Monotonicity and regularity of the speed for excited random walks in higher dimensions[*]

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Abstract

We introduce a method for studying monotonicity of the speed of excited random walks in high dimensions, based on a formula for the speed obtained via cut-times and Girsanov's transform. While the method gives rise to similar results as have been or can be obtained via the expansion method of van der Hofstad and Holmes, it may be more palatable to a general probabilistic audience. We also revisit the law of large numbers for stationary cookie environments. In particular, we introduce a new notion of e_1-exchangeable cookie environment and prove the law of large numbers for this case.

1 Introduction

1.1 Excited random walk with random cookies (ERWRC)

Excited random walks (ERW) were introduced in [2] by I. Benjamini and D. Wilson. After that, M. Zerner generalized ERW when he introduced in [12], [13] cookie random walks, which are also called multi-excited random walks. In our paper, we consider a model of random walk called the excited random walk with m random cookies which we denote by ERWRC or m-ERWRC when more explicitly needed. This is a generalisation of multi-excited random walk and also a particular case of the excited random walk in random environment introduced and considered in [10] and [9].

Let us describe $m-$ERWRC. Let m be a positive integer or $m = +\infty$. We place m cookies on every site of the lattice $\mathbb{Z}^d$. Moreover, m random variables $(\beta_k(y))_{1 \leqslant k \leqslant m}$ with values in $[-1, 1]$ are attached to each site y of $\mathbb{Z}^d$. The process $\beta := \{(\beta_k(y))_{1 \leqslant k \leqslant m}\}_{y \in \mathbb{Z}^d}$ serves as a random environment whose law is denoted by $\mathbb{Q}$. Let $\mathbb{B} := ([-1, 1]^m)^{\mathbb{Z}^d}$ be the set of random environments. The excited random walk with m cookies $\beta = \{(\beta_k(y))_{1 \leqslant k \leqslant m}\}_{y \in \mathbb{Z}^d}$ is a discrete time nearest neighbor random walk $(Y_n)_{n \geq 0}$ on the

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lattice $\mathbb{Z}^d$ obeying the following rule: when the walk visits y for the k-th time, $1 \leq k \leq m$, then it eats one cookie and jumps with probability $(1 + \beta_k(y))/2d$ to the right, probability $(1 - \beta_k(y))/2d$ to the left, and probability $1/(2d)$ to the other nearest neighbor sites. On the other hand, when the walk is at a site y where there is no more cookie, then it jumps uniformly at random with probability $1/(2d)$ to one of the $2d$ neighboring sites. When $m = 1$ and the environment β is constant, we recover the excited random walk.

Throughout this paper, we denote by $\{Y_n \notin^k\}$ the event that Y_n has been visited fewer than k times before time n and denote by $\{Y_n \in^k\}$ the complement of $\{Y_n \notin^k\}$. When $k = 1$ we also use the notations $\{Y_n \notin\} := \{Y_n \notin^1\}$ and $\{Y_n \in\} := \{Y_n \in^1\}$. Moreover, the event that Y_n has been exactly visited $k - 1$ times before time n is denoted by $\{Y_n \notin_k\}$ and its complement is denoted by $\{Y_n \in_k\}$.

From the description of $m-$ERWRC, when β is fixed, the "quenched" law $\mathbb{P}_\beta$ of excited random walk with m random cookies β is the probability on the path space $(\mathbb{Z}^d)^{\mathbb{N}}$, defined by:

- $\mathbb{P}_\beta(Y_0 = 0) = 1$,
- $\mathbb{P}_\beta[Y_{n+1} - Y_n = \pm e_i | Y_0, ..., Y_n] = \frac{1}{2d}$ for $2 \leqslant i \leqslant d$,
- if Y_n has been visited exactly $k - 1$ times before time n, i.e. on the event $\{Y_n \notin_k\}$

$$\mathbb{P}_\beta[Y_{n+1} - Y_n = \pm e_1 | Y_0, ..., Y_n] = \left\{ \begin{array}{ll} \frac{1 \pm \beta_k(Y_n)}{2d} & \text{for } 1 \leq k \leq m, \\ \frac{1}{2d} & \text{for } k > m. \end{array} \right.$$

The "annealed" law P is then defined as the semi-direct product on $\mathbb{B} \times (\mathbb{Z}^d)^{\mathbb{N}}$: $P = \mathbb{Q} \otimes \mathbb{P}_\beta$. We say that the cookies are "identical" if

$$\forall k \text{ such that } 1 \leqslant k \leqslant m, \ \forall y \in \mathbb{Z}^d, \ \beta_k(y) = \beta(y). \tag{IDEN}$$

In this model, the random cookie environment $\beta = \{\beta(y)\}_{y \in \mathbb{Z}^d}$ is assumed to be:

- stationary: $\beta(y + \cdot) \stackrel{law}{=} \beta$ for any y in $\mathbb{Z}^d$,
- e_1-exchangeable: to define this notion, we consider a family $\Delta = \{\delta_z\}_{z \in \mathbb{Z}^{d-1}}$ of bijective mappings from $\mathbb{Z}$ to $\mathbb{Z}$. The mapping $\sigma_\Delta : \mathbb{Z}^d \to \mathbb{Z}^d$ defined by $\sigma_\Delta(x, z) = (\delta_z(x), z)$ for all $x \in \mathbb{Z}, z \in \mathbb{Z}^{d-1}$, is then a bijection from $\mathbb{Z}^d$ to $\mathbb{Z}^d$, acting on the set $\mathbb{B}$ of environments by $\sigma_\Delta(\beta)(y) = \beta(\sigma_\Delta(y))$. The environment is said to be e_1-exchangeable if and only if $\sigma_\Delta(\beta) \stackrel{law}{=} \beta$ for any family Δ. In other words, an environment is e_1-exchangeable if its law does not change when performing permutations of the environment on each horizontal line.

An i.i.d. cookie environment is of course stationary and e_1-exchangeable. Another simple example is provided by a stationary environment not depending on the horizontal component: for all $y = (x, z) \in \mathbb{Z} \times \mathbb{Z}^{d-1}$, $\beta(y) = \beta(z)$, where $(\beta(z))_{z \in \mathbb{Z}^{d-1}}$ is stationary.

To describe our main result about this model, we introduce a partial ordering on the laws of environments. Generally speaking, let Q_1, Q_2 be two probability measures on a partially ordered set $(E, \leqslant)$. We say that a probability measure Q on $E \times E$ is a monotone coupling of Q_1 and Q_2, if when denoting by l_1 and l_2 the coordinate maps from $E \times E$ to E:

for $i = 1, 2$, for all B events of E, $Q(l_i \in B) = Q_i(B)$ and $Q(l_1 \leqslant l_2) = 1$.

When such a monotone coupling exist, we say that $Q_1 \prec Q_2$.

The set $\mathbb{B}$ of environment is provided with the partial ordering:

$$\beta_1 \leqslant \beta_2 \text{ if and only if } \beta_{1,k}(y) \leqslant \beta_{2,k}(y), \ 1 \leqslant k \leqslant m, \ y \in \mathbb{Z}^d.$$

Let $(Z_n)_{n \geqslant 0}$ (resp. $(X_n)_{n \geqslant 0}$) be the vertical (resp. horizontal) component of $m-$ERWRC $(Y_n)_{n \geqslant 0}$:

$$Z_n := (Y_n \cdot e_2, ..., Y_n \cdot e_d), \ X_n := Y_n \cdot e_1.$$

Then $(Z_n)_{n \geq 0}$ is a simple random walk on $\mathbb{Z}^{d-1}$. We can extend this simple random walk to times integer to obtain the simple random walk $(Z_n)_{n \in \mathbb{Z}}$ (see (2.1) in Section 2.1). For $d-1 \geqslant 5$, E. Bolthausen, A-S. Sznitman and O. Zeitouni [4] proved the existence of cut times, i.e. times splitting the trajectory into two non-intersecting paths. Moreover, these cut times are integrable for $d-1 \geqslant 5$. Let $\mathcal{D}$ be the set of cut times, write $\mathcal{D} = \{... < T_{-2} < T_{-1} < T_0 \leqslant 0 < T_1 < T_2 < ...\}$. We denote $T := T_1$ and $\hat{P} = P(\cdot|0 \in \mathcal{D})$. Our main result reads then as follows:

Theorem 1.1. Let $Y = (Y_n)$ be $m-$ERWRC, assume that the random cookie environment is stationary and e_1-exchangeable. We denote $X_n = Y_n \cdot e_1$ the projection of the random walk on the first coordinate.

- Law of large numbers:

    For $d \geqslant 6$, $\frac{X_n}{n}$ converges $P-$a.s. to a random variable V, whose expectation under $\hat{P}$ is denoted by $v(\mathbb{Q})$ satisfying $v(\mathbb{Q}) = \hat{E}[V] = \frac{\hat{E}(X_T)}{\hat{E}(T)}$. In the particular case that the cookie environment is $i.i.d.$ then V is constant and $V = v(\mathbb{Q})$.

- Monotonicity:

    1. If the cookies are identical *i.e.* $\forall y \in \mathbb{Z}^d$, $\beta_1(y) = \beta_2(y) = ... = \beta_m(y) = \beta(y)$ then there exists $d_0 \in \mathbb{N}^*$ such that $v(\mathbb{Q})$ is increasing w.r.t. $\mathbb{Q}$ for $d \geqslant d_0$ (w.r.t. the partial ordering $\prec$).
    2. If the cookies are identical, there exists $\sigma_0 \in (0,1)$ such that for any $d \geqslant 10$, $v(\mathbb{Q})$ is increasing w.r.t. $\mathbb{Q}$ on the set $\{\mathbb{Q} \text{ such that } \mathbb{Q}(0 \leq |\beta(y)| \leq \sigma_0, \forall y \in \mathbb{Z}^d) = 1\}$.

About the law of large numbers (LLN):

To prove the law of large numbers, we use the technique of cut times as in [4], [8], [7]. Our contribution is to use it for e_1-exchangeable stationary environment. In the i.i.d. setting, the LLN of Theorem 1.1 is a consequence of Theorem 1.1 of [8]. However, the proof of the LLN for i.i.d. setting in our paper is not totally the same as in [4], [8] (see Section 2.3.2). The formula $v(\mathbb{Q}) = \frac{\hat{E}(X_T)}{\hat{E}(T)}$ obtained in our proof is different from the formula of the speed in [4]. To use cut times, the dimension d is required to be not smaller than 6. This implies the existence of cut times of the projection Z of the random walk Y on the d_1 last coordinates ($d_1 = d - 1 \geqslant 5$). In [9], the LLN of Theorems 4.6 and 4.8 is proved for all dimension $d \geqslant 1$ using renewal structure. However, to use this technique, the conditions of uniform ellipticity of the cookie environment and the transience of the random walk in some direction $l \in \mathbb{R}^d$ are needed.

About the monotonicity of the speed:

Our result is to prove for the case of e_1-exchangeable and stationary cookie environment. For the i.i.d. setting, M. Holmes and R. Sun [8] considered random walks in partially random environment which is similar to random walks with an infinite number of identical random cookies, $m = +\infty$. In this model the probability of stepping in d_1 last coordinates is random (i.e. this probability depends on the cookie environment) and $d_0 = d - d_1 \geqslant 1$. The question of monotonicity is considered under the assumption that there is an explicit coupling of two laws of the random environments. In this case, the laws of the random environments are allowed to take two values, say ν_1 and ν_2 with probabilities β and $1 - \beta$ (where β is a constant in $[0, 1]$). They proved the monotonicity of the speed with respect to β.

In the paper, we prove the monotonicity for all $1 \leqslant m \leqslant +\infty$. In fact, there is an intersection between our model and the model used in [8] that the projected random walk on $\mathbb{Z}^{d_1}$ is a simple random walk, $m = +\infty$ and $d_0 = 1$. In this case, the monotonicity can be easily proved by coupling argument for stochastic domination (see [8], page 5).

About the methodology, for the case of the probability of stepping in d_1 last coordinates is not random, with the method cut times and Girsanov's transform, the explicit coupling of the laws of random environments used in [8] is not needed in our proof. We notice that the lace expansion method can be applied to prove the monotonicity of the speed of Theorem 1.1. More precisely, using the stationary coupling $\beta_t = (1-t)\beta_1 + t\beta_2$, we can prove the existence of the speed by the law of large numbers. Together with boundedness and convergence of the lace expansion series, the lace expansion formula for the expectation of the speed then follows. These techniques can be found in [8].

In [7], M. Holmes asked about monotonicity of the speed with respect to stochastic domination. He considered the model with $1 \leqslant m \leqslant +\infty$, $d_0 = 1$ and the probability of stepping in d_1 last coordinates is not random. The author proved the following result:

Theorem (Theorem 2.3, [7]). Set $\delta_i := \mathbb{E}[\beta_i(0)]$. Let A be a finite set of integers $A \subset N$. If $\beta_i(o)$ is independent of $(\beta_j(o))_{j \neq i}$ for each $i \in A$, then for each fixed joint distribution of $\beta_{A^c}(o) = (\beta_i(o))_{i \notin A}$, the annealed speed v in dimension d is a continuous function of $(\delta_i)_{i \in A}$ when $d \geqslant 6$ and is differentiable in δ_i for each $i \in A$ when $d \geqslant 8$. If $1 \in A$, then v is strictly increasing in δ_1 when $d \geqslant 12$.

Under the conditions of this theorem, for $i \in A$ and $i > 0$, the speed depends on the β_i via the mean $\delta_i = \mathbb{E}[\beta_i(x)]$ where $x \in \mathbb{Z}^d$. This means that the law of the random walk does not change when we replace $\beta_i, i \in A$ by the constant δ_i. Here the speed is monotone in the first drift δ_1 when the i-th cookie is independent of the others for $i \in A$ and $1 \in A$. This is a special case of stochastic domination. The model in our paper is quite similar to the model in [7] except the conditions of the random cookie environment. We prove the monotonicity of the speed with respect to the law of the random cookie environment $\mathbb{Q}$ for the special case of m identical random cookies.

1.2 Excited random walk with m identical deterministic cookies (m-ERW)

This model is a partial model of $m-$ERWRC when the cookie environment is not random and identical, i.e. the cookies are the same for every site:

$$\forall k \text{ such that } 1 \leqslant k \leqslant m \,, \ \forall y \in \mathbb{Z}^d \,, \ \beta_k(y) = \beta \,,$$

for some real number $\beta \in [0,1]$. We see that the $m-$ERW is also a partial model of the model called multi-excited random walk which was introduced in [12]. Let $\mathbb{P}_{m,\beta}$ denote the law of m-ERW. As m is large, the m-ERW is more and more like a simple random walk with bias β. Let $v(m,\beta)$ be the speed of the m-ERW whose existence is proven for $d \geqslant 2$ in [3], [10], [9]. We prove in Section 4 the following result:

Theorem 1.2. For $d \geqslant 8$, the speed $v(m,\beta)$ is differentiable w.r.t β in $[0,1]$. Moreover, the derivative converges to $\frac{1}{d}$, uniformly in β on compact subsets of $[0,1]$: for any $\beta_0 \in [0,1)$,

$$\lim_{m \to \infty} \sup_{\beta \in [0,\beta_0]} \left| \frac{\partial}{\partial \beta} v(m,\beta) - \frac{1}{d} \right| = 0 \,.$$

Hence, there exists $m(\beta_0)$ such that for $m \geqslant m(\beta_0)$ the speed of the m-ERW is increasing in β on $[0,\beta_0]$.

