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Renewal theorems for random walks in random scenery*

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Abstract

Random walks in random scenery are processes defined by $Z_n := \sum_{k=1}^n \xi_{X_1+\ldots+X_k}$, where $(X_k, k \geq 1)$ and $(\xi_y, y \in \mathbb{Z})$ are two independent sequences of i.i.d. random variables. We suppose that the distributions of X_1 and ξ_0 belong to the normal domain of attraction of strictly stable distributions with index $\alpha \in [1,2]$ and $\beta \in (0,2]$ respectively. We are interested in the asymptotic behaviour as $|a|$ goes to infinity of quantities of the form $\sum_{n\geq 1} \mathbb{E}[h(Z_n - a)]$ (when $(Z_n)_n$ is transient) or $\sum_{n\geq 1} \mathbb{E}[h(Z_n) - h(Z_n - a)]$ (when $(Z_n)_n$ is recurrent) where h is some complex-valued function defined on $\mathbb{R}$ or $\mathbb{Z}$.

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1 Introduction

Renewal theorems in probability theory deal with the asymptotic behaviour when $|a| \to +\infty$ of the potential kernel formally defined as

$$K_a(h) := \sum_{n=1}^{\infty} \mathbb{E}[h(Z_n - a)]$$

where h is some complex-valued function defined on $\mathbb{R}$ and $(Z_n)_{n\geq 1}$ a real transient random process. The above kernel $K_a(.)$ is not well-defined for recurrent process $(Z_n)_{n\geq 1}$, in that case, we would rather study the kernel

$$G_{n,a}(h) := \sum_{k=1}^{n} \Big\{ \mathbb{E}[h(Z_k)] - \mathbb{E}[h(Z_k - a)] \Big\}$$

for n and $|a|$ large. In the classical case when Z_n is the sum of n non-centered independent and identically distributed real random variables, renewal theorems were proved

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by Erdös, Feller and Pollard [10], Blackwell [1, 2]. Extensions to multi-dimensional real random walks or additive functionals of Markov chains were also obtained (see [12] for statements and references).

In the particular case where the process $(Z_n)_{n\geq 1}$ takes its values in $\mathbb{Z}$ and h is the Dirac function at 0, the study of the corresponding kernels

$$K_a(\delta_0) = \sum_{n=1}^{\infty} \mathbb{P}[Z_n = a]$$

and

$$G_{n,a}(\delta_0) = \sum_{k=1}^{n} \Big\{ \mathbb{P}[Z_k = 0] - \mathbb{P}[Z_k = a] \Big\}$$

have a long history (see [18]). In the case of aperiodic recurrent random walks on $\mathbb{Z}$ with finite variance, the potential kernel is known to behave asymptotically as $|a|$ when $|a|$ goes to infinity and, for some particular random walks as the simple random walk, an explicit formula can be given (see Chapter VII in [18]).

In this paper we are interested in renewal theorems for random walk in random scenery (RWRS). Random walk in random scenery is a simple model of process in disordered media with long-range correlations. They have been used in a wide variety of models in physics to study anomalous dispersion in layered random flows [16], diffusion with random sources, or spin depolarization in random fields (we refer the reader to Le Doussal's review paper [14] for a discussion of these models). On the mathematical side, motivated by the construction of new self-similar processes with stationary increments, Kesten and Spitzer [13] and Borodin [4, 5] introduced RWRS in dimension one and proved functional limit theorems. Their work has been further developed in [3] and [8]. These processes are defined as follows. We consider two independent sequences $(X_k, k \geq 1)$ and $(\xi_y, y \in \mathbb{Z})$ of independent identically distributed random variables with values in $\mathbb{Z}$ and $\mathbb{R}$ respectively. We define

$$\forall n \geq 1, \quad S_n := \sum_{k=1}^{n} X_k \text{ and } S_0 := 0.$$

The *random walk in random scenery* Z is then defined for all $n \geq 1$ by

$$Z_n := \sum_{k=1}^{n} \xi_{S_k}.$$

The symbol $\#$ stands for the cardinality of a finite set. Denoting by $N_n(y)$ the local time of the random walk S :

$$N_n(y) = \#\big\{k = 1, ..., n \ : \ S_k = y\big\}$$

the random variable Z_n can be rewritten as

$$Z_n = \sum_{y\in\mathbb{Z}} \xi_y N_n(y). \tag{1.1}$$

The distribution of ξ_0 is assumed to belong to the normal domain of attraction of a strictly stable distribution $\mathcal{S}_\beta$ of index $\beta \in (0, 2]$, with characteristic function ϕ given by

$$\phi(u) = e^{-|u|^\beta (A_1 + iA_2 \text{ sgn}(u))}, \quad u \in \mathbb{R}, \tag{1.2}$$

where $0 < A_1 < \infty$ and $|A_1^{-1} A_2| \leq |\tan(\pi\beta/2)|$. When $\beta = 1$, A_2 is null. We will denote by φ_ξ the characteristic function of the random variables ξ_x. When $\beta > 1$, this implies that $\mathbb{E}[\xi_0] = 0$. Under these conditions, we have, for $\beta \in (0, 2]$,

$$\forall t > 0\,, \ \mathbb{P}\left[|\xi_0| \geq t\right] \leq \frac{C(\beta)}{t^\beta}\,. \tag{1.3}$$

Concerning the random walk $(S_n)_{n\geq 1}$, the distribution of X_1 is assumed to belong to the normal domain of attraction of a strictly stable distribution $\mathcal{S}'_\alpha$ of index α. Since, when $\alpha < 1$, the behaviour of $(Z_n)_n$ is very similar to the behaviour of the sum of the ξ_k's, $k = 1, \dots, n$, we restrict ourselves to the study of the case when $\alpha \in [1, 2]$. Under the previous assumptions, the following weak convergences hold in the space of càdlàg real-valued functions defined on $[0, \infty)$ and on $\mathbb{R}$ respectively, endowed with the Skorohod topology :

$$\left(n^{-\frac{1}{\alpha}} S_{\lfloor nt \rfloor}\right)_{t\geq 0} \overset{\mathcal{L}}{\underset{n\to\infty}{\Longrightarrow}} (U(t))_{t\geq 0}$$

$$\text{and} \quad \left(n^{-\frac{1}{\beta}} \sum_{k=0}^{\lfloor nx \rfloor} \xi_k\right)_{x\geq 0} \overset{\mathcal{L}}{\underset{n\to\infty}{\Longrightarrow}} (Y(x))_{x\geq 0},$$

where U and Y are two independent Lévy processes such that $U(0) = 0$, $Y(0) = 0$, $U(1)$ has distribution $\mathcal{S}'_\alpha$ and $Y(1)$ has distribution $\mathcal{S}_\beta$. For $\alpha \in]1, 2]$, we will denote by $(L_t(x))_{x\in\mathbb{R}, t\geq 0}$ a continuous version with compact support of the local time of the process $(U(t))_{t\geq 0}$ and by $|L|_\beta$ the random variable $\left(\int_{\mathbb{R}} L_1^\beta(x)\, dx\right)^{1/\beta}$. Next let us define

$$\delta := 1 - \frac{1}{\alpha} + \frac{1}{\alpha\beta} = 1 + \frac{1}{\alpha}(\frac{1}{\beta} - 1). \tag{1.4}$$

In [13], Kesten and Spitzer proved the convergence in distribution of $((n^{-\delta} Z_{nt})_{t\geq 0})_n$, when $\alpha > 1$, to a process $(\Delta_t)_{t\geq 0}$ defined as

$$\Delta_t = \int_{\mathbb{R}} L_t(x)\, dY(x),$$

by considering a process $(Y(-x))_{x\geq 0}$ with the same distribution as $(Y(x))_{x\geq 0}$ and independent of U and $(Y(x))_{x\geq 0}$.
In [8], Deligiannidis and Utev considered the case when $\alpha = 1$ and $\beta = 2$ and proved the convergence in distribution of $((Z_{nt}/\sqrt{n \log(n)})_{t\geq 0})_n$ to a Brownian motion. This result is obtained by an adaptation of the proof of the same result by Bothausen in [3] in the case when $\beta = 2$ and for a square integrable two-dimensional random walk $(S_n)_n$.
In [7], Castell, Guillotin-Plantard and Pène completed the study of the case $\alpha = 1$ by proving the convergence of $(n^{-\frac{1}{\beta}} (\log(n))^{\frac{1}{\beta}-1} Z_{nt})_{t\geq 0})_n$ to $c^{\frac{1}{\beta}} (Y(t))_{t\in\mathbb{R}}$, with

$$c := (\pi a_0)^{1-\beta} \Gamma(\beta + 1), \tag{1.5}$$

where a_0 is such that $t \mapsto e^{-a_0|t|}$ is the characteristic function of the limit of $(n^{-1} S_n)_n$ and Γ denotes the Euler's gamma function.
Let us indicate that, when $\alpha \geq 1$, the process $(Z_n)_n$ is transient (resp. recurrent) if $\beta < 1$ (resp. $\beta > 1$) (see [6, 17]).
We recall the definition of the Fourier transform $\hat{h}$ as follows. For every $h : \mathbb{R} \to \mathbb{C}$ (resp. $h : \mathbb{Z} \to \mathbb{C}$) integrable with respect to the Lebesgue measure on $\mathbb{R}$ (resp. with respect to the counting measure on $\mathbb{Z}$), we denote by $I[h]$ the integral of h and by $\hat{h} : \mathcal{I} \to \mathbb{C}$ its Fourier transform defined by

$$\forall x \in \mathcal{I}, \quad \hat{h}(x) := I[h(\cdot)e^{ix\cdot}], \quad \text{with } \mathcal{I} = \mathbb{R} \ \ (\text{resp. } \mathcal{I} = [-\pi; \pi]).$$

1.1 Recurrent case : $\beta \in [1, 2]$

We consider two distinct cases:

- **Lattice case:** The random variables $(\xi_x)_{x\in\mathbb{Z}}$ are assumed to be $\mathbb{Z}$-valued and non-arithmetic i.e. $\{u; |\varphi_\xi(u)|=1\}=2\pi\mathbb{Z}$.
  The distribution of ξ_0 belongs to the normal domain of attraction of $\mathcal{S}_\beta$ with characteristic function ϕ given by (1.2).
- **Strongly non-lattice case:** The random variables $(\xi_x)_{x\in\mathbb{Z}}$ are assumed to be strongly non-lattice i.e.
$$\limsup_{|u|\to+\infty}|\varphi_\xi(u)|<1.$$
  The distribution of ξ_0 belongs to the normal domain of attraction of $\mathcal{S}_\beta$ with characteristic function ϕ given by (1.2).

For any $a\in\mathbb{R}$ (resp. $a\in\mathbb{Z}$), we consider the kernel $K_{n,a}$ defined as follows : for any $h:\mathbb{R}\to\mathbb{C}$ (resp. $h:\mathbb{Z}\to\mathbb{C}$) in the strongly non-lattice (resp. in the lattice) case, we write
$$K_{n,a}(h):=\sum_{k=1}^{n}\Big\{\mathbb{E}[h(Z_k)]-\mathbb{E}[h(Z_k-a)]\Big\}$$
when it is well-defined.

