2. *For every $q, q' \in \mathbb{N}$ with $q \geqslant q'$, and $p \geqslant 2$ there exists $l \in \mathbb{N}^*$, $C \geqslant 1$ which depend on $r_*, \varepsilon_*, m_*, q, p$ and the moments of Z, but not on n,such that*

$$\sup_{t\in\pi^T_{T,n}} \sup_{0\leqslant|\alpha|\leqslant q-q'} \|\partial^\alpha_x X^n_t(x)\|_{q',p} \leqslant C\mathfrak{K}_{q+2}(\psi)^l, \tag{4.7}$$

$$\sup_{t\in\pi^T_{T,n}} \|L X^n_t\|_{q,p} \leqslant C\mathfrak{K}_{q+4}(\psi)^l, \tag{4.8}$$

where $\mathfrak{K}_r(\psi)$ is defined in (4.5) and is given by

$$\mathfrak{K}_r(\psi) = (1 + \|\psi\|_{1,r,\infty}) \exp(\|\psi\|^2_{1,3,\infty}).$$

The proof is long and technical so we postpone it to Section 6.

4.1 The Malliavin covariance matrix

We turn now to the covariance matrix. We will work under the probability $\mathbb{P}_\Theta$ defined in (3.29). We recall that $T > 0$ and $n \in \mathbb{N}$ are given and we have denoted $\Lambda_S = \{\frac{1}{\lfloor nS/T\rfloor}\sum_{k=1}^{\lfloor nS/T\rfloor} \chi_k \geqslant \frac{m_*}{2}\}$. The localization random variable $\Theta_{S,n}$ is defined in (3.27) and we have proved in (3.28) that, for every $l \in \mathbb{N}$,

$$\mathbb{P}(\Theta_{S,n} = 0) \leqslant \exp(-m_*^2 \lfloor nS/T\rfloor/2) + \frac{M_{4(l+1)}(Z)}{n^l}.$$

We also have

$$\{\Theta_{S,n} \neq 0\} \subset \{\frac{1}{n}\sum_{k=1}^{\lfloor nS/T\rfloor} \chi_k \geqslant \frac{\lfloor nS/T\rfloor m_*}{2n}\} \cap \{|Z_k| \leqslant n^{1/4},\ k = 1,\ldots,n\}.$$

Using the computational rules for $k \in \{0,\ldots,m-1\}$ and $m \leqslant n$, we obtain

$$D_{(k+1,i)} X^n_{t^n_m} = I_{k,i} + \sum_{l=k+1}^{m-1} J_l D_{(k+1,i)} X^n_{t^n_l} \tag{4.9}$$

with

$$\begin{aligned}
I_{k,i} &= \frac{1}{\sqrt{n}}\chi_{k+1}\big(a^i_k + \sum_{j=1}^{N} H^j_{k+1} b^{i,j}_k + \sum_{j,q=1}^{N} H^j_{k+1} H^q_{k+1} c^{i,j,q}_k + \delta^n_{k+1}\tilde{c}^i_k\big), &(4.10)\\
c^{i,j,q}_k &= \frac{1}{\sqrt{n}}\chi_{k+1}\int_0^1 \lambda(1-\lambda)\partial_{z_i}\partial_{z_j}\partial_{z_q}\psi_k(X^n_{t^n_k}, \lambda H_{k+1}, 0)d\lambda &\\
\tilde{c}^i_k &= \int_0^1 \partial_{z_i}\partial_t \psi_k(X^n_{t^n_k}, H_{k+1}, \lambda\delta^n_{k+1})d\lambda &(4.11)
\end{aligned}$$

and the $d \times d$ dimensional matrices J_l, defined by

$$J^{p,r}_l = J^{p,r}_{l,0} + \sum_{j=1}^{N} J^{p,r}_l(j) + \sum_{j,q=1}^{N} J^{p,r}_l(j,q)$$

with

$$\begin{aligned} J_{l,0}^{p,r} &= \delta_{l+1}^n \int_0^1 \partial_{x_p}\partial_t(\psi_l(X_{t_l^n}^n, H_{l+1}, \lambda\delta_{l+1}^n))_r d\lambda, \\ J_l^{p,r}(j) &= H_{l+1}^j \partial_{x_p}\partial_{z_j}(\psi_l(X_{t_l^n}^n, 0, 0))_r, \\ J_l^{p,r}(j,q) &= H_{l+1}^j H_{l+1}^q \int_0^1 (1-\lambda)\partial_{x_p}\partial_{z_j}\partial_{z_q}(\psi_l(X_{t_l^n}^n, \lambda H_{l+1}, 0))_r d\lambda. \end{aligned}$$

We first aim to express $D_{(k+1,i)}X_t^n$ using the variance of constants method. We consider the tangent flow $Y_t^n = \nabla_x X_t^n(x)$, $t \in \pi_{T,n}$, which is the $d \times d$ dimensional matrix solution of

$$Y_{t_m^n}^n = I + \sum_{l=0}^{m-1} J_l Y_{t_l^n}^n,$$

where I is the identity matrix. The explicit solution of the above equation is given by $Y_{t_m^n}^n = \prod_{k=0}^{m-1}(I + J_k)$. If each of the matrices $I + J_k$, $k = 1, \ldots, m$, is invertible then, $Y_{t_m^n}^n$ is also invertible. On the set $\{\Theta_{t_m^n,n} \neq 0\}$, we have $|H_k| = |n^{-1/2} Z_k| \leqslant n^{-1/4}$ so that $\|J_k\|_\infty := \sup_{i,j\leqslant d} \|J_k^{i,j}\|_\infty \leqslant 3\|\psi\|_{1,3,\infty} n^{-1/4}$. It follows that, among others, if $\|\psi\|_{1,3,\infty} n^{-1/4} \leqslant 1/6$, then the lower eigenvalue of $I + J_k$ is larger then $1/2$, so we have the invertibility property. We denote by $(\widehat{Y}_t^n)_{t\in\pi_{T,n}}$ the inverse of $(Y_t^n)_{t\in\pi_{T,n}}$ and it is easy to check that $\widehat{Y}^n$ solves the equation:

$$\widehat{Y}_{t_m^n}^n = I - \sum_{l=0}^{m-1} \widehat{Y}_{t_l^n}^n (I + J_l)^{-1} J_l.$$

The following representation of the Malliavin derivative, known as the "variance of constants method", is given by

$$\forall t \in \pi_{T,n}, t \geqslant t_{k+1}^n \qquad D_{(k+1,i)}X_t^n = Y_t^n \widehat{Y}_{t_{k+1}^n}^n I_{k,i}, \tag{4.12}$$

and is zero if $t < t_{k+1}^n$. We will use the following estimates.

Lemma 4.3. *Let $p \geqslant 2$. There exists a constant $C \geqslant 1$, which depends on p and T, such that the following holds. Suppose that n and $t \in \pi_{T,n}^{0,T}$ are sufficiently large in order to have*

$$\frac{3\|\psi\|_{1,3,\infty}}{n^{1/4}} + \frac{M_8(Z)}{n} + \exp(-m_*^2 nt/(2T)) \leqslant \frac{1}{2}. \tag{4.13}$$

Then,

$$\mathbb{E}_{\Theta_{t,n}}[\sup_{s\in\pi_{T,n}^T} \|Y_s^n\|^p] \leqslant 2\exp\left(C(M_{2p}(Z)^{2/p} + M_4(Z)^2)(1 + \|\psi\|_{1,3,\infty}^2)\right), \tag{4.14}$$

and

$$\mathbb{E}_{\Theta_{t,n}}[\sup_{s\in\pi_{T,n}^T} \|\widehat{Y}_s^n\|^p] \leqslant 2\exp\left(C(M_{2p}(Z)^{2/p} + M_4(Z)^2)(1 + \|\psi\|_{1,3,\infty}^2)\right), \tag{4.15}$$

with

$$\|Y_t^n\| := \sup_{i,j\leqslant d} |(Y_t^n)_{i,j}|.$$

Proof. **Step 1**. We notice that on the set $\{\Theta_{t,n} \neq 0\}$ we have $H_l = \overline{H}_l := H_l \mathbb{1}_{\{|Z_l|\leqslant n^{1/4}\}}$. Consequently $J_l = \overline{J}_l := J_l \mathbb{1}_{\{|Z_{l+1}|\leqslant n^{1/4}\}}$ and $\widehat{Y}^n = \overline{Y}^n$ where $(\overline{Y}_t^n)_{t\in\pi_{T,n}}$ is the solution of the equation

$$\overline{Y}_{t_m^n}^n = I - \sum_{l=0}^{m-1} \overline{Y}_{t_l^n}^n (I + \overline{J}_l)^{-1} \overline{J}_l.$$

Moreover, we have

$$\mathbb{E}_{\Theta_{t,n}}[\sup_{s\in\pi_{T,n}} \|\widehat{Y}^n_s\|^p] \leqslant \frac{1}{\mathbb{E}[\Theta_{t,n}]}\mathbb{E}[\sup_{s\in\pi_{T,n}} \|\overline{Y}^n_s\|^p] \leqslant C\mathbb{E}[\sup_{s\in\pi_{T,n}} \|\overline{Y}^n_s\|^p],$$

the last inequality is a consequence of (3.28). Indeed

$$\mathbb{E}[\Theta_{t,n}] \geqslant 1-\mathbb{P}(\Theta_{t,n}=0) \geqslant 1-\exp(-m_*^2 nt/(2T)) - \frac{M_8(Z)}{n} \geqslant \frac{1}{2}.$$

The last inequality is true under the hypothesis (4.13). So, our task is now to estimate $\mathbb{E}[\sup_{s\in\pi^T_{T,n}} \|\overline{Y}^n_s\|^p]$.

Step 2. Let

$$\mathcal{F}_l = \sigma(\chi_i, U_i, V_i, i=1,\ldots,l).$$

Since, from (4.13), the lower eigenvalue of $(I+\overline{J}_l)$ is larger than $1/2$, then $\|(I+\overline{J}_l)^{-1}\| \leqslant 2$. It follows that $\|\overline{Y}^n_{t^n_l}(I+\overline{J}_l)^{-1}\overline{J}_l\| \leqslant 2\|\overline{Y}^n_{t^n_l}\|\|\overline{J}_l\|$ and since $\overline{Y}^n_{t^n_l}$ is $\mathcal{F}_l$ measurable, we obtain

$$\|\mathbb{E}[\overline{Y}^n_{t^n_l}(I+\overline{J}_l)^{-1}\overline{J}_l \mid \mathcal{F}_l]\| \leqslant 2\|\overline{Y}^n_{t^n_l}\|\mathbb{E}[\|\overline{J}_l\| \mid \mathcal{F}_l].$$

Now, we notice that $\mathbb{E}[\|\overline{J}^{p,r}_{l,0}\| \mid \mathcal{F}_l] \leqslant C\|\psi\|_{1,2,\infty}/n$ and

$$\begin{aligned}\mathbb{E}[\|\overline{J}^{p,r}_l(j)\| \mid \mathcal{F}_l] &\leqslant \frac{C}{\sqrt{n}}\|\psi\|_{1,2,\infty}\mathbb{E}[|Z^j_{l+1}|\mathbb{1}_{\{|Z_{l+1}|\geqslant n^{1/4}\}} \mid \mathcal{F}_l] \\ &\leqslant \frac{C}{n}\|\psi\|_{1,2,\infty}\mathbb{E}[|Z_{l+1}|^3] \leqslant \frac{C\|\psi\|_{1,2,\infty}M_3(Z)}{n}.\end{aligned}$$

Moreover, using the Hölder inequality, we obtain

$$\mathbb{E}[\|\overline{J}^{p,r}_l(j,q)\| \mid \mathcal{F}_l]| \leqslant C\frac{M_4(Z)^{1/2}\|\psi\|_{1,3,\infty}}{n}.$$

It follows that $\mathbb{E}[\|\overline{J}_l\| \mid \mathcal{F}_l] \leqslant CM_4(Z)\|\psi\|_{1,3,\infty}/n$ so, finally, we obtain

$$\|\mathbb{E}[\overline{Y}^n_{t^n_l}(I+\overline{J}_l)^{-1}\overline{J}_l \mid \mathcal{F}_l)\| \leqslant CM_4(Z)(1+\|\psi\|_{1,3,\infty})\|\overline{Y}^n_{t^n_l}\|/n. \tag{4.16}$$

Step 3. We are now ready to start our proof. We write

$$(\overline{Y}^n_{t^n_m})^{i,j} = \delta_{i,j} - \sum_{l=0}^{m-1}\theta^{i,j}_l \tag{4.17}$$

with

$$\theta^{i,j}_l = (\overline{Y}^n_{t^n_l}(I+\overline{J}_l)^{-1}\overline{J}_l)^{i,j}.$$