The differentiability of the speed was proved in [7] Theorem 2.3. The rest could also be obtained by minor modification of the proof of Theorem 2.3 of [7].

1.3 Excited random walk

Excited random walk is introduced in [2], this model is a partial case of $m-$ERW when $m = 1$. Our main result for the excited random walk is the following:

Theorem 1.3. Let $v(\beta)$ be the speed of ERW with bias β.

1. $v(\beta)$ is differentiable in $\beta \in [0, 1)$ for $d \geq 8$. For $d \geqslant 6$, the derivative at the critical point 0 exists, is positive and satisfies :

$$\lim_{\beta \to 0} \frac{v(\beta)}{\beta} = \frac{1}{d} R(0)\,,$$

where $R(0) := \lim_{n\to\infty} (R_n/n)$, R_n is the number of points visited at time n by the symmetric simple random walk on $\mathbb{Z}^d$.

2. There exist $d_0 \in \mathbb{N}^*$, $\beta_0 \in (0, 1)$ such that the speed of the excited random walk is strictly increasing in $\beta \in [0, 1]$ for $d \geqslant d_0$ and strictly increasing in $\beta \in [0, \beta_0)$ for $d \geq 8$.

For the monotonicity of the speed in a neighborhood of 0, we need $d \geqslant 10$ in Theorem 1.1, but in Theorem 1.3 here, we need only $d \geqslant 8$. In the point 1 of Theorem 1.3, the differentiability of $v(\beta)$ on $[0, 1)$ for $d \geqslant 8$ is contained in Theorem 2.3 of [8]. However, we add the differentiability at the critical point 0 for $d \geqslant 6$. The point 2 is proved in [6] for $d_0 = 9$ by the lace expansion method.

In our paper, we prove the results by using cut times and Girsanov's transform. Our proof is based on two ingredients:

- Using stationary properties, it is possible to express the expectation of the speed in the direction e_1 as follows:

$$v(\mathbb{Q}) = \frac{\mathbb{Q}\mathbb{E}_\beta(X_T | 0 \in \mathcal{D})}{E_\beta(T | 0 \in \mathcal{D})}, \tag{1.1}$$

where $\mathbb{E}_\beta$ is the expectation under the "quenched" law $\mathbb{P}_\beta$ of ERWRC, $\mathcal{D}$ is the set of cut times, and $X_T = Y_T \cdot e_1$. In the case of i.i.d. cookies, $v(\mathbb{Q})$ is also the speed when the speed is deterministic.

- Starting from (1.1), we consider two random cookies β_1 and β_2, and a stationary coupling $\beta_t = (1 - t)\beta_1 + t\beta_2, t \in [0, 1]$. We get the expectation of the speed for random cookies β_t (see (2.20)) as follows:

$$f(t) := \frac{\mathbb{Q}\mathbb{E}_{\beta_t}(X_T \, 1_{0\in\mathcal{D}})}{E(T \, 1_{0\in\mathcal{D}})}. \tag{1.2}$$

Where, E is the expectation w.r.t. P. We use Girsanov's transforms to make the dependence of $f(t)$ w.r.t to t more explicit. This enables us to compute the derivative of $f(t)$ when $d \geqslant 8$ and to prove that this derivative is positive for d high enough or if random cookie is small enough to 0.

All results of differentiability and monotonicity in [6], [7], [8] were proved by using the lace expansion. We do not use this method in this paper. In [8], the authors used cut times to prove the law of large numbers. The existence of the speed and the convergence of the lace expansion series allow to express the speed by the lace expansion formula. This formula was used in calculating the derivative and showing that the derivative is positive. In this paper, to prove the law of large number we also use the cut times. However, with different arguments, we obtain the formulas of the speed (see (1.1) and (1.2)), which are more explicit than the formulas in the previous works.

In other to prove the monotonicity of the speed, we do not use the lace expansion formula, we use directly the formulas (1.1) and (1.2) of the speed via cut time T. These formulas have the advantage that the denominator $\mathbb{E}_\beta(T|0 \in \mathcal{D})$ does not depend on random cookie β. Girsanov's transform gives an expression of the derivative $\frac{\partial f}{\partial t}(t)$ via the cut time T (see (2.23)). Using this formula, we estimate the derivative of the speed and obtain that the derivative is positive when d large enough depending on the moments of T. Here, the condition $d \geqslant 6$ is needed for the existence of cut time, and we have $\sup_{d \geqslant 6} \hat{E}T = \sup_{d \geqslant 6} \frac{1}{P(0 \in \mathcal{D})} < +\infty$.

In the proof of the monotonicity in Theorem 1.1, in the estimation of the derivative, there is the appearance of the third moment of cut time T (see (2.30)). Therefore, we need $d \geqslant 10$ to get $\sup_{d \geqslant 10} \hat{E}(T^3) < +\infty$. For the particular case of ERW, the second moment of cut time T appears (see (3.1)). Hence, we need $d \geqslant 8$ to have that $\sup_{d \geqslant 8} \hat{E}(T^2) < +\infty$.

Notice that the constant d_0 in our method depends on the moments of T. While by using lace expansion method, M. Holmes and co-authors gave a explicit integer d_0, example in [6] $d_0 = 9$, in Theorem 2.3 of [7] $d_0 = 12$.

The paper is organized as follows: in Section 2, we prove Theorem 1.1. First we give a construction of $m-$ERWRC. We then prove the law of large numbers and obtain an expression of the speed by using cut times for stationary and e_1-exchangeable cookies. In the particular case of i.i.d. cookies, we prove that the speed is deterministic. Using Girsanov's transforms, we get the derivative of the speed and estimate it to obtain the differentiability and monotonicity of the speed. Section 3 is devoted to the proof of Theorem 1.3 based on that of Theorem 1.1. In Section 4, we prove Theorem 1.2. The key of the proof is Lemma 4.1. We use this lemma to show that the derivative of the speed tends uniformly in the drift β to a positive constant when the number of cookies tends to the infinity.

2 Proof of Theorem 1.1

2.1 A construction of m-ERWRC

We begin this section by constructing the $m-$ERWRC from some independent sequences of random variables. This plays an important role to prove the monotonicity. Fix $\beta(y) = (\beta_1(y), \beta_2(y), ..., \beta_m(y)), y \in \mathbb{Z}^d$. First, we consider a simple random walk (SRW) $\{\tilde{Z}_n\}_{n \in \mathbb{Z}}$ on $\mathbb{Z}^{d-1}$ where $\tilde{Z}_0 := 0$. Let three sequences of random variables and random vectors $\{\eta_i\}_{i \geqslant 0}, \{\xi_i\}_{i \geqslant 0}$ and $\{\zeta_1(y), ..., \zeta_m(y)\}_{y \in \mathbb{Z}^d}$ such that every random variable in these sequences is independent of each other, independent of $\tilde{Z}$ and having distribution

$$\eta_i \sim Ber\left(\frac{1}{d}\right), \quad \xi_i \sim Ber\left(\frac{1}{2}\right), \quad \zeta_k(y) \sim Ber\left((\beta_k(y) + 1)/2\right) \text{ where } 1 \leqslant k \leqslant m.$$

$\{\tilde{Z}_n\}_{n \geqslant 0}$ will give the sequence of vertical moves of the excited random walk, $\eta_i = +1$ will mean that at time i, the excited random walk performs an horizontal move. The direction of this move is given by ξ_i when the $m-$ERWRC is at a site that has been visited more than $m - 1$ times before the time i, and by $\zeta_k(y), k \in \{1, 2, ..., m\}, y \in \mathbb{Z}^d$ otherwise. More precisely, set $A_i^n := \{\sum_{j=0}^{n-1}(1 - \eta_j) = i\}, (0 \leqslant i \leqslant n)$ for $n > 0$ and $A_0^0 := \Omega$. Then for every $n \geqslant 0$, we have $\bigcup_{i=0}^n A_i^n = \Omega$ and $A_i^n \bigcap A_j^n = \emptyset$ for $i \neq j$. We define the vertical component Z of Y by:

$$\forall n \in \mathbb{Z}, Z_n = \begin{cases} \tilde{Z}_0 & \text{if } n = 0, \\ \tilde{Z}_{\sum_{i=0}^{n-1}(1-\eta_i)} & \text{if } n > 0, \\ \tilde{Z}_{-\sum_{i=n}^{-1}(1-\eta_i)} & \text{if } n < 0. \end{cases} \tag{2.1}$$

We now construct the horizontal component X of Y. Set $Y_0 := 0$ and assume that $(Y_j, 0 \leqslant j \leqslant i)$ are constructed. Let us define Y_{i+1}. On the event $Y_i \notin_k$ i.e. Y_i has been exactly visited $k-1$ times before time i, set

$$\mathcal{E}_i := \begin{cases} (2\zeta_k(Y_i) - 1)\, \mathbb{1}_{\eta_i=1} & \text{if } 1 \leqslant k \leqslant m\,, \\ (2\xi_i - 1)\, \mathbb{1}_{\eta_i=1} & \text{if } k > m\,. \end{cases}$$

We then set $X_{i+1} := X_i + \mathcal{E}_i$, and $Y_{i+1} := (X_{i+1}, Z_{i+1})$. With this construction, we obtain:

Lemma 2.1. Y is a $m-$ERWRC of the quenched law $\mathbb{P}_\beta$.

Proof. For the proof of Lemma 2.1, we need the following lemma:

Lemma 2.2. Let $\mathcal{F}$ and $\mathcal{G}$ be two sigma-algebras and $C \in \mathcal{F} \cap \mathcal{G}$ such that $\mathcal{F}|_C := \{A \cap C \text{ with } A \in \mathcal{F}\} \subset \mathcal{G}$. For any integrable random variable V, we get

$$\mathbb{E}(V1_C|\mathcal{F}) = \mathbb{E}\left[\mathbb{E}(V1_C|\mathcal{G})|\mathcal{F}\right].$$

The proof of Lemma 2.2 is easy. Now, we return to the proof of Lemma 2.1. Set

$$\mathcal{F}_n^Y := \sigma(Y_j, 0 \leqslant j \leqslant n)$$
$$\mathcal{F}_n := \sigma(Z_j, 0 \leqslant j \leqslant n, \eta_j, \xi_j, \zeta_k(y), 0 \leqslant j \leqslant n-1, 1 \leqslant k \leqslant m, y \in \mathbb{Z}^d)$$
$$\mathcal{G}_{ni} := \sigma(\tilde{Z}_j, 0 \leqslant j \leqslant i, \xi_j, \eta_j, \zeta_k(y), 0 \leqslant j \leqslant n-1, 1 \leqslant k \leqslant m, y \in \mathbb{Z}^d)$$

It is clear that $\mathcal{F}_n^Y \subset \mathcal{F}_n$ and $A_i^n \in \mathcal{F}_n \cap \mathcal{G}_{ni}$. Moreover, $\mathcal{F}_n|_{A_i^n} \subset \mathcal{G}_{ni}$. Now, using Lemma 2.2, we have for $j \geqslant 2$,

$$\mathbb{P}(Y_{n+1} - Y_n = \pm e_j | \mathcal{F}_n^Y)$$
$$= \sum_{i=0}^n \mathbb{P}\left(\tilde{Z}_{i+1} - \tilde{Z}_i = \pm e_j, A_i^n, \eta_n = 0 | \mathcal{F}_n^Y\right)$$
$$= \sum_{i=0}^n \mathbb{P}\left[\mathbb{P}\left(\tilde{Z}_{i+1} - \tilde{Z}_i = \pm e_j, A_i^n, \eta_n = 0 | \mathcal{F}_n\right) | \mathcal{F}_n^Y\right]$$
$$= \sum_{i=0}^n \mathbb{P}\left[\mathbb{P}\left(\tilde{Z}_{i+1} - \tilde{Z}_i = \pm e_j, A_i^n, \eta_n = 0 | \mathcal{G}_{ni}\right) | \mathcal{F}_n^Y\right]$$
$$= \sum_{i=0}^n \mathbb{P}(\tilde{Z}_{i+1} - \tilde{Z}_i = \pm e_j)\mathbb{P}(\eta_n = 0)\mathbb{P}(A_i^n | \mathcal{F}_n^Y)$$
$$= \mathbb{P}(\tilde{Z}_{i+1} - \tilde{Z}_i = \pm e_j)\mathbb{P}(\eta_n = 0) = \frac{1}{2(d-1)} \cdot \left(1 - \frac{1}{d}\right) = \frac{1}{2d}.$$

For the case $e_j = e_1$, on the event $Y_n \notin_k$ where $k \leqslant m$,

$$\mathbb{P}(Y_{n+1} - Y_n = +e_1 | \mathcal{F}_n^Y) = \mathbb{P}(\eta_n = 1, \mathcal{E}_n = 1 | \mathcal{F}_n^Y)$$
$$= \mathbb{P}(\eta_n = 1, \zeta_k(Y_n) = 1 | \mathcal{F}_n^Y) = \mathbb{P}(\eta_n = 1).\mathbb{P}(\zeta_k(Y_n) = 1 | \mathcal{F}_n^Y)$$
$$= \frac{1}{d} \cdot \frac{1 + \beta_k(Y_n)}{2} = \frac{1 + \beta_k(Y_n)}{2d}.$$

The cases $e_j = -e_1$ and $k > m$ are treated similarly. Lemma 2.1 is now proved. □

Now, set $\mathcal{D} := \{n \in \mathbb{Z} \text{ such that } Z_{(-\infty,n)} \cap Z_{[n,+\infty)} = \emptyset\}$ to be the set of cut times of Z and similarly let $\tilde{\mathcal{D}}$ be the set of cut times of $\tilde{Z}$. The sequence of cut times of Z is then defined by induction:

$$T_1 := \inf\{n > 0 \text{ such that } n \in \mathcal{D}\},$$
$$T_{i+1} := \inf\{n > T_i \text{ such that } n \in \mathcal{D}\}\,, \text{ for } i \geqslant 1,$$
$$T_{i-1} := \sup\{n < T_i \text{ such that } n \in \mathcal{D}\}, \text{ for } i \leqslant 1.$$

By construction, $T_0 \leq 0 < T_1$ and we set $T := T_1$. We define similarly $\tilde{T}_i$ and $\tilde{T}$ for $\tilde{Z}$. Observe that the laws of T and $\tilde{T}$ do not depend on the environment β, since they depend only on Z and $\tilde{Z}$. Moreover, it follows from (2.1) that

$$\tilde{T} = \sum_{i=0}^{T-1} (1 - \eta_i), \quad \text{and} \quad \{T > k\} = \left\{ \tilde{T} > \sum_{i=0}^{k-1} (1 - \eta_i) \right\}. \tag{2.2}$$

We consider $W := \{\omega \in \Omega : \forall j \, , T_j(\omega) < \infty\}$. E. Bolthausen, A-S. Sznitman and O. Zeitouni [4] proved that $\mathbb{P}(W) = 1$ and $\mathbb{P}(0 \in \mathcal{D}) > 0$ for $d - 1 \geqslant 5$. Let $\hat{\mathbb{P}} := \mathbb{P}(.|0 \in \mathcal{D})$ be the Palm measure.

