# Yule processes with rare mutation and their applications to percolation on $b$-ary trees[*]

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### Abstract

We consider supercritical Bernoulli bond percolation on a large $b$-ary tree, in the sense that with high probability, there exists a giant cluster. We show that the size of the giant cluster has non-gaussian fluctuations, which extends a result due to Schweinsberg [15] in the case of random recursive trees. Using ideas in the recent work of Bertoin and Uribe Bravo [5], the approach developed in this work relies on the analysis of the sub-population with ancestral type in a system of branching processes with rare mutations, which may be of independent interest. This also allows us to establish the analogous result for scale-free trees.

## 1 Introduction and main result

Consider a tree of large but finite size $n$ and perform Bernoulli bond percolation with parameter $p_n \in (0,1)$ that depends on the size of that tree. So each edge is removed with probability $1 - p_n$ and independently of the other edges, inducing a partition of the set of vertices into connected clusters. We are interested in the supercritical regime, in the sense that with high probability, there exists a giant cluster of size comparable to $n$, and its complement has also a size of order $n$. In fact, it has been shown recently in [4] that for fairly general families of trees, the supercritical regime corresponds to parameters of the form $p_n = 1 - c/\ell(n)$, where $\ell(n)$ is an estimate of the height of a typical vertex in the structure.

In the case of the uniform random recursive trees (i.e. trees on an ordered set of vertices where the smallest vertex serves as the root, and the sequence of vertices along any branch from the root to a leaf is increasing) it easily seen that $\ell(n) = \ln n$, so choosing the percolation parameter so that

$$p_n = 1 - \frac{c}{\ln n}, \tag{1.1}$$

where $c > 0$ is fixed, corresponds to the supercritical regime. More precisely, if $\Gamma_n$ denotes the size of the largest cluster, then $\lim_{n\to\infty} n^{-1}\Gamma_n = e^{-c}$ in probability. This can be viewed as the law of large numbers for the giant cluster, and it is then natural to investigate its fluctuations. Schweinsberg [15] (see also Bertoin [3] for an alternative approach) has shown that in this particular case, the fluctuations are non-Gaussian. Specifically

$$(n^{-1}\Gamma_n - e^{-c})\ln n - ce^{-c}\ln\ln n \Rightarrow -ce^{-c}(\mathcal{Z} + \ln c), \tag{1.2}$$

where $\Rightarrow$ means convergence in law as $n \to \infty$ and the variable $\mathcal{Z}$ has the continuous Luria-Delbrück distribution, i.e. its characteristic function is given by

$$\mathbb{E}(e^{i\theta\mathcal{Z}}) = \exp\left(-\frac{\pi}{2}|\theta| - i\theta\ln|\theta|\right), \qquad \theta \in \mathbb{R}.$$

The main purpose of this work is to investigate the case of large random $b$-ary recursive trees ($b \geq 2$). The process to build a $b$-ary recursive tree starts at $n = 1$ from the tree $T_1$ with one internal vertex (which corresponds to the root) and $b$ external vertices. Then, we suppose that $T_n$ has been constructed for some $n \geq 1$ that is a tree with $n$ internal vertices and $(b-1)n + 1$ external ones (also called leaves). Then choose an external vertex uniformly at random and replace it by an internal vertex to which $b$ new leaves are attached. In the case $b = 2$, the algorithm yields a so-called binary search tree (see for instance Mahmoud [13], Drmota [8]). We consider that the size of the tree is the number of internal vertices.

Then we perform Bernoulli bond percolation with parameter given by (1.1) on a random $b$-ary recursive tree of size $n$, which corresponds precisely to the supercritical regime as the case of the random recursive trees. Roughly speaking, since the $b$-ary recursive trees have also logarithmic height, i.e. the height of typical vertex is approximately $\ell(n) = (b\ln n)/(b-1)$ (see Javanian and Vahidi-Asl [12]), one can verify that percolation then produces a giant cluster whose size $C_0^{(p)}$ (number of internal vertices) satisfies

$$\lim_{n\to\infty} n^{-1}C_0^{(p)} = e^{-\frac{b}{b-1}c} \qquad \text{in probability.}$$

We now state the central result of this work, which shows that the fluctuations of the giant cluster in the case of the $b$-ary recursive trees are also described by the continuous Luria-Delbrück distribution. We stress that this distribution was further observed in relation with a random algorithm for the isolation of the root, in the context of uniform random recursive tree by Iksanov and Möhle [11], and for random binary search tree by Holmgren [10].

**Theorem 1.1.** *Set $\beta = b/(b-1)$, and assume that the percolation parameter $p_n$ is given by (1.1). Then as $n \to \infty$, there is the weak convergence*

$$(n^{-1}C_0^{(p)} - e^{-\beta c})\ln n - \beta c e^{-\beta c}\ln\ln n \Rightarrow -\beta c e^{-\beta c}\mathcal{Z}_{c,\beta}$$

*where*

$$\mathcal{Z}_{c,\beta} = \mathcal{Z} - \kappa_\beta + \ln(\beta c) \tag{1.3}$$

*with $\mathcal{Z}$ having the continuous Luria-Delbrück distribution,*

$$\kappa_\beta = 1 - \frac{1}{\beta} + \frac{1}{\beta}\sum_{k=2}^{\infty} \frac{(\beta)_k}{k!}\frac{(-1)^k}{k-1}, \tag{1.4}$$

*and $(x)_k = x(x-1)\cdots(x-k+1)$, for $k \in \mathbb{N}$ and $x \in \mathbb{R}$, is the Pochhammer function. In particular, for $b = 2$, i.e. for the binary search tree case, $\kappa_2 = 1$.*

It should be noted the close similarity with the result for uniform recursive trees. It is remarkable that the normalizing functions and the limit in Theorem 1.1 only depend on the parameter $\beta = b/(b-1)$ through some constants. Observe that the left-hand side of (1.2) is the same as in Theorem 1.1 for $\beta = 1$; however the expressions (1.3) and (1.4) are not defined for $\beta = 1$ !

The basic idea of Schweinsberg [15], for establishing the result (1.2) for uniform recursive trees relies on the estimation of the rate of decrease of the number of blocks in the Bolthausen-Sznitman coalescent, using the construction due to Goldschmidt and Martin [9] of the latter in terms of uniform recursive trees. On the other hand, the alternative approach of Bertoin [3] makes use on the remarkable coupling due to Iksanov and Möhle [11] connecting the Meir and Moon algorithm for the isolation of the root, with a certain random walk in the domain of attraction of the completely asymmetric Cauchy process. These approaches depend crucially on the *splitting property* (see Section 3.1 in Bertoin [2]) which fails for the $b$-ary recursive trees. We thus have to use a different argument, although some guiding lines are similar to [3].

Essentially, we consider a continuous time version of the growth algorithm of the $b$-ary tree which bears close relations to Yule processes. The connection between recursive trees and branching processes is well-known, we make reference to Chauvin, et. al. [6] for the binary search trees and Bertoin and Uribe Bravo [5] for the case of scale-free trees. In this way, we adapt the recent strategy of [5]. Roughly speaking, incorporating percolation to the algorithm yields systems of branching processes with mutations, where a mutation event corresponds to disconnecting a leaf from its parent, and simultaneously replacing it by an internal vertex to which $b$ new leaves are attached. Each percolation cluster size can then be thought of as a sub-population with some given genetic type. Hence the problem is reduced to study the fluctuations of the size of the sub-population with the ancestral type, which corresponds to the number of internal vertices connected to the root cluster.

The work is organized as follows. In Section 2, we introduce the system of branching processes with rare mutations. We investigate the fluctuations of the size of the sub-population with the ancestral type, when the total population goes to infinity and the mutation parameter $1 - p_n$ satisfies (1.1). Then in Section 3, we make the link with percolation on $b$-ary recursive trees in order to prove Theorem 1.1. Finally, we briefly show in Section 4 that the present approach also applies to study the fluctuations of the size of the giant cluster for percolation on scale-free trees.

## 2 Yule process with rare mutations

The purpose of this section is to introduce a system of branching process with rare mutations, which is quite similar to the one considered in [5], although there are also some key differences (in particular, death may occur causing the extinction of sub-population with the ancestral type). Then we focus on estimating the size of the sub-population with the ancestral type, when the total population in the system grows and the mutation parameter depends of the size of the latter.

We consider a population in which each individual is either a clone (i.e. an individual with the ancestral type) or a mutant with some genetic type. A clone individual lives for an exponential time of parameter 1, and gives birth at its death to $b$ clones with probability $p \in (0, 1)$, or $b$ mutants that share the same genetic type with probability

$1 - p$. A mutant individual lives for an exponential time of parameter 1, and gives birth at its death to $b$ children of the same genetic of its parent. More precisely, the evolution of the population system is described by the process $\mathbf{Z}^{(p)} = (\mathbf{Z}^{(p)}(t) : t \geq 0)$, where

$$\mathbf{Z}^{(p)}(t) = (Z_0^{(p)}(t), Z_1(t), \dots), \quad \text{for} \quad t \geq 0,$$

is a collection of nonnegative variables which represents the current size of the sub-populations. At the initial time, the sub-populations $Z_i(0)$ of type $i \geq 1$ are zero, and $Z_0^{(p)}(0) = b$ which is the size of the ancestral population. Formally, we take $\mathbf{Z}^{(p)}$ to be a pure-jump Markov chain whose transitions are described as follows. When at state $\mathbf{z} = (z_i : i \geq 0)$, our process jumps to a state $\tilde{\mathbf{z}} = (\tilde{z}_i : i \geq 0)$ where $\tilde{z}_j = z_j$ for $j \neq k$ and $\tilde{z}_k = z_k + (b - 1)$ at rate

$$\begin{cases} p z_0 & \text{if} \quad k = 0, \\ z_k & \text{if} \quad k \neq 0. \end{cases}$$

This corresponds to a reproduction event in the sub-population with type $k$. Otherwise, the process jumps from $\mathbf{z}$ to $\hat{\mathbf{z}} = (\hat{z}_i : i \geq 0)$ at rate $(1 - p)z_0$ where, if $k$ is the first index such that $z_k = 0$, then $\hat{z}_0 = z_0 - 1$, $\hat{z}_k = b$, and $\hat{z}_j = z_j$ for $j \neq 0, k$. This corresponds to a mutation event of the sub-population with the ancestral type.