We denote

$$\widehat{\theta}_l = \mathbb{E}[\theta_l \mid \mathcal{F}_l], \qquad \widetilde{\theta}_l = \theta_l - \widehat{\theta}_l$$

and we write

$$\begin{aligned}\overline{Y}^n_{t^n_m} &= M_m + A_m \qquad \text{with} \\ M_m &= -\sum_{l=0}^{m-1}\widetilde{\theta}_l, \qquad A^{i,j}_m = \delta_{i,j} - \sum_{l=0}^{m-1}\widehat{\theta}^{i,j}_l.\end{aligned}$$

By (4.16) we have $\|n\widehat{\theta}_l\| \leqslant CM_4(Z)(1+\|\psi\|_{1,3,\infty})\|\overline{Y}^n_{t^n_l}\|$ and using the triangle inequality, we deduce that

$$\sup_{t^n_k\leqslant t^n_m}\|A_k\| \leqslant 1 + CM_4(Z)(1+\|\psi\|_{1,3,\infty})\frac{1}{n}\sum_{l=0}^{m-1}\|\overline{Y}^n_{t^n_l}\|.$$

So that,

$$\mathbb{E}[\sup_{t^n_k\leqslant t^n_m}\|A_k\|^p]^{1/p} \leqslant 1 + CM_4(Z)(1+\|\psi\|_{1,3,\infty})\frac{1}{n}\sum_{l=0}^{m-1}\|\overline{Y}^n_{t^n_l}\|_p.$$

We recall that $\|\theta_l\| \leqslant 2\|\overline{J}_l\|\|\overline{Y}^n_{t^n_l}\|$ and it follows that

$$\|\widetilde{\theta}_l\| \leqslant \|\theta_l\| + \|\widehat{\theta}_l\| \leqslant C(|Z_{l+1}|^2 + M_4(Z))(1+\|\psi\|_{1,3,\infty})\|\overline{Y}^n_{t^n_l}\|/n^{1/2},$$

and then,

$$\|\widetilde{\theta}_l\|_p^2 \leqslant C(M_{2p}(Z)^{2/p} + M_4(Z)^2)(1+\|\psi\|^2_{1,3,\infty})\|\overline{Y}^n_{t^n_l}\|_p^2/n.$$

Moreover, $(M_m)_{m\in\mathbb{N}^*}$ is a martingale so, using Burkholder's inequality (see (6.2)), we have

$$\mathbb{E}[\sup_{t^n_k\leqslant t^n_m}\|M_k\|^p]^{1/p} \leqslant C(\sum_{l=0}^{m-1}\|\widetilde{\theta}_l\|_p^2)^{1/2}.$$

We conclude that

$$\mathbb{E}[\sup_{t^n_k\leqslant t^n_m}\|\overline{Y}_{t^n_k}\|^p]^{1/p} \leqslant 1 + C(M_{2p}(Z)^{1/p} + M_4(Z))(1+\|\psi\|_{1,3,\infty})(\frac{1}{n}\sum_{l=0}^{m-1}\|\overline{Y}^n_{t^n_l}\|_p^2)^{1/2}.$$

Now, we are going to use the Gronwall's lemma. We put $Q_l = \|\overline{Y}^n_{t^n_l}\|^2$, so that, $\|\overline{Y}^n_{t^n_l}\|^2_p = \|Q_l\|_{p/2}$. It follows that

$$\mathbb{E}[\sup_{k\leqslant m}Q_k^{p/2}]^{1/p} \leqslant 1 + C(M_{2p}(Z)^{1/p} + M_4(Z))(1+\|\psi\|^2_{1,3,\infty})(\frac{1}{n}\sum_{l=0}^{m-1}\|Q_l\|_{p/2})^{1/2},$$

which gives,

$$\begin{aligned}\|\sup_{k\leqslant m}Q_k\|_{p/2} &\leqslant 1 + C(M_{2p}(Z)^{2/p} + M_4(Z)^2)(1+\|\psi\|^2_{1,3,\infty})\frac{1}{n}\sum_{l=0}^{m-1}\|Q_l\|_{p/2}\\ &\leqslant 1 + C(M_{2p}(Z)^{2/p} + M_4(Z)^2)(1+\|\psi\|^2_{1,3,\infty})\frac{1}{n}\sum_{l=0}^{m-1}\|\sup_{k\leqslant l}Q_k\|_{p/2}.\end{aligned}$$

Then, by Gronwall's lemma,

$$\|\sup_{k\leqslant m}Q_k\|_{p/2} \leqslant \exp\big(C(M_{2p}(Z)^{2/p} + M_4(Z)^2)(1+\|\psi\|^2_{1,3,\infty})\big).$$

The estimate of $\mathbb{E}_{\Theta_{t,n}}[\sup_{s\in\pi_{T,n}}\|Y^n_s\|^p]$ is similar but simpler, so we leave it out. □

We have the following estimate for the covariance matrix of X^n :

Proposition 4.4. *Suppose that there exists $\lambda_* > 0$ such that*

$$\inf_{\kappa\in\mathbb{R}}\inf_{x\in\mathbb{R}^d}\inf_{|\xi|=1}\sum_{i=1}^{N}\left\langle \partial_{z_i}\psi(\kappa,x,0,0),\xi\right\rangle^2 \geqslant \lambda_* \tag{4.18}$$

Assume also that n and $t\in\pi_{T,n}^{0,T}$ are sufficiently large such that (4.13) holds and that

$$n^{1/2}\geqslant \frac{8(N^3+N^2+1)}{\lambda_*}\|\psi\|_{1,3,\infty}^2. \tag{4.19}$$

Let $\sigma_{X_t^n}$ be the Malliavin covariance matrix of X_t^n defined in (3.10) for $t\in\pi_{T,n}$. There exists a constant $C\geqslant 1$, which depends on p, T and the moment of Z up to order $8p$, such that

$$\mathbb{E}_{\Theta_{t,n}}[(\det\sigma_{X_t^n})^{-p}]^{1/p}\leqslant C\frac{\exp(C\|\psi\|_{1,3,\infty}^2)}{\lambda_* m_* t/T}. \tag{4.20}$$

.

Proof. Let $t\in\pi_{T,n}^{0,T}$ and $m\in\mathbb{N}^*$ such that $t_m^n=t$. By (4.12), $\sigma_{X_t^n}=Y_t^n\widehat{\sigma}(Y_t^n)^*$, with $(Y_t^n)^*$ the transpose matrix of Y_t^n and $\widehat{\sigma}=\sum_{k=1}^m(\widehat{Y}_{t_k^n}^nI_{k-1})\times(\widehat{Y}_{t_k^n}^nI_{k-1})^*$. It follows that $\det\sigma_{X_t^n}=(\det Y_t^n)^2\det\widehat{\sigma}$ andt

$$\mathbb{E}_{\Theta_{t,n}}[(\det\sigma_{X_t^n})^{-p}]\leqslant \mathbb{E}_{\Theta_{t,n}}[(\det Y_t^n)^{-4p}]^{1/2}\mathbb{E}_{\Theta_{t,n}}[(\det\widehat{\sigma})^{-2p}]^{1/2}.$$

Since $(\det Y_t^n)^{-1}=\det\widehat{Y}_t^n$, we use (4.15) and we obtain $\mathbb{E}_{\Theta_{t,n}}[(\det Y_t^n)^{-4p}]^{1/2}\leqslant \exp(C(M_{8p}(Z)^{1/(2p)}+M_4(Z)^2)(1+\|\psi\|_{1,3,\infty}^2))$. We estimate now the lower eigenvalue of $\widehat{\sigma}$ given by

$$\begin{aligned}\widehat{\lambda}&=\inf_{|\xi|=1}\sum_{k=1}^m\sum_{i=1}^N\left\langle(\widehat{Y}_{t_k^n}^nI_{k-1,i})\times(\widehat{Y}_{t_k^n}^nI_{k-1,i})^*\xi,\xi\right\rangle\\ &=\inf_{|\xi|=1}\sum_{k=1}^m\sum_{i=1}^N\left\langle(I_{k-1,i}I_{k-1,i})^*(\widehat{Y}_{t_k^n}^n)^*\xi,(\widehat{Y}_{t_k^n}^n)^*\xi\right\rangle. \end{aligned}\tag{4.21}$$

Recall that, $I_{k,i}$ is given in (4.10):

$$I_{k,i}=\frac{1}{\sqrt{n}}\chi_{k+1}(a_k^i+\sum_{j=1}^N H_{k+1}^jb_k^{i,j}+\frac{1}{\sqrt{n}}\sum_{j,q=1}^N H_{k+1}^jH_{k+1}^qc_k^{i,j,q}+\delta_{k+1}^n\tilde{c}_k^i).$$

Then, for $\eta\in\mathbb{R}^d$ and $k\in\{0,\dots,M-1\}$ we have

$$\begin{aligned}\sum_{i=1}^N\left\langle(I_{k,i}I_{k,i})^*\eta,\eta\right\rangle=\sum_{i=1}^N\left\langle I_{k,i},\eta\right\rangle^2\geqslant\frac{1}{2n}\sum_{i=1}^N\chi_{k+1}\left\langle a_k^i,\eta\right\rangle^2-2(N^3+N^2+1)\\ \times\sup_{i,j,q}\{\,|\left\langle H_{k+1}^jb_k^{i,j},\eta\right\rangle|^2,|\left\langle H_{k+1}^jH_{k+1}^qc_k^{i,j,q},\eta\right\rangle|^2,|\langle\delta_{k+1}^n\tilde{c}_k^i,\eta\rangle|^2\}.\end{aligned}$$

Since we are on the set $\{\Theta_{t,n}\neq 0\}$, we have $\sup_{k\in\{1,..,n\}}|H_k|\leqslant n^{-1/4}$. Moreover, $\sup_{i,j,q}\{|b_k^{i,j}|,|c_k^{i,j,q}|,|\tilde{c}_k^i|\}\leqslant\|\psi\|_{1,3,\infty}$, for all $k\in\{0,\dots,n-1\}$, so that

$$\sup_{i,j,q}\{\,|\left\langle H_{k+1}^jb_k^{i,j},\eta\right\rangle|,|\left\langle H_{k+1}^jH_{k+1}^qc_k^{i,j,q},\eta\right\rangle|,|\langle\delta_{k+1}^n\tilde{c}_k^i,\eta\rangle|\}\leqslant\frac{1}{n^{1/4}}\|\psi\|_{1,3,\infty}|\eta|.$$

We recall that we have the hypothesis (4.18)

$$\sum_{i=1}^N\left\langle a_k^i,\eta\right\rangle^2=\sum_{i=1}^N\left\langle\partial_{z_i}\psi_k(X_{t_k^n}^n,0,0),\eta\right\rangle^2\geqslant\lambda_*|\eta|^2.$$

Using (4.19), we have $\lambda_*/2 - 2(N^3+N^2+1)\|\psi\|_{1,3,\infty}^2/n^{1/2} \geqslant \lambda_*/4$, and we obtain

$$\sum_{i=1}^{N} \langle (I_{k,i}I_{k,i})^*\eta,\eta\rangle \geqslant \frac{\chi_{k+1}}{n}(\frac{\lambda_*}{2} - 2\frac{(N^3+N^2+1)\|\psi\|_{1,3,\infty}^2}{n^{1/2}})|\eta|^2 \geqslant \chi_{k+1}\frac{1}{4n}\lambda_*|\eta|^2$$

We come back to (4.21) and we take $\eta = (\widehat{Y}_{t_k^n}^n)^*\xi$. Since on the set $\{\Theta_{t,n}\neq 0\}$ and $\lfloor nt/T\rfloor = nt/T$, we have $\frac{T}{nt}\sum_{k=1}^{nt/T}\chi_k \geqslant \frac{1}{2}m_*$, it follows that

$$\begin{aligned}
\widehat{\lambda} &\geqslant \frac{\lambda_*}{4}\frac{1}{n}\inf_{|\xi|=1}\sum_{k=1}^{nt/T}\chi_k\|(\widehat{Y}_{t_k^n}^n)^*\xi\|^2 \geqslant \frac{\lambda_*}{4n}\sum_{k=1}^{nt/T}\chi_k\inf_{|\xi|=1}\|(\widehat{Y}_{t_k^n}^n)^*\xi\|^2\\
&\geqslant \frac{\lambda_* m_* nt/T}{8n}\inf_{s\in\pi_{T,n}^T;s\leqslant t}\inf_{|\xi|=1}\|(\widehat{Y}_s^n)^*\xi\|^2 \geqslant \frac{\lambda_* m_* t}{8T}(\sup_{s\in\pi_{T,n}^T;s\leqslant t}\|Y_s^n\|)^{-2}
\end{aligned}$$

