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Penalizing null recurrent diffusions

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Abstract

We present some limit theorems for the normalized laws (with respect to functionals involving last passage times at a given level a up to time t) of a large class of null recurrent diffusions. Our results rely on hypotheses on the Lévy measure of the diffusion inverse local time at 0. As a special case, we recover some of the penalization results obtained by Najnudel, Roynette and Yor in the (reflected) Brownian setting.

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1 Introduction

1.1 A few notation

We consider a linear regular null recurrent diffusion $(X_t, t \geq 0)$ taking values in $\mathbb{R}^+$, with 0 an instantaneously reflecting boundary and $+\infty$ a natural boundary. Let $\mathbb{P}_x$ and $\mathbb{E}_x$ denote, respectively, the probability measure and the expectation associated with X when started from $x \geq 0$. We assume that X is defined on the canonical space $\Omega := \mathcal{C}(\mathbb{R}^+ \to \mathbb{R}^+)$ and we denote by $(\mathcal{F}_t, t \geq 0)$ its natural filtration, with $\mathcal{F}_\infty := \bigvee_{t\geq 0} \mathcal{F}_t$.

We denote by s its scale function, with the normalization $s(0) = 0$, and by $m(dx)$ its speed measure, which is assumed to have no atoms. It is known that $(X_t, t \geq 0)$ admits a transition density $q(t, x, y)$ with respect to m, which is jointly continuous and symmetric in x and y, that is: $q(t, x, y) = q(t, y, x)$. This allows us to define, for $\lambda > 0$, the resolvent kernel of X by:

$$u_\lambda(x, y) = \int_0^\infty e^{-\lambda t} q(t, x, y)dt. \tag{1.1}$$

We also introduce, for every $a \in \mathbb{R}^+$, $(L_t^a, t \geq 0)$ the local time of X at a, with the normalization:

$$L_t^a := \lim_{\varepsilon \downarrow 0} \frac{1}{m([a, a+\varepsilon[)} \int_0^t 1_{[a,a+\varepsilon[}(X_s)ds$$

and $(\tau_l^{(a)}, l \geq 0)$ the right-continuous inverse of $(L_t^a, t \geq 0)$:

$$\tau_l^{(a)} := \inf\{t \geq 0; L_t^a > l\}.$$

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As is well-known, when $X_0 = a$, $(\tau_l^{(a)}, l \geq 0)$ is a subordinator, and we denote by $\nu^{(a)}$ its Lévy measure. To simplify the notation, we shall write in the sequel τ_l for $\tau_l^{(0)}$ and ν for $\nu^{(0)}$. We shall also denote sometimes by $\overline{\mu}(t) = \mu([t, +\infty[)$ the tail of the measure μ.

1.2 Motivations

Our aim in this paper is to establish some penalization results involving null recurrent diffusions. Let us start by giving a definition of penalization:

Definition 1.1. *Let $(\Gamma_t, t \geq 0)$ be a measurable process taking positive values, and such that $0 < \mathbb{E}_x[\Gamma_t] < \infty$ for any $t > 0$ and every $x \geq 0$. We say that the process $(\Gamma_t, t \geq 0)$ satisfies the penalization principle if there exists a probability measure $\mathbb{Q}_x^{(\Gamma)}$ defined on $(\Omega, \mathcal{F}_\infty)$ such that:*

$$\forall s \geq 0,\ \forall \Lambda_s \in \mathcal{F}_s, \qquad \lim_{t \to +\infty} \frac{\mathbb{E}_x[1_{\Lambda_s}\Gamma_t]}{\mathbb{E}_x[\Gamma_t]} = \mathbb{Q}_x^{(\Gamma)}(\Lambda_s).$$

This problem has been widely studied by Roynette, Vallois and Yor when $\mathbb{P}_x$ is the Wiener measure or the law of a Bessel process (see [21] for a synthesis and further references). They showed in particular that Brownian motion may be penalized by a great number of functionals involving local times, supremums, additive functionals, numbers of downcrossings on an interval... Most of these results were then unified by Najnudel, Roynette and Yor (see [15]) in a general penalization theorem, whose proof relies on the construction of a remarkable measure $\mathcal{W}$.

