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A forest-fire model on the upper half-plane*

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Abstract

We consider a discrete forest-fire model on the upper half-plane of the two-dimensional square lattice. Each site can have one of the following two states: “vacant” or “occupied by a tree”. At the starting time all sites are vacant. Then the process is governed by the following random dynamics: Trees grow at rate 1, independently for all sites. If an occupied cluster reaches the boundary of the upper half-plane or if it is about to become infinite, the cluster is instantaneously destroyed, i.e. all of its sites turn vacant. Additionally, we demand that the model is invariant under translations along the x-axis.

We prove that such a model exists and arises naturally as a subsequential limit of forest-fire processes in finite boxes when the box size tends to infinity.

Moreover, the model exhibits a phase transition in the following sense: There exists a critical time t_c (which corresponds with the critical probability p_c in ordinary site percolation by $1 - e^{-t_c} = p_c$) such that before t_c, only sites close to the boundary have been affected by destruction, whereas after t_c, sites on the entire half-plane have been affected by destruction.

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1 Introduction and statement of the main results

Forest-fire models were first introduced in the physics literature by B. Drossel and F. Schwabl in [6] and subsequently studied by various mathematicians, e.g. by J. van den Berg and R. Brouwer (see [17], [18]), by M. Dürre (see [7], [8], [9]), by A. Stahl (see [16]), by B. Ráth and B. Tóth (see [14]), and by X. Bressaud and N. Fournier (see [3], [4]). They were devised as an example of self-organized criticality, a concept brought up by P. Bak, C. Tang and K. Wiesenfeld in their seminal paper [2]. Let us begin with a brief description of critical states and self-organized criticality. Models in equilibrium statistical mechanics such as independent site percolation or the Ising model usually have a model parameter which greatly influences their behaviour (the density p of open sites in the case of percolation and the inverse temperature β in the case of the Ising

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model). As the parameter varies, the system experiences a phase transition at the critical value of the parameter (the critical probability p_c for percolation and the critical temperature β_c for the Ising model). Formally, the critical value can be defined as the threshold between the regime with no infinite cluster and the regime with one infinite cluster. The critical state is typically characterized by a power law behaviour of the cluster size distribution and fractal structures in the scaling limit. These phenomena are frequently observed in nature, which seems strange in the context of equilibrium statistical mechanics because this would require the model parameter to be tuned at exactly the right value. To explain this paradoxon, the authors of [2] predicted the existence of dynamical systems which are governed by local interactions and which are inherently driven towards an unstable critical state. This critical state shows properties similar to the equilibrium case. What is more, at the critical state the local interactions build up to trigger global "catastrophic" events and thus possibly return the system back into a more stable state. These kinds of systems are said to exhibit self-organized criticality. A detailed introduction to this concept with many examples of dynamical systems can be found in [1] and [12].

The forest-fire models we consider in this paper will all be defined on subsets of the square lattice $\mathbb{Z}^2$. We always assume the vertex set $\mathbb{Z}^2$ to be equipped with the standard lattice edge set, where two sites in $\mathbb{Z}^2$ are connected by an edge if and only if they have Euclidean distance 1. For practical purposes we will identify $\mathbb{Z}^2 \subset \mathbb{R}^2$ with $\mathbb{Z} + i\mathbb{Z} \subset \mathbb{C}$ (where $i := \sqrt{-1} = (0,1)$) and mostly use the complex number notation even though we do not use the multiplicative structure of $\mathbb{C}$. The finite volume versions of the model will be defined on boxes

$$B_n(w) := w + [-n,n]^2 \cap \mathbb{Z}^2$$

with centre $w \in \mathbb{Z}^2$ and radius $n \in \mathbb{N}$. To begin with, we endow the vertex set $B_n(w)$ with the standard edges inherited from the square lattice $\mathbb{Z}^2$ and we denote this by writing $B_n^{\mathrm{s}}(w)$ instead of $B_n(w)$. Later on, for each $k \in \{-n, -n+1, \dots, n\}$, we will insert an additional edge between the vertex $w - n + ik$ on the left and the vertex $w + n + ik$ on the right in order to make the setup periodic in the x-direction; in this case we write $B_n^{\mathrm{p}}(w)$ instead of $B_n(w)$. The graph $B_n^{\mathrm{p}}(w)$ is best visualized as a cylinder. The infinite volume version of the forest-fire model will be defined on the "closed" upper half-plane

$$\overline{\mathbb{H}} := \{x + iy \in \mathbb{Z} + i\mathbb{Z} : y \geq 0\},$$

which we endow with the edges inherited from the square lattice $\mathbb{Z}^2$. We will also denote by

$$\mathbb{H} := \{x + iy \in \mathbb{Z} + i\mathbb{Z} : y > 0\}$$

the "open" upper half-plane.

In order to explain some more notation, let us for a moment consider an arbitrary connected graph with vertex set V. (In practice, this will usually be one of the graphs $B_n^{\mathrm{s}}(w)$, $\mathbb{Z}^2$, $B_n^{\mathrm{p}}(w)$ or $\overline{\mathbb{H}}$.) For a subset $S \subset V$, we write

$$\partial S := \{v \in V \setminus S : (\exists w \in S : v \text{ and } w \text{ are neighbours})\}$$

for the **(outer) boundary** of S in V. For the subset $\mathbb{H} \subset \overline{\mathbb{H}}$, for instance, we simply have $\partial \mathbb{H} = \mathbb{Z}$. At any given time, the forest-fire model will be described by a random configuration $(\alpha_v)_{v \in V} \in \{0,1\}^V$, which induces a subgraph of V on the vertex set $\{v \in V : \alpha_v = 1\}$. For $z \in V$ the maximal connected component of this subgraph containing z is called the **cluster** of z in the configuration $(\alpha_v)_{v \in V}$. If $\alpha_z = 0$, then the cluster of z is just the empty set.

We are now ready to describe the forest-fire model on the box $B_n^{\mathrm{s}}(0)$ ($n \in \mathbb{N}$) which is the starting point of our work. It is a continuous-time Markov process on the state space $\{0,1\}^{B_n^{\mathrm{s}}(0)}$, where a site with "1" is said to be "occupied by a tree" and a site with "0" is said to be "vacant". At the starting time all sites are vacant. Then the process is governed by the following two conflicting mechanisms:

[GROWTH] Sites turn from "vacant" to "occupied" according to independent rate 1 Poisson processes.

[DESTRUCTION] If an occupied cluster reaches the inner boundary $B_n(0) \setminus B_{n-1}(0)$ of the box[1], it is instantaneously destroyed, i.e. all of its sites turn vacant.

The most interesting aspect about this model is the question of what happens in the limit $n \to \infty$ (provided that it exists in a suitable sense). It is in this limit that the model is expected to exhibit self-organized criticality, and the intuitive reasoning goes as follows: For large n, small clusters are unlikely to get destroyed but sufficiently large clusters are still vulnerable to destruction. So a hypothetical limit process on $\mathbb{Z}^2$ might have the following dynamics: At the starting time all sites are vacant. Then the process is governed by the following two conflicting mechanisms:

[GROWTH] Sites turn from "vacant" to "occupied" according to independent rate 1 Poisson processes.

[DESTRUCTION] If an occupied cluster becomes infinite, it is instantaneously destroyed, i.e. all of its sites turn vacant.

In such a process the states with an emergent infinite cluster could be dubbed self-organized critical, and the destruction of the infinite cluster would correspond to the global "catastrophic" events mentioned above. However, it is not even clear that such a process exists at all (see [17] for a discussion of that question), and a mathematically rigorous treatment of the question of convergence for $n \to \infty$ currently seems hard to achieve.

A first step towards a better understanding of the $n \to \infty$ limit probably lies in the analysis of the behaviour of the sites close to the inner boundary when n is large. We therefore change our perspective in the following way:

- Instead of keeping the centre of the box fixed and letting the box tend to infinity in all four directions, we keep the bottom side fixed and let the box tend to infinity in the remaining three directions. In other words, we consider the process on the box $B_n(in)$ instead of the box $B_n(0)$. In the (subsequential) limit $n \to \infty$ we thus get a process on the upper half-plane $\overline{\mathbb{H}}$.

Additionally, we make the following changes, which are natural for the new setting:

- We restrict the destruction mechanism [DESTRUCTION] to clusters which reach the fixed bottom side instead of destroying clusters at all four sides.
- We use periodic boundary conditions in the x-direction, i.e. we work on $B_n^{\mathrm{p}}(in)$ instead of $B_n^{\mathrm{s}}(in)$.

Let us define this new process more formally, in a fashion similar to the definition of the Max Dürre forest-fire model in [7]. We include the underlying Poisson growth processes into our notation and thus obtain a continuous-time process on the state space $(\{0,1\} \times \mathbb{N}_0)^{B_n^{\mathrm{p}}(in)}$. For convenience we henceforth abbreviate $B_n := B_n^{\mathrm{p}}(in)$. Figure 1 depicts the box B_n and its edges, embedded into the upper half-plane $\overline{\mathbb{H}}$. In

[1] where we set $B_0(0) := \{0\}$

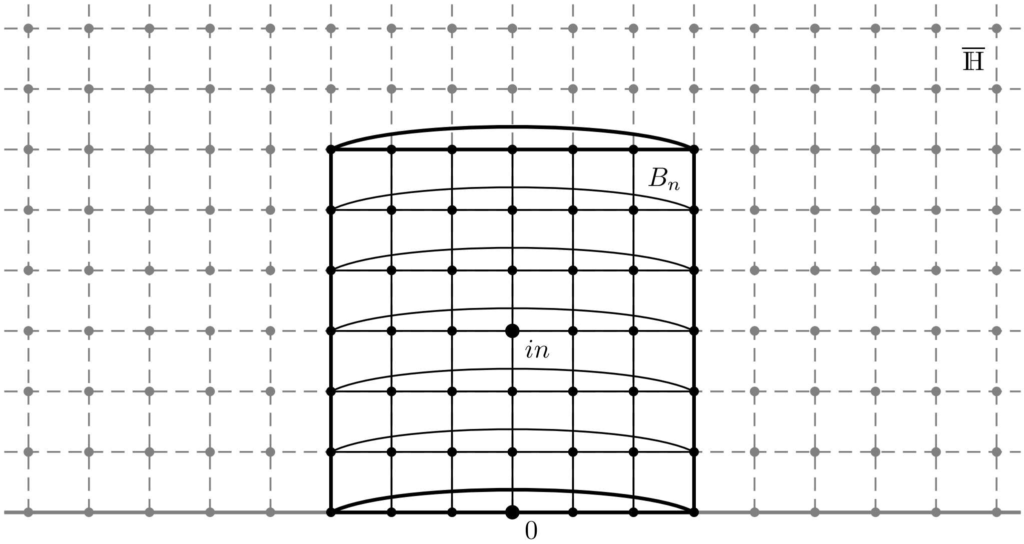

Figure 1: The box $B_n := B_n^{\mathrm{p}}(in)$ for $n = 3$ (black) and the upper half-plane $\overline{\mathbb{H}}$ (grey)

accordance with the periodic boundary conditions of B_n, for $z \in B_n$ and $x \in \mathbb{Z}$ we define "periodic addition" by

$$z \oplus x := [((\operatorname{Re} z + x) + n) \bmod (2n+1)] - n + i \operatorname{Im} z \in B_n.$$

Moreover, for a function $[0,\infty) \ni t \mapsto f_t \in \mathbb{R}$ we write $f_{t^-} := \lim_{s \uparrow t} f_s$ for the left-sided limit at $t > 0$, provided the limit exists.

Definition 1.1. *Let $n \in \mathbb{N}$. Let $(\eta^n_{t,z}, G^n_{t,z})_{t \geq 0, z \in B_n}$ be a process[2] with values in $(\{0,1\} \times \mathbb{N}_0)^{[0,\infty) \times B_n}$, initial condition $\eta^n_{0,z} = 0$ for $z \in B_n$ and boundary condition $\eta^n_{t,x} = 0$ for $t \geq 0, x \in \partial\mathbb{H} \cap B_n$. Suppose that for all $z \in B_n$ the process $(\eta^n_{t,z}, G^n_{t,z})_{t \geq 0}$ is càdlàg, i.e. right-continuous with left limits. For $z \in B_n$ and $t > 0$, let $C^n_{t^-,z}$ denote the cluster of z in the configuration $(\eta^n_{t^-,w})_{w \in B_n}$.*

*Then $(\eta^n_{t,z}, G^n_{t,z})_{t \geq 0, z \in B_n}$ is called a B_n**-forest-fire process** if the following conditions are satisfied:*

[POISSON] *The processes $(G^n_{t,z})_{t \geq 0}$, $z \in B_n$, are independent Poisson processes with rate 1.*

[ROT-INV] *The distribution of $(\eta^n_{t,z}, G^n_{t,z})_{t \geq 0, z \in B_n}$ is invariant under rotations of the cylinder B_n, i.e. the processes $(\eta^n_{t,z}, G^n_{t,z})_{t \geq 0, z \in B_n}$ and $(\eta^n_{t,z\oplus 1}, G^n_{t,z\oplus 1})_{t \geq 0, z \in B_n}$ have the same distribution.*

[GROWTH] *For all $t > 0$ and all $z \in \mathbb{H} \cap B_n$ the following implications hold:*

(i) $G^n_{t^-,z} < G^n_{t,z} \Rightarrow \eta^n_{t,z} = 1$,
i.e. the growth of a tree at the site z at time t implies that the site z is occupied at time t;

(ii) $\eta^n_{t^-,z} < \eta^n_{t,z} \Rightarrow G^n_{t^-,z} < G^n_{t,z}$,
i.e. if the site z gets occupied at time t, there must have been the growth of a tree at the site z at time t.

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[2] A more precise but more cumbersome notation would be $\big((\eta^n_{t,z}, G^n_{t,z})_{z \in B_n}\big)_{t \geq 0}$.

[DESTRUCTION] *For all $t > 0$ and all $x \in \partial\mathbb{H} \cap B_n$, $z \in \mathbb{H} \cap B_n$ the following implications hold:*

(i) $G^n_{t^-,x} < G^n_{t,x} \Rightarrow \forall w \in C^n_{t^-,x+i} : \eta^n_{t,w} = 0$,
i.e. if the cluster at $x + i$ grows to the boundary $\partial\mathbb{H} \cap B_n$ at time t, it is destroyed at time t;

(ii) $\eta^n_{t^-,z} > \eta^n_{t,z} \Rightarrow \exists u \in \partial C^n_{t^-,z} \cap \partial\mathbb{H} : G^n_{t^-,u} < G^n_{t,u}$,
i.e. if the site z is destroyed at time t, its cluster must have grown to the boundary $\partial\mathbb{H} \cap B_n$ at time t.

