Excursions of excited random walks on integers

Elena Kosygina[*]   Martin P. W. Zerner[†]

Abstract

Several phase transitions for excited random walks on the integers are known to be characterized by a certain drift parameter $\delta \in \mathbb{R}$. For recurrence/transience the critical threshold is $|\delta| = 1$, for ballisticity it is $|\delta| = 2$ and for diffusivity $|\delta| = 4$. In this paper we establish a phase transition at $|\delta| = 3$. We show that the expected return time of the walker to the starting point, conditioned on return, is finite iff $|\delta| > 3$. This result follows from an explicit description of the tail behaviour of the return time as a function of δ, which is achieved by diffusion approximation of related branching processes by squared Bessel processes.

Keywords: branching process; cookie walk; diffusion approximation; excited random walk; excursion; squared Bessel process; return time; strong transience.
AMS MSC 2010: Primary 60G50; 60K37, Secondary 60F17; 60J70; 60J80; 60J85.
Submitted to EJP on July 25, 2013, final version accepted on January 21, 2014.

1 Introduction

A transient random walk (RW) is called *strongly transient* if the expectation of its return time R to the starting point, conditioned on the event $\{R < \infty\}$, is finite, see e.g. [Hug95, §3.2.6] and the references therein. The simple symmetric RW on $\mathbb{Z}^d$ is strongly transient iff $d \geq 5$, see [Hug95, §3.3.4, Table 3.4]. "Under fairly general conditions, biased walks are strongly transient" [Hug95, p. 127]. In the present paper we study the tail behavior of the depth and the duration of excursions of excited random walks (ERWs). In particular, we show that ERWs can be biased, in the sense of satisfying a strong law of large numbers with non-zero speed, and at the same time be not strongly transient. Precise statements are given later in this section after we describe our model of ERW. (For a recent survey on ERW we refer the reader to [KZ13].)

An ERW evolves in a so-called cookie environment. The latter is an element $\omega = (\omega(z,i))_{z \in \mathbb{Z}, i \geq 1}$ of $\Omega := [0,1]^{\mathbb{Z} \times \mathbb{N}}$. Given $\omega \in \Omega, z \in \mathbb{Z}$ and $i \in \mathbb{N}$ we call $\omega(z,i)$ the i-th cookie at site z and $\omega(z,\cdot)$ the stack of cookies at z. The cookie $\omega(z,i)$ serves as transition probability from z to $z + 1$ of the ERW upon its i-th visit to z. More precisely,

---
[*]Department of Mathematics, Baruch College, New York, USA.
  E-mail: elena.kosygina@baruch.cuny.edu
[†]Mathematisches Institut, Universität Tübingen, Tübingen, Germany.
  E-mail: martin.zerner@uni-tuebingen.de http://www.math.uni-tuebingen.de/user/zerner

given $\omega \in \Omega$ and $x \in \mathbb{Z}$ an ERW starting at x in the environment ω is a process $(X_n)_{n \geq 0}$ on a suitable probability space $(\Omega', \mathcal{F}', P_{x,\omega})$ which satisfies for all $n \geq 0$:

$$
\begin{aligned}
P_{x,\omega}[X_0 = x] &= 1, \\
P_{x,\omega}[X_{n+1} = X_n + 1 \,|\, (X_i)_{0 \leq i \leq n}] &= \omega(X_n, \#\{i \leq n \,|\, X_i = X_n\}), \qquad (1.1) \\
P_{x,\omega}[X_{n+1} = X_n - 1 \,|\, (X_i)_{0 \leq i \leq n}] &= 1 - \omega(X_n, \#\{i \leq n \,|\, X_i = X_n\}).
\end{aligned}
$$

The environment ω is chosen at random according to some probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$, where $\mathcal{F}$ is the canonical product Borel σ-field. Throughout the paper we assume that $\mathbb{P}$ satisfies the following hypotheses (IID), (WEL), and (BD$_M$) for some $M \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

$$\text{The family } (\omega(z, \cdot))_{z \in \mathbb{Z}} \text{ of cookie stacks is i.i.d. under } \mathbb{P}. \qquad \text{(IID)}$$

We denote the distribution of $\omega(0, \cdot)$ under $\mathbb{P}$ by ν, so that $\mathbb{P} = \bigotimes_{\mathbb{Z}} \nu$. To avoid degenerate cases we assume the following (weak) ellipticity hypothesis:

$$\mathbb{P}\left[\forall i \in \mathbb{N}:\ \omega(z, i) > 0\right] > 0, \mathbb{P}\left[\forall i \in \mathbb{N}:\ \omega(z, i) < 1\right] > 0 \quad \text{for all } z \in \mathbb{Z}. \qquad \text{(WEL)}$$

If we assumed only (IID) and (WEL) the model would include RWs in random i.i.d. environments (RWRE), since for them $\mathbb{P}$-a.s. $\omega(0, i) = \omega(0, 1)$ for all $i \geq 1$. However, for the ERW model considered in this paper we assume that there is a non-random $M \geq 0$ such that after M visits to any site the ERW behaves on any subsequent visit to that site like a simple symmetric RW:

$$\mathbb{P}\text{-a.s. } \omega(z, i) = 1/2 \text{ for all } z \in \mathbb{Z} \text{ and } i > M. \qquad \text{(BD}_M\text{)}$$

If we average the so-called *quenched* measure $P_{x,\omega}$ defined above over the environment ω we obtain the *averaged* (often also called *annealed*) measure $P_x[\cdot] := \mathbb{E}[P_{x,\omega}[\cdot]]$ on $\Omega \times \Omega'$. The expectation operators with respect to $P_{x,\omega}$, $\mathbb{P}$, and P_x are denoted by $E_{x,\omega}$, $\mathbb{E}$, and E_x, respectively.

Several features of the ERW can be characterized by the parameter

$$
\delta := \mathbb{E}\left[\sum_{i \geq 1} (2\omega(0, i) - 1)\right] = \mathbb{E}\left[\sum_{i=1}^{M} (2\omega(0, i) - 1)\right], \qquad (1.2)
$$

which represents the expected total average displacement of the ERW after consumption of all the cookies at any given site. Most prominently, the ERW $(X_n)_{n \geq 0}$

- is transient, i.e. tends P_0-a.s. to $\pm\infty$, iff $|\delta| > 1$ (see [KZ13, Th. 3.10] and the references therein),
- is ballistic, i.e. has P_0-a.s. a deterministic non-zero speed $\lim_{n \to \infty} X_n/n$, iff $|\delta| > 2$ (see [KZ13, Th. 5.2] and the references therein),
- converges after diffusive scaling under P_0 to a Brownian motion iff $|\delta| > 4$ or $\delta = 0$ (see [KZ13, Theorems 6.1, 6.3, 6.5, 6.7] and the references therein).

In this paper we establish a transition at $|\delta| = 3$. We are concerned with the finite excursions of ERWs. Let

$$R := \inf\{n \geq 1:\ X_n = X_0\}$$

be the time at which the ERW returns to its starting point. Denote for $k \in \mathbb{Z}$ the first passage time of k by

$$T_k := \inf\{n \geq 0: X_n = k\}.$$

Theorem 1.1 (Averaged excursion depth, duration, and return time)**.** *Let $\delta \in \mathbb{R}\backslash\{1\}$. There are constants $c_1(\nu), c_2(\nu), c_3(\nu) \in (0, \infty)$ such that*

$$\lim_{n\to\infty} n^{|\delta-1|} P_1[T_n < T_0 < \infty] \;=\; c_1, \tag{1.3}$$

$$\lim_{n\to\infty} n^{|\delta-1|/2} P_1[n < T_0 < \infty] \;=\; c_2, \tag{1.4}$$

$$\lim_{n\to\infty} n^{||\delta|-1|/2} P_0[n < R < \infty] \;=\; c_3. \tag{1.5}$$

Moreover, for $\delta = 1$ and every $\varepsilon > 0$,

$$\lim_{n\to\infty} n^{\varepsilon} P_1[T_n < T_0] = \lim_{n\to\infty} n^{\varepsilon} P_1[T_0 > n] = \lim_{n\to\infty} n^{\varepsilon} P_0[R > n] = \infty. \tag{1.6}$$

An immediate consequence of (1.5) and (1.6) is the following result.

Corollary 1.2 (Averaged strong transience)**.** $E_0[R, R < \infty] < \infty$ iff $|\delta| > 3$.

Remark 1.3 (Case $\delta = 1$)**.** Relations (1.6) are an easy consequence of (1.3)-(1.5) (see the proof in Section 6). We believe that for $\delta = 1$ the quantities $P_1[T_n < T_0], P_1[T_0 > n]$, and $P_0[R > n]$ have a logarithmic decay. In the special case described in Remark 2.2 below, the existence of a nontrivial limit of $(\ln n)P_1[T_n < T_0]$ as $n \to \infty$ follows from connections with branching processes with immigration and [Zub72, second part of (21)], see also [FYK90, Th. 1, part 2], quoted in [KZ08, Th. A (ii)].

Remark 1.4 (Once-excited RWs)**.** In the case of once-excited RWs with identical cookies (i.e. $M = 1$, $\mathbb{P}$-a.s. $\omega(z, 1) = \omega(0, 1) \in (0, 1)$ for all $z \in \mathbb{Z}$), results (1.3) and (1.4) have been obtained in [AR05, Section 3.3]. Note that the case $M = 1$ is very special, since at time T_k all the cookies $\omega(z, i) \neq 1/2$ between the starting point and the current location k of the ERW have been "eaten". This allows to use simple symmetric RW calculations between the starting point and k. For $M \geq 2$ such simplification is not available.

Problem 1.5. Find the analog of Theorem 1.1 in the quenched setting.

Problem 1.6. Find necessary and sufficient criteria under which averaged/quenched RWRE in one dimension is strongly transient.

Our approach is based on the connection between ERWs and a class of critical branching processes (BPs) with random migration (see Section 2 for details). It is close in spirit to the (second) Ray-Knight theorem (see, for example, [Tó96], where similar ideas were used for other types of self-interacting RWs). This approach was proposed for ERWs in [BS08] and, since then, seemed to dominate the study of one-dimensional ERWs under the (IID) assumption. The main benefits of this connection are:

(i) BPs associated to ERWs are markovian, while the original processes do not enjoy this property;

(ii) after rescaling, these BPs are well approximated by squared Bessel processes of generalized dimension.

From these diffusion approximations one can immediately conjecture such important properties of BPs as survival versus extinction, the tail asymptotics of the extinction time and of the total progeny (conditioned on extinction where appropriate). Rigorous proofs of these conjectures are somewhat technical, but, in a nutshell, they are based on standard martingale techniques applied to gambler-ruin-like problems.

Diffusion approximations for BPs associated to ERWs and the mentioned above martingale techniques were used in [KM11] to study the tail behavior of regeneration times of transient ERWs, which led to theorems about limit laws for these processes. In the

current work we extend some of the results and techniques of [KM11] and, in addition, apply the Doob transform to treat BPs conditioned on extinction. The results for conditioned BPs are then readily translated into the proof of Theorem 1.1.

While the majority of results in the BPs literature rely on a generating functions approach, diffusion approximation of BPs is also a well-developed subject. Its history goes back to [Fel51] (see [EK86, Ch. 9] for precise statements and additional references). But it seems that diffusion approximations for our kind of BPs are not available in the literature. Moreover, among a wealth of results (obtained by any approach) about conditioned BPs we could not find those which would cover our needs (but see the related work [Mel83] and the references therein).

We would like to point out one more aspect of the relationship between ERWs and BPs. At first (see, for example, [BS08], [KZ08]) there was a tendency to use known results for BPs to infer results about ERWs. Gradually, as we mentioned above, the study of ERWs required additional results about BPs, not covered by the literature. In [KM11] all BP results needed for ERWs were obtained directly. In this work we continue the trend. Theorem 5.1 gives asymptotics of the tails of extinction time and the total progeny of a class of critical BPs with random migration and geometric offspring distribution conditioned on extinction. We believe that this result might be of independent interest and that our methods are sufficiently robust to be applicable to more general critical BPs with random migration.

Let us now describe how the present article is organized. We close the introduction with some notation. In the next section we recall how excursions of ERWs are related to certain BPs. Section 3 deals with diffusion approximations of these BPs. In Section 4 we prove that BPs conditioned on extinction can be approximated by the diffusions from Section 3 conditioned on hitting zero. In Section 5 we use these results to obtain tail asymptotics of the extinction time and of the total progeny of BPs conditioned on extinction. Short Section 6 translates the obtained asymptotics into the proof of Theorem 1.1. In the Appendix we collect and extend as necessary several auxiliary results from the literature, which we quote throughout the paper and which do not depend on the results from Sections 3–6.

Notation. For any $I \subseteq [0, \infty)$ and $f : I \to \mathbb{R}$ we let $\sigma_y^f := \inf \{t \in I : f(t) \leq y\}$ and $\tau_y^f := \inf \{t \in I : f(t) \geq y\}$ be the entrance time of f into $(-\infty, y]$ and $[y, \infty)$, respectively. (Here $\inf \emptyset := \infty$.) If Z is a process with $P[\sigma_0^Z < \infty] > 0$ then we shall denote by $\overline{Z}$ any process which has the same distribution as Z under $P[\ \cdot \ | \ \sigma_0^Z < \infty]$. Whenever we have a Markov process starting at time 0 we may indicate the starting point by a subscript to the probability measure. Convergence in distribution is indicated by $\Rightarrow$. The space of real-valued càdlàg functions on $[0, \infty)$ is denoted by $D[0, \infty)$ and equipped with Skorokhod's J_1 topology. Convergence in distribution with respect to this topology is denoted by $\overset{J_1}{\Longrightarrow}$.

2 Excursions of RWs and branching processes

We recall a relationship between nearest neighbor paths from 1 to 0, representing RW excursions to the right, and BPs. Among the first descriptions of this relation is [Har52, Section 6]. We refer to [KZ08, Sections 3, 4] and [Pet13, Section 2.1] for detailed explanations in the context of ERW.

Assume that the nearest neighbor random walk $(X_n)_{n \geq 0}$ starts at $X_0 = 1$, set $U_0 := 1$ and let for $k \geq 1$,

$$U_k := \#\{n \geq 1 : \ n < T_0, \ X_{n-1} = k, \ X_n = k + 1\} \tag{2.1}$$

be the number of upcrossings from k to $k + 1$ by the walk before time T_0. If we set

$\Delta_0^{(k)} := 0$ then

$$\Delta_m^{(k)} := \inf \left\{ n \geq 1 : \Delta_{m-1}^{(k)} < n \leq T_0, \ X_{n-1} = k, X_n = k - 1 \right\}, \qquad k, m \geq 1,$$

is, if finite, the time of the completion of the m-th downcrossing from k to $k-1$ prior to T_0. Define

$$\zeta_m^{(k)} := \# \left\{ n \geq 1 : \Delta_{m-1}^{(k)} < n < \Delta_m^{(k)}, \ X_{n-1} = k, \ X_n = k + 1 \right\}, \qquad k, m \geq 1.$$

If $\Delta_m^{(k)}$ is finite $\zeta_m^{(k)}$ is the number of upcrossings from k to $k+1$ between the $(m-1)$-th and the m-th downcrossing from k to $k-1$ before T_0. Then U_{k+1} can be represented in BP form as

$$U_{k+1} = \sum_{m=1}^{U_k} \zeta_m^{(k+1)}.$$

Here $\zeta_m^{(k+1)}$ can be interpreted as the number of children of the m-th individual in the k-generation. The joint distribution of these numbers depends on the RW model under consideration. In the case of ERW it may be quite complicated, especially in the case where $T_0 = \infty$ with positive probability. Therefore, we study instead of U a slightly different BP V, the so-called *forward BP* described in the following statement.