Theorem 1.1. *The following assertions hold for every integrable function h on $\mathbb{R}$ with Fourier transform integrable on $\mathbb{R}$ in the strongly non-lattice case and for every integrable function h on $\mathbb{Z}$ in the lattice case.*

- *when $\alpha>1$ and $\beta>1$,*
$$\lim_{a\to+\infty}a^{1-\frac{1}{\delta}}\lim_{n\to+\infty}K_{n,a}(h)=C_1I[h],$$
  with
$$C_1:=\frac{\Gamma(\frac{1}{\delta\beta})\Gamma(2-\frac{1}{\delta})\mathbb{E}\big[|L|_\beta^{-1/\delta}\big]}{\pi\beta(1-\delta)(A_1^2+A_2^2)^{1/2\delta\beta}}\sin\left(\frac{1}{\delta}\left(\frac{\pi}{2}-\frac{1}{\beta}\arctan\left(\frac{A_2}{A_1}\right)\right)\right).$$
- *when $\alpha\ge1$ and $\beta=1$,*
$$\lim_{a\to+\infty}(\log a)^{-1}\lim_{n\to+\infty}K_{n,a}(h)=C_2I[h],$$
  with $C_2:=(\pi A_1)^{-1}$.
- *when $\alpha=1$ and $\beta\in(1,2)$,*
$$\lim_{a\to+\infty}\left(a^{-1}\log(a^\beta)\right)^{\beta-1}\lim_{n\to+\infty}K_{n,a}(h)=D_1I[h],$$
  with
$$D_1:=\frac{\Gamma(2-\beta)}{\pi c(\beta-1)(A_1^2+A_2^2)^{1/2}}\sin\left(\frac{\pi\beta}{2}-\arctan\left(\frac{A_2}{A_1}\right)\right).$$
- *when $\alpha=1$ and $\beta=2$, assume that h is even and that the distribution of the $\xi_x's$ is symmetric, then*
$$\lim_{a\to+\infty}\left(a^{-1}\log(a^2)\right)\lim_{n\to+\infty}K_{n,a}(h)=D_2I[h],$$
  with $D_2:=(2A_1c)^{-1}$.

Remark 1.2. *1- It is worth noting that since $|A_2/A_1|\le|\tan(\pi\beta/2)|$, the constants C_1 and D_1 are strictly positive.*
2- The limit as a goes to $-\infty$ is not considered in Theorem 1.1 and Theorem 1.3 since it can be easily obtained from the limit as a goes to infinity. Indeed, the problem is then equivalent to study the limit as a goes to infinity with the random variables $(\xi_x)_x$ replaced by $(-\xi_x)_x$ and the function h by $x\mapsto h(-x)$. The limits can easily be deduced from the above limit constants by changing A_2 to $-A_2$.

1.2 Transient case : $\beta \in (0,1)$

Let $\mathcal{H}_1$ denote the set of all the complex-valued Lebesgue-integrable functions h such that its Fourier transform $\hat{h}$ is continuously differentiable on $\mathbb{R}$, with in addition $\hat{h}$ and $(\hat{h})'$ Lebesgue-integrable.

Theorem 1.3. *Assume that $\alpha \in (1,2]$ and that the characteristic function of the random variable ξ_0 is equal to ϕ given by (1.2).*
Then, for all $h \in \mathcal{H}_1$, we have

$$\lim_{a\to+\infty} a^{1-\frac{1}{\delta}} \sum_{n\geq 1} \mathbb{E}[h(Z_n - a)] = C_0\ I[h]$$

with

$$C_0 := \frac{\Gamma(\frac{1}{\delta\beta})\Gamma(2-\frac{1}{\delta})\mathbb{E}\left[|L|_\beta^{-1/\delta}\right]}{\pi\beta(\delta-1)(A_1^2+A_2^2)^{1/2\delta\beta}} \sin\left(\frac{1}{\delta}\left(\frac{\pi}{2} - \frac{1}{\beta}\arctan\left(\frac{A_2}{A_1}\right)\right)\right).$$

1.3 Preliminaries to the proofs

In our proofs, we will use Fourier transforms for some $h : \mathbb{R} \to \mathbb{C}$ or $h : \mathbb{Z} \to \mathbb{C}$ and, more precisely, the following fact

$$2\pi\mathbb{E}[h(Z_n - a)] = \int_{\mathcal{I}} \hat{h}(t)\mathbb{E}[e^{itZ_n}]e^{-iat}\, dt.$$

This will lead us to the study of $\sum_{n\geq 1}\mathbb{E}[e^{itZ_n}]$. Therefore it will be crucial to observe that we have

$$\forall t \in \mathbb{R},\ \forall n \geq 1,\ \mathbb{E}[e^{itZ_n}] = \mathbb{E}\left[\prod_{y\in\mathbb{Z}} e^{it\xi_y N_n(y)}\right] = \mathbb{E}\left[\prod_{y\in\mathbb{Z}} \varphi_\xi(tN_n(y))\right],$$

since, taken $(S_k)_{k\leq n}$, $(\xi_y)_y$ is a sequence of iid random variables with characteristic function φ_ξ. Let us note that, in the particular case when ξ_0 has the stable distribution given by characteristic function (1.2), the quantity $\sum_{n\geq 1}\mathbb{E}[e^{itZ_n}]$ is equal to

$$\psi(t) := \sum_{n\geq 1}\mathbb{E}\left[\prod_{y\in\mathbb{Z}} e^{-|t|^\beta N_n(y)^\beta(A_1+iA_2\mathrm{sgn}(t))}\right].$$

Section 2 is devoted to the study of this series thanks to which, we prove Theorem 1.1 in Section 3 and Theorem 1.3 in Section 4.

2 Study of the series ψ

Let us note that we have, for every real number $t \neq 0$,

$$\psi(t) = \sum_{n\geq 1}\mathbb{E}[e^{-|t|^\beta V_n(A_1+iA_2\mathrm{sgn}(t))}], \tag{2.1}$$

with $V_n := \sum_{y\in\mathbb{Z}} N_n(y)^\beta$. Let us observe that $\frac{N_n(y)}{n} \leq (\frac{N_n(y)}{n})^\beta \leq \frac{N_n(y)}{n^\beta}$ if $\beta \leq 1$, and $\frac{N_n(y)}{n^\beta} \leq (\frac{N_n(y)}{n})^\beta \leq \frac{N_n(y)}{n}$ if $\beta > 1$. Combining this with the fact that $\sum_{y\in\mathbb{Z}} N_n(y) = n$, we obtain:

$$\beta \leq 1 \ \Rightarrow\ n^\beta \leq V_n \leq n \tag{2.2a}$$

$$\beta \geq 1 \ \Rightarrow\ n \leq V_n \leq n^\beta. \tag{2.2b}$$

Proposition 2.1. *When $\beta \in (0,2]$, for every $r \in (0,+\infty)$, the function ψ is bounded on the set $\{t \in \mathbb{R} : |t| \geq r\}$.*
When $\beta \in (0,1)$, the function ψ is differentiable on $\mathbb{R} \setminus \{0\}$, and for every $r \in (0,+\infty)$, its derivative ψ' is bounded on the set $\{t \in \mathbb{R} : |t| \geq r\}$.

Proof. Let $r > 0$. Then: $|t| \geq r \;\Rightarrow\; |\psi(t)| \leq \sum_{n\geq 1} e^{-A_1 r^\beta n^{1\wedge\beta}}$, so the first assertion is proved. Next, when $\beta \in (0,1)$, since $\sum_{n\geq 1} n\, e^{-A_1 (rn)^\beta} < \infty$, it easily follows from Lebesgue's theorem that ψ is differentiable on $\{t \in \mathbb{R} : |t| \geq r\}$, with $|\psi'(t)| \leq \beta r^{\beta-1}(A_1 + |A_2|) \sum_{n\geq 1} n\, e^{-A_1(rn)^\beta}$ when $|t| \geq r$. □

In the particular case when $\beta = 1$, we have $A_2 = 0$ and

$$\psi(t) = \frac{1}{e^{A_1|t|} - 1} \sim_{t\to 0} \gamma(t), \quad \text{with} \quad \gamma(t) := {A_1}^{-1}|t|^{-1}.$$

When $\beta \neq 1$, the expression of $\psi(t)$ is not so simple. We will need some estimates to prove our results. Recall that the constant c is defined in (1.5) if $\alpha = 1$ and set

$$C := \frac{1}{\delta\beta}\Gamma(\frac{1}{\delta\beta})\mathbb{E}[|L|_\beta^{-1/\delta}].$$

Proposition 2.2. *When $\alpha > 1, \beta \neq 1$, we have*

$$\lim_{t\to 0} \frac{\psi(t)}{\gamma(t)} = 1 \tag{2.3}$$

and

$$\lim_{t\to 0} \frac{\psi'(t)}{\gamma'(t)} = 1, \tag{2.4}$$

where γ is the function defined by

$$\gamma(t) := C|t|^{-1/\delta}(A_1 + iA_2 \operatorname{sgn}(t))^{-1/(\delta\beta)}.$$

When $\alpha = 1$ and $\beta > 1$, we have

$$\lim_{t\to 0} \frac{\psi(t)}{\gamma(t)} = 1, \tag{2.5}$$

where γ is the function defined by

$$\gamma(t) := \frac{(-\log(|t|^\beta))^{1-\beta}}{c|t|^\beta(A_1 + iA_2 \operatorname{sgn}(t))}.$$

To prove Proposition 2.2, we need some preliminaries lemmas. Let us define

$$b_n := n^\delta \text{ if } \alpha > 1 \quad \text{and} \quad b_n := n^{\frac{1}{\beta}}(\log(n))^{1-\frac{1}{\beta}} \text{ if } \alpha = 1. \tag{2.6}$$

We first recall some facts on the behaviour of the sequence $\left(b_n^{-1}V_n^{1/\beta}\right)_n$.

Lemma 2.3 (Lemma 6 in [13], Lemma 5 in [7])**.** *When $\alpha > 1$, the sequence of random variables $\left(b_n^{-1}V_n^{1/\beta}\right)_n$ converges in distribution to $|L|_\beta = \left(\int_{\mathbb{R}} L_1^\beta(x)\,dx\right)^{1/\beta}$.*

When $\alpha = 1$, the sequence of random variables $\left(b_n^{-1}V_n^{1/\beta}\right)_n$ converges almost surely to $c^{\frac{1}{\beta}}$.

Lemma 2.4 (Lemma 11 in [6], Lemma 16 in [7]). *If $\beta > 1$, then*

$$\sup_n \mathbb{E}\left[\left(\frac{b_n}{V_n^{1/\beta}}\right)^{\beta/(\beta-1)}\right] < +\infty.$$

If $\beta \leq 1$, then for every $p \geq 1$,

$$\sup_n \mathbb{E}\left[\left(\frac{b_n}{V_n^{1/\beta}}\right)^{p}\right] < +\infty.$$

The idea will be that V_n is of order b_n^β. Therefore the study of $\sum_{n\geq 1} e^{-|tb_n|^\beta(A_1+iA_2\, sgn(t))}$ will be useful in the study of $\psi(t)$. For any function $g : \mathbb{R}_+ \to \mathbb{R}$, we denote by $\mathcal{L}(g)$ the Laplace transform of g given, for every $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$, by

$$\mathcal{L}(g)(z) = \int_0^{+\infty} e^{-zt} g(t)dt,$$

when it is well defined.