Due to the finiteness of the box B_n, the existence and uniqueness (in distribution) of a B_n-forest-fire process is clear: Given independent rate 1 Poisson processes $(G^n_{t,z})_{t\geq 0}$, $z \in B_n$, a unique corresponding càdlàg process $(\eta^n_{t,z})_{t\geq 0}$, $z \in B_n$, which has the required initial and boundary conditions and satisfies [GROWTH] and [DESTRUCTION] can be obtained by a so-called graphical construction, and [ROT-INV] then follows automatically by the rotation-invariance of the cylinder B_n. For more details on graphical constructions, the reader is referred to [13].

Above, we raised the question of what happens with forest-fire processes on boxes of size n when $n \to \infty$. As far as the dynamics are concerned, this question is partially answered for B_n-forest-fire processes by the following result, where $\mathbb{Q}^+_0 := \mathbb{Q} \cap [0, \infty)$ denotes the set of non-negative rational numbers:

Theorem 1.2. *For $n \in \mathbb{N}$ let $(\eta^n_{t,z}, G^n_{t,z})_{t\geq 0, z\in B_n}$ be a B_n-forest-fire process. Embed this process into the upper half-plane $\overline{\mathbb{H}}$ by setting $(\eta^n_{t,z}, G^n_{t,z}) := (0, 0)$ for $z \in \overline{\mathbb{H}} \setminus B_n$ and all $t \geq 0$. Then for any strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ of natural numbers, there exists a subsequence $(n_{k_l})_{l\in\mathbb{N}}$ such that $(\eta^{n_{k_l}}_{t,z}, G^{n_{k_l}}_{t,z})_{t\in\mathbb{Q}^+_0, z\in\overline{\mathbb{H}}}$ converges weakly to some random variable $(\eta^{\mathbb{Q}}_{t,z}, G^{\mathbb{Q}}_{t,z})_{t\in\mathbb{Q}^+_0, z\in\overline{\mathbb{H}}}$, where convergence is understood in the space $(\{0, 1\} \times \mathbb{N}_0)^{\mathbb{Q}^+_0 \times\overline{\mathbb{H}}}$ endowed with the product topology. Moreover, the right-sided limit*

$$(\eta_{t,z}, G_{t,z}) := \lim_{s\downarrow t, s\in\mathbb{Q}^+_0} (\eta^{\mathbb{Q}}_{s,z}, G^{\mathbb{Q}}_{s,z}), \qquad t \geq 0, z \in \overline{\mathbb{H}},$$

exists a.s., and restricted to the complement of a null set, the resulting process $(\eta_{t,z}, G_{t,z})_{t\geq 0, z\in\overline{\mathbb{H}}}$ is an $\overline{\mathbb{H}}$-forest-fire process in the sense of Definition 1.3 below.

Definition 1.3. *Let $(\eta_{t,z}, G_{t,z})_{t\geq 0, z\in\overline{\mathbb{H}}}$ be a process[3] with values in $(\{0, 1\} \times \mathbb{N}_0)^{[0,\infty)\times\overline{\mathbb{H}}}$, initial condition $\eta_{0,z} = 0$ for $z \in \overline{\mathbb{H}}$ and boundary condition $\eta_{t,x} = 0$ for $t \geq 0, x \in \partial\mathbb{H}$. Suppose that for all $z \in \overline{\mathbb{H}}$ the process $(\eta_{t,z}, G_{t,z})_{t\geq 0}$ is càdlàg. For $z \in \overline{\mathbb{H}}$ and $t > 0$, let $C_{t^-,z}$ denote the cluster of z in the configuration $(\eta_{t^-,w})_{w\in\overline{\mathbb{H}}}$.*

*Then $(\eta_{t,z}, G_{t,z})_{t\geq 0, z\in\overline{\mathbb{H}}}$ is called an $\overline{\mathbb{H}}$-**forest-fire process** if the following conditions are satisfied:*

[POISSON] *The processes $(G_{t,z})_{t\geq 0}$, $z \in \overline{\mathbb{H}}$, are independent Poisson processes with rate 1.*

[TRANSL-INV] *The distribution of $(\eta_{t,z}, G_{t,z})_{t\geq 0, z\in\overline{\mathbb{H}}}$ is invariant under translations along the real line, i.e. the processes $(\eta_{t,z}, G_{t,z})_{t\geq 0, z\in\overline{\mathbb{H}}}$ and $(\eta_{t,z+1}, G_{t,z+1})_{t\geq 0, z\in\overline{\mathbb{H}}}$ have the same distribution.*

[GROWTH] *For all $t > 0$ and all $z \in \mathbb{H}$ the following implications hold:*

[3] Again, a more precise but more cumbersome notation would be $\big((\eta_{t,z}, G_{t,z})_{z\in\overline{\mathbb{H}}}\big)_{t\geq 0}$.

(i) $G_{t^-,z} < G_{t,z} \Rightarrow \eta_{t,z} = 1$,
i.e. the growth of a tree at the site z at time t implies that the site z is occupied at time t;

(ii) $\eta_{t^-,z} < \eta_{t,z} \Rightarrow G_{t^-,z} < G_{t,z}$,
i.e. if the site z gets occupied at time t, there must have been the growth of a tree at the site z at time t.

[DESTRUCTION] *For all $t > 0$ and all $x \in \partial\mathbb{H}$, $z \in \mathbb{H}$ the following implications hold:*

(i) $\big(G_{t^-,x} < G_{t,x} \Rightarrow \forall w \in C_{t^-,x+i} : \eta_{t,w} = 0\big) \wedge$
$\big(|C_{t^-,z}| = \infty \Rightarrow \forall w \in C_{t^-,z} : \eta_{t,w} = 0\big)$,
i.e. if the cluster at $x + i$ grows to the boundary $\partial\mathbb{H}$ at time t, it is destroyed at time t, and if the cluster at z is about to become infinite at time t, it is destroyed at time t;

(ii) $\eta_{t^-,z} > \eta_{t,z}$
$\Rightarrow \big(\big(\exists u \in \partial C_{t^-,z} \cap \partial\mathbb{H} : G_{t^-,u} < G_{t,u}\big) \vee |C_{t^-,z}| = \infty\big)$,
i.e. if the site z is destroyed at time t, its cluster either must have grown to the boundary $\partial\mathbb{H}$ at time t or it must have been about to become infinite at time t.

For the remainder of this section, let $(\eta_{t,z}, G_{t,z})_{t\geq 0, z\in\overline{\mathbb{H}}}$ be any $\overline{\mathbb{H}}$-forest-fire process (not necessarily the specific process constructed in Theorem 1.2). A closely related auxiliary process is the **pure growth process** $(\sigma_{t,z})_{t\geq 0, z\in\overline{\mathbb{H}}}$, which is obtained when the destruction mechanism [DESTRUCTION] in Definition 1.3 is omitted, and which is formally defined by

$$\sigma_{t,z} := 1_{\{G_{t,z}>0\}}, \qquad t \geq 0, z \in \overline{\mathbb{H}}, \tag{1.1}$$

where we write 1_A for the indicator function of an event A. Obviously, $(\sigma_{t,z})_{t\geq 0, z\in\overline{\mathbb{H}}}$ is monotone increasing in t and dominates $(\eta_{t,z})_{t\geq 0, z\in\overline{\mathbb{H}}}$ in the sense that

$$\sigma_{t,z} \geq \sigma_{s,z} \geq \eta_{s,z}, \qquad 0 \leq s \leq t, z \in \overline{\mathbb{H}}, \tag{1.2}$$

holds. For a fixed time t, the configuration $(\sigma_{t,z})_{z\in\overline{\mathbb{H}}}$ is simply independent site percolation on $\overline{\mathbb{H}}$, where each site is open with probability $1 - e^{-t}$. In particular, if p_c denotes the critical probability of independent site percolation on $\overline{\mathbb{H}}$ (or equivalently $\mathbb{Z}^2$), then the critical time t_c, defined by $1 - e^{-t_c} = p_c$, has the property that a.s. for $t \leq t_c$, there exists no infinite cluster in the configuration $(\sigma_{t,z})_{z\in\overline{\mathbb{H}}}$, while for $t > t_c$, there exists exactly one infinite cluster in the configuration $(\sigma_{t,z})_{z\in\overline{\mathbb{H}}}$.

However, [DESTRUCTION] in Definition 1.3 and the fact that the paths of $(\eta_{t,z}, G_{t,z})_{t\geq 0, z\in\overline{\mathbb{H}}}$ are càdlàg imply that for all $t \geq 0$ there exists no infinite cluster in the configuration $(\eta_{t,z})_{z\in\overline{\mathbb{H}}}$. This gives rise to the question to what extent the processes $(\eta_{t,z})_{t\geq 0, z\in\overline{\mathbb{H}}}$ and $(\sigma_{t,z})_{t\geq 0, z\in\overline{\mathbb{H}}}$ differ, which motivates the following definition:

Definition 1.4. *For $t \geq 0$, $x \in \partial\mathbb{H}$ let*

$$Y_{t,x} := \sup\{y \in \mathbb{N} : (\exists 0 < t' < t'' \leq t : \eta_{t',x+iy} = 1, \eta_{t'',x+iy} = 0)\} \vee 0 \ \in\ \mathbb{N}_0 \cup \{\infty\}$$

*be the height up to which points with real part x have been destroyed up to time t. We call $Y_{t,x}$ the **height of destruction** at the point x up to time t.*

Note that for $t \geq 0$ and $x \in \partial\mathbb{H}$

$$\{Y_{t,x} < \infty\} \subset \{\forall 0 \leq s \leq t\, \forall y \geq Y_{t,x} + 1 : \sigma_{s,x+iy} = \eta_{s,x+iy}\} \tag{1.3}$$

holds. It turns out that as a function of time, the height of destruction experiences a phase transition:

Theorem 1.5. *Let $(\eta_{t,z}, G_{t,z})_{t\geq 0, z\in\overline{\mathbb{H}}}$ be an $\overline{\mathbb{H}}$-forest-fire process and let $(Y_{t,x})_{t\geq 0, x\in\partial\mathbb{H}}$ be the corresponding heights of destruction. Then for all $x \in \partial\mathbb{H}$, the following holds a.s.: $Y_{t,x} < \infty$ for $t < t_c$ and $Y_{t,x} = \infty$ for $t > t_c$.*

Informally speaking, this means that after the critical time t_c, the influence of the destruction mechanism [DESTRUCTION] in Definition 1.3 is not just confined to areas close to the boundary $\partial\mathbb{H}$ but is global on all of $\mathbb{H}$.

We will prove Theorems 1.2 and 1.5 in Sections 3 and 4, respectively. In Section 2 we draw the reader's attention to some obvious but open questions about $\overline{\mathbb{H}}$-forest-fire processes.

2 Open problems

The following natural questions about $\overline{\mathbb{H}}$-forest-fire processes $(\eta_{t,z}, G_{t,z})_{t\geq 0, z\in\overline{\mathbb{H}}}$ and the corresponding heights of destruction $(Y_{t,x})_{t\geq 0, x\in\partial\mathbb{H}}$ remain open:

- Are $\overline{\mathbb{H}}$-forest-fire processes unique in distribution? Is $(\eta_{t,z}, G_{t,z})_{t\geq 0, z\in\overline{\mathbb{H}}}$ adapted to the filtration generated by the growth processes $(G_{t,z})_{t\geq 0, z\in\overline{\mathbb{H}}}$?
- Does there exist $z \in \mathbb{H}$ such that the event $\{\exists t > 0 : |C_{t^-,z}| = \infty\}$ has positive probability (where $C_{t^-,z}$ is defined as in Definition 1.3), i.e. do infinite clusters in the left-sided limit occur with positive probability?
- How does the height of destruction behave at the critical time t_c? For instance, does $Y_{t_c,x} < \infty$ a.s. hold for $x \in \partial\mathbb{H}$?

3 Proof of Theorem 1.2

The construction of the limit process in Theorem 1.2 is partly analogous to the construction of the infinite volume Max Dürre forest-fire model in [7]. However, a new strategy is needed when it comes to infinite clusters in the process. This is where we will make use of the translation-invariance property [TRANSL-INV] of the process. We will only give a brief sketch of the parts that are similar to [7] in Sections 3.1, 3.2 and 3.3 and then focus on the issue of infinite clusters in Sections 3.4 and 3.5.

For the remainder of this section, consider the following setup: For $n \in \mathbb{N}$ let $(\eta^n_{t,z}, G^n_{t,z})_{t\geq 0, z\in B_n}$ be a B_n-forest-fire process. Embed this process into the upper half-plane $\overline{\mathbb{H}}$ by setting $(\eta^n_{t,z}, G^n_{t,z}) := (0,0)$ for $z \in \overline{\mathbb{H}} \setminus B_n$ and all $t \geq 0$. Let $(n_k)_{k\in\mathbb{N}}$ be a strictly increasing sequence of natural numbers.

3.1 Construction of the limit process and easy properties

Lemma 3.1. *(i) The sequence $(\eta^n_{t,z}, G^n_{t,z})_{t\in\mathbb{Q}_0^+, z\in\overline{\mathbb{H}}}$, $n \in \mathbb{N}$, is tight in the space $(\{0,1\}\times\mathbb{N}_0)^{\mathbb{Q}_0^+\times\overline{\mathbb{H}}}$ endowed with the product topology.*

(ii) There exists a subsequence $(n_{k_l})_{l\in\mathbb{N}}$ of natural numbers such that $(\eta^{n_{k_l}}_{t,z}, G^{n_{k_l}}_{t,z})_{t\in\mathbb{Q}_0^+, z\in\overline{\mathbb{H}}}$ converges weakly to some random variable $(\eta^{\mathbb{Q}}_{t,z}, G^{\mathbb{Q}}_{t,z})_{t\in\mathbb{Q}_0^+, z\in\overline{\mathbb{H}}}$.