The process of the total size of the population in the system

$$Z(t) = Z_0^{(p)}(t) + \sum_{i \geq 1} Z_i(t), \quad t \geq 0,$$

is distributed as a Yule process, where each individual lives for an exponential time of parameter 1 and gives birth at its death to $b$ children, which then evolve independently of one another according to the same dynamics as their parent, no matter the choice of $p$. Clearly, the process of the size of the sub-population with the ancestral type $Z_0^{(p)}$ is a continuous time branching process, with reproduction law given by the distribution of $b\epsilon_p$, where $\epsilon_p$ stands for a Bernoulli random variable with parameter $p$. Moreover, if for $i \geq 1$, we write

$$a_i^{(p)} = \inf\{t \geq 0 : Z_i(t) > 0\},$$

for the birth time of the sub-population with type $i$, then each process

$$(Z_i(t - a_i^{(p)}) : t \geq a_i^{(p)})$$

is a branching process with the same reproduction law as $Z$ starting from $b$. Indeed, the different populations present in the system (i.e., those with strictly positive sizes) evolve independently of one another. The following statement is just a formal formulation of the previous observation which should be plain from the construction of $\mathbf{Z}^{(p)}$; it is essentially Lemma 1 in [5].

**Lemma 2.1.** *The processes $(Z_i(t - a_i^{(p)}) : t \geq a_i^{(p)})$ for $i \geq 1$ form a sequence of i.i.d. branching process with the same law as $Z$ and with starting value $b$. Further, this sequence is independent of that of the birth-times $(a_i^{(p)})_{i \geq 1}$ and the process $Z_0^{(p)}$ of the sub-population with ancestral type.*

We are now ready to present the main result of this section. We henceforth assume that the parameter $p = p_n$ is given by (1.1) and for simplicity, we write $p$ rather than $p_n$, omitting the integer $n$ from the notation. We consider the time

$$\tau(n) = \inf\{t \geq 0 : Z(t) = (b - 1)n + 1\},$$

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when the total population has size $(b-1)n+1$. The size of the sub-population with the ancestral type at this time is given by

$$G_n := Z_0^{(p)}(\tau(n)).$$

**Theorem 2.2.** *Set $\beta = b/(b-1)$. As $n \to \infty$, there is the weak convergence*

$$\left( n^{-1}G_n - \frac{1}{\beta-1}e^{-\beta c} \right) \ln n - \frac{\beta}{\beta-1} c e^{-\beta c} \ln \ln n \Rightarrow -\frac{\beta}{\beta-1} c e^{-\beta c} \left( \mathscr{Z}_{c,\beta} + 1 - \frac{1}{\beta} \right),$$

*where $\mathscr{Z}_{c,\beta}$ is the random variable defined in (1.3).*

We stress that this result also allows us to deduce the fluctuations of the number of mutants in the total population, since this quantity is given by $(b-1)n+1-G_n$.

The rest of this section is devoted to the proof of Theorem 2.2. Our approach is similar to that in [3]. Broadly speaking, we divide the study of the fluctuations in two well-defined phases. The crucial point is to obtain a precise estimate of the number $\Delta_n$ of mutants when the total population of the system attains the size $(b-1)\lfloor \ln^4 n \rfloor + 1$; this can be viewed as the germ of the fluctuations of $(b-1)n+1-G_n$. Then, we resume the growth of the system from size $(b-1)\lfloor \ln^4 n \rfloor + 1$ to the size $(b-1)n+1$ and observe that the sub-population with the ancestral type grows essentially regularly. We point out that even though the study of these two phases plays a key role in [3], the tools developed here to deal with each phase are much different from those used there.

## 2.1 The germ of fluctuations

In this first phase, we observe the growth of the system of branching processes until the time

$$\tau(\ln^4 n) = \inf\{ t \geq 0 : Z(t) = (b-1)\lfloor \ln^4 n \rfloor + 1 \},$$

which is when the total size of the population reaches $(b-1)\lfloor \ln^4 n \rfloor + 1$, and our purpose in this section is to estimate precisely the number $\Delta_n$ of mutants in the total population at this time, that is

$$\Delta_n = (b-1)\lfloor \ln^4 n \rfloor + 1 - Z_0^{(p)}(\tau(\ln^4 n)).$$

We stress that the threshold $(b-1)\ln^4 n + 1$ is somewhat arbitrary, and any power close to $4$ of $\ln n$ would work just as well. However, as is remarked by Bertoin in [3], it is crucial to choose a threshold which is both sufficiently high so that fluctuations are already visible, and sufficiently low so that one can estimate the germ with the desired accuracy.

We start by setting down the key results that lead us to the main result of this section, in order to give an easier articulation of the argument. In this direction, it is convenient to introduce the number $\Delta_{0,n}$ of mutants at time

$$\tau_0(\ln^4 n) = \inf\{ t \geq 0 : Z_0^{(p)}(t) = (b-1)\lfloor \ln^4 n \rfloor + 1 \},$$

which is when the size of the sub-population with the ancestral type reaches $(b-1)\lfloor \ln^4 n \rfloor + 1$, i.e.

$$\Delta_{0,n} = Z(\tau_0(\ln^4 n)) - (b-1)\lfloor \ln^4 n \rfloor - 1.$$

This will be useful since the distribution of $\Delta_{0,n}$ is easier to estimate than that of $\Delta_n$. Then, we establish the following limit theorem in law that relates the fluctuations of $\Delta_{0,n}$ with the continuous Luria-Delbrück variable $\mathscr{Z}$.

**Proposition 2.3.** *As $n \to \infty$, there is the weak convergence*

$$\frac{\Delta_{0,n}}{\ln^3 n} - 3\frac{\beta}{\beta-1}c\ln\ln n \Rightarrow \frac{\beta}{\beta-1}c\left(\mathcal{Z}_{c,\beta} + 1 - \frac{1}{\beta}\right)$$

*where $\mathcal{Z}_{c,\beta}$ is the random variable defined in (1.3).*

As we are interested in estimate the number $\Delta_n$ of mutants in the total population at time $\tau(\ln^4 n)$, and we know the behavior of $\Delta_{0,n}$, the purpose of the next lemma is to point out that these two quantities are close enough when $n \to \infty$. We need to introduce the notation:

$$A_n = B_n + o(f(n)) \quad \text{in probability,}$$

where $A_n$ and $B_n$ are two sequences of random variables and $f : \mathbb{N} \to (0, \infty)$ is a function, to indicate that $|A_n - B_n|/f(n) \to 0$ in probability when $n \to \infty$.

**Lemma 2.4.** *We have*

$$\Delta_n = \Delta_{0,n} + o(\ln^3 n) \quad \text{in probability.}$$

It then follows from Proposition 2.3 that $\Delta_n$ and $\Delta_{0,n}$ have the same asymptotic behavior. Specifically:

**Corollary 2.5.** *As $n \to \infty$, there is the weak convergence*

$$\frac{\Delta_n}{\ln^3 n} - 3\frac{\beta}{\beta-1}c\ln\ln n \Rightarrow \frac{\beta}{\beta-1}c\left(\mathcal{Z}_{c,\beta} + 1 - \frac{1}{\beta}\right)$$

*where $\mathcal{Z}_{c,\beta}$ is the random variable defined in (1.3).*

The above result will be sufficient for our purpose. We now prepare the ground for the proofs of Proposition 2.3 and Lemma 2.4. Recall that we wish to study the behavior of the number $\Delta_{0,n}$ of mutants at time $\tau_0(\ln^4 n)$, which is easier than that of $\Delta_n$, thanks to Lemma 2.1. In words, at time $\tau_0(\ln^4 n)$ there is an independence property between the mutant sub-populations, and the process that counts the number of mutation events, which allows us to express $\Delta_{0,n}$ as a random sum of independent Yule processes. Clearly, the above is not possible at time $\tau(\ln^4 n)$ due to the lack of independence within the sub-populations. Formally, we start by writing

$$M(t) = \max\{i \geq 1 : Z_i(t) > 0\}$$

for the number of mutations that have occurred before time $t \geq 0$. Lemma 2.1 ensures that $M$ is independent of the processes $(Z_i(t - a_i^{(p)}) : t \geq a_i^{(p)})$ for $i \geq 1$. In addition, we note that the jump times of $M$ are in fact $a_1^{(p)} < a_2^{(p)} < \cdots$. This enables us to express the total mutant population at time $t$ as,

$$Z_{\mathrm{m}}(t) = \sum_{i=1}^{M(t)} Z_i(t - a_i^{(p)}),$$

and we are thus interested in

$$\Delta_{0,n} = Z_{\mathrm{m}}(\tau_0(\ln^4 n)). \tag{2.1}$$

We now turn our attention to study the fluctuations of $\Delta_{0,n}$ through the analysis of its characteristic function. In this direction, we will be mainly interested in the following feature of $Z_m(t)$.

**Lemma 2.6.** *We have for $t \geq 0$ and $\theta \in \mathbb{R}$.*

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*i)* *The characteristic function of $Z(t)$ started from $Z(0) = b$,*

$$\varphi_t(\theta) = \mathbb{E}\left[e^{i\theta Z(t)}\big| Z(0) = b\right] = \left(\frac{e^{i\theta(b-1)}e^{-(b-1)t}}{1 - e^{i\theta(b-1)} + e^{i\theta(b-1)}e^{-(b-1)t}}\right)^{\frac{b}{b-1}}. \tag{2.2}$$

*ii)* *We have*

$$\mathbb{E}[e^{i\theta Z_\mathrm{m}(t)}] = \mathbb{E}\left[\exp\left((1-p)\int_0^t Z_0^{(p)}(t-s)\left(\varphi_s(\theta) - 1\right)\mathrm{d}s\right)\right]. \tag{2.3}$$

*Proof.* Recall that the processes $(Z_i(t - a_i^{(p)}) : t \geq a_i^{(p)})$ for $i \geq 1$ are i.i.d. branching process with the same law as $Z$ with starting value $b$. Then according to page 109 in Chapter III of Athreya and Ney [1], their characteristic function is given by the expression (2.2). We now observe from the dynamics of $\mathbf{Z}^{(p)}$ that the counting process $M$ has jumps at rate $(1-p)Z_0^{(p)}$. Moreover, conditionally on $Z_0^{(p)}$, the process $Z_\mathrm{m}$ is a non homogeneous filtered Poisson process whose characteristic function can be written in terms of the characteristic function of $Z_i$. By extending equation (5.43) of Parzen [14] slightly to allow the underlying Poisson process to be non homogeneous, we obtain

$$\mathbb{E}\left[e^{i\theta Z_\mathrm{m}(t)}\big| \left(Z_0^{(p)}(s) : 0 \leq s \leq t\right)\right] = \exp\left((1-p)\int_0^t Z_0^{(p)}(s)\left(\varphi_{t-s}(\theta) - 1\right)\mathrm{d}s\right), \tag{2.4}$$

for $t \geq 0$ and $\theta \in \mathbb{R}$. Hence our claim follows after taking expectations on both sides of the equation and make a simple change of variables. $\square$