Lemma 2.5. *When $\alpha > 1$, for every complex number z such that $\mathrm{Re}(z) > 0$ and every $p \geq 0$, we have*

$$K_{p,\alpha}(z) := \sup_{u>0}\left|\sum_{n\geq 1} e^{-zu^{\delta\beta}b_n^\beta}(u^{\delta\beta}b_n^\beta)^p - \frac{1}{u\delta\beta}\Gamma(p+\frac{1}{\delta\beta})z^{-(p+\frac{1}{\delta\beta})}\right| < +\infty.$$

When $\alpha = 1$, for every complex number z such that $\mathrm{Re}(z) > 0$ and every $p \geq 0$, we have

$$K_{p,1}(z) := \sup_{u>0}\left|\sum_{n\geq 1} e^{-zub_n^\beta}(ub_n^\beta)^p - u^p\mathcal{L}(\tilde{w}_p)(zu)\right| < +\infty,$$

where $\tilde{w}_p(t) := \tilde{w}_0(t)t^p$ with

$$\tilde{w}_0(t) := \frac{(\beta-1)^{1-\beta} w\left(\frac{t^{\frac{1}{\beta-1}}}{\beta-1}\right)^{2-\beta}}{1+w\left(\frac{t^{\frac{1}{\beta-1}}}{\beta-1}\right)} \quad \textit{if} \quad \beta > 1$$

and

$$\tilde{w}_0(t) := \mathbf{1}_{[(e/(1-\beta))^{1-\beta};+\infty)}(t)\frac{\Delta((1-\beta)t^{\frac{1}{1-\beta}})^{2-\beta}(1-\beta)^{1-\beta}}{\Delta((1-\beta)t^{\frac{1}{1-\beta}})-1} \quad \textit{if} \quad \beta < 1.$$

Here w is the Lambert function defined on $[0;+\infty)$ as the inverse function of $y \mapsto ye^y$ (defined on $[0;+\infty)$) and Δ is the function defined on $[e;+\infty)$ as the inverse function of $y \mapsto e^y/y$ defined on $[1;+\infty)$.

Proof. First, we consider the case when $\alpha > 1$. With the change of variable $y = (ux)^{\delta\beta}$, we get

$$\int_0^{+\infty} e^{-(ux)^{\delta\beta}z}(ux)^{p\delta\beta}\, dx = \frac{1}{u\delta\beta}\int_0^{\infty} e^{-yz}y^{\frac{1}{\delta\beta}+p-1}dy = \frac{1}{u\delta\beta}\Gamma\left(p+\frac{1}{\delta\beta}\right)z^{-(p+\frac{1}{\delta\beta})}.$$

Let $\lfloor x \rfloor$ denote the integer part of x. Observe that

$$\sum_{n\geq 1} e^{-(un)^{\delta\beta}z}(un)^{p\delta\beta} = \int_1^{+\infty} e^{-(u\lfloor x\rfloor)^{\delta\beta}z}(u\lfloor x\rfloor)^{p\delta\beta}\, dx.$$

Let us write
$$E_p(u,x)=\left|e^{-(u\lfloor x\rfloor)^{\delta\beta}z}(u\lfloor x\rfloor)^{p\delta\beta}-e^{-(ux)^{\delta\beta}z}(ux)^{p\delta\beta}\right|.$$
Applying Taylor's inequality to the function $v\mapsto e^{-vz}v^p$ on the interval $[(u\lfloor x\rfloor)^{\delta\beta},(ux)^{\delta\beta}]$, we obtain for every $x>2$ (use $\lfloor x\rfloor\ge x/2$)
$$E_p(u,x)\le(1+|z|)(1+p)(1+(ux)^{p\delta\beta})e^{-(ux/2)^{\delta\beta}\mathrm{Re}(z)}\,u^{\delta\beta}\big(x^{\delta\beta}-\lfloor x\rfloor^{\delta\beta}\big).$$
Next, by applying Taylor's inequality to the function $t\mapsto t^{\delta\beta}$ according that $\delta\beta>1$ or $\delta\beta<1$ (again use $\lfloor x\rfloor\ge x/2$ in the last case), we have
$$E_p(u,x)\le(1+|z|)(1+p)(1+(ux)^{p\delta\beta})\max\left(1,2^{1-\delta\beta}\right)\delta\beta\,u^{\delta\beta}\,e^{-(ux/2)^{\delta\beta}\mathrm{Re}(z)}\,x^{\delta\beta-1}.$$
Therefore, with the change of variable $t=(ux/2)^{\delta\beta}$, we get
$$\int_2^{+\infty}E_p(u,x)\,dx\le(1+|z|)(1+p)\max\left(2^{\delta\beta},2\right)\int_0^{+\infty}(1+2^{p\delta\beta}t^p)e^{-\mathrm{Re}(z)t}\,dt.$$

Now, we suppose that $\alpha=1$ and follow the same scheme. We observe that $\delta\beta=1$. With the change of variable $t=x(\log x)^{\beta-1}$, we get
$$\int_1^{+\infty}e^{-zux(\log x)^{\beta-1}}(ux(\log x)^{\beta-1})^p\,dx=u^p\mathcal{L}(\tilde w_p)(zu).$$
Indeed, if $\beta>1$, we have $t=[(\beta-1)x^{\frac{1}{\beta-1}}(\log(x^{\frac{1}{\beta-1}}))]^{\beta-1}$ and so $x=\left[\exp\left(w\left(\frac{t^{\frac{1}{\beta-1}}}{\beta-1}\right)\right)\right]^{\beta-1}$ which gives $dx=\tilde w_0(t)\,dt$ (since $w'(y)=\frac{w(y)}{y(1+w(y))}$ and since $e^{w(x)}=\frac{x}{w(x)}$). Moreover, if $\beta<1$, we have
$$t=\left[\frac{x^{\frac{1}{1-\beta}}}{(1-\beta)\log(x^{\frac{1}{1-\beta}})}\right]^{1-\beta}\quad\text{and so } x=\left[\exp\left((1-\beta)\Delta\left((1-\beta)t^{\frac{1}{1-\beta}}\right)\right)\right],$$
which gives $dx=\tilde w_0(t)\,dt$ (since $\Delta'(y)=\frac{\Delta(y)}{y(\Delta(y)-1)}$ and since $e^{\Delta(y)}=y\Delta(y)$).

Let $x_0:=\max(4,e^{2(1-\beta)})$. We have
$$\left|\sum_{n\ge1}e^{-zun(\log n)^{\beta-1}}(un(\log n)^{\beta-1})^p-u^p\mathcal{L}(\tilde w_p)(zu)\right|\le2x_0\sup_y(e^{-Re(z)y}y^p)+\int_{x_0}^{+\infty}E_p(u,x)\,dx,$$
where
$$E_p(u,x):=\left|e^{-(zu\lfloor x\rfloor)(\log\lfloor x\rfloor)^{\beta-1}}(u\lfloor x\rfloor(\log\lfloor x\rfloor)^{\beta-1})^p-e^{-zux(\log x)^{\beta-1}}(ux(\log x)^{\beta-1})^p\right|.$$
Applying Taylor's inequality, we get
$$E_p(u,x)\le(1+|z|)(1+p)e^{-\frac{\mathrm{Re}(z)ux}{4}(\log x)^{\beta-1}}2^p(1+(2ux(\log x)^{\beta-1})^p)8u(\beta-1+\log(x))(\log x)^{\beta-2}$$
(using the fact that $\lfloor x\rfloor\ge x/2$ and $\log(x)\ge\log\lfloor x\rfloor\ge(\log x)/2\ge1-\beta$ if $x\ge x_0$). Hence there exists some c depending only on p and $|z|$ such that $\int_{x_0}^{+\infty}E_p(u,x)\,dx$ is less than
$$c\int_{x_0}^{+\infty}(1+(2ux(\log x)^{\beta-1})^p)e^{-\frac{\mathrm{Re}(z)ux}{4}(\log x)^{\beta-1}}u(\beta-1+\log(x))(\log x)^{\beta-2}\,dx.$$
With the change of variable $t=ux(\log x)^{\beta-1}$, for which we have $dt=u(\beta-1+\log x)(\log x)^{\beta-2}\,dx$, we get
$$\int_{x_0}^{+\infty}E_p(u,x)\,dx\le c\int_0^{+\infty}(1+(2t)^p)e^{-\frac{\mathrm{Re}(z)t}{4}}\,dt.$$
The last integral is finite. □

Lemma 2.6. *For every complex number z such that* $\mathrm{Re}(z) > 0$ *and every* $p \geq 0$*, we have*

$$\lim_{u \to 0^+} [(zu)^{p+1} \mathcal{L}(\tilde{w}_p)(zu) - \Gamma(p+1)(-\log|u|)^{1-\beta}] = 0$$

and for every $u_0 > 0$*,*

$$\sup_{0<u<u_0} \frac{u^{p+1}\mathcal{L}(\tilde{w}_p)(u)}{\Gamma(p+1)(-\log|u|)^{1-\beta}} < \infty.$$

Hence, if $\alpha = 1$*, for every* z *such that Re*$(z) > 0$*, we have*

$$\sum_{n\geq 1} e^{-zu^\beta b_n^\beta} \left(u^\beta b_n^\beta\right)^p \sim_{u\to 0+} \Gamma(p+1) \frac{u^{-\beta}(-\log(|u|^\beta))^{1-\beta}}{z^{p+1}}.$$

Proof. We know that $w(x) \sim_{+\infty} \log x$ and $\Delta(x) \sim_{+\infty} \log x$. Hence, for every $p \geq 0$, we have

$$\tilde{w}_p(t) \sim_{t\to+\infty} t^p (\log t)^{1-\beta}.$$

Now, we apply Tauberian theorems (Theorems p. 443–446 in [11]) to Laplace transforms $\mathcal{L}(.)$ defined for complex numbers such that $\mathrm{Re}(z) > 0$. The lemma follows. ☐

Lemma 2.7. *There exist a sequence of random variables* $(a_n)_n$ *and a random variable* A *defined on* $(0,1)$ *endowed with the Lebesgue measure* λ *such that, for every* $n \geq 1$, a_n *and* $(b_n V_n^{-\frac{1}{\beta}})^{\frac{1}{\delta}}$ *have the same distribution, such that* $\mathbb{E}_\lambda[\sup_{n\geq 1} a_n] < +\infty$ *and such that* $(a_n)_n$ *converges almost surely and in* L^1 *to the random variable* A*.*

Proof. Following the the Skorohod representation theorem, we define

$$a_n(x) := \inf\left\{u > 0 \ : \ \mathbb{P}\left((b_n V_n^{-\frac{1}{\beta}})^{\frac{1}{\delta}} \leq u\right) \geq x\right\}$$

and A as follows :

$$A(x) := c^{-1} \ \text{ if } \alpha = 1$$

and

$$A(x) := \inf\left\{u > 0 \ : \ \mathbb{P}\left(|L|_\beta^{-1/\delta} \leq u\right) \geq x\right\} \ \text{ if } \alpha > 1.$$