Since we have (4.13), (4.14) follows and we conclude that

$$\begin{aligned}
\mathbb{E}_{\Theta_{t,n}}[\widehat{\lambda}^{-p}]^{1/p} \leqslant& \frac{8n}{\lambda_* m_*(nt/T)}\mathbb{E}_{\Theta_{t,n}}[\sup_{s\in\pi_{T,n}^T}\|Y_s^n\|^{2p}]^{1/p}\\
\leqslant& C\frac{\exp\big(C(M_{4p}(Z)^{1/p}+M_4(Z)^2)(1+\|\psi\|_{1,3,\infty}^2)\big)T}{\lambda_* m_* t}. \qquad \square
\end{aligned}$$

4.2 The regularization property

We still fix $T>0$ and $n\in\mathbb{N}^*$ and we consider the Markov chain $(X_t^n)_{t\in\pi_{T,n}}$, defined in (4.1). We also recall that $\Theta_{S,n}$ is defined in (3.27) and we introduce $(Q_t^{n,\Theta})_{t\in\pi_{T,n}}$ such that,

$$\forall t\in\pi_{T,n},\quad Q_t^{n,\Theta}f(x) := \mathbb{E}_{\Theta_{t,n}}[f(X_t^n(x))] = \frac{1}{\mathbb{E}[\Theta_{t,n}]}\mathbb{E}[\Theta_{t,n}f(X_t^n(x))]. \tag{4.22}$$

Notice that $(Q_t^{n,\Theta})_{t\in\pi_{T,n}}$, is not a semigroup, but this is not necessary. We will not be able to prove the regularization property for Q^n but for $Q^{n,\Theta}$ and every $t\leqslant T$.

Proposition 4.5. ***A.*** *Let $T>0$ and $n\in\mathbb{N}^*$. We assume that n and $t\in\pi_{T,n}^{0,T}$ are sufficiently large in order to have (4.13):*

$$\frac{3\|\psi\|_{1,3,\infty}}{n^{1/4}} + \frac{M_8(Z)}{n} + \exp(-m_*^2 nt/(2T)) \leqslant \frac{1}{2}$$

and (4.19). Moreover we assume that (4.18) holds true. Then for every $q\in\mathbb{N}$ and multi index α,β with $|\alpha|+|\beta|\leqslant q$, there exists $l\in\mathbb{N}^*$ and $C\geqslant 1$ which depend on m_*,r_* and the moments of Z such that

$$\|\partial_\alpha Q_t^{n,\Theta}\partial_\beta f\|_\infty \quad\leqslant\quad C\frac{\mathfrak{K}_{q+3}(\psi)^l}{(\lambda_* t)^{q(q+1)}}\|f\|_\infty \tag{4.23}$$

with $\mathfrak{K}_r(\psi)$ defined in (4.5). In particular, $Q_t^{n,\Theta}(x,dy) = p_t^{n,\Theta}(x,y)dy$ and $(x,y)\mapsto p_t^{n,\Theta}(x,y)$ belongs to $\mathcal{C}^\infty(\mathbb{R}^d\times\mathbb{R}^d)$.

B. *There exists $C\geqslant 1$, such that for every $l\in\mathbb{N}$ and $t\in\pi_{T,n}^T$, we have*

$$\|Q_t^n f - Q_t^{n,\Theta}f\|_\infty \leqslant 4(\exp(-m_*^2 nt/(2T)) + \frac{M_{4(l+1)}(Z)}{n^l})\|f\|_\infty. \tag{4.24}$$

Remark 4.6. *(4.23) means that the strong regularization property $\overline{R}_{q,\eta}$ (see (2.12)), with $\eta(q)=q(q+1)$, holds for $Q^{n,\Theta}$.*

Proof. We fix $t \in \pi^{0,T}_{T,n}$. Let us prove **A**.

$$\partial_\alpha Q^{n,\Theta}_t \partial_\beta f(x) = \sum_{|\beta| \leqslant |\gamma| \leqslant q} \mathbb{E}_{\Theta_{t,n}}[\partial_\gamma f(X^n_t(x)) \mathcal{P}_\gamma(X^n_t)], \tag{4.25}$$

where $\mathcal{P}_\gamma(X^n_t)$ is a universal polynomial of $\partial^\rho_x X^n_t(x), 0 \leqslant |\rho| \leqslant q - |\gamma| + 1$. Using the integration by parts formula (3.30) and the estimate (3.32) (together with $\mathbb{E}[\Theta_{t,n}] \geqslant 1/2$ using (4.13)) we obtain

$$\begin{aligned} |\mathbb{E}_{\Theta_{t,n}}[\partial_\gamma f(X^n_t(x)) \mathcal{P}_\gamma(X^n_t(x))]| &= |\mathbb{E}_{\Theta_{t,n}}[f(X^n_t(x)) H^{\Theta_{t,n}}_\gamma(X^n_t(x), \mathcal{P}_\gamma(X^n_t(x))]| \\ &\leqslant \|f\|_\infty \mathbb{E}_{\Theta_{t,n}}[|H^{\Theta_{t,n}}_\gamma(X^n_t(x), \mathcal{P}_\gamma(X^n_t(x))|] \\ &\leqslant C\|f\|_\infty \times A_1 \times A_2 \times A_3 \end{aligned} \tag{4.26}$$

with

$$\begin{aligned} A_1 &= 1 \vee \mathbb{E}_{\Theta_{t,n}}[((\det \sigma_{X^n_t(x)})^{-1})^{2q(q+1)}]^{1/2} \\ A_2 &= 1 + \mathbb{E}[|X^n_t(x)|^{8qd(q+2)}_{1,q+1}]^{1/4} + \mathbb{E}[|LX^n_t(x)|^{8q}_{q-1}]^{1/4} \\ A_3 &= \mathbb{E}[|\mathcal{P}_\gamma(X^n_t(x))|^4_{|\gamma|}]^{1/4}. \end{aligned}$$

Using the results from Theorem 4.2, we obtain

$$A_2 \times A_3 \leqslant C\mathfrak{K}_{q+3}(\psi)^l.$$

We use now (4.20) and it follows

$$A_1 = 1 \vee \mathbb{E}_{\Theta_{t,n}}[(\det \sigma_{X^n_t(x)})^{-2q(q+1)}]^{1/2} \leqslant 1 \vee C(\lambda_* t)^{-q(q+1)} \exp(Cq(q+1)\|\psi\|^2_{1,3,\infty}).$$

Now, we gather all the terms together,

$$|\partial_\alpha Q^{n,\Theta}_t \partial_\beta f(x))| \leqslant C \frac{\mathfrak{K}_{q+3}(\psi)^l}{(\lambda_* t)^{q(q+1)}} \|f\|_\infty.$$

B. We have

$$\begin{aligned} |Q^n_t f(x) - Q^{n,\Theta}_t f(x)| &\leqslant |Q^n_t f(x)||1 - \frac{1}{\mathbb{E}[\Theta_{t,n}]}| + \frac{1}{\mathbb{E}[\Theta_{t,n}]} |\mathbb{E}[f(X^n_t(x))(1 - \Theta_{t,n})]| \\ &\leqslant 2\|f\|_\infty \frac{\mathbb{E}[|1 - \Theta_{t,n}|]}{\mathbb{E}[\Theta_{t,n}]} \leqslant 2\|f\|_\infty \frac{\mathbb{P}(\Theta_{t,n} = 0)}{1 - \mathbb{P}(\Theta_{t,n} = 0)}. \end{aligned}$$

By (3.28) we have, for every $l \in \mathbb{N}$, $\mathbb{P}(\Theta_{t,n} = 0) \leqslant \exp(-m_*^2 nt/(2T)) + M_{4(l+1)}(Z) n^{-l}$ and we conclude using (4.13) in order to obtain $1 - \mathbb{P}(\Theta_{t,n} = 0) \geqslant 1/2$. □

We give now an alternative way to regularize the semigroup Q^n (by convolution). We consider a d dimensional standard normal random variable G which is independent from $Z_k, k \in \mathbb{N}^*$, and for $\theta > 0$, we introduce $(X^{n,\theta}_t)_{t \in \pi_{T,n}}$ as follows

$$X^{n,\theta}_t(x) = \frac{1}{n^\theta} G + X^n_t(x). \tag{4.27}$$

We denote by $p^{n,\theta}_t(x,y)$ the density of the law of $X^{n,\theta}_t(x)$ and for $t \in \pi_{T,n}$, we define

$$Q^{n,\theta}_t f(x) := \mathbb{E}[f(\frac{1}{n^\theta} G + X^n_t(x))]. \tag{4.28}$$

Corollary 4.7. *Under the hypothesis of the previous proposition we have:*

A. *For every multi index α, β with $|\alpha| + |\beta| \leqslant q$, and every $q \in \mathbb{N}^*$, there exists $l \in \mathbb{N}^*$, $C \geqslant 1$, which depend on q, T and the moments of Z such that for all $l' \in \mathbb{N}$ and $t \in \pi^{0,T}_{T,n}$ sufficiently large in order to have (4.13) and (4.19), the following estimate holds:*

$$\begin{aligned}&\|\partial_\alpha Q^{n,\theta}_t \partial_\beta f\|_\infty \\ &\quad\leqslant C\Big(\frac{\mathfrak{K}_{q+3}(\psi)^l}{(\lambda_* t)^{q(q+1)}} + n^{q\theta}\mathfrak{K}_{q+3}(\psi)^l(\exp(-m_*^2 nt/(4T)) + \frac{M_{4(l'+1)}(Z)^{1/2}}{n^{l'/2}})\Big)\|f\|_\infty, \end{aligned} \tag{4.29}$$

with $\mathfrak{K}_r(\psi)$ defined in (4.23).

B. *There exists $l \in \mathbb{N}^*$, $C \geqslant 1$, such that for every $l' \in \mathbb{N}$ and $t \in \pi^T_{T,n}$*

$$\|Q^n_t f(x) - Q^{n,\theta}_t f(x)\|_\infty \leqslant C\big(\frac{\mathfrak{K}_4(\psi)^l}{(\lambda_* t)^2 n^\theta} + 2(\exp(-m_*^2 nt/(2T)) + \frac{M_{4(l'+1)}(Z)}{n^{l'}})\big)\|f\|_\infty. \tag{4.30}$$

Proof. We fix $t \in \pi^{0,T}_{T,n}$. Let us prove **A**. As in (4.25), we write

$$\partial_\alpha Q^{n,\theta}_t \partial_\beta f(x) = \sum_{|\beta|\leqslant|\gamma|\leqslant q} \mathbb{E}[(\partial_\gamma f)(n^{-\theta}G + X^n_t(x))\mathcal{P}_\gamma(X^n_t(x))],$$

where $\mathcal{P}_\gamma(X^n_t)$ is a universal polynomial of $\partial^\rho_x X^n_t(x), 0 \leqslant |\rho| \leqslant q - |\gamma| + 1$. We decompose

$$\mathbb{E}[(\partial_\gamma f)(n^{-\theta}G + X^n_t(x))\mathcal{P}_\gamma(X^n_t(x))] = I + J$$

with

$$\begin{aligned} I &= \mathbb{E}[\Theta_{t,n}]\mathbb{E}_{\Theta_{t,n}}[\partial_\gamma f(n^{-\theta}G + X^n_t(x))\mathcal{P}_\gamma(X^n_t(x))], \\ J &= \mathbb{E}[(\partial_\gamma f)(n^{-\theta}G + X^n_t(x))\mathcal{P}_\gamma(X^n_t(x))(1-\Theta_{t,n})]. \end{aligned}$$

The reasoning from the previous proof shows that

$$I \leqslant C\frac{\mathfrak{K}_{q+3}(\psi)^l}{(\lambda_* t)^{q(q+1)}}\|f\|_\infty.$$

And since G follows the standard normal law and is independent from X^n and $\Theta_{t,n}$, we have

$$J = \mathbb{E}\big[\mathcal{P}_\gamma(X^n_t(x))(1-\Theta_{t,n})\int_{\mathbb{R}^d}(\partial_\gamma f)(n^{-\theta}y + X^n_t(x))(2\pi)^{-d/2}e^{-|y|^2/2}dy\big].$$

Moreover, one has

$$(\partial_\gamma f)(n^{-\theta}y + X^n_t(x)) = n^{|\gamma|\theta}\partial^\gamma_y(f(n^{-\theta}y + X^n_t(x))),$$

so that, using standard integration by parts, we have

$$J = n^{|\gamma|\theta}\mathbb{E}\big[\mathcal{P}_\gamma(X^n_t(x))(1-\Theta_{t,n})\int_{\mathbb{R}^d} f(n^{-\theta}y + X^n_t(x))H_\gamma(y)(2\pi)^{-d/2}e^{-|y|^2/2}dy\big],$$

where H_γ is the Hermite polynomial corresponding to the multi-index γ. Finally we obtain

$$|J| \leqslant Cn^{|\gamma|\theta}\mathfrak{K}_{q+3}(\psi)^l\|f\|_\infty\mathbb{E}[1-\Theta_{t,n}]^{1/2}$$

$$\leqslant Cn^{|\gamma|\theta}\mathfrak{K}_{q+3}(\psi)^l\|f\|_\infty(\exp(-m_*^2nt/(4T))+\frac{M_{4(l'+1)}(Z)^{1/2}}{n^{l'/2}})$$

the last inequality being a consequence of (3.28).