Exactly in the same way, we can prove (see Lemma 1.1 of [4]) the following lemma:

Lemma 2.3. Let f be a non-negative measurable function, for $d \geqslant 6$ we have

$$\int f d\mathbb{P} = \frac{\int \sum_{k=0}^{T-1} f \circ \theta_k \, d\hat{\mathbb{P}}}{\int T d\hat{\mathbb{P}}} \tag{2.3}$$

with convention that one of two sides equals to $+\infty$ so the other equals to $+\infty$. A simple instance of this formula is to take $f = \mathbb{1}_{0 \in \mathcal{D}}$, so that $\sum_{k=0}^{T-1} f \circ \theta_k = 1$, leading to

$$\mathbb{P}(0 \in \mathcal{D}) = (\hat{\mathbb{E}}T)^{-1} \text{ and } \mathbb{E}[T 1_{0 \in \mathcal{D}}] = 1. \tag{2.4}$$

Proof. Indeed, by Lemma 1.1 of [4], (2.3) is true for $f.1_{f \leqslant c}$ for some positive constant c. Take c to tend to $+\infty$ we get (2.3). $\qquad \square$

It is proved in [4] that $\hat{\mathbb{E}}T = 1/\mathbb{P}(0 \in \mathcal{D}) < \infty$ for $d \geqslant 6$, $\mathbb{E}T < +\infty$ when $d \geqslant 8$ and $\mathbb{E}(T^2) < +\infty$ when $d \geqslant 10$. Hence we can take $f = T$ in (2.3). Observe that $T \circ \theta_k = T - k$ for $k \in \{0, 1, 2, ..., T-1\}$, (2.3) reads

$$\hat{\mathbb{E}}T \, \mathbb{E}T = \int [T + (T-1)... + 1)] \, d\hat{\mathbb{P}} = \hat{\mathbb{E}} \left(\frac{T^2 + T}{2} \right). \tag{2.5}$$

Now, we take $f = T^2$, observe that $T^2 \circ \theta_k = (T-k)^2$ for $k \in \{0, 1, 2, ..., T-1\}$, (2.3) reads

$$\hat{\mathbb{E}}T \, \mathbb{E}(T^2) = \int \left[T^2 + (T-1)^2... + 1^2 \right] d\hat{\mathbb{P}} = \hat{\mathbb{E}} \left[\frac{T(T+1)(2T+1)}{6} \right]. \tag{2.6}$$

Therefore,

$$\hat{\mathbb{E}}(T^2) < +\infty \text{ for } d \geqslant 8, \; \hat{\mathbb{E}}(T^3) < +\infty \text{ for } d \geqslant 10. \tag{2.7}$$

Actually, Lemma 2.4 asserts the stronger result that

$$c_1 := \sup_{d \geqslant 8} \hat{\mathbb{E}}(T^2) < +\infty.$$

To prove monotonicity of the speed, we need the moments of T are bounded as in the following lemma:

Lemma 2.4.

$$c_1 := \sup_{d \geqslant 8} \hat{\mathbb{E}} \left(T^2 \right) < +\infty$$

and

$$c_2 := \sup_{d \geqslant 10} \hat{\mathbb{E}}(T^3) < +\infty$$

Proof. From (2.5), we have

$$\hat{\mathbb{E}}(T^2) = 2\mathbb{E}T\hat{\mathbb{E}}T - \hat{\mathbb{E}}T = \frac{2\mathbb{E}T - 1}{\mathbb{P}(0 \in \mathcal{D})}.$$

Because $\lim_{d\to+\infty} \mathbb{P}(0 \in \tilde{\mathcal{D}}) = 1$ and $\mathbb{P}(0 \in \mathcal{D}) = \frac{d-1}{d}\mathbb{P}(0 \in \tilde{\mathcal{D}})$ (see [5], remark 3, page 248), to show that $c_1 < +\infty$ (resp. $c_2 < +\infty$), it is enough to prove that $\sup_{d\geqslant 8} \mathbb{E}(T) < +\infty$ (resp. $\sup_{d\geqslant 10} \mathbb{E}(T^2) < +\infty$).

Choose ε such that $0 < \varepsilon < 1$. We consider a simple random walk Z^ε on $\mathbb{Z}^{d-1}$ such that:

$$\mathbb{P}\left(Z_{n+1}^\varepsilon - Z_n^\varepsilon = e | \mathcal{F}_n^{Z^\varepsilon}\right) = \frac{\varepsilon}{2(d-1)}, \text{ for } e \in \{\pm e_2, \pm e_2, ..., \pm e_d\},$$

$$\mathbb{P}\left(Z_{n+1}^\varepsilon - Z_n^\varepsilon = 0 | \mathcal{F}_n^{Z^\varepsilon}\right) = 1 - \varepsilon. \tag{2.8}$$

Note that, we can construct Z^ε from the sequences $(\tilde{Z}_n)_{n\in\mathbb{Z}}$, $(\eta_n^\varepsilon)_{n\in\mathbb{Z}}$, where $\eta_n^\varepsilon \sim Ber(1-\varepsilon)$ as in the construction of Z. Set $\mathcal{J} := \{n \text{ such that } Z_n^\varepsilon \neq Z_{n-1}^\varepsilon\}$ and write $\mathcal{J} = \{... < j_{-1} < j_0 \leqslant 0 < j_1 < ...\}$. Set $\mu_n := j_n - j_{n-1}$ for $n > 1$ and $\mu_1 := j_1$. Then, the $(\mu_n)_{n\geqslant 0}$ are i.i.d. , Geometric(ε) random variables. We call $\{T_n^\varepsilon\}_{n\in\mathbb{Z}}$ the cut times of Z^ε, $T^\varepsilon := T_1^\varepsilon$ and $\mathcal{D}^\varepsilon$ is the set of cut times. Then $\mathbb{P}(0 \in \mathcal{D}^\varepsilon) = \varepsilon\mathbb{P}(0 \in \tilde{\mathcal{D}})$ converges to ε when $d \to \infty$ and $\mathbb{P}(0 \in \mathcal{D}^\varepsilon)$ is bounded by ε. We also have $T^\varepsilon = \sum_{i=1}^{\tilde{T}} \mu_i$. Then

$$\begin{aligned}
\mathbb{E}(T^\varepsilon) &= \sum_{k\geqslant 1} \mathbb{E}(\sum_{i=1}^k \mu_i)\mathbb{P}[\tilde{T} = k] \\
&= \sum_{k\geqslant 1} \frac{k}{\varepsilon}\mathbb{P}[\tilde{T} = k] \\
&= \frac{\mathbb{E}\tilde{T}}{\varepsilon}.
\end{aligned} \tag{2.9}$$

We compute similarly and get that

$$\mathbb{E}[(T^\varepsilon)^2] = \frac{\mathbb{E}(\tilde{T}^2) + (1-\varepsilon)\mathbb{E}(\tilde{T})}{\varepsilon^2}.$$

T is T^ε with $\varepsilon = \frac{d-1}{d}$ then $\mathbb{E}T = \frac{d}{d-1}\mathbb{E}\tilde{T}$, so that $\mathbb{E}T = \frac{d\varepsilon}{d-1}\mathbb{E}(T^\varepsilon)$. Therefore, in order to prove that $\sup_{d\geqslant 8} \mathbb{E}T < +\infty$ (resp. $\sup_{d\geqslant 10} \mathbb{E}(T^2) < +\infty$), it is enough to prove that $\sup_{d\geqslant 8} \mathbb{E}(T^\varepsilon) < +\infty$ (resp. $\sup_{d\geqslant 10} \mathbb{E}[(T^\varepsilon)^2] < +\infty$) for some fixed ε.

Now, repeating the proof of (1.12) in [4], we obtain for $k_j = 1 + Lj$, $j \geqslant 0$ ($L \geqslant 1, J \geqslant 1$ are two fixed integers),

$$\mathbb{P}\left(T^\varepsilon > k_{2J}\right) \leqslant \mathbb{P}(0 \in \mathcal{D}^\varepsilon)^J + (2J+1)\sum_{k\geqslant L} k\mathbb{P}\left(Z_k^\varepsilon = 0\right)$$

$$\leqslant \varepsilon^J + (2J+1)\sum_{k\geqslant L} k\mathbb{P}\left(Z_k^\varepsilon = 0\right). \tag{2.10}$$

Using the fact that $\mathbb{P}\left(Z_n^\varepsilon = 0\right)$ decreases with $d \geqslant 2$ (we delay the proof to the end), let $D \geqslant 6$, we have

$$\mathbb{P}[T^\varepsilon > k_{2J}] \text{ (with } d \geqslant D) \leqslant \varepsilon^J + (2J+1)\sum_{k\geqslant L} k\mathbb{P}\left(Z_k^\varepsilon = 0\right) \text{ (when } d = D)$$

$$\leqslant \varepsilon^J + (2J+1) \text{ const } L^{-\frac{D-5}{2}}. \tag{2.11}$$

Choosing a large enough γ depending on ε, and setting $J = [\gamma \log n]$, $L = [\frac{n}{3J}]$ then

$$\mathbb{P}[T^\varepsilon > n] \leqslant c(\log n)^{1+\frac{D-5}{2}} n^{-\frac{D-5}{2}}, \quad n \geqslant 1, d \geqslant D, \tag{2.12}$$

and

$$n\mathbb{P}[T^\varepsilon > n] \leqslant c(\log n)^{1+\frac{D-5}{2}} n^{-\frac{D-7}{2}}, \quad n \geqslant 1, d \geqslant D, \tag{2.13}$$

where c depends only on D and ε. This implies that choosing $D = 8$ we get $\sup_{d\geqslant 8}\mathbb{E}T^\varepsilon < \infty$ and choose $D = 10$ we get $\sup_{d\geqslant 10}\mathbb{E}[(T^\varepsilon)^2] < \infty$.

Now, in order to finish the proof of Lemma 2.4, we have to prove that $\mathbb{P}[Z_n^\varepsilon = 0]$ decreases with $d \geqslant 2$.

Remark that for n odd $\mathbb{P}[Z_n^\varepsilon = 0] = 0$, so we consider n even. Using characteristic functions, we obtain

$$\mathbb{P}(Z_n^\varepsilon = 0) = \frac{1}{(2\pi)^{d-1}} \int_{-\pi}^{\pi} ... \int_{-\pi}^{\pi} \left(\frac{\varepsilon}{d-1} \sum_{i=1}^{d-1} \cos\theta_i + 1 - \varepsilon \right)^n d\theta_1...d\theta_{d-1}$$

$$= \frac{1}{(2\pi)^{d-1}} \int_{-\pi}^{\pi} ... \int_{-\pi}^{\pi} \left(\frac{\varepsilon}{d-1} \sum_{i=1}^{d-1} \left(\cos\theta_i + \frac{1-\varepsilon}{\varepsilon} \right) \right)^n d\theta_1...d\theta_{d-1}$$

$$= \mathbb{E}\left[\left(\frac{1}{d-1} \sum_{i=1}^{d-1} (\varepsilon\cos\Theta_i + 1 - \varepsilon) \right)^n \right]. \tag{2.14}$$

where we consider a sequence $\{\Theta_i\}_{i=1}^{d-1}$ of i.i.d. random variables having uniform distribution $U[-\pi, \pi]$. Now, we consider the function $f(x) = x^n$, n is even, f is a convex function on $\mathbb{R}$ and

$$f\left(\frac{x_1 + x_2 + ... + x_d}{d} \right) \leqslant \frac{f(x_1) + f(x_2) + ... + f(x_d)}{d}, \quad \forall x_1, x_2, ..., x_d \in \mathbb{R}.$$

For $a_1, a_2, ..., a_d \in \mathbb{R}$, choose

$$x_1 = \frac{a_1 + a_2 + ... + a_{d-1}}{d-1}, x_2 = \frac{a_2 + a_3 + ... + a_d}{d-1}, ..., x_d = \frac{a_d + a_1 + ... + a_{d-2}}{d-1},$$

then we get

$$\left(\frac{a_1 + a_2 + ... + a_d}{d} \right)^n$$
$$\leqslant \frac{1}{d}\left\{ \left(\frac{a_1 + a_2 + ... + a_{d-1}}{d-1} \right)^n + \left(\frac{a_2 + a_3 + ... + a_d}{d-1} \right)^n + ... + \left(\frac{a_d + a_1 + ... + a_1}{d-1} \right)^n \right\}. \tag{2.15}$$

Now, take $a_i = \varepsilon\cos\Theta_i + 1 - \varepsilon$ for $i = 1, \cdots, d$ and take the expectation. It comes

$$\mathbb{E}\left[\left(\frac{1}{d} \sum_{i=1}^{d} (\varepsilon\cos\Theta_i + 1 - \varepsilon) \right)^n \right] \leqslant \mathbb{E}\left[\left(\frac{1}{d-1} \sum_{i=1}^{d-1} (\varepsilon\cos\Theta_i + 1 - \varepsilon) \right)^n \right]. \tag{2.16}$$

It means that $\mathbb{P}[Z_n^\varepsilon = 0]$ decreases with $d \geqslant 2$. $\qquad\square$

2.2 Girsanov's transform

This section is devoted to the Girsanov's transform connecting $\mathbb{P}_\beta$ and $\mathbb{P}_0$ where $\beta = \{(\beta_1, \beta_2, ..., \beta_m)(y)\}_{y\in\mathbb{Z}^d}$ is fixed environment. We begin by introducing several σ-algebras. For $n \in \mathbb{Z}$, let $\mathcal{F}_n^Z = \sigma(Z_k, k \leqslant n)$. For $n \geqslant 0$, let $\mathcal{F}_n^Y = \sigma(Y_k, 0 \leqslant k \leqslant n)$, $\mathcal{F}_n = \sigma(\mathcal{F}_n^Z, \mathcal{F}_n^Y) = \sigma(\mathcal{F}_{-1}^Z, \mathcal{F}_n^Y)$, and $\mathcal{G}_n = \sigma(\mathcal{F}_n^Y, \sigma(Z_k, k \in \mathbb{Z}))$. We get $\mathcal{F}_n \subset \mathcal{G}_n$. Moreover T is not a $(\mathcal{F}_n)$-stopping time, but is obviously a $(\mathcal{G}_n)$-stopping time, so that we can define the σ-algebra $\mathcal{G}_T$ of the events prior to T. Recall that $\mathcal{E}_j = (Y_{j+1} - Y_j).e_1$ and $\{Y_j \notin_k\}$

means that Y_j has been visited exactly k times at time j. We define for $n \geq 0$, and $\beta \in ([-1,1]^m)^{\mathbb{Z}^d}$:

$$M_n(\beta) = \prod_{j=0}^{n-1} \prod_{k=1}^{m} \left[1 + \mathcal{E}_j \beta_k(Y_j) 1_{Y_j \notin_k} \right],$$

with the convention the product $\prod_{j=0}^{n-1}(...) = 1$ and $M_n(\beta) = 1$ for $n = 0$.