Remark that Lemma 2.4 insures the uniform integrability of $\left((b_n V_n^{-\frac{1}{\beta}})^{\frac{1}{\delta}}\right)_n$. Therefore, from Lemma 2.3, the sequence $(a_n)_n$ converges almost surely and in L^1 to the random variable A as n goes to infinity.
Moreover, from the formula $\mathbb{E}_\lambda[\sup_{n\geq 1} a_n] = \int_0^{+\infty} \lambda(\sup_n a_n > t)\, dt$ and the fact that

$$\sup_n a_n(x) > t \ \Leftrightarrow \ \inf_n \mathbb{P}\left((b_n V_n^{-\frac{1}{\beta}})^{\frac{1}{\delta}} \leq t\right) < x,$$

we get

$$\lambda(\sup_n a_n > t) = \sup_n \mathbb{P}\left((b_n V_n^{-\frac{1}{\beta}})^{\frac{1}{\delta}} > t\right) \leq t^{-\gamma} \sup_n \mathbb{E}\left[b_n^{\frac{\gamma}{\delta}} V_n^{-\frac{\gamma}{\delta\beta}}\right],$$

with $\gamma := 2$ when $\beta \leq 1$; $\gamma := \delta\beta/(\beta-1)$ when $\alpha > 1$ and $\beta > 1$ or when $\alpha = 1$ and $\beta \in (1,2)$. In each of this case, this gives $\mathbb{E}_\lambda[\sup_{n\geq 1} a_n] < +\infty$ from Lemma 2.4. If $\alpha = 1$ and $\beta = 2$, we take $\gamma = 2$ and use the fact that

$$\sup_n \mathbb{E}\left[b_n^{\frac{\gamma}{\delta}} V_n^{-\frac{\gamma}{\delta\beta}}\right] = \sup_n \mathbb{E}\left[\left(\frac{n\log n}{V_n}\right)^2\right] \leq \sup_n \frac{(\log n)^2}{n^2} \mathbb{E}[R_n^2] < \infty,$$

with $R_n := \#\{y \in \mathbb{Z} \ : \ N_n(y) > 0\}$ (according to inequality (3b) in [15]) and we conclude analogously. ☐

Therefore, using the previous lemma, the series ψ can be rewritten, for every real number $t \neq 0$, as

$$\psi(t) = \mathbb{E}_\lambda \left[\sum_{n\geq 1} e^{-|t|^\beta b_n^\beta a_n^{-\delta\beta}(A_1 + iA_2 \mathrm{sgn}(t))} \right]. \tag{2.7}$$

Lemma 2.8. *There exists $t_0 > 0$ such that when $\alpha > 1$ or ($\alpha = 1$ and $\beta > 1$), the family of random variables*

$$\left(\frac{1}{\gamma(t)} \sum_{n\geq 1} e^{-|t|^\beta b_n^\beta a_n^{-\delta\beta}(A_1 + iA_2 sgn(t))} \right)_{0<|t|<t_0}$$

is uniformly integrable and such that, if $\alpha > 1$, the family

$$\left(\frac{|t|^\beta}{\gamma(t)} \sum_{n\geq 1} b_n^\beta a_n^{-\delta\beta} e^{-|t|^\beta (a_n^{-1}n)^{\delta\beta}(A_1 + iA_2 sgn(t))} \right)_{0<|t|<t_0}$$

is also uniformly integrable.

Proof. If $\alpha > 1$, thanks to lemma 2.5, we know that, for every real number $t \in (0,1)$ and every complex number z such that $\mathrm{Re}(z) > 0$, we have

$$\left| \sum_{n\geq 1} e^{-|t|^\beta (a_n^{-1}n)^{\delta\beta} z} \right| \leq |t|^{-\frac{1}{\delta}} \frac{\sup_n a_n}{\delta\beta} \Gamma\left(\frac{1}{\delta\beta}\right) |z|^{-\frac{1}{\delta\beta}} + K_{0,\alpha}(z), \tag{2.8}$$

and,

$$|t|^\beta \left| \sum_{n\geq 1} (a_n^{-1}n)^{\delta\beta} e^{-|t|^\beta (a_n^{-1}n)^{\delta\beta} z} \right| \leq |t|^{-\frac{1}{\delta}} \frac{\sup_n a_n}{\delta\beta} \Gamma\left(1 + \frac{1}{\delta\beta}\right) |z|^{-(1+\frac{1}{\delta\beta})} + K_{1,\alpha}(z), \tag{2.9}$$

from which we conclude.

Now, let us consider the case $\alpha = 1$ and $\beta > 1$. According to lemmas 2.5 and 2.6, since $0 < A_1|t|^\beta (1 + \sup_n a_n)^{-1} \leq A_1$, we have

$$\left| \sum_{n\geq 1} e^{-|t|^\beta b_n^\beta a_n^{-1}(A_1 + iA_2 \mathrm{sgn}(t))} \right| \leq K_{0,1}(A_1) + \mathcal{L}(\tilde{w}_0)\left(A_1|t|^\beta (1 + \sup_n a_n)^{-1} \right)$$
$$\leq K_{0,1}(A_1) + c_0|t|^{-\beta}\ (1 + \sup_n a_n) \left(-\log\left(A_1|t|^\beta (1 + \sup_n a_n)^{-1} \right) \right)^{1-\beta}$$

for some positive constant $c_0 > 0$. Hence, there exists $t_1 \in (0,1)$ such that for every $0 < |t| < t_1$, the quantity $\frac{1}{|\gamma(t)|} \left| \sum_{n\geq 1} e^{-|t|^\beta b_n^\beta a_n^{-1}(A_1 + iA_2 \mathrm{sgn}(t))} \right|$ is less than

$$c_1 \left[1 + (1 + \sup_n a_n) \left(\frac{-\log\left(A_1|t|^\beta (1 + \sup_n a_n)^{-1} \right)}{-\log\left(A_1|t|^\beta c \right)} \right)^{1-\beta} \right]$$

for some positive constant $c_1 > 0$.

If $\beta > 1$, since $c(1 + \sup_n(a_n)) \geq 1$ a.s., the right-hand side of the above inequality is almost surely less than $c_1(2 + \sup_n a_n)$ for every $|t| < (cA_1)^{-\frac{1}{\beta}}$. Then, we can choose t_0 as the infimum of $(cA_1)^{-\frac{1}{\beta}}$ and t_1. The uniform integrability then follows from Lemma 2.7. $\square$

Lemma 2.9. *If $(\alpha > 1, \beta \neq 1)$ or $(\alpha = 1, \beta > 1)$, we have*

$$\lim_{t\to 0} \frac{1}{\gamma(t)} \sum_{n\geq 1} \left(e^{-|t|^\beta (a_n^{-\delta\beta} b_n^\beta)(A_1 + iA_2 sgn(t))} - e^{-|t|^\beta (A^{-\delta\beta} b_n^\beta)(A_1 + iA_2 sgn(t))} \right) = 0 \ a.s..$$

Moreover, when $\alpha > 1$ and $\beta < 1$,

$$\lim_{t\to 0} \frac{|t|^\beta}{\gamma(t)} \sum_{n\geq 1} \left(\frac{b_n^\beta}{a_n^{\delta\beta}} \ e^{-|t|^\beta (a_n^{-\delta\beta} b_n^\beta)(A_1 + iA_2 sgn(t))} - \frac{b_n^\beta}{A^{\delta\beta}} e^{-|t|^\beta (A^{-\delta\beta} b_n^\beta)(A_1 + iA_2 sgn(t))} \right) = 0 \ \ a.s..$$

Proof. We only prove the first assertion, the proof of the second one following the same scheme. Let $\beta \neq 1$ and $\alpha \geq 1$. Let $\varepsilon \in (0, 1/\delta)$,

$$\sum_{n=1}^{\lfloor |t|^{\varepsilon - 1/\delta} \rfloor} \left| \left(e^{-|t|^\beta a_n^{-\delta\beta} b_n^\beta (A_1 + iA_2 \text{sgn}(t))} - e^{-|t|^\beta A^{-\delta\beta} b_n^\beta (A_1 + iA_2 \text{sgn}(t))} \right) \right| = \mathcal{O}(|t|^{\varepsilon - 1/\delta}) = o(\gamma(t)).$$

Now it remains to prove the almost sure convergence to 0 as t goes to 0 of the following quantity :

$$\varepsilon_t := \frac{1}{\gamma(t)} \sum_{n > [|t|^{\varepsilon - 1/\delta}]} \left(e^{-|t|^\beta a_n^{-\delta\beta} b_n^\beta (A_1 + iA_2 \text{sgn}(t))} - e^{-|t|^\beta A^{-\delta\beta} b_n^\beta (A_1 + iA_2 \text{sgn}(t))} \right).$$

By applying Taylor's inequality to the function $v \mapsto e^{-|t|^\beta |v|^{\delta\beta} b_n^\beta (A_1 + iA_2 \text{sgn}(t))}$, we have

$$\begin{aligned} |\varepsilon_t| &\leq \delta\beta(A_1 + |A_2|) \frac{|t|^\beta}{\gamma(t)} \sum_{n > [|t|^{\varepsilon - 1/\delta}]} (\inf_{n > [|t|^{\varepsilon - 1/\delta}]} a_n)^{-\delta\beta} b_n^\beta e^{-A_1 |t|^\beta (\sup_{n > [|t|^{\varepsilon - 1/\delta}]} a_n)^{-\delta\beta} b_n^\beta} \times \\ &\qquad \times \left| \frac{a_n^{-1} - A^{-1}}{(\sup_{n > [|t|^{\varepsilon - 1/\delta}]} a_n)^{-1}} \right| \\ &= o(1) \ \text{ a.s.}, \end{aligned}$$

using lemmas 2.5 and 2.6 and according to the fact that $(a_n)_n$ converges almost surely to A. ☐

Proof of Proposition 2.2. First consider the case $\alpha > 1$ and $\beta \neq 1$. Thanks to lemmas 2.5 and 2.9, we get that

$$\frac{1}{\gamma(t)} \left[\sum_{n\geq 1} e^{-|t|^\beta a_n^{-\delta\beta} b_n^\beta (A_1 + iA_2 \text{sgn}(t))} - \frac{A}{\delta\beta} \Gamma(\frac{1}{\delta\beta})(A_1 + iA_2 \text{sgn}(t))^{-1/\delta\beta} |t|^{-1/\delta} \right] \to 0 \text{ a.s.}$$

as t goes to 0. Therefore, due to (2.7) and to the uniform integrability (Lemma 2.8), we deduce (2.3). The proof of (2.5) is similar (using Lemma 2.6) and is omitted.
Again, to prove (2.4), we use (2.7). Since for $t \neq 0$,

$$\psi'(t) = -\beta \text{sgn}(t)(A_1 + iA_2 \text{sgn}(t))|t|^{\beta - 1} \sum_{n\geq 1} \mathbb{E} \left[a_n^{-\delta\beta} b_n^\beta e^{-|t|^\beta (a_n^{-\delta\beta} b_n^\beta (A_1 + iA_2 \text{sgn}(t))} \right],$$

and

$$\gamma'(t) = -\frac{C}{\delta}(A_1 + iA_2 \text{sgn}(t))^{-1/\delta\beta} |t|^{-1/\delta - 1},$$

we decompose

$$\left(\frac{C}{\delta\beta}(A_1 + iA_2 \text{sgn}(t))^{-(1 + 1/\delta\beta)} \right) \left[\frac{\psi'(t)}{\gamma'(t)} - 1 \right]$$

as the sum of

$$|t|^{1/\delta}\mathbb{E}\left[\sum_{n\geq 1}\left(e^{-|t|^\beta(a_n^{-\delta\beta}b_n^\beta(A_1+iA_2\mathrm{sgn}(t))}\frac{|t|^\beta}{a_n^{\delta\beta}}b_n^\beta - e^{-|t|^\beta A^{-\delta\beta}b_n^\beta(A_1+iA_2\mathrm{sgn}(t))}\frac{|t|^\beta}{A^{\delta\beta}}b_n^\beta\right)\right]$$

and of

$$|t|^{1/\delta}\mathbb{E}\left[\sum_{n\geq 1}e^{-|t|^\beta A^{-\delta\beta}b_n^\beta(A_1+iA_2\mathrm{sgn}(t))}\frac{|t|^\beta}{A^{\delta\beta}}b_n^\beta - \frac{A}{|t|^{1/\delta}\delta\beta}\frac{\Gamma(1+\frac{1}{\delta\beta})}{(A_1+iA_2\mathrm{sgn}(t))^{1+1/\delta\beta}}\right]$$