Proof. First note that since the index set $\mathbb{Q}_0^+ \times \overline{\mathbb{H}}$ is countable, the product spaces $\{0,1\}^{\mathbb{Q}_0^+\times\overline{\mathbb{H}}}$, $\mathbb{N}_0^{\mathbb{Q}_0^+\times\overline{\mathbb{H}}}$ and $(\{0,1\}\times\mathbb{N}_0)^{\mathbb{Q}_0^+\times\overline{\mathbb{H}}}$ are metrizable and, in fact, are Polish spaces. By Tychonoff's theorem, the space $\{0,1\}^{\mathbb{Q}_0^+\times\overline{\mathbb{H}}}$ is compact and hence the sequence $(\eta^n_{t,z})_{t\in\mathbb{Q}_0^+, z\in\overline{\mathbb{H}}}$, $n \in \mathbb{N}$, is trivially tight. Moreover, the sequence $(G^n_{t,z})_{t\in\mathbb{Q}_0^+, z\in\overline{\mathbb{H}}}$, $n \in \mathbb{N}$, is clearly convergent and therefore tight by Prokhorov's theorem. As we work in the product topology, we conclude that the joint sequence $(\eta^n_{t,z}, G^n_{t,z})_{t\in\mathbb{Q}_0^+, z\in\overline{\mathbb{H}}}$, $n \in \mathbb{N}$, is tight, as well. This proves (i). Part (ii) then follows from (i) by another application of Prokhorov's theorem (in the opposite direction). $\square$

It is easy to see that the limit random variable $(\eta^{\mathbb{Q}}_{t,z}, G^{\mathbb{Q}}_{t,z})_{t\in\mathbb{Q}_0^+, z\in\overline{\mathbb{H}}}$ can be extended to a process $(\eta_{t,z}, G_{t,z})_{t\geq 0, z\in\overline{\mathbb{H}}}$, which we henceforth call the limit process:

Lemma 3.2. *A.s. the right-sided limit*

$$(\eta_{t,z}, G_{t,z}) := \lim_{s\downarrow t, s\in\mathbb{Q}_0^+} (\eta^{\mathbb{Q}}_{s,z}, G^{\mathbb{Q}}_{s,z}), \qquad t\geq 0, z\in\overline{\mathbb{H}},$$

exists.

Proof. This is proved analogously to Lemma 7 in [7]. □

We now realize the processes $(\eta^n_{t,z}, G^n_{t,z})_{t\geq 0, z\in\overline{\mathbb{H}}}$, $n\in\mathbb{N}$, and $(\eta_{t,z}, G_{t,z})_{t\geq 0, z\in\overline{\mathbb{H}}}$ on a joint probability space $(\Omega, \mathcal{A}, \mathbf{P})$, where $\mathcal{A}$ is the completion of the σ-field

$$\sigma\left(\eta^n_{t,z}, G^n_{t,z}; \eta_{t,z}, G_{t,z} : t\geq 0, z\in\overline{\mathbb{H}}, n\in\mathbb{N}\right).$$

There is a very useful relation between the limit process $(\eta_{t,z}, G_{t,z})_{t\geq 0, z\in\overline{\mathbb{H}}}$ and the B_n-forest-fire processes $(\eta^n_{t,z}, G^n_{t,z})_{t\geq 0, z\in\overline{\mathbb{H}}}$, which allows to transfer properties from the B_n-forest-fire processes to the limit process:

Lemma 3.3. *Let A be an event which is described by the configuration of finitely many sites and finitely many points in time, i.e. there exist $h\in\mathbb{N}$ and a finite set $S\subset\overline{\mathbb{H}}$ such that $A\in\mathcal{P}((\{0,1\}\times\mathbb{N}_0)^{[h]\times S})$, where $\mathcal{P}(X)$ denotes the power set of a set X and $[h] := \{1,2,\ldots,h\}$. If there exists $N\in\mathbb{N}$ such that for all $0\leq t_1 < t_2 < \ldots < t_h$ and all $n\geq N$*

$$\mathbf{P}\left[(\eta^n_{t_j,z}, G^n_{t_j,z})_{j\in[h], z\in S}\in A\right] = 0$$

holds, then

$$\mathbf{P}\left[\exists 0\leq t_1 < t_2 < \ldots < t_h : (\eta_{t_j,z}, G_{t_j,z})_{j\in[h], z\in S}\in A\right] = 0$$

also holds.

Proof. This is proved analogously to Lemma 9 in [7]. □

The construction of the limit process in Lemma 3.2 immediately implies that a.s. for all $z\in\overline{\mathbb{H}}$ the process $(\eta_{t,z}, G_{t,z})_{t\geq 0}$ is càdlàg. For $z\in\overline{\mathbb{H}}$ and $t\geq 0$, let $C_{t,z}$ denote the cluster of z in the configuration $(\eta_{t,w})_{w\in\overline{\mathbb{H}}}$, and for $z\in\overline{\mathbb{H}}$ and $t>0$, let $C_{t^-,z}$ denote the cluster of z in the configuration $(\eta_{t^-,w})_{w\in\overline{\mathbb{H}}}$. Then the following properties of the limit process are straightforward:

Lemma 3.4. *A.s. the process $(\eta_{t,z}, G_{t,z})_{t\geq 0, z\in\overline{\mathbb{H}}}$ satisfies the initial condition $\eta_{0,z} = 0$ for $z\in\overline{\mathbb{H}}$ and the boundary condition $\eta_{t,x} = 0$ for $t\geq 0, x\in\partial\mathbb{H}$. Moreover, it a.s. has the properties [POISSON] and [GROWTH] (ii) of Definition 1.3 and satisfies [TRANSL-INV].*

Proof. The proofs for the initial condition and the property [GROWTH] (ii) are easy consequences of Lemma 3.3 above and are analogous to the proofs of Lemmas 26 and 10 in [7]. The proof of the property [POISSON] is identical to the proof of Lemma 5 in [7]. The zero boundary condition for the limit process is trivial since the same boundary condition is satisfied by the B_n-forest-fire processes for all n. Finally, the translation-invariance [TRANSL-INV] of the limit process is a consequence of the rotation-invariance [ROT-INV] of the B_n-forest-fire processes for all n. □

3.2 Some auxiliary lemmas

It thus remains to show that the process $(\eta_{t,z}, G_{t,z})_{t\geq 0, z\in\overline{\mathbb{H}}}$ also a.s. has the properties [GROWTH] (i) and [DESTRUCTION] (i), (ii) of Definition 1.3. In this section we state some auxiliary lemmas, which are in a sense weaker versions of these properties.

We first introduce some further notation: For $0 \leq s \leq t$, $z \in \overline{\mathbb{H}}$ and $n \in \mathbb{N}$, let

$$\mathrm{G}_{s,t,z} := \{G_{s,z} < G_{t,z}\}, \qquad \mathrm{G}^n_{s,t,z} := \left\{G^n_{s,z} < G^n_{t,z}\right\}$$

be the events that the growth of a tree occurs at the site z in the time interval $(s,t]$, and for $0 < s \leq t$, $z \in \overline{\mathbb{H}}$ and $n \in \mathbb{N}$, let

$$\mathrm{G}_{s^-,t,z} := \left\{G_{s^-,z} < G_{t,z}\right\}, \qquad \mathrm{G}^n_{s^-,t,z} := \left\{G^n_{s^-,z} < G^n_{t,z}\right\}$$

be the events that the growth of a tree occurs at the site z in the time interval $[s,t]$. Moreover, if $X \ni x \mapsto f_x \in U$ is any function from a set X to a set U, then for $X' \subset X$, $u \in U$ we abbreviate the expression $\forall x \in X' : f_x = u$ by $f_{X'} = u$. Finally, if A, B are two events, we will denote the complement of A by $\complement A$, and (in slight abuse of notation) we will write $\{A, B\}$ instead of $A \cap B$.

Lemma 3.5 is a weaker version of [GROWTH] (i):

Lemma 3.5. *Suppose that $w, z \in \mathbb{H}$ are neighbouring sites. Then*

$$\mathbf{P}\left[\exists t > 0 : \eta_{t,w} = 1, \mathrm{G}_{t^-,t,z}, \eta_{t,z} = 0\right] = 0$$

holds; in other words: A.s. if there is the growth of a tree at the site z at some time t and a neighbouring site w is occupied at time t, then the site z is also occupied at time t.

Proof. Let $w, z \in \mathbb{H}$ be neighbouring sites. Since $(G_{t,w})_{t\geq 0}$ and $(G_{t,z})_{t\geq 0}$ are independent Poisson processes (see Lemma 3.4), a.s. they do not have jumps at the same time. Using this and the fact that Poisson process paths are a.s. piecewise constant and càdlàg, we obtain

$$\begin{aligned}\left\{\exists t > 0 : \eta_{t,w} = 1, \mathrm{G}_{t^-,t,z}, \eta_{t,z} = 0\right\} &\overset{\text{a.s.}}{\subset} \left\{\exists t > 0 : \complement\, \mathrm{G}_{t^-,t,w}, \eta_{t,w} = 1, \mathrm{G}_{t^-,t,z}, \eta_{t,z} = 0\right\} \\ &\overset{\text{a.s.}}{\subset} \left\{\exists 0 \leq s < t : \complement\, \mathrm{G}_{s,t,w}, \eta_{t,w} = 1, \mathrm{G}_{t^-,t,z}, \eta_{t,z} = 0\right\} \\ &\overset{\text{a.s.}}{\subset} \left\{\exists 0 \leq s < t : \complement\, \mathrm{G}_{s,t,w}, \eta_{t,w} = 1, \mathrm{G}_{s,t,z}, \eta_{t,z} = 0\right\}.\end{aligned}$$

Now for all sufficiently large n (such that $w, z \in B_n$) and arbitrary $0 \leq s < t$, it is easy to deduce from [GROWTH] and [DESTRUCTION] in Definition 1.1 that B_n-forest-fire processes satisfy

$$\mathbf{P}\left[\complement\, \mathrm{G}^n_{s,t,w}, \eta^n_{t,w} = 1, \mathrm{G}^n_{s,t,z}, \eta^n_{t,z} = 0\right] = 0.$$

The result therefore follows from Lemma 3.3. □

Lemmas 3.6 and 3.7 are about the destruction of occupied clusters:

Lemma 3.6. *For all $w, z \in \mathbb{H}$ we have*

$$\mathbf{P}\left[\exists 0 \leq s < t : w \in C_{s,z}, \eta_{t,w} = 0, \eta_{t,z} = 1, \complement\, \mathrm{G}_{s,t,z}\right] = 0;$$

in other words: A.s. if a site w was occupied at some time s but is vacant at some later time $t > s$, then any other site z which was in the cluster of w at time s must be vacant at time t unless there is the growth of a tree at that site in the time interval $(s,t]$.

Proof. This is a consequence of Lemma 3.3 above and is proved analogously to Lemma 12 in [7]. □

Lemma 3.7 is the first half of [DESTRUCTION] (i) in Definition 1.3:

Lemma 3.7. *For all $x \in \partial\mathbb{H}$ we have*

$$\mathbf{P}\left[\exists t > 0 : \mathrm{G}_{t^-,t,x}, \exists w \in C_{t^-,x+i} : \eta_{t,w} = 1\right] = 0;$$

in other words: A.s. if the cluster at $x+i$ grows to the boundary $\partial\mathbb{H}$ at some time t, it is destroyed at time t.

Proof. Let $x \in \partial\mathbb{H}$. For $z \in \mathbb{H}$, let C_z^{fin} denote the (countable) set of all finite connected subsets of $\mathbb{H}$ which contain the site z. Due to the equality

$$\begin{aligned}&\left\{\exists t > 0 : \mathrm{G}_{t^-,t,x}, \exists w \in C_{t^-,x+i} : \eta_{t,w} = 1\right\}\\ &\qquad = \bigcup_{S \in C_{x+iy}^{\mathrm{fin}}} \bigcup_{w \in S} \left\{\exists t > 0 : \mathrm{G}_{t^-,t,x}, \eta_{t^-,S} = 1, \eta_{t,w} = 1\right\}\end{aligned}$$

it suffices to show that for all $S \in C_{x+iy}^{\mathrm{fin}}$ and $w \in S$

$$\mathbf{P}\left[\exists t > 0 : \mathrm{G}_{t^-,t,x}, \eta_{t^-,S} = 1, \eta_{t,w} = 1\right] = 0$$

holds. So let $S \in C_{x+iy}^{\mathrm{fin}}$ and $w \in S$ be fixed. Since $(G_{t,x})_{t\geq 0}$ and $(G_{t,w})_{t\geq 0}$ are independent Poisson processes (see Lemma 3.4), a.s. they do not have jumps at the same time. Using this and the fact that the paths of the limit process are a.s. piecewise constant and càdlàg, we obtain

$$\left\{\exists t > 0 : \mathrm{G}_{t^-,t,x}, \eta_{t^-,S} = 1, \eta_{t,w} = 1\right\} \overset{\text{a.s.}}{\subset} \left\{\exists 0 \leq s < t : \mathrm{G}_{s,t,x}, \eta_{s,S} = 1, \eta_{t,w} = 1, \complement\, \mathrm{G}_{s,t,w}\right\}.$$

Now for all sufficiently large n (such that $\{x\} \cup S \subset B_n$) and arbitrary $0 \leq s < t$, it is easy to deduce from [GROWTH] and [DESTRUCTION] in Definition 1.1 that B_n-forest-fire processes satisfy

$$\mathbf{P}\left[\mathrm{G}_{s,t,x}^n, \eta_{s,S}^n = 1, \eta_{t,w}^n = 1, \complement\, \mathrm{G}_{s,t,w}^n\right] = 0.$$

The result therefore follows from Lemma 3.3. □

3.3 A Markov-type property of the limit process

Let $(\mathcal{F}_t)_{t\geq 0}$ be the completion of the canonical filtration of the limit process $(\eta_{t,z}, G_{t,z})_{t\geq 0, z\in\overline{\mathbb{H}}}$, i.e. $\mathcal{F}_t$ is the completion of the σ-field

$$\sigma\left((\eta_{s,z}, G_{s,z}) : 0 \leq s \leq t, z \in \overline{\mathbb{H}}\right)$$

generated up to time $t \geq 0$. As is customary, if T is a stopping time with respect to $(\mathcal{F}_t)_{t\geq 0}$, we define the σ-field up to time T by

$$\mathcal{F}_T := \{A \in \mathcal{A} : (\forall t \geq 0 : A \cap \{T \leq t\} \in \mathcal{F}_t)\},$$

where $\mathcal{A}$ is the full σ-field introduced in the paragraph below Lemma 3.2. Then the limit process satisfies the following Markov-type property:

Lemma 3.8. *Let T be a stopping time with respect to $(\mathcal{F}_t)_{t\geq 0}$. Then for all $A \in \mathcal{F}_T$*

$$\mathbf{P}\left[(G_{T+t,z} - G_{T,z})_{t\geq 0, z\in\overline{\mathbb{H}}} \in \cdot\,, T < \infty, A\right] = \mathbf{P}\left[(G_{t,z})_{t\geq 0, z\in\overline{\mathbb{H}}} \in \cdot\,\right] \mathbf{P}\left[T < \infty, A\right]$$

holds; in other words: On the event $\{T < \infty\}$, the increments $(G_{T+t,z} - G_{T,z})_{t\geq 0}$, $z \in \overline{\mathbb{H}}$, of the Poisson processes after time T are independent of the σ-field $\mathcal{F}_T$ and are again independent Poisson processes.