Proposition 2.1 (Coupling of ERW and forward BP). *Suppose we are given $M \in \mathbb{N}$ and an ERW $X = (X_n)_{n \geq 0}$ which satisfies* (IID), (WEL) *and* (BD$_M$). *Then without loss of generality we may assume that there are, on the same probability space with averaged measure P_1, $\mathbb{N}_0$-valued random variables $\xi_m^{(k)}$, $m, k \geq 1$, which define a Markov chain $V = (V_k)_{k \geq 0}$ by $V_0 := 1$ and*

$$V_{k+1} := \sum_{m=1}^{V_k} \xi_m^{(k+1)}, \qquad k \geq 0, \tag{2.2}$$

such that the following holds:

The random quantities $(\xi_1^{(k)}, \ldots, \xi_M^{(k)}), \xi_m^{(k)}$ $(m > M, k \geq 1)$ are independent. (2.3)

The random vectors $(\xi_1^{(k)}, \ldots, \xi_M^{(k)})$ $(k \geq 1)$ are identically distributed, $\mathbb{N}_0^M$-valued, vanish with positive probability, and have a finite fourth moment. (2.4)

$$\sum_{m=1}^{M} (\xi_m^{(1)} - 1) \quad \text{has expected value } \delta \text{ (see (1.2)).} \tag{2.5}$$

The random variables $\xi_m^{(k)}$ $(m > M, k \geq 1)$ are geometrically distributed with parameter $1/2$ and expected value 1. (2.6)

$U_k \leq V_k \qquad \text{for all } k \geq 0 \text{ and}$ (2.7)

$U = V \qquad \text{on the event } \{\sigma_0^U < \infty\} \cup \{\sigma_0^V < \infty\},$ (2.8)

where U is defined by (2.1).

Proposition 2.1 follows from the so-called coin-toss construction of ERW described in [KZ08, Section 4], see also [Pet13, Section 2]. Note that the above conditions (2.2)–(2.6) do not completely characterize the distribution of V. For this statement (2.4) would have to be made stronger. However, we refrain from doing so, since the conditions (2.2)–(2.6) are the only ones we need for our proofs to work. (The moment condition in (2.4) is inherited from the proof of [KM11, Lemma 5.2] and could be relaxed.) Indeed, we only make the following assumptions on V.

Assumptions on the BP V. For the remainder of the paper we assume that the Markov chain V is defined by (2.2), where the offspring variables $\xi_m^{(k)}$, $m, k \geq 1$, satisfy (2.3)–(2.6).

Remark 2.2 (Cookies of strength 1 and BPs with immigration). In [KZ08, p. 1960] we describe how the above process V can be viewed as a BP with migration, i.e. emigration and immigration. If (IID) and (WEL) hold, but not necessarily (BD$_M$), and if there is $\mathbb{P}$-a.s. some random $K \in \mathbb{N} \cup \{\infty\}$ such that $\omega(0, i) = 1$ for all $1 \leq i < K$ and $\omega(0, i) = 1/2$ for all $i \geq K$ then one can couple the ERW in a way similar to the one described in Proposition 2.1 to a BP with immigration without emigration, see e.g. [Bau13, Section 3]. This kind of BP seems to be more tractable than BPs with immigration and emigration and several results are available in the BP literature which have direct implications for such ERWs. For example, the recurrence/transience phase transition in δ can be obtained from [Pak71, Th. 1] or [Zub72, Th. 3]. For other examples see Remarks 4.4 and 5.2.

Remark 2.3 (Other uses of the forward BP). The above mentioned relationship between excursions of RWs and BPs has been mainly used so far to translate results about BPs into results about RWs. The RW is then called the contour process associated to the BP. We list a few examples.
(a) Solomon's recurrence/transience theorem [Sol75, second part of Th. (1.7)] for RW in i.i.d. environment follows from results by Smith and Wilkinson [SW69] about the extinction of Galton-Watson processes in i.i.d. environment. (b) Similarly, the generalization of Solomon's result to RWs in stationary and ergodic environments by Alili [Ali99, Th. 2.1] can be deduced from the generalization of the above mentioned result of Smith and Wilkinson to Galton-Watson processes in stationary and ergodic environments obtained by Athreya and Karlin [AK71], see also [AN72, Ch. VI.5]. (c) In [Afa9, p. 268] this relationship is shown to imply that for recurrent RWRE $P_1[T_0 > n] \sim c/\log n$ as $n \to \infty$ for some constant $0 < c < \infty$. (d) In [KZ08, Th. 1] we used this correspondence and results from [FYK90] for a proof of the recurrence/transience result about ERW mentioned above, see also Corollary 7.10 below. (e) In [Bau13] and [Bau14] this connection allowed to determine how many cookies (of maximal value $\omega(x, i) = 1$) are needed to change the recurrence/transience behavior of RWRE. (f) And in [Pet13, Th. 1.7] strict monotonicity with respect to the environment of the return probability of a transient ERW is shown to be inherited from monotonicity properties of the forward BP.

Remark 2.4 (Backward BP). There is yet another family of branching processes associated to random walk paths, sometimes called the *backward BPs*, see [BS08], [KZ08, Section 6], [KM11], [KZ13, Th. 5.2], and [Pet13, Section 2.2].

We notice that all results of [KM11] about backward BPs have the corresponding analogs for forward BPs, which are obtained by replacing δ (which is assumed to be positive in [KM11]) with $1 - \delta < 1$ throughout. The proofs carry over essentially word for word without any additional changes. In what follows we simply quote such results. All additional results about forward BPs, in particular, for $\delta > 1$, are supplied with detailed proofs or comments as appropriate.

3 Diffusion approximation of unconditioned branching processes

The main result of this section is Theorem 3.4 about diffusion approximation of the process V. It extends [KM11, Lemma 3.1], which only considered the process V stopped at $\sigma_{\varepsilon n}^V$ with $\varepsilon > 0$.

The limiting processes are defined in terms of solutions of the stochastic differential equation (SDE)

$$dY(t) = \delta \, dt + \sqrt{2Y^+(t)} \, dB(t), \tag{3.1}$$

where $(B(t))_{t \geq 0}$ is a one-dimensional Brownian motion. For discussions of this particular SDE see e.g. [RW00, Ch. V.48] and [IW89, Example IV-8.2]. By [EK86, Th. 3.10, p.

299] the SDE (3.1) has a weak solution $Y = (Y(t))_{t \geq 0}$ for any initial distribution μ on $\mathbb{R}$ and any $\delta \in \mathbb{R}$. Due to the Yamada-Watanabe uniqueness theorem [YW71, Th. 1] (see also [RW00, Th. 40.1, p. 265]) pathwise uniqueness holds for (3.1). By [YW71, Prop. 1] (see also [EK86, Th. 3.6, p. 296]) distributional uniqueness holds as well. For $\delta \geq 0$, $2Y$ is a squared Bessel processes of dimension 2δ, see e.g. [RY99, Ch. XI, §1]. For $\delta < 0$, $2Y$ coincides with squared Bessel processes of negative dimension (see [GY03, Section 3]) up to time σ_0^Y and continues degenerately after time σ_0^Y since by the strong Markov property a.s. $Y(\sigma_0^Y + t) = \delta t$ for $t \geq 0, \delta < 0$.

In order to obtain the diffusion approximation we first introduce a modification $\widetilde{V}$ of the original process V and state in Proposition 3.2 a functional limit theorem for this process. The advantage of the process $\widetilde{V}$ is that it admits some nice martingales. Note that (2.2) can be rewritten as

$$V_{k+1} = V_k + \sum_{m=1}^{V_k} (\xi_m^{(k+1)} - 1).$$

This recursion is modified below in (3.2).

Lemma 3.1. *Let $x \in \mathbb{Z}$, $\widetilde{V}_0 := x$ and let ξ satisfy (2.3)–(2.6) under some probability measure P. Set $v := \operatorname{Var}\left[\sum_{m=1}^{M} \xi_m^{(1)}\right]$. For $k \in \mathbb{N}_0$ define*

$$\widetilde{V}_{k+1} \quad := \quad \widetilde{V}_k + \sum_{m=1}^{\widetilde{V}_k \vee M} (\xi_m^{(k+1)} - 1), \tag{3.2}$$

$$M_k \quad := \quad \widetilde{V}_k - k\delta, \quad and \tag{3.3}$$

$$A_k \quad := \quad vk + 2\sum_{m=0}^{k-1} (\widetilde{V}_m - M)^+. \tag{3.4}$$

Then $(M_k)_{k \geq 0}$ and $(M_k^2 - A_k)_{k \geq 0}$ are martingales with respect to the filtration $(\mathcal{F}_k)_{k \geq}$, where $\mathcal{F}_k$ is generated by $\xi_m^{(i)}, m \geq 1, 1 \leq i \leq k$.

Proof. By (2.4)–(2.6),

$$E\left[\sum_{m=1}^{i} (\xi_m^{(k+1)} - 1)\right] = \delta \quad \text{for all } i \geq M \text{ and } k \geq 0. \tag{3.5}$$

This implies the first statement. To find the Doob decomposition of the submartingale $(M_k^2)_{k \geq 0}$ we compute

$$E[M_{k+1}^2 - M_k^2 \mid \mathcal{F}_k] = E[(M_{k+1} - M_k + M_k)^2 - M_k^2 \mid \mathcal{F}_k]$$

$$= E[(M_{k+1} - M_k)^2 \mid \mathcal{F}_k] + 2M_k E[M_{k+1} - M_k \mid \mathcal{F}_k]$$

$$\overset{(3.3)}{=} E[(\widetilde{V}_{k+1} - \widetilde{V}_k - \delta)^2 \mid \mathcal{F}_k]$$

$$\overset{(3.2)}{=} E\left[\left(\left(\sum_{m=1}^{\widetilde{V}_k \vee M} \left(\xi_m^{(k+1)} - 1\right)\right) - \delta\right)^2 \,\middle|\, \mathcal{F}_k\right] \overset{(3.5)}{=} \operatorname{Var}\left[\sum_{m=1}^{\widetilde{V}_k \vee M} \left(\xi_m^{(k+1)} - 1\right) \,\middle|\, \mathcal{F}_k\right]$$

$$\overset{(2.3)}{=} \operatorname{Var}\left[\sum_{m=1}^{M} \left(\xi_m^{(k+1)} - 1\right)\right] + \sum_{m=M+1}^{\widetilde{V}_k \vee M} \operatorname{Var}\left[\xi_m^{(k+1)} - 1\right] \overset{(2.6)}{=} v + 2\left(\widetilde{V}_k - M\right)^+.$$

Recalling (3.4) we obtain the second claim. $\qquad \square$

Proposition 3.2. *Let $(x_n)_{n\geq 1}$ be a sequence of positive numbers which converges to $x > 0$, $\delta \in \mathbb{R}$, and ξ satisfy (2.3)–(2.6) under some probability measure P. For each $n \in \mathbb{N}$ define $\widetilde{V}_n = (\widetilde{V}_{n,k})_{k\geq 0}$ and $\widetilde{Y}_n = (\widetilde{Y}_n(t))_{t\geq 0}$ by setting $\widetilde{V}_{n,0} := \lfloor nx_n \rfloor$ and*

$$\widetilde{V}_{n,k+1} \quad := \quad \widetilde{V}_{n,k} + \sum_{m=1}^{\widetilde{V}_{n,k}\vee M} (\xi_m^{(k+1)} - 1) \qquad \text{for } k \geq 0 \text{ and}$$

$$\widetilde{Y}_n(t) \quad := \quad \frac{\widetilde{V}_{n,\lfloor nt \rfloor}}{n} \qquad \text{for } t \in [0,\infty).$$

Let $Y = (Y(t))_{t\geq 0}$ solve (3.1) with $Y(0) = x$. Then $\widetilde{Y}_n \overset{J_1}{\Longrightarrow} Y$ as $n \to \infty$.

Proof. We are going to apply [EK86, Th. 4.1, p. 354]. To check the assumptions of this theorem we first let μ be a distribution on $\mathbb{R}$ and consider the $C_{\mathbb{R}}[0,\infty)$ martingale problem for (A, μ) with $A = \{(f, Gf) : f \in C_c^\infty(\mathbb{R})\}$, where $Gf := (a/2)f'' + \delta f'$ and $a(x) := 2x^+$. This martingale problem is well posed due to [EK86, Cor. 3.4, p. 295] and our discussion after (3.1) concerning the existence and distributional uniqueness of solutions of (3.1).

Now define for each $n \in \mathbb{N}$, $(M_{n,k})_{k\geq 0}$ and $(A_{n,k})_{k\geq 0}$ in terms of $\widetilde{V}_{n,k}$ as in (3.3) and (3.4), respectively. For $t \in [0,\infty)$ set

$$M_n(t) := \frac{M_{n,\lfloor tn \rfloor}}{n}, \quad A_n(t) := \frac{A_{n,\lfloor tn \rfloor}}{n^2}, \quad B_n(t) := \frac{\lfloor tn \rfloor \delta}{n}.$$

We are now going to check conditions (4.1)-(4.7) of [EK86, Th. 4.1, p. 354]. By Lemma 3.1, M_n and $M_n^2 - A_n$ are martingales for all $n \in \mathbb{N}$, i.e. conditions (4.1) and (4.2) are satisfied. To verify the remaining conditions (4.3)–(4.7) we fix $r, T \in (0,\infty)$ and set $\tau_{n,r} := \inf\{t > 0 : |\widetilde{Y}_n(t)| \vee |\widetilde{Y}_n(t-)| \geq r\}$. To check condition (4.3), we have to show that

$$\lim_{n\to\infty} E\left[\sup_{t\leq T\wedge\tau_{n,r}} |\widetilde{Y}_n(t) - \widetilde{Y}_n(t-)|^2\right] = 0. \tag{3.6}$$

This is a consequence of (2.6) and the fact that the geometric distribution has exponential tails. More precisely,

$$E\left[\sup_{t\leq T\wedge\tau_{n,r}} |\widetilde{Y}_n(t) - \widetilde{Y}_n(t-)|^2\right] = \frac{1}{n^2} E\left[\max_{1\leq k\leq (Tn)\wedge\tau_{\lfloor rn \rfloor}^{\widetilde{V}_n}} \Big| \sum_{m=1}^{\widetilde{V}_{n,k-1}\vee M} (\xi_m^{(k)} - 1)\Big|^2\right]$$

$$\leq \frac{2}{n^2} E\left[\max_{1\leq k\leq Tn} \Big| \sum_{m=1}^{M} (\xi_m^{(k)} - 1)\Big|^2 + \max_{1\leq k\leq (Tn)\wedge\tau_{\lfloor rn \rfloor}^{\widetilde{V}_n}} \Big| \sum_{m=M+1}^{\widetilde{V}_{n,k-1}} (\xi_m^{(k)} - 1)\Big|^2\right]$$

$$\leq \frac{2T}{n} E\left[\Big| \sum_{m=1}^{M} (\xi_m^{(0)} - 1)\Big|^2\right] + \frac{2}{n^2} E\left[\max_{1\leq k\leq Tn} \max_{M+1\leq j\leq rn} \Big| \sum_{m=M+1}^{j} (\xi_m^{(k)} - 1)\Big|^2\right].$$