The second assertion in Lemma 2.9 and the uniform integrability in Lemma 2.8 implies that the first sum goes to 0 as t goes to 0. From Lemma 2.5, we get that the second one goes to 0 as t goes to 0. □

3 Proof of Theorem 1.1

We first begin to prove that for every $a \in \mathbb{R}$, the sequence of

$$K_{n,a}(h) = \sum_{k=1}^{n}\{\mathbb{E}[h(Z_k)] - \mathbb{E}[h(Z_k - a)]\}$$

converges as n tends to infinity. Indeed, for every $a \in \mathbb{R}$, we have

$$K_{n,a}(h) = \frac{1}{2\pi}\int_{\mathcal{I}}\hat{h}(t)\left(\sum_{k=1}^{n}\mathbb{E}[e^{itZ_k}]\right)\left(1-e^{-ita}\right)dt. \tag{3.1}$$

Proposition 3.1. *i)- The series*

$$\sum_{n\geq 1}|\mathbb{E}[e^{itZ_n}]|$$

is bounded on any set $[r, +\infty[$ *with* $r > 0$ *and so the series*

$$\tilde{\psi}(t) := \sum_{n\geq 1}\mathbb{E}[e^{itZ_n}]$$

is well defined for every $t \neq 0$.

ii)- We have

$$\lim_{t\to 0}\frac{1}{\gamma(t)}\sum_{n\geq 1}\left|\mathbb{E}[e^{itZ_n}] - \mathbb{E}\left[e^{-|t|^\beta V_n(A_1+iA_2 sgn(t))}\right]\right| = 0,$$

and so

$$\lim_{t\to 0}\frac{1}{\gamma(t)}[\tilde{\psi}(t) - \psi(t)] = 0,$$

Proof. In order to prove *ii)*, we show that

$$\lim_{t\to 0}\frac{1}{\gamma(t)}\sum_{n\geq 1}\left|\mathbb{E}\left[\prod_{y\in\mathbb{Z}}\varphi_\xi(tN_n(y))\right] - \mathbb{E}\left[e^{-|t|^\beta V_n(A_1+iA_2 sgn(t))}\right]\right| = 0.$$

From Lemma 6 in [6] and Lemma 12 in [7], for every $\eta > 0$ and every $n \geq 1$, there exists a subset Ω_n such that for every $p > 1$, $\mathbb{P}(\Omega_n) = 1 - o(n^{-p})$ and such that, on Ω_n, we have

$$N_n^* = \sup_x N_n(x) \leq n^{1-\frac{1}{\alpha}+\eta} \quad \text{and} \quad V_n \geq n^{\delta\beta-\eta'},$$

with $\eta' = \frac{\eta\beta}{2}$ if $\alpha > 1, \beta > 1$; $\eta' = \eta(1-\beta)$ if $\alpha > 1, \beta \le 1$ and $\eta' = \eta(1-\beta)_+$ if $\alpha = 1$. Hence, it is enough to prove that

$$\sum_{n\ge 1} |\mathbb{E}\left[E_n(t)\mathbf{1}_{\Omega_n}\right]| = o(\gamma(t)) \ \text{ as } \ t \to 0,$$

with $E_n(t) := \prod_{y\in\mathbb{Z}} \varphi_\xi(tN_n(y)) - e^{-|t|^\beta V_n(A_1+iA_2 sgn(t))}$.
In [6, 7], we also define some $\bar\eta \le \eta \max(1, \beta^{-1})$ and we take some η such that $\eta+\bar\eta < \frac{1}{\alpha\beta}$. Hence, for every $\varepsilon_0 > 0$, there exists n_1 such that for every $n \ge n_1$, we have $n^{\eta+\bar\eta-\frac{1}{\alpha\beta}} \le \varepsilon_0$.
In the proofs of propositions 8, 9 and 10 of [6] (and propositions 14, 15 of [7]) or using the strong lattice property, we prove that there exist $c > 0$, $\theta > 0$ and n_0 such that for every t and every integer $n \ge n_0$ and such that $|t| > n^{-\delta+\bar\eta}$, we have, on Ω_n,

$$\prod_{y\in\mathbb{Z}} |\varphi_\xi(tN_n(y))| \le e^{-cn^\theta} \ \text{ and } \ \prod_{y\in\mathbb{Z}} |\phi(tN_n(y))| \le e^{-cn^\theta}.$$

Now, let t and $n \ge n_1$ be such that $|t| \le n^{-\delta+\bar\eta}$. Recall that we have

$$\left|\varphi_\xi(u) - \phi(u)\right| \le |u|^\beta h(|u|),$$

with h a continuous and monotone function on $[0;+\infty)$ vanishing in 0. Therefore there exist $\varepsilon_0 > 0$ and $\sigma > 0$ such that, for every $u \in [-\varepsilon_0;\varepsilon_0]$, we have

$$\max(|\phi(u)|, |\varphi_\xi(u)|) \le \exp(-\sigma|u|^\beta).$$

We have

$$|E_n(t)| \le \sum_y \left(\prod_{z<y} |\varphi_\xi(tN_n(z))|\right) |\varphi_\xi(tN_n(y)) - \phi(tN_n(y))| \left(\prod_{z>y} |\phi(tN_n(z))|\right).$$

Now, since $|t| \le n^{-\delta+\bar\eta}$, on Ω_n, for every $y \in \mathbb{Z}$, we have $|t|N_n(y) \le n^{\eta+\bar\eta-\frac{1}{\alpha\beta}} \le \varepsilon_0$, we get

$$\begin{aligned} |E_n(t)| &\le \sum_y h(n^{\eta+\bar\eta-\frac{1}{\alpha\beta}})|t|^\beta N_n(y)^\beta \exp(-\sigma|t|^\beta V_n)\exp(\sigma\varepsilon_0^\beta) \\ &\le h(n^{\eta+\bar\eta-\frac{1}{\alpha\beta}})|t|^\beta V_n \exp(-\sigma|t|^\beta V_n)\exp(\sigma\varepsilon_0^\beta). \end{aligned}$$

Now, we fix some $t \ne 0$. Let us write

$$\mathcal{N}_1(t) := \{n \ge 1 \ : \ n \ge n_0, \ |t| > n^{-\delta+\bar\eta}\}$$

and

$$\mathcal{N}_2(t) := \{n \ge 1 \ : \ n \ge n_1, \ |t| \le n^{-\delta+\bar\eta}, \ n > |t|^{-\frac{1}{2\delta}}\}.$$

We have

$$\sum_{n\le \max(n_0,n_1)} |E_n(t)| \le 2\max(n_0, n_1),$$

$$\sum_{n\le |t|^{-\frac{1}{2\delta}}} |E_n(t)| \le 2|t|^{-\frac{1}{2\delta}} = o(\gamma(t)), \ \text{ as } \ t \to 0,$$

$$\sum_{n\in\mathcal{N}_1(t)} |E_n(t)| \le 2\sum_{n\ge 1} e^{-cn^\theta}$$

and

$$\sum_{n\in\mathcal{N}_2(t)} \mathbb{E}[|E_n(t)|\mathbf{1}_{\Omega_n}] \quad \leq \quad e^{\sigma\varepsilon_0^\beta} h\left(t^{-\frac{1}{2\delta}\left(\eta+\bar\eta-\frac{1}{\alpha\beta}\right)}\right)|t|^\beta \mathbb{E}\Big[\sum_{n\geq 1} V_n \exp(-\sigma|t|^\beta V_n)\Big] = o(\gamma(t)),$$

as $t\to 0$, using Proposition 2.2 and the continuity of the function h at 0 (and the fact that $|t|^\beta V_n\exp(-\sigma|t|^\beta V_n)\leq k_0\exp(-\frac{1}{2}\sigma|t|^\beta V_n)$ for some $k_0>0$). Then, ii)- is proved and i)- can easily be deduced from the above arguments. □

The integrand in (3.1) is bounded by $\Theta(t):=|\hat h(t)||1-e^{-ita}|\ \sum_{n\geq 1}|\mathbb{E}[e^{itZ_n}]|$.

Let $r>0$, on the set $\{t;|t|\geq r\}$, by i)- from Proposition 3.1, since $\hat h$ is integrable, Θ is integrable. From Propositions 2.2 and 3.1 (item ii)-) and from the fact that $\hat h$ is continuous at 0, $\Theta(t)$ is in $O\left(|t|\gamma(t)\right)$ (at $t=0$), which is integrable in the neighborhood of 0 in all cases considered in Theorem 1.1 except $(\alpha,\beta)=(1,2)$. From the dominated convergence theorem, we deduce that

$$\lim_{n\to+\infty} K_{n,a}(h)=\frac{1}{2\pi}\int_{\mathcal{I}}\hat h(t)\tilde\psi(t)\left(1-e^{-ita}\right)dt. \tag{3.2}$$

In the case $(\alpha,\beta)=(1,2)$, by assumption, for every integer $n\geq 1$, the function $t\to \hat h(t)\sum_{k=1}^n\mathbb{E}[e^{itZ_k}]$ being even, we have

$$K_{n,a}(h)=\frac{1}{2\pi}\int_{\mathcal{I}}\hat h(t)\left(\sum_{k=1}^n\mathbb{E}[e^{itZ_k}]\right)(1-\cos(ta))\,dt. \tag{3.3}$$

The integrand in (3.3) is uniformly bounded in n by a function in $O\left(\log(1/|t|)^{-1}\right)$ (at $t=0$), which is integrable in the neighborhood of 0. From the dominated convergence theorem, we deduce that

$$\lim_{n\to+\infty} K_{n,a}(h)=\frac{1}{2\pi}\int_{\mathcal{I}}\hat h(t)\tilde\psi(t)\,(1-\cos(ta))\,dt. \tag{3.4}$$

In the rest of the proof we only consider the strongly non-lattice case, the lattice case can be handled in the same way.