Proof. This is proved analogously to Lemma 19 in [7]. □

3.4 Non-existence of infinite clusters

The aim of this section is to prove Lemma 3.12, which states that a.s. there do not exist infinite clusters in the process $(\eta_{t,z})_{t\geq 0, z\in\overline{\mathbb{H}}}$.

Lemma 3.9. *For all $t \geq 0$, $z \in \mathbb{H}$ and $R \in \mathbb{N}$ we have*

$$\mathbf{P}\left[|\{w \in C_{t,z} : \operatorname{Im} w = R\}| = \infty\right] = 0;$$

in other words: For any fixed time t there a.s. does not exist a cluster which contains infinitely many sites with the same distance R from $\partial\mathbb{H}$.

Intuitively, the reason why Lemma 3.9 should hold is the following: Suppose that the cluster $C_{t,z}$ contains infinitely many sites w with distance $\operatorname{Im} w = R$ from $\partial\mathbb{H}$. Lemma 3.6 and the fact that the paths of the limit process are a.s. piecewise constant and càdlàg imply that a.s. the cluster $C_{t,z}$ persists during time $[t, t+\epsilon]$ for some $\epsilon > 0$. However, for any $\epsilon > 0$ there a.s. is a growth sequence within $[t, t+\epsilon]$ from one of the sites w with $\operatorname{Im} w = R$ downto the boundary $\partial\mathbb{H}$, which causes the cluster at z to be destroyed before time $t+\epsilon$ - a contradiction. We now make this argument rigorous.

Proof. Let $t \geq 0$, $z \in \mathbb{H}$ and $R \in \mathbb{N}$. We abbreviate

$$E_{t,z} := \{|\{w \in C_{t,z} : \operatorname{Im} w = R\}| = \infty\}.$$

On the event $E_{t,z}$, let $(W_k)_{k\in\mathbb{Z}}$ be a disjoint enumeration of the sites $w \in C_{t,z}$ with $\operatorname{Im} w = R$. Moreover, for $w \in \mathbb{H}$ and $s \geq 0$, $\gamma > 0$ let

$$\text{V-GROWTH-SEQ}(w, s, \gamma) := \left\{\forall j \in \{1, \dots, \operatorname{Im} w\} : \mathrm{G}_{s+\frac{j-1}{\operatorname{Im} w}\gamma, s+\frac{j}{\operatorname{Im} w}\gamma, w-ji}\right\}$$

denote the event that there is a vertical growth sequence from the site $w-i$ to the boundary $\partial\mathbb{H}$ between times s and $s+\gamma$ (with the jth growth event between times $s+\frac{j-1}{\operatorname{Im} w}\gamma$ and $s+\frac{j}{\operatorname{Im} w}\gamma$ for $j = 1, \dots, \operatorname{Im} w$).

Since the paths of the limit process are a.s. piecewise constant and càdlàg, we have

$$E_{t,z} \overset{\text{a.s.}}{\subset} \left\{E_{t,z}, \exists\epsilon \in \mathbb{Q}\cap(0,\infty) : \eta_{[t,t+\epsilon],z} = 1, \complement\,\mathrm{G}_{t,t+\epsilon,z}\right\}.$$

It therefore suffices to show

$$\mathbf{P}\left[E_{t,z}, \eta_{[t,t+\epsilon],z} = 1, \complement\,\mathrm{G}_{t,t+\epsilon,z}\right] = 0 \tag{3.1}$$

for arbitrary $\epsilon > 0$.

So pick $\epsilon > 0$. Lemma 3.8 implies that conditional on $E_{t,z}$ (we can assume $\mathbf{P}[E_{t,z}] > 0$ without loss of generality), the events V-GROWTH-SEQ(W_k, t, ϵ), $k \in \mathbb{Z}$, are independent with

$$\mathbf{P}\left[\text{V-GROWTH-SEQ}(W_k, t, \epsilon) | E_{t,z}\right] = \mathbf{P}\left[\text{V-GROWTH-SEQ}(iR, 0, \epsilon)\right] > 0$$

for all $k \in \mathbb{Z}$. We therefore conclude from the Borel-Cantelli lemma that

$$E_{t,z} \overset{\text{a.s.}}{\subset} \{E_{t,z}, \text{V-GROWTH-SEQ}(W_k, t, \epsilon) \text{ for infinitely many } k\} \tag{3.2}$$

holds.

For the moment, let $k \in \mathbb{Z}$ be fixed. Considering the first $R-1$ growth events (in $\mathbb{H}$) of the event V-GROWTH-SEQ(W_k, t, ϵ) and applying Lemmas 3.5 and 3.6 repeatedly, we see that

$$\begin{aligned}&\left\{E_{t,z}, \eta_{[t,t+\epsilon],z} = 1, \complement\,\mathrm{G}_{t,t+\epsilon,z}, \text{V-GROWTH-SEQ}(W_k, t, \epsilon)\right\} \\ &\qquad \overset{\text{a.s.}}{\subset} \left\{E_{t,z}, \forall s \in [t + \frac{R-1}{R}\epsilon, t+\epsilon] : \underbrace{W_k - (R-1)i}_{=\operatorname{Re} W_k + i} \in C_{s,z}\right\}.\end{aligned} \tag{3.3}$$

However, considering the last growth event (in $\partial\mathbb{H}$) of the event V-GROWTH-SEQ(W_k, t, ϵ) and using (3.3) and Lemma 3.7, it follows that

$$\begin{aligned}\left\{E_{t,z}, \eta_{[t,t+\epsilon],z} = 1, \complement\, \mathrm{G}_{t,t+\epsilon,z}, \text{V-GROWTH-SEQ}(W_k, t, \epsilon)\right\} \\ \overset{\text{a.s.}}{\subset} \left\{E_{t,z}, \exists s \in [t + \frac{R-1}{R}\epsilon, t+\epsilon] : \eta_{s,z} = 0\right\}.\end{aligned} \tag{3.4}$$

Since the condition $\exists s \in [t + \frac{R-1}{R}\epsilon, t+\epsilon] : \eta_{s,z} = 0$ on the right side of (3.4) contradicts the condition $\eta_{[t,t+\epsilon],z} = 1$ on its left side, we conclude that

$$\mathbf{P}\left[E_{t,z}, \eta_{[t,t+\epsilon],z} = 1, \complement\, \mathrm{G}_{t,t+\epsilon,z}, \text{V-GROWTH-SEQ}(W_k, t, \epsilon)\right] = 0 \tag{3.5}$$

for all $k \in \mathbb{Z}$.

Equation (3.1) is now a direct consequence of (3.2) and (3.5). □

Definition 3.10. *For $t \geq 0$ let $N_t \in \mathbb{N}_0 \cup \{\infty\}$ denote the number of infinite clusters in the configuration $(\eta_{t,z})_{z\in\overline{\mathbb{H}}}$.*

Lemma 3.11. *For all $t \geq 0$ we have $\mathbf{P}[N_t = 0] = 1$; in other words: For any fixed time t there a.s. does not exist an infinite cluster in the configuration $(\eta_{t,z})_{z\in\overline{\mathbb{H}}}$.*

Intuitively, the reason why Lemma 3.11 should hold is the following: Due to the translation-invariance [TRANSL-INV] of the limit process we expect $N_t \in \{0, 1, \infty\}$ a.s. If $N_t = 1$, then the translation-invariance implies that a.s. there exists $R \in \mathbb{N}$ such that there are infinitely many sites w with $\operatorname{Im} w = R$ in the unique infinite cluster at time t, but this is ruled out by Lemma 3.9. On the other hand, if $N_t = \infty$, then the translation-invariance implies that a.s. there exists $R \in \mathbb{N}$ such that there are infinitely many infinite clusters with distance R from $\partial\mathbb{H}$ at time t. But due to the translation-invariance and the limited amount of space these clusters must be very close to one another. Using this observation and the fact that the paths of the limit process are a.s. piecewise constant and càdlàg, we find that a.s. there exists $\epsilon > 0$ such that by time $t + \epsilon$, the above-mentioned clusters have grown together to form one infinite cluster containing infinitely many sites with distance R from $\partial\mathbb{H}$. Yet once more, this is ruled out by Lemma 3.9. It should be noted that the classical Burton-Keane argument to rule out the case $N_t = \infty$ cannot be applied here because we work on the half-plane $\overline{\mathbb{H}}$ and not on $\mathbb{Z}^2$, and because the translation-invariance [TRANSL-INV] only holds in the x-direction. We now make the above heuristics rigorous.

Proof. Let $t \geq 0$. In the following, for a subset $S \subset \overline{\mathbb{H}}$ we write

$$\operatorname{dist}(S, \partial\mathbb{H}) := \min\{\operatorname{Im} w : w \in S\}$$

for its vertical distance from $\partial\mathbb{H}$. Let us call a site $z \in \mathbb{H}$ the rightmost lowest point of its cluster $C_{t,z}$ (hereinafter abbreviated by $z = \operatorname{RLP}(C_{t,z})$) if

- $\operatorname{Im} z$ is minimal in $C_{t,z}$, i.e. $\operatorname{Im} z = \operatorname{dist}(C_{t,z}, \partial\mathbb{H})$, and
- $\operatorname{Re} z$ is maximal among all $w \in C_{t,z}$ with $\operatorname{Im} w = \operatorname{dist}(C_{t,z}, \partial\mathbb{H})$.

Lemma 3.9 implies that a.s. every non-empty cluster in the configuration $(\eta_{t,z})_{z\in\overline{\mathbb{H}}}$ has a rightmost lowest point, so that

$$\{N_t \geq 1\} \overset{\text{a.s.}}{\subset} \{\exists x \in \mathbb{Z}\, \exists y \in \mathbb{N} : x + iy = \operatorname{RLP}(C_{t,x+iy}), |C_{t,x+iy}| = \infty\}.$$

Let $y \in \mathbb{N}$ be fixed, and set

$$A_{t,x} := \{x + iy = \operatorname{RLP}(C_{t,x+iy}), |C_{t,x+iy}| = \infty\}$$

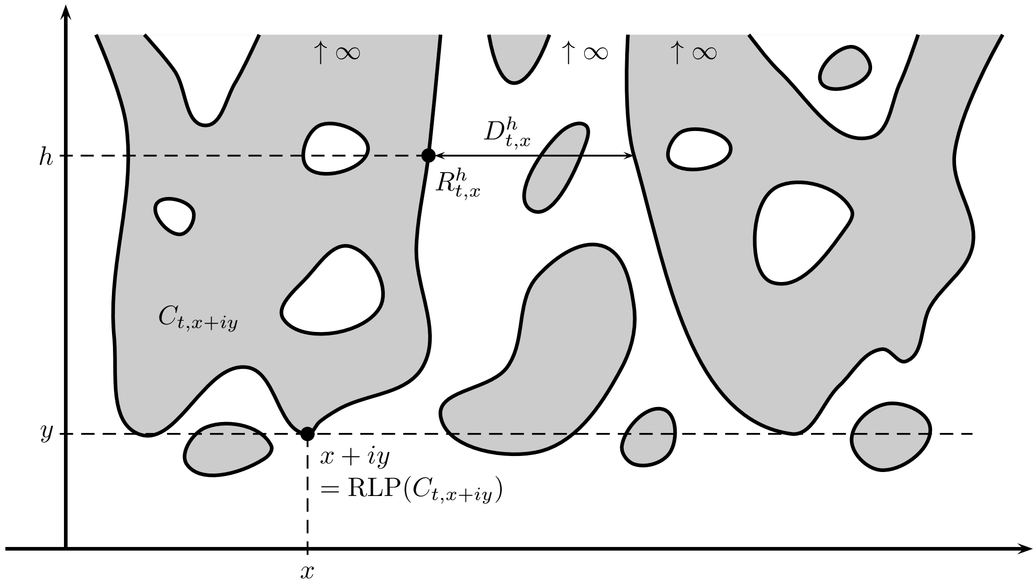

Figure 2: A visualisation of the event $A_{t,x}$ and the associated random variables $R^h_{t,x}$, $D^h_{t,x}$

for all $x \in \mathbb{Z}$. Due to the translation-invariance [TRANSL-INV] of the limit process we have $\mathbf{P}[A_{t,x}] = \mathbf{P}[A_{t,0}]$ for all $x \in \mathbb{Z}$, and it thus suffices to prove $\mathbf{P}[A_{t,0}] = 0$.

Step 1: Using the translation-invariance [TRANSL-INV] of the limit process again, we see that for all $x \in \mathbb{Z}$

$$A_{t,x} \overset{\text{a.s.}}{\subset} \{A_{t,u} \text{ for infinitely many } u \in \mathbb{N}_0\} \tag{3.6}$$

holds by the Poincaré recurrence theorem (see e.g. [15], Section V.1, Theorem 1). Since the rightmost lowest point of a cluster is unique (if it exists), (3.6) in particular implies that on the event $A_{t,x}$, there a.s. exist infinitely many infinite clusters at time t which are to the right of the cluster $C_{t,x+iy}$ and have vertical distance y from $\partial\mathbb{H}$. For $x \in \mathbb{Z}$ and integer $h \geq y$, on the event $A_{t,x}$, let

$$R^h_{t,x} := \max\{r \in \mathbb{Z} : r + ih \in C_{t,x+iy}\} + ih$$

be the rightmost point of the cluster $C_{t,x+iy}$ at height h, and let

$$D^h_{t,x} := \min\left\{d \in \mathbb{N} : |C_{t,R^h_{t,x}+d}| = \infty, \mathrm{dist}(C_{t,R^h_{t,x}+d}, \partial\mathbb{H}) = y\right\}$$

be the horizontal distance from $R^h_{t,x}$ to the “next” infinite cluster with vertical distance y from $\partial\mathbb{H}$. On the event $A_{t,x}$, $R^h_{t,x}$ and $D^h_{t,x}$ are a.s. well-defined because obviously

$$A_{t,x} \overset{\text{a.s.}}{\subset} \{A_{t,x}, \forall h \geq y \, \exists r \in \mathbb{Z} : r + ih \in C_{t,x+iy}\}$$

holds, and because of Lemma 3.9 and the observation below equation (3.6). See Figure 2 for a visualisation of the event $A_{t,x}$ and the associated random variables $R^h_{t,x}$, $D^h_{t,x}$. The aim of Step 1 is to prove that

$$A_{t,x} \overset{\text{a.s.}}{\subset} \left\{A_{t,x}, \liminf_{h\to\infty} D^h_{t,x} < \infty\right\} \tag{3.7}$$

holds for all $x \in \mathbb{Z}$.