The first term in the last line goes to 0 as $n \to \infty$ and the second term is equal to

$$\frac{2}{n^2} \sum_{y\geq 0} P\left[\max_{1\leq k\leq Tn} \max_{M+1\leq j\leq rn} \Big| \sum_{m=M+1}^{j} (\xi_m^{(k)} - 1)\Big|^2 > y\right]$$

$$\leq \frac{2r^{3/2}}{n^{1/2}} + \frac{2}{n^2} \sum_{y>(rn)^{3/2}} P\left[\max_{1\leq k\leq Tn} \max_{M+1\leq j\leq rn} \Big| \sum_{m=M+1}^{j} (\xi_m^{(k)} - 1)\Big|^2 > y\right].$$

The first term in the last line vanishes as $n \to \infty$. Applying the union bound and Lemma 7.4 to the last probability we find that the second term does not exceed

$$4rT \sum_{y > (rn)^{3/2}} e^{-y/(6(rn \vee \sqrt{y}))} \leq 4rT \sum_{y > (rn)^{3/2}} e^{-y^{1/3}/6} \to 0 \quad \text{as} \ \ n \to \infty.$$

This finishes the proof of (3.6). Conditions (4.4) and (4.6) of [EK86, Th. 4.1, p. 354] (with $b \equiv \delta$) hold obviously. Condition (4.5) is fulfilled, since

$$\sup_{t \leq T \wedge \tau_{n,r}} |A_n(t) - A_n(t-)| \leq \frac{v + 2(nr + M)}{n^2}.$$

For (4.7) we consider for all $t \leq T \wedge \tau_{n,r}$,

$$\left| A_n(t) - 2 \int_0^t \widetilde{Y}_n^+(s) \, ds \right|$$

$$= \left| \frac{v \lfloor tn \rfloor}{n^2} + \frac{2}{n^2} \sum_{m=0}^{\lfloor tn \rfloor - 1} (\widetilde{V}_{n,m} - M)^+ - \frac{2}{n} \int_0^{\frac{\lfloor tn \rfloor}{n}} \widetilde{V}_{n,\lfloor sn \rfloor}^+ \, ds - 2 \int_{\frac{\lfloor tn \rfloor}{n}}^t Y_n^+(s) \, ds \right|$$

$$\leq \frac{vt}{n} + \frac{2}{n^2} \sum_{m=0}^{\lfloor tn \rfloor - 1} \left| (\widetilde{V}_{n,m} - M)^+ - \widetilde{V}_{n,m}^+ \right| + \frac{2}{n} \sup_{s < t} Y_n^+(s) \ \leq \ \frac{(v + 2M)T + 2r}{n},$$

which does not depend on t and converges to 0 as $n \to \infty$. Thus, (4.7) holds as well. The proposition follows now from [EK86, Th. 4.1, p. 354]. $\qquad \square$

To be able to apply the continuous mapping theorem to Proposition 3.2 we need the following statement. For every $f \in D[0, \infty)$ and $y \in \mathbb{R}$ let

$$\varphi_y(f) := f(\cdot \wedge \sigma_y^f) \tag{3.7}$$

be the function f stopped upon entering $(-\infty, y]$.

Lemma 3.3. *Let $\delta \in \mathbb{R}$, $0 < \varepsilon < x < \infty$, and let ψ be any of the following three mappings defined on $D[0, \infty)$:*

$$f \mapsto \sigma_\varepsilon^f \in [0, \infty], \quad f \mapsto \varphi_\varepsilon(f) \in D[0, \infty), \quad f \mapsto \int_0^{\sigma_\varepsilon^f} f^+(s) \, ds \in [0, \infty].$$

Denote by $\mathrm{Cont}(\psi) := \{f \in D[0, \infty) : \psi \text{ is continuous at } f\}$ the set of continuity points of ψ. Then the solution Y of (3.1) satisfies $P_x[Y \in \mathrm{Cont}(\psi)] = 1$.

Proof. For $0 < \varepsilon < x < \infty$ let

$$F \quad := \quad \left\{ f \in C[0, \infty) \ \middle| \ f(0) = x, \quad \sigma_\varepsilon^f < \infty \Rightarrow f \text{ has no local minimum at } \sigma_\varepsilon^f \right\}.$$

Then under the conditions of the lemma $P_x[Y \in F] = 1$. Indeed, it follows from the strong Markov property and [RW00, Lemma (46.1) (i), p. 273] that Y a.s. does not have a local minimum at σ_ε^Y.

Consequently, it suffices to show that $F \subseteq \mathrm{Cont}(\psi)$. For $\psi = \sigma_\varepsilon^{\cdot}$ and $\psi = \varphi_\varepsilon$ this follows from [JS87, Ch. VI, Prop. 2.11] and [JS87, Ch. VI, Prop. 2.12], respectively. (In the notation of [JS87, Ch. VI, 2.9], $\sigma_\varepsilon^f = S_a(\alpha)$ with $\alpha := e^{-f}$ and $a := e^{-\varepsilon}$. Note that $f \mapsto e^{-f}$ is continuous w.r.t. the J_1-topology.)

For $\psi(f) = \int_0^{\sigma_\varepsilon^f} f^+(s) \, ds$, we assume that $D[0, \infty) \ni f_n \xrightarrow{J_1} f \in F$. We need to show that $\psi(f_n) \to \psi(f)$. Since $\sigma_\varepsilon^{\cdot}$ is continuous, as shown above, $\sigma_\varepsilon^{f_n} \to \sigma_\varepsilon^f$. If $\sigma_\varepsilon^f < \infty$ then

$\sigma_\varepsilon^{f_n} < \sigma_\varepsilon^f + 1 =: T$ for n large, and hence $|\psi(f_n) - \psi(f)| \le T \sup_{t \in [0,T]} |f_n(t) - f(t)|$, which converges to 0 as $n \to \infty$, see e.g. [JS87, Ch. VI, Prop. 1.17b]. If $\sigma_\varepsilon^f = \infty$, then for any $T < \infty$, $\sigma_\varepsilon^{f_n} \ge T$ for n large and thus

$$\psi(f_n) \ge \int_0^T f_n^+(s) \, ds \xrightarrow[n \to \infty]{} \int_0^T f^+(s) \, ds \xrightarrow[T \to \infty]{} \int_0^\infty f^+(s) \, ds = \infty$$

since $f(s) > \varepsilon$ for all $s \ge 0$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Theorem 3.4 (Convergence of unconditioned processes). *Let $(x_n)_{n \ge 1}$ be a sequence of positive numbers which converges to $x > 0$, let $\delta \in \mathbb{R} \backslash \{1\}$, and assume (2.3)–(2.6) under some probability measure P. For each $n \in \mathbb{N}$ define $V_n = (V_{n,k})_{k \ge 0}$ and $Y_n = (Y_n(t))_{t \ge 0}$ by setting $V_{n,0} := \lfloor n x_n \rfloor$ and*

$$
\begin{aligned}
V_{n,k+1} &:= \sum_{m=1}^{V_{n,k}} \xi_m^{(k+1)} &&\text{for } k \ge 0 \text{ and} \\
Y_n(t) &:= \frac{V_{n, \lfloor nt \rfloor}}{n} &&\text{for } t \in [0, \infty).
\end{aligned}
$$

Let Y be a solution of (3.1) with $Y(0) = x$. Then $Y_n \overset{J_1}{\Longrightarrow} Y\left(\cdot \wedge \sigma_0^Y\right)$ as $n \to \infty$.

Proof. Let $\widetilde{V}_n$ and $\widetilde{Y}_n$ be defined as in Proposition 3.2, where $\widetilde{V}_{n,0} = V_{n,0}$ for all n. We denote by d_∞° the J_1-metric on $D[0, \infty)$ as defined in [Bil99, (16.4)].

We first consider the case $\delta > 1$. In this case $P_x[\sigma_0^Y = \infty] = 1$ and, hence, $Y = Y\left(\cdot \wedge \sigma_0^Y\right)$ a.s., see e.g. [RW00, (48.5) (i), p. 286]. Moreover, on the event $\{\sigma_M^{V_n} = \infty\}$ we have $\widetilde{V}_n = V_n$ and thus $\widetilde{Y}_n = Y_n$. Consequently, we have for every $\varepsilon > 0$, $P\left[d_\infty^\circ\left(\widetilde{Y}_n, Y_n\right) > \varepsilon\right] \le P\left[\sigma_M^{V_n} < \infty\right] \to 0$ as $n \to \infty$ due to Corollary 7.9. Consequently, $d_\infty^\circ\left(\widetilde{Y}_n, Y_n\right)$ converges in distribution to 0 as $n \to \infty$. Therefore, by Proposition 3.2 and [Bil99, Th. 3.1] (a "convergence together" theorem), $Y_n \overset{J_1}{\Longrightarrow} Y$ as $n \to \infty$. This completes the proof in case $\delta > 1$.

Now we consider the case $\delta < 1$. Recall (3.7). Our first goal is to show that

$$\varphi_0\left(\widetilde{Y}_n\right) \overset{J_1}{\Longrightarrow} \varphi_0(Y) \qquad \text{as } n \to \infty. \tag{3.8}$$

We aim to use [Bil99, Th. 3.2], quoted as Lemma 7.1 in the Appendix, for this purpose. First observe that for all $m \in \mathbb{N}$, $\varphi_{1/m}(\widetilde{Y}_n) \overset{J_1}{\Longrightarrow} \varphi_{1/m}(Y)$ as $n \to \infty$ due to Proposition 3.2, Lemma 3.3 and the continuous mapping theorem. Moreover, $\varphi_{1/m}(Y) \Longrightarrow \varphi_0(Y)$ as $m \to \infty$ since Y has a.s. continuous paths. For the proof of (3.8) it therefore suffices to show, due to Lemma 7.1, that

$$\lim_{m \to \infty} \limsup_{n \to \infty} P\left[d_\infty^\circ\left(\varphi_{1/m}\left(\widetilde{Y}_n\right), \varphi_0\left(\widetilde{Y}_n\right)\right) > 2\varepsilon\right] = 0 \quad \text{for every } \varepsilon > 0. \tag{3.9}$$

For the proof of (3.9) we use [Bil99, (12.16)] and see that for all $\varepsilon > 0$, $n \in \mathbb{N}$, and $y \in (0, x \wedge \varepsilon)$,

$$
\begin{aligned}
P\left[d_\infty^\circ\left(\varphi_y\left(\widetilde{Y}_n\right), \varphi_0\left(\widetilde{Y}_n\right)\right) > 2\varepsilon\right] &\le P\left[\left\|\varphi_y\left(\widetilde{Y}_n\right) - \varphi_0\left(\widetilde{Y}_n\right)\right\|_\infty > 2\varepsilon\right] \\
&\le P\left[\sigma_0^{\widetilde{V}_n} < \infty, \widetilde{V}_{n, \sigma_0^{\widetilde{V}_n}} < -n\varepsilon\right] + P\left[\sup\left\{\widetilde{Y}_n(s) : \sigma_y^{\widetilde{Y}_n} \le s \le \sigma_0^{\widetilde{Y}_n}\right\} > \varepsilon\right] \tag{3.10}
\end{aligned}
$$

The first term in (3.10) is 0 for large enough n since $\widetilde{V}_{n, \sigma_0^{\widetilde{V}}} \ge -M$. The second term is $\le P_{\lfloor yn \rfloor}\left[\tau_{\varepsilon n}^{\widetilde{V}} < \sigma_0^{\widetilde{V}}\right]$. Lemma 7.8 now yields (3.8) if we choose $y = 1/m$. If we choose

$y = M/n$ then the above estimate and Lemma 7.8 give that

$$d_\infty^\circ \left(\varphi_{M/n} \left(\widetilde{Y}_n \right), \varphi_0 \left(\widetilde{Y}_n \right) \right) \to 0 \quad \text{in distribution as } n \to \infty. \tag{3.11}$$

Consequently, by (3.8) and [Bil99, Th. 3.1], $\varphi_{M/n} \left(\widetilde{Y}_n \right) \overset{J_1}{\Longrightarrow} \varphi_0(Y)$ as $n \to \infty$. However, recall that $\widetilde{V}_{n,k} = V_{n,k}$ for all $0 \leq k \leq \sigma_M^{V_n}$ and therefore $\varphi_{M/n} \left(\widetilde{Y}_n \right) = \varphi_{M/n} (Y_n)$. Hence, $\varphi_{M/n} (Y_n) \overset{J_1}{\Longrightarrow} \varphi_0(Y)$ as $n \to \infty$. As in (3.11), $d_\infty^\circ(\varphi_{M/n}(Y_n), \varphi_0(Y_n))$ tends to 0 in distribution as $n \to \infty$. The claim for $\delta < 1$ now follows from another application of [Bil99, Th. 3.1]. (Note that $\varphi_0 (Y_n) = Y_n$ since 0 is absorbing for V.) $\qquad \square$

4 Diffusion approximation of conditioned branching processes

The main result of this section is the following. Recall that $\overline{V}$ is obtained from V by conditioning on $\{\sigma_0^V < \infty\}$. In particular, by Corollary 7.9, $\overline{V} = V$ if $\delta < 1$.

Theorem 4.1 (Convergence of conditioned processes). *Assume the conditions of Theorem 3.4 and let* $\overline{Y} = (\overline{Y}(t))_{t \geq 0}$ *be a solution of*

$$d\overline{Y}(t) = (1 - |\delta - 1|) \, dt + \sqrt{2\overline{Y}^+(t)} \, dB_t, \qquad \overline{Y}(0) = x. \tag{4.1}$$

Then as $n \to \infty$,

$$\overline{Y}_n \quad \overset{J_1}{\Longrightarrow} \quad \overline{Y} \left(\cdot \wedge \sigma_0^{\overline{Y}} \right), \tag{4.2}$$

$$\sigma_0^{\overline{Y}_n} \quad \Longrightarrow \quad \sigma_0^{\overline{Y}}, \quad \text{and} \tag{4.3}$$

$$\int_0^{\sigma_0^{\overline{Y}_n}} \overline{Y}_n(s) \, ds \quad \Longrightarrow \quad \int_0^{\sigma_0^{\overline{Y}}} \overline{Y}(s) \, ds. \tag{4.4}$$

The (harmonic) function h defined by

$$h(n) := P_n[\sigma_0^V < \infty], \qquad n \in \mathbb{N}_0, \tag{4.5}$$

will play an important role in the proof of Theorem 4.1. Recall that according to our notation $V(0) = n$ under P_n. Then it follows from (2.2) that $h(n)$ is non-increasing in n.