Later on, Salminen and Vallois managed in [28] to extend the class of diffusions for which penalization results hold. They proved in particular that under the assumption that the (restriction of the) Lévy measure $\frac{1}{\nu([1,+\infty[)}\nu_{|[1,+\infty[}$ of the subordinator $(\tau_l, l \geq 0)$ is subexponential, the penalization principle holds for the functional $(\Gamma_t = h(L_t^0), t \geq 0)$ with h a non-negative and non-increasing function with compact support.
Let us recall that a probability measure μ is said to be subexponential (μ belongs to class $\mathcal{S}$) if, for every $t \geq 0$,

$$\lim_{t \to +\infty} \frac{\mu^{*2}([t, +\infty[)}{\mu([t, +\infty[)} = 2,$$

where μ^{*2} denotes the convolution of μ with itself. The main examples of subexponential distributions are given by measures having a regularly varying tail (see Chistyakov [4] or Embrechts, Goldie and Veraverbek [6]):

$$\mu([t, +\infty[) \underset{t \to +\infty}{\sim} \frac{\eta(t)}{t^\beta}$$

where $\beta \geq 0$ and η is a slowly varying function. When $\beta \in]0, 1[$, we shall say that such a measure belongs to class $\mathcal{R}$. Let us also remark that a subexponential measure always satisfies the following property:

$$\forall x \in \mathbb{R}, \quad \lim_{t \to +\infty} \frac{\mu([t + x, +\infty[)}{\mu([t, +\infty[)} = 1.$$

The set of such measures shall be denoted by $\mathcal{L}$, hence:

$$\mathcal{R} \subset \mathcal{S} \subset \mathcal{L}.$$

Now, following Salminen and Vallois, one may reasonably wonder what kind of penalization results may be obtained for diffusions whose normalized Lévy measure belongs to classes $\mathcal{R}$ or $\mathcal{L}$. This is the main purpose of this paper, i.e. we shall prove that the results of Najnudel, Roynette and Yor remain true for diffusions whose normalized Lévy measure belongs to $\mathcal{R}$, and we shall give an “integrated version” when it belongs to $\mathcal{L}$[1].

1.3 Statement of the main results

Let $a \geq 0$, $g_a^{(t)} := \sup\{u \leq t; X_u = a\}$ and $(F_t, t \geq 0)$ be a positive and predictable process such that

$$0 < \mathbb{E}_x\left[\int_0^{+\infty} F_u dL_u^a\right] < \infty.$$

Theorem 1.2.

1. *If ν belongs to class $\mathcal{L}$, then*

$$\forall a \geq 0, \qquad \int_0^t \nu^{(a)}([s, +\infty[)ds \underset{t\to+\infty}{\sim} \int_0^t \nu([s, +\infty[)ds$$

and

$$\mathbb{E}_x\left[\int_0^t F_{g_a^{(s)}} ds\right] \underset{t\to+\infty}{\sim} \left(\mathbb{E}_x[F_0](s(x) - s(a))^+ + \mathbb{E}_x\left[\int_0^{+\infty} F_u dL_u^a\right]\right) \int_0^t \nu([s, +\infty[)ds.$$

2. *If ν belongs to class $\mathcal{R}$:*

$$\forall a \geq 0, \qquad \nu^{(a)}([t, +\infty[) \underset{t\to+\infty}{\sim} \nu([t, +\infty[)$$

and if F is decreasing:

$$\mathbb{E}_x\left[F_{g_a^{(t)}}\right] \underset{t\to+\infty}{\sim} \left(\mathbb{E}_x[F_0](s(x) - s(a))^+ + \mathbb{E}_x\left[\int_0^{+\infty} F_u dL_u^a\right]\right) \nu([t, +\infty[)$$

Remark 1.3. *Point 2. does not hold for every $\nu \in \mathcal{L}$. Indeed, otherwise, taking $a = 0$ and $F_t = 1_{\{L_t^0 \leq \ell\}}$ with $\ell > 0$, one would obtain:*

$$\mathbb{P}_0(L_t^0 \leq \ell) = \mathbb{P}_0(\tau_\ell > t) \underset{t\to+\infty}{\sim} \ell\nu([t, +\infty[),$$

a relation which is known to hold if and only if $\nu \in \mathcal{S}$, see [6] or [27, p.164].