Let us first consider the case $\alpha>1,\beta\in(1,2]$. We recall that, in this case, we have set

$$C=(\delta\beta)^{-1}\Gamma(\frac{1}{\delta\beta})\mathbb{E}[|L|_\beta^{-\frac{1}{\delta}}].$$

Since the function $t\to\hat h(t)\tilde\psi(t)$ is integrable on $\mathcal{I}\setminus[-\pi,\pi]$ (note that $\hat h$ is integrable and $\tilde\psi$ is bounded on $\mathcal{I}\setminus[-\pi,\pi]$ by Proposition 3.1), we have

$$\lim_{a\to+\infty}\frac{a^{1-1/\delta}}{2\pi}\int_{\{|t|\geq\pi\}}\left|\hat h(t)\tilde\psi(t)\left(1-e^{-ita}\right)\right|dt=0.$$

We define the functions

$$g(t):=(1-e^{-it})|t|^{-1/\delta}(A_1+iA_2\mathrm{sgn}(t))^{-1/\delta\beta},\ \ g_a(t):=ag(at) \tag{3.5}$$

and $f(t):=\mathbf{1}_{[-\pi,\pi]}(t)\ \hat h(t)|t|^{1/\delta}\tilde\psi(t)(A_1+iA_2\mathrm{sgn}(t))^{1/\delta\beta}$. We have:

$$\frac{a^{1-1/\delta}}{2\pi}\int_{\{|t|\leq\pi\}}\hat h(t)\tilde\psi(t)\left(1-e^{-ita}\right)dt=\frac{1}{2\pi}\int_{\mathbb{R}}f(t)\,g_a(t)dt=\frac{1}{2\pi}\left(f*g_a\right)(0).$$

Since g is integrable on $\mathbb{R}$ and f is bounded on $\mathbb{R}$ and continuous at $t=0$ with $f(0)=C\hat h(0)$ (by Propositions 4 and 12), it follows from classical arguments of approximate identity that

$$\lim_{a\to+\infty}\left(f*g_a\right)(0)=C\hat h(0)\int_{\mathbb{R}}g(t)dt.$$

Let us observe that

$$\int_{\mathbb{R}} g(t)dt = 2\mathrm{Re}\left[(A_1+iA_2)^{-\frac{1}{\delta\beta}}\int_0^\infty \frac{1-e^{-it}}{t^{1/\delta}}\,dt\right].$$

Apply the residue theorem to the function $z\mapsto z^{-1/\delta}(1-e^{-iz})$ with the contour in the complex plane defined by the line segment from $-ir$ to $-iR$ ($r<R$), the circular arc connecting $-iR$ to R, the line segment from R to r and the circular arc from r to $-ir$. Then, by letting r tend to 0 and R to $+\infty$, we obtain

$$\int_0^\infty \frac{1-e^{-it}}{t^{1/\delta}}dt = \frac{\delta}{1-\delta}\Gamma\Big(2-\frac{1}{\delta}\Big)e^{i\frac{\pi}{2\delta}(1-\delta)}.$$

From this formula we easily deduce the first statement of theorem 1.1 using the fact that

$$(A_1+iA_2)^{-\frac{1}{\delta\beta}} = \frac{e^{-i\frac{\theta}{\delta\beta}}}{(A_1^2+A_2^2)^{\frac{1}{2\delta\beta}}},\quad\text{with }\;\theta=\arctan\left(\frac{A_2}{A_1}\right).$$

Now assume $\alpha\geq 1,\beta=1$ or $\alpha=1,\beta\in(1,2)$. We have $\gamma(t)=b_t|t|^{-\beta}(-\log|t|)^{1-\beta}$ (with $b_t=A_1^{-1}$ if $\beta=1$ and with $b_t=c^{-1}(A_1+iA_2sgn(t))^{-1}$ if $\alpha=1,\beta\in(1,2)$). Moreover, by combining propositions 2.2 and 3.1, we have

$$\lim_{t\to 0}\left|(\gamma(t))^{-1}\tilde\psi(t)-1\right| = 0.$$

Hence, for every $\varepsilon\in(0,1)$, there exists $0<A_\varepsilon<1$ such that

$$\forall t,\ \ |t|\leq A_\varepsilon\ \Rightarrow\ [\ |\tilde\psi(t)-\gamma(t)|<\varepsilon\gamma(t)\text{ and }|\hat h(t)-\hat h(0)|<\varepsilon\ ]. \tag{3.6}$$

Since $\tilde\psi$ is bounded on $[A_\varepsilon,+\infty[$ and $\hat h$ is integrable on $\mathcal{I}$, we have

$$\left|\frac{1}{2\pi}\int_{t\in\mathcal{I},\ |t|\geq A_\varepsilon}\hat h(t)\tilde\psi(t)(1-e^{-ita})dt\right|\leq C(\varepsilon).$$

Let a be such that $a\geq A_\varepsilon^{-1/\beta}$. We have

$$\left|\frac{1}{2\pi}\int_{\{|t|<a^{-\beta}\}}\hat h(t)\tilde\psi(t)(1-e^{-ita})dt\right|\leq \frac{a}{\pi}||\hat h||_\infty\int_0^{a^{-\beta}} t\,|\gamma(t)|\,dt,$$

that can be neglected as a goes to infinity since

$$\int_0^{a^{-\beta}}|at\gamma(t)|\,dt = \mathcal{O}\left(a^{(\beta-1)^2}\log(a)^{1-\beta}\right) = o(a^{\beta-1}\log(a)^{1-\beta})\text{ as }a\to\infty\ \text{ if }\ \alpha=1,\ \beta\in(1,2)$$

as a goes to infinity and since

$$\int_0^{a^{-\beta}} at|\gamma(t)|\,dt = \mathcal{O}(1) = o(\log(a))\ \text{ as }a\to\infty\text{ if }\ \beta=1.$$

It remains to estimate $\frac{1}{2\pi}\int_{\{a^{-\beta}\leq|t|\leq A_\varepsilon\}}\hat h(t)\tilde\psi(t)(1-e^{-ita})dt$ that we decompose into two parts:

$$I_1(a) := \frac{1}{2\pi}\int_{\{a^{-\beta}\leq|t|<A_\varepsilon\}}[\hat h(t)\tilde\psi(t)-\hat h(0)\gamma(t)](1-e^{-ita})dt$$

and

$$I_2(a) := \frac{\hat h(0)}{2\pi}\int_{\{a^{-\beta}\leq|t|<A_\varepsilon\}}(1-e^{-ita})\gamma(t)dt.$$

•We first estimate $I_2(a)$ for a large. Remark that by the change of variables $u = at$,

$$I_2(a) = \frac{\hat{h}(0)}{2\pi a} \int_{\{a^{1-\beta}<|u|<aA_\varepsilon\}} (1-e^{-iu})\gamma\Big(\frac{u}{a}\Big)\,du.$$

We treat separately the cases $\beta = 1$ and $\alpha = 1, \beta \in (1,2)$. If $\beta = 1$, we have

$$\frac{1}{2\pi a}\int_{\{1<|u|<aA_\varepsilon\}} (1-e^{-iu})\gamma\Big(\frac{u}{a}\Big)\,du = \frac{1}{A_1\pi}\int_{\{1<u<aA_\varepsilon\}} \frac{1-\cos u}{u}\,du \sim \frac{1}{A_1\pi}\log(a)$$

since

$$\lim_{x\to+\infty} \frac{1}{\log(x)}\int_1^x \frac{1-\cos(u)}{u} du = 1.$$

This comes from the fact that $\left(\int_1^x \frac{\cos(t)}{t}\,dt\right)_x$ is bounded.
If $\alpha = 1$ and $\beta \in (1,2)$, we have

$$\begin{aligned}
&\frac{1}{2\pi a}\int_{\{a^{1-\beta}<|u|<aA_\varepsilon\}} (1-e^{-iu})\gamma\Big(\frac{u}{a}\Big)\,du = \\
&\quad = \frac{a^{\beta-1}\beta^{1-\beta}}{2\pi c}\int_{\{a^{1-\beta}<|u|<aA_\varepsilon\}} (1-e^{-iu})(A_1+iA_2\mathrm{sgn}(u))^{-1}|u|^{-\beta}(\log(a)-\log|u|)^{1-\beta}\,du \\
&\quad = \frac{a^{\beta-1}(\log(a^\beta))^{1-\beta}}{2\pi c}\int_{\mathbb{R}} f_a(u)\,du,
\end{aligned}$$

with

$$f_a(u) := \mathbf{1}_{[a^{1-\beta},aA_\varepsilon]}(|u|)(1-e^{-iu})(A_1+iA_2\mathrm{sgn}(u))^{-1}|u|^{-\beta}\left(1-\frac{\log|u|}{\log a}\right)^{1-\beta}.$$

We observe that

$$|f_a(u)| \le F(u) := \min(1,|u|)|A_1+iA_2|^{-1}|u|^{-\beta}\beta^{1-\beta}$$

(with F integrable on $\mathbb{R}$ since $\beta \in (1,2)$) and that we have

$$\forall u \neq 0,\ \lim_{a\to+\infty} f_a(u) = (1-e^{-iu})(A_1+iA_2\mathrm{sgn}(u))^{-1}|u|^{-\beta} =: g(u).$$

So,

$$\lim_{a\to+\infty}\frac{(\log(a^\beta))^{\beta-1}}{2\pi a^\beta}\int_{\{a^{1-\beta}<|u|<aA_\varepsilon\}}(1-e^{-iu})\gamma(u/a)\,du = \frac{1}{2\pi c}\int_{\mathbb{R}} g(u)\,du.$$

We recall that

$$\int_{\mathbb{R}} g(t)dt = 2\mathrm{Re}\left[(A_1+iA_2)^{-1}\int_0^\infty \frac{1-e^{-it}}{t^\beta}\,dt\right]$$

and that

$$\int_0^\infty \frac{1-e^{-it}}{t^\beta} dt = \frac{\Gamma(2-\beta)}{\beta-1}e^{\frac{i}{2}(\beta-1)\pi}.$$

This gives

$$\lim_{a\to+\infty}\frac{(\log(a^\beta))^{\beta-1}}{a^{\beta-1}} I_2(a) = D_1.$$

•Second, we estimate $I_1(a)$. From (3.6), we have

$$\begin{aligned}
|I_1(a)| &\le \frac{\varepsilon(\mathcal{O}(1)+|\hat{h}(0)|)}{\pi}\int_{\{a^{-\beta}\le|t|<A_\varepsilon\}} |1-e^{-ita}||\gamma(t)|\,dt \\
&\le C\frac{\varepsilon}{a}\int_{\{a^{1-\beta}<|u|<aA_\varepsilon\}} |1-e^{-iu}|\gamma\Big(\frac{u}{a}\Big)\,du.
\end{aligned}$$

When $\beta = 1$, $|I_1(a)| \le \varepsilon \log(a)$. When $\alpha = 1$ and $\beta \in (1,2)$, from the above computations, we also have

$$\lim_{a\to+\infty} \frac{(\log a)^{\beta-1}}{2\pi a^\beta} \int_{\{a^{1-\beta}<|u|<aA_\varepsilon\}} |1-e^{-iu}|\gamma\Big(\frac{u}{a}\Big)\,du = \frac{1}{2\pi c}\int_{\mathbb{R}} |g(u)|\,du.$$

Therefore, we get $|I_1(a)| \le C\varepsilon a^{\beta-1}(\log a)^{1-\beta}$.
The case $(\alpha,\beta) = (1,2)$ can be handled in the same way as $\alpha = 1,\ \beta \in (1,2)$ using the inequality $1-\cos(t) \le \min(2,t^2)$. Details are omitted.