Remark 4.2 (Doob transform). Recall that $\overline{V}$ is Doob's h-transform of V with h as defined in (4.5), see e.g. [LPW09, Ch. 17.6.1]. By this we mean that $\overline{V}$ is a Markov chain with transition probabilities $P_x[\overline{V}_n = y] = P_x[V_n = y]\frac{h(y)}{h(x)}$. More generally, it follows from the strong Markov property, that for any stopping time $\sigma \leq \sigma_0^V$ and all $x, y \in \mathbb{N}_0$,

$$P_x[\overline{V}_\sigma = y] = \frac{P_x[V_\sigma = y, \sigma_0^V < \infty]}{P_x[\sigma_0^V < \infty]} = P_x[V_\sigma = y]\frac{h(y)}{h(x)}. \tag{4.6}$$

In many cases a Doob transform of a process belongs to the same class of processes as the process itself. For example, the asymmetric simple RW on $\mathbb{N}_0$ with probability $p \in (1/2, 1)$ of stepping to the right, start at 1 and absorption at 0 is, conditioned on hitting 0, an asymmetric simple RW on $\mathbb{N}_0$ with probability p of stepping to the left and absorption at 0. (In this case $h(x) = ((1-p)/p)^x$ for $x \geq 0$.) Similarly, a supercritical Galton-Watson process conditioned on extinction is a subcritical Galton-Watson process, see e.g. [AN72, Ch. I.12, Th. 3]. Moreover, with an appropriate definition, squared Bessel processes of dimension $d > 2$ conditioned on hitting 0 are squared Bessel processes of dimension $4 - d$.

Similarly, if $\overline{X}$, i.e. X conditioned on hitting 0, were under P_1 an ERW satisfying (IID), (WEL) and (BD$_{\overline{M}}$) for some $\overline{M}$, or if $\overline{V}$ were of the form described in Proposition 2.1, then Theorem 4.1 would follow from Theorem 3.4. However, we do not expect that the conditioned processes $\overline{X}$ and $\overline{V}$ are of this form on the microscopic level. Theorem 4.1 shows that, nevertheless, on a macroscopic scale $\overline{V}$ does behave as V with drift parameter $\overline{\delta} = 1 - |\delta - 1|$.

First we investigate the tail behavior of h.

Proposition 4.3 (Asymptotics of h). *Let $\delta > 1$. Then there is $c_4 \in (0, \infty)$ such that*

$$\lim_{n \to \infty} n^{\delta-1} h(n) = c_4. \tag{4.7}$$

Remark 4.4. In the special case described in Remark 2.2, formula (4.7) also follows from [Pak72, Th. 4], see also the discussion in [Höp85, pp. 921–922].

Proof of Proposition 4.3. The proof is similar to that of [KM11, Lemma 8.1]. Let $\delta > 1$. First we show that it suffices to prove that

$$g(a) := \lim_{m \to \infty} a^{m(\delta-1)} h(\lfloor a^m \rfloor) \in (0, \infty) \quad \text{for all } a \in (1, 2]. \tag{4.8}$$

Let $a \in (1, 2]$ and denote $m_n := \lfloor \log_a n \rfloor$ for $n \in \mathbb{N}$. Then, by monotonicity of h,

$$a^{m_n(\delta-1)} h\left(\lfloor a^{m_n+1} \rfloor\right) \le n^{\delta-1} h(n) \le a^{(m_n+1)(\delta-1)} h\left(\lfloor a^{m_n} \rfloor\right)$$

for all $n \in \mathbb{N}$ and, hence, by (4.8),

$$a^{1-\delta} g(a) \le \liminf_{n \to \infty} n^{\delta-1} h(n) \le \limsup_{n \to \infty} n^{\delta-1} h(n) \le a^{\delta-1} g(a).$$

Since $0 < g(a) < \infty$ this implies

$$1 \le \frac{\limsup_{n \to \infty} n^{\delta-1} h(n)}{\liminf_{n \to \infty} n^{\delta-1} h(n)} \le a^{2(\delta-1)}.$$

Letting $a \searrow 1$ proves the claim of the proposition.

It remains to show (4.8). Fix $a \in (1, 2]$, $\lambda \in (0, 1/8)$, and choose ℓ_0 according to Lemma 7.7. Then for all $m > \ell \ge \ell_0$,

$$
\begin{aligned}
a^{m(\delta-1)} h(\lfloor a^m \rfloor) &= a^{m(\delta-1)} P_{\lfloor a^m \rfloor}[\sigma_{a^\ell}^V \le \sigma_0^V < \infty] \\
&\ge \quad a^{m(\delta-1)} P_{\lfloor a^m \rfloor}[\sigma_{a^\ell}^V < \infty] P_{\lfloor a^\ell \rfloor}[\sigma_0^V < \infty] \\
&\stackrel{(7.9)}{\ge} \quad a^{(m-\ell)(\delta-1)} h_\ell^-(m) a^{\ell(\delta-1)} h\left(\lfloor a^\ell \rfloor\right) \stackrel{(7.8)}{\ge} K_1(\ell) a^{\ell(\delta-1)} h\left(\lfloor a^\ell \rfloor\right) > 0.
\end{aligned}
$$

Hence, since $K_1(\ell) \to 1$,

$$\liminf_{m \to \infty} a^{m(\delta-1)} h(\lfloor a^m \rfloor) \ge \limsup_{\ell \to \infty} a^{\ell(\delta-1)} h(\lfloor a^\ell \rfloor) > 0,$$

which establishes the existence of $g(a) > 0$. To rule out $g(a) = \infty$ observe that

$$
\begin{aligned}
a^{m(\delta-1)} h(\lfloor a^m \rfloor) &\le \quad a^{m(\delta-1)} P_{\lfloor a^m \rfloor}\left[\sigma_{a^{\ell_0}}^V < \infty\right] \stackrel{(7.9)}{\le} a^{(m-\ell_0)(\delta-1)} h_{\ell_0}^+(m) a^{\ell_0(\delta-1)} \\
&\stackrel{(7.8)}{\le} \quad K_2(\ell_0) a^{\ell_0(\delta-1)} < \infty
\end{aligned}
$$

for all $m > \ell_0$. $\qquad\square$

Lemma 4.5. *Let $\delta \in \mathbb{R}\backslash\{1\}$. Then there is $c_5 \in (0,1)$ such that for all $k, x \in \mathbb{N}$ and $n \geq 0$,*

$$P_n \left[\# \left\{ i \in \left\{ 1, \ldots, \sigma_0^{\overline{V}} \right\} : \overline{V}_i \in [x, 2x] \right\} > 2xk \right] \leq P_n \left[\rho_0 < \sigma_0^{\overline{V}} \right] c_5^k, \tag{4.9}$$

where $\rho_0 := \inf\{i \geq 0 : \overline{V}_i \in [x, 2x]\}$.

Proof. For $\delta < 1$, where $\overline{V} = V$, this is the statement of [KM11, Prop. 6.1].

For the case $\delta > 1$ we slightly modify the proof of [KM11, Prop. 6.1] as follows. First we show that there is $c_6 > 0$ such that for all $x \in \mathbb{N}$,

$$\min_{x \leq z < 2x} P_z \left[\sigma_{x/2}^{\overline{V}} \leq x \right] > c_6. \tag{4.10}$$

By the strong Markov property and monotonicity with respect to the starting point we have for all $2M \leq x \leq z < 2x$,

$$
\begin{aligned}
P_z \left[\sigma_{x/2}^{\overline{V}} \leq x \right] &= \frac{P_z[\sigma_{x/2}^V \leq x, \sigma_0^V < \infty]}{h(z)} \geq \frac{h(\lceil x/2 \rceil)}{h(z)} P_z[\sigma_{x/2}^V \leq x] \\
&\geq P_z[\sigma_{x/2}^V \leq x] \geq P_{2x}[\sigma_{x/2}^V \leq x] = P_{2x}[\sigma_{x/2}^{\widetilde{V}} \leq x].
\end{aligned}
$$

The last expression is strictly positive for all $x > 0$ and converges due to Proposition 3.2 and Lemma 3.3 as $x \to \infty$ to $P_2[\sigma_{1/2}^Y < 1] > 0$, where Y solves the SDE (3.1). This proves (4.10).

Next we show that there is $c_7 > 0$ such that for all $x \in \mathbb{N}$,

$$\min_{0 \leq z \leq x/2} P_z \left[\sigma_0^{\overline{V}} < \tau_x^{\overline{V}} \right] > c_7. \tag{4.11}$$

For the proof of (4.11) note that by the strong Markov property and monotonicity of h for all $0 \leq z \leq x/2, z \in \mathbb{N}_0$,

$$
\begin{aligned}
P_z \left[\tau_x^{\overline{V}} < \sigma_0^{\overline{V}} \right] &= \frac{P_z[\tau_x^V < \sigma_0^V < \infty]}{h(z)} \leq \frac{h(x)}{h(z)} P_z[\tau_x^V < \sigma_0^V] \\
&\leq \frac{h(x)}{h(z)} \leq \frac{h(x)}{h(\lfloor x/2 \rfloor)} \leq 2^{1-\delta} \frac{x^{\delta-1} h(x)}{\lfloor x/2 \rfloor^{\delta-1} h(\lfloor x/2 \rfloor)},
\end{aligned}
$$

which converges due to Proposition 4.3 to $2^{1-\delta} < 1$ as $x \to \infty$. Since the left hand side of (4.11) is strictly positive for all x this implies (4.11).

Now define $\rho_k := \inf\{n \geq \rho_{k-1} + 2x : \overline{V}_n \in [x, 2x]\}$ for all $k \in \mathbb{N}$. Then the left hand side of (4.9) is less than or equal to

$$P_n \left[\rho_k < \sigma_0^{\overline{V}} \right] = P_n \left[\rho_k < \sigma_0^{\overline{V}} \mid \rho_{k-1} < \sigma_0^{\overline{V}} \right] P_n \left[\rho_{k-1} < \sigma_0^{\overline{V}} \right]. \tag{4.12}$$

By the strong Markov property for $\overline{V}$,

$$
\begin{aligned}
P_n \left[\rho_k < \sigma_0^{\overline{V}} \mid \rho_{k-1} < \sigma_0^{\overline{V}} \right] &\leq \max_{x \leq z < 2x} P_z \left[\rho_1 < \sigma_0^{\overline{V}} \right] \\
&= \max_{x \leq z < 2x} \left(P_z \left[\rho_1 < \sigma_0^{\overline{V}}, \sigma_{x/2}^{\overline{V}} \leq x \right] + P_z \left[\rho_1 < \sigma_0^{\overline{V}}, \sigma_{x/2}^{\overline{V}} > x \right] \right) \\
&\leq \max_{x \leq z < 2x} \left(P_z \left[\rho_1 < \sigma_0^{\overline{V}} \mid \sigma_{x/2}^{\overline{V}} \leq x \right] P_z \left[\sigma_{x/2}^{\overline{V}} \leq x \right] + 1 - P_z \left[\sigma_{x/2}^{\overline{V}} \leq x \right] \right) \\
&= \max_{x \leq z < 2x} \left(1 - P_z \left[\sigma_{x/2}^{\overline{V}} \leq x \right] \left(1 - P_z \left[\rho_1 < \sigma_0^{\overline{V}} \mid \sigma_{x/2}^{\overline{V}} \leq x \right] \right) \right) \\
&= 1 - \min_{x \leq z \leq 2x} \left(P_z \left[\sigma_{x/2}^{\overline{V}} \leq x \right] P_z \left[\rho_1 > \sigma_0^{\overline{V}} \mid \sigma_{x/2}^{\overline{V}} \leq x \right] \right) \\
&\leq 1 - \left(\min_{x \leq z < 2x} P_z \left[\sigma_{x/2}^{\overline{V}} \leq x \right] \right) \left(\min_{0 \leq y \leq x/2} P_y \left[\rho_0 > \sigma_0^{\overline{V}} \right] \right) \leq 1 - c_6 c_7
\end{aligned}
$$

by (4.10) and (4.11). Substituting this into (4.12) and iterating gives the claim with $c_5 := 1 - c_6 c_7$. $\qquad\square$

The next lemma states that Lemma 7.8 also holds for $\overline{V}$.

Lemma 4.6. *Let $\delta > 1$. For every $\gamma > 0$ there is $c_8(\gamma) \in (0, \infty)$ such that $P_n\left[\tau_{c_8 n}^{\overline{V}} < \sigma_0^{\overline{V}}\right] < \gamma$ for all $n \in \mathbb{N}$.*

Proof. By the strong Markov property and monotonicity of h, for all $t > 1$,

$$\limsup_{n \to \infty} P_n\left[\tau_{tn}^{\overline{V}} < \sigma_0^{\overline{V}}\right] = \limsup_{n \to \infty} \frac{P_n\left[\tau_{tn}^{V} < \sigma_0^{V} < \infty\right]}{h(n)} \leq \limsup_{n \to \infty} \frac{h(\lceil tn \rceil)}{h(n)} = t^{1-\delta}$$

due to Proposition 4.3. Since $t^{1-\delta} \to 0$ as $t \to \infty$ and $P_n[\sigma_0^{\overline{V}} < \infty] = 1$ for all $n \in \mathbb{N}$ by definition of $\overline{V}$ this finishes the proof. $\qquad\square$

Lemma 4.7. *Let $\delta \in \mathbb{R}\backslash\{1\}$. For each $\varepsilon > 0$ there is $c_9(\varepsilon) \in (0, \infty)$ such that $P_n[\sigma_0^{\overline{V}} > c_9(\varepsilon)n] < \varepsilon$ for all $n \in \mathbb{N}$.*

Proof. The proof is the same as the one of [KM11, Prop. 7.1]. It uses Lemmas 4.5 and 4.6 instead of (6.1) and (5.5) of [KM11], respectively. $\qquad\square$

Proof of Theorem 4.1. We first prove (4.2). For $\delta < 1$, (4.2) follows from Theorem 3.4 since $\overline{V} = V$.

Now assume $\delta > 1$. By [EK86, Ch. 4, Th. 2.12] it suffices to show that the one-dimensional marginal distributions converge, i.e. that for all $t > 0$,

$$\overline{Y}_n(t) \Longrightarrow \overline{Y}\left(t \wedge \sigma_0^{\overline{Y}}\right) \qquad \text{as } n \to \infty. \tag{4.13}$$

(This theorem is applicable since the semigroup corresponding to $\overline{Y}(\cdot \wedge \sigma_0^{\overline{Y}})$ is Feller on the space of continuous functions on $[0, \infty)$ vanishing at infinity, see [EK86, Ch. 8, Th. 1.1, Cor. 1.2].)

For the proof of (4.13) let us assume for the moment that we have already shown that for all $t > 0$ and $\varepsilon \in (0, x)$,

$$\overline{Y}_n\left(t \wedge \sigma_\varepsilon^{\overline{Y}_n}\right) \Longrightarrow \overline{Y}\left(t \wedge \sigma_\varepsilon^{\overline{Y}}\right) \qquad \text{as } n \to \infty. \tag{4.14}$$

Since $\overline{Y}$ has continuous trajectories we have for all $t > 0$, $\overline{Y}\left(t \wedge \sigma_\varepsilon^{\overline{Y}}\right) \Rightarrow \overline{Y}\left(t \wedge \sigma_0^{\overline{Y}}\right)$ as $\varepsilon \searrow 0$. Moreover, for all $\eta > 0$,

$$\lim_{\varepsilon \searrow 0} \limsup_{n \to \infty} P\left[\left|\overline{Y}_n\left(t \wedge \sigma_\varepsilon^{\overline{Y}_n}\right) - \overline{Y}_n(t)\right| > \eta\right] \leq \lim_{\varepsilon \searrow 0} \limsup_{n \to \infty} P_{\lfloor n\varepsilon \rfloor}\left[\tau_{n\eta}^{\overline{V}} < \sigma_0^{\overline{V}}\right] = 0$$

due to Lemma 4.6. Therefore, we can apply Lemma 7.1 and obtain (4.13).