Now we prove **B**. Let $l'\in\mathbb{N}^*$. Using (3.28) and (4.23), there exists $C,l\geqslant 1$ such that we have

$$\begin{aligned}|Q_t^nf(x)-Q_t^{n,\theta}f(x)| &\leqslant \mathbb{E}[\Theta_{t,n}]|\mathbb{E}_{\Theta_{t,n}}[f(X_t^n(x))-f(X_t^n(x)+n^{-\theta}G)]|+2\|f\|_\infty\mathbb{E}[1-\Theta_{t,n}]\\ &\leqslant n^{-\theta}\sum_{j=1}^d\int_0^1|\mathbb{E}_{\Theta_{t,n}}[\partial_jf(X_t^n(x)+\lambda n^{-\theta}G)G^j]|d\lambda+2\|f\|_\infty\mathbb{E}[1-\Theta_{t,n}]\\ &\leqslant Cn^{-\theta}\frac{\mathfrak{K}_4(\psi)^l}{(\lambda_*t)^2}\|f\|_\infty+2(\exp(-m_*^2nt/(2T))+\frac{M_{4(l'+1)}(Z)}{n^{l'}})\|f\|_\infty. \quad \square\end{aligned}$$

4.3 Approximation result

In this section we give the approximation result for a Markov semigroup $(P_t)_{t\geqslant 0}$. We recall that $T>0$ and $n\in\mathbb{N}$ are fixed. We denote $\mu_k^n(x,dy)=P_{T/n}(x,dy)$ for all $k\in\mathbb{N}^*$. We consider now an approximation scheme based on the Markov chain introduced in the previous section (see (4.1). Therefore, we consider two sequences of independent random variables $Z_{k+1}\in\mathbb{R}^N,\ \kappa_k\in\mathbb{R},\ k\in\mathbb{N}$ and we take $(\delta_k^n)_{k\in\mathbb{N}^*}$ such that $\sup_{k\in\mathbb{N}^*}\delta_k^n\leqslant C/n$ for a constant $C\geqslant 1$. We assume that $Z_1,\ldots,Z_n$ verifies (3.1) and have finite moments of any order: For every $p\geqslant 1$,

$$M_p(Z)=1\vee\sup_{k\leqslant n}\mathbb{E}[|Z_k|^p]<\infty. \tag{4.31}$$

Moreover, we take $\psi\in\mathcal{C}^\infty(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^N\times\mathbb{R}_+;\mathbb{R}^d)$ such that $\psi(\kappa,x,0,0)=x$ and we construct $X^n_{t^n_{k+1}}(x)=\psi(\kappa_k,X^n_{t^n_k}(x),Z_{k+1}/\sqrt{n},\delta^n_{k+1})$ with $X_0^n(x)=x$. We denote $\nu^n_{k+1}(x,dy)=\mathbb{P}(X^n_{t^n_{k+1}}\in dy\mid X^n_{t^n_k}=x)$ and we construct the discrete semigroup $Q^n_{t^n_{k+1}}=Q^n_{t^n_k}\nu^n_{k+1}$ on the time grid $\pi_{T,n}$. We recall that the notation $\|\psi\|_{1,r,\infty}$ is introduced in (4.3) and we assume that, for every $r\in\mathbb{N}$,

$$\|\psi\|_{1,r,\infty}<\infty. \tag{4.32}$$

We also assume that there exists $\lambda_*>0$ such that

$$\inf_{\kappa\in R}\inf_{x\in\mathbb{R}^d}\inf_{|\xi|=1}\sum_{i=1}^N\langle\partial_{z_i}\psi(\kappa,x,0,0),\xi\rangle^2\geqslant\lambda_*. \tag{4.33}$$

Now we are able to prove our main result.

Theorem 4.8. *We recall that $T>0$. We fix $q\in\mathbb{N}$, $h>0$ and $S\in(0,T/2)$. For a given $n\in\mathbb{N}^*$, we consider the Markov semigroup $(P_t)_{t\geqslant 0}$, and the approximation Markov chain $(Q_t^n)_{t\in\pi_{T,n}}$, defined above. Moreover, we assume that there exists $n_0\in\mathbb{N}^*$ such that $T/n_0\leqslant S$ and, (4.13) and (4.19) hold with $n=n_0$ and $t=S$. Then, for all $n\geqslant n_0$, we have the following properties.*

- ***A.*** *We assume that (4.31), (4.32) and (4.33) hold. Moreover we assume that $E_m(h,q)$ (see (2.3)) and $E_m^*(h,q)$ (see (2.6)) hold between $(P_t^m)_{t\in\pi_{T,m}}=(P_t)_{t\in\pi_{T,m}}$ and $(Q_t^m)_{t\in\pi_{T,m}}$ for every $m\geqslant n$. Then, there exists $l\in\mathbb{N}^*$, $C\geqslant 1$, which depend on q, T and the moments of Z, such that*

$$\sup_{t\in\pi^{2S,T}_{T,n}}\|P_tf-Q_t^nf\|_\infty\leqslant C\frac{\mathfrak{K}_{q+3}(\psi)^l}{(\lambda_*S)^{\eta(q)}}\|f\|_\infty/n^h. \tag{4.34}$$

  with $\eta(q)=q(q+1)$.

B. *Moreover, for every $t>0, P_t(x,dy)=p_t(x,y)dy$ with $(x,y)\mapsto p_t(x,y)$ belonging to $\mathcal{C}^\infty(\mathbb{R}^d\times\mathbb{R}^d)$.*

C. *We recall the $Q^{n,\Theta}$ is defined in (4.22) and verifies $Q_t^{n,\Theta}(x,dy)=p_t^{n,\Theta}(x,y)dy$. Then, there exists $l\in\mathbb{N}^*$ such that for every $R>0,\varepsilon\in(0,1)$, $x_0,y_0\in\mathbb{R}^d$, and every multi-index $\alpha,\ \beta$ with $|\alpha|+|\beta|=u$, we also have*

$$\sup_{t\in\pi_{T,n}^{2S,T}}\ \sup_{(x,y)\in\overline{B}_R(x_0,y_0)}|\partial_x^\alpha\partial_y^\beta p_t(x,y)-\partial_x^\alpha\partial_y^\beta p_t^{n,\Theta}(x,y)|\leqslant C\frac{\mathfrak{K}_{q+3}(\psi)^l}{(\lambda_*S)^{\eta(p_{u,\varepsilon}\vee q)}}/n^{h(1-\varepsilon)} \tag{4.35}$$

with a constant C which depends on R,x_0,y_0,T and on $|\alpha|+|\beta|$ and $p_{u,\varepsilon}=(u+2d+1+2\lceil(1-\varepsilon)(u+d)/(2\varepsilon)\rceil)$.

D. *Let $\theta>h+1$. We recall the $Q^{n,\theta}$ is defined in (4.28) and verifies $Q_t^{n,\theta}(x,dy)=p_t^{n,\theta}(x,y)dy$. Then, there exists $l\in\mathbb{N}^*$ such that for every $R>0,\varepsilon\in(0,1)$, $x_0,y_0\in\mathbb{R}^d$, and every multi-index $\alpha,\ \beta$ with $|\alpha|+|\beta|=u$, we also have*

$$\sup_{t\in\pi_{T,n}^{2S,T}}\ \sup_{(x,y)\in\overline{B}_R(x_0,y_0)}|\partial_x^\alpha\partial_y^\beta p_t(x,y)-\partial_x^\alpha\partial_y^\beta p_t^{n,\theta}(x,y)|\leqslant C\frac{\mathfrak{K}_{q+3}(\psi)^l}{(\lambda_*S)^{\eta(p_{u,\varepsilon}\vee q)}}/n^{h(1-\varepsilon)} \tag{4.36}$$

Proof. **A-B**. We use Proposition 2.4: We have proved in Proposition 4.5 that $Q^{n,\Theta}$ verifies the regularization properties. The proof of (4.34) and (4.35) is an immediate consequence of Theorem 2.6. **C**. In order prove (4.36) one uses Corollary 4.7 instead of Proposition 4.5. □

Remark 4.9. *The simulation of an approximation scheme given by $Q^{n,\Theta}$ may be cumbersome, so the estimate obtained in (4.35) is not very useful. This is why we propose the regularized scheme $X^{n,\theta}$ which is easier to simulate.*

5 The Ninomiya Victoir scheme

We illustrate Theorem 4.8 when X^n is the Ninomiya Victoir scheme for a diffusion process. This is a variant of the result already obtained by Kusuoka [23] in the case where Z_k has a Gaussian distribution (and so the standard Malliavin calculus is available). Since in our paper Z_k has an arbitrary distribution (except for the property (3.1)), our result may be seen as an invariance principle as well. We consider the d dimensional diffusion process

$$dX_t=\sum_{i=1}^N V_i(X_t)\circ dW_t^i+V_0(X_t)dt \tag{5.1}$$

with $V_0,V_i\in\mathcal{C}_b^\infty(\mathbb{R}^d;\mathbb{R}^d),\ i=1,\ldots,N$ and $W=(W^1,\ldots,W^N)$ a standard Brownian motion and $\circ dW_t^i$ denotes the Stratonovich integral with respect to W^i. The infinitesimal operator of this Markov process is given by

$$A=V_0+\frac{1}{2}\sum_{k=1}^N V_k^2 \tag{5.2}$$

with the notation $Vf(x)=\langle V(x),\nabla f(x)\rangle$. Let us define $\exp(V)(x):=\Phi_V(x,1)$ where Φ_V solves the deterministic equation

$$\Phi_V(x,t)=x+\int_0^t V(\Phi_V(x,s))ds. \tag{5.3}$$

By a change of variables, it is possible to show that $\Phi_{\varepsilon V}(x,t)=\Phi_V(x,\varepsilon t)$, so we have

$$\exp(\varepsilon V)(x):=\Phi_{\varepsilon V}(x,1)=\Phi_V(x,\varepsilon).$$

We also notice that the semigroup of the above Markov process is given by $P_t^V f(x)=f(\Phi_V(x,t))$ and has the infinitesimal operator $A_V f(x)=Vf(x)$. In particular the relation $P_t^V A_V=A_V P_t^V$ reads

$$Vf(\Phi_V(x,t))=A_V P_t^V f=P_t^V A_V f=\langle V(x),\nabla_x\left(f(\Phi_V(x,t))\right)\rangle.$$

Using m times Dynkin's formula $P_t^V f(x)=f(x)+\int_0^t P_s^V A_V f(x)ds$ we obtain

$$f(\Phi_V(x,t)))=f(x)+\sum_{r=1}^{m}\frac{t^r}{r!}V^r f(x)+\frac{1}{m!}\int_0^t (t-s)^m V^{m+1}P_s^V f(x)ds. \tag{5.4}$$

We present now the Ninomiya Victoir scheme. We consider a sequence $\rho_k,\ k\in\mathbb{N}$ of independent Bernoulli random variables and we define $\psi_k:\mathbb{R}^d\times\mathbb{R}^{N+1}\to\mathbb{R}^d$ in the following way

$$\psi_k(x,w^1,w^0)=\exp(w^0V_0)\circ\exp(w^{1,1}V_1)\circ\cdot\circ\exp(w^{1,N}V_N)\circ\exp(w^0V_0)(x),\ \text{if } \rho_k=1, \tag{5.5}$$

$$\psi_k(x,w^1,w^0)=\exp(w^0V_0)\circ\exp(w^{1,N}V_N)\circ\cdot\circ\exp(w^{1,1}V_1)\circ\exp(w^0V_0)(x),\ \text{if } \rho_k=-1. \tag{5.6}$$

The Ninomiya Victoir scheme uses these functions with $w_k^0=T/2n$ and $w_k^{1,i}=\sqrt{T}Z_k^i/\sqrt{n}$, for $i=1,\ldots,N$. Moreover $Z_k^i,\ i=1,\ldots,d,\ k\in\mathbb{N}^*$ are independent random variables which verify (3.1) and moreover satisfy the following moment conditions:

$$\mathbb{E}[Z_k^i]=\mathbb{E}[(Z_k^i)^3]=\mathbb{E}[(Z_k^i)^5]=0,\qquad \mathbb{E}[(Z_k^i)^2]=1,\qquad \mathbb{E}[(Z_k^i)^4]=6. \tag{5.7}$$

In the original paper of Ninomiya Victoir, the random variables Z_k^i are standard normally distributed, and then verify (3.1). The new point here is that we do not require that Z_k follows this particular law anymore but only the weaker assumptions (3.1) and (5.7). We recall that $t_k^n=Tk/n$. One step of our scheme is given by

$$X^n_{t^n_{k+1}}=\psi_k(X^n_{t^n_k},w^1_{k+1},w^0_{k+1}). \tag{5.8}$$

We have the first following result.