Remark 1.4. *If $(X_t, t \geq 0)$ is a positively recurrent diffusion, then $\int_0^{+\infty} \nu([s, +\infty[)ds = m(\mathbb{R}^+)$ (see Remark 3.2 below) and the limit in Point 1. equals:*

$$\lim_{t\to+\infty} \mathbb{E}_x\left[\int_0^t F_{g_a^{(s)}} ds\right] = \mathbb{E}_x\left[\int_0^{+\infty} F_{g_a^{(s)}} ds\right] = \mathbb{E}_x[F_0]\mathbb{E}_x[T_a] + \mathbb{E}_x\left[\int_0^{+\infty} F_u dL_u^a\right] m(\mathbb{R}^+).$$

In the following penalization result, we shall choose the weighting functional Γ according to ν:

Theorem 1.5. *Assume that:*

$a)$ *either ν belongs to class $\mathcal{L}$, and $\Gamma_t = \displaystyle\int_0^t F_{g_a^{(s)}} ds$,*

[1] In the remainder of the paper, we shall make a slight abuse of notation and say that the measure ν belongs to $\mathcal{L}$ or $\mathcal{R}$ instead of $\frac{1}{\nu([1,+\infty[)}\nu_{|[1,+\infty[}$ belongs to $\mathcal{L}$ or $\mathcal{R}$. This is of no importance since the fact that a probability measure belongs to classes $\mathcal{L}$ or $\mathcal{R}$ only involves the behavior of its tail at $+\infty$.

b) or ν belongs to class $\mathcal{R}$ and $\Gamma_t = F_{g_a^{(t)}}$ with F decreasing.

Then, the penalization principle is satisfied by the functional $(\Gamma_t, t \geq 0)$, i.e. there exists a probability measure $\mathbb{Q}_x^{(F)}$ on $(\Omega, \mathcal{F}_\infty)$, which is the same in both cases, such that,

$$\forall s \geq 0,\ \forall \Lambda_s \in \mathcal{F}_s, \quad \lim_{t \to +\infty} \frac{\mathbb{E}_x\left[1_{\Lambda_s} \Gamma_t\right]}{\mathbb{E}_x\left[\Gamma_t\right]} = \mathbb{Q}_x^{(F)}(\Lambda_s).$$

Furthermore:

1. *The measure $\mathbb{Q}_x^{(F)}$ is weakly absolutely continuous with respect to $\mathbb{P}_x$:*

$$\mathbb{Q}_{x|\mathcal{F}_t}^{(F)} = \frac{M_t(F_{g_a})}{\mathbb{E}_x[F_0](s(x)-s(a))^+ + \mathbb{E}_x\left[\int_0^{+\infty} F_u dL_u^a\right]} \cdot \mathbb{P}_{x|\mathcal{F}_t}$$

   where the martingale $(M_t(F_{g_a}), t \geq 0)$ is given by:

$$M_t(F_{g_a}) = F_{g_a^{(t)}}(s(X_t) - s(a))^+ + \mathbb{E}_x\left[\int_t^{+\infty} F_u dL_u^a | \mathcal{F}_t\right].$$

2. *Define $g_a := \sup\{s \geq 0,\ X_s = a\}$. Then, under $\mathbb{Q}_x^{(F)}$:*

   i) g_a is finite a.s.,

   ii) conditionally to g_a, the processes $(X_t, t \leq g_a)$ and $(X_{g_a+t}, t \geq 0)$ are independent,

   iii) the process $(X_{g_a+u}, u \geq 0)$ is transient, goes towards $+\infty$ and its law does not depend on the functional F.

We shall give in Theorem 5.1 a precise description of $\mathbb{Q}_x^{(F)}$ through an integral representation.

Remark 1.6. *The main example of diffusion satisfying Theorems 1.2 and 1.5 is of course the Bessel process with dimension $\delta \in]0, 2[$ reflected at 0. Indeed, setting $\beta = 1 - \frac{\delta}{2} \in]0, 1[$, the tail of its Lévy measure at 0 equals:*

$$\nu([t, +\infty[) = \frac{2^{1-\beta}}{\Gamma(\beta)} \frac{1}{t^\beta}$$

i.e. $\nu \in \mathcal{R}$. This may be obtained by integrating Formula (3.28) of [28] (where the computations are made via Bessel processes killed at 0), or by inverting the Laplace transform of Lemma 3.1 below with $u_\lambda(0,0