It remains to verify (4.14). Fix $t > 0$ and $\varepsilon \in (0, x)$. We need to show that for any bounded and continuous function $f : \mathbb{R} \to \mathbb{R}$,

$$\lim_{n \to \infty} E\left[f\left(\overline{Y}_n\left(t \wedge \sigma_\varepsilon^{\overline{Y}_n}\right)\right)\right] = E\left[f\left(\overline{Y}\left(t \wedge \sigma_\varepsilon^{\overline{Y}}\right)\right)\right]. \tag{4.15}$$

By (4.6),

$$E\left[f\left(\overline{Y}_n\left(t \wedge \sigma_\varepsilon^{\overline{Y}_n}\right)\right)\right] = E\left[f\left(Y_n\left(t \wedge \sigma_\varepsilon^{Y_n}\right)\right) \frac{h\left(nY_n\left(t \wedge \sigma_\varepsilon^{Y_n}\right)\right)}{h(\lfloor x_n n \rfloor)}\right] = A_n + B_n,$$

where

$$A_n \;:=\; E\left[f\left(Y_n\left(t\wedge\sigma_\varepsilon^{Y_n}\right)\right)\left(\frac{h\left(nY_n\left(t\wedge\sigma_\varepsilon^{Y_n}\right)\right)}{h(\lfloor x_n n\rfloor)} - \left(\frac{x_n}{Y_n\left(t\wedge\sigma_\varepsilon^{Y_n}\right)\vee(\varepsilon/2)}\right)^{\delta-1}\right)\right],$$

$$B_n \;:=\; x_n^{\delta-1}E\left[f\left(Y_n\left(t\wedge\sigma_\varepsilon^{Y_n}\right)\right)\left(Y_n\left(t\wedge\sigma_\varepsilon^{Y_n}\right)\vee(\varepsilon/2)\right)^{1-\delta}\right].$$

Note that by Theorem 3.4, Lemma 3.3, and the continuous mapping theorem, $Y_n(t\wedge\sigma_\varepsilon^{Y_n})$ converges in distribution as $n\to\infty$ to $\varphi_\varepsilon(t)$, where $\varphi_\varepsilon := Y(\cdot\wedge\sigma_\varepsilon^Y)$ and Y solves (3.1) with initial condition $Y(0)=x$. Hence, since the function $g(x):=f(x)(x\vee(\varepsilon/2))^{1-\delta}$ is bounded and continuous,

$$\lim_{n\to\infty}B_n \;=\; x^{\delta-1}E\left[f\left(\varphi_\varepsilon(t)\right)\left(\varphi_\varepsilon(t)\vee(\varepsilon/2)\right)^{1-\delta}\right] \;=\; x^{\delta-1}E\left[f\left(\varphi_\varepsilon(t)\right)\left(\varphi_\varepsilon(t)\right)^{1-\delta}\right]$$

$$=\; E\left[f\left(\overline{Y}\left(t\wedge\sigma_\varepsilon^Y\right)\right)\right], \tag{4.16}$$

where the last identity follows from a change of measure as follows. Since Y solves (3.1) with $Y(0)=x$, φ_ε solves

$$d\varphi_\varepsilon(s) = \delta\mathbb{1}_{\varphi_\varepsilon(s)>\varepsilon}\,ds + \sqrt{2\varphi_\varepsilon(s)}\mathbb{1}_{\varphi_\varepsilon(s)>\varepsilon}\,dB(s), \quad \varphi_\varepsilon(0)=x, \tag{4.17}$$

see e.g. [IW89, Prop. II-1.1 (iv)]. By Itô's formula,
using that $\varphi_\varepsilon\ge\varepsilon$,

$$d\ln\varphi_\varepsilon(s) \;=\; \frac{1}{\varphi_\varepsilon(s)}\,d\varphi_\varepsilon(s) - \frac{1}{2}\frac{1}{\varphi_\varepsilon^2(s)}\left(d\varphi_\varepsilon(s)\right)^2$$

$$\overset{(4.17)}{=}\; \frac{\mathbb{1}_{\varphi_\varepsilon(s)>\varepsilon}}{\varphi_\varepsilon(s)}\left(\delta\,ds + \sqrt{2\varphi_\varepsilon(s)}\,dB(s)\right) - \frac{\mathbb{1}_{\varphi_\varepsilon(s)>\varepsilon}}{\varphi_\varepsilon(s)}\,ds$$

$$=\; \frac{\delta-1}{\varphi_\varepsilon(s)}\mathbb{1}_{\varphi_\varepsilon(s)>\varepsilon}\,ds + \sqrt{\frac{2}{\varphi_\varepsilon(s)}}\mathbb{1}_{\varphi_\varepsilon(s)>\varepsilon}\,dB(s).$$

Therefore,

$$Z_\varepsilon(t) \;:=\; \exp\left((1-\delta)\left(\int_0^t\frac{\delta-1}{\varphi_\varepsilon(s)}\mathbb{1}_{\varphi_\varepsilon(s)>\varepsilon}\,ds + \int_0^t\sqrt{\frac{2}{\varphi_\varepsilon(s)}}\mathbb{1}_{\varphi_\varepsilon(s)>\varepsilon}\,dB(s)\right)\right)$$

$$=\; \exp\left((1-\delta)\left(\ln\varphi_\varepsilon(t)-\ln x\right)\right) \;=\; x^{\delta-1}\left(\varphi_\varepsilon(t)\right)^{1-\delta}.$$

Now consider the measure $\widetilde{P}_\varepsilon$ with $d\widetilde{P}_\varepsilon/dP = Z_\varepsilon(t)$. By Girsanov's transformation, see e.g. [IW89, Th. IV-4.2] with $\alpha(t,x):=\sqrt{2x}\mathbb{1}_{x>\varepsilon}$, $\beta(t,x):=\delta\mathbb{1}_{x>\varepsilon}$ and $\gamma(t,x):=(1-\delta)\sqrt{2/x}\mathbb{1}_{x>\varepsilon}$, the process $(\varphi_\varepsilon(s))_{0\le s\le t}$ satisfies

$$d\varphi_\varepsilon(s) = (2-\delta)\mathbb{1}_{\varphi_\varepsilon(s)>\varepsilon}\,ds + \sqrt{2\varphi_\varepsilon(s)}\mathbb{1}_{\varphi_\varepsilon(s)>\varepsilon}\,d\widetilde{B}(s), \quad \varphi_\varepsilon(0)=x,$$

where $\widetilde{B}$ is a standard Brownian motion w.r.t. $\widetilde{P}_\varepsilon$. Hence $\varphi_\varepsilon(t)$ has under $\widetilde{P}$ the same distribution as $\overline{Y}(t\wedge\sigma_\varepsilon^Y)$ under P. This implies (4.16).

For the proof of (4.15) it remains to show that $A_n\to 0$ as $n\to\infty$. Let $\kappa:=\|f\|_\infty$ and abbreviate $\varphi_{n,\varepsilon}:=Y_n\left(t\wedge\sigma_\varepsilon^{Y_n}\right)$. Then

$$|A_n| \le \kappa E\left[\left|\frac{h\left(n\varphi_{n,\varepsilon}\right)}{h(\lfloor x_n n\rfloor)} - \left(\frac{x_n}{\varphi_{n,\varepsilon}\vee(\varepsilon/2)}\right)^{\delta-1}\right|\right]$$

$$= \kappa x_n^{\delta-1}E\left[\left|\frac{h\left(n\varphi_{n,\varepsilon}\right)\left((\varphi_{n,\varepsilon}\vee(\varepsilon/2))\,n\right)^{\delta-1} - (x_n n)^{\delta-1}h(\lfloor x_n n\rfloor)}{(x_n n)^{\delta-1}h(\lfloor x_n n\rfloor)\left(\varphi_{n,\varepsilon}\vee(\varepsilon/2)\right)^{\delta-1}}\right|\right]$$

$$\le \kappa\left(\frac{2x_n}{\varepsilon}\right)^{\delta-1}E\left[\left|\frac{h\left(n\varphi_{n,\varepsilon}\right)\left((\varphi_{n,\varepsilon}\vee(\varepsilon/2))\,n\right)^{\delta-1} - (x_n n)^{\delta-1}h(\lfloor x_n n\rfloor)}{(x_n n)^{\delta-1}h(\lfloor x_n n\rfloor)}\right|\right]. \tag{4.18}$$

Now let $\gamma \in (0, c_4/2)$, where c_4 is the constant from Proposition 4.3. By Proposition 4.3 there is $n_0 \in \mathbb{N}$ such that

$$\left| s^{\delta-1} h(\lfloor s \rfloor) - c_4 \right| < \gamma \quad \text{for all } s \in [n_0, \infty).$$

Thus for large enough n the right-hand side of (4.18) is less than or equal to

$$\frac{\kappa}{c_4 - \gamma} \left(\frac{2x_n}{\varepsilon} \right)^{\delta-1} E\left[\left| h\left(n\varphi_{n,\varepsilon}\right) \left(\left(\varphi_{n,\varepsilon} \vee (\varepsilon/2)\right) n\right)^{\delta-1} - (x_n n)^{\delta-1} h(\lfloor x_n n \rfloor) \right| \right]$$

$$\leq \frac{2\kappa}{c_4} \left(\frac{2x_n}{\varepsilon} \right)^{\delta-1} \left(E\left[\left| h\left(n\varphi_{n,\varepsilon}\right) \left(\left(\varphi_{n,\varepsilon} \vee (\varepsilon/2)\right) n\right)^{\delta-1} - c_4 \right| \right] + \gamma \right)$$

$$= \frac{2\kappa}{c_4} \left(\frac{2x_n}{\varepsilon} \right)^{\delta-1} \left(E\left[\left| h\left(n\varphi_{n,\varepsilon}\right) \left(\varphi_{n,\varepsilon} n\right)^{\delta-1} - c_4 \right|, \varphi_{n,\varepsilon} \geq \varepsilon/2 \right] \right.$$

$$\left. + E\left[\left| h\left(n\varphi_{n,\varepsilon}\right) \left(\varepsilon n/2\right)^{\delta-1} - c_4 \right|, \varphi_{n,\varepsilon} < \varepsilon/2 \right] + \gamma \right)$$

$$\leq \frac{2\kappa}{c_4} \left(\frac{2x_n}{\varepsilon} \right)^{\delta-1} \left(\gamma + \left((\varepsilon n/2)^{\delta-1} + c_4 \right) P\left[\varphi_{n,\varepsilon} < \varepsilon/2 \right] + \gamma \right) \leq \frac{6\gamma\kappa}{c_4} \left(\frac{2x_n}{\varepsilon} \right)^{\delta-1}.$$

Here we used in the last step that $P\left[\varphi_{n,\varepsilon} < \varepsilon/2 \right]$ decays exponentially fast as $n \to \infty$ due to (7.6). Letting $n \to \infty$ and then $\gamma \searrow 0$ yields $\lim_n A_n = 0$. This proves (4.2).

For the next statement, (4.3), we shall use Lemma 7.1. It follows from (4.2), Lemma 3.3, the continuous mapping theorem, and the a.s. continuity of $\overline{Y}$ that

$$\sigma_{1/m}^{\overline{Y}_n} \underset{n}{\Longrightarrow} \sigma_{1/m}^{\overline{Y}} \underset{m}{\Longrightarrow} \sigma_0^{\overline{Y}}.$$

To verify the condition corresponding to (7.1) fix $\varepsilon > 0$ and let $c_9(\cdot)$ be the function of Lemma 4.7. Choose $m_0 \in \mathbb{N}$ and $g : \mathbb{N} \to (0, \infty)$ such that $c_9(g(m)) \leq \varepsilon m$ for all $m \geq m_0$ and $g(m) \to 0$ as $m \to \infty$. Then for all $n \geq m \geq m_0$,

$$P\left[\sigma_0^{\overline{Y}_n} - \sigma_{1/m}^{\overline{Y}_n} > \varepsilon \right] \leq P_{\lfloor \frac{n}{m} \rfloor}\left[\sigma_0^{\overline{V}} > \varepsilon n \right] \tag{4.19}$$

$$\leq P_{\lfloor \frac{n}{m} \rfloor}\left[\sigma_0^{\overline{V}} > c_9(g(m)) \left\lfloor \frac{n}{m} \right\rfloor \right] \leq g(m)$$

due to Lemma 4.7. We conclude that the condition corresponding to (7.1) is satisfied and (4.3) follows.

The final statement, (4.4), is obtained similarly. As above, it follows from (4.2), Lemma 3.3, the continuous mapping theorem, and the a.s. continuity of $\overline{Y}$ that

$$\int_0^{\sigma_{1/m}^{\overline{Y}_n}} \overline{Y}_n(s) \, ds \underset{n \to \infty}{\Longrightarrow} \int_0^{\sigma_{1/m}^{\overline{Y}}} \overline{Y}(s) \, ds \underset{m \to \infty}{\Longrightarrow} \int_0^{\sigma_0^{\overline{Y}}} \overline{Y}(s) \, ds.$$

Moreover, for all $\varepsilon \in (0, 1)$ and $n \geq m \geq m_0$,

$$P\left[\left| \int_0^{\sigma_0^{\overline{Y}_n}} \overline{Y}_n(s) \, ds - \int_0^{\sigma_{1/m}^{\overline{Y}_n}} \overline{Y}_n(s) \, ds \right| > \varepsilon \right] = P\left[\int_{\sigma_{1/m}^{\overline{Y}_n}}^{\sigma_0^{\overline{Y}_n}} \overline{Y}_n(s) \, ds > \varepsilon \right]$$

$$\leq P\left[\sup\left\{ \overline{Y}_n(s) : \sigma_{1/m}^{\overline{Y}_n} \leq s \leq \sigma_0^{\overline{Y}_n} \right\} \left(\sigma_0^{\overline{Y}_n} - \sigma_{1/m}^{\overline{Y}_n} \right) > \varepsilon \right]$$

$$\leq P\left[\sup\left\{ \overline{Y}_n(s) : \sigma_{1/m}^{\overline{Y}_n} \leq s \leq \sigma_0^{\overline{Y}_n} \right\} > \sqrt{\varepsilon} \right] + P\left[\sigma_0^{\overline{Y}_n} - \sigma_{1/m}^{\overline{Y}_n} > \sqrt{\varepsilon} \right]$$

$$\leq P_{\lfloor \frac{n}{m} \rfloor}\left[\tau_{\varepsilon n}^{\overline{V}} < \sigma_0^{\overline{V}} \right] + g(m)$$

due to (4.19). Letting first $n \to \infty$ and then $m \to \infty$ we see that this converges to 0 due to Lemma 4.6 and the choice of g. Having verified the assumptions of Lemma 7.1 we obtain statement (4.4). $\square$

5 Asymptotics of diffusions and branching processes

In this section we derive certain asymptotics of the conditioned BP $\overline{V}$ from those of the approximating squared Bessel process as stated in Lemma 7.3. Our goal is the following result:

Theorem 5.1. *Let* $V = (V_n)_{n \geq 0}$ *be defined as in* (2.2) *such that* (2.3)–(2.6) *hold with* $V_0 = 1$ *and let* $\delta \in \mathbb{R} \backslash \{1\}$. *Then there are* $c_{10}, c_{11} \in (0, \infty)$ *such that*

$$\lim_{n \to \infty} n^{|\delta - 1|} P_1[\sigma_0^{\overline{V}} > n] = c_{10} \quad \text{and} \tag{5.1}$$

$$\lim_{n \to \infty} n^{|\delta - 1|/2} P_1 \left[\sum_{i=0}^{\sigma_0^{\overline{V}}} \overline{V}_i > n \right] = c_{11}. \tag{5.2}$$

Remark 5.2. In the special case described in Remark 2.2, formula (5.1) follows for $0 < \delta < 1$ from the tail behavior of the extinction time of recurrent critical BPs with immigration described in [Zub72, Th. 2, first part of (21)]. For $\delta > 1$ it is reminiscent of the unproven claims made in [IS85, middle of page 225] about the extinction time of transient critical BPs with immigration. More general results concerning BPs with migration for the case $\delta \in (-\infty, 1) \backslash \{0, -1\}$ are given in [FYK90, Theorems 1,3,4], see also [KZ08, Th. A (iv)].