We recall some important properties of the branching processes $Z$ and $Z_0^{(p)}$, which will be useful later on. The process

$$W(t) := e^{-(b-1)t}Z(t), \qquad t \geq 0$$

is a nonnegative square-integrable martingale which converges a.s. and in $L^2(\mathbb{P})$, and we write $W(\infty)$ for its terminal value. Furthermore $W(\infty) > 0$ a.s. since $Z$ can not become extinct (we also pointed out that $Z$ never explodes a.s.). Similarly, the process

$$W_0^{(p)}(t) = e^{-(bp-1)t}Z_0^{(p)}(t), \qquad t \geq 0$$

is a martingale which terminal value is denoted by $W_0^{(p)}(\infty)$. In addition, following the proof of Lemma 3 in [5] we have

**Lemma 2.7.** *It holds that*

$$\lim_{p \to 1, t \to \infty} \mathbb{E}_z\left[\sup_{s \geq t}\left|W_0^{(p)}(s) - W(\infty)\right|^2\right] = 0.$$

*In particular, $W_0^{(p)}(\infty)$ converges to $W(\infty)$ in $L^2(\mathbb{P})$ as $p \to 1$.*

We next estimate the characteristic function of $Z_\mathrm{m}(t)$ given in (2.3), but we still need some additional notation. For $t \geq 0$,

$$I^{(p)}(t) = (1-p)\int_0^t Z_0^{(p)}(t-s)(\varphi_s(u) - 1)\mathrm{d}s$$

and

$$I_\mathrm{m}^{(p)}(t) = (1-p)W_0^{(p)}(\infty)e^{(b-1)t}\int_0^t e^{-(b-1)s}(\varphi_s(u) - 1)\mathrm{d}s,$$

where $u = \theta/\left(\beta c \ln^3 n\right)$ for $\theta \in \mathbb{R}$ and $\beta = b/(b-1)$.

**Lemma 2.8.** *We have*

$$\lim_{n \to \infty} \left( I^{(p)}(\tau_0(\ln^4 n)) - I_{\mathrm{m}}^{(p)}(\tau_0(\ln^4 n)) \right) = 0 \qquad \textit{in probability.}$$

*Proof.* Define the function

$$I_{\mathrm{a}}^{(p)}(t) = (1-p)W_0^{(p)}(\infty)e^{(bp-1)t} \int_0^t e^{-(bp-1)s}(\varphi_s(u) - 1)\mathrm{d}s, \qquad t \geq 0,$$

which is simply obtained by replacing $b$ by $bp$ in the exponential terms of $I_{\mathrm{m}}^{(p)}(t)$. We first prove that

$$\lim_{n \to \infty} \left( I^{(p)}(\tau_0(\ln^4 n)) - I_{\mathrm{a}}^{(p)}(\tau_0(\ln^4 n)) \right) = 0 \quad \text{in probability.} \tag{2.5}$$

In this direction, we observe from the triangle inequality that

$$\left| I^{(p)}(\tau_0(\ln^4 n)) - I_{\mathrm{a}}^{(p)}(\tau_0(\ln^4 n)) \right|$$

$$\leq (1-p)e^{(bp-1)\tau_0(\ln^4 n)} \int_0^{\tau_0(\ln^4 n)} |W_0^{(p)}(\tau_0(\ln^4 n) - s) - W_0^{(p)}(\infty)||\varphi_s(u) - 1|e^{-(bp-1)s}\mathrm{d}s. \tag{2.6}$$

We define

$$A^{(p)} := \frac{3}{2(bp-1)} \sup_{s \geq 0} e^{(bp-1)s/3}|W_0^{(p)}(s) - W_0^{(p)}(\infty)|,$$

and since Lemma 2 in [5] shows that $A^{(p)}$ is bounded in $L^2(\mathbb{P})$, we have by the Markov inequality that

$$\lim_{n \to \infty} \left( \ln^{-\frac{1}{3}} n \right) A^{(p)} = 0 \quad \text{in probability.} \tag{2.7}$$

We set $t_n = (b-1)^{-1} \ln \ln n$ and recall that $\varphi_t(u)$ fulfills (2.2). Hence from the inequality $|e^{ix} - 1| \leq 2$ for $x \in \mathbb{R}$, we have that

$$|\varphi_t(u) - 1| \leq 2. \tag{2.8}$$

Then,

$$(1-p)e^{(bp-1)\tau_0(\ln^4 n)} \int_{\tau_0(\ln^4 n)-t_n}^{\tau_0(\ln^4 n)} |W_0^{(p)}(\tau_0(\ln^4 n) - s) - W_0^{(p)}(\infty)||\varphi_s(u) - 1|e^{-(bp-1)s}\mathrm{d}s$$

$$\leq 2(1-p) \int_0^{t_n} |W_0^{(p)}(s) - W_0^{(p)}(\infty)|e^{(bp-1)s}\mathrm{d}s$$

$$\leq 2(1-p) \left( \ln^{\frac{2}{3}} n \right) A^{(p)}. \tag{2.9}$$

On the other hand, from (2.2) and the inequality $|e^{ix} - 1| \leq |x|$ for $x \in \mathbb{R}$, we get that

$$|\varphi_t(u) - 1| \leq b|u|e^{(b-1)t}, \tag{2.10}$$

which implies that

$$(1-p)e^{(bp-1)\tau_0(\ln^4 n)} \int_0^{\tau_0(\ln^4 n)-t_n} |W_0^{(p)}(\tau_0(\ln^4 n) - s) - W_0^{(p)}(\infty)||\varphi_s(u) - 1|e^{-(bp-1)s}\mathrm{d}s$$

$$\leq (1-p)b|u|e^{(b-1)\tau_0(\ln^4 n)} \int_{t_n}^{\tau_0(\ln^4 n)} |W_0^{(p)}(s) - W_0^{(p)}(\infty)|e^{-(b-1)s}e^{(bp-1)s}\mathrm{d}s$$

$$\leq 2b|u|(1-p) \left( \ln^{-\frac{1}{3}} n \right) A^{(p)}e^{(b-1)\tau_0(\ln^4 n)}. \tag{2.11}$$

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We recall that $u = \theta/\left(\beta c \ln^3 n\right)$, and $p = p_n$ is given by (1.1). Then from (2.6), (2.9), and (2.11) follow that

$$\left| I^{(p)}(\tau_0(\ln^4 n)) - I_{\mathrm{a}}^{(p)}(\tau_0(\ln^4 n)) \right| \leq 2 \left( c + (b-1)|\theta| \left(\ln^{-4} n\right) e^{(b-1)\tau_0(\ln^4 n)} \right) \left( \ln^{-\frac{1}{3}} n \right) A^{(p)}.$$

We observe that since $Z_0^{(p)}(\tau_0(\ln^4 n)) = (b-1)\lfloor \ln^4 n \rfloor + 1$, Lemma 2.7 ensures that

$$\lim_{n \to \infty} \left( \ln^{-4} n \right) e^{(b-1)\tau_0(\ln^4 n)} = \frac{b-1}{W(\infty)} \qquad \text{in probability,} \tag{2.12}$$

where $W(\infty)$ is strictly positive almost surely. Thus, we deduce (2.5) from (2.7) by letting $n \to \infty$ in the last inequality.

Next, we prove that

$$\lim_{n \to \infty} \left( I_{\mathrm{m}}^{(p)}(\tau_0(\ln^4 n)) - I_{\mathrm{a}}^{(p)}(\tau_0(\ln^4 n)) \right) = 0 \qquad \text{in probability,} \tag{2.13}$$

by proceeding similarly as the proof of (2.5). We observe for the triangle inequality that

$$\left| I_{\mathrm{m}}^{(p)}(\tau_0(\ln^4 n)) - I_{\mathrm{a}}^{(p)}(\tau_0(\ln^4 n)) \right|$$
$$\leq (1-p)W_0^{(p)}(\infty) \int_0^{\tau_0(\ln^4 n)} \left| 1 - e^{b(1-p)(s-\tau_0(\ln^4 n))} \right| |\varphi_s(u) - 1| e^{-(b-1)(s-\tau_0(\ln^4 n))} \mathrm{d}s. \tag{2.14}$$

We set $t_n = (b-1)^{-1} \ln \ln n$ again. Hence from the inequality (2.8) we have that

$$(1-p)W_0^{(p)}(\infty) \int_{\tau_0(\ln^4 n)-t_n}^{\tau_0(\ln^4 n)} \left| 1 - e^{b(1-p)(s-\tau_0(\ln^4 n))} \right| |\varphi_s(u) - 1| e^{-(b-1)(s-\tau_0(\ln^4 n))} \mathrm{d}s$$
$$\leq 2(1-p)W_0^{(p)}(\infty) \int_{\tau_0(\ln^4 n)-t_n}^{\tau_0(\ln^4 n)} \left| 1 - e^{b(1-p)(s-\tau_0(\ln^4 n))} \right| e^{-(b-1)(s-\tau_0(\ln^4 n))} \mathrm{d}s$$
$$= 2(1-p)W_0^{(p)}(\infty) \int_0^{t_n} \left| 1 - e^{-b(1-p)s} \right| e^{(b-1)s} \mathrm{d}s.$$

Then by making the change of variables $x = e^{(b-1)s}$, we get that

$$(1-p)W_0^{(p)}(\infty) \int_{\tau_0(\ln^4 n)-t_n}^{\tau_0(\ln^4 n)} \left| 1 - e^{b(1-p)(s-\tau_0(\ln^4 n))} \right| |\varphi_s(u) - 1| e^{-(b-1)(s-\tau_0(\ln^4 n))} \mathrm{d}s$$
$$\leq \frac{2}{b-1}(1-p)W_0^{(p)}(\infty) \int_1^{\ln n} \left( 1 - x^{-\beta(1-p)} \right) \mathrm{d}x$$
$$\leq \frac{2}{b-1}(1-p)W_0^{(p)}(\infty) \left( 1 - (\ln n)^{-\beta(1-p)} \right) \ln n. \tag{2.15}$$