Theorem 5.1. *There exists some universal constants $l\in\mathbb{N}^*$, $C\geqslant 1$ such that for every $f\in\mathcal{C}_b^6(\mathbb{R}^d)$, we have*

$$\sup_{t\in\pi^T_{T,n}}|\mathbb{E}[f(X_t)]-\mathbb{E}[f(X^n_t)]|\leqslant CC_6(V)^l\|f\|_{6,\infty}/n^2 \tag{5.9}$$

with $C_q(V):=\sup_{i=0,.,N}\|V_i\|_{q,\infty}$.

Remark 5.2. *The same estimate has already been proved by Alfonsi [1] using short time expansions on the solution of the Feynman Kac partial differential equation associated to the diffusion process.*

Under an ellipticity condition we are able to give an estimate of the total variation distance between a diffusion process of the form (5.1) and its Ninomiya Victoir scheme.

Theorem 5.3. *We assume that*

$$\inf_{|\xi|=1}\sum_{i=1}^{N}\langle V_i(x),\xi\rangle^2 \geqslant \lambda_* > 0 \qquad \forall x\in\mathbb{R}^d. \tag{5.10}$$

Let $S\in(0,T/2)$. Then there exists $n_0\in\mathbb{N}^*$ such that for every $n\geqslant n_0$, there exists $l\in\mathbb{N}^*$, $C\geqslant 1$ such that for every bounded and measurable function $f:\mathbb{R}^d\to\mathbb{R}$,

$$\sup_{t\in\pi_{T,n}^{2S,T}}|\mathbb{E}[f(X_t)]-\mathbb{E}[f(X_t^n)]|\leqslant C\frac{C_6(V)^l\mathfrak{K}_9(\psi)^l}{(\lambda_*S)^{42}}\left\|f\right\|_\infty/n^2. \tag{5.11}$$

Remark 5.4. *This estimate has already been proved by Kusuoka [23] (with a different approach). He considers a much more general non degeneracy assumptions (of Hörmander type) and uses Malliavin calculus in order to prove his result. Here the noise Z_k^i is no more Gaussian so the standard Malliavin calculus does not work anymore, but, since we have the property (3.1), we may use the abstract integration by parts formula introduced in Section 3.*

Proof of Theorem 5.1. We have to show $E_n(3,6)$ (see (2.3)) and (2.2) for Q^n. Indeed, the proof will then follow from Proposition 2.2. First, we notice that (2.2) is satisfied with $q=6$ for the semigroup Q^n using Theorem 4.2 (see (4.7)). Now, we focus on the proof of $E_n(3,6)$. In order to simplify the notations, we fix $T=1$ without loss of generality. We denote

$$\mathcal{T}_0f(x)=\mathcal{T}_{N+1}f(x)=f(\exp(\frac{1}{2n}V_0)(x)),\quad \mathcal{T}_if(x)=f(\exp(\frac{Z}{\sqrt{n}}V_1)(x)), i=1,\ldots,N.$$

Notice that, with the notation introduced in the beginning of this section, $\mathcal{T}_if(x)=P_i^{U_i}f(x)$ with $U_i=ZV_i/\sqrt{n}$, if $i=1,\ldots,N$ and $U_0=U_{N+1}=V_0/(2n)$. Using (5.4) with $t=1$ and $V=U_i, i=1,\ldots,N$ we obtain

$$\mathcal{T}_if(x)=f(x)+\sum_{r=1}^{m}\frac{Z^r}{n^{r/2}}\frac{1}{r!}V_i^rf(x)+\frac{Z^{m+1}}{n^{(m+1)/2}}R_{m+1,i}f(x) \tag{5.12}$$

with

$$R_{m+1,i}f(x)=\frac{1}{m!}\int_0^1(1-\lambda)^mV_i^{m+1}P_\lambda^{U_i}f(x)d\lambda \tag{5.13}$$

and we recall that $P_\lambda^{U_i}f(x)=f(\exp(\lambda ZV_i/\sqrt{n}))$. We have a similar expansion if we put $V=V_0/(2n)$ in (5.4). We aim to give an expansion of order 3 (with respect to $1/n$) for $\mathbb{E}[f(\psi_k(x,w_{k+1}^1,w_{k+1}^0)]$ (see (5.14) below). In order to do it, we replace each $\mathcal{T}_i, i=1,\ldots,N$, with an expansion of order $m\leqslant 5$ given above with $Z=Z_{k+1}^i$ (and we proceed in the same way when $V=V_0/(2n)$). Then, we calculate the products of the miscellaneous expansions, each with a well chosen order m such that there is no term with factor n^{-r}, $r>3$, appearing in those products. Moreover, all the terms containing n^{-3} go in the remainder. The last step consists in computing the expectancy. We notice that $\mathbb{E}[P_t^{U_i}]=P_t^{V_i^2/(2n)}$ and $\mathbb{E}[(Z_{k+1}^i)^r]=0$ for odd $r\leqslant 5$. Finally, since $\mathbb{E}[(Z_{k+1}^i)^2]=1$, $\mathbb{E}[(Z_{k+1})^4]=6$, the calculus is completed and we obtain:

$$\begin{aligned}\mathbb{E}[f(\psi_k(x,w_{k+1}^1,w_{k+1}^0)] &= \mathbb{E}[\mathcal{T}_0\mathcal{T}_1\ldots.\mathcal{T}_{N+1}f(x)] \qquad (5.14)\\ &= f(x)+\frac{1}{n}(V_0f(x)+\frac{1}{2}\sum_{i=1}^{N}V_i^2f(x))+\frac{1}{2n^2}V_0^2f(x)+\frac{1}{8n^2}\sum_{i=1}^{N}V_i^4f(x)\\ &\quad+\frac{1}{4n^2}\sum_{i<j}V_i^2V_j^2f(x)+\frac{1}{4n^2}\sum_{i=1}^{N}(V_0V_i^2f(x)+V_i^2V_0f(x))+\frac{1}{n^3}Rf(x).\end{aligned}$$

The remainder R is a sum of terms of the following form:

$$C\mathcal{T}_{0,\alpha_0},\ldots,\mathcal{T}_{N+1,\alpha_{N+1}}f(x) \tag{5.15}$$

with $\alpha=(\alpha_0,\ldots,\alpha_{N+1})\in\{0,\ldots,3\}^{N+2}$, $|\alpha|=\alpha_0+\ldots+\alpha_{N+1}=3$, and using the notation given in (5.13),

$$\begin{array}{ll} \mathcal{T}_{0,k},\mathcal{T}_{N+1,k}\in\{V_0^k,R_{k,0}\}, & \mathcal{T}_{i,k}\in\{V_i^{2k},R_{2k,i}\}, i\in\{1,\ldots,N\} \quad k=0,\ldots,2,\\ \mathcal{T}_{0,3}=\mathcal{T}_{N+1,3}=R_{3,0}, & \mathcal{T}_{i,3}=\mathbf{R}_{6,i}, i\in\{1,\ldots,N\}, \end{array}$$

with for $i=1,\ldots,N$,

$$\mathbf{R}_{6,i}=\mathbb{E}[(Z^i)^6R_{6,i}]=\int_0^1(1-\lambda)^5\mathbb{E}[Z^6V_i^6P_\lambda^{U_1}f(x)]d\lambda.$$

It is easy to check that for every $g\in\mathcal{C}^{k+p}(\mathbb{R})$, we have the following property

$$\|\mathcal{T}_{i,k}g\|_{p,\infty}\leqslant CC_{2k+p}(V)^l\|g\|_{k+p,\infty}$$

for some constants $l\in\mathbb{N}^*$, $C\geqslant 1$. So, it follows that

$$\|Rf\|_\infty\leqslant CC_6(V)^l\|f\|_{6,\infty}. \tag{5.16}$$

We turn now to the diffusion process X_t. For any $t>0$, we have the expansion

$$\mathbb{E}[f(X_t(x))]=P_t^Af(x)=f(x)+tAf(x)+\frac{t^2}{2}A^2f(x)+\frac{t^3}{3!}R'_tf(x).$$

with

$$R'_tf(x)=t^{-1}\int_0^tP_\lambda^AA^3f(x)(1-\lambda/t)^2d\lambda. \tag{5.17}$$

We take $t=n^{-1}$ and make the difference between (5.17) and (5.14). All the terms cancel except for the remainders so we obtain

$$\begin{aligned} &\forall k\in\{0,\ldots,n-1\},\\ &\qquad\mathbb{E}[f(X_{t^n_{k+1}})-f(X^n_{t^n_{k+1}})\mid X_{t^n_k}=X^n_{t^n_k}=x]=(R'_{1/n}f(x)/3!-Rf(x))/n^3. \end{aligned} \tag{5.18}$$

We clearly have $\|R'_{1/n}f\|_\infty\leqslant CC_6(V)^l\|f\|_{6,\infty}$. This, together with (5.16) completes the proof. □

Proof of Theorem 5.3. This will be a consequence of Theorem 4.8 as soon as we check that the ellipticity assumption (4.18) holds true. We fix k and we look at $\psi_k(x,w^1,w^0)$ defined in (5.6). We suppose that $\rho_k=1$ (the proof for $\rho_k=-1$ is similar). We denote $w^1=(w^{1,1},\cdots,w^{1,N}\}$ and $\tilde{w}=(w^1,w^0)$ with $w^0\in\mathbb{R}_+$. Using the notation $T_i=i$, we consider the process $x_t(w),0\leqslant t\leqslant T_{N+2}$ solution of the following equation:

$$\begin{aligned} x_t(\tilde{w}) &= x+\frac{w^0}{2}\int_{T_0}^tV_0(x_s(\tilde{w}))ds, \qquad T_0\leqslant t\leqslant T_1,\\ x_t(\tilde{w}) &= x_{T_i}(\tilde{w})+w^{1,i}\int_{T_i}^tV_i(x_s(\tilde{w}))ds, \qquad T_i\leqslant t\leqslant T_{i+1}, \qquad i=1,\ldots,N,\\ x_t(\tilde{w}) &= x_{T_{N+1}}(\tilde{w})+\frac{w^0}{2}\int_{T_{N+1}}^tV_0(x_s(\tilde{w}))ds, \qquad T_{N+1}\leqslant t\leqslant T_{N+2}. \end{aligned}$$

Then, $\psi_k(x,\tilde w)=x_{T_{N+2}}(\tilde w)$ and consequently for $r\in\{1,\ldots,N\}$, we have $\partial_{w^{1,r}}\psi_k(x,\tilde w)=\partial_{w^{1,r}}x_{T_{N+2}}(\tilde w)$. Moreover $\partial_{w^{1,r}}x_t(\tilde w)=0$ for $t\leqslant T_r$ and

$$\begin{aligned}\partial_{w^{1,r}}x_t(\tilde w)=\partial_{w^{1,r}}x_{T_{r+1}}(\tilde w)+\sum_{i=r+1}^{N}w^{1,i}\int_{T_i\wedge t}^{T_{i+1}\wedge t}\nabla V_i(x_s(\tilde w))\partial_{w^{1,r}}x_s(\tilde w)ds\\+\frac{w_0}{2}\int_{T_{N+1}\wedge t}^{t}\nabla V_0(x_s(\tilde w))\partial_{w^{1,r}}x_s(\tilde w)ds,\end{aligned}$$

for $t\geqslant T_{r+1}$, in particular for $t=T_{N+1}$. For $T_r<t\leqslant T_{r+1}$, $\partial_{w^{1,r}}x_t(\tilde w)$ solves the equation

$$\partial_{w^{1,r}}x_t(\tilde w)=\int_{T_r}^{t}V_r(x_s(\tilde w))ds+w^{1,r}\int_{T_r}^{t}\nabla V_r(x_s(\tilde w))\partial_{w^{1,r}}x_s(\tilde w)ds.$$

It follows that

$$\partial_{w^{1,r}}x_t(\tilde w)\mid_{\tilde w=0}=\int_{T_r}^{t}V_r(x_s(0))ds=V_r(x)(t-T_r).$$

Notice that $T_{r+1}-T_r=1$. Then, we have

$$\partial_{w^{1,r}}x_{T_{N+2}}(\tilde w)\mid_{\tilde w=0}=\partial_{w^{1,r}}x_{T_{r+1}}(\tilde w)\mid_{\tilde w=0}=V_r(x).$$

and then, by (5.10),

$$\sum_{r=1}^{N}\langle\partial_{w^{1,r}}x_{T_{N+2}}(0),\xi\rangle^2\geqslant\lambda_*|\xi|^2.$$

□

6 Proof of Theorem 4.2 on Sobolev norms

In this section, we will obtain estimates of the Sobolev norms of X^n and LX^n which appear in Theorem 4.2. The method we adopt here is to prove the estimates for a generic class of processes which involves the Malliavin derivatves of X^n and LX^n.