Lemma 5.3. *Let* $\delta \in \mathbb{R} \backslash \{1\}$. *Then there is* $c_{12} \in (0, \infty)$ *such that*
$\lim_{n \to \infty} n^{|\delta - 1|} P_1 \left[\tau_n^{\overline{V}} < \sigma_0^{\overline{V}} \right] = c_{12}$.

Proof. For $\delta < 1$ the statement follows from [KM11, (C), Lemma 8.1].

Now let $\delta > 1$. Then by the strong Markov property and monotonicity of h,

$$\limsup_{n \to \infty} n^{\delta - 1} P_1 \left[\tau_n^{\overline{V}} < \sigma_0^{\overline{V}} \right] \leq \limsup_{n \to \infty} n^{\delta - 1} \frac{P_1 \left[\tau_n^V < \infty \right] h(n)}{h(1)} = \frac{1 - h(1)}{h(1)} c_4 =: c_{12}$$

due to Proposition 4.3. On the other hand, for all $\gamma > 1$ by the strong Markov property and monotonicity of h,

$$\liminf_{n \to \infty} n^{\delta - 1} P_1 \left[\tau_n^{\overline{V}} < \sigma_0^{\overline{V}} \right] \geq \liminf_{n \to \infty} n^{\delta - 1} \frac{P_1 \left[\tau_n^V < \sigma_0^V < \infty, V_{\tau_n^V} \leq \gamma n \right]}{h(1)}$$

$$\geq \liminf_{n \to \infty} n^{\delta - 1} \frac{P_1 \left[\tau_n^V < \infty, V_{\tau_n^V} \leq \gamma n \right] h(\lfloor \gamma n \rfloor)}{h(1)} = \frac{1 - h(1)}{h(1)} c_4 \gamma^{1 - \delta} \xrightarrow[\gamma \searrow 1]{} c_{12},$$

where in the last identity we used (7.5) and Proposition 4.3. $\qquad \square$

Lemma 5.4. *Let* $\delta \in \mathbb{R} \backslash \{1\}$. *Then for every* $\varepsilon > 0$,

$$\lim_{x \searrow 0} \limsup_{n \to \infty} n^{|\delta - 1|} P_1 \left[\# \left\{ i \in \{1, \ldots, \sigma_0^{\overline{V}}\} : \overline{V}_i < xn \right\} > \varepsilon n \right] = 0.$$

Proof. The proof is the same as the one of [KM11, Prop. 6.2]. It uses Lemmas 4.5 and 5.3 instead of (6.1) and (5.4) of [KM11], respectively. $\qquad \square$

Proof of Theorem 5.1. We first prove (5.1) with $c_{10} := c_{12} b_1 (1 - |\delta - 1|)$, where c_{12} is as in Lemma 5.3 and the function b_1 is as in Lemma 7.3. The proof generalizes the one of [KM11, Th. 2.1], which covers the case $\delta < 1$. Let $A_n := \{\sigma_0^{\overline{V}} > n\}$ for $n \geq 0$.

Lower bound in (5.1). Fix $x > 0$. Then by the strong Markov property for $\overline{V}$ we have for all $n \geq 0$,

$$
\begin{aligned}
n^{|\delta-1|} P_1\left[A_n\right] &\geq n^{|\delta-1|} P_1\left[\sigma_0^{\overline{V}} > n + \tau_{xn}^{\overline{V}}, \overline{V}_{\tau_{xn}^{\overline{V}}} \leq 2xn\right] \\
&= n^{|\delta-1|} E_1\left[P_{\overline{V}_{\tau_{xn}^{\overline{V}}}}[A_n],\ \sigma_0^{\overline{V}} > \tau_{xn}^{\overline{V}}, \overline{V}_{\tau_{xn}^{\overline{V}}} \leq 2xn\right] \quad\quad (5.3) \\
&\geq n^{|\delta-1|} P_1\left[\sigma_0^{\overline{V}} > \tau_{xn}^{\overline{V}}, \overline{V}_{\tau_{xn}^{\overline{V}}} \leq 2xn\right] \inf_{y \in [x,2x]} P_{\lfloor yn\rfloor}[A_n]. \quad\quad (5.4)
\end{aligned}
$$

Now we choose suitable $y_n \in [x, 2x]$ where the infimum in (5.4) is attained. Then the expression in (5.4) can be estimated from below by

$$
\left((xn)^{|\delta-1|} P_1\left[\sigma_0^{\overline{V}} > \tau_{xn}^{\overline{V}}\right]\right) x^{-|\delta-1|} P_{\lfloor y_n n\rfloor}[A_n] - n^{|\delta-1|} P_1\left[\tau_{xn}^{\overline{V}} < \infty, \overline{V}_{\tau_{xn}^{\overline{V}}} > 2xn\right].
$$

The second term above vanishes as $n \to \infty$ due to (7.7). Therefore, by Lemma 5.3, for all $x > 0$,

$$
\begin{aligned}
\liminf_{n\to\infty} n^{|\delta-1|} P_1\left[A_n\right] &\geq c_{12} x^{-|\delta-1|} \liminf_{n\to\infty} P_{\lfloor y_n n\rfloor}[A_n] \\
&= c_{12} x^{-|\delta-1|} \lim_{k\to\infty} P_{\lfloor y_{n_k} n_k\rfloor}[A_{n_k}] \quad\quad (5.5)
\end{aligned}
$$

for some increasing sequence $(n_k)_{k\geq 0}$. Choose a subsequence $(m_k)_{k\geq 0}$ of $(n_k)_{k\geq 0}$ along which $(y_{m_k})_{k\geq 0}$ converges to some $y \in [x, 2x]$. Then by (4.3), the limit in (5.5) is equal to $P_y[\sigma_0^{\overline{Y}} > 1]$ where $\overline{Y}$ solves the SDE (4.1). By scaling (Lemma 7.2),

$$
P_y[\sigma_0^{\overline{Y}} > 1] = P_1[\sigma_0^{\overline{Y}} > 1/y] \geq P_1[\sigma_0^{\overline{Y}} > 1/x] \sim x^{|\delta-1|} b_1(1 - |\delta - 1|)
$$

as $x \searrow 0$ by (7.2). This proves the lower bound in (5.1).

Upper bound in (5.1). Fix $\varepsilon \in (0,1)$ and estimate for all $x > 0$ and $n \geq 0$,

$$
\begin{aligned}
n^{|\delta-1|} P_1\left[A_n\right] &\leq n^{|\delta-1|} P_1\left[A_n, \tau_{xn}^{\overline{V}} \leq \varepsilon n\right] + n^{|\delta-1|} P_1\left[\tau_{xn}^{\overline{V}} \wedge \sigma_0^{\overline{V}} > \varepsilon n\right] \quad\quad (5.6) \\
&\leq n^{|\delta-1|} P_1\left[A_n, \tau_{xn}^{\overline{V}} \leq \varepsilon n,\ \overline{V}_{\tau_{xn}^{\overline{V}}} \leq (1+\varepsilon)xn\right] \quad\quad (5.7) \\
&\quad + n^{|\delta-1|} P_1\left[\tau_{xn}^{\overline{V}} < \infty, \overline{V}_{\tau_{xn}^{\overline{V}}} > (1+\varepsilon)xn\right] \quad\quad (5.8) \\
&\quad + n^{|\delta-1|} P_1\left[\#\left\{i \in \{1,\ldots,\sigma_0^{\overline{V}}\} : \overline{V}_i < xn\right\} > \varepsilon n\right]. \quad\quad (5.9)
\end{aligned}
$$

The expression in (5.8) vanishes as $n \to \infty$ due to (7.7). The term in (5.9) vanishes as well due to Lemma 5.4 if we let first $n \to \infty$ and then $x \searrow 0$. For the treatment of (5.7) let $B_n := \left\{\sigma_0^{\overline{V}} > (1-\varepsilon)n\right\}$ for $n \geq 0$. Then by the strong Markov property the quantity in (5.7) is less than or equal to

$$
\begin{aligned}
n^{|\delta-1|} E_1&\left[P_{\tau_{xn}^{\overline{V}}}[B_n], \tau_{xn}^{\overline{V}} < \sigma_0^{\overline{V}},\ \overline{V}_{\tau_{xn}^{\overline{V}}} \leq (1+\varepsilon)xn\right] \quad\quad (5.10) \\
&\leq n^{|\delta-1|} P_1\left[\tau_{xn}^{\overline{V}} < \sigma_0^{\overline{V}}\right] \sup_{y\in[x,(1+\varepsilon)x]} P_{\lfloor yn\rfloor}[B_n] \\
&= \left((xn)^{|\delta-1|} P_1\left[\tau_{xn}^{\overline{V}} < \sigma_0^{\overline{V}}\right]\right) x^{-|\delta-1|} P_{\lfloor y_n(x)\,n\rfloor}[B_n]
\end{aligned}
$$

for suitable $y_n(x) \in [x, (1+\varepsilon)x]$. The first factor in the last expression converges to c_{12} as $n \to \infty$ by Lemma 5.3. Summarizing we get that for all $\varepsilon \in (0,1)$,

$$
\begin{aligned}
\limsup_{n\to\infty} n^{|\delta-1|} P_1\left[A_n\right] &\leq c_{12} \limsup_{x\searrow 0} x^{-|\delta-1|} \limsup_{n\to\infty} P_{\lfloor y_n(x)\,n\rfloor}[B_n] \\
&= c_{12} \limsup_{x\searrow 0} x^{-|\delta-1|} \lim_{k\to\infty} P_{\lfloor y_{n_{k,x}}(x)\,n_{k,x}\rfloor}[B_{n_{k,x}}] \quad\quad (5.11)
\end{aligned}
$$

for suitable increasing sequences $(n_{k,x})_{k\geq 0}$, $x > 0$. Choose for all $x > 0$ a subsequence $(m_{k,x})_{k\geq 0}$ of $(n_{k,x})_{k\geq 0}$ along which $(y_{m_{k,x}}(x))_{k\geq 0}$ converges to some $y(x) \in [x, (1+\varepsilon)x]$ and let $\overline{Y}$ solve the SDE (4.1). Then by (4.3), the expression in (5.11) is equal to

$$c_{12} \limsup_{x\searrow 0} x^{-|\delta-1|} P_{y(x)} \left[\sigma_0^{\overline{Y}} > 1 - \varepsilon\right] \stackrel{\text{L. 7.2}}{=} c_{12} \limsup_{x\searrow 0} x^{-|\delta-1|} P_1 \left[\sigma_0^{\overline{Y}} > \frac{1-\varepsilon}{y(x)}\right]$$

$$\leq c_{12} \limsup_{x\searrow 0} x^{-|\delta-1|} P_1 \left[\sigma_0^{\overline{Y}} > \frac{1-\varepsilon}{(1+\varepsilon)x}\right] \stackrel{(7.2)}{=} c_{10} \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{|\delta-1|} \xrightarrow[\varepsilon\searrow 0]{} c_{10}.$$

This completes the proof of (5.1).

Now we turn to the proof of (5.2) with $c_{11} := c_{12} b_2(1 - |\delta - 1|)$, where c_{12} is as in Lemma 5.3 and the function b_2 is as in Lemma 7.3. The proof is similar to the one of [KM11, Lemma 4.1]. It is enough to consider convergence along the subsequence $(n^2)_{n\geq 0}$. Moreover, it suffices to show

$$\lim_{x\searrow 0} \lim_{n\to\infty} n^{|\delta-1|} P_1 \left[A_n, \tau_{xn}^{\overline{V}} < \sigma_0^{\overline{V}}\right] = c_{11}, \quad \text{where } A_n := \left\{\sum_{i=0}^{\sigma_0^{\overline{V}}} \overline{V}_i > n^2\right\}. \tag{5.12}$$

Indeed, for all $x > 0$ due to (5.1)

$$n^{|\delta-1|} P_1 \left[A_n, \sigma_0^{\overline{V}} < \tau_{xn}^{\overline{V}}\right] \leq n^{|\delta-1|} P_1 \left[xn\sigma_0^{\overline{V}} > n^2\right] \xrightarrow[n\to\infty]{} c_{10} x^{|\delta-1|} \xrightarrow[x\searrow 0]{} 0.$$

Lower bound in (5.12). Fix $x > 0$. Then

$$n^{|\delta-1|} P_1 \left[A_n, \tau_{xn}^{\overline{V}} < \sigma_0^{\overline{V}}\right] \geq n^{|\delta-1|} P_1 \left[\sum_{i=\tau_{xn}^{\overline{V}}}^{\sigma_0^{\overline{V}}} \overline{V}_i > n^2, \tau_{xn}^{\overline{V}} < \sigma_0^{\overline{V}}, \overline{V}_{\tau_{xn}^{\overline{V}}} \leq 2xn\right]. \tag{5.13}$$

This can be estimated from below by the expression in (5.3). By the same argument as after (5.3), using (4.4) instead of (4.3), we obtain for a suitable $y \in [x, 2x]$ and a solution $\overline{Y}$ of (4.1),

$$\liminf_{n\to\infty} n^{|\delta-1|} P_1 \left[A_n, \tau_{xn}^{\overline{V}} < \sigma_0^{\overline{V}}\right] \geq c_{12} x^{-|\delta-1|} P_y \left[\int_0^{\sigma_0^{\overline{Y}}} \overline{Y}(s)\,ds > 1\right]$$

$$\stackrel{\text{L. 7.2}}{=} c_{12} x^{-|\delta-1|} P_1 \left[\int_0^{\sigma_0^{\overline{Y}}} \overline{Y}(s)\,ds > \frac{1}{y^2}\right] \geq c_{12} x^{-|\delta-1|} P_1 \left[\int_0^{\sigma_0^{\overline{Y}}} \overline{Y}(s)\,ds > \frac{1}{x^2}\right]$$

$$\stackrel{\text{L. 7.3}}{\xrightarrow[x\searrow 0]{}} c_{12} b_2(1 - |\delta - 1|) = c_{11}.$$