Lemma 2.5. For any $\beta \in ([-1,1]^m)^{\mathbb{Z}^d}$, $d \geqslant 6$, $n \geq 0$,

$$M_n(\beta) = \frac{d\mathbb{P}_\beta|_{\mathcal{F}_n}}{d\mathbb{P}_0|_{\mathcal{F}_n}}, \ \ M_n(\beta) = \frac{d\mathbb{P}_\beta|_{\mathcal{G}_n}}{d\mathbb{P}_0|_{\mathcal{G}_n}}, \ \ M_T(\beta) = \frac{d\mathbb{P}_\beta|_{\mathcal{G}_T}}{d\mathbb{P}_0|_{\mathcal{G}_T}}.$$

Proof. Since $\mathcal{F}_n \subset \mathcal{G}_n$, $M_n(\beta)$ is $\mathcal{F}_n$-measurable, and T is a finite $(\mathcal{G}_n)$-stopping time, it is enough to prove that $M_n(\beta) = \frac{d\mathbb{P}_\beta|_{\mathcal{G}_n}}{d\mathbb{P}_0|_{\mathcal{G}_n}}$. Let $A \in \mathcal{F}_{-1}^Z$, $y_1, ..., y_n \in (\mathbb{Z}^d)^n$, and $B \in \sigma(Z_{n+k} - Z_n, k \geq 0)$ be fixed. Since $(Z_{n+.} - Z_n)$ is independent from $\mathcal{F}_n$, we get:

$$\mathbb{P}_\beta(A, Y_0 = 0, Y_1 = y_1, ..., Y_n = y_n, B) = \mathbb{P}_\beta(A, Y_0 = 0, Y_1 = y_1, ..., Y_n = y_n)\mathbb{P}_\beta(B).$$

Note that the law of Z does not depend on β, so that $\mathbb{P}_\beta(B) = \mathbb{P}_0(B)$. Now by the definition of $m-$ERWRC,

$$\mathbb{P}_\beta[Y_n = y_n \,|A, Y_0 = 0, Y_1 = y_1, ..., Y_{n-1} = y_{n-1}] = \frac{1}{2d} \prod_{k=1}^{m} \left[1 + \varepsilon_{n-1}\beta_k(y_{n-1})1_{y_{n-1} \notin_k} \right],$$

where $\varepsilon_{n-1} = (y_n - y_{n-1}).e_1$. Then we get by induction that for any $\beta \in ([-1,1]^m)^{\mathbb{Z}^d}$,

$$\mathbb{P}_\beta[A, Y_0 = 0, Y_1 = y_1, ..., Y_n = y_n] = \left(\frac{1}{2d} \right)^n \prod_{j=0}^{n-1} \prod_{k=1}^{m} \left[1 + \varepsilon_j \beta_k(y_j)1_{y_j \notin_k} \right] \mathbb{P}_\beta[A]$$

$$= \left(\frac{1}{2d} \right)^n \prod_{j=0}^{n-1} \prod_{k=1}^{m} \left[1 + \varepsilon_j \beta_k(y_j)1_{y_j \notin_k} \right] \mathbb{P}_0[A],$$

where the last equality comes from the fact that $A \in \mathcal{F}_{-1}^Z$. Hence,

$$\frac{\mathbb{P}_\beta[A, Y_0 = 0, Y_1 = y_1, ..., Y_n = y_n, B]}{\mathbb{P}_0[A, Y_0 = 0, Y_1 = y_1, ..., Y_n = y_n, B]} = \prod_{j=0}^{n-1} \prod_{k=1}^{m} \left[1 + \varepsilon_j \beta(y_j)1_{y_j \notin_k} \right].$$

We have just proved that for all $A \in \mathcal{F}_{-1}^Z$, $y_1, ..., y_n \in (\mathbb{Z}^d)^n$, and $B \in \sigma(Z_{n+k} - Z_n, k \geq 0)$,

$$\mathbb{P}_\beta[A, Y_0 = 0, Y_1 = y_1, ..., Y_n = y_n, B] = \mathbb{E}_0[1_A \, 1_{Y_0 = 0, Y_1 = y_1, ..., Y_n = y_n} \, 1_B \, M_n(\beta)].$$

The result follows since $\mathcal{G}_n = \sigma(\mathcal{F}_{-1}^Z, \mathcal{F}_n^Y, \sigma(Z_{n+k} - Z_n, k \geq 0))$. □

2.3 Existence of the speed.

2.3.1 e_1-exchangeable and stationary environment

We begin with some notations used throughout the section. For $z \in (\mathbb{Z}^{d-1})^{\mathbb{Z}}$, and $k, l \in \mathbb{Z}, k \leqslant l$, $z_{[k,l]} := (z_k, z_{k+1}, \cdots, z_l)$. The expectation w.r.t. the law $\mathbb{Q}$ of the environment is still denoted by $\mathbb{Q}$. We also use the notation $\hat{P}(\cdot) = P(\cdot|0 \in \mathcal{D})$, and for β fixed, $\hat{\mathbb{P}}_\beta(\cdot) = \mathbb{P}_\beta(\cdot|0 \in \mathcal{D})$. Since $\mathbb{P}_\beta(0 \in \mathcal{D})$ does not depend on β, we get $\hat{P}(\cdot) = \mathbb{Q}(\hat{\mathbb{P}}_\beta(\cdot))$. Let A be any Borel set of $(\mathbb{Z}^d)^\mathbb{N}$, then

$$\hat{P}(Y_{T+.} - Y_T \in A) = \mathbb{Q}[\hat{\mathbb{P}}_\beta(Y_{T+.} - Y_T \in A)]$$

$$= \sum_{k \geqslant 1} \sum_{z_{[1,k]}} \sum_{x \in \mathbb{Z}} \mathbb{Q}[\hat{\mathbb{P}}_\beta(Y_{k+.} - Y_k \in A|T = k, Z_{[1,k]} = z_{[1,k]}, X_k = x)$$

$$\times \hat{\mathbb{P}}_\beta(X_k = x|T = k, Z_{[1,k]} = z_{[1,k]})]\hat{P}(T = k, Z_{[1,k]} = z_{[1,k]}). \quad (2.17)$$

Monotonicity and regularity of the speed for excited random walks

Note that by the definition of the cut times, the trajectory of Y between T_n and $T_{n+1} - 1$ does not intersect the trajectory of Y before T_n. Hence $\hat{\mathbb{P}}_\beta(Y_{k+.} - Y_k \in A | T = k, Z_{[1,k]} = z_{[1,k]}, X_k = x)$ depends only on $\{\beta(., z)\}_{z \notin z_{[1,k]}}$, while $\hat{\mathbb{P}}_\beta(X_k = x | T = k, Z_{[1,k]} = z_{[1,k]})$ depends only on $\{\beta(., z)\}_{z \in z_{[1,k]}}$. $z_{[1,k]}$ and $x \in \mathbb{Z}$ being given, we consider the mapping $\delta : \mathbb{Z}^d \to \mathbb{Z}^d$ defined by:

$$\forall (u,v) \in \mathbb{Z} \times \mathbb{Z}^{d-1}, \; \delta(u,v) = \begin{cases} (u,v) & \text{if } v \in z_{[1,k]}, \\ (u-x, v) & \text{if } v \notin z_{[1,k]}. \end{cases}$$

It follows from the preceding remark that:

$$\begin{aligned}
&\hat{\mathbb{P}}_{\delta\beta}(Y_{k+.} - Y_k \in A | T = k, Z_{[1,k]} = z_{[1,k]}, X_k = x) \\
&= \hat{\mathbb{P}}_{\theta_{(-x,0)}\beta}(Y_{k+.} - Y_k \in A | T = k, Z_{[1,k]} = z_{[1,k]}, X_k = x) \\
&= \hat{\mathbb{P}}_{\theta_{(0,z_k)}\beta}(Y_. \in A | T_{-1} = -k, Z_{[-k,-1]} = \overline{z}_{[-k,-1]}),
\end{aligned}$$

where $\theta_{(x,z)}\beta(u,v) = \beta(u+x, v+z)$, and $\overline{z}_{[-k,-1]} = (-z_k, z_1 - z_k, \cdots, z_{k-1} - z_k)$. Moreover,

$$\hat{\mathbb{P}}_{\delta\beta}(X_k = x | T = k, Z_{[1,k]} = z_{[1,k]}) = \hat{\mathbb{P}}_\beta(X_k = x | T = k, Z_{[1,k]} = z_{[1,k]}).$$

The random environment being e_1-exchangeable, $\delta(\beta)$ has the same law as β. Hence,

$$\begin{aligned}
&\hat{P}(Y_{T+.} - Y_T \in A) \\
&= \sum_{k \geqslant 1} \sum_{z_{[1,k]}} \sum_{x \in \mathbb{Z}} \mathbb{Q}[\hat{\mathbb{P}}_{\delta\beta}(Y_{k+.} - Y_k \in A | T = k, Z_{[1,k]} = z_{[1,k]}, X_k = x) \\
&\qquad\qquad \times \hat{\mathbb{P}}_{\delta\beta}(X_k = x | T = k, Z_{[1,k]} = z_{[1,k]})]\hat{P}(T = k, Z_{[1,k]} = z_{[1,k]}) \\
&= \sum_{k \geqslant 1} \sum_{z_{[1,k]}} \sum_{x \in \mathbb{Z}} \mathbb{Q}[\hat{\mathbb{P}}_{\theta_{(0,z_k)}\beta}(Y_. \in A | T_{-1} = -k, Z_{[-k,-1]} = \overline{z}_{[-k,-1]}) \\
&\qquad\qquad \times \hat{\mathbb{P}}_\beta(X_k = x | T = k, Z_{[1,k]} = z_{[1,k]})]\hat{P}(T = k, Z_{[1,k]} = z_{[1,k]}) \\
&= \sum_{k \geqslant 1} \sum_{z_{[1,k]}} \mathbb{Q}[\hat{\mathbb{P}}_{\theta_{(0,z_k)}\beta}(Y_. \in A | T_{-1} = -k, Z_{[-k,-1]} = \overline{z}_{[-k,-1]})]\hat{P}(T = k, Z_{[1,k]} = z_{[1,k]}).
\end{aligned}$$

$$(2.18)$$

Using the stationarity of the environment, we get then

$$\begin{aligned}
&\hat{P}(Y_{T+.} - Y_T \in A) \\
&= \sum_{k \geqslant 1} \sum_{z_{[1,k]}} \mathbb{Q}[\hat{\mathbb{P}}_\beta(Y_. \in A | T_{-1} = -k, Z_{[-k,-1]} = \overline{z}_{[-k,-1]})]\hat{P}(T_{-1} = -k, Z_{[-k,-1]} = \overline{z}_{[-k,-1]}) \\
&= \hat{P}(Y_. \in A).
\end{aligned}$$

$$(2.19)$$

Now, set $H_n = X_{T_n} - X_{T_{n-1}}$ for $n \geqslant 1$. We have just seen that the sequence $\{H_n\}_{n \geqslant 1}$ is stationary under $\hat{P}$. Furthermore, $\hat{E}|H_n| \leqslant \hat{E}T < \infty$ for $d \geqslant 6$. By the ergodic theorem, $\hat{P} - a.s.$

$$\lim_{n \to \infty} \frac{H_1 + H_2 + \ldots + H_n}{n} = \hat{E}(H_1 | \mathcal{F}_H),$$

where $\mathcal{F}_H$ is the σ-algebra generated by the invariant sets of the sequence $\{H_n\}$. Therefore $\lim_{n \to \infty} \frac{X_{T_n}}{n} = \hat{E}(X_T | \mathcal{F}_H)$. On the other hand, we also have $\hat{P} - as \lim_{n \to \infty} \frac{T_n}{n} = \hat{E}(T)$, so that $\hat{P} - as$, $V := \lim_{n \to \infty} \frac{X_n}{n}$ exists for $d \geqslant 6$, and

$$V = \frac{\hat{E}(X_T | \mathcal{F}_H)}{\hat{E}(T)}.$$

2.3.2 i.i.d random environment.

We consider now the case of an i.i.d environment with m cookies. In this situation, we can prove that the speed is deterministic. To this end, we construct an ergodic dynamical system on which the $m-$ERWRC is defined. Let μ be the law of $\beta = (\beta_1, \beta_2, ..., \beta_m)(0) \in [-1, 1]^m$.

We consider the probability space

$$W := \Gamma \times (\mathbb{Z}^{d-1})^{\mathbb{Z}} \times \{0,1\}^{\mathbb{Z}} \times \{\{0,1\}^m\}^{\mathbb{Z}} \text{ where } \Gamma = ([-1,1]^m)^{\mathbb{Z}},$$

endowed with the probability semi-product $P_s := \mathbb{Q}_s \times \mathbb{P}_\gamma$, where $\mathbb{Q}_s = \mu^{\otimes \mathbb{Z}}$ and for $\gamma \in \Gamma$,

$$\mathbb{P}_\gamma = q^{\otimes \mathbb{Z}} \otimes p_1^{\otimes \mathbb{Z}} \otimes \bigotimes_{n \in \mathbb{Z}} \bigotimes_{1 \leqslant k \leqslant m} p_{kn}(\gamma),$$

where

- q is the law of the increments of Z,
- p_1 is a Bernoulli distribution of parameter 1/2,
- $p_{kn}(\gamma)$ is a Bernoulli distribution with $p_{kn}\{1\} = \frac{1+\gamma_n(k)}{2}$, $p_{kn}\{0\} = \frac{1-\gamma_n(k)}{2}$.