On the other hand, from the inequality (2.10) we have that

$$(1-p)W_0^{(p)}(\infty) \int_0^{\tau_0(\ln^4 n)-t_n} \left| 1 - e^{b(1-p)(s-\tau_0(\ln^4 n))} \right| |\varphi_s(u) - 1| e^{-(b-1)(s-\tau_0(\ln^4 n))} \mathrm{d}s$$
$$\leq (1-p)b|u|e^{(b-1)\tau_0(\ln^4 n)} W_0^{(p)}(\infty) \int_{t_n}^{\tau_0(\ln^4 n)} \left( 1 - e^{-b(1-p)s} \right) \mathrm{d}s$$
$$\leq \frac{b^2}{2}(1-p)^2 |u| e^{(b-1)\tau_0(\ln^4 n)} W_0^{(p)}(\infty)(\tau_0(\ln^4 n))^2. \tag{2.16}$$

Recall that $u = \theta/\left(\beta c \ln^3 n\right)$, and $p = p_n$ is given by (1.1). Then from (2.14), (2.15), and (2.16) follow that

$$\left| I_{\mathrm{m}}^{(p)}(\tau_0(\ln^4 n)) - I_{\mathrm{a}}^{(p)}(\tau_0(\ln^4 n)) \right|$$

$$\leq 2b(b-1)cW_0^{(p)}(\infty)\left(1 - e^{-\beta c\frac{\ln\ln n}{\ln n}} + |\theta|(\tau_0(\ln^4 n))^2 e^{(b-1)\tau_0(\ln^4 n)}\ln^{-5} n\right).$$

We deduce from (2.12) that

$$\lim_{n\to\infty}\frac{\tau_0(\ln^4 n)}{4(b-1)^{-1}\ln\ln n} = 1 \qquad \text{in probability,}$$

and since $\lim_{n\to\infty} W_0^{(p)}(\infty) = W(\infty)$ in $L^2(\mathbb{P})$ from Lemma 2.7, we get (2.13) from (2.12) by letting $n \to \infty$ in the last inequality. Finally, our claim follows by combining (2.5) and (2.13). $\qquad\square$

We observe that thanks to (2.2), the integral $I_{\mathrm{m}}^{(p)}$ can be computed explicitly.

**Lemma 2.9.** *We have for $t \geq 0$,*

$$\int_0^t e^{-(b-1)s}(\varphi_s(u) - 1)\mathrm{d}s = \frac{1 - e^{iu(b-1)}}{(b-1)e^{iu(b-1)}}\left(\beta\ln(1 - e^{iu(b-1)} + e^{iu(b-1)}e^{-(b-1)t}) + \kappa_{b,u}(t)\right),$$

*where*

$$\kappa_{b,u}(t) = \sum_{k=2}^{\infty}\frac{(\beta)_k}{k!}\frac{(e^{iu(b-1)} - 1)^{k-1}}{k-1}\left(1 - \frac{1}{(1 - e^{iu(b-1)} + e^{iu(b-1)}e^{-(b-1)t})^{k-1}}\right), \quad (2.17)$$

*and $(\cdot)_k$ is the Pochhammer function.*

*Proof.* Define the function

$$f(\lambda) = \int_0^t e^{-(b-1)r}\left(\left(\frac{e^{-\lambda(b-1)}e^{-(b-1)r}}{1 - e^{-\lambda(b-1)} + e^{-\lambda(b-1)}e^{-(b-1)r}}\right)^{\beta} - 1\right)\mathrm{d}r, \quad \lambda \geq 0.$$

Hence by setting $x = 1 - e^{-\lambda(b-1)} + e^{-\lambda(b-1)}e^{-(b-1)r}$ and $y_\lambda = e^{-\lambda(b-1)}$ for convenience, we have that

$$f(\lambda) = \frac{1}{(b-1)y_\lambda}\int_{1-y_\lambda+y_\lambda e^{-(b-1)t}}^{1}\left(\left(\frac{x + y_\lambda - 1}{x}\right)^{\beta} - 1\right)\mathrm{d}x.$$

Moreover, using a well-known extension of Newton's binomial formula, we get

$$f(\lambda) = \frac{1}{(b-1)y_\lambda}\sum_{k=1}^{\infty}\frac{(\beta)_k}{k!}(y_\lambda - 1)^k\int_{1-y_\lambda+y_\lambda e^{-(b-1)t}}^{1}x^{-k}\mathrm{d}x,$$

where $(\cdot)_k$ is the Pochhammer function. Note that the series converges absolutely since $\beta > 0$ and $|y_\lambda - 1|/x \leq 1$, for $1 - y_\lambda + y_\lambda e^{-(b-1)t} \leq x \leq 1$. Then straightforward calculations yield

$$f(\lambda) = \frac{1 - y_\lambda}{(b-1)y_\lambda}\left(\beta\ln(1 - y_\lambda + y_\lambda e^{-(b-1)t}) + \kappa'_{b,\lambda}(t)\right),$$

where

$$\kappa'_{b,\lambda}(t) = \sum_{k=2}^{\infty}\frac{(\beta)_k}{k!}\frac{(y_\lambda - 1)^{k-1}}{k-1}\left(1 - \frac{1}{(1 - y_\lambda + y_\lambda e^{-(b-1)t})^{k-1}}\right).$$

Yule processes with rare mutation and their applications to percolation on $b$-ary trees

We note that the function $f$ allows an analytic extension to $\{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \geq 0\}$. Then by taking into account the principal branch of the complex logarithm, we conclude that

$$f(-iu) = \int_0^t e^{-(b-1)r} \left( \left( \frac{e^{iu(b-1)}e^{-(b-1)r}}{1 - e^{iu(b-1)} + e^{iu(b-1)}e^{-(b-1)r}} \right)^\beta - 1 \right) \mathrm{d}r,$$

and our assertion follows by observing that $\kappa'_{b,\lambda}(t) = \kappa_{b,u}(t)$ when $\lambda = -iu$. $\qquad\square$

We are now able to establish Proposition 2.3.

*Proof of Proposition 2.3.* Fix $\theta \in \mathbb{R}$ and define $m_n = \beta c \ln^3 n$. From the identity (2.1) and Lemma 2.6, the characteristic function of $m_n^{-1}\Delta_{0,n} - (\beta - 1)^{-1}\ln m_n$ is given by

$$\mathbb{E}\left[ e^{i\theta(m_n^{-1}\Delta_{0,n} - (\beta-1)^{-1}\ln m_n)} \right] = \mathbb{E}\left[ \exp\left( I^{(p)}(\tau_0(\ln^4 n)) - i\theta(\beta-1)^{-1}\ln m_n \right) \right].$$

Recall that by Lemma 2.8 we have

$$\lim_{n\to\infty} \left( I^{(p)}(\tau_0(\ln^4 n)) - I_{\mathrm{m}}^{(p)}(\tau_0(\ln^4 n)) \right) = 0 \quad \text{in probability.}$$

Then, we must verify that

$$\lim_{n\to\infty} \left( I_{\mathrm{m}}^{(p)}(\tau_0(\ln^4 n)) - i\theta(\beta-1)^{-1}\ln m_n \right)$$

$$= -i\theta(\beta-1)^{-1}\left( \kappa_\beta - 1 + \frac{1}{\beta} \right) - i\theta(\beta-1)^{-1}\ln|(\beta-1)^{-1}\theta| - \frac{1}{2}\pi|(\beta-1)^{-1}\theta| \quad (2.18)$$

in probability. In this direction, we define $y_u = e^{iu(b-1)}$ for convenience and recall also from Lemma 2.9 that

$$I_{\mathrm{m}}^{(p)}(\tau_0(\ln^4 n))$$

$$= (1-p)W_0^{(p)}(\infty)e^{(b-1)\tau_0(\ln^4 n)}\frac{1-y_u}{(b-1)y_u}\left( \beta\ln(1 - y_u + y_u e^{-(b-1)\tau_0(\ln^4 n)}) + \kappa_{b,u}(\tau_0(\ln^4 n)) \right),$$

where $\kappa_{b,u}(\tau_0(\ln^4 n))$ is defined in (2.17) and $u = \theta/\left( \beta c \ln^3 n \right)$. We know from Lemma 2.7 that

$$\lim_{n\to\infty} e^{-(b-1)\tau_0(\ln^4 n)}Z(\tau_0(\ln^4 n)) = \lim_{n\to\infty} e^{-(bp-1)\tau_0(\ln^4 n)}Z_0^{(p)}(\tau_0(\ln^4 n)) = W(\infty)$$

in probability, and $\lim_{n\to\infty} W_0^{(p)}(\infty) = W(\infty)$ in $L^2(\mathbb{P})$. Hence since $p = p_n$ fulfilled (1.1), we deduce that

$$\lim_{n\to\infty} \frac{(1-p)W_0^{(p)}(\infty)e^{(b-1)\tau_0(\ln^4 n)}}{(b-1)c\ln^3 n} = 1 \quad \text{in probability.}$$

On the other hand,

$$y_u = 1 + O\left( \frac{1}{m_n} \right) \quad \text{and} \quad m_n(1 - y_u) = -i\theta(b-1) + O\left( \frac{1}{m_n} \right),$$

and since $b - 1 = (\beta - 1)^{-1}$, we deduce that

$$\lim_{n\to\infty} \left( I_m^{(p)}(\tau_0(\ln^4 n)) - i\theta(\beta-1)^{-1}\ln m_n \right)$$

$$= -i\theta(\beta-1)^{-1}\ln(-i\theta(\beta-1)^{-1}) - i\theta(\beta-1)^{-1}\left( \kappa_\beta - 1 + \frac{1}{\beta} \right)$$

in probability, which implies (2.18). Finally, we observe from (2.4) and the modulus inequality for conditional expectation that

$$\left| \exp\left( I^{(p)}(\tau_0(\ln^4 n)) - i\theta(\beta-1)^{-1} \ln m_n \right) \right| \le 1.$$

Therefore, by the dominated convergence theorem we conclude that the Fourier transform of $m_n^{-1}\Delta_n^0 - (\beta-1)^{-1} \ln m_n$ converges pointwise as $n$ tends to infinity to the continuous function

$$\theta \mapsto \exp\left( -i\theta(\beta-1)^{-1}\left(\kappa_\beta - 1 + \frac{1}{\beta}\right) - i\theta(\beta-1)^{-1}\ln|(\beta-1)^{-1}\theta| - \frac{1}{2}\pi|(\beta-1)^{-1}\theta| \right),$$

and then our claim follows for the continuity theorem for Fourier transforms. $\qquad\square$

We now turn our attention to the proof of Lemma 2.4.