Suppose that (3.7) is not true. Then there exist sequences $(c_h)_{h\geq y}$ and $(d_h)_{h\geq y}$ of natural numbers with $d_h \uparrow \infty$ as $h \to \infty$ such that the events

$$B_{t,x} := \left\{A_{t,x}, \forall h \geq y : \left|\operatorname{Re} R^h_{t,x} - x\right| \leq c_h, D^h_{t,x} \geq d_h\right\}$$

have positive probability for all $x \in \mathbb{Z}$. (Of course, the translation-invariance [TRANSL-INV] of the limit process implies $\mathbf{P}[B_{t,x}] = \mathbf{P}[B_{t,0}]$ for all $x \in \mathbb{Z}$.) From the translation-invariance [TRANSL-INV] of the limit process and the Birkhoff ergodic theorem (see e.g. [15], Section V.3, Theorem 1) we thus deduce that there exists $\beta > 0$ such that

$$\mathbf{P}\left[\sum_{x=0}^{n-1} 1_{B_{t,x}} > \beta n \text{ eventually as } n \to \infty\right] > 0.$$

On the event $\left\{\sum_{x=0}^{n-1} 1_{B_{t,x}} > \beta n \text{ eventually as } n \to \infty\right\}$, for large n the sites $iy, 1+iy, \ldots, (n-1)+iy$ are part of at least $\lceil \beta n\rceil$ different infinite clusters for which the following holds: For $h \geq y$ their rightmost points at height h are all contained in the interval $[-c_h, (n-1)+c_h] + ih$ and have at least horizontal distance d_h from one another. Hence the horizontal distance between the right-most points at height h of the first and the $\lceil \beta n\rceil$th cluster is less than $n + 2c_h$ but greater than or equal to $(\lceil \beta n\rceil - 1)d_h$. In particular, it holds that

$$\frac{\lceil \beta n\rceil - 1}{n} d_h \leq 1 + \frac{2c_h}{n}.$$

Letting $n \to \infty$, we obtain $\beta d_h \leq 1$ for all $h \geq y$. But since $\beta > 0$, this contradicts the condition that $d_h \uparrow \infty$ for $h \to \infty$. We have thus proven (3.7).

Step 2: We now prove $\mathbf{P}[A_{t,0}] = 0$. Let $\epsilon > 0$ be arbitrary; since the paths of the limit process are a.s. piecewise constant and càdlàg, it suffices to show

$$\mathbf{P}\left[A_{t,0}, \eta_{[t,t+\epsilon],iy} = 1, \complement\, \mathrm{G}_{t,t+\epsilon,iy}\right] = 0.$$

In fact, we will prove

$$\left\{A_{t,0}, \eta_{[t,t+\epsilon],iy} = 1, \complement\, \mathrm{G}_{t,t+\epsilon,iy}\right\} \overset{\text{a.s.}}{\subset} \left\{\left|\left\{w \in C_{t+\epsilon,iy} : \operatorname{Im} w = y\right\}\right| = \infty\right\}, \tag{3.8}$$

and the latter is a null set by Lemma 3.9. Let $K \in \mathbb{N}$ be arbitrary; the inclusion (3.8) then follows if we can show

$$\left\{A_{t,0}, \eta_{[t,t+\epsilon],iy} = 1, \complement\, \mathrm{G}_{t,t+\epsilon,iy}\right\} \overset{\text{a.s.}}{\subset} \left\{\left|\left\{w \in C_{t+\epsilon,iy} : \operatorname{Im} w = y\right\}\right| > K\right\}. \tag{3.9}$$

On the event $A_{t,0}$, we recursively define

$$X_1 := 0,\ Z_1 := iy$$

and for $k \geq 2$

$$X_k := \min\left\{x > X_{k-1} : 1_{A_{t,x}} = 1\right\},\ Z_k := X_k + iy,$$

which is a.s. well-defined by (3.6). Informally speaking, for $k \in \mathbb{N}$, C_{t,Z_k} is "the kth infinite cluster with distance y from $\partial\mathbb{H}$", where we count clusters from left to right, starting with the cluster at iy. Since the paths of the limit process are a.s. piecewise constant and càdlàg, and because of (3.7), we have

$$A_{t,0} \overset{\text{a.s.}}{\subset} \Big\{A_{t,0}, \exists \tilde\epsilon \in \mathbb{Q} \cap (0,\epsilon), \exists d \in \mathbb{N}\, \forall k \in \{1,\ldots,K\} : \eta_{[t,t+\tilde\epsilon],Z_{k+1}} = 1, \complement\, \mathrm{G}_{t,t+\tilde\epsilon,Z_{k+1}}, \liminf_{h\to\infty} D^h_{t,X_k} \leq d\Big\}.$$

So let $0 < \tilde{\epsilon} < \epsilon$ and $d \in \mathbb{N}$ be arbitrary. For the proof of (3.9) it then suffices to show

$$\begin{aligned}&\Big\{A_{t,0}, \eta_{[t,t+\epsilon],iy} = 1, \complement\, \mathrm{G}_{t,t+\epsilon,iy}, \forall k \in \{1,\dots,K\} : \eta_{[t,t+\tilde{\epsilon}],Z_{k+1}} = 1, \complement\, \mathrm{G}_{t,t+\tilde{\epsilon},Z_{k+1}},\\ &\liminf_{h\to\infty} D^h_{t,X_k} \le d\Big\} \overset{\text{a.s.}}{\subset} \{|\{w \in C_{t+\epsilon,iy} : \operatorname{Im} w = y\}| > K\}.\end{aligned} \tag{3.10}$$

During the next two paragraphs, let $k \in \{1,\dots,K\}$ be fixed. On the event $\Big\{A_{t,0}, \liminf\limits_{h\to\infty} D^h_{t,X_k} \le d\Big\}$, we recursively define

$$H_{k,1} := \min\left\{h \ge y : D^h_{t,X_k} \le d\right\}$$

and for $l \ge 2$

$$H_{k,l} := \min\left\{h > H_{k,l-1} : D^h_{t,X_k} \le d\right\}.$$

(This is well-defined since D^h_{t,X_k} is integer-valued.) Then for $l \in \mathbb{N}$, $R^{H_{k,l}}_{t,X_k}$ is the "lth rightmost point of the cluster C_{t,Z_k} whose horizontal distance to the cluster $C_{t,Z_{k+1}}$ is less than or equal to d", where we count these points from bottom to top. Moreover, for $w \in \mathbb{H}$, $c \in \mathbb{N}$ and $s \ge 0$, $\gamma > 0$ let

$$\text{H-GROWTH-SEQ}(w,c,s,\gamma) := \left\{\forall j \in \{1,\dots,c\} : \mathrm{G}_{s+\frac{j-1}{c}\gamma, s+\frac{j}{c}\gamma, w+j}\right\}$$

denote the event that there is a horizontal growth sequence from the site $w+1$ to the site $w+c$ between times s and $s+\gamma$ (with the jth growth event between times $s+\frac{j-1}{c}\gamma$ and $s+\frac{j}{c}\gamma$ for $j = 1,\dots,c$).

Lemma 3.8 implies that conditional on $A_{t,0}$ (we can assume $\mathbf{P}[A_{t,0}] > 0$ without loss of generality), the events $\text{H-GROWTH-SEQ}(R^{H_{k,l}}_{t,X_k}, d, t, \tilde{\epsilon})$, $l \in \mathbb{N}$, are independent with

$$\mathbf{P}\left[\text{H-GROWTH-SEQ}(R^{H_{k,l}}_{t,X_k}, d, t, \tilde{\epsilon}) \middle| A_{t,0}\right] = \mathbf{P}\left[\text{H-GROWTH-SEQ}(i, d, 0, \tilde{\epsilon})\right] > 0$$

for all $l \in \mathbb{N}$. We therefore conclude from the Borel-Cantelli lemma that

$$A_{t,0} \overset{\text{a.s.}}{\subset} \left\{A_{t,0}, \text{H-GROWTH-SEQ}(R^{H_{k,l}}_{t,X_k}, d, t, \tilde{\epsilon}) \text{ for infinitely many } l\right\} \tag{3.11}$$

holds.

But for any fixed numbers $l_1,\dots,l_K \in \mathbb{N}$ repeated applications of Lemmas 3.5 and 3.6 yield

$$\begin{aligned}&\Big\{A_{t,0}, \eta_{[t,t+\epsilon],iy} = 1, \complement\, \mathrm{G}_{t,t+\epsilon,iy}, \forall k \in \{1,\dots,K\} : \eta_{[t,t+\tilde{\epsilon}],Z_{k+1}} = 1, \complement\, \mathrm{G}_{t,t+\tilde{\epsilon},Z_{k+1}},\\ &\liminf_{h\to\infty} D^h_{t,X_k} \le d, \text{H-GROWTH-SEQ}(R^{H_{k,l_k}}_{t,X_k}, d, t, \tilde{\epsilon})\Big\}\\ &\qquad \overset{\text{a.s.}}{\subset} \left\{A_{t,0}, C_{t+\epsilon,iy} \supset \bigcup_{k=1}^{K+1} C_{t,Z_k}\right\}\\ &\qquad \overset{\text{a.s.}}{\subset} \{|\{w \in C_{t+\epsilon,iy} : \operatorname{Im} w = y\}| \ge K+1\}.\end{aligned} \tag{3.12}$$

Equation (3.10) is now a direct consequence of (3.11) and (3.12). □

Lemma 3.12. *We have* $\mathbf{P}[\forall t \ge 0 : N_t = 0] = 1$*; in other words: A.s. there does not exist an infinite cluster in the configuration* $(\eta_{t,z})_{z\in\overline{\mathbb{H}}}$ *for any time* $t \ge 0$.

Proof. Using the fact that the paths of the limit process are a.s. piecewise constant and càdlàg, and then applying Lemma 3.6, we obtain

$$\begin{aligned}\{\exists t \geq 0 : N_t \geq 1\} &\overset{\text{a.s.}}{\subset} \left\{\exists t \geq 0\, \exists z \in \mathbb{H}\, \exists \epsilon > 0 : |C_{t,z}| = \infty, \eta_{[t,t+\epsilon],z} = 1, \complement\, \mathrm{G}_{t,t+\epsilon,z}\right\} \\ &\overset{\text{a.s.}}{\subset} \left\{\exists t \geq 0\, \exists z \in \mathbb{H}\, \exists \epsilon > 0 : |C_{t,z}| = \infty, \forall w \in C_{t,z} : \eta_{[t,t+\epsilon],w} = 1\right\} \\ &\subset \left\{\exists t \in \mathbb{Q}_0^+ : N_t \geq 1\right\}.\end{aligned}$$

Since the set $\mathbb{Q}_0^+$ is countable, the last event is a null set by Lemma 3.11. □

3.5 Infinite clusters in the left-sided limit

The aim of this section is to prove Lemma 3.18, which states that a.s. clusters in the process $(\eta_{t,z})_{t\geq 0, z\in\overline{\mathbb{H}}}$ are destroyed if they are about to become infinite. We start with the following weaker version of this statement:

Lemma 3.13. *For all $z \in \mathbb{H}$ we have*

$$\mathbf{P}\left[\exists t > 0 : |C_{t^-,z}| = \infty, \complement\, \mathrm{G}_{t^-,t,z}, \eta_{t,z} = 1\right] = 0;$$

in other words: A.s. if the left-sided limit of the cluster at z is infinite at some time t, then the site z gets destroyed at time t unless there is the growth of a tree at z at time t.

Proof. Let $z \in \mathbb{H}$. Since a.s. $|C_{t,z}| < \infty$ holds for any time $t \geq 0$ (Lemma 3.12) and since the paths of the limit process are a.s. piecewise constant and càdlàg, it follows that

$$\begin{aligned}&\left\{\exists t > 0 : |C_{t^-,z}| = \infty, \complement\, \mathrm{G}_{t^-,t,z}, \eta_{t,z} = 1\right\} \\ &\quad\overset{\text{a.s.}}{\subset} \left\{\exists t > 0 : |C_{t^-,z}| = \infty, \complement\, \mathrm{G}_{t^-,t,z}, \eta_{t,z} = 1, \exists w \in C_{t^-,z} : \eta_{t,w} = 0\right\} \\ &\quad\overset{\text{a.s.}}{\subset} \left\{\exists 0 \leq s < t : \complement\, \mathrm{G}_{s,t,z}, \eta_{t,z} = 1, \exists w \in C_{s,z} : \eta_{t,w} = 0\right\}.\end{aligned}$$

But the latter is a null set by Lemma 3.6. □

Lemma 3.14. *(i) For all $z \in \mathbb{H}$ we have*

$$\mathbf{P}\left[\exists 0 < s < t : |C_{s^-,z}| = \infty, |C_{t^-,z}| = \infty, \complement\, \mathrm{G}_{s^-,t,z}\right] = 0; \tag{3.13}$$

in other words: A.s. if the left-sided limit of the cluster at z is infinite at some time s, the left-sided limit of the cluster cannot be infinite at some later time $t > s$ unless there is the growth of a tree at z in the time interval $[s,t]$.

(ii) For all $z \in \mathbb{H}$ the set $\left\{t > 0 : |C_{t^-,z}| = \infty\right\}$ of times at which the left-sided limit of the cluster at z is infinite a.s. has no accumulation points.

Proof. Let $z \in \mathbb{H}$. Lemma 3.13 and the property [GROWTH] (ii) of the limit process (see Lemma 3.4) imply

$$\begin{aligned}&\left\{\exists 0 < s < t : |C_{s^-,z}| = \infty, |C_{t^-,z}| = \infty, \complement\, \mathrm{G}_{s^-,t,z}\right\} \\ &\quad\overset{\text{a.s.}}{\subset} \left\{\exists 0 < s < t : \eta_{s,z} = 0, \complement\, \mathrm{G}_{s,t,z}, |C_{t^-,z}| = \infty\right\} \\ &\quad\overset{\text{a.s.}}{\subset} \left\{\exists 0 < s < t : \eta_{[s,t],z} = 0, |C_{t^-,z}| = \infty\right\}.\end{aligned}$$

But since the conditions $\eta_{[s,t],z} = 0$ and $|C_{t^-,z}| = \infty$ in the last event obviously contradict each other, we conclude that (3.13) holds indeed.