4 Proof of Theorem 1.3 (transient case)

We suppose that $\alpha > 1$ and $\beta < 1$. So $\delta > 1$. We will again use the notation

$$C = (\delta\beta)^{-1}\Gamma(\frac{1}{\delta\beta})\mathbb{E}[|L|_\beta^{-\frac{1}{\delta}}].$$

Let $h:\mathbb{R}\to\mathbb{C}$ be a Lebesgue-integrable function such that its Fourier transform $\hat{h}$ is differentiable, with $\hat{h}$ and $(\hat{h})'$ Lebesgue-integrable. Then, using the Fourier inversion formula, we obtain for every $n \ge 1$,

$$2\pi\mathbb{E}[h(Z_n-a)] = \int_{\mathbb{R}} \hat{h}(t)\mathbb{E}[e^{itZ_n}]e^{-ita}\,dt.$$

We get

$$2\pi\sum_{n\ge1}\mathbb{E}[h(Z_n-a)] = \sum_{n\ge1}\int_{\mathbb{R}} \hat{h}(t)\mathbb{E}[e^{itZ_n}]e^{-ita}\,dt.$$

Since here $\beta < 1$ (thus $\delta > 1$), the function $t \mapsto \hat{h}(t)\sum_{n\ge1}\left|\mathbb{E}[e^{itZ_n}]\right|$ is integrable (note that $\sum_{n\ge1}\left|\mathbb{E}[e^{itZ_n}]\right|$ corresponds to the case $A_2 = 0$, then use Proposition 2.1 and (2.3)). Therefore, from (2.1), we have

$$2\pi\sum_{n\ge1}\mathbb{E}[h(Z_n-a)] = \int_{\mathbb{R}} \hat{h}(t)\,\psi(t)\,e^{-ita}\,dt.$$

Let $\mathcal{S}(\mathbb{R})$ denote the so-called Schwartz space. Let $r \in (0,+\infty)$ and let $\chi \in \mathcal{S}(\mathbb{R})$ be such that

$$|\chi| \le 1 \quad \text{and} \quad \forall t \in [-r;r],\ \chi(t) = 1. \tag{4.1}$$

We have

$$2\pi\sum_{n\ge1}\mathbb{E}[h(Z_n-a)] = I_1(a)+I_2(a)+I_3(a),$$

with

$$I_1(a) := C\,\hat{h}(0)\int_{\mathbb{R}} \chi(t)\,|t|^{-\frac{1}{\delta}}(A_1+iA_2\mathrm{sgn}(t))^{-\frac{1}{\delta\beta}}\,e^{-ita}\,dt,$$
$$I_2(a) := \int_{\mathbb{R}} \chi(t)\left\{\hat{h}(t)\psi(t) - C\,\hat{h}(0)|t|^{-\frac{1}{\delta}}(A_1+iA_2\mathrm{sgn}(t))^{-\frac{1}{\delta\beta}}\right\}e^{-ita}\,dt,$$
$$I_3(a) := \int_{\{|t|>r\}} (1-\chi(t))\,\hat{h}(t)\,\psi(t)\,e^{-ita}\,dt.$$

The study of $I_3(a)$ is easy. Set $g_3(t) = (1-\chi(t))\hat{h}(t)\psi(t)$. From (4.1), we have $I_3(a) = \widehat{g_3}(a)$, and from Propositions 2.1 and 4, g_3 and g_3' are Lebesgue-integrable on $\mathbb{R}$. An integration by parts then gives

$$I_3(a) = O(a^{-1}) = o(a^{1/\delta-1}) \text{ as } a \text{ goes to } \infty.$$

The next two subsections are devoted to the study of $I_1(a)$ and $I_2(a)$.

4.1 Study of $I_1(a)$

Let us prove that:

$$\lim_{a\to+\infty} a^{1/\delta-1}\, I_1(a) = C\,\hat{h}(0)\, c^-_{\delta,\beta}$$

where $c^-_{\delta,\beta}$ is a constant defined in Lemma 4.1 below. The last property follows from Lemma 4.2 below. Before let us establish the following.

Lemma 4.1. *For every function $g\in\mathcal{S}(\mathbb{R})$,*

$$\int_{\mathbb{R}} \frac{\hat{g}(u)}{|u|^{1/\delta}(A_1+iA_2\ \mathrm{sgn}(u))^{\frac{1}{\delta\beta}}}\, du = \int_{\mathbb{R}} \frac{g(v)}{|v|^{1-\frac{1}{\delta}}}\left(c^+_{\delta,\beta}\mathbf{1}_{\mathbb{R}_+}(v) + c^-_{\delta,\beta}\mathbf{1}_{\mathbb{R}_-}(v)\right) dv,$$

where

$$c^+_{\delta,\beta} := \frac{2\,\Gamma(1-\frac{1}{\delta})}{(A_1^2+A_2^2)^{\frac{1}{2\delta\beta}}}\sin\left(\frac{1}{\delta}\Big(\frac{\pi}{2}+\frac{1}{\beta}\arctan\Big(\frac{A_2}{A_1}\Big)\Big)\right)$$

and

$$c^-_{\delta,\beta} := \frac{2\,\Gamma(1-\frac{1}{\delta})}{(A_1^2+A_2^2)^{\frac{1}{2\delta\beta}}}\sin\left(\frac{1}{\delta}\Big(\frac{\pi}{2}-\frac{1}{\beta}\arctan\Big(\frac{A_2}{A_1}\Big)\Big)\right).$$

Note that, since $\delta>1$, the functions $w\mapsto|w|^{-1/\delta}$ and $w\mapsto|w|^{-(1-\frac{1}{\delta})}$ are Lebesgue-integrable on any neighborhood of $w=0$, so that the two previous integrals are well defined.

Proof. For every $u\neq0$, we have

$$|u|^{-\frac{1}{\delta}}(A_1+iA_2\mathrm{sgn}(u))^{-\frac{1}{\delta\beta}} = \int_0^{+\infty} e^{-x|u|^{\frac{1}{\delta}}(A_1+iA_2\mathrm{sgn}(u))^{\frac{1}{\delta\beta}}}\, dx.$$

For any $x>0$, let us denote by f_x the Fourier transform of $u\mapsto e^{-x|u|^{\frac{1}{\delta}}(A_1+iA_2\mathrm{sgn}(u))^{\frac{1}{\delta\beta}}}$. By Fubini's theorem and Parseval's identity, we have

$$\begin{aligned}\int_{\mathbb{R}}\frac{\hat{g}(u)}{|u|^{\frac{1}{\delta}}(A_1+iA_2\mathrm{sgn}(u))^{\frac{1}{\delta\beta}}}\,du &= \int_0^{+\infty}\left(\int_{\mathbb{R}}\hat{g}(u)e^{-x|u|^{\frac{1}{\delta}}(A_1+iA_2\mathrm{sgn}(u))^{\frac{1}{\delta\beta}}}\,du\right)dx\\ &= \int_0^{+\infty}\left(\int_{\mathbb{R}} g(v)f_x(v)\,dv\right)dx.\end{aligned}$$

Next, since we have: $\forall x>0,\ \forall v\in\mathbb{R},\ f_x(v)=x^{-\delta}f_1\left(\frac{v}{x^\delta}\right)$, we obtain, from Fubini's theorem, with the change of variable $y=|v|/x^\delta$ and finally by the dominated convergence theorem (since $c^\pm_{\delta,\beta}$ are well defined, see below), that

$$\begin{aligned}\int_{\mathbb{R}}\frac{\hat{g}(u)}{|u|^{\frac{1}{\delta}}(A_1+iA_2\mathrm{sgn}(u))^{\frac{1}{\delta\beta}}}\,du &= \lim_{A\to0}\int_{\mathbb{R}} g(v)\left[\int_A^{+\infty}x^{-\delta}f_1\left(\frac{v}{x^\delta}\right)dx\right]dv\\ &= \lim_{A\to0}\int_{\mathbb{R}} g(v)|v|^{1/\delta-1}\left[\int_0^{|v|A^{-\delta}}\frac{f_1(\mathrm{sgn}(v)y)}{\delta y^{1/\delta}}\,dy\right]dv\\ &= \int_{\mathbb{R}}\frac{g(v)}{|v|^{1-1/\delta}}\left(c^+_{\delta,\beta}\mathbf{1}_{\mathbb{R}_+}(v)+c^-_{\delta,\beta}\mathbf{1}_{\mathbb{R}_-}(v)\right)dv,\end{aligned}$$

with

$$c^+_{\delta,\beta} := \int_0^{+\infty}\frac{f_1(y)}{\delta y^{1/\delta}}\,dy \ \text{ and }\ c^-_{\delta,\beta} := \int_0^{+\infty}\frac{f_1(-y)}{\delta y^{1/\delta}}\,dy.$$

Let us compute $c^+_{\delta,\beta}$. We have

$$\begin{aligned}
c^+_{\delta,\beta} &= \lim_{A\to+\infty} \frac{1}{\delta}\int_0^A f_1(y)\, y^{-1/\delta}\, dy \\
&= \lim_{A\to+\infty} \frac{1}{\delta}\int_0^A y^{-1/\delta}\left(\int_{\mathbb{R}} e^{ixy} e^{-|x|^{\frac{1}{\delta}}(A_1+iA_2\mathrm{sgn}(x))^{\frac{1}{\delta\beta}}}\, dx\right) dy \\
&= \lim_{A\to+\infty} \int_{\mathbb{R}} |u|^{-\frac{1}{\delta}} e^{iu}\left(\int_{\frac{|u|}{A}}^{+\infty} \frac{1}{\delta} v^{\frac{1}{\delta}-1} e^{-v^{\frac{1}{\delta}}(A_1+iA_2\mathrm{sgn}(u))^{\frac{1}{\delta\beta}}}\, dv\right) du \\
&= \lim_{A\to+\infty} \int_{\mathbb{R}} |u|^{-\frac{1}{\delta}} e^{iu} \frac{e^{-|u|^{\frac{1}{\delta}} A^{-\frac{1}{\delta}}(A_1+iA_2\mathrm{sgn}(u))^{\frac{1}{\delta\beta}}}}{(A_1+iA_2\mathrm{sgn}(u))^{\frac{1}{\delta\beta}}}\, du \\
&= \lim_{A\to+\infty} 2\,\mathrm{Re}\left[\int_0^{+\infty} u^{-\frac{1}{\delta}} e^{iu} \frac{e^{-u^{\frac{1}{\delta}} A^{-\frac{1}{\delta}}(A_1+iA_2)^{\frac{1}{\delta\beta}}}}{(A_1+iA_2)^{\frac{1}{\delta\beta}}}\, du\right],
\end{aligned}$$

using the change of variables $(u,v)=(yx,x)$. Now applying the residue theorem to the function $z\mapsto z^{-\frac{1}{\delta}} e^{iz} e^{-z^{\frac{1}{\delta}} A^{-\frac{1}{\delta}}(A_1+iA_2)^{\frac{1}{\delta\beta}}}$ with the contour in the complexe plane defined as follows : the line segment from r to R ($r<R$), the circular arc connecting R to iR, the line segment from iR to ir and the circular arc from ir to r and letting $r\to 0, R\to+\infty$, we get that

$$\int_0^{+\infty} u^{-\frac{1}{\delta}} e^{iu} e^{-u^{\frac{1}{\delta}} A^{-\frac{1}{\delta}}(A_1+iA_2)^{\frac{1}{\delta\beta}}}\, du = e^{i\left(\frac{\pi}{2}-\frac{\pi}{2\delta}\right)} \int_0^{+\infty} t^{-\frac{1}{\delta}} e^{-t} e^{-t^{\frac{1}{\delta}} e^{\frac{i\pi}{2\delta}} A^{-\frac{1}{\delta}}(A_1+iA_2)^{\frac{1}{\delta\beta}}}\, dt.$$

Taking $A\to+\infty$, we get the expression of $c^+_{\delta,\beta}$. □

Lemma 4.2. *We have:* $\lim_{a\to+\infty} a^{1-1/\delta}\int_{\mathbb{R}} \chi(t)|t|^{-\frac{1}{\delta}}(A_1+iA_2\ \mathrm{sgn}(t))^{-\frac{1}{\delta\beta}} e^{-ita}\, dt = c^-_{\delta,\beta}$.