Before doing it, we give some preliminary results. We consider a separable Hilbert space U, we denote $|a|_U$ the norm of U and, for a random variable $F\in U$, we denote $\|F\|_{U,p}=(\mathbb{E}[|F|_U^p)]^{1/p}$. Moreover we consider a martingale $M_n\in U$, $n\in\mathbb{N}$ and we recall Burkholder's inequality in this framework: For each $p\geqslant 2$ there exists a constant $b_p\geqslant 1$ such that

$$\forall n\in\mathbb{N},\quad \|M_n\|_{U,p}\leqslant b_p\mathbb{E}[(\sum_{k=1}^{n}|M_k-M_{k-1}|_U^2)^{p/2}]^{1/p}. \tag{6.1}$$

As an immediate consequence

$$\|M_n\|_{U,p}\leqslant b_p(\sum_{k=1}^{n}\|M_k-M_{k-1}\|_{U,p}^2)^{1/2}. \tag{6.2}$$

Indeed

$$\begin{aligned}\|M_n\|_{U,p}^2 &\leqslant b_p^2\mathbb{E}[(\sum_{k=1}^{n}|M_k-M_{k-1}|_U^2)^{p/2}]^{2/p}=b_p^2\|\sum_{k=1}^{n}|M_k-M_{k-1}|_U^2\|_{p/2}\\ &\leqslant b_p^2\sum_{k=1}^{n}\||M_k-M_{k-1}|_U^2\|_{p/2}=b_p^2\sum_{k=1}^{n}\|M_k-M_{k-1}\|_{U,p}^2.\end{aligned}$$

We consider the scheme defined in the previous sections (see (4.6)):

$$X^n_{t^n_{k+1}}=x+\sum_{i=1}^{N}\sum_{k=0}^{m-1}H^i_{k+1}a^i_k(X^n_{t^n_k})+\sum_{k=0}^{m-1}\delta^n_{k+1}\tilde b_k(X^n_{t^n_k},H_{k+1},\delta^n_{k+1})$$

$$+\frac{1}{2}\sum_{i,j=1}^{N}\sum_{k=0}^{m-1}H^i_{k+1}H^j_{k+1}b^{i,j}_k(X^n_{t^n_k},H_{k+1})$$

with $H_k=n^{-1/2}Z_k$ and

$$\begin{aligned}a^i_k(x)&=\partial_{z_i}\psi(\kappa_k,x,0,0),\\ b^{i,j}_k(x,z)&=\int_0^1(1-\lambda)\partial_{z_i}\partial_{z_j}\psi(\kappa_k,x,\lambda z,0)d\lambda,\\ \tilde{b}_k(x,z,t)&=\int_0^1\partial_t\psi(\kappa_k,x,z,\lambda t)d\lambda.\end{aligned}$$

We also denote

$$\begin{aligned}A_k=&\sum_{i=1}^{N}H^i_{k+1}\nabla_x a^i_k(X^n_{t^n_k})+\delta^n_{k+1}\nabla_x\tilde{b}_k(X^n_{t^n_k},H_{k+1},\delta^n_{k+1})\\ &+\frac{1}{2}\sum_{i,j=1}^{N}H^i_{k+1}H^j_{k+1}\nabla_x b^{i,j}_k(X^n_{t^n_k},H_{k+1}).\end{aligned}$$

Notice that $X^n_t,a^i_k,b^{i,j}_k,\tilde{b}_k\in\mathbb{R}^d$ and A_k is a $d\times d$ dimensional matrix.

Now, we focus on the estimates of the Sobolev norms. As before, U is a separable Hilbert space. We say that, a U valued random variable F belongs to $\mathcal{S}(U)$ if for every $h\in U$ we have $\langle h,F\rangle\in\mathcal{S}$ (see (3.8)) and we define DF by $\langle h,DF\rangle=D\langle h,F\rangle$ for every $h\in U$. Then, we define the norms (see (3.20) and (3.21))

$$|F|^2_{U,m}=\sum_{0\leqslant|\alpha|\leqslant m}|D_\alpha F|^2_U,\qquad \|F\|_{U,m,p}=\left\||F|_{U,m}\right\|_p=\mathbb{E}[|F|^p_{U,m}]^{1/p}.$$

The Hilbert space U being given, we denote $V=U^d$ (recall that $X^n_{t^n_k}\in\mathbb{R}^d$ so, in this case, $U=\mathbb{R}$ and $V=\mathbb{R}^d$). We consider now some processes $(\alpha_k)_{k\in\mathbb{N}},(\beta_k)_{k\in\mathbb{N}},(\Gamma_k)_{k\in\mathbb{N}}$ with $\alpha_k=(\alpha^1_k,\ldots,\alpha^N_k)\in V^N$, $\beta_k=(\beta^1_k,\ldots,\beta^N_k)\in V^N$, $\Gamma_k\in V$. We assume that $\alpha^i_k=\alpha^i_k(Z_1,\ldots,Z_k)$ and $\langle h,\alpha^i_k\rangle\in\mathcal{C}^\infty_b(\mathbb{R}^{kN})$ for every $h\in V,i=1,\ldots,N$ (we recall that $Z_k\in\mathbb{R}^N$). So $\alpha_k\in\mathcal{S}(V)$. The same is assumed on β_k and Γ_k. We look at a process $Y_k\in V=U^d$, $k\in\mathbb{N}$ which satisfies the equation

$$Y_m=Y_0+\sum_{k=0}^{m-1}A_kY_k+\sum_{i=1}^{N}\sum_{k=0}^{m-1}H^i_{k+1}\alpha^i_k+\sum_{i=1}^{N}\sum_{k=0}^{m-1}LH^i_{k+1}\beta^i_k+\Gamma_m. \tag{6.3}$$

Notice that we do not discuss about existence and uniqueness of the solution of such an equation. We just suppose that, the process Y at hand satisfies this equation (which naturally appears in our calculus). We aim to estimate the Sobolev norms of Y_m. Let $q\in\mathbb{N}$ and $p\geqslant 2$. We denote

$$C_{q,p}(\alpha,\beta,\Gamma)=\sup_{0\leqslant m\leqslant n-1}\sup_{i=1,..,N}(1+\|\alpha^i_m\|_{V,q,p}+\|\beta^i_m\|_{V,q,p}+\|\Gamma_{m+1}\|_{V,q,p}) \tag{6.4}$$

Proposition 6.1. *For every $q\in\mathbb{N}$ and $p\geqslant 2$ there exists some constants $l\in\mathbb{N}^*$, $C\geqslant 1$ (depending on q and p) such that*

$$\sup_{m\leqslant n}\|Y_m\|_{V,q,p}\leqslant C(M_l(Z)+\frac{m_*^{1/l}}{r_*}(1+r_*^{-q}))C_{q,l}(\alpha,\beta,\Gamma)\mathfrak{K}_{q+2}(CM_l(Z)\psi)^l. \tag{6.5}$$

with $\mathfrak{K}_r(\psi)$ and $M_l(Z)$ defined in (4.5) and (3.2).

Proof. **Step 1**. Let $q=0$, so that $\|Y_m\|_{V,q,p}=\|Y_m\|_{V,p}$. We will check that

$$\sup_{m\leqslant n}\|Y_m\|_{V,p}\leqslant C\big(M_p(Z)^{1/p}C_{0,p}(\alpha,0,0)+\frac{m_*^{1/p}}{r_*}C_{0,p}(0,\beta,0)+C_{0,p}(0,0,\Gamma)\big)$$
$$\times\exp(CM_{2p}(Z)^{2/p}\|\psi\|_{1,3,\infty}^2). \tag{6.6}$$

We study the terms which appear in the right hand side of (6.3). Notice that β_k^i is $\sigma(Z_1,\ldots,Z_k)$ measurable and $\mathbb{E}[LH_{k+1}^i]=0$ (see (3.23)). It follows that, $M_m=\sum_{k=0}^{m-1}LH_{k+1}^i\beta_k^i$ is a martingale and consequently, by (6.2)

$$\|M_m\|_{V,p}\leqslant b_p(\sum_{k=0}^{m-1}\|LH_{k+1}^i\beta_k^i\|_{V,p}^2)^{1/2}.$$

Since LH_{k+1}^i and β_k^i are independent, using (3.24) we obtain

$$\|LH_{k+1}^i\beta_k^i\|_{V,p}^2=\|LH_{k+1}^i\|_p^2\|\beta_k^i\|_{V,p}^2\leqslant\frac{Cm_*^{2/p}}{r_*^2}\|\beta_k^i\|_{V,p}^2/n.$$

We conclude that

$$\sup_{m\leqslant n}\|M_m\|_{V,p}\leqslant\frac{Cm_*^{1/p}}{r_*}(\frac{1}{n}\sum_{k=0}^{n-1}\|\beta_k^i\|_{V,p}^2)^{1/2}\leqslant\frac{Cm_*^{1/p}}{r_*}\sup_{k\leqslant n-1}\|\beta_k^i\|_{V,p}.$$

Since H_{k+1}^i is independent from α_k^i and $\mathbb{E}[H_{k+1}^i]=0$, it follows that $M_m=\sum_{k=0}^{m-1}H_{k+1}^i\alpha_k^i$ is a martingale. We have $\|H_k^i\|_p\leqslant n^{-1/2}M_p(Z)^{1/p}$ so the same reasoning as above proves that the previous inequality holds for M_m (with $m_*^{1/p}r_*^{-1}$ replaced by $M_p(Z)^{1/p}$ and $\|\beta_k^i\|_{V,p}$ replaced by $\|\alpha_k^i\|_{V,p}$). We use the same reasoning for $M_m=\sum_{k=0}^{m-1}H_{k+1}^i\nabla_x a_k^i(X_{t_k^n}^n)Y_k\in V$ and we obtain

$$\|M_m\|_{V,p}\leqslant b_p(\sum_{k=0}^{m-1}\|H_{k+1}^i\nabla_x a_k^i(X_{t_k^n}^n)Y_k\|_{V,p}^2)^{1/2}\leqslant CM_p(Z)^{1/p}\|\psi\|_{1,2,\infty}(\frac{1}{n}\sum_{k=0}^{m-1}\|Y_k\|_{V,p}^2)^{1/2}.$$

Finally, using the triangle inequality

$$\begin{aligned}\|\sum_{k=0}^{m-1}H_{k+1}^iH_{k+1}^j\nabla_x b_k^{i,j}(X_{t_k^n}^n,H_{k+1})Y_k\|_{V,p} &\leqslant \sum_{k=0}^{m-1}\|H_{k+1}^iH_{k+1}^j\nabla_x b_k^{i,j}(X_{t_k^n}^n,H_{k+1})Y_k\|_{V,p}\\ &\leqslant CM_{2p}(Z)^{1/p}\|\psi\|_{1,3,\infty}\frac{1}{n}\sum_{k=0}^{m-1}\|Y_k\|_{V,p},\end{aligned}$$

and in the same way $\|\sum_{k=0}^{m-1}\delta_{k+1}^n\nabla_x\tilde{b}_k(X_{t_k^n}^n,H_{k+1},\delta_{k+1}^n)Y_k\|_{V,p}\leqslant C\|\psi\|_{1,3,\infty}\times\sum_{k=0}^{m-1}\|Y_k\|_{V,p}/n$. We gather all the terms and we obtain

$$\begin{aligned}\|Y_m\|_{V,p} &\leqslant \|Y_0\|_{V,p}+CM_{2p}(Z)^{1/p}\|\psi\|_{1,3,\infty}(\frac{1}{n}\sum_{k=0}^{m-1}\|Y_k\|_{V,p}^2)^{1/2}\\ &\quad +C(M_p(Z)^{1/p}\sup_{k\leqslant n-1}\|\alpha_k^i\|_{V,p}+\frac{m_*^{1/p}}{r_*}\sup_{k\leqslant n-1}\|\beta_k^i\|_{V,p})+\|\Gamma_m\|_{V,p}\end{aligned}$$

Using Gronwall's lemma we obtain (6.6).