Upper bound in (5.12). We estimate the first term in (5.13) from above for all $n \geq 0$ and $x, \varepsilon \in (0, 1)$ by

$$n^{|\delta-1|} \left(P_1 \left[\sum_{i=\tau_{xn}^{\overline{V}}}^{\sigma_0^{\overline{V}}} \overline{V}_i > (1-\varepsilon x)n^2, \tau_{xn}^{\overline{V}} < \sigma_0^{\overline{V}}\right] + P_1 \left[\sum_{i=0}^{\tau_{xn}^{\overline{V}}-1} \overline{V}_i > \varepsilon x n^2, \tau_{xn}^{\overline{V}} < \sigma_0^{\overline{V}}\right]\right)$$

$$\leq n^{|\delta-1|} P_1 \left[\sum_{i=\tau_{xn}^{\overline{V}}}^{\sigma_0^{\overline{V}}} \overline{V}_i > (1-\varepsilon)n^2, \tau_{xn}^{\overline{V}} < \sigma_0^{\overline{V}}, V_{\tau_{xn}^{\overline{V}}} \leq (1+\varepsilon)xn\right] \tag{5.14}$$

$$+ n^{|\delta-1|} P_1 \left[\tau_{xn}^{\overline{V}} < \infty, \overline{V}_{\tau_{xn}^{\overline{V}}} > (1+\varepsilon)xn\right] + n^{|\delta-1|} P_1 \left[\varepsilon n \leq \tau_{xn}^{\overline{V}} < \sigma_0^{\overline{V}}\right]. \tag{5.15}$$

As above, see (5.8) and the second term on the right-hand side of (5.6), the sum in (5.15) is negligible. The expression in (5.14) can be estimated from above by the strong Markov property by the expression in (5.10), where $B_n := \left\{ \sum_{i=0}^{\sigma_0^{\overline{V}}} \overline{V}_i > (1-\varepsilon)n^2 \right\}$. By the same argument as after (5.10), using (4.4) instead of (4.3), we obtain for suitable $y(x) \in [x, (1+\varepsilon)x]$ and $\overline{Y}$ a solution of (4.1),

$$\limsup_{x \searrow 0} \limsup_{n \to \infty} n^{|\delta-1|} \, P_1 \left[A_n, \, \tau_{xn}^{\overline{V}} < \sigma_0^{\overline{V}} \right]$$

$$\leq \quad c_{12} \limsup_{x \searrow 0} x^{-|\delta-1|} P_{y(x)} \left[\int_0^{\sigma_0^{\overline{Y}}} \overline{Y}(s) \, ds > 1 - \varepsilon \right]$$

$$\overset{\text{L. 7.2}}{\leq} \quad c_{12} \limsup_{x \searrow 0} x^{-|\delta-1|} P_1 \left[\int_0^{\sigma_0^{\overline{Y}}} \overline{Y}(s) \, ds > \frac{1-\varepsilon}{(1+\varepsilon)^2 x^2} \right]$$

$$\overset{\text{L. 7.3}}{=} \quad c_{12} b_2 (1 - |\delta - 1|) \left(\frac{1+\varepsilon}{\sqrt{1-\varepsilon}} \right)^{|\delta-1|},$$

which converges to c_{11} as $\varepsilon \searrow 0$. $\qquad\square$

6 Proof of the main result

Proof of Theorem 1.1. Observe that in the setting of Proposition 2.1, $\sigma_0^U = \sigma_0^V$ due to (2.8) and therefore, by definition (2.1),

$$\sup\{X_n : \, n < T_0\} = \sigma_0^V \quad \text{and} \quad T_0 \mathbb{1}_{T_0 < \infty} = \left(-1 + 2 \sum_{i=0}^{\sigma_0^V} V_i \right) \mathbb{1}_{\sigma_0^V < \infty}. \qquad (6.1)$$

(The second identity has already been observed in [Har52, Th. 5].) Claims (1.3) and (1.4) now follow from Theorem 5.1.

Now let $\delta = 1$, $0 < \varepsilon < 1$, and consider the first two limits in (1.6). The divergence of these limits would be obvious if we had $P_1[T_0 = \infty] > 0$. Therefore, we assume $P_1[T_0 < \infty] = 1$. Let $\hat{\mathbb{P}}$ be another probability measure which satisfies (IID), (WEL), and (BD$_\mathrm{M}$) and $\hat{\mathbb{P}}[\omega(0,i) \leq t] \geq \mathbb{P}[\omega(0,i) \leq t]$ for all $i \geq 1$ and $t \in [0,1]$ and whose corresponding parameter $\hat{\delta}$, defined as in (1.2), satisfies $1 - \varepsilon < \hat{\delta} < 1$. Then by monotonicity as stated in [Zer05, Lemma 15] we obtain $\hat{\mathbb{E}}[P_{1,\omega}[T_n < T_0]] \leq P_1[T_n < T_0]$ and $\hat{\mathbb{E}}[P_{1,\omega}[T_0 > n]] \leq P_1[T_0 > n]$. (Note that the additional condition of a.s. positivity of $2\omega(x,i) - 1$ which was assumed in [Zer05] is not needed for the proof of [Zer05, Lemma 15].) Applying (1.3) and (1.4) to $\hat{\mathbb{P}}$ then yields the first two statements in (1.6).

As for the remaining claims about the tail of R consider the first step of the walk and use the fact that $P_{x,\omega}[n \leq T_0 < \infty]$ does not depend on $\omega(0, \cdot)$ to obtain that

$$\begin{aligned} P_0[n < R < \infty] &= \mathbb{E}\left[P_{0,\omega}[n < R < \infty] \right] \\ &= \mathbb{E}\left[\omega(0,1) P_{1,\omega}[n \leq T_0 < \infty] \right] + \mathbb{E}\left[(1 - \omega(0,1)) P_{-1,\omega}[n \leq T_0 < \infty] \right] \\ &= \mathbb{E}[\omega(0,1)] P_1[n \leq T_0 < \infty] + \mathbb{E}[1 - \omega(0,1)] P_{-1}[n \leq T_0 < \infty] \end{aligned} \qquad (6.2)$$

due to (IID). Because of (WEL) both $\mathbb{E}[\omega(0,1)] > 0$ and $\mathbb{E}[1 - \omega(0,1)] > 0$. Moreover, $P_{-1}[n \leq T_0 < \infty] = \widetilde{P}_1[n \leq T_0 < \infty]$, where $\widetilde{P}_1[\cdot] = \widetilde{\mathbb{E}}[P_{1,\omega}[\cdot]]$ and ω is distributed under $\widetilde{\mathbb{P}}$ as $1 - \omega$ under $\mathbb{P}$. The parameter $\widetilde{\delta}$ for $\widetilde{\mathbb{P}}$ is equal to $-\delta$. Therefore, (1.4) implies $n^{(|\delta-1| \wedge |-\delta-1|)/2} P_0[n < R < \infty] \to c_3$ as $n \to \infty$, i.e. (1.5), since $|\delta-1| \wedge |-\delta-1| = ||\delta|-1|$. Moreover, (6.2) and the second statement of (1.6) imply the third one. $\qquad\square$

7 Appendix: Further results from the literature

The results in this section do not depend on those of Sections 3–6.

Lemma 7.1. ([Bil99, Th. 3.2]) *Let (S, d) be a metric space. Suppose that $X_{m,n}, X_n,$ Z_m $(n, m \in \mathbb{N})$ and X are S-valued random variables such that $X_{n,m}$ and X_n are defined on the same probability space with probability measure P_n for all $n, m \in \mathbb{N}$. If $X_{m,n} \underset{n \to \infty}{\Longrightarrow} Z_m \underset{m \to \infty}{\Longrightarrow} X$ and*

$$\lim_{m \to \infty} \limsup_{n \to \infty} P_n \left[d(X_{m,n}, X_n) > \varepsilon \right] = 0 \tag{7.1}$$

for each $\varepsilon > 0$ then $X_n \underset{n \to \infty}{\Longrightarrow} X$.

Lemma 7.2 (Scaling of squared Bessel processes). *For $i = 1, 2$ let $x_i > 0$ and let $Y_i = (Y_i(t))_{t \geq 0}$ solve the SDE (3.1) with initial condition $Y_i(0) = x_i$. Then*

$$Y_1 \left(\cdot \wedge \sigma_0^{Y_1} \right) \quad \text{and} \quad \frac{x_1}{x_2} Y_2 \left(\frac{x_2}{x_1} \cdot \wedge \sigma_0^{Y_2} \right)$$

have the same distribution.

Proof. For $\delta \geq 0$ see [RY99, Ch. XI, (1.6) Prop.]. For general δ see [GY03, A.3] and [KM11, Lemma 3.2 (i)]. ∎

Lemma 7.3. *Let $\delta < 1$ and let $Y = (Y(t))_{t \geq 0}$ solve the SDE (3.1) with $Y(0) = 1$. Then there are constants $b_1(\delta), b_2(\delta) \in (0, \infty)$ such that*

$$\lim_{t \to \infty} t^{1-\delta} P_1[\sigma_0^Y > t] = b_1(\delta) \quad \text{and} \tag{7.2}$$

$$\lim_{t \to \infty} t^{(1-\delta)/2} P_1 \left[\int_0^{\sigma_0^Y} Y(s) \, ds > t \right] = b_2(\delta). \tag{7.3}$$

Proof. Statements (7.2) and (7.3) follow from Lemma 7.2 and are the contents of [KM11, Lemma 3.3] and [KM11, Lemma 3.5], respectively.

In fact, with respect to (7.2) much more is known. In [GY03, (15)] a formula for the density of the first passage time to 0 of a Bessel process with dimension in $[0, 2)$ is given. This formula implies (7.2) in the case $0 \leq \delta < 1$. It has been noticed in the remark after [Ale11, (4.24)] that the same formula also holds for negative dimensions. ∎

Lemma 7.4. (cf. [KM11, Lemma A.1]) *Let ξ_i, $i \in \mathbb{N}$, be independent random variables which are geometrically distributed with parameter $1/2$ and $E[\xi_i] = 1$. Then for all $x, y \in \mathbb{N}$,*

$$P \left[\left| \sum_{i=1}^x (\xi_i - 1) \right| \geq y \right] \leq 2 \exp \left(\frac{-y^2}{6(x \vee y)} \right).$$

Proof. We are going to use a special case of Azuma's inequality, which states that for the simple symmetric RW $(S_n)_{n \geq 0}$ on $\mathbb{Z}$ starting at 0 and any $a, n \geq 0$, $P[S_n \geq a] \leq \exp(-a^2/(2n))$, see e.g. [AS00, Th. A.1.1]. Let $(Y_i)_{i \geq 1}$ be an independent sequence of Bernoulli(1/2)-distributed random variables. By interpreting $\xi_i + 1$ as the time of the first appearance of "heads" in a sequence of independent fair coin flips we obtain

$$
\begin{aligned}
P \left[\sum_{i=1}^x (\xi_i - 1) \geq y \right] &= P \left[\sum_{i=1}^x (\xi_i + 1) \geq 2x + y \right] = P \left[\sum_{i=1}^{2x+y-1} Y_i < x \right] \\
&= P \left[\sum_{i=1}^{2x+y-1} (2Y_i - 1) < -(y - 1) \right] = P \left[S_{2x+y-1} \geq y \right] \\
&\leq e^{-y^2/(2(2x+y-1))} \leq e^{-y^2/(6(x \vee y))}
\end{aligned}
$$

by Azuma's inequality. Similarly, for $x \geq y$,

$$
\begin{aligned}
P\left[\sum_{i=1}^{x}(\xi_i - 1) \leq -y\right] &= P\left[\sum_{i=1}^{x}(\xi_i + 1) \leq 2x - y\right] = P\left[\sum_{i=1}^{2x-y} Y_i \geq x\right] \quad (7.4)\\
&= P\left[S_{2x-y} \geq y\right] \leq e^{-y^2/(2(2x-y))} \leq e^{-y^2/(6(x \vee y))}
\end{aligned}
$$

again by Azuma's inequality. For $x < y$ the quantities in (7.4) are 0. A union bound now yields the claim. $\qquad \square$

The following lemmas about the BPs $V, \overline{V}$, and $\widetilde{V}$ are slight modifications of results from [KM11]. The first one controls how much the BPs V and $\overline{V}$ "overshoot" x at the times τ_x and σ_x.

Lemma 7.5 (Overshoot). *There are constants $c_{13}, c_{14} > 0$ and $N \in \mathbb{N}$ such that for every $x \geq N$, $y \geq 0$, and $\varepsilon > 0$,*

$$
\max_{0 < z < x} P_z\left[V_{\tau_x^V} > x + y \mid \tau_x^V < \sigma_0^V\right] \leq c_{13}(e^{-c_{14}y^2/x} + e^{-c_{14}y}), \quad (7.5)
$$

$$
\max_{x < z < 4x} P_z[V_{\sigma_x^V \wedge \tau_{4x}^V} < x - y] \leq c_{13}e^{-c_{14}y^2/x}, \qquad and \quad (7.6)
$$

$$
\max_{0 < z < x} P_z\left[\tau_x^{\overline{V}} < \infty, \, \overline{V}_{\tau_x^{\overline{V}}} > (1 + \varepsilon)x\right] \leq c_{13}(e^{-c_{14}\varepsilon^2 x} + e^{-c_{14}\varepsilon x}). \quad (7.7)
$$

Proof. The proofs of the first two statements repeat those of [KM11, Lemma 5.1]. For the third statement note that by definition of $\overline{V}$,

$$
\begin{aligned}
P_z\left[\tau_x^{\overline{V}} < \infty, \, \overline{V}_{\tau_x^{\overline{V}}} > (1 + \varepsilon)x\right] &= P_z\left[\tau_x^V < \infty, \, V_{\tau_x^V} > (1 + \varepsilon)x \mid \sigma_0^V < \infty\right]\\
&= \frac{P_z\left[\sigma_0^V < \infty \mid \tau_x^V < \infty, \, V_{\tau_x^V} > (1 + \varepsilon)x\right]}{P_z\left[\sigma_0^V < \infty\right]} P_z\left[\tau_x^V < \infty, \, V_{\tau_x^V} > (1 + \varepsilon)x\right]\\
&\leq P_z\left[\tau_x^V < \infty, \, V_{\tau_x^V} > (1 + \varepsilon)x\right]
\end{aligned}
$$

since by the strong Markov property and monotonicity with respect to the starting point, the fraction above does not exceed 1 for all $0 < z < x$. An application of (7.5) with $y = \varepsilon x$ completes the proof. $\qquad \square$

Lemma 7.6. *Let $\delta > 1$, $a \in (1, 2]$, and $\gamma_n := \inf\{k \geq 0 : V_k \notin (a^{n-1}, a^{n+1})\}$ for $n \in \mathbb{N}$. Then for all sufficiently large n and all $x \in \mathbb{N}$ such that $|x - a^n| \leq a^{2n/3}$,*

$$
P_x\left[\operatorname{dist}(V_{\gamma_n}, (a^{n-1}, a^{n+1})) \geq a^{2(n-1)/3}\right] \leq \exp(-a^{n/4}) \quad and
$$

$$
\left|P_x[V_{\gamma_n} \leq a^{n-1}] - \frac{1}{a^{\delta-1} + 1}\right| \leq a^{-n/4}.
$$

Proof. The proof is identical to that of [KM11, Lemma 5.2] if one replaces δ with $1 - \delta$ throughout. $\qquad \square$

Lemma 7.7. *Let $\delta > 1, a \in (1, 2], \lambda \in (0, 1/8)$, and*

$$
h_\ell^\pm(m) := \prod_{r=\ell+1}^{m} \left(a^{\delta-1} \mp a^{-\lambda r}\right)^{-1} \qquad for\ all\ \ell, m \in \mathbb{N},\ \ell < m.
$$

Then there are $K_1, K_2 : \mathbb{N} \to (0, \infty)$ such that $K_1(\ell), K_2(\ell) \to 1$ as $\ell \to \infty$ and

$$
K_1(\ell) \leq h_\ell^\pm(m)\, a^{(m-\ell)(\delta-1)} \leq K_2(\ell) \quad (7.8)
$$

for all $m > \ell \geq 0$. Moreover, there is $\ell_0 \in \mathbb{N}$ such that if $\ell, m, x \in \mathbb{N}$ satisfy $\ell_0 \leq \ell < m$ and $|x - a^m| \leq a^{2m/3}$ then

$$
h_\ell^-(m) \leq P_x[\sigma_{a^\ell}^V < \infty] \leq h_\ell^+(m). \quad (7.9)
$$

Proof. To see (7.8) we notice that $h_\ell^-(m) \leq h_\ell^+(m)$ for all $m > \ell$ and

$$h_\ell^+(m)a^{(m-\ell)(\delta-1)} = \prod_{r=\ell+1}^{m}(1-a^{-\lambda r - \delta+1})^{-1} \leq \prod_{r=\ell+1}^{\infty}(1-a^{-\lambda r})^{-1} =: K_1(\ell) < \infty;$$

$$h_\ell^-(m)a^{(m-\ell)(\delta-1)} = \prod_{r=\ell+1}^{m}(1+a^{-\lambda r - \delta+1})^{-1} \geq \prod_{r=\ell+1}^{\infty}(1+a^{-\lambda r})^{-1} =: K_2(\ell) > 0.$$

Clearly, $K_1(\ell), K_2(\ell) \to 1$ as $\ell \to \infty$.