Proof. Let $\gamma\in\mathcal{S}(\mathbb{R})$ such that $\hat\gamma=\chi$, and define: $\forall x\in\mathbb{R},\ \tilde\gamma_a(x) := a\gamma(-ax)$. From Lemma 4.1 and from the change of variable $v=wa$, we get

$$\begin{aligned}
\int_{\mathbb{R}} \chi(t)|t|^{-1/\delta}(A_1+iA_2\mathrm{sgn}(t))^{-\frac{1}{\delta\beta}} e^{-ita}\, dt &= \int_{\mathbb{R}} \widehat{\gamma(\cdot+a)}(t)|t|^{-1/\delta}(A_1+iA_2\mathrm{sgn}(t))^{-\frac{1}{\delta\beta}}\, dt \\
&= \int_{\mathbb{R}} \gamma(v+a)|v|^{1/\delta-1}(c^+_{\delta,\beta}\mathbf{1}_{\mathbb{R}_+}(v)+c^-_{\delta,\beta}\mathbf{1}_{\mathbb{R}_-}(v))\, dv \\
&= a^{1/\delta-1}\int_{\mathbb{R}} a\,\gamma\left(a(w+1)\right) g_\delta(w)\, dw \\
&= a^{1/\delta-1}\int_{\mathbb{R}} \tilde\gamma_a(-1-w)\, g_\delta(w)\, dw \\
&= a^{1/\delta-1}\left(\tilde\gamma_a * g_\delta\right)(-1),
\end{aligned}$$

where $*$ denotes the convolution product on $\mathbb{R}$ and

$$g_\delta(v) := |v|^{1/\delta-1}(c^+_{\delta,\beta}\mathbf{1}_{\mathbb{R}_+}(v)+c^-_{\delta,\beta}\mathbf{1}_{\mathbb{R}_-}(v)).$$

Observe that we have

$$\int_{\mathbb{R}} \tilde\gamma_a(w)dw = \int_{\mathbb{R}} \gamma(y)dy = \chi(0) = 1.$$

Now, from the fact that $\tilde\gamma\in\mathcal{S}(\mathbb{R})$ (actually use $\sup_{x\in\mathbb{R}}(1+x^2)|\tilde\gamma(x)|<\infty$), that g_δ is continuous at -1 and that the function $w\to w^{-2}g_\delta(w)$ is Lebesgue-integrable at infinity, it can be easily deduced from classical arguments of approximate identity that we have (see Prop. 1.14 in D. Guibourg's thesis [12] for details): $\lim_{a\to+\infty}(\tilde\gamma_a * g_\delta)(-1) = g_\delta(-1) = c^-_{\delta,\beta}$. □

4.2 Study of $I_2(a)$

Let us prove that:

$$I_2(a) = o(a^{1/\delta - 1}) \text{ as } a \text{ goes to } \infty.$$

Set $\Phi(t) := \psi(t) - C|t|^{-\frac{1}{\delta}}(A_1 + iA_2\mathrm{sgn}(t))^{-\frac{1}{\delta\beta}}$. We have

$$I_2(a) = \int_{\mathbb{R}} \chi(t)\left\{\hat{h}(t)\psi(t) - C\,\hat{h}(0)|t|^{-\frac{1}{\delta}}(A_1 + iA_2\mathrm{sgn}(t))^{-\frac{1}{\delta\beta}}\right\} e^{-ita}\,dt = J_1(a) + J_2(a)$$

with

$$J_1(a) := \int_{\mathbb{R}} \chi(t)\left\{\hat{h}(t) - \hat{h}(0)\right\}\psi(t)e^{-ita}\,dt \quad \text{and} \quad J_2(a) := \hat{h}(0)\int_{\mathbb{R}} \chi(t)\Phi(t)e^{-ita}\,dt.$$

Note that $J_1(a) = \widehat{g_1}(-a)$, with $g_1 := \chi(\hat{h} - \hat{h}(0))\psi$. From Proposition 2.2 and since $\hat{h}$ is continuously differentiable, we have $\psi(t) = O(|t|^{-1/\delta})$ and $(\hat{h}(t) - \hat{h}(0))\psi'(t) = O(|t|^{-1/\delta})$ when $t \to 0$. Hence g_1 and g_1' are Lebesgue-integrable on $\mathbb{R}$, so that we obtain by integration by parts:

$$J_1(a) = O(a^{-1}) = o(a^{1-1/\delta}) \text{ as } a \text{ goes to } \infty.$$

To study $J_2(a)$, let us set $G(t) := \chi(t)\Phi(t)$, and write

$$\begin{aligned} J_2(a) &= \hat{h}(0)\int_{\left\{|t|\le\frac{2\pi}{a}\right\}} G(t)\,e^{-ita}\,dt + \hat{h}(0)\int_{\left\{|t|>\frac{2\pi}{a}\right\}} G(t)\,e^{-ita}\,dt \\ &=: \hat{h}(0)J_{2,1}(a) + \hat{h}(0)J_{2,2}(a) \end{aligned} \tag{4.3}$$

where $J_{2,1}(a)$ and $J_{2,2}(a)$ are above defined in an obvious way. From Proposition 2.2 we have $\Phi(t) = \vartheta_0(t)|t|^{-\frac{1}{\delta}}$, with $\lim_{u\to 0}\vartheta_0(u) = 0$. Since $|\chi| \le 1$, we obtain:

$$\left|J_{2,1}(a)\right| \le \int_{\left\{|t|\le\frac{2\pi}{a}\right\}} \left|\Phi(t)\right|dt \le \frac{2}{1-\frac{1}{\delta}}\left(\frac{2\pi}{a}\right)^{1-\frac{1}{\delta}} \sup_{|t|\le\frac{2\pi}{a}} |\vartheta_0(t)| = o(a^{\frac{1}{\delta}-1}), \tag{4.4}$$

as a goes to infinity. Next we have $J_{2,2}(a) = -\int_{\left\{|t|>\frac{2\pi}{a}\right\}} G(t)\,e^{-i\left(t-\frac{\pi}{a}sgn(t)\right)a}\,dt$, hence

$$J_{2,2}(a) = \frac{1}{2}\left\{\int_{\left\{|t|>\frac{2\pi}{a}\right\}} G(t)\,e^{-ita}\,dt - \int_{\left\{|t|>\frac{\pi}{a}\right\}} G\left(t + \frac{\pi}{a}sgn(t)\right)\,e^{-ita}\,dt\right\},$$

from which we deduce:

$$\left|J_{2,2}(a)\right| \le \frac{1}{2}\int_{\left\{|t|>\frac{\pi}{a}\right\}} \left|G(t) - G\left(t + \frac{\pi}{a}sgn(t)\right)\right|\,dt + \int_{\left\{\frac{\pi}{a}<|t|<\frac{2\pi}{a}\right\}} |G(t)|\,dt. \tag{4.5}$$

The last integral in (4.5) is $o(a^{\frac{1}{\delta}-1})$ (use the second inequality in (4.4)). Next, by using Proposition 2.2, one can easily see that there exists $\vartheta_1 : \mathbb{R}\setminus\{0\} \to \mathbb{C}$ such that

$$G'(u) = |u|^{-1-\frac{1}{\delta}}\,\vartheta_1(u) \quad \text{with} \quad \lim_{u\to 0}\vartheta_1(u) = 0.$$

Let $\varepsilon > 0$, and let $\alpha = \alpha(\varepsilon) > 0$ be such that $\sup_{|s|<\alpha}|\vartheta_1(s)| \le \frac{\varepsilon}{2\delta}$. Note that

$$\left[a > \frac{2\pi}{\alpha} \text{ and } |t| < \frac{\alpha}{2}\right] \Rightarrow |t| \le \left|t + \frac{\pi}{a}sgn(t)\right| < \alpha.$$

Then, by applying Taylor's inequality to G, we obtain for all a such that $a > \frac{2\pi}{\alpha}$

$$\int_{\left\{\frac{\pi}{a}<|t|<\frac{\alpha}{2}\right\}} \left|G(t) - G\left(t + \frac{\pi}{a}sgn(t)\right)\right|\,dt \le \frac{\varepsilon}{\delta}\,\frac{\pi}{a}\int_{\frac{\pi}{a}}^{+\infty} t^{-1-\frac{1}{\delta}}\,dt \le \varepsilon\left(\frac{\pi}{a}\right)^{1-\frac{1}{\delta}}. \tag{4.6}$$

Moreover, since Φ and Φ' are bounded on $\mathbb{R}\setminus[-\frac{\alpha}{2};\frac{\alpha}{2}]$ (by Proposition 2.1), and from $\chi\in\mathcal{S}(\mathbb{R})$, there exists a positive constant D_α such that:

$$\forall x\in\mathbb{R}\setminus\left[-\frac{\alpha}{2};\frac{\alpha}{2}\right],\quad \left|G'(x)\right|\le\frac{D_\alpha}{x^2}.$$

Thus, if a is large enough, namely if a is such that $\frac{4D_\alpha}{\alpha}\left(\frac{\pi}{a}\right)^{\frac{1}{\delta}}\le\varepsilon$, then we have

$$\int_{\left\{|t|\ge\frac{\alpha}{2}\right\}}\left|G(t)-G\left(t+\frac{\pi}{a}sgn(t)\right)\right|\,dt\;\le\;2D_\alpha\,\frac{\pi}{a}\int_{\frac{\alpha}{2}}^{+\infty}t^{-2}\,dt\;\le\;\varepsilon\,\left(\frac{\pi}{a}\right)^{1-\frac{1}{\delta}} \tag{4.7}$$

From (4.5) (4.6) (4.7), it follows that we have when a is sufficiently large: $J_{2,2}(a)\le\varepsilon\,\left(\frac{\pi}{a}\right)^{1-\frac{1}{\delta}}$. From this fact and from (4.3) (4.4), we have:

$$J_2(a)=o(a^{1/\delta-1})\ \text{ as } a \text{ goes to } \infty.$$

The desired property for $I_2(a)$ is then established. This completes the proof of Theorem 1.3. □

Remark 4.3. *The generalization of our proof to the more general context when the distribution of ξ_0 belongs to the normal domain of attraction of a stable distribution of index β is not as simple as in the recurrent case. Indeed we used precise estimation of the derivative of ψ that should require the existence of the derivative of φ_ξ outside 0, which does not appear as a natural hypothesis when $\beta<1$ since ξ_0 is not integrable.*

References

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