Step 2. Let

$$H=\{h:\{1,\ldots,n\}\times\{1,\ldots,N\}\to\mathbb{R}:|h|_H^2=\sum_{k=1}^n\sum_{i=1}^N h^2(k,i)<\infty\}.$$

so that $DX^n_{t^n_m} \in H^d$. We are going to prove that

$$\sup_{m\leqslant n} \|DX^n_{t^n_m}\|_{H^d,p} \leqslant CM_{2p}(Z)^{1/p}\|\psi\|_{1,3,\infty} \exp(CM_{2p}(Z)^{2/p}\|\psi\|^2_{1,3,\infty}). \tag{6.7}$$

For $h \in H$ we denote

$$D_hF = \langle DF, h\rangle = \sum_{k=1}^{n}\sum_{i=1}^{N} h(k,i)D_{(k,i)}F.$$

Since

$$D_{(r,j)}H^i_k = \frac{1}{\sqrt{n}}\delta_{r,k}\delta_{j,i}\chi_k,$$

we use (4.6) to obtain

$$\begin{aligned}
D_hX^n_{t^n_{k+1}} \quad = \quad & D_hX^n_{t^n_k} + A_kD_hX^n_{t^n_k} + \frac{1}{\sqrt{n}}\sum_{i=1}^{N}\chi_{k+1}h(k+1,i)a^i_k(X^n_{t^n_k}) \\
& + \frac{1}{\sqrt{n}}\sum_{i,j=1}^{N}\chi_{k+1}(h(k+1,i)H^j_{k+1} + h(k+1,j)H^i_{k+1})b^{i,j}_k(X^n_{t^n_k}, H_{k+1}) \\
& + \frac{1}{\sqrt{n}}\sum_{i,j,q=1}^{N}\chi_{k+1}H^i_{k+1}H^j_{k+1}\partial_{z^q}b^{i,j}_k(X^n_{t^n_k}, H_{k+1})h(k+1,q) \\
& + \frac{1}{\sqrt{n}}\chi_{k+1}\delta^n_{k+1}\sum_{q=1}^{N}\partial_{z^q}\tilde{b}_k(X^n_{t^n_k}, H_{k+1}, \delta^n_{k+1})h(k+1,q)
\end{aligned}$$

Iterating this formula over k we obtain

$$D_hX^n_{t^n_m} = \sum_{k=0}^{m-1} A_kD_hX^n_{t^n_k} + \langle h, \Gamma_m\rangle$$

with $\Gamma_m(k,i) = 0$ for $k > m$ and, for $k \leqslant m$

$$\begin{aligned}
\Gamma_m(k,i) = & \frac{\chi_k}{\sqrt{n}}\big(a^i_{k-1}(X^n_{t^n_{k-1}}) + \sum_{j=1}^{N} H^j_kb^{i,j}_{k-1}(X^n_{t^n_{k-1}}, H_k) + \sum_{j,l=1}^{N} H^j_kH^l_k\partial_{z^i}b^{l,j}_{k-1}(X^n_{t^n_{k-1}}, H_k) \\
& + \delta^n_k\partial_{z^i}\tilde{b}_{k-1}(X^n_{t^n_{k-1}}, H_k, \delta^n_k)\big).
\end{aligned}$$

One has

$$|\Gamma_m|^2_{H^d} = \sum_{k=1}^{m}\sum_{i=1}^{N}|\Gamma_m(k,i)|^2 \leqslant C\|\psi\|^2_{1,3,\infty}\frac{1}{n}\sum_{k=1}^{n}(1+|Z_k|^4)$$

so, using (6.6) (with V replaced by H^d and $\alpha_k = \beta_k = 0$), we obtain

$$\begin{aligned}
\sup_{m\leqslant n}\|DX^n_{t^n_m}\|_{H^d,p} \quad & \leqslant \quad C\sup_{m\leqslant n}\|\Gamma_m\|_{H^d,p}\exp(CM_{2p}(Z)^{2/p}\|\psi\|^2_{1,3,\infty}) \\
& \leqslant \quad CM_{2p}(Z)^{1/p}\|\psi\|_{1,3,\infty}\exp(CM_{2p}(Z)^{2/p}\|\psi\|^2_{1,3,\infty}).
\end{aligned}$$

Step 3. We estimate the derivatives of Y_m, solution of (6.3). We have

$$DY_m = \sum_{k=0}^{m-1} A_kDY_k + \sum_{i=1}^{N}\sum_{k=0}^{m-1} H^i_{k+1}\overline{\alpha}^i_k + \sum_{i=1}^{N}\sum_{k=0}^{m-1} LH^i_{k+1}\overline{\beta}^i_k + \overline{\Gamma}_m$$

with

$$\begin{aligned}\overline{\alpha}_k^i &= \nabla_x \nabla_x a_k^i(X_{t_k^n}^n) DX_{t_k^n}^n Y_k + D\alpha_k^i,\\ \overline{\beta}_k^i &= D\beta_k^i\end{aligned}$$

and

$$\begin{aligned}\overline{\Gamma}_m &= \sum_{k=0}^{m-1}\sum_{i=1}^{N} \nabla_x a_k^i(X_{t_k^n}^n) DH_{k+1} Y_k + \frac{1}{2}\sum_{i,j=1}^{N}\sum_{k=0}^{m-1} D(H_{k+1}^i H_{k+1}^j \nabla_x b_k^{i,j}(X_{t_k^n}^n, H_{k+1}))Y_k\\ &+ \sum_{k=0}^{m-1} \delta_{k+1}^n D(\nabla_x \tilde{b}_k(X_{t_k^n}^n, H_{k+1}, \delta_{k+1}^n))Y_k + \sum_{i=1}^{N}\sum_{k=0}^{m-1} \alpha_k^i DH_{k+1}^i\\ &+ \sum_{i=1}^{N}\sum_{k=0}^{m-1} \beta_k^i DLH_{k+1}^i + D\Gamma_m.\end{aligned}$$

Notice that DY_m is a process with values in H^d. We will prove that

$$\begin{aligned}&M_p(Z)^{1/p} C_{0,p}(\overline{\alpha}, 0, 0) + \frac{m_*^{1/p}}{r_*} C_{0,p}(0, \overline{\beta}, 0) + C_{0,p}(0, 0, \overline{\Gamma}) \qquad (6.8)\\ &\quad\leqslant CM_{4p}(Z)^{1/p}\big(M_{2p}(Z)^{1/2p} C_{0,2p}(\alpha, 0, \Gamma) + \frac{m_*^{1/2p}}{r_*}(1 + r_*^{-1}) C_{0,2p}(0, \beta, 0)\big)\\ &\qquad\times \|\psi\|_{1,4,\infty}^2 \exp(CM_{4p}(Z)^{1/p}\|\psi\|_{1,4,\infty}^2) + M_p(Z)^{1/p} C_{1,p}(\alpha, 0, \Gamma) + \frac{m_*^{1/p}}{r_*} C_{1,p}(0, \beta, 0).\end{aligned}$$

Once (6.8) is proved, the whole proof is concluded. Indeed, using (6.8) and the result from the first step (that is (6.5) with $q = 0$ and Y_m replaced by DY_m), we obtain (6.5) with $q = 1$. Consequently, using recursively the same reasoning we obtain (6.5) for every $q \in \mathbb{N}$.

We estimate each of the terms which appear in the right hand side of (6.8). First, we write

$$\begin{aligned}\|\nabla_x \nabla_x a_k^i(X_{t_k^n}^n) DX_{t_k^n}^n Y_k\|_{H^d,p} &\leqslant C\|\psi\|_{1,3,\infty} \big\| |DX_{t_k^n}^n|_{H^d} |Y_k|_V \big\|_p\\ &\leqslant C\|\psi\|_{1,3,\infty} \|DX_{t_k^n}^n\|_{H^d,2p} \|Y_k\|_{V,2p}\\ &\leqslant CM_{4p}(Z)^{1/2p}\|\psi\|_{1,3,\infty}^2 \big(M_{2p}(Z)^{1/2p} C_{0,2p}(\alpha, 0, 0)\\ &\quad + \frac{m_*^{1/2p}}{r_*} C_{0,2p}(0, \beta, 0)\\ &\quad + C_{0,2p}(0, 0, \Gamma)\big) \exp(CM_{4p}(Z)^{1/p}\|\psi\|_{1,3,\infty}^2),\end{aligned}$$

the last inequality being a consequence of (6.6) and (6.7). It follows that

$$\begin{aligned}\|\overline{\alpha}_k^i\|_{H^d,p} &\leqslant CM_{4p}(Z)^{1/2p}\big(M_{2p}(Z)^{1/2p} C_{0,2p}(\alpha, 0, \Gamma) + \frac{m_*^{1/2p}}{r_*} C_{0,2p}(0, \beta, 0)\big)\\ &\quad\times \|\psi\|_{1,3,\infty}^2 \exp(CM_{4p}(Z)^{1/p}\|\psi\|_{1,3,\infty}^2) + C_{1,p}(\alpha, 0, 0).\end{aligned}$$

And

$$\|\overline{\beta}_k^i\|_{H^d,p} = \|D\beta_k^i\|_{H^d,p} \leqslant C_{1,p}(0, \beta, 0).$$

We analyse now $\overline{\Gamma}_m$. We treat first $I_m := \sum_{k=0}^{m-1} \beta_k^i DLH_{k+1}^i$. Since $\beta_k^i D_{(p,j)} LH_{k+1}^i = 0$ if $p \neq k+1$, we obtain

$$|I_m|_{H^d}^2 \leqslant \sum_{j=1}^{N}\sum_{k=0}^{m-1} |D_{(k+1,j)} LH_{k+1}^i|^2 |\beta_k^i|_V^2$$

so that, using (3.24), and the independency of LH_{k+1} and β_k, we have

$$\begin{aligned}
\||I_m|_{H^d}\|_p &= \big\||I_m|^2_{H^d}\big\|^{1/2}_{p/2} \leqslant (\sum_{j=1}^{N}\sum_{k=0}^{m-1}\big\||D_{(k+1,j)}LH^i_{k+1}|^2|\beta^i_k|^2_V\big\|_{p/2})^{1/2} \\
&= (\sum_{j=1}^{N}\sum_{k=0}^{m-1}\big\||D_{(k+1,j)}LH^i_{k+1}||\beta^i_k|_V\big\|^2_p)^{1/2} \\
&= (\sum_{j=1}^{N}\sum_{k=0}^{m-1}\|D_{(k+1,j)}LH^i_{k+1}\|^2_p\big\||\beta^i_k|_V\big\|^2_p)^{1/2} \\
&\leqslant \frac{Cm_*^{1/p}}{r_*}(1+r_*^{-1})\sup_{k\leqslant m-1}\big\||\beta^i_k|_V\big\|_p = \frac{Cm_*^{1/p}}{r_*}(1+r_*^{-1})\sup_{k\leqslant m-1}\|\beta^i_k\|_{V,p}.
\end{aligned}$$

Since DH^i_k has properties which are similar to the ones of DLH^i_k, the same reasoning as above gives

$$\|\sum_{k=0}^{m-1}\alpha^i_k DH^i_{k+1}\|_{H^d,p} \leqslant C\sup_{k\leqslant m-1}\|\alpha^i_k\|_{V,p}$$

and we have

$$\begin{aligned}
|\sum_{k=0}^{m-1}\nabla_x a^i_k(X^n_{t^n_k})Y_k DH^i_{k+1}|^2_{H^d} &\leqslant \|\psi\|^2_{1,2,\infty}\sum_{k=0}^{m-1}\sum_{j=1}^{N}|Y_k|^2_V|D_{k+1,j}H^i_{k+1}|^2 \\
&\leqslant \frac{C}{n}\|\psi\|^2_{1,2,\infty}\sum_{k=0}^{n-1}|Y_k|^2_V.
\end{aligned}$$