It now follows from (3.13) that a.s. if the set $\left\{t > 0 : |C_{t^-,z}| = \infty\right\}$ has an accumulation point, then the set $\left\{t > 0 : \mathrm{G}_{t^-,t,z}\right\}$ of times at which a tree grows at the site z also has an accumulation point. But since $(G_{t,z})_{t\geq 0}$ is a Poisson process, this happens with probability zero. □

For $z \in \mathbb{H}$, we recursively define $T_{0,z} := 0$ and for $k \in \mathbb{N}$

$$T_{k,z} := \inf \left\{ t > T_{k-1,z} : |C_{t^-,z}| = \infty \right\} \in (0, \infty].$$

Lemma 3.14 (ii) implies that a.s. the inclusion

$$\left\{ t > 0 : |C_{t^-,z}| = \infty \right\} \subset \{T_{k,z} : k \in \mathbb{N}\} \tag{3.14}$$

holds. This allows us to treat the issue of infinite left-sided clusters at z by considering the countable sequence of random times $T_{k,z}$, $k \in \mathbb{N}$. In fact, these random times are predictable stopping times with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ introduced in Section 3.3, where predictability is defined as follows:

Definition 3.15. *A stopping time T with respect to $(\mathcal{F}_t)_{t\geq 0}$ is called **predictable** if there exists an increasing sequence $(T_n)_{n\in\mathbb{N}}$ of stopping times with respect to $(\mathcal{F}_t)_{t\geq 0}$ which a.s. satisfy $T_n \uparrow T$ for $n \to \infty$ and $T_n < T$ for all $n \in \mathbb{N}$. In this case, the sequence $(T_n)_{n\in\mathbb{N}}$ is said to announce the stopping time T.*

Lemma 3.16. *For all $z \in \mathbb{H}$ and $k \in \mathbb{N}$, $T_{k,z}$ is a predictable stopping time with respect to $(\mathcal{F}_t)_{t\geq 0}$.*

Proof. Let $z \in \mathbb{H}$ and $k \in \mathbb{N}$. Then $T_{k,z}$ is obviously a stopping time with respect to $(\mathcal{F}_t)_{t\geq 0}$. We now prove that $T_{k,z}$ is announced by the sequence

$$T_{k,z,n} := \inf \{t > T_{k-1,z} : |C_{t,z}| \geq n\} \wedge n, \qquad n \in \mathbb{N}.$$

Clearly, for each $n \in \mathbb{N}$, $T_{k,z,n}$ is a stopping time with respect to $(\mathcal{F}_t)_{t\geq 0}$, which a.s. satisfies $T_{k,z,n} < T_{k,z}$ and $T_{k,z,n} \leq T_{k,z,n+1}$. Consequently, the limit $\tilde{T}_{k,z} := \lim_{n\to\infty} T_{k,z,n}$ exists a.s. and satisfies $\tilde{T}_{k,z} \leq T_{k,z}$ a.s. From the latter we deduce that

$$\tilde{T}_{k,z} = T_{k,z} \text{ a.s. on the event } \left\{ \tilde{T}_{k,z} = \infty \right\}$$

holds. On the other hand, the definition of $T_{k,z,n}$ and the fact that the paths of the limit process are a.s. piecewise constant and càdlàg imply that for all $n \in \mathbb{N}$

$$\begin{aligned} &\left\{ \tilde{T}_{k,z} < \infty \right\} \\ &\quad \overset{\text{a.s.}}{\subset} \left\{ \tilde{T}_{k,z} < \infty, \exists s \in [0, \tilde{T}_{k,z}) : |C_{s,z}| \geq n, \forall t \in [s, \tilde{T}_{k,z})\, \forall w \in H_n(z) : \eta_{t,w} = \eta_{s,w} \right\} \end{aligned}$$

holds, where $H_n(z) := z + [-n, n]^2 \cap \mathbb{H}$. Since n is arbitrary, this yields

$$\left\{ \tilde{T}_{k,z} < \infty \right\} \overset{\text{a.s.}}{\subset} \left\{ \tilde{T}_{k,z} < \infty, |C_{\tilde{T}_{k,z}^-,z}| = \infty \right\}. \tag{3.15}$$

Moreover, since a.s. no two growth events occur at the same time, Lemma 3.13 in particular shows that on the event $\left\{ \tilde{T}_{k,z} < \infty \right\}$, we a.s. have $|C_{T_{k-1,z},z}| \leq 1$ and hence $\tilde{T}_{k,z} > T_{k-1,z}$. We can therefore conclude from (3.15) that

$$\tilde{T}_{k,z} = T_{k,z} \text{ a.s. on the event } \left\{ \tilde{T}_{k,z} < \infty \right\}$$

holds, which completes the proof of the lemma. □

The Markov-type property stated in Lemma 3.8 now implies the following:

Lemma 3.17. *(i) Let T be a predictable stopping time with respect to $(\mathcal{F}_t)_{t\geq 0}$. Then for all $z \in \overline{\mathbb{H}}$ we have*

$$\mathbf{P} \left[T < \infty, \mathrm{G}_{T^-,T,z} \right] = 0. \tag{3.16}$$

(ii) For any $w \in \mathbb{H}$, $z \in \overline{\mathbb{H}}$ it holds that

$$\mathbf{P}\left[\exists t > 0 : |C_{t^-,w}| = \infty, \mathrm{G}_{t^-,t,z}\right] = 0.$$

Proof. Part (i): Let T be a predictable stopping time which is announced by some sequence $(T_n)_{n\in\mathbb{N}}$ of stopping times. Let $z \in \overline{\mathbb{H}}$. Pick $\epsilon > 0$ arbitrary. Then the definition of predictability yields

$$\mathbf{P}\left[T < \infty, \mathrm{G}_{T^-,T,z}\right] = \lim_{n\to\infty} \mathbf{P}\left[T < \infty, T - T_n < \epsilon, \mathrm{G}_{T^-,T,z}\right].$$

Fixing $n \in \mathbb{N}$, we obtain

$$\begin{aligned}\mathbf{P}\left[T < \infty, T - T_n < \epsilon, \mathrm{G}_{T^-,T,z}\right] &\leq \mathbf{P}\left[T < \infty, T - T_n < \epsilon, \mathrm{G}_{T_n,T_n+\epsilon,z}\right] \\ &\leq \mathbf{P}\left[\mathrm{G}_{T_n,T_n+\epsilon,z}\right] \\ &= \mathbf{P}\left[\mathrm{G}_{0,\epsilon,z}\right] = 1 - e^{-\epsilon},\end{aligned}$$

where we used Lemma 3.8 for the penultimate equality. It thus follows that

$$\mathbf{P}\left[T < \infty, \mathrm{G}_{T^-,T,z}\right] \leq 1 - e^{-\epsilon}.$$

Since $\epsilon > 0$ is arbitrary, this proves equation (3.16).

Part (ii): Let $w \in \mathbb{H}$, $z \in \overline{\mathbb{H}}$. Equation (3.14) implies

$$\left\{\exists t > 0 : |C_{t^-,w}| = \infty, \mathrm{G}_{t^-,t,z}\right\} \overset{\text{a.s.}}{\subset} \left\{\exists k \in \mathbb{N} : T_{k,w} < \infty, \mathrm{G}_{T_{k,w}^-,T_{k,w},z}\right\}.$$

But the latter is a null set by part (i) because $T_{k,w}$ is a predictable stopping time for all $k \in \mathbb{N}$ by Lemma 3.16. ☐

We have thus proved that the limit process a.s. satisfies the second half of [DESTRUCTION] (i) in Definition 1.3:

Lemma 3.18. *For all $z \in \mathbb{H}$ we have*

$$\mathbf{P}\left[\exists t > 0 : |C_{t^-,z}| = \infty, \eta_{t,z} = 1\right] = 0;$$

in other words: A.s. if the left-sided limit of the cluster at z is infinite at some time t, the site z becomes vacant at time t.

Proof. This is an immediate consequence of Lemma 3.13 and Lemma 3.17 (ii). ☐

3.6 Completion of the proof of Theorem 1.2

We next prove that the limit process $(\eta_{t,z}, G_{t,z})_{t\geq 0, z\in\overline{\mathbb{H}}}$ a.s. satisfies [DESTRUCTION] (ii) in Definition 1.3:

Lemma 3.19. *For all $z \in \mathbb{H}$ we have*

$$\mathbf{P}\left[\exists t > 0 : \eta_{t^-,z} > \eta_{t,z}, |C_{t^-,z}| < \infty, \forall u \in \partial C_{t^-,z} \cap \partial\mathbb{H} : \complement\, \mathrm{G}_{t^-,t,u}\right] = 0;$$

in other words: A.s. if the site z becomes vacant at some time t and its cluster was not about to become infinite at time t, its cluster must have grown to the boundary at time t.

Proof. The following argument is similar to the proof of Lemma 23 in [7]. Let $z \in \mathbb{H}$. As in Lemma 3.7, let C_z^{fin} denote the (countable) set of all finite connected subsets of $\mathbb{H}$ which contain the site z. Then the relation

$$\begin{aligned}&\left\{\exists t > 0 : \eta_{t^-,z} > \eta_{t,z}, |C_{t^-,z}| < \infty, \forall u \in \partial C_{t^-,z} \cap \partial\mathbb{H} : \complement\, \mathrm{G}_{t^-,t,u}\right\} \\ &\qquad = \bigcup_{S\in C_z^{\mathrm{fin}}} \left\{\exists t > 0 : \eta_{t^-,z} > \eta_{t,z}, C_{t^-,z} = S, \forall u \in \partial S \cap \partial\mathbb{H} : \complement\, \mathrm{G}_{t^-,t,u}\right\} \\ &\qquad \subset \bigcup_{S\in C_z^{\mathrm{fin}}} \left\{\exists t > 0 : D_{S,t,z}\right\}\end{aligned}$$

holds, where we abbreviate

$$D_{S,t,z} := \left\{\eta_{t-,z} > \eta_{t,z}, \forall w \in \partial S : \eta_{t-,w} = 0, \forall u \in \partial S \cap \partial \mathbb{H} : \complement\, \mathrm{G}_{t-,t,u}\right\}.$$

So let $S \in C_z^{\mathrm{fin}}$; it then suffices to show $\mathbf{P}\left[\exists t > 0 : D_{S,t,z}\right] = 0$. We distinguish whether or not at time t there is the growth of a tree at some site in $\partial S \cap \mathbb{H}$ and thus obtain

$$\{\exists t > 0 : D_{S,t,z}\} = A_{S,z} \cup B_{S,z}$$

with

$$\begin{aligned} A_{S,z} &:= \left\{\exists t > 0 : D_{S,t,z}, \forall v \in \partial S \cap \mathbb{H} : \complement\, \mathrm{G}_{t-,t,v}\right\}, \\ B_{S,z} &:= \left\{\exists t > 0 : D_{S,t,z}, \exists v \in \partial S \cap \mathbb{H} : \mathrm{G}_{t-,t,v}\right\}. \end{aligned}$$

We first consider the event $A_{S,z}$: Since the paths of the limit process are a.s. piecewise constant and càdlàg, and since the set ∂S is finite, it follows that

$$A_{S,z} \overset{\text{a.s.}}{\subset} \left\{\exists 0 \le s < t : \eta_{s,z} > \eta_{t,z}, \forall w \in \partial S : \eta_{s,w} = 0, \complement\, \mathrm{G}_{s,t,w}\right\}.$$

Now for all sufficiently large n (such that $S \cup \partial S \subset B_n$, where the boundary ∂S is taken in $\overline{\mathbb{H}}$) and arbitrary $0 \le s < t$, it is easy to deduce from [GROWTH] and [DESTRUCTION] in Definition 1.1 that B_n-forest-fire processes satisfy

$$\mathbf{P}\left[\eta^n_{s,z} > \eta^n_{t,z}, \forall w \in \partial S : \eta^n_{s,w} = 0, \complement\, \mathrm{G}^n_{s,t,w}\right] = 0.$$

Hence Lemma 3.3 yields $\mathbf{P}[A_{S,z}] = 0$.

We now consider the event $B_{S,z}$: Resorting to Lemma 3.17 (ii), we obtain

$$\begin{aligned} B_{S,z} &\overset{\text{a.s.}}{\subset} \left\{\exists t > 0 : D_{S,t,z}, \exists v \in \partial S \cap \mathbb{H} : \mathrm{G}_{t-,t,v}, \forall x \in \mathbb{H} : |C_{t-,x}| < \infty\right\} \\ &\subset \left\{\exists t > 0 : D_{S,t,z}, \exists v \in \partial S \cap \mathbb{H} : \mathrm{G}_{t-,t,v}, \exists S' \in C_z^{\mathrm{fin}} : S' = S \cup \{v\} \cup \bigcup_{x \in \partial\{v\}} C_{t-,x}\right\} \\ &\subset \bigcup_{S' \in C_z^{\mathrm{fin}}} \left\{\exists t > 0 : \eta_{t-,z} > \eta_{t,z}, \forall w \in \partial S' : \eta_{t-,w} = 0, \exists v \in S' : \mathrm{G}_{t-,t,v}\right\} \\ &\overset{\text{a.s.}}{\subset} \bigcup_{S' \in C_z^{\mathrm{fin}}} \left\{\exists t > 0 : \eta_{t-,z} > \eta_{t,z}, \forall w \in \partial S' : \eta_{t-,w} = 0, \complement\, \mathrm{G}_{t-,t,w}\, \exists v \in S' : \mathrm{G}_{t-,t,v}\right\} \\ &\subset \bigcup_{S' \in C_z^{\mathrm{fin}}} A_{S',z}, \end{aligned}$$

where in the penultimate step we used that a.s. no two growth events occur at the same time. So the above implies $\mathbf{P}[B_{S,z}] = 0$. □

Finally, we show that the limit process also a.s. satisfies [GROWTH] (i) in Definition 1.3:

Lemma 3.20. *For all $z \in \mathbb{H}$ we have*

$$\mathbf{P}\left[\exists t > 0 : \mathrm{G}_{t-,t,z}, \eta_{t,z} = 0\right] = 0;$$

in other words: A.s. if a tree grows at the site z at some time t, then the site z is occupied at time t.