Now, we take $w = (\gamma, u, l, h) \in W$ with $\gamma \in \Gamma$, $u \in (\mathbb{Z}^{d-1})^{\mathbb{Z}}$, $l \in \{0,1\}^{\mathbb{Z}}$, $h \in \{\{0,1\}^m\}^{\mathbb{Z}}$. For $n \in \mathbb{Z}$, let $(\beta_n, I_n, \zeta_n, \xi_n)$ be the canonical process on W:

$$\beta_n(w) = \gamma_n \in [-1,1]^m, \ I_n(w) = u_n \in \mathbb{Z}^{d-1}, \ \zeta_n(w) = l_n \in \{0,1\}, \ \xi_{n,j}(w) = h_{n,j} \in \{0,1\}.$$

From $(I_n)_{n \in \mathbb{Z}}$, we define $Z, \tilde{Z}$ as follows:

$$Z_k = \begin{cases} I_1 + ... + I_k & \text{if } k > 0, \\ 0 & \text{if } k = 0, \\ -(I_{k+1} + ... + I_0) & \text{if } k < 0, \end{cases} \qquad \eta_k := 1_{Z_k = Z_{k+1}}.$$

Set $U(k) := \inf\{n, \sum_{i=0}^{n-1}(1 - \eta_i) = k\}$ and $\tilde{Z}_k := Z_{U(k)}$. It is clear that this definition of $Z, \tilde{Z}$ satisfies (2.1). Once Z is defined, we construct the horizontal part's increment $\mathcal{E}_i = X_{i+1} - X_i \in \{-1, 0, 1\}$ for $i \geqslant 0$, as follows. Set $Y_0 = 0$ and assume that $(Y_0, ..., Y_i)$ have been constructed. Then,

- On the event $\{Y_i \notin_k\}$ $(1 \leqslant k \leqslant m)$ (i.e. Y_i has been exactly visited k times at time i),

$$\mathcal{E}_i = (2\xi_{n_1(Y_0,...,Y_i),k} - 1) 1_{Z_i = Z_{i+1}},$$

  where $n_1(Y_0, ..., Y_i) = \inf\{n \leqslant i, \text{ such that } Y_n = Y_i\}$.
- On the event $\{Y_i \in^m\}$ (i.e. Y_i has been visited more than m times at time i),

$$\mathcal{E}_i = (2\zeta_i - 1) 1_{Z_i = Z_{i+1}}.$$

It is proved similarly as in Section 2.1 about the construction $m-$ERWRC to have that the construction of Y above satisfies,

$$P_\gamma(Y_{n+1} - Y_n = \pm e_1 | \mathcal{F}_n^Y, Y_n \notin_k) = \frac{1 \pm \gamma_k(n_1(Y_0, ..., Y_n))}{2d} \text{ for } k \leqslant m,$$

$$P_\gamma(Y_{n+1} - Y_n = \pm e_i | \mathcal{F}_n^Y, Y_n \notin_k) = \frac{1}{2d} \text{ for } i > 1 \text{ or } k > m.$$

Lemma 2.6. Under P, the sequence $(Y_n)_{n \geqslant 0}$ is an m-ERWRC with i.i.d environment $\beta = (\beta(y))_{y \in \mathbb{Z}^d}$ of common law μ.

Proof. We begin with giving an expression for the law of the m-ERWRC with i.i.d environment β. Fix $y_0 = 0, y_1, ..., y_n \in \mathbb{Z}^d$ and set $\varepsilon_i = (y_{i+1} - y_i).e_1 \in \{0, \pm 1\}$. Then, for an m-ERWRC with i.i.d environment β, we have

$$\mathbb{Q}\mathbb{P}_\beta[Y_0 = y_0, Y_1 = y_1, ..., Y_n = y_n]$$
$$= \mathbb{Q}\left[\left(\frac{1}{2d}\right)^n \prod_{i=0}^{n-1} \prod_{k=1}^{m} (1 + \beta_k(y_i)\varepsilon_i \, 1_{y_i \notin k})\right].$$

We decompose the first product according to the value of the first visit to y_i.

$$\mathbb{Q}\mathbb{P}_\beta[Y_0 = y_0, Y_1 = y_1, ..., Y_n = y_n]$$
$$= \left(\frac{1}{2d}\right)^n \mathbb{Q}\left\{\prod_{n_1=0}^{n-1} \prod_{k=1}^{m} \prod_{j=n_1}^{n-1} \left[1 + 1_{y_{n_1} \notin} \beta_k(y_{n_1})\, \varepsilon_j \, 1_{y_j = y_{n_1}} \, 1_{y_j \notin k}\right]\right\}$$
$$= \left(\frac{1}{2d}\right)^n \prod_{n_1=0}^{n-1} \mathbb{Q}\left\{\prod_{k=1}^{m} \prod_{j=n_1}^{n-1} \left[1 + 1_{y_{n_1} \notin} \beta_k(y_{n_1})\, \varepsilon_j \, 1_{y_j = y_{n_1}} \, 1_{y_j \notin k}\right]\right\}.$$

The last equation comes from the independence of the random variables $\beta_k(y_i)$ for $y_i \notin$. On the other hand, using the construction above,

$$P_s[Y_0 = y_0, Y_1 = y_1, ..., Y_n = y_n] = \mathbb{Q}_s \mathbb{P}_\gamma[Y_0 = y_0, Y_1 = y_1, ..., Y_n = y_n]$$
$$= \left(\frac{1}{2d}\right)^n \mathbb{Q}_s\left\{\prod_{n_1=0}^{n-1} \prod_{k=1}^{m} \prod_{j=n_1}^{n-1} \left[1 + 1_{y_{n_1} \notin} \gamma_k(n_1)\, \varepsilon_i \, 1_{y_j = y_i} \, 1_{y_i \in k}\right]\right\}$$
$$= \left(\frac{1}{2d}\right)^n \prod_{n_1=0}^{n-1} \mathbb{Q}_s\left\{\prod_{k=1}^{m} \prod_{j=n_1}^{n-1} \left[1 + 1_{y_i \notin} \gamma_k(n_1)\, \varepsilon_i \, 1_{y_j = y_i} \, 1_{y_i \in k}\right]\right\}.$$

This finishes the proof of Lemma 2.6 since $\{1_{y_{n_1} \notin} \beta(y_{n_1})\}_{n_1=0,...,n-1}$ and $\{1_{y_{n_1} \notin} \gamma(n_1)\}_{n_1=0,...,n-1}$ are two sequences of i.i.d. random vectors with common law μ. $\qquad \square$

Now, we denote by $(\theta_k)_{k \in \mathbb{Z}}$ the canonical shift on W, i.e. $\theta_k(w.) = (w_{k+.})$. We set

$$\hat{W} = \Gamma \times \left[(\mathbb{Z}^{d-1})^\mathbb{Z} \cap \{0 \in \mathcal{D}\}\right] \times \{0,1\}^\mathbb{Z} \times (\{0,1\}^m)^\mathbb{Z}.$$

On $\hat{W}$ we define $\hat\theta := \hat\theta_1 = \theta_T$ and $\hat{P}_s(\cdot) = P_s(\cdot | 0 \in \mathcal{D})$.

Lemma 2.7. (W, θ, P_s) is an ergodic system. As a consequence, $(\hat{W}, \hat\theta, \hat{P}_s)$ is also an ergodic system.

Proof. The idea of proof comes from [4]. Firstly, we prove that θ is a measure-preserving transformation. Consider a measurable set $A \times B$ of W, where $A \subset \Gamma$, and $B \subset (\mathbb{Z}^{d-1})^\mathbb{Z} \times \{0,1\}^\mathbb{Z} \times (\{0,1\}^m)^\mathbb{Z}$. We have that

$$\theta_k \circ P_s(A \times B) = P_s(\theta_k^{-1}A \times \theta_k^{-1}B)$$
$$= \int_{\theta_k^{-1}A} \mathbb{P}_\gamma(\theta_k^{-1}B)d\mathbb{Q} = \int_{\theta_k^{-1}A} \mathbb{P}_{\theta_k\gamma}(B)d\mathbb{Q}_s$$
$$= \int_A \mathbb{P}_\gamma(B)\,(\theta_k^{-1}\mathbb{Q}_s)(d\gamma) = \int_A \mathbb{P}_\gamma(B)\mathbb{Q}_s(d\gamma)$$
$$= P_s(A \times B).$$

Now, we prove that θ is ergodic. Let A be a measurable subset of W, invariant under θ and $\varepsilon > 0$. There exists an integer $m_\varepsilon > 0$ and a measurable subset A_ε depending only on $(w_m)_{|m| \leqslant m_\varepsilon}$ such that

$$|E_{P_s}[1_A - 1_{A_\varepsilon}]| \leqslant \varepsilon.$$

Then, for $L \geqslant 0$,

$$P_s(A) = \mathbb{E}_{P_s}[1_A 1_A \circ \theta_L] = \mathbb{E}_{P_s}[1_{A_\varepsilon} 1_{A_\varepsilon} \circ \theta_L] + c_\varepsilon,$$

with $|c_\varepsilon| \leqslant 2\varepsilon$.

Because that $p_{kn}(\gamma)$ depends only on γ_n, we prove that the sequence $(\gamma_n, I_n, \zeta_n, \xi_n)_{n \in \mathbb{Z}}$ is the sequence of independent variables under P_s. Indeed, let $i < j$, $i, j \in \mathbb{Z}$, we take two measurable sets $A_i \times B_i$ and $A_j \times B_j$, where $A_i, A_j \subset [0,1]^m$, and $B_i, B_j \subset \mathbb{Z}^{d-1} \times \{0,1\} \times \{0,1\}^m$. We have

$$P_s \left\{ [(\gamma_i, I_i, \zeta_i, \xi_i) \in A_i \times B_i] \bigcap [(\gamma_j, I_j, \zeta_j, \xi_j) \in A_j \times B_j] \right\}$$

$$= \int_\Gamma \mathbb{Q}_s(d\gamma) 1_{\gamma_i(\gamma) \in A_i, \gamma_j(\gamma) \in A_j} \mathbb{P}_\gamma \left\{ [(I_i, \zeta_i, \xi_i) \in B_i] \bigcap [(I_j, \zeta_j, \xi_j) \in B_j] \right\}$$

$$= \int_\Gamma \mathbb{Q}_s(d\gamma) 1_{\gamma_i(\gamma) \in A_i, \gamma_j(\gamma) \in A_j} \mathbb{P}_\gamma \left\{ (I_i, \zeta_i, \xi_i) \in B_i \right\} \mathbb{P}_\gamma \left\{ [(I_j, \zeta_j, \xi_j) \in B_j] \right\}$$

$$= \int_\Gamma \mathbb{Q}_s(d\gamma) 1_{\gamma_i(\gamma) \in A_i} \mathbb{P}_\gamma \left\{ (I_i, \zeta_i, \xi_i) \in B_i \right\} . \int_\Gamma \mathbb{Q}_s(d\gamma) 1_{\gamma_j(\gamma) \in A_j} \mathbb{P}_\gamma \left\{ (I_j, \zeta_j, \xi_j) \in B_j \right\}$$

$$= P_s \left\{ [(\gamma_i, I_i, \zeta_i, \xi_i) \in A_i \times B_i] \right\} . P_s \left\{ [(\gamma_j, I_j, \zeta_j, \xi_j) \in A_j \times B_j] \right\}$$

So, for $L > 2m_\varepsilon$, we get $\mathbb{E}_{P_s}[1_{A_\varepsilon} 1_{A_\varepsilon} \circ \theta_L] = P_s(A_\varepsilon) P_s(A_\varepsilon \circ \theta_L) = P_s(A_\varepsilon)^2$. Therefore

$$|P_s(A) - P_s(A)^2| \leqslant |P_s(A) - P_s(A_\varepsilon)^2| + 2\varepsilon \leqslant 4\varepsilon.$$

Letting ε tend to 0, we have that $P_s(A) = 0$ or 1. □

Lemma 2.8. Let Y is a $m-$ERWRC such that the environment cookie is i.i.d., X is the horizontal component $X_n = Y_n \cdot e_1$. For any $d \geqslant 6$ then $P - as$, $\lim_{n \to \infty} \frac{X_n}{n} = v(\mathbb{Q}) := \frac{\hat{E}(X_T)}{\hat{E}(T)}$.

Proof. The existence of the limit, the fact that it is deterministic and the expression of $v(\mathbb{Q})$ for $d \geqslant 6$ follow from the ergodicity of $(\hat{W}, \hat{\theta}, \hat{P}_s)$, and the integrability of T w.r.t $\hat{P}_s$ when $d \geqslant 6$. □

2.4 Monotonicity and differentiability of the speed.

Now, we prove that the expectation $v(\mathbb{Q}) = \hat{E}[V] = \frac{\hat{E}(X_T)}{\hat{E}(T)}$ is increasing in $\mathbb{Q}$.

Consider $\beta_1 = \{\beta_1(y)\}_{y \in \mathbb{Z}^d}, \beta_2 = \{\beta_2(y)\}_{y \in \mathbb{Z}^d}$ defined on $(\Omega, \mathcal{A}, Q) \to \mathbb{B} = ([-1,1]^m)^{\mathbb{Z}^d}$ such that $Q(\beta_1 \leqslant \beta_2) = 1$. It is proved in D. Aldous and R. Lyons [1], that if there exists a monotone coupling of $\mathbb{Q}_1$ and $\mathbb{Q}_2$, then there also exists a stationary monotone coupling of $\mathbb{Q}_1$ and $\mathbb{Q}_2$, as soon as $\mathbb{Q}_1$ and $\mathbb{Q}_2$ are stationary.

Therefore we can suppose that $\{(\beta_1, \beta_2)(y)\}_{y \in \mathbb{Z}^d}$ is stationary. Set $\beta_t(y) = (1-t)\beta_1(y) + t\beta_2(y)$ for $t \in [0,1]$. $\beta_t = \{\beta_t(y)\}_{y \in \mathbb{Z}^d}$ is a stationary environment . Consider

$$f(t) := \frac{\mathbb{Q}\mathbb{E}_{\beta_t}(X_T 1_{0 \in \mathcal{D}})}{E(T 1_{0 \in \mathcal{D}})}. \tag{2.20}$$

Note that β_t is not necessarily exchangeable, so that we can not assert that $f(t)$ is the mean of the speed of the ERW in the random environment β_t. Nevertheless, β_1 and β_2

being exchangeable, we get $f(0) = v(\mathbb{Q}_1)$, $f(1) = v(\mathbb{Q}_2)$, so that it is enough to prove that $f(t)$ is increasing in t. First of all, we need the Girsanov's transform. We have

$$M_n(\beta_t) = \prod_{j=0}^{n-1} [1 + \mathcal{E}_j \beta_t(Y_j) 1_{Y_j \notin^m}],$$

where $Y_j \notin^m$ denotes the event that Y_j has not been visited more than $m-1$ times before time j. As in section 2.2, we have Girsanov's transforms:

$$\frac{dP_{\beta_t}|_{\mathcal{F}_n}}{dP_0|_{\mathcal{F}_n}} = \frac{dP_{\beta_t}|_{\mathcal{G}_n}}{dP_0|_{\mathcal{G}_n}} = M_n(\beta_t) \,, \quad \frac{dP_{\beta_t}|_{\mathcal{G}_T}}{dP_0|_{\mathcal{G}_T}} = M_T(\beta_t) \,.$$

2.4.1 Differentiability of $f(t)$.

We begin by giving another expression of the numerator in (2.20).