Using (6.6) and the triangle inequality, we obtain

$$\begin{aligned}
\|\sum_{k=0}^{m-1}\nabla_x a^i_k(X^n_{t^n_k})Y_k DH^i_{k+1}\|_{H^d,p} &\leqslant C\|\psi\|_{1,2,\infty}\big(n^{-1}\sum_{k=0}^{n-1}\|Y_k\|^2_{V,p}\big)^{1/2} \\
&\leqslant C\big(M_p(Z)^{1/p}C_{0,p}(\alpha,0,\Gamma)+\frac{m_*^{1/p}}{r_*}C_{0,p}(0,\beta,0)\big) \\
&\quad \times\|\psi\|_{1,2,\infty}\exp(CM_{2p}(Z)^{2/p}\|\psi\|^2_{1,3,\infty}).
\end{aligned}$$

We write now

$$\sum_{k=0}^{m-1}D(H^i_{k+1}H^j_{k+1}\nabla_x b^{i,j}_k(X^n_{t^n_k},H_{k+1}))Y_k = I+J$$

with

$$\begin{aligned}
I &= \sum_{k=0}^{m-1}(H^i_{k+1}DH^j_{k+1}+H^j_{k+1}DH^i_{k+1})\nabla_x b^{i,j}_k(X^n_{t^n_k},H_{k+1})Y_k, \\
J &= \sum_{k=0}^{m-1}H^i_{k+1}H^j_{k+1}D(\nabla_x b^{i,j}_k(X^n_{t^n_k},H_{k+1}))Y_k.
\end{aligned}$$

We have

$$|I|^2_{H^d} \leqslant C\|\psi\|^2_{1,3,\infty}n^{-1}\sum_{k=0}^{m-1}(|H^i_{k+1}|^2+|H^j_{k+1}|^2)|Y_k|^2_V,$$

and using the independence between Y_k and H_{k+1}, it follows that

$$\|I\|_{H^d,p} \leqslant Cn^{-1/2}\big(M_p(Z)^{1/p}C_{0,p}(\alpha,0,\Gamma) \\ + \frac{m_*^{1/p}}{r_*}C_{0,p}(0,\beta,0)\big)M_p(Z)^{1/p}\|\psi\|_{1,3,\infty}\exp(CM_{2p}(Z)^{2/p}\|\psi\|^2_{1,3,\infty}).$$

Considering the estimates of $DX^n_{t^n_k}$, we obtain in a similar way

$$\begin{aligned}\|J\|_{H^d,p} \leqslant{}& Cn^{-1}\big(M_p(Z)^{1/p}C_{0,p}(\alpha,0,\Gamma) \\ &+ \frac{m_*^{1/p}}{r_*}C_{0,p}(0,\beta,0)\big)M_{2p}(Z)^{1/p}\|\psi\|_{1,3,\infty}\exp(CM_{2p}(Z)^{2/p}\|\psi\|^2_{1,3,\infty}) \\ &+ Cn^{-1/2}\big(M_{2p}(Z)^{1/2p}C_{0,2p}(\alpha,0,\Gamma) + \frac{m_*^{1/2p}}{r_*}C_{0,2p}(0,\beta,0)\big)M_{4p}(Z)^{1/p} \\ &\times \|\psi\|^2_{1,4,\infty}\exp(CM_{4p}(Z)^{1/p}\|\psi\|^2_{1,3,\infty}) \\ \leqslant{}& Cn^{-1/2}\big(M_{2p}(Z)^{1/2p}C_{0,2p}(\alpha,0,\Gamma) + \frac{m_*^{1/2p}}{r_*}C_{0,2p}(0,\beta,0)\big)M_{4p}(Z)^{1/p} \\ &\times \|\psi\|^2_{1,4,\infty}\exp(CM_{4p}(Z)^{1/p}\|\psi\|^2_{1,3,\infty})\end{aligned}$$

It follows that a similar estimate holds for $\sum_{k=0}^{m-1} D(H^i_{k+1}H^j_{k+1}\nabla_x b_{i,j}(X^n_{t^n_k}))Y_k$ as for J. Finally, in the same way, we obtain

$$\begin{aligned}&\|\sum_{k=0}^{m-1}\delta^n_{k+1}D(\nabla_x\tilde{b}_k(X^n_{t^n_k},H_{k+1},\delta^n_{k+1}))Y_k\|_{H^d,p} \\ &\leqslant Cn^{-1/2}\big(M_{2p}(Z)^{1/2p}C_{0,2p}(\alpha,0,\Gamma) \\ &\quad + \frac{m_*^{1/2p}}{r_*}C_{0,2p}(0,\beta,0)\big)\|\psi\|^2_{1,4,\infty}\exp(CM_{4p}(Z)^{1/p}\|\psi\|^2_{1,3,\infty}).\end{aligned}$$

We gather all these terms and we obtain (6.8). □

Now, we are in a position to prove Theorem 4.2. For the reader's convenience we recall the statement of this result.

Theorem 6.2. *For every $q,q'\in\mathbb{N}$, $q'\leqslant q$, and $p\geqslant 2$ there exists some constants $l\in\mathbb{N}^*$, $C\geqslant 1$ (depending on $r_*,\varepsilon_*,m_*,q,p$ and the moments of Z but not on n) such that*

$$\sup_{t\in\pi^T_{T,n}}\sup_{0\leqslant|\alpha|\leqslant q-q'}\|\partial^\alpha_x X^n_t(x)\|_{q',p} \leqslant C\mathfrak{K}_{q+2}(\psi)^l, \tag{6.9}$$

$$\sup_{t\in\pi^T_{T,n}}\|LX^n_t\|_{q,p} \leqslant C\mathfrak{K}_{q+4}(\psi)^l. \tag{6.10}$$

where $\mathfrak{K}_r(\psi)$ is defined in (4.5) and is given by

$$\mathfrak{K}_r(\psi) = (1+\|\psi\|_{1,r,\infty})\exp(\|\psi\|^2_{1,3,\infty}).$$

Proof. We estimate first $\|X^n_t\|_{q,p}$. We have already checked that

$$DX^n_{t^n_m} = \sum_{k=0}^{m-1}A_k DX^n_{t^n_k} + \Gamma_m$$

with

$$\Gamma_m(k,i) = \mathbb{1}_{\{k\leqslant m\}}\frac{\chi_k}{\sqrt{n}}\big(a^i_{k-1}(X^n_{t^n_{k-1}}) + \sum_{j=1}^N H^j_k b^{i,j}_{k-1}(X^n_{t^n_{k-1}},H_k)$$

$$+\sum_{j,l=1}^{N} H_k^j H_k^l \partial_{z^i} b_{k-1}^{l,j}(X_{t_{k-1}^n}^n, H_k) + \delta_k^n \partial_{z^i} \tilde{b}_{k-1}(X_{t_{k-1}^n}^n, H_k, \delta_k^n)\big).$$

Using (6.5), the only thing to prove is that $\|\Gamma_m\|_{q-1,p} \leqslant C\mathfrak{K}_{q+2}(\psi)^l$. We have already done it for the first order derivatives (that is $q=1$). For higher order derivatives, the proof follows the same line (using a recurrence argument).

Now, we study $\nabla_x X_t^n(x)$ which solves the equation

$$\nabla_x X_{t_m^n}^n(x) = I + \sum_{k=1}^{m-1} A_k \nabla_x X_{t_k^n}^n(x).$$

This equation is similar to (6.3) so the upper bound of $\|\nabla_x X_{t_m^n}^n(x)\|_{q,p}$ follows from (6.6). For higher order derivatives the reasoning is the same.

Let us now deal with LX_t^n. Notice that $\left\langle DH_k^j, DH_k^i \right\rangle = 0$ for $i \neq j$. Then, using the computational rules (see (3.15)), we obtain

$$LX_{t_{k+1}^n}^n = A_k L X_{t_k^n}^n + \sum_{i=1}^N H_{k+1}^i \alpha_k^i + \sum_{i=1}^N LH_{k+1}^i \beta_k^i + \sum_{i,j=1}^N \gamma_k^{i,j}$$

with

$$\alpha_k^i = \sum_{l,r=1}^d \partial_{x_l}\partial_{x_r} a_k^i(X_{t_k^n}^n)\langle (DX_{t_k^n}^n)^r, (DX_{t_k^n}^n)^l\rangle, \qquad \beta_k^i = a_k^i(X_{t_k^n}^n)$$

and

$$\begin{aligned}
\gamma_k^{i,j} &= \frac{1}{2} LH_{k+1}^i H_{k+1}^j b_k^{i,j}(X_{t_k^n}^n, H_{k+1}) + \frac{1}{2} LH_{k+1}^i H_{k+1}^j b_k^{i,j}(X_{t_k^n}^n, H_{k+1}) \\
&+ \frac{1}{2} H_{k+1}^i H_{k+1}^j \Big(\sum_{l,r=1}^d \partial_{x_l}\partial_{x_r} b_k^{i,j}(X_{t_k^n}^n, H_{k+1}) \langle (DX_{t_k^n}^n)^l, (DX_{t_k^n}^n)^r\rangle \\
&+ \sum_{r=1}^N \partial_{z_r} b_k^{i,j}(X_{t_k^n}^n, H_{k+1}) LH_{k+1}^r + \frac{\chi_{k+1}}{n} \sum_{r=1}^N \partial_{z_r}^2 b_k^{i,j}(X_{t_k^n}^n, H_{k+1}) \Big) \\
&+ \mathbb{1}_{i=j} \frac{\chi_{k+1}}{n} b_k^{i,i}(X_{t_k^n}^n, H_{k+1}) + \frac{\chi_{k+1}}{n} \big(H_{k+1}^i \partial_{z_j} b_k^{i,j}(X_{t_k^n}^n, H_{k+1}) \\
&+ H_{k+1}^j \partial_{z_i} b_k^{i,j}(X_{t_k^n}^n, H_{k+1}) \big) \\
&+ \frac{1}{2}\delta_{k+1}^n \Big(\sum_{l,r=1}^d \partial_{x_l}\partial_{x_r} \tilde{b}_k(X_{t_k^n}^n, H_{k+1}, \delta_{k+1}^n) \langle (DX_{t_k^n}^n)^l, (DX_{t_k^n}^n)^r\rangle \\
&+ \sum_{r=1}^N \partial_{z_r} \tilde{b}_k(X_{t_k^n}^n, H_{k+1}, \delta_{k+1}^n) LH_{k+1}^r + \frac{\chi_{k+1}}{n} \sum_{r=1}^N \partial_{z_r}^2 \tilde{b}_k(X_{t_k^n}^n, H_{k+1}, \delta_{k+1}^n) \Big).
\end{aligned}$$

We have

$$\|\alpha_k^i\|_{q,p} \leqslant C\|\psi\|_{1,q+3,\infty} \|X_{t_k^n}^n\|_{q+1,p}^2 \leqslant C\mathfrak{K}_{q+3}(\psi)^l$$

and a similar estimate holds for $\|\beta_k^i\|_{q,p}$. Moreover, we have $\Gamma_m = \sum_{i,j=1}^N \sum_{k=0}^{m-1} \gamma_k^{i,j}$ so we have to analyse each of the terms in $\gamma_k^{i,j}$. We look first at

$$\begin{aligned}
\|LH_{k+1}^i H_{k+1}^j b_k^{i,j}(X_{t_k^n}^n, H_{k+1})\|_{q,p} &\leqslant \|LH_{k+1}^i H_{k+1}^j\|_{q,2p} \|b_k^{i,j}(X_{t_k^n}^n, H_{k+1})\|_{q,2p} \\
&\leqslant \|LH_{k+1}^i\|_{q,4p} \|H_{k+1}^j\|_{q,4p} \|\psi\|_{1,q+2,\infty}^l (\|X_{t_k^n}^n\|_{q,2p}^l + \|H_k^j\|_{q,2p}^l) \\
&\leqslant C\mathfrak{K}_{q+2}(\psi)^l/n.
\end{aligned}$$

The other terms in $\gamma_k^{i,j}$ verify similar estimates. So we obtain

$$\|\Gamma_m\|_{q,p} \leqslant \sum_{i,j=1}^{N} \sum_{k=0}^{m-1} \|\gamma_k^{i,j}\|_{q,p} \leqslant C\mathfrak{K}_{q+4}(\psi)^l.$$

We conclude that

$$C_{q,p}(\alpha,\beta,\Gamma) \leqslant C\mathfrak{K}_{q+4}(\psi)^l$$

and the proof is competed. □

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