Proof. The argument to come is similar to the proof of Lemma 24 in [7]. Let $z \in \mathbb{H}$. Then the following inclusions hold:

$$\begin{aligned}
&\left\{\exists t>0 : \mathrm{G}_{t-,t,z}, \eta_{t,z}=0\right\} \\
&\quad \overset{\text{a.s.}}{\subset} \left\{\exists t>0 : \mathrm{G}_{t-,t,z}, \eta_{t,z}=0, \forall w \in \overline{\mathbb{H}} \setminus \{z\} : \complement\, \mathrm{G}_{t-,t,w}, |C_{t-,w}|<\infty\right\} \\
&\quad \overset{\text{a.s.}}{\subset} \left\{\exists t>0 : \mathrm{G}_{t-,t,z}, \eta_{t,z}=0, \forall w \in \partial\{z\} : \eta_{t-,w}=\eta_{t,w}, \complement\, \mathrm{G}_{t-,t,w}\right\} \\
&\quad \overset{\text{a.s.}}{\subset} \left\{\exists 0 \le s<t : \mathrm{G}_{s,t,z}, \eta_{t,z}=0, \forall w \in \partial\{z\} : \eta_{s,w}=\eta_{t,w}, \complement\, \mathrm{G}_{s,t,w}\right\}.
\end{aligned}$$

Indeed, the first inclusion follows from Lemma 3.17 (ii) and the fact that a.s. no two growth events occur at the same time, the second inclusion is a consequence of the properties [GROWTH] (ii) and [DESTRUCTION] (ii) in Definition 1.3 (which have already been proved for the limit process in Lemmas 3.4 and 3.19), and the third inclusion is due to the fact that the paths of the limit process are a.s. piecewise constant and càdlàg. (The case $w \in \partial\mathbb{H}$ in these events is somewhat separate but trivial due to the zero boundary condition proved in Lemma 3.4.) Now for all sufficiently large n (such that $\{z\} \cup \partial\{z\} \subset B_n$, where the boundary $\partial\{z\}$ is taken in $\overline{\mathbb{H}}$) and arbitrary $0 \le s < t$, it is easy to deduce from [GROWTH] and [DESTRUCTION] in Definition 1.1 that B_n-forest-fire processes satisfy

$$\mathbf{P}\left[\mathrm{G}^n_{s,t,z}, \eta^n_{t,z}=0, \forall w \in \partial\{z\} : \eta^n_{s,w}=\eta^n_{t,w}, \complement\, \mathrm{G}^n_{s,t,w}\right]=0.$$

The result therefore follows from Lemma 3.3. □

Lemmas 3.1, 3.2, 3.4, 3.7, 3.18, 3.19 and 3.20 combined thus provide the proof of Theorem 1.2.

4 Proof of Theorem 1.5

Throughout this section, let $(\eta_{t,z}, G_{t,z})_{t\ge0, z\in\overline{\mathbb{H}}}$ be an $\overline{\mathbb{H}}$-forest-fire process (see Definition 1.3), let $(\sigma_{t,z})_{t\ge0,z\in\overline{\mathbb{H}}}$ be the associated pure growth process defined by equation (1.1), and let $(Y_{t,x})_{t\ge0,x\in\partial\mathbb{H}}$ be the heights of destruction of the process $(\eta_{t,z}, G_{t,z})_{t\ge0,z\in\overline{\mathbb{H}}}$ (see Definition 1.4). As we already noted in Section 1, for fixed $t \ge 0$ the distribution of $\sigma_t := (\sigma_{t,z})_{z\in\overline{\mathbb{H}}}$ is independent site percolation on $\overline{\mathbb{H}}$, where each site is open with probability $1-e^{-t}$.

In the following, it will also be convenient to consider independent site percolation on the whole lattice $\mathbb{Z}^2$. So for $t \ge 0$, let $\xi_t := (\xi_{t,z})_{z\in\mathbb{Z}^2}$ be distributed according to independent site percolation on $\mathbb{Z}^2$, where each site is open with probability $1-e^{-t}$. We realize both $(\eta_{t,z}, G_{t,z})_{t\ge0,z\in\overline{\mathbb{H}}}$ and ξ_t, $t \ge 0$, on a probability space $(\Omega, \mathcal{A}, \mathbf{P})$.

A key concept for the treatment of site percolation on the square lattice $\mathbb{Z}^2$ is the so-called matching lattice $\mathbb{Z}^{2*}$, which is obtained from the square lattice $\mathbb{Z}^2$ by adding diagonal edges to all faces in $\mathbb{Z}^2$. In this way, certain statements about open sites on the square lattice $\mathbb{Z}^2$ can be reformulated as statements about closed sites on the matching lattice $\mathbb{Z}^{2*}$; see [11], Section 3.1, for more details. We therefore extend our terminology as follows: Let W be a subset of $\mathbb{Z}^2$ and let $\alpha := (\alpha_w)_{w\in W} \in \{0,1\}^W$ be any configuration on W. Let $\mathbb{Z}^2|_{\alpha,1}$ denote the subgraph of the square lattice $\mathbb{Z}^2$ induced by the vertex set $\{w \in W : \alpha_w = 1\}$. Then by a **$1$-path** in the configuration α we simply mean any path on the graph $\mathbb{Z}^2|_{\alpha,1}$. Similarly, let $\mathbb{Z}^{2*}|_{\alpha,0}$ denote the subgraph of the matching lattice $\mathbb{Z}^{2*}$ induced by the vertex set $\{w \in W : \alpha_w = 0\}$. Then a **$0*$-path** in the configuration α is simply any path on the graph $\mathbb{Z}^{2*}|_{\alpha,0}$.

For $w \in \mathbb{Z}^2$ and $n \in \mathbb{N}$, let

$$B_n(w) := w + [-n,n]^2 \cap \mathbb{Z}^2 = \left\{z \in \mathbb{Z}^2 : |\operatorname{Re}(z-w)| \le n, |\operatorname{Im}(z-w)| \le n\right\} \tag{4.1}$$

denote the box with centre w and radius n, and let

$$S_n(w) := \{z \in \mathbb{Z}^2 : |\operatorname{Re}(z-w)| = n, |\operatorname{Im}(z-w)| \le n\} \\ \cup \{z \in \mathbb{Z}^2 : |\operatorname{Re}(z-w)| \le n, |\operatorname{Im}(z-w)| = n\}$$

denote the inner boundary of that box. For later reference we also define the left side

$$L_n(w) := \{z \in \mathbb{Z}^2 : \operatorname{Re}(z-w) = -n, |\operatorname{Im}(z-w)| \le n\} \tag{4.2}$$

and the right side

$$R_n(w) := \{z \in \mathbb{Z}^2 : \operatorname{Re}(z-w) = n, |\operatorname{Im}(z-w)| \le n\} \tag{4.3}$$

of the box $B_n(w)$.

We will need the following two well-known results from percolation theory:

Correlation length. For all $t > t_c$ the "inverse correlation length"

$$c(t) := \lim_{n\to\infty} \frac{\log \mathbf{P}\left[\xi_t \text{ contains a } 0{*}\text{-path from } 0 \text{ to } S_n(0)\right]}{-n}$$

is well-defined, and there exist universal constants $\rho, \sigma > 0$ such that

$$\rho n^{-1} e^{-c(t)n} \le \mathbf{P}\left[\xi_t \text{ contains a } 0{*}\text{-path from } 0 \text{ to } S_n(0)\right] \le \sigma n e^{-c(t)n} \tag{4.4}$$

holds for all $t > t_c$ and all $n \in \mathbb{N}$ (see [11], Section 6.1, for instance[4]). We will only use the left inequality in (4.4).

Percolation on subsets of the half-plane. Let $t > t_c$. Define the bijective function $h_t : [e, \infty) \to [\frac{1}{c(t)}, \infty)$ by

$$h_t(y) := \frac{1}{c(t)} \left(\log y + 3 \log\log y\right), \qquad y \ge e, \tag{4.5}$$

and let $g_t : [\frac{1}{c(t)}, \infty) \to [e, \infty)$ be its inverse function. Extend g_t continuously to $[0, \infty)$ by setting

$$g_t(x) := e, \qquad 0 \le x < \frac{1}{c(t)}.$$

(The specific way of the extension is immaterial.) Then

$$\mathbf{P}\left[(\sigma_{t,x+iy})_{x\ge 0, y \ge g_t(x)} \text{ contains an infinite cluster}\right] = 1 \tag{4.6}$$

holds; in other words: A.s. the restriction of σ_t to the area $\{x+iy \in \overline{\mathbb{H}} : x \ge 0, y \ge g_t(x)\}$ (endowed with the edges inherited from $\overline{\mathbb{H}}$) contains an infinite cluster. A more detailed account of this topic can be found in [11], Section 11.5, or in the original papers [10], [5][5].

[4] In this reference the statement is proved for independent bond percolation on $\mathbb{Z}^2$ but the proof is identical for independent site percolation on $\mathbb{Z}^2$.

[5] Again, in these references the statement is proved for independent bond percolation on $\mathbb{Z}^2$ but the proof carries over to independent site percolation on $\mathbb{Z}^2$ when duality of lattices is replaced by matching of lattices.

Remark 4.1. *A closer look shows that the core of the proof of Theorem 1.5 only relies on the following weaker versions of equations (4.4) and (4.6): For all $t > t_c$ there exists $a(t) > 0$ such that for all $n \in \mathbb{N}$*

$$\mathbf{P}\left[\xi_t \text{ contains a } 0{*}\text{-path from } 0 \text{ to } S_n(0)\right] \geq e^{-a(t)n}$$

holds and there exists $b(t) > 0$ such that

$$\mathbf{P}\left[(\sigma_{t,x+iy})_{x\geq 0, y\geq e^{b(t)x}} \text{ contains an infinite cluster}\right] = 1$$

holds. However, if we used only these equations, the statements of some of the lemmas to come would have to be weakened accordingly, e.g. the width of the semi-infinite tube in Lemma 4.3 would then also depend on t.

As a direct consequence of (4.6), we deduce the following lemma:

Lemma 4.2. *For $t > t_c$ define the function $f_t : (0,\infty) \to (0,\infty)$ by*

$$f_t(x) := \frac{1}{(c(t)x)^3} e^{c(t)x}, \qquad x > 0.$$

Then for all $t > t_c$ we have

$$\mathbf{P}\left[Y_{t,x} \geq f_t(x) \text{ for infinitely many } x \in \mathbb{N}\right] = 1. \tag{4.7}$$

Proof. Let $t > t_c$. From equations (1.3) and (4.6), together with the fact that the configuration $(\eta_{t,z})_{z\in\overline{\mathbb{H}}}$ a.s. does not contain an infinite cluster, we conclude

$$\mathbf{P}\left[Y_{t,x} \geq g_t(x) \text{ for infinitely many } x \in \mathbb{N}\right] = 1.$$

It is therefore enough to show that $g_t(x) \geq f_t(x)$ holds for all sufficiently large $x \in \mathbb{N}$. Indeed, the definition of g_t (below (4.5)) yields

$$x = \frac{1}{c(t)}\left(\log g_t(x) + 3\log\log g_t(x)\right), \qquad x \geq \frac{1}{c(t)},$$

and applying f_t on both sides of this equation gives

$$f_t(x) = \left(\frac{\log g_t(x)}{\log g_t(x) + 3\log\log g_t(x)}\right)^3 g_t(x), \qquad x \geq \frac{1}{c(t)}.$$

Since $g_t(x) \geq e$ for $x \geq \frac{1}{c(t)}$, we have

$$\left(\frac{\log g_t(x)}{\log g_t(x) + 3\log\log g_t(x)}\right)^3 \leq 1, \qquad x \geq \frac{1}{c(t)},$$

which completes the proof. □

The first inequality in (4.4) also implies the following:

Lemma 4.3. *For $t > t_c$ and $x \in \mathbb{N}$ let*

$$T_{t,x} := \left[\tfrac{3}{4}x, \tfrac{5}{4}x\right] \times \left[\tfrac{1}{2}f_t(x), \infty\right) \cap \mathbb{H}$$

be the semi-infinite tube with vertical midline at x, width $2\lfloor\frac{x}{4}\rfloor$ and starting height $\lceil\frac{1}{2}f_t(x)\rceil$, and let

$$D_{t,x} := \left[\tfrac{3}{4}x, \tfrac{5}{4}x\right] \times \left\{\lceil\tfrac{1}{2}f_t(x)\rceil\right\} \cap \mathbb{H}$$

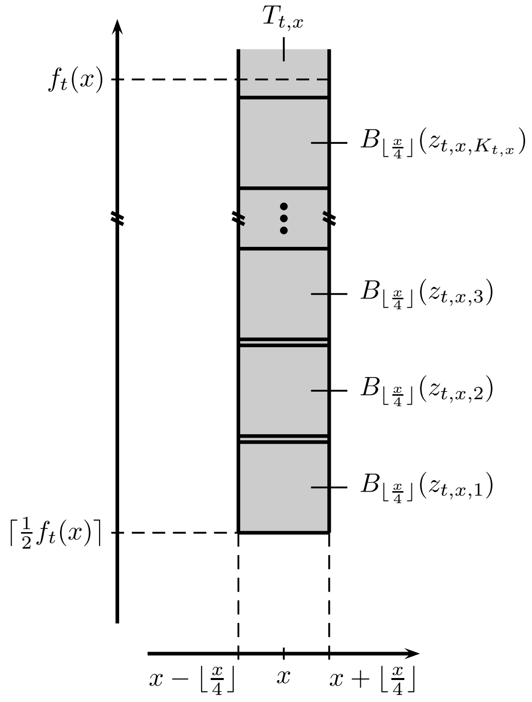

Figure 3: Partition of the tube $T_{t,x}$ into $K_{t,x}$ boxes

be its baseline. Additionally, let

$$\text{PATH-IN-TUBE}_{t,x} := \{\exists y \geq f_t(x) : \sigma_t \textit{ contains a } 1\textit{-path from } x + iy \textit{ to } D_{t,x} \textit{ within } T_{t,x}\}$$

be the event that in the configuration σ_t there exists a site with real part x and imaginary part at least $f_t(x)$ which is connected by a 1-path to the baseline $D_{t,x}$ within the tube $T_{t,x}$. Then for all $t > t_c$ we have

$$\mathbf{P}\left[\text{PATH-IN-TUBE}_{t,x} \textit{ for infinitely many } x \in \mathbb{N}\right] = 0. \tag{4.8}$$