We imagine that we begin our observation of the system of branching processes with rare mutations $\mathbf{Z}^{(p)}$ once it has reached the size $(b-1)\lfloor \ln^4 n \rfloor + 1$, that is, from the time $\tau(\ln^4 n)$. We thus write $Z' = (Z'(t) : t \ge 0)$ for the process of the total size of the population started from $Z'(0) = (b-1)\lfloor \ln^4 n \rfloor + 1$, which has the same law of the Yule process $Z$ described at the beginning of Section 2. We introduce the time

$$\tau'(\ln^4 n) = \inf\{t \ge 0 : Z'(t) = Z(\tau_0(\ln^4 n))\},$$

at which it hits $Z(\tau_0(\ln^4 n))$. Equivalently, $\tau'(\ln^4 n)$ is the time needed to have a population with the ancestral type of size $(b-1)\lfloor \ln^4 n \rfloor + 1$. We shall first estimate this quantity.

**Lemma 2.10.** *We have*

$$\lim_{n \to \infty} \tau'(\ln^4 n) = 0 \qquad \text{in probability}.$$

*Proof.* We know that

$$\lim_{n \to \infty} e^{-(b-1)\tau(\ln^4 n)} Z(\tau(\ln^4 n)) = W(\infty) \qquad \text{in probability}.$$

By definition of the time $\tau(\ln^4 n)$, we have $Z(\tau(\ln^4 n)) = (b-1)\lfloor \ln^4 n \rfloor + 1$, hence

$$\lim_{n \to \infty} \frac{\tau(\ln^4 n)}{4(b-1)^{-1}\ln\ln n} = 1 \qquad \text{in probability}.$$

On the other hand, from Lemma 2.7 we have that

$$\lim_{n \to \infty} e^{-(bp-1)\tau_0(\ln^4 n)} Z_0^{(p)}(\tau_0(\ln^4 n)) = W(\infty) \qquad \text{in probability}.$$

Recall that $p = p_n$ is given by (1.1), and observe that $Z_0^{(p)}(\tau_0(\ln^4 n)) = (b-1)\lfloor \ln^4 n \rfloor + 1$. Hence

$$\lim_{n \to \infty} e^{-(bp-1)(\tau(\ln^4 n) - \tau_0(\ln^4 n))} = 1 \qquad \text{in probability},$$

and our claim follows from the identity $\tau'(\ln^4 n) = \tau_0(\ln^4 n) - \tau(\ln^4 n)$. $\qquad\square$

We observe that the population at time $\tau(\ln^4 n)$ when we start our observation consists of $\Delta_n$ mutants and $(b-1)\lfloor \ln^4 n \rfloor + 1 - \Delta_n$ individuals of the ancestral type. Then, we write $Z_0'^{(p)} = (Z_0'^{(p)}(t) : t \ge 0)$ for the process that counts the number of individuals with the ancestral type, which has the same law as $Z_0^{(p)}$ but starting from $Z_0'^{(p)}(0) = (b-1)\lfloor \ln^4 n \rfloor + 1 - \Delta_n$. We recall that

$$W'(t) := e^{-(b-1)t} Z'(t) \quad \text{and} \quad W_0'^{(p)}(t) := e^{-(bp-1)t} Z'^{(p)}(t), \quad t \ge 0$$

are nonnegative square-integrable martingales which converge a.s. and in $L^2(\mathbb{P})$.

*Proof of Lemma 2.4.* An application of Doob's inequality (see, e.g., Equation (6) in [7]) shows for all $\eta > 0$ that

$$\lim_{n\to\infty} \mathbb{P}\left(\left|e^{-(b-1)\tau'(\ln^4 n)}Z'(\tau'(\ln^4 n)) - Z'(0)\right| > \eta \ln^3 n\right) = 0$$

and using the fact that $Z_0^{\prime(p)}(0) \leq (b-1)\lfloor\ln^4 n\rfloor + 1$, we also get

$$\lim_{n\to\infty} \mathbb{P}\left(\left|e^{-(bp-1)\tau'(\ln^4 n)}Z_0^{\prime(p)}(\tau'(\ln^4 n)) - Z_0^{\prime(p)}(0)\right| > \eta \ln^3 n\right) = 0.$$

Then, since $\Delta_{0,n} = Z'(\tau'(\ln^4 n)) - (b-1)\lfloor\ln^4 n\rfloor - 1$, $\Delta_n = (b-1)\lfloor\ln^4 n\rfloor + 1 - Z_0^{\prime(p)}(0)$, and $Z'(0) = (b-1)\lfloor\ln^4 n\rfloor + 1$, one readily gets

$$\Delta_n - \Delta_{0,n}$$
$$= Z'(\tau'(\ln^4 n))\left(e^{-(b-1)\tau'(\ln^4 n)} - 1\right) - Z_0^{\prime(p)}(\tau'(\ln^4 n))\left(e^{-(bp-1)\tau'(\ln^4 n)} - 1\right) + o(\ln^3 n)$$

in probability. We next note from Lemma 2.10 that

$$Z_0^{\prime(p)}(\tau'(\ln^4 n))\left(e^{(1-p)\tau'(\ln^4 n)} - 1\right) = o(\ln^3 n) \quad \text{in probability,}$$

which yields

$$\Delta_n - \Delta_{0,n} = \left(W'(\tau'(\ln^4 n)) - W_0^{\prime(p)}(\tau'(\ln^4 n))\right)\left(1 - e^{(b-1)\tau'(\ln^4 n)}\right) + o(\ln^3 n)$$

in probability. Since by Lemma 2.4 we have that

$$\lim_{n\to\infty}\left(1 - e^{(b-1)\tau'(\ln^4 n)}\right) = 0 \quad \text{in probability,}$$

we must verify that

$$W'(\tau'(\ln^4 n)) - W_0^{\prime(p)}(\tau'(\ln^4 n)) = o(\ln^3 n) \quad \text{in probability,}$$

in order to get the result of Lemma 2.4. We observe from properties of square-integrable martingales that

$$\mathbb{E}\left[\left(W'(\tau'(\ln^4 n))\right)^2\right] = \mathbb{E}\left[[W']_{\tau'(\ln^4 n)}\right]$$

where

$$[W']_t = \sum_{0\leq s\leq t} e^{-2(b-1)s}|Z'(s) - Z'(s-)|^2 \quad \text{for} \quad t \geq 0.$$

A straightforward calculation shows that the compensator of jump process $[W']$ is

$$\langle W'\rangle_t = (b-1)^2 \int_0^t e^{-2(b-1)s}Z'(s)\mathrm{d}s \quad \text{for} \quad t \geq 0,$$

that is $[W']_t - \langle W'\rangle_t$ is a local martingale. Thus,

$$\mathbb{E}\left[\left(W'(\tau'(\ln^4 n))\right)^2\right] = \mathbb{E}\left[\langle W'\rangle_{\tau'(\ln^4 n)}\right] \leq \mathbb{E}\left[\langle W'\rangle_\infty\right] = (b-1)((b-1)\lfloor\ln^4 n\rfloor + 1).$$

Hence by the Markov inequality we have that

$$W'(\tau'(\ln^4 n)) = o(\ln^3 n) \quad \text{in probability.}$$

Similarly one gets

$$W_0^{\prime(p)}(\tau'(\ln^4 n)) = o(\ln^3 n) \quad \text{in probability,}$$

from where our claim follows. $\square$

## 2.2 The spread of fluctuations

The purpose here is to resume the growth of the system of branching processes with rare mutation from the size $(b-1)\lfloor \ln^4 n \rfloor + 1$ to the size $(b-1)n+1$ and observe that the germ of the fluctuations $\Delta_n$ spreads regularly. In this direction, we proceed similarly as the last part of the preceding section. We recall that $Z'$ denotes the process of the total population started from $Z'(0) = (b-1)\lfloor \ln^4 n \rfloor + 1$. We consider

$$\tau'(n) = \inf\{t \geq 0 : Z'(t) = (b-1)n+1\},$$

the time needed for the total population to reach size $(b-1)n+1$. Hence, in the notation of Theorem 2.2

$$G_n = Z_0'^{(p)}(\tau'(n)),$$

where as the previous section, we write $Z_0'^{(p)}$ for the process that counts the number of individuals with the ancestral type starting from $Z_0'^{(p)}(0) = (b-1)\lfloor \ln^4 n \rfloor + 1 - \Delta_n$.

We have now all the ingredients to establish Theorem 2.2.

*Proof of Theorem 2.2.* Again from the estimate of Equation (6) in [7], we get for all $\eta > 0$ that

$$\lim_{n\to\infty} \mathbb{P}\left(\left|((b-1)n+1)e^{-(b-1)\tau'(n)} - (b-1)\ln^4 n - 1\right| > \eta \ln^3 n\right) = 0,$$

this yields

$$e^{(b-1)\tau'(n)} = \frac{n}{\ln^4 n} + o\left(\frac{1}{\ln n}\right) \qquad \text{in probability.} \tag{2.19}$$

On the other hand, using the fact that $Z_0'^{(p)}(0) \leq (b-1)\lfloor \ln^4 n \rfloor + 1$, we also get

$$\lim_{n\to\infty} \mathbb{P}\left(\left|e^{-(bp-1)\tau'(n)}Z_0'^{(p)}(\tau'(n)) - Z_0'^{(p)}(0)\right| > \eta \ln^3 n\right) = 0,$$

and deduce that

$$G_n = e^{(bp-1)\tau'(n)}((b-1)\ln^4 n - \Delta_n) + o\left(\frac{n}{\ln n}\right) \qquad \text{in probability.}$$