Lemma 2.9. For $n \geq 1$, then

$$\mathbb{E}_{\beta_t}(X_T 1_{0 \in \mathcal{D}}) = \mathbb{E}_{\beta_t} \left[\sum_{j=0}^{T-1} \beta_t(Y_j) \, 1_{0 \in \mathcal{D}} \, 1_{Y_j \notin^m} \, 1_{Z_j = Z_{j+1}} \right]$$

$$= \mathbb{E}_0 \left[\sum_{j=0}^{T-1} \beta_t(Y_j) \, 1_{0 \in \mathcal{D}} \, 1_{Y_j \notin^m} \, 1_{Z_j = Z_{j+1}} \, M_T(\beta_t) \right]. \qquad (2.21)$$

Proof. Observe that

$$\mathbb{P}_{\beta_t}[\mathcal{E}_j = \pm 1 | \mathcal{G}_j] = \frac{1 \pm \beta_t(Y_j)}{2} 1_{Y_j \notin^m} 1_{Z_j = Z_{j+1}} + \frac{1}{2} 1_{Y_j \in^m} 1_{Z_j = Z_{j+1}}$$

$$= \left(\frac{1}{2} \pm \frac{\beta_t(Y_j)}{2} 1_{Y_j \notin^m} \right) 1_{Z_j = Z_{j+1}}. \qquad (2.22)$$

Hence,

$$\mathbb{E}_{\beta_t}(X_T 1_{0 \in \mathcal{D}}) = \mathbb{E}_{\beta_t}(\sum_{j=0}^{+\infty} \mathcal{E}_j 1_{T > j} 1_{0 \in \mathcal{D}}) = \sum_{j=0}^{+\infty} \mathbb{E}_{\beta_t}(\mathcal{E}_j 1_{T > j} 1_{0 \in \mathcal{D}}) \,,$$

where the last equality follows from the integrability of T w.r.t $\hat{\mathbb{P}}$ for $d \geq 6$. Note that $\{0 \in \mathcal{D}\}$ and $\{T > j\}$ belong to $\mathcal{G}_j$. Therefore,

$\mathbb{E}_{\beta_t}(\mathcal{E}_j 1_{T > j} 1_{0 \in \mathcal{D}})$
$= \mathbb{E}_{\beta_t} [1_{T > j} 1_{0 \in \mathcal{D}} \, \mathbb{P}_{\beta_t}(\mathcal{E}_j = 1 | \mathcal{G}_j)] - \mathbb{E}_{\beta_t} [1_{T > j} 1_{0 \in \mathcal{D}} \, \mathbb{P}_{\beta_t}(\mathcal{E}_j = -1 | \mathcal{G}_j)]$
$= \mathbb{E}_{\beta_t} \left[\frac{1 + \beta_t(Y_j)}{2} 1_{T > j} 1_{0 \in \mathcal{D}} 1_{Y_j \notin^m} 1_{Z_j = Z_{j+1}} \right] - \mathbb{E}_{\beta_t} \left[\frac{1 - \beta_t(Y_j)}{2} 1_{T > j} 1_{0 \in \mathcal{D}} 1_{Y_j \notin^m} 1_{Z_j = Z_{j+1}} \right]$
$= \mathbb{E}_{\beta_t} \left[\beta_t(Y_j) \, 1_{T > j} \, 1_{0 \in \mathcal{D}} \, 1_{Y_j \notin^m} \, 1_{Z_j = Z_{j+1}} \right].$

Thus,

$$\mathbb{E}_{\beta_t}(X_T 1_{0 \in \mathcal{D}}) = \sum_{j=0}^{+\infty} \mathbb{E}_{\beta_t} \left[\beta_t(Y_j) \, 1_{T > j} 1_{0 \in \mathcal{D}} \, 1_{Y_j \notin^m} \, 1_{Z_j = Z_{j+1}} \right]$$

$$= \mathbb{E}_{\beta_t} \left[\sum_{j=0}^{T-1} \beta_t(Y_j) \, 1_{0 \in \mathcal{D}} \, 1_{Y_j \notin^m} \, 1_{Z_j = Z_{j+1}} \right]$$

$$= \mathbb{E}_0 \left[\sum_{j=0}^{T-1} \beta_t(Y_j) \, 1_{0 \in \mathcal{D}} \, 1_{Y_j \notin^m} \, 1_{Z_j = Z_{j+1}} \, M_T(\beta_t) \right].$$

This proves the first equality. The second one follows from the fact that $\sum_{j=0}^{T-1} \beta_t(Y_j)$ $1_{0\in\mathcal{D}} 1_{Y_j \notin^m} 1_{Z_j = Z_{j+1}}$ is $\mathcal{G}_T$-measurable, and Lemma 2.5. $\qquad\square$

We turn now to the derivative of $f(t)$. We study now the sign of the derivative on the set of bounded environment, and from now on we assume that for $i = 1, 2$, $|\beta_i(y)| \leqslant \sigma < 1$ a.s. for any y of $\mathbb{Z}^d$ where σ is a constant in $(0, 1)$.

Lemma 2.10. For $d \geq 8$, the function $t \in [0, 1] \to \mathbb{Q}[\mathbb{E}_{\beta_t}(X_T 1_{0\in\mathcal{D}})]$ is differentiable and,

$$E(T 1_{0\in\mathcal{D}}).\frac{\partial f}{\partial t}(t) = \frac{\partial}{\partial t}\mathbb{Q}[\mathbb{E}_{\beta_t}(X_T 1_{0\in\mathcal{D}})]$$

$$= \mathbb{Q}\mathbb{E}_{\beta_t}\left[\sum_{j=0}^{T-1}(\beta_2 - \beta_1)(Y_j)\, 1_{0\in\mathcal{D}}\, 1_{Y_j\notin^m}\, 1_{Z_j=Z_{j+1}}\right]$$

$$+ \mathbb{Q}\mathbb{E}_{\beta_t}\left[\sum_{j=0}^{T-1}\beta_t(Y_j)\, 1_{0\in\mathcal{D}}\, 1_{Y_j\notin^m}\, 1_{Z_j=Z_{j+1}} \sum_{i=1}^{T-1}\frac{(\beta_2 - \beta_1)(Y_i)\mathcal{E}_i 1_{Y_i\notin^m}}{1 + \beta_t(Y_i)\mathcal{E}_i}\, 1_{Z_i=Z_{i+1}}\right]. \quad (2.23)$$

Proof. We have $M_T(\beta_t) = \prod_{j=0}^{T-1}\left[1 + \mathcal{E}_j\beta_t(Y_j)1_{Y_j\notin^m}\right]$ then

$$\frac{\partial}{\partial t}M_T(\beta_t) = \left(\sum_{j=0}^{T-1}\frac{(\beta_2 - \beta_1)(Y_j)\mathcal{E}_j}{1 + \beta_t(Y_j)\mathcal{E}_j}1_{Y_j\notin^m}\right)M_T(\beta_t)$$

$$= \left(\sum_{j=0}^{T-1}\frac{(\beta_2 - \beta_1)(Y_j)\mathcal{E}_j}{1 + \beta_t(Y_j)\mathcal{E}_j}1_{Y_j\notin^m}1_{Z_j=Z_{j+1}}\right)M_T(\beta_t)$$

the last equality is followed by the fact that $Z_j = Z_{j+1}$ when $\mathcal{E}_j \neq 0$. Set

$$N_T(t) := \sum_{j=0}^{T-1}\beta_t(Y_j)\, 1_{0\in\mathcal{D}}\, 1_{Y_j\notin^m}\, 1_{Z_j=Z_{j+1}}.$$

Then

$$\frac{\partial}{\partial t}N_T(t) = \sum_{j=0}^{T-1}(\beta_2 - \beta_1)(Y_j)\, 1_{0\in\mathcal{D}}\, 1_{Y_j\notin^m}\, 1_{Z_j=Z_{j+1}}.$$

and

$$U_T(\beta_t) := \frac{\partial}{\partial t}\left[N_T(t)M_T(\beta_t)\right] = \frac{\partial}{\partial t}N_T(t)M_T(\beta_t) + N_T(t)\frac{\partial}{\partial t}M_T(\beta_t)$$

$$= \sum_{j=0}^{T-1}(\beta_2 - \beta_1)(Y_j)\, 1_{0\in\mathcal{D}}\, 1_{Y_j\notin^m}\, 1_{Z_j=Z_{j+1}}M_T(\beta_t)$$

$$+ N_T(t)\left(\sum_{j=0}^{T-1}\frac{(\beta_2 - \beta_1)(Y_j)\mathcal{E}_j}{1 + \beta_t(Y_j)\mathcal{E}_j}1_{Y_j\notin^m}1_{Z_j=Z_{j+1}}\right)M_T(\beta_t). \quad (2.24)$$

We have

$$\mathbb{Q}\mathbb{E}_0\left[N_T(t)M_T(\beta_t)\right] = \mathbb{Q}\mathbb{E}_0[N_T(0)M_T(\beta_0)] + \mathbb{Q}\mathbb{E}_0\left[\int_0^t U_T(\beta_x)dx\right]. \quad (2.25)$$

Since $N_T(t) \leqslant T1_{0 \in \mathcal{D}}$, $\frac{\partial}{\partial t} N_T(t) \leqslant 2T1_{0 \in \mathcal{D}}$ and $\left| \frac{\mathcal{E}_j}{1 + x\mathcal{E}_j} \right| \leqslant \frac{1}{1-\sigma}, \forall x \leqslant \sigma$, we get

$$\int_0^t \mathbb{QE}_0 |U_T(\beta_x)| dx \leqslant 2 \int_0^t \mathbb{QE}_0(T1_{0 \in \mathcal{D}} M_T(\beta_x)) dx + \frac{2\sigma}{1-\sigma} \int_0^t \mathbb{QE}_0(T^2 1_{0 \in \mathcal{D}} M_T(\beta_x)) dx$$

$$= 2 \int_0^t \hat{\mathbb{E}}_0(T) dx + \frac{2\sigma}{1-\sigma} \int_0^t \mathbb{QE}_{\beta_x}(T^2 1_{0 \in \mathcal{D}}) dx$$

$$= 2 \int_0^t \hat{\mathbb{E}}_0(T) dx + \frac{2\sigma}{1-\sigma} \int_0^t \mathbb{QE}_0(T^2 1_{0 \in \mathcal{D}}) dx,$$

(since T and $\{0 \in \mathcal{D}\}$ belong to $\sigma(Z)$, then they do not depend on x)

$$= 2t\hat{\mathbb{E}}_0 T + \frac{2t\sigma}{1-\sigma} \hat{\mathbb{E}}_0(T^2) < +\infty \text{ when } \hat{\mathbb{E}}_0(T^2) < +\infty.$$

It follows from Lemma 2.4 that $\hat{\mathbb{E}}_0(T^2) < +\infty$ for $d \geqslant 8$. Fubini's theorem leads then to

$$\mathbb{QE}_0[N_T(t) M_T(\beta_t)] = \mathbb{QE}_0[N_T(0) M_T(\beta_0)] + \int_0^t \mathbb{QE}_0[U_T(\beta_x)] dx. \qquad (2.26)$$

Now, we prove that $\mathbb{QE}_0[U_T(\beta_x)]$ is continuous in $x \in [0,1]$. To this end, we recall a general result about uniform integrability of positive random variables (see for instance Theorem 5 page 189 in Shiryaev [11]).

Lemma 2.11. Let J be an interval of $\mathbb{R}$, and $(X(\beta), \beta \in J)$ be a family of positive integrable random variables. Assume that $\{X(\beta)\}_{\beta \in J}$ is a.s. continuous in β. Then, the function $\varphi(\beta) = \mathbb{E}[X(\beta)]$ is continuous in β if only if the family $\{X(\beta)\}_{\beta \in J}$ is uniformly integrable.

Observe from (2.24) that

$$|U_T(\beta_x)| \leqslant 2\sigma T M_T(\beta_x) 1_{0 \in \mathcal{D}} + \frac{2\sigma}{1-\sigma} T^2 M_T(\beta_x) 1_{0 \in \mathcal{D}} \leqslant \frac{4}{1-\sigma} T^2 M_T(\beta_x) 1_{0 \in \mathcal{D}}. \qquad (2.27)$$

For $x_0 \in [0,1]$, we have:

1. $\lim_{x \to x_0} T^2 M_T(\beta_x) 1_{0 \in \mathcal{D}} = T^2 M_T(\beta_{x_0}) 1_{0 \in \mathcal{D}}$ a.s.,
2. $T^2 M_T(\beta_x) 1_{0 \in \mathcal{D}} \geqslant 0$,
3. $\forall x$, $\mathbb{QE}_0[T^2 M_T(\beta_x) 1_{0 \in \mathcal{D}}] = \mathbb{QE}_{\beta_x}(T^2 1_{0 \in \mathcal{D}}) = \mathbb{E}_0(T^2 1_{0 \in \mathcal{D}}) < +\infty$, since $\hat{\mathbb{E}}_0(T^2) < +\infty$ for $d \geqslant 8$.

It follows then from Lemma 2.11 that the family $\{T^2 M_T(\beta_x) 1_{0 \in \mathcal{D}}\}_{x \in [0,1]}$ is uniformly integrable. By (2.27), this is also true for the family $\{U_T(\beta_x)\}_{x \to x_0}$ in a neighborhood of $x_0 \in [0,1]$. Therefore, we obtain,

$$\lim_{x \to x_0} \mathbb{QE}_0(U_T(\beta_x)) = \mathbb{QE}_0(U_T(\beta_{x_0})) \text{ i.e. } \mathbb{QE}_0(U_T(\beta_x)) \text{ is continuous.}$$

Then, we get

$$\frac{\partial}{\partial t} \mathbb{QE}_0[N_T(t) M_T(\beta_t)] = \mathbb{QE}_0[U_T(\beta_t)].$$

This finishes the proof of Lemma 2.10. $\qquad \square$

2.4.2 Monotonicity of the speed

We remind the reader that $\tilde{Z}$ is defined as the walk Z when it moves, and $\tilde{\mathcal{D}}$ denotes the cut times of $\tilde{Z}$. Since $T \geq 1$, the first term is bounded from below by its first item corresponding to $j = 0$.

$$\mathbb{Q}\mathbb{E}_{\beta_t}\left[\sum_{j=0}^{T-1}(\beta_2-\beta_1)(Y_j)\,1_{0\in\mathcal{D}}\,1_{Y_j\notin^m}\,1_{Z_j=Z_{j+1}}\right] \geqslant \mathbb{Q}\left[(\beta_2-\beta_1)(0)\right]P(0\in\mathcal{D}, Z_0=Z_1)$$

$$=\frac{1}{d}\mathbb{Q}\left[(\beta_2-\beta_1)(0)\right]P(0\in\mathcal{D}). \qquad (2.28)$$

The equality (2.28) follows, since $\mathcal{D}:=\{n\in\mathbb{Z} \text{ such that } Z_{(-\infty,n)}\cap Z_{[n,+\infty)}=\emptyset\}$ and, therefor, $\{0\in\mathcal{D}\}=\{Z_{-1}\neq Z_0, 0\in\tilde{\mathcal{D}}\}=\{\eta_{-1}=0, 0\in\tilde{\mathcal{D}}\}$. So we have $P(0\in\mathcal{D}, Z_0=Z_1)=P(0\in\tilde{\mathcal{D}},\eta_0=1,\eta_{-1}=0)=P(0\in\tilde{\mathcal{D}},\eta_{-1}=0).P(\eta_0=1)=\frac{1}{d}P(0\in\mathcal{D})$.