Proof. Let $t > t_c$ and $x \in \mathbb{N}$. As depicted in Figure 3, we partition the tube $T_{t,x}$ upto height $f_t(x)$ into disjoint boxes of radius $\lfloor \frac{x}{4} \rfloor$ such that adjacent boxes have vertical distance 1. Let $K_{t,x} := \left\lfloor \frac{f_t(x) - \lceil \frac{1}{2} f_t(x) \rceil + 1}{2\lfloor \frac{x}{4} \rfloor + 1} \right\rfloor$ be the number of these boxes, and for $k \in \{1, \ldots, K_{t,x}\}$ let

$$z_{t,x,k} := x + i\left(\lceil \tfrac{1}{2} f_t(x) \rceil + (2k-1)\lfloor \tfrac{x}{4} \rfloor + (k-1)\right)$$

be the centre of the kth such box. Recalling the notation introduced in equations (4.1), (4.2) and (4.3), and passing from the lattice $\mathbb{Z}^2$ to the matching lattice $\mathbb{Z}^{2*}$, we obtain

$$\begin{aligned}
&\mathbf{P}\left[\text{PATH-IN-TUBE}_{t,x}\right] \\
&\quad \leq \mathbf{P}\Big[\forall k \in \{1, \ldots, K_{t,x}\} : \sigma_t \text{ contains no } 0{*}\text{-path from } L_{\lfloor \frac{x}{4} \rfloor}(z_{t,x,k}) \\
&\qquad\quad \text{to } R_{\lfloor \frac{x}{4} \rfloor}(z_{t,x,k}) \text{ within } B_{\lfloor \frac{x}{4} \rfloor}(z_{t,x,k})\Big] \\
&\quad = \Big(1 - \mathbf{P}\Big[\xi_t \text{ contains a } 0{*}\text{-path from } L_{\lfloor \frac{x}{4} \rfloor}(0) \text{ to } R_{\lfloor \frac{x}{4} \rfloor}(0) \text{ within } B_{\lfloor \frac{x}{4} \rfloor}(0)\Big]\Big)^{K_{t,x}}.
\end{aligned}$$

Now an argument similar to the proof of Theorem 11.55 in [11] gives

$$\begin{aligned}
&\mathbf{P}\left[\xi_t \text{ contains a } 0*\text{-path from } L_{\lfloor \frac{x}{4} \rfloor}(0) \text{ to } R_{\lfloor \frac{x}{4} \rfloor}(0) \text{ within } B_{\lfloor \frac{x}{4} \rfloor}(0)\right] \\
&\qquad \geq \mathbf{P}\left[\xi_t \text{ contains a } 0*\text{-path from } L_{\lfloor \frac{x}{4} \rfloor}(0) \text{ through } 0 \text{ to } R_{\lfloor \frac{x}{4} \rfloor}(0) \text{ within } B_{\lfloor \frac{x}{4} \rfloor}(0)\right] \\
&\qquad \geq \left(\frac{1}{4}\mathbf{P}\left[\xi_t \text{ contains a } 0*\text{-path from } 0 \text{ to } S_{\lfloor \frac{x}{4} \rfloor}(0)\right]\right)^2 \\
&\qquad \geq \frac{1}{16}\rho^2 \frac{1}{\lfloor \frac{x}{4} \rfloor^2} e^{-2c(t)\lfloor \frac{x}{4} \rfloor} \\
&\qquad \geq \Omega\left(e^{-\frac{4c(t)}{6}x}\right) \qquad \text{for } x \to \infty.
\end{aligned}$$

Here the second inequality is obtained by an application of the FKG inequality, the third inequality is a consequence of (4.4), and in the last inequality we use Landau notation. In addition, it is evident from the definition of $K_{t,x}$ that

$$K_{t,x} \geq \Omega\left(e^{\frac{5c(t)}{6}x}\right) \qquad \text{for } x \to \infty$$

holds. From all this we conclude that $\mathbf{P}\left[\text{PATH-IN-TUBE}_{t,x}\right]$ decays at least exponentially for $x \to \infty$, in particular

$$\sum_{x=1}^{\infty} \mathbf{P}\left[\text{PATH-IN-TUBE}_{t,x}\right] < \infty$$

holds. Equation (4.8) therefore follows from the Borel-Cantelli lemma. □

In Lemma 4.2 we saw that for any time t there are a.s. infinitely many $x \in \mathbb{N}$ with $Y_{t,x} \geq f_t(x)$. Very roughly speaking, we now want to prove that if $Y_{t,x} \geq f_t(x)$ holds for some $x \in \mathbb{N}$, then for all $\tilde{x}$ of order x the corresponding height of destruction $Y_{t,\tilde{x}}$ is also of order at least $f_t(x)$, i.e. there cannot be "large fluctuations" in the heights of destruction at time t. The precise statement is as follows:

Lemma 4.4. *For $t > t_c$ and $x \in \mathbb{N}$ let*

$$\begin{aligned}
\text{LARGE-FLUCT}_{t,x} := \big\{ &Y_{t,x} \geq f_t(x), \exists x_1, x_2 \in \mathbb{N} : \tfrac{3}{4}x \leq x_1 < x < x_2 \leq \tfrac{5}{4}x, \\
&Y_{t,x_1} < \tfrac{1}{2} f_t(x), Y_{t,x_2} < \tfrac{1}{2} f_t(x) \big\}
\end{aligned}$$

denote the event that the height of destruction at x up to time t is at least $f_t(x)$ but there exist $\frac{3}{4}x \leq x_1 < x < x_2 \leq \frac{5}{4}x$ such that the height of destruction at x_1 and x_2 up to time t is less than $\frac{1}{2}f_t(x)$ (see Figure 4). Then for all $t > t_c$ we have

$$\mathbf{P}\left[\text{LARGE-FLUCT}_{t,x} \text{ for infinitely many } x \in \mathbb{N}\right] = 0. \tag{4.9}$$

Proof. Let $t > t_c$ and $x \in \mathbb{N}$. We are going to prove

$$\text{LARGE-FLUCT}_{t,x} \overset{\text{a.s.}}{\subset} \text{PATH-IN-TUBE}_{t,x}, \tag{4.10}$$

from which equation (4.9) follows by Lemma 4.3. So assume that the event $\text{LARGE-FLUCT}_{t,x}$ occurs. Then by the definition of the height of destruction, there exist $y \geq f_t(x)$ and $0 < s \leq t$ such that

$$\eta_{s^-,x+iy} = 1, \eta_{s,x+iy} = 0$$

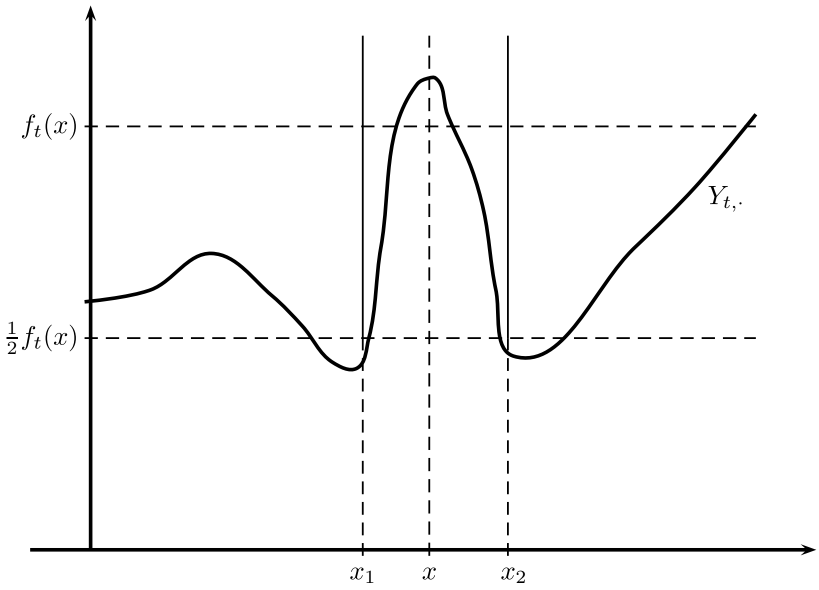

Figure 4: A visualisation of the event $\mathrm{LARGE\text{-}FLUCT}_{t,x}$

holds. According to the property [DESTRUCTION] in Definition 1.3, this means that one of the following two cases occurs:
Case 1: $|C_{s^-,x+iy}| = \infty$.
Case 2: $C_{s^-,x+iy}$ contains a site in $\partial\mathbb{H} + i$.
However, the condition $Y_{t,x_1} < \frac{1}{2}f_t(x), Y_{t,x_2} < \frac{1}{2}f_t(x)$ in the event $\mathrm{LARGE\text{-}FLUCT}_{t,x}$ implies that all sites of the form $x_1 + iy_1, x_2 + iy_2$ with $y_1, y_2 \geq \frac{1}{2}f_t(x)$ cannot be part of $C_{s^-,x+iy}$. It is easy to see that a.s. in both cases this implies that the configuration $(\eta_{s^-,z})_{z\in\overline{\mathbb{H}}}$ contains a 1-path which runs from $x + iy$ to the baseline

$$(x_1, x_2) \times \left\{\lceil \tfrac{1}{2}f_t(x)\rceil\right\} \cap \mathbb{H}$$

within the half-infinite tube

$$(x_1, x_2) \times \left[\tfrac{1}{2}f_t(x), \infty\right) \cap \mathbb{H}. \tag{4.11}$$

(For case 1 observe that a.s. there exists $v \geq y$ with $\eta_{s^-,u+iv} = 0$ for all $u \in \{x_1, x_1 + 1, \dots, x_2\}$ so that the cluster $C_{s^-,x+iy}$ cannot stretch to infinity within the tube (4.11).) Since the tube (4.11) is a subset of the tube $T_{t,x}$ and because of the basic inequality (1.2), this proves the inclusion (4.10). □

Lemmas 4.2 and 4.4 enable us to prove the following lemma, which is only slightly weaker than Theorem 1.5:

Lemma 4.5. *For all* $t > t_c$ *we have* $\mathbf{P}[Y_{t,0} = \infty] = 1$.

Proof. Let $t > t_c$. Suppose that the lemma is not true; then there exists $y \in \mathbb{N}_0$ with $\mathbf{P}[Y_{t,0} = y] > 0$. The translation-invariance of $\overline{\mathbb{H}}$-forest-fire processes ([TRANSL-INV] in Definition 1.3) and the Birkhoff ergodic theorem (see e.g. [15], Section V.3, Theorem 1) imply that the sequence $\frac{1}{n}\sum_{x=0}^{n-1} 1_{\{Y_{t,x}=y\}}$, $n \in \mathbb{N}$, is a Cauchy sequence a.s. and that there exists $\epsilon > 0$ such that the event

$$A := \left\{ \frac{1}{n}\sum_{x=0}^{n-1} 1_{\{Y_{t,x}=y\}} > \epsilon \text{ eventually as } n \to \infty \right\}$$

satisfies $\mathbf{P}[A] > 0$. Consequently, on the event A there a.s. exists $n_0 \in \mathbb{N}$ such that for all $n_1, n_2 \geq n_0$

$$\left| \frac{1}{n_1} \sum_{x=0}^{n_1-1} 1_{\{Y_{t,x}=y\}} - \frac{1}{n_2} \sum_{x=0}^{n_2-1} 1_{\{Y_{t,x}=y\}} \right| < \frac{1}{9}\epsilon \tag{4.12}$$

and

$$\frac{1}{n_1} \sum_{x=0}^{n_1-1} 1_{\{Y_{t,x}=y\}} > \epsilon$$

hold.

However, given n_0 on the event A, it follows from Lemmas 4.2 and 4.4 that there a.s. exists $n_1 \geq \max\{n_0, 8\}$ such that for all $x \in \left\{n_1, \ldots, n_1 + \lfloor \frac{n_1}{4} \rfloor\right\}$

$$1_{\{Y_{t,x}=y\}} = 0$$

holds. With this n_1 and $n_2 := n_1 + \lfloor \frac{n_1}{4} \rfloor$ we obtain

$$\begin{aligned} \frac{1}{n_1} \sum_{x=0}^{n_1-1} 1_{\{Y_{t,x}=h\}} - \frac{1}{n_2} \sum_{x=0}^{n_2-1} 1_{\{Y_{t,x}=h\}} &= \frac{1}{n_1} \sum_{x=0}^{n_1-1} 1_{\{Y_{t,x}=h\}} \left(1 - \frac{n_1}{n_1 + \lfloor \frac{n_1}{4} \rfloor}\right) \\ &> \epsilon \cdot \left(1 - \frac{1}{\frac{5}{4} - \frac{1}{n_1}}\right) \geq \frac{1}{9}\epsilon, \end{aligned}$$

which is opposed to (4.12). Hence $\mathbf{P}[A] > 0$ cannot hold - a contradiction. $\square$

Theorem 1.5 is now an immediate consequence of Lemma 4.5: The translation-invariance [TRANSL-INV] implies that we only need to consider the case $x = 0$ in Theorem 1.5. Since $Y_{t,0}$ is obviously monotone increasing in t, we have

$$\{\forall t > t_c : Y_{t,0} = \infty\} = \{\forall t \in (t_c, \infty) \cap \mathbb{Q} : Y_{t,0} = \infty\}$$

so that

$$\mathbf{P}\left[\forall t > t_c : Y_{t,0} = \infty\right] = 1 \tag{4.13}$$

follows from Lemma 4.5. Moreover, if $0 \leq t < t_c$ and $y \in \mathbb{N}_0$, then the definition of the height of destruction, the condition [DESTRUCTION] in Definition 1.3 and equation (1.2) yield

$$\{Y_{t,0} \geq y\} \subset \{\exists v \geq y : \sigma_t \text{ contains a 1-path from } iv \text{ to } \partial\mathbb{H}\}.$$

As a consequence of the exponential decay of the radius for subcritical independent site percolation on $\mathbb{Z}^2$, the probability of the latter event decays to zero as $y \to \infty$ so that

$$\mathbf{P}\left[Y_{t,0} = \infty\right] = \lim_{y\to\infty} \mathbf{P}\left[Y_{t,0} \geq y\right] = 0$$

holds for $0 \leq t < t_c$. Herefrom we readily deduce

$$\mathbf{P}\left[\exists 0 \leq t < t_c : Y_{t,0} = \infty\right] = 0 \tag{4.14}$$

by a similar monotonicity argument as above. Equations (4.13) and (4.14) complete the proof of Theorem 1.5.

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