Next, it is convenient to apply Skorokhod's representation theorem and assume that the weak convergence in Corollary 2.5 holds almost surely. Hence

$$G_n = e^{(bp-1)\tau'(n)}\left((b-1)\ln^4 n - \frac{\beta}{\beta-1}c\ln^3 n\left(3\ln\ln n + \left(\mathcal{Z}_{c,\beta} + 1 - \frac{1}{\beta}\right)\right)\right) + o\left(\frac{n}{\ln n}\right)$$

in probability. We next note from (2.19) that

$$e^{(bp-1)\tau'(n)} = e^{-\beta c}\frac{n}{\ln^4 n} + 4\beta c e^{-\beta c} n\frac{\ln\ln n}{\ln^5 n} + o\left(\frac{n}{\ln^5 n}\right)$$

in probability. Recall that $\beta = b/(b-1)$, then

$$G_n = \frac{1}{\beta-1}e^{-\beta c}n + \frac{\beta}{\beta-1}ce^{-\beta c}n\frac{\ln\ln n}{\ln n} - \frac{\beta}{\beta-1}ce^{-\beta c}\frac{n}{\ln n}\left(\mathcal{Z}_{c,\beta} + 1 - \frac{1}{\beta}\right) + o\left(\frac{n}{\ln n}\right)$$

in probability, which completes the proof. $\qquad\square$

## 3   Proof of Theorem 1.1

Our approach is based in the introduction of a continuous version of the construction of a $b$-ary recursive tree that enables us to superpose Bernoulli bond percolation dynamically in the tree structure. We begin at time $0$ from the tree with just one internal vertex which corresponds to the root having $b$ external vertices. Once the random tree with size $n \geq 1$ has been constructed, we equip each of the $(b-1)n+1$ external vertices with independent exponential random variables $\zeta_i$ of parameter $1$. Then, after a waiting time equal to $\min_{i \in \{1,\ldots,(b-1)n+1\}} \zeta_i$, one of the external vertices is chosen uniformly at random and is replaced it by the internal vertex $n+1$ to which $b$ new leaves are attached. We observe that $\min_{i \in \{1,\ldots,(b-1)n+1\}} \zeta_i$ is exponentially distributed with parameter $(b-1)n+1$.

We denote by $T(t)$ the tree which has been constructed at time $t \geq 0$, and by $|T(t)|$ its size, i.e. the number of internal vertices. The process of the size $(|T(t)| : t \geq 0)$ is clearly Markovian and if we define

$$\gamma(n) = \inf\{t \geq 0 : |T(t)| = n\}, \qquad n \geq 1,$$

then $T(\gamma(n))$ is a version of the $b$-ary recursive tree of size $n$, $T_n$. However for our purpose it will be more convenient work with the process $Y$ defined by

$$Y(t) = (b-1)|T(t)| + 1, \quad t \geq 0 \tag{3.1}$$

with starting value $Y(0) = b$. It should be clear that $Y$ is a Yule process as described in Section 2, i.e. it has jumps of size $b-1$ and unit birth rate per unit population size. We also point out that the process $Y$ gives us the number of external vertices on the tree.

We next superpose Bernoulli bond percolation with parameter $p = p_n$ defined in (1.1) to the growth algorithm in continuous time of the $b$-ary recursive tree. We follow the approach developed by Bertoin and Uribe Bravo [5] but with a slight modification. We draw an independent Bernoulli random variable $\epsilon_p$ with parameter $p$, each time an internal vertex is inserted. The edge which connects this new internal vertex is cut at its midpoint when $\epsilon_p = 0$ and remains intact otherwise. This disconnects the tree into connected clusters which motivates the following. We write $T^{(p)}(t)$ for the resulting combinatorial structure at time $t$. So, the percolation clusters of $T(t)$ are the connected components by a path of intact edges of $T^{(p)}(t)$.

Let $T_0^{(p)}(t)$ be the subtree that contains the root. We write $H_0^{(p)}(t)$ for the number of half-edges pertaining to the root cluster at time $t$. So that, its number of external vertices is given by

$$Y_0^{(p)}(t) = (b-1)|T_0^{(p)}(t)| + 1 - H_0^{(p)}(t).$$

We are now be able to observe the connection with the system of branching processes with rare mutations described in the preceding section. It should be plain from the construction that the size of the root-cluster at time $t$, i.e. $Y_0^{(p)}(t)$, of $T(t)$ after percolation with parameter $p$, coincides with the number of individuals with the ancestral type $Z_0^{(p)}(t)$ in the system $\mathbf{Z}^{(p)}$ of branching processes with rare mutations of Section 2. In fact, we already mentioned that the process $Y$ has the same random evolution as the process of the total size in the system $Z$. Recall that the algorithm for constructing a $b$-ary recursive tree is run until the time

$$\gamma(n) = \inf\{t \geq 0 : |T(t)| = n\} = \inf\{t \geq 0 : Y(t) = (b-1)n + 1\}$$

when the structure has $n$ internal vertices. Then, the size $C_0^{(p)}$ of the percolation cluster containing the root when the tree has $n$ internal vertices satisfies

$$C_0^{(p)} = |T_0^{(p)}(\gamma(n))|.$$

In addition, it should be plain that

$$Y_0^{(p)}(\gamma(n)) = (b-1)C_0^{(p)} + 1 - H_0^{(p)}(\gamma(n)),$$

coincides with the number of individuals with the ancestral type in the branching system $\mathbf{Z}^{(p)}$, at time when the total population reaches the size $(b-1)n+1$, i.e. $G_n$, according to the notation of Theorem 2.2. Hence in order to establish Theorem 1.1, it is sufficient to get an estimate of the number of half-edges pertaining to the root-subtree at time $\gamma(n)$.

**Lemma 3.1.** *We have*

$$\lim_{n\to\infty} \frac{\ln n}{n} H_0^{(p)}(\gamma(n)) = ce^{-\beta c} \quad \text{in probability.}$$

*Proof.* We observe that the processes

$$H_0^{(p)}(t) - (1-p)\int_0^t Y_0^{(p)}(s)\mathrm{d}s \quad \text{and} \quad Y_0^{(p)}(t) - (bp-1)\int_0^t Y_0^{(p)}(s)\mathrm{d}s, \quad t \geq 0$$

are martingales. Thus,

$$L^{(p)}(t) := H_0^{(p)}(t) - \frac{1-p}{bp-1}Y_0^{(p)}(t), \quad t \geq 0$$

is also a martingale. Observe that since $p = p_n$ satisfies (1.1), for $n$ large enough such that $2/(b+1) \leq p \leq 1$, its jumps $|L^{(p)}(t) - L^{(p)}(t-)|$ have size at most $b$. Since there are at most $n$ jumps up to time $\gamma(n)$, the bracket of $L^{(p)}$ can be bounded by $[L^{(p)}]_{\gamma(n)} \leq b^2 n$. Hence we have

$$\lim_{n\to\infty} \mathbb{E}\left(\left|\frac{\ln n}{n}L^{(p)}(\gamma(n))\right|^2\right) = 0. \tag{3.2}$$

On the other hand, we know from Lemma 2.7 that

$$\lim_{n\to\infty} e^{-(b-1)\gamma(n)}Y(\gamma(n)) = \lim_{n\to\infty} e^{-(bp-1)\gamma(n)}Y_0^{(p)}(\gamma(n)) = W(\infty) \quad \text{in probability}$$

which implies that

$$\lim_{n\to\infty} \frac{Y_0^{(p)}(\gamma(n))}{n} = (b-1)e^{-\beta c} \quad \text{in probability,}$$

and the result follows readily from (3.2), the above limit and the fact that $1-p = o(1)$. $\quad\square$

Therefore, from the identity

$$C_0^{(p)} = \frac{Y_0^{(p)}(\gamma(n)) - 1 + H_0^{(p)}(\gamma(n))}{(b-1)},$$

Theorem 2.2 applies to $Y_0^{(p)}(\gamma(n))$ and Lemma 3.1 yields the result of Theorem 1.1.

## 4  Percolation on scale-free trees

We conclude this work by showing that the approach used in the proof of Theorem 1.1 can be also applied to study percolation on scale-free trees, which form a family of random trees on a set of ordered vertices, say $\{0, 1, \dots, n\}$, that grow following a preferential attachment algorithm. Specifically, fix a parameter $a \in (-1, \infty)$, and start for $n = 1$ from the tree $T_1^{(a)}$ on $\{0, 1\}$ which has a single edge connecting $0$ and $1$. Suppose that $T_n^{(a)}$ has been constructed for some $n \geq 2$, and for every $i \in \{0, 1, \dots, n\}$, denote by $d_n(i)$ the degree of the vertex $i$ in $T_n^{(a)}$. Then conditionally given $T_n^{(a)}$, the tree $T_{n+1}^{(a)}$ is built by adding an edge between the new vertex $n + 1$ and a vertex $v_n$ in $T_n^{(a)}$ chosen at random according to the law

$$\mathbb{P}\left(v_n = i | T_n^{(a)}\right) = \frac{d_n(i) + a}{2n + a(n+1)}, \quad i \in \{0, 1, \dots, n\}.$$

Clearly, the preceding expression defines a probability measure since the sum of the degrees of a tree with $n + 1$ vertices equals $2n$. Note also that when one lets $a \to \infty$ the algorithm yields an uniform recursive tree since $v_n$ becomes uniformly distributed on $\{0, 1, \dots, n\}$. We then perform Bernoulli bond percolation with parameter given by (1.1), i.e. $p_n = 1 - c/\ln n$, where $c > 0$ is fixed. It has been observed by Bertoin and Uribe Bravo [5] that this choice of the percolation parameter corresponds to the supercritical regime. More precisely, the size of the cluster $\Gamma_n^{(\alpha)}$ containing the root satisfies

$$\lim_{n \to \infty} n^{-1} \Gamma_n^{(\alpha)} = e^{-\alpha c} \quad \text{in probability,}$$

where $\alpha = (1 + a)/(2 + a)$. We are interested in the fluctuations of $\Gamma_n^{(\alpha)}$, and show that an analogous result to Theorem 1.1 holds for large scale-free random trees.

**Theorem 4.1.** *Set $\alpha = (1 + a)/(2 + a)$, and assume that the percolation parameter $p_n$ is given by (1.1). Then as $n \to \infty$, there is the weak convergence*

$$\left(n^{-1} \Gamma_n^{(\alpha)} - e^{-\alpha c}\right) \ln n - \alpha c e^{-\alpha c} \ln \ln n \Rightarrow -\alpha c e^{-\alpha c} \mathcal{Z}'_{c,\alpha}$$

*where*

$$\mathcal{Z}'_{c,\alpha} = \mathcal{Z} - \kappa'_\alpha + \ln(\alpha c) \tag{4.1}$$

*with $\mathcal{Z}$ the continuous Luria-Delbrück distribution and*

$$\kappa'_\alpha = 1 - \frac{1}{\alpha} + \frac{1}{\alpha} \sum_{k=2}^{\infty} \frac{(\alpha)_k}{k!} \frac{(-1)^k}{k-1}.$$

We now focus on the proof of Theorem 4.1. We follow the route used in the proof of Theorem 1.1, and we analyze a system of branching processes with rare neutral mutations. We point out that in order to avoid repetitions, some technical details will be skipped.