Now, we focus on the second term. Since $\mathbb{E}_{\beta_t}[\mathcal{E}_k 1_{Y_k\notin^m}/(1+\beta_t(Y_k)\mathcal{E}_k)|\mathcal{G}_k]=0$, then

$$\mathbb{Q}\mathbb{E}_{\beta_t}\left[\sum_{0\leqslant j\leqslant i\leqslant T-1}\beta_t(Y_j)\,1_{0\in\mathcal{D}}\,1_{Y_j\notin^m}\,1_{Z_j=Z_{j+1}}\frac{(\beta_2-\beta_1)(Y_i)\mathcal{E}_i}{1+\beta_t(Y_i)\mathcal{E}_i}1_{Y_i\notin^m}1_{Z_i=Z_{i+1}}\right]=$$

$$\mathbb{Q}\mathbb{E}_{\beta_t}\left[\sum_{0\leqslant j\leqslant i\leqslant T-1}\beta_t(Y_j)\,1_{0\in\mathcal{D}}\,1_{Y_j\notin^m}\,1_{Z_j=Z_{j+1}}(\beta_2-\beta_1)(Y_i)1_{Z_i=Z_{i+1}}\mathbb{E}_{\beta_t}\left(\frac{\mathcal{E}_i 1_{Y_i\notin^m}}{1+\beta_t(Y_i)\mathcal{E}_i}|\mathcal{G}_i\right)\right]$$

$$=0. \qquad (2.29)$$

Then the second term of (2.23) is equal to:

$$\mathbb{Q}\mathbb{E}_{\beta_t}\left[\sum_{0\leqslant i<j\leqslant T-1}\beta_t(Y_j)\,1_{0\in\mathcal{D}}\,1_{Y_j\notin^m}\,1_{Z_j=Z_{j+1}}\frac{(\beta_2-\beta_1)(Y_i)\mathcal{E}_i}{1+\beta_t(Y_i)\mathcal{E}_i}1_{Y_i\notin^m}1_{Z_i=Z_{i+1}}\right]$$

$$\geqslant -\sigma\mathbb{Q}\mathbb{E}_{\beta_t}\left[\sum_{0\leqslant i<j\leqslant T-1}1_{0\in\mathcal{D}}\,1_{Z_j=Z_{j+1}}(\beta_2-\beta_1)(Y_i)\mathbb{E}_{\beta_t}\left(\frac{|\mathcal{E}_i|}{1+\beta_t(Y_i)\mathcal{E}_i}|\mathcal{G}_i\right)1_{Y_i\notin^m}1_{Z_i=Z_{i+1}}\right]$$

(since $|\beta_t|\leqslant\sigma$)

$$\geqslant -\sigma\mathbb{Q}\mathbb{E}_{\beta_t}\left[\sum_{0\leqslant i<j\leqslant T-1}1_{0\in\mathcal{D}}\,1_{Z_j=Z_{j+1}}(\beta_2-\beta_1)(Y_i)1_{Z_i=Z_{i+1}}1_{Y_i\notin^m}1_{Z_i=Z_{i+1}}\right]$$

$$\geqslant -\sigma\mathbb{Q}\mathbb{E}_{\beta_t}\left[\sum_{0\leqslant i<j}(\beta_2-\beta_1)(Y_i)1_{Z_j=Z_{j+1}}1_{Z_i=Z_{i+1}}1_{0\in\mathcal{D}}1_{T>j}\right],$$

$$\geqslant -\sigma\mathbb{Q}\mathbb{E}_{\beta_t}\left[\sum_{0\leqslant i<j}(\beta_2-\beta_1)(Y_i)1_{\eta_j=1}1_{\eta_i=1}1_{0\in\tilde{D}}1_{\tilde{T}>\sum_{k=0}^{j-1}(1-\eta_k)}\right]$$

$$\geqslant -\sigma\mathbb{Q}\mathbb{E}_{\beta_t}\left[\sum_{0\leqslant i<j}(\beta_2-\beta_1)(Y_i)1_{\eta_j=1}1_{\eta_i=1}1_{0\in\tilde{D}}1_{\tilde{T}>\sum_{k=0,k\neq i}^{j-1}(1-\eta_k)}\right]$$

$$=-\frac{\sigma}{d^2}\mathbb{Q}\mathbb{E}_{\beta_t}\left[\sum_{0\leqslant i<j}(\beta_2-\beta_1)(Y_i)1_{0\in\mathcal{D}}1_{\tilde{T}>\sum_{k=0,k\neq i}^{j-1}(1-\eta_k)}\right]$$

since η_j is independent of $\tilde{Z}, \mathcal{F}_i^Y, \eta_1,...,\eta_{j-1}$ and η_i is independent of $\tilde{Z}, \mathcal{F}_i^Y, \{\eta_k\}_{k\neq i}$,

$$=-\frac{\sigma}{d^2}\mathbb{Q}\mathbb{E}_{\beta_t}\left[\sum_{0\leqslant i<j}(\beta_2-\beta_1)(Y_i)1_{0\in\mathcal{D}}1_{\tilde{T}>\sum_{k=0}^{j-2}(1-\eta_k)}\right]$$

with the convention that the sum over an empty set equals to 0

$$= -\frac{\sigma}{d^2} \mathbb{Q}\mathbb{E}_{\beta_t} \left[\sum_{0 \leqslant i < j} (\beta_2 - \beta_1)(Y_i) 1_{0 \in \mathcal{D}} 1_{T > j-1} \right], \tag{EQ}$$

$$= -\frac{\sigma}{d^2} \sum_{i=1}^{+\infty} \mathbb{Q}\mathbb{E}_{\beta_t} \left[(\beta_2 - \beta_1)(Y_i)(T - i) 1_{T > i} 1_{0 \in \mathcal{D}} \right],$$

$$= -\frac{\sigma}{d^2} \sum_{i=1}^{+\infty} \mathbb{Q}\mathbb{E}_{\beta_t} \left[\sum_{z \in \mathbb{Z}^{d-1}} \sum_{x \in \mathbb{Z}} \frac{(\beta_2 - \beta_1)(y)}{d^2} 1_{Z_i = z} 1_{X_i = x} (T - i) 1_{T > i} 1_{0 \in \mathcal{D}} \right] \text{ with } y = (x, z),$$

$$= -\frac{\sigma}{d^2} \sum_{i \geqslant 1} \mathbb{Q} \left[(\beta_2 - \beta_1)(0) \sum_{z \in \mathbb{Z}^{d-1}} \sum_{x \in \mathbb{Z}} \mathbb{E}_{\theta_y \beta_t} (1_{Y_i = y} (T - i) 1_{T > i} 1_{0 \in \mathcal{D}}) \right]$$

because β is stationary,

$$\geqslant -\frac{\sigma}{d^2} \sum_{i \geqslant 1} \mathbb{Q} \left\{ (\beta_2 - \beta_1)(0) \sum_{z \in \mathbb{Z}^{d-1}} \mathbb{E}_{\theta_y \beta_t} [(2i + 1)(1_{Z_i = z} (T - i) 1_{T > i} 1_{0 \in \mathcal{D}})] \right\}$$

$$\text{for } X_i = x \Rightarrow |x| \leqslant i,$$

$$\geqslant -\frac{\sigma}{d^2} \sum_{i \geqslant 1} \mathbb{Q} \left\{ (\beta_2 - \beta_1)(0) \sum_{z \in \mathbb{Z}^{d-1}} \mathbb{E}_0 [(2T + 1) 1_{Z_i = z} (T - i) 1_{T > i} 1_{0 \in \mathcal{D}}] \right\},$$

$$\geqslant -\frac{\sigma}{d^2} \mathbb{Q} \left[(\beta_2 - \beta_1)(0) \right] \mathbb{E}_0 \left[\frac{(2T + 1)T(T + 1)}{2} 1_{0 \in \mathcal{D}} \right].$$

Therefore, we get

$$\hat{E}(T) \frac{\partial}{\partial t} f(t) \geqslant \frac{1}{d} \mathbb{Q}[(\beta_2 - \beta_1)(0)] \left[1 - \frac{1}{d} \sigma \hat{E} \left[\frac{(2T + 1)T(T + 1)}{2} \right] \right].$$

This implies that $\frac{\partial}{\partial t} f(t) \geqslant 0$ when $d \geqslant \sigma \hat{E} \left[\frac{(2T+1)T(T+1)}{2} \right]$. Lemma 2.4 asserts that

$$d_0 := \max \left\{ \left\lfloor \sup_{d \geqslant 10} \hat{E} \left[\frac{(2T + 1)T(T + 1)}{2} \right] \right\rfloor + 1, 10 \right\} < +\infty. \tag{2.30}$$

Then, for $d \geqslant d_0 > \sigma d_0$, we have $(\partial/\partial t) f(t) \geqslant 0$, which implies that $f(0) \leqslant f(1)$ so that $v(\mathbb{Q}_{\beta_1}) \leqslant v(\mathbb{Q}_{\beta_2})$ on the set of probability measures on bounded environment. Choose $\sigma_0 = \frac{10}{d_0}$, then we have the monotonicity for environments bounded by σ_0 for any $d \geqslant 10$. For $d \geqslant d_0$, we have proved the monotonicity on the set of environments bounded by $\sigma < 1$, take σ tending to 1, this finishes the proof.

3 Proof of Theorem 1.3

The proof of Theorem 1.3 is based on that of Theorem 1.1.

3.1 The differentiability of the speed $v(\beta)$

In the proof of Theorem 1.1, Section 2.4.1 about the differentiability of $f(t)$ for $d \geqslant 8$, we consider $m = 1, \beta_1(y) = 0, \beta_2(y) = \beta_2, \beta_t = t\beta_2, t \in [0, 1]$ for all $y \in \mathbb{Z}^d$ and β_2 is constant in $(0, 1)$. The function $f(t)$ is difined by the couple of the environments β_1, β_2 so we denote $f_c(t)$ to be the function defined by $\beta_1 = 0, \beta_2 = c$ for some constant $c \in [0, 1)$. Then we have $v(\beta) = f_{\beta_2}(\frac{\beta}{\beta_2})$, moreover $f(t)$ is differentiable in $t \in [0, 1]$, this implies that $v(\beta)$ is differentiable in $\beta \in [0, \beta_2)$ for all $\beta_2 < 1$ i.e. it is differentiable in $[0,1)$ when $d \geqslant 8$.

We are now interested in proving the existence and computing the derivative at the critical point 0. By Lemma 2.9, with $N_n := d \sum_{j=0}^{n-1} 1_{Y_j \notin} 1_{Z_j = Z_{j+1}}$, we get

$$\frac{v(\beta)}{\beta} = \frac{1}{d} \frac{\mathbb{E}_0(N_T 1_{0 \in \mathcal{D}} M_T(\beta))}{\mathbb{E}_0(T 1_{0 \in \mathcal{D}})} = \frac{1}{d} \mathbb{E}_0(N_T 1_{0 \in \mathcal{D}} M_T(\beta)).$$

Note that

- $T 1_{0 \in \mathcal{D}} M_T(\beta) \geqslant 0$,
- $\lim_{\beta \to 0} (T 1_{0 \in \mathcal{D}} M_T(\beta)) = T 1_{0 \in \mathcal{D}}$,
- $\mathbb{E}_0(T 1_{0 \in \mathcal{D}} M_T(\beta)) = \mathbb{E}_\beta(T 1_{0 \in \mathcal{D}}) = \mathbb{E}_0(T 1_{0 \in \mathcal{D}}) = 1$ for $d \geqslant 6$.

Therefore, by Lemma 2.11, $\{T 1_{0 \in \mathcal{D}} M_T(\beta)\}_\beta$ is uniformly integrable in a neighborhood of 0. This is also true for $\{N_T 1_{0 \in \mathcal{D}} M_T(\beta)\}_{\beta \to 0}$ since $N_T \leqslant dT$. Therefore, we get

$$\lim_{\beta \to 0} \mathbb{E}_0(N_T 1_{0 \in \mathcal{D}} M_T(\beta)) = \mathbb{E}_0(N_T 1_{0 \in \mathcal{D}}).$$

On the other hand, with R_n is the range of the simple symmetric random walk on $\mathbb{Z}^d$ and denote $\{Y_i \notin\} := \{Y_i \notin \{Y_0, Y_1, ..., Y_{i-1}\}\}$ then

$$R(0) := \lim_{n \to \infty} \frac{R_n}{n} = \lim_{n \to \infty} \frac{R_{T_n}}{T_n} = \lim_{n \to \infty} \frac{R_{T_1} + (R_{T_2} - R_{T_1}) + ... + (R_{T_n} - R_{T_{n-1}})}{T_n}$$

$$= \lim_{n \to \infty} \frac{(1_{Y_0 \notin} + ... + 1_{Y_{T_1-1} \notin}) + (1_{Y_{T_1} \notin} + ... + 1_{Y_{T_2-1} \notin}) + ... + (1_{Y_{T_{n-1}} \notin} + ... + 1_{Y_{T_n-1} \notin})}{T_n}$$

$$= \frac{\mathbb{E}_0(R_T 1_{0 \in \mathcal{D}})}{\mathbb{E}_0(T 1_{0 \in \mathcal{D}})} = \mathbb{E}_0(R_T 1_{0 \in \mathcal{D}}), \text{(because } \mathbb{E}_0(T 1_{0 \in \mathcal{D}}) = \hat{\mathbb{E}}_0(T) \mathbb{P}_0(0 \in \mathcal{D}) = 1).$$

Similarly, with $N_n = d \sum_{j=0}^{n-1} 1_{Y_j \notin} 1_{Z_j = Z_{j+1}}$ then

$$\lim_{n \to \infty} \frac{N_n}{n} = \frac{\mathbb{E}_0(N_T 1_{0 \in \mathcal{D}})}{\mathbb{E}_0(T 1_{0 \in \mathcal{D}})} = \mathbb{E}_0(N_T 1_{0 \in \mathcal{D}}).$$

Note that

$$\mathbb{E}_0(N_n) = d \sum_{j=0}^{n-1} \mathbb{E}_0(1_{Y_j \notin} 1_{Z_j = Z_{j+1}}) = d \sum_{j=0}^{n-1} \mathbb{E}_0(1_{Y_j \notin}) \mathbb{P}_0(Z_j = Z_{j+1})$$

$$= \mathbb{E}_0 \left(\sum_{j=0}^{n-1} 1_{Y_j \notin} \right) = \mathbb{E}_0(R_n).$$

Therefore

$$R(0) := \lim_{n \to \infty} \frac{R_n}{n} = \lim_{n \to \infty} \mathbb{E}_0 \left(\frac{R_n}{n} \right) = \lim_{n \to \infty} \mathbb{E}_0 \left(\frac{N_n}{n} \right) = \mathbb{E}_0(N_T 1_{0 \in \mathcal{D}}).$$

3.2 Monotonicity of $v(\beta)$

In Section 2.4.2, we consider the particular case $m = 1$ and $\beta_1(y) = \beta_1, \beta_2(y) = \beta_2$ for all $y \in \mathbb{Z}^d$, where β_1 and β_2 are two constants in $[0, 1)$ such that $\beta_1 \leqslant \beta_2 \leqslant \sigma < 1$. By (2.28) and (EQ) we get that

$$\hat{E}(T) \frac{\partial}{\partial t} f(t) \geqslant \frac{1}{d}(\beta_2 - \beta_1) \left[P(0 \in \mathcal{D}) - \frac{\sigma}{d} \mathbb{Q} \mathbb{E}_{\beta_t} \left(\sum_{j \geqslant 1} j 1_{0 \in \mathcal{D}} 1_{T > j-1} \right) \right]$$

$$\geqslant \frac{1}{d}(\beta_2 - \beta_1)\left[P(0 \in \mathcal{D}) - \frac{\sigma}{d}E\left(\sum_{j\geqslant 1} j 1_{0\in\mathcal{D}} 1_{T>j-1}\right)\right]$$

$$\geqslant \frac{1}{d}(\beta_2 - \beta_1)\left[1 - \frac{\sigma}{d}\hat{E}\left(\frac{T^2+T}{2}\right)\right]. \tag{3.1}$$

Set $d_0 := \max\left\{\left\lfloor \sup_{d\geqslant 8}\hat{E}\left(\frac{T^2+T}{2}\right)\right\rfloor + 1, 8\right\}$ then $\frac{\partial}{\partial t}f(t) \geqslant 0$ i.e. f(t) is increasing in $t \in [0,1]$ and $v(\beta)$ is increasing in $\beta \in [0,1]$ when $d \geqslant d_0$ or $\sigma \leqslant \frac{8}{d_0}$ for all $d \geqslant 8$.