Inequalities (7.9) will follow if we show that for all $u > m$,

$$\frac{h_\ell^-(m) - h_\ell^-(u)}{1 - h_\ell^-(u)} \leq P_x[\sigma_{a^\ell}^V < \tau_{a^u}^V] \leq \frac{h_\ell^+(m) - h_\ell^+(u)}{1 - h_\ell^+(u)}$$

and then let $u \to \infty$. (Note that (7.8) implies that $h_\ell^\pm(u) \to 0$ as $u \to \infty$.) The inequalities in the previous line are equivalent to

$$\frac{1 - h_\ell^+(m)}{1 - h_\ell^+(u)} \leq P_x[\sigma_{a^\ell}^V > \tau_{a^u}^V] \leq \frac{1 - h_\ell^-(m)}{1 - h_\ell^-(u)}.$$

The proof of the last inequalities follows essentially word for word the one of [KM11, Lemma 5.3] and uses Lemma 7.5 ((7.5) and (7.6)) and Lemma 7.6 in place of Lemma 5.1 and Lemma 5.2 of [KM11], respectively. □

Lemma 7.8. *Let $\delta < 1$. For every $\gamma > 0$ there is $c_{15}(\gamma) \in (0, \infty)$ such that $P_n[\tau_{c_{15}n}^V < \sigma_0^V] < \gamma$ for all $n \in \mathbb{N}$. Moreover, $P_n[\sigma_0^V < \infty] = 1 = P_n[\sigma_0^{\widetilde{V}} < \infty]$ for all $n \in \mathbb{N}$. Similarly, for every $\gamma > 0$ there is $c_{16}(\gamma) \in (0, \infty)$ such that $P_n[\tau_{c_{16}n}^{\widetilde{V}} < \sigma_0^{\widetilde{V}}] < \gamma$ for all $n \in \mathbb{N}$.*

Proof. For the proof of the first claim about V see the proof of [KM11, (5.5)] using Lemmas 7.5 and 7.7 instead of Lemma 5.1 and Lemma 5.3 of [KM11], respectively, and replacing δ with $1 - \delta$ throughout.

For the second statement observe that P_n-a.s. $\sigma_0^V \wedge \tau_{c_{15}n}^V < \infty$ due to (2.3) and (2.6). This together with the first statement implies P_n-a.s. $\sigma_0^V < \infty$.

For the third assertion observe that due to (2.3), (2.4), and (2.6), $P_n[\sigma_0^{\widetilde{V}} = 1] > 0$ for all $n \geq 1$. Therefore, $P_n[\sigma_0^{\widetilde{V}} = \infty] \leq P_n[\widetilde{V}_m \to \infty]$, which is equal to 0 since $\widetilde{V} = V$ up to time $\sigma_M^{\widetilde{V}}$ and therefore for all $i \geq 1$, $P_i[\sigma_M^{\widetilde{V}} < \infty] = P_i[\sigma_M^V < \infty] \geq P_i[\sigma_0^V < \infty] = 1$ as we have already shown.

For the final statement let $t = c_{15}(\gamma/2)$. Then

$$
\begin{aligned}
\limsup_{n\to\infty} P_n\left[\tau_{tn}^{\widetilde{V}} < \sigma_0^{\widetilde{V}}\right] &\leq \limsup_{n\to\infty} P_n\left[\tau_{tn}^{\widetilde{V}} < \sigma_M^{\widetilde{V}}\right] + \limsup_{n\to\infty} P_n\left[\sigma_M^{\widetilde{V}} < \tau_{tn}^{\widetilde{V}} < \sigma_0^{\widetilde{V}}\right] \\
&\leq \limsup_{n\to\infty} P_n\left[\tau_{tn}^V < \sigma_M^V\right] + \max_{i=1,\ldots,M}\limsup_{n\to\infty} P_i\left[\tau_{tn}^{\widetilde{V}} < \sigma_0^{\widetilde{V}}\right] \\
&\leq \gamma/2 + \max_i P_i[\sigma_0^{\widetilde{V}} = \infty].
\end{aligned}
$$

Since $P_i[\sigma_0^{\widetilde{V}} = \infty] = 0$ for all i this implies the claim. □

Corollary 7.9 (Extinction and survival of V)**.** *If $\delta < 1$ then $P_1[\sigma_0^V < \infty] = 1$. If $\delta > 1$ then for all $N \geq 0$ it holds that $P_n[\sigma_N^V < \infty] \to 0$ as $n \to \infty$ and, moreover, $P_1[\sigma_0^V = \infty] > 0$.*

Proof. The first claim is contained in Lemma 7.8. Since $P_n[\sigma_N^V < \infty]$ is non-decreasing in N for all $n > N$, Lemma 7.7 implies the second claim. Finally, choose $n \in \mathbb{N}$ such that $P_n[\sigma_0^V < \infty] < 1$. Then $P_1[\sigma_0^V = \infty] \geq P_n[\sigma_0^V = \infty] > 0$. □

As a result we recover, except for the critical case $|\delta| = 1$, the recurrence-transience criterion [KZ08, Th. 1] in a more self-contained way without using [FYK90].

Corollary 7.10 (Recurrence and transience of ERW)**.** *If $|\delta| < 1$ then the ERW X returns a.s. infinitely often to its starting point. If $\delta > 1$ then $X_n \to \infty$ a.s., whereas $X_n \to -\infty$ a.s. if $\delta < -1$.*

Proof. The proof goes along the lines of proof of [KZ08, Th. 1] except that [KZ08, Prop. 9] (for $|\delta| \neq 1$) now follows from Corollary 7.9 and $P_1[T_0 < \infty] = P_1[\sigma_0^V < \infty]$, see (2.1) and (2.8). $\qquad\square$

References

[Afa9]   V. I. Afanasyev. On the maximum of a critical branching process in a random environment. *Discrete Math. Appl.*, 9(3):267–284, 1999. MR-1712160

[Ale11]   K. Alexander. Excursions and local limit theorems for Bessel-like random walks. *Electron. J. Probab.*, 16(1):1–44, 2011. MR-2749771

[Ali99]   S. Alili. Asymptotic behaviour for random walks in random environments. *J. Appl. Probab.*, 36(2):334–349, 1999. MR-1724844

[AS00]   N. Alon and J. H. Spencer. *The probabilistic method.* 2nd ed. Wiley, New York, 2000. MR-1885388

[AR05]   T. Antal and S. Redner. The excited random walk in one dimension. *J. Phys. A*, 38(12):2555–2577, 2005. MR-2132073

[AK71]   K. B. Athreya and S. Karlin. On branching processes with random environments. I. Extinction probabilities. *Ann. Math. Statist.* 42:1499–1520, 1971. MR-0298780

[AN72]   K. B. Athreya and P. E. Ney. *Branching processes.* Reprint of the 1972 original. Mineola, NY: Dover Publications. xi, 287 p., 2004. MR-2047480

[BS08]   A.-L. Basdevant and A. Singh. On the speed of a cookie random walk. *Probab. Theory Related Fields*, 141(3-4):625–645, 2008. MR-2391167

[Bau13]   E. Bauernschubert. Perturbing transient random walk in a random environment with cookies of maximal strength. *Ann. Inst. H. Poincaré Probab. Statist.*, 49(3):638–653, 2013. MR-3112429

[Bau14]   E. Bauernschubert. Recurrence and transience of critical branching processes in random environment with immigration and an application to excited random walks. To appear in *Adv. in Appl. Probab.*, 46(3), 2014. arXiv:1301.5450

[Bil99]   P. Billingsley. *Convergence of probability measures.* 2nd ed. Wiley, New York, 1999. MR-1700749

[EK86]   S. Ethier and T. Kurtz. *Markov processes. Characterization and convergence.* John Wiley & Sons Inc., New York, 1986. MR-0838085

[Fel51]   W. Feller. Diffusion processes in genetics. *Proc. of the Second Berkeley Symposium on Mathematical Statistics and Probability*, 1950, pp. 227–246. University of California Press, Berkeley and Los Angeles, 1951. MR-0046022

[FYK90]   Sh. K. Formanov, Makhmud Takha Yasin, and S. V. Kaverin. Life spans of Galton-Watson processes with migration. (Russian) In *Asymptotic problems in probability theory and mathematical statistics,* edited by T. A. Azlarov and Sh. K. Formanov. Tashkent: Fan. 176 p., pp. 117–135, 1990. MR-1142599

[GY03]   A. Göing-Jaeschke and M. Yor. A survey and some generalizations of Bessel processes. *Bernoulli* 9(2):313–349, 2003. MR-1997032

[Har52]   T. E. Harris. First passage and recurrence distributions. *Trans. Amer. Math. Soc.*, 73:471–486, 1952. MR-0052057

[Höp85]   R. Höpfner. A note on the probability of extinction in a class of population-size-dependent Galton-Watson processes. *J. Appl. Probab.*, 22(4):920–925, 1985. MR-0808871

[Hug95]   B. D. Hughes. *Random walks and random environments.* Vol. 1. Random walks. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1995. MR-1341369

[IW89]   N. Ikeda and S. Watanabe. *Stochastic differential equations and diffusion processes.* Second edition. North-Holland, Amsterdam, 1989. MR-1011252

[IS85]   B. G. Ivanoff and E. Seneta. The critical branching process with immigration stopped at zero. *J. Appl. Probab.*, 22(1):223–227, 1985. MR-0776902

[JS87]   J. Jacod and A. N. Shiryaev. *Limit theorems for stochastic processes.* Springer, Berlin, 1987. MR-0959133

[KM11]   E. Kosygina and T. Mountford. Limit laws of transient excited random walks on integers. *Ann. Inst. H. Poincaré Probab. Stat.*, 47(2):575–600, 2011. MR-2814424

[KZ08]   E. Kosygina and M. P. W. Zerner. Positively and negatively excited random walks on integers, with branching processes. *Electron. J. Probab.*, 13:no. 64, 1952–1979, 2008. MR-2453552

[KZ13]   E. Kosygina and M. P. W. Zerner. Excited random walks: results, methods, open problems. *Bull. Inst. Math. Acad. Sin. (N.S.)*, 8(1):105–157, 2013. MR-3097419

[LPW09]   D. A. Levin, Y. Peres, and E. L. Wilmer. *Markov chains and mixing times.* AMS, Providence, 2009. MR-2466937

[Mel83]   B. Mellein. Kac functionals of diffusion processes approximating critical branching processes. *Bol. Soc. Mat. Mexicana (2)* 28(2):95–107, 1983. MR-0791092

[Pak71]   A. G. Pakes. On the critical Galton-Watson process with immigration. *J. Austral. Math. Soc.* 12:476–482, 1971. MR-0307370

[Pak72]   A. G. Pakes. Further results on the critical Galton-Watson process with immigration. *J. Austral. Math. Soc.* 13:277–290, 1972. MR-0312585

[Pet13]   J. Peterson. Strict monotonicity properties in one-dimensional excited random walks. *Markov Process. Related Fields.* 19(4):721–734, 2013. arXiv:1210.4518

[RY99]   D. Revuz and M. Yor. *Continuous martingales and Brownian motion.* Third edition. Springer, Berlin, 1999. MR-1725357

[RW00]   L. C. G. Rogers and D. Williams. *Diffusions, Markov processes and martingales.* Vol. 2, Cambridge University Press, Reprint of the second (1994) edition, 2000. MR-1780932

[SW69]   W. L. Smith and W. E. Wilkinson. On branching processes in random environments. *Ann. Math. Statist.* 40(3):814–827, 1969. MR-0246380

[Sol75]   F. Solomon. Random walks in a random environment. *Ann. Probab.* 3(1):1–31, 1975. MR-0362503

[Tó96]   B. Tóth. Generalized Ray-Knight theory and limit theorems for self-interacting random walks on $\mathbb{Z}$. *Ann. Probab.*, 24:1324–1367, 1996. MR-1411497

[YW71]   T. Yamada and S. Watanabe. On the uniqueness of solutions of stochastic differential equations. *J. Math. Kyoto Univ.* 11:155–167, 1971. MR-0278420

[Zer05]   M. P. W. Zerner. Multi-excited random walks on integers. *Probab. Theory Related Fields*, 133(1):98–122, 2005. MR-2197139

[Zub72]   A. M. Zubkov. Life-periods of a branching process with immigration. *Theory Probab. Appl.*, 17(1):174–183, 1972. MR-0300351

Electronic Journal of Probability
Electronic Communications in Probability

Advantages of publishing in EJP-ECP

- Very high standards

- Free for authors, free for readers

- Quick publication (no backlog)

Economical model of EJP-ECP

- Low cost, based on free software (OJS[1])

- Non profit, sponsored by IMS[2], BS[3], PKP[4]

- Purely electronic and secure (LOCKSS[5])

Help keep the journal free and vigorous

- Donate to the IMS open access fund[6] (click here to donate!)

- Submit your best articles to EJP-ECP

- Choose EJP-ECP over for-profit journals

---

[1]OJS: Open Journal Systems http://pkp.sfu.ca/ojs/
[2]IMS: Institute of Mathematical Statistics http://www.imstat.org/
[3]BS: Bernoulli Society http://www.bernoulli-society.org/
[4]PK: Public Knowledge Project http://pkp.sfu.ca/
[5]LOCKSS: Lots of Copies Keep Stuff Safe http://www.lockss.org/
[6]IMS Open Access Fund: http://www.imstat.org/publications/open.htm