We start by considering a pure birth branching process $Z^{(a)} = (Z^{(a)}(t) : t \geq 0)$ in continuous space, that has only jumps of size $2 + a$, and with unit birth rate per unit population size. We shall be mainly interested in a class of population systems which arises by incorporating mutations to the preceding branching process. More precisely, we describe the evolution of such a system by a process $\mathbf{Z}^{(p,a)} = (\mathbf{Z}^{(p,a)} : t \geq 0)$, where for each $t \geq 0$, $\mathbf{Z}^{(p,a)}(t) = (Z_0^{(p,a)}(t), Z_1^{(a)}(t), \dots)$ is a collection of nonnegative variables. At the initial time, all the sub-populations $Z_i^{(a)}(0)$ of type $i \geq 1$ are taken to be equal

to zero, and $Z_0^{(p,a)}(0) = 2 + 2a$ which is the size of the ancestral (or clone) population. We consider that at rate $p$ per unit population size, the clone population produces $2 + a$ new clones, and that at rate $1 - p$ per unit population size, they always create a single mutant population of a new type of size $1 + a$. The new mutant populations behave as the process $Z^{(a)}$ but starting from $1 + a$. Clearly, the sum over all sub-populations

$$Z^{(a)}(t) = Z_0^{(p,a)}(t) + \sum_{i \geq 1} Z_i^{(a)}(t), \qquad t \geq 0,$$

evolves as the pure birth branching process described at the beginning of this paragraph.

We next observe the growth of the system of branching process $\mathbf{Z}^{(p,a)}$ until the time

$$\tau^{(a)}(\ln^4 n) = \inf\{t \geq 0 : Z^{(a)}(t) = (2 + a)\lfloor \ln^4 n \rfloor + a\},$$

which is when the total size of the population reaches $(2+a)\lfloor \ln^4 n \rfloor + a$. Our first purpose is to estimate precisely the number $\Delta_n^{(\alpha)}$ of mutants at this time. This stage corresponds to the analysis of the germ, and approach line will be similar to that in Section 2.1. In this direction, it will be useful to study the number of mutants $\Delta_{0,n}^{(\alpha)}$ at time

$$\tau_0^{(a)}(\ln^4 n) = \inf\{t \geq 0 : Z_0^{(p,a)}(t) = (2 + a)\lfloor \ln^4 n \rfloor + a\}$$

whose distribution is easier to estimate than that of $\Delta_n^{(\alpha)}$. We shall establish the following limit theorem in law that is equivalent to the Proposition 2.3.

**Proposition 4.2.** *As $n \to \infty$, there is the weak convergence*

$$\frac{\Delta_{0,n}^{(\alpha)}}{\ln^3 n} - 3\frac{\alpha}{1 - \alpha} c \ln \ln n \Rightarrow \frac{\alpha}{1 - \alpha} c \left( \mathcal{Z}'_{c,\alpha} + 1 - \frac{1}{\alpha} \right)$$

*where $\mathcal{Z}'_{c,\alpha}$ is the random variable defined in (4.1).*

Before proving the Proposition 4.2, it is convenient to introduce the following representation of the total mutant population as we have done in Section 2.1. For $i \geq 1$, we write

$$b_i^{(p)} = \inf\{t \geq 0 : Z_i^{(a)}(t) > 0\},$$

for the birth time of the sub-population with type $i$. Then the processes $(Z_i^{(a)}(t - b_i^{(p)}) : t \geq b_i^{(p)})$ form a sequence of i.i.d. branching processes with the same law as $Z^{(a)}$ but starting value $1 + a$, which is independent of the birth-times $(b_i^{(p)})_{i \geq 1}$ and the process $Z_0^{(p,a)}$. Moreover, this sequence is also independent of the process that counts the number of mutation events which is defined by $M^{(a)}(t) = \max\{i \geq 1 : Z_i^{(a)}(t) > 0\}$. Thus, since the jump times of $M^{(a)}$ are in fact $b_1^{(p)} < b_2^{(p)} < \dots$, we can express the total mutant population at time $t \geq 0$ as,

$$Z_{\mathrm{m}}^{(a)}(t) = \sum_{i=1}^{M^{(a)}(t)} Z_i^{(a)}(t - b_i^{(p)}).$$

We observe that for $i \geq 1$, the process $((2 + a)^{-1} Z_i^{(a)}(t - b_i^{(p)}) : t \geq b_i^{(p)})$ is a Yule branching process in continuous space with birth rate $2 + a$ per unit population size. Then similarly as we obtained the result in Lemma 2.6, we get for $t \geq 0$ and $\theta \in \mathbb{R}$,

$$\mathbb{E}[e^{i\theta Z_{\mathrm{m}}^{(a)}(t)}] = \mathbb{E}\left[ \exp\left( (1 - p) \int_0^t Z_0^{(p,a)}(t - s)(\varphi_s^{(a)}(\theta) - 1)\mathrm{d}s \right) \right] \tag{4.2}$$

where

$$\varphi_t^{(a)}(\theta) = \mathbb{E}\left[e^{i\theta Z^{(a)}(t)} \big| Z^{(a)}(0) = 1 + a\right] = \left(\frac{e^{i\theta(2+a)}e^{-(2+a)t}}{1 - e^{i\theta(2+a)} + e^{i\theta(2+a)}e^{-(2+a)t}}\right)^\alpha$$

with $\alpha = (1+a)/(2+a)$.

At this point, the difference between the constants in Theorem 1.1 and Theorem 4.1 must be evident, mostly due to the different behavior of the branching processes associated to the $b$-ary recursive trees and scale-free random trees. Essentially, the constant $\kappa_\beta$ of Theorem 1.1 depends of the characteristic function (2.2) through the computations made in Lemma 2.9, which is clearly distinct from (4.2).

We are now able to establish Proposition 4.2.

*Proof of Proposition 4.2.* We fix $\theta \in \mathbb{R}$ and define $m_n = \alpha c \ln^3 n$. Since we have the identity $\Delta_{0,n}^{(\alpha)} = Z_{\mathrm{m}}^{(a)}(\tau_0^{(a)}(\ln^4 n))$, it follows from (4.2) that the characteristic function of $m_n^{-1}\Delta_{0,n}^{(\alpha)} - (1-\alpha)^{-1}\ln m_n$ is given by

$$\mathbb{E}\left[e^{i\theta(m_n^{-1}\Delta_{0,n}^{(\alpha)} - (1-\alpha)^{-1}\ln m_n)}\right] = \mathbb{E}\left[\exp\left(I^{(p,a)}(\tau_0^{(a)}(\ln^4 n)) - (1-\alpha)^{-1}\ln m_n\right)\right], \quad (4.3)$$

where

$$I^{(p,a)}(t) = (1-p)\int_0^t Z_0^{(p,a)}(t-s)(\varphi_s^{(a)}(u) - 1)\mathrm{d}s \quad \text{for } t \geq 0,$$

and $u = \theta/(\alpha c \ln^3 n)$. Next, a similar computation as the proof of Lemma 2.8 shows that

$$\lim_{n\to\infty}\left(I^{(p,a)}(\tau_0^{(a)}(\ln^4 n)) - I_{\mathrm{m}}^{(p,a)}(\tau_0^{(a)}(\ln^4 n))\right) = 0 \quad \text{in probability,} \quad (4.4)$$

where

$$I_{\mathrm{m}}^{(p,a)}(t) = (1-p)W_0^{(p,a)}(\infty)e^{(2+a)t}\int_0^t e^{-(2+a)s}(\varphi_s^{(a)}(u) - 1)\mathrm{d}s \quad \text{for } t \geq 0.$$

and $W_0^{(p,a)}(\infty)$ is the terminal value of the martingale $W_0^{(p,a)}(t) = e^{-(1+p(1+a))t}Z_0^{(p,a)}(t)$. Moreover, the integral of the previous expression can be computed explicitly,

$$\int_0^t e^{-(2+a)s}(\varphi_s^{(a)}(u) - 1)\mathrm{d}s = \frac{1 - e^{iu(2+a)}}{(2+a)e^{iu(2+a)}}\left(\alpha\ln(1 - e^{iu(2+a)} + e^{iu(2+a)}e^{-(2+a)t}) + \kappa'_{\alpha,u}(t)\right),$$

with

$$\kappa'_{\alpha,u}(t) = \sum_{k=2}^\infty \frac{(\alpha)_k}{k!}\frac{(e^{iu(2+a)} - 1)^{k-1}}{k-1}\left(1 - \frac{1}{(1 - e^{iu(2+a)} + e^{iu(2+a)}e^{-(2+a)t})^{k-1}}\right).$$

Hence from Lemma 3 in [5] (which is the analog of Lemma 2.7), we conclude after some computations that

$$\lim_{n\to\infty}\left(I_{\mathrm{m}}^{(p,a)}(\tau_0^{(a)}(\ln^4 n)) - i\theta(1-\alpha)^{-1}\ln m_n\right)$$

$$= -i\theta(1-\alpha)^{-1}\left(\kappa'_\alpha - 1 + \frac{1}{\alpha}\right) - i\theta(1-\alpha)^{-1}\ln|(1-\alpha)^{-1}\theta| - \frac{1}{2}\pi|(1-\alpha)^{-1}\theta|$$

in probability, and our claim follows from (4.4), by letting $n \to \infty$ in (4.3). $\qquad\square$

It follows now readily from the same arguments that we have developed to show Lemma 2.4 that $\Delta_n^{(\alpha)}$ and $\Delta_{0,n}^{(\alpha)}$ have the same asymptotic behavior. Specifically, we have:

**Corollary 4.3.** *As $n \to \infty$, there is the weak convergence*

$$\frac{\Delta_n^{(\alpha)}}{\ln^3 n} - 3\frac{\alpha}{1-\alpha}c\ln\ln n \Rightarrow \frac{\alpha}{1-\alpha}c\left(\mathscr{Z}'_{c,\alpha} + 1 - \frac{1}{\alpha}\right)$$