4 Proof of Theorem 1.2

For $m-$ERW, we denote the function $f(t)$ by $f_c(m,t)$ in the case of the couple environments such that $\beta_1 = 0, \beta_2 = c$ where c is a constant in $[0,1)$ and $\beta_t = tc, t \in [0,1]$.

Set

$$N_n^m = d\sum_{j=0}^{n-1} 1_{Y_j\notin^m} 1_{Z_j=Z_{j+1}}\,.$$

Then, from the formula (2.21) we get

$$\mathbb{E}_{m,\beta}(X_T 1_{0\in\mathcal{D}}) = \frac{\beta}{d}\mathbb{E}_{m,\beta}(N_T^m 1_{0\in\mathcal{D}}).$$

$m-$ERW is the particular case of $m-$ERW with i.i.d. random cookies, then the law of large numbers gives the following formula of the speed when $d \geqslant 6$:

$$v(m,\beta) = \frac{\mathbb{E}_{m,\beta}(X_T 1_{0\in\mathcal{D}})}{\mathbb{E}_{m,\beta}(T 1_{0\in\mathcal{D}})} = \frac{\beta}{d}\frac{\mathbb{E}_0(N_T^m 1_{0\in\mathcal{D}})}{\mathbb{E}_0(T 1_{0\in\mathcal{D}})}. \tag{4.1}$$

We see that $v(m,\beta) = f_c(m, \frac{\beta}{c})$ (where $t = \frac{\beta}{c}$), then

$$\frac{\partial v}{\partial \beta}(m,\beta) = \frac{\partial f_c}{\partial t}(m, \frac{\beta}{c}).\frac{1}{c}\,,$$

and combine with the formula (2.23) we obtain the derivative of the speed:

$$\frac{\partial v}{\partial \beta}(m,\beta) = \frac{1}{d}\frac{\mathbb{E}_0[N_T^m M_T^m(\beta) 1_{0\in\mathcal{D}}]}{\mathbb{E}_0(T 1_{0\in\mathcal{D}})} + \frac{\beta}{d}\frac{\mathbb{E}_0[N_T^m M_T^m(\beta) U_T^m(\beta) 1_{0\in\mathcal{D}}]}{\mathbb{E}_0(T 1_{0\in\mathcal{D}})}, \text{ for } \beta \in [0,1) \tag{4.2}$$

where

$$U_T^m(\beta) = \sum_{j=0}^{T-1} \frac{\mathcal{E}_j}{1+\beta\mathcal{E}_j} 1_{Y_j\notin^m\{Y_0,\dots Y_{j-1}\}} 1_{Z_j=Z_{j+1}}\,,$$

$$M_T^m(\beta) = \prod_{j=0}^{T-1}\left[1 + \varepsilon_j\beta 1_{Y_j\notin^m\{Y_0,\dots Y_{j-1}\}}\right]\,.$$

In order to prove the uniform convergence of $(\partial v/\partial \beta)(m,\beta)$ as m goes to $+\infty$, we use the following lemma, whose proof is given below:

Lemma 4.1. Let J be an interval of $\mathbb{R}$, and $\{X_n(\beta)\}_{\beta\in J, n\geqslant 1}$, $\{X(\beta)\}_{\beta\in J}$ be families of non-negative random variables. Assume that

1. for every n, $\{X_n(\beta)\}_{\beta\in J}$ is uniformly integrable,
2. $\{X(\beta)\}_{\beta\in J}$ is uniformly integrable,
3. $X_n(\beta)$ converges in probability to $X(\beta)$, uniformly in β: for any $\varepsilon > 0$,

$$\lim_{n\to+\infty}\sup_{\beta\in J}\mathbb{P}(|X_n(\beta) - X(\beta)| > \varepsilon) = 0\,.$$

Then, $\lim_{n \to +\infty} \sup_{\beta \in J} |\mathbb{E}(X_n(\beta)) - \mathbb{E}(X(\beta))| = 0$ if and only if $\{X_n(\beta)\}_{n \in \mathbb{N}, \beta \in J}$ is uniformly integrable.

Set

$$N_T^\infty = d \sum_{j=0}^{T-1} 1_{Z_j = Z_{j+1}}, \ U_T^\infty(\beta) = \sum_{j=0}^{T-1} \frac{\mathcal{E}_j}{1 + \beta \mathcal{E}_j} 1_{Z_j = Z_{j+1}}, \ M_T^\infty(\beta) = \prod_{j=0}^{T-1} (1 + \varepsilon_j \beta) .$$

One can check that the following inequalities hold: $\forall m \in \mathbb{N} \cup \{+\infty\}$, $\forall \beta \in [0, \beta_0)$ $(\beta_0 < 1)$,

$$N_T^m \le dT, \ M_T^m(\beta) \le 2^T, \ V_T^m(\beta) \le \frac{T}{1 - \beta_0},$$

$$|N_T^m - N_T^\infty| \le d(T - m)_+,$$

$$\sup_{\beta \in [0,1]} |M_T^m(\beta) - M_T^\infty(\beta)| \le 2^T (T - m)_+,$$

$$\sup_{\beta \in [0,\beta_0]} |V_T^m(\beta) - V_T^\infty(\beta)| \le \frac{1}{1 - \beta_0}(T - m)_+.$$

We deduce from these inequalities that $\sup_{\beta \in [0,1]} |N_T^m M_T^m(\beta) - N_T^\infty M_T^\infty(\beta)|$ converges a.s. to 0 when m tends to ∞. The same is true for

$$\sup_{\beta \in [0,\beta_0]} |N_T^m M_T^m(\beta) V_T^m(\beta) - N_T^\infty M_T^\infty(\beta) V_T^\infty(\beta)|.$$

Using Lemma 2.11, we can also show that for every $m \ge 1$ the family $\{T M_T^m(\beta) 1_{0 \in \mathcal{D}}\}_{\beta \in [0,1]}$ is uniformly integrable w.r.t. index β for $d \ge 6$. Indeed, it is a.s. continuous in β for every $m \ge 1$, and for $d \ge 6$,

$$\mathbb{E}_0(T M_T^m(\beta) 1_{0 \in \mathcal{D}}) = \mathbb{E}_{m,\beta}(T 1_{0 \in \mathcal{D}}) = \mathbb{E}_0(T 1_{0 \in \mathcal{D}}) = 1.$$

Since $N_T^m \le dT$, for every $m \ge 1$ the family $\{N_T^m M_T^m(\beta) 1_{0 \in \mathcal{D}}\}_{\beta \in [0,1]}$ is uniformly integrable for $d \ge 6$.

In the same way, Lemma 2.11 implies that for every $m \ge 1$ the family $\{T^2 M_T^m(\beta) 1_{0 \in \mathcal{D}}\}_{\beta \in [0,1]}$ is uniformly integrable for $d \ge 8$. Since $N_T^m \le T$ and $V_T^m(\beta) \le \frac{1}{1-\beta_0} T$ for $0 \le \beta \le \beta_0 < 1$, for every $m \ge 1$ the family $\{N_T^m V_T^m(\beta) M_T^m(\beta) 1_{0 \in \mathcal{D}}\}_{\beta \in [0,\beta_0]}$ is also uniformly integrable. To apply Lemma 4.1, it remains to prove that $\{N_T^\infty M_T^\infty(\beta) 1_{0 \in \mathcal{D}}\}_{\beta \in [0,1]}$, (resp. $\{N_T^\infty M_T^\infty(\beta) V_T^\infty(\beta) 1_{0 \in \mathcal{D}}\}_{\beta \in [0,1]}$) are uniformly integrable. This is true for $d \ge 6$ (resp. $d \ge 8$) using again Lemma 2.11.

By Lemma 4.1, we conclude that for $d \ge 8$, and $0 \le \beta_0 < 1$,

$$\lim_{m \to +\infty} \sup_{\beta \in [0,\beta_0]} \left| \frac{\partial v}{\partial \beta}(m, \beta) - \frac{\partial v}{\partial \beta}(\infty, \beta) \right| = 0.$$

Note that $\mathbb{P}_{\infty,\beta}$ is the law of simple random walk with drift β. Therefore, $v(\infty, \beta) = \beta/d$ and $(\partial v / \partial \beta)(\infty, \beta) = 1/d$, leading to the statement in Theorem 1.2. This in turn implies that for $d \ge 8$, for all $\beta_0 \in [0,1)$ there exists $m(\beta_0)$ such that for $m \ge m(\beta_0)$ the speed of ERW with m cookies is increasing in β on $[0, \beta_0]$.

To finish the proof of Theorem 1.2, we prove Lemma 4.1.

Proof of Lemma 4.1.

($\Leftarrow$) We prove the sufficiency. Since $\{X_n(\beta)\}_{n,\beta}$ and $\{X(\beta)\}_\beta$ are uniformly integrable, for all $\varepsilon > 0$, there exists c_0 such that for all $c \ge c_0$, we have:

$$\sup_{n,\beta} \mathbb{E}[X_n(\beta) 1_{X_n(\beta) \ge c}] < \varepsilon, \ \sup_\beta \mathbb{E}[X(\beta) 1_{X(\beta) \ge c}] < \varepsilon.$$

Therefore

$$|\mathbb{E}[X_n(\beta)] - \mathbb{E}[X(\beta)]| \tag{4.3}$$
$$\leqslant \varepsilon + \mathbb{E}[|X_n(\beta)|1_{|X_n(\beta)-X(\beta)|>\varepsilon}] + \mathbb{E}[|X(\beta)|1_{|X_n(\beta)-X(\beta)|>\varepsilon}]$$
$$\leqslant \varepsilon + \mathbb{E}[X_n(\beta)1_{X_n(\beta)\geqslant c_0}] + \mathbb{E}[X_n(\beta)1_{X_n(\beta)<c_0}1_{|X_n(\beta)-X(\beta)|>\varepsilon}]$$
$$\quad + \mathbb{E}[X(\beta)1_{X(\beta)\geqslant c_0}] + \mathbb{E}[X(\beta)1_{X(\beta)<c_0}1_{|X_n(\beta)-X(\beta)|>\varepsilon}]$$
$$\leqslant 3\varepsilon + 2c_0 \sup_\beta \mathbb{P}[|X_n(\beta) - X(\beta)| > \varepsilon]. \tag{4.4}$$

By assumption 3, we get that for all $\varepsilon > 0$,

$$\limsup_{n\to+\infty} \sup_\beta |\mathbb{E}[X_n(\beta)] - \mathbb{E}[X(\beta)]| \leqslant 3\varepsilon \,.$$

($\Rightarrow$) We prove now the necessity. For any $C > 0$,

$$\mathbb{E}(X_n(\beta) 1_{X_n(\beta)\geqslant C})$$
$$= \mathbb{E}(X_n(\beta) - X(\beta)) + \mathbb{E}(X(\beta) 1_{X(\beta)\geqslant C-1}) + \mathbb{E}(X(\beta) 1_{X(\beta)<C-1} - X_n(\beta) 1_{X_n(\beta)<C}) \,.$$

Using the positivity of $X_n(\beta)$, for any $\varepsilon \in (0,1)$,

$$X(\beta) 1_{X(\beta)<C-1} - X_n(\beta) 1_{X_n(\beta)<C}$$
$$\leqslant [X(\beta) - X_n(\beta)]1_{X(\beta)<C-1, X_n(\beta)<C} + X(\beta) 1_{X(\beta)<C-1}1_{|X_n(\beta)-X(\beta)|\geqslant\varepsilon}$$
$$\leqslant \varepsilon + 2C1_{|X_n(\beta)-X(\beta)|\geqslant\varepsilon} \,.$$

Therefore, for any $C > 0$ and any $\varepsilon \in (0,1)$,

$$\sup_\beta \mathbb{E}[X_n(\beta) 1_{X_n(\beta)\geqslant C}]$$
$$\leqslant \sup_\beta |\mathbb{E}[X_n(\beta) - X(\beta)]| + \sup_\beta \mathbb{E}[X(\beta) 1_{X(\beta)\geqslant C-1}] + \varepsilon + 2C \sup_\beta \mathbb{P}(|X_n(\beta) - X(\beta)| \geqslant \varepsilon) \,.$$

Taking the limit $n \to \infty$, then $\varepsilon \to 0$ leads to

$$\limsup_{n\to\infty} \sup_\beta \mathbb{E}[X_n(\beta) 1_{X_n(\beta)\geqslant C}] \leqslant \sup_\beta \mathbb{E}[X(\beta) 1_{X(\beta)\geqslant C-1}] \,. \tag{4.5}$$

Let $\varepsilon > 0$. Using the uniform integrability of the family $\{X(\beta)\}_\beta$, one can find $C_0(\varepsilon)$ such that $\sup_\beta \mathbb{E}[X(\beta) 1_{X(\beta)\geqslant C_0(\varepsilon)-1}] \leqslant \varepsilon$. By (4.5), there exists $n_0(\varepsilon)$ such that for all $n \geqslant n_0(\varepsilon)$,

$$\sup_\beta \mathbb{E}[X_n(\beta) 1_{X_n(\beta)\geqslant C_0(\varepsilon)}] \leqslant 2\varepsilon \,.$$

For $n < n_0(\varepsilon)$, we use the uniform integrability of the family $\{X_n(\beta)\}_\beta$ to get $C_1(\varepsilon)$ such that for any $C \geqslant C_1(\varepsilon)$, $\sup_{n\leqslant n_0(\varepsilon),\beta} \mathbb{E}[X_n(\beta)1_{X_n(\beta)>C}] < \varepsilon$. Now, choosing $C_2(\varepsilon) = \max\{C_0(\varepsilon), C_1(\varepsilon)\}$, we get $\sup_{n,\beta} \mathbb{E}[X_n(\beta)1_{X_n(\beta)>C}] < 2\varepsilon$ for all $C > C_2(\varepsilon)$.

$\square$

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