*where $\mathscr{Z}'_{c,\alpha}$ is the random variable defined in (4.1).*

Similarly as in Section 2.2, we now resume the growth of the system of branching process with rare mutation from the size $(2+a)\lfloor\ln^4 n\rfloor + a$ to the size $(2+a)n + a$, and show that the fluctuations of $\Delta_n^{(\alpha)}$ spread regularly. In this direction, we write $Z'^{(a)} = (Z'^{(a)}(t) : t \geq 0)$ for the process of the total size of the population started from $Z'^{(a)}(0) = (2+a)\lfloor\ln^4 n\rfloor + a$, which has the same law as the branching process $Z^{(a)}$. We observe that the population at the time when we restart our observation consists of $\Delta_n^{(\alpha)}$ mutants and $(2+a)\lfloor\ln^4 n\rfloor + a - \Delta_n^{(\alpha)}$ individuals with the ancestral type. Then, we write $Z_0'^{(p,a)} = (Z_0'^{(p,a)}(t) : t \geq 0)$ for the process that counts the number of clone individuals, which has the same law as $Z_0^{(p,a)}$ but starting from $Z_0'^{(p,a)}(0) = (2+a)\lfloor\ln^4 n\rfloor + a - \Delta_n^{(\alpha)}$. We consider the time

$$\tau'^{(a)}(n) = \inf\{t \geq 0 : Z'^{(a)}(t) = (2+a)n + a\}.$$

Then the number of individuals with the ancestral type at time when the total population generated by the branching process reaches $(2+a)n + a$ is given by

$$G_n^{(\alpha)} = Z_0'^{(p,a)}(\tau'^{(a)}(n)).$$

We are now able to state the following analog of Theorem 2.2.

**Theorem 4.4.** *Set $\alpha = (1+a)/(2+a)$. As $n \to \infty$, there is the weak convergence*

$$\left(n^{-1}G_n^{(\alpha)} - \frac{1}{1-\alpha}e^{-\alpha c}\right)\ln n - \frac{\alpha}{1-\alpha}ce^{-\alpha c}\ln\ln n \Rightarrow -\frac{\alpha}{1-\alpha}ce^{-\alpha c}\left(\mathscr{Z}'_{c,\alpha} + 1 - \frac{1}{\alpha}\right),$$

*where $\mathscr{Z}'_{c,\alpha}$ is the random variable defined in (4.1).*

*Proof.* We recall that

$$W'^{(a)}(t) := e^{-(2+a)t}Z'^{(a)}(t) \quad \text{and} \quad W_0'^{(p,a)}(t) := e^{-(1+p(1+a))t}Z'^{(p,a)}(t), \quad t \geq 0$$

are nonnegative square-integrable martingales which converge a.s. and in $L^2(\mathbb{P})$. Hence from the estimate of Equation (6) in [7], we get for all $\eta > 0$ that

$$\lim_{n\to\infty} \mathbb{P}\left(\left|((2+a)n+a)e^{-(2+a)\tau'^{(a)}(n)} - ((2+a)\lfloor\ln^4 n\rfloor + a)\right| > \eta\ln^3 n\right) = 0,$$

this yields

$$e^{(2+a)\tau'^{(a)}(n)} = \frac{n}{\ln^4 n} + o\left(\frac{1}{\ln n}\right) \quad \text{in probability.}$$

On the other hand, using the fact that $Z_0'^{(p,a)}(0) \leq (2+a)\lfloor\ln^4 n\rfloor + a$, we also get

$$\lim_{n\to\infty} \mathbb{P}\left(\left|e^{-(1+p(1+a))\tau'^{(a)}(n)}Z_0'^{(p,a)}(\tau'^{(a)}(n)) - Z_0'^{(p,a)}(0)\right| > \eta\ln^3 n\right) = 0,$$

and deduce that

$$G_n^{(\alpha)} = e^{(1+p(1+a))\tau'^{(a)}(n)}((2+a)\ln^4 n - \Delta_n^{(\alpha)}) + o\left(\frac{n}{\ln n}\right) \quad \text{in probability.}$$

Skorokhod's representation theorem enables us to assume that the weak convergence in Corollary 4.3 holds almost surely. Hence

$$G_n^{(\alpha)} = e^{(1+p(1+a))\tau'^{(a)}(n)} \left( (2+a) \ln^4 n - \frac{\alpha}{1-\alpha} c \ln^3 n \left( 3 \ln \ln n + \mathcal{Z}'_{c,\alpha} + 1 - \frac{1}{\alpha} \right) \right) + o\left( \frac{n}{\ln n} \right)$$

in probability. It follows that

$$G_n^{(\alpha)} = \frac{1}{1-\alpha} e^{-\alpha c} n + \frac{\alpha}{1-\alpha} c e^{-\alpha c} n \frac{\ln \ln n}{\ln n} - \frac{\alpha}{1-\alpha} c e^{-\alpha c} \frac{n}{\ln n} \left( \mathcal{Z}'_{c,\alpha} + 1 - \frac{1}{\alpha} \right) + o\left( \frac{n}{\ln n} \right)$$

in probability, which completes the proof. $\qquad\square$

We have now all the ingredients to establish Theorem 4.1.

*Proof of Theorem 4.1.* We follow Bertoin and Uribe Bravo [5], and we consider a continuous time version of the growth algorithm with preferential attachment as we have done for the $b$-ary recursive trees. We start at $0$ from the tree $\{0,1\}$, and once the random tree with size $n \geq 2$ has been constructed, we equip each vertex $i \in \{0,1,\ldots,n\}$ with an exponential random variable $\zeta_i$ of parameter $d_n(i) + a$, independently of the other vertices. Then the next vertex $n+1$ is attached after time $\min_{i \in \{0,1,\ldots,n\}} \zeta_i$ at the vertex $v_n = \operatorname{argmin}_{i \in \{0,1,\ldots,n\}} \zeta_i$. Let us denote by $T^{(a)}(t)$ the tree which has been constructed at time $t$, and by $|T^{(a)}(t)|$ its size, i.e. its number of vertices. It should be plain that if we define

$$\gamma^{(a)}(n) = \inf\{t \geq 0 : |T^{(a)}(t)| = n+1\},$$

then $T^{(a)}\left(\gamma^{(a)}(n)\right)$ is a version of a scale-free tree of size $n+1$, $T_n^{(a)}$. Furthermore, the process $Y^{(a)}$ defined by

$$Y^{(a)}(t) = (2+a)|T^{(a)}(t)| - 2, \qquad t \geq 0,$$

is a pure branching process with initial value $Y^{(a)}(0) = 2a+2$ that has only jumps of size $2+a$, and with unit birth rate per unit population size. Then we incorporate Bernoulli bond percolation to the algorithm similarly to how we did in Section 2 for the $b$-ary recursive trees. We draw an independent Bernoulli random variable $\epsilon_p$ with parameter $p$, each time an edge is inserted. If $\epsilon_p = 1$, the edge is left intact, otherwise we cut this edge at its midpoint. We write $T^{(p,a)}(t)$ for the resulting combinatorial structure at time $t$. Hence the percolation clusters of $T^{(a)}(t)$ are the connected components by a path of intact edges of $T^{(p,a)}(t)$. Let $T_0^{(p,a)}(t)$ be the subtree that contains the root. We write $H_0^{(p,a)}(t)$ for the number of half-edges pertaining to the root cluster at time $t$ and set

$$Y_0^{(p,a)}(t) = (2+a)|T_0^{(p,a)}(t)| + H_0^{(p,a)}(t) - 2.$$

We now observe the connection with the system of branching processes with rare mutations $\mathbf{Z}^{(p,a)}$. It should be plain from the construction that $Y_0^{(p,a)}$ has the same random evolution as the process of the number of individuals with the ancestral type $Z_0^{(p,a)}(t)$. In fact, the process $Y^{(a)}$ coincides with the process of the total size $Z^{(a)}$. Then, the size $\Gamma_n^{(\alpha)}$ of the percolation cluster containing the root when the structure has size $n+1$ satisfies $\Gamma_n^{(\alpha)} = |T_0^{(p,a)}(\gamma^{(a)}(n))|$. In addition, it should be plain that

$$Y_0^{(p,a)}(\gamma^{(a)}(n)) = (2+a)\Gamma_n^{(\alpha)} + H_0^{(p,a)}(\gamma^{(a)}(n)) - 2, \tag{4.5}$$

coincides with the number of individuals with the ancestral type in the branching system $\mathbf{Z}^{(p,a)}$, at time when the total population reaches the size $(2+a)n + a$, i.e. $G_n^{(\alpha)}$. Then,

in order to establish Theorem 4.1, it is sufficient to get an estimate of the number of half-edges pertaining to the root-subtree at time $\gamma^{(a)}(n)$. We follow the route of Lemma 3.1 and observe that the process

$$L^{(p,a)}(t) := H_0^{(p,a)}(t) - \frac{1-p}{1+p+pa}Y_0^{(p,a)}(t), \quad t \geq 0$$

is a martingale whose jumps have size at most $2 + a$. Since there are at most $n$ jumps up to time $\gamma^{(a)}(n)$, the bracket of $L^{(p,a)}$ can be bounded by $[L^{(p,a)}]_{\gamma^{(a)}(n)} \leq (2+a)^2 n$. Hence we have

$$\lim_{n\to\infty} \mathbb{E}\left( \left| \frac{\ln n}{n} L^{(p,a)}(\gamma^{(a)}(n)) \right|^2 \right) = 0.$$

On the other hand, from Lemma 3 in [5] we get that

$$\lim_{n\to\infty} e^{-(2+a)\gamma^{(a)}(n)} Y^{(a)}(\gamma^{(a)}(n)) = \lim_{n\to\infty} e^{-(1+p(1+a))\gamma^{(a)}(n)} Y_0^{(p,a)}(\gamma^{(a)}(n)) = W^{(a)}(\infty)$$

in probability, where $W^{(a)}(\infty)$ is defined as the terminal value of the martingale $W^{(a)}(t) = e^{-(2+a)t}Y^{(a)}(t)$. Thus, we have that

$$\lim_{n\to\infty} \frac{\ln n}{n} H_0^{(p,a)}(\gamma^{(a)}(n)) = ce^{-\alpha c} \quad \text{in probability,}$$

and the result in Theorem 4.1 follows from Theorem 4.4 and the identity (4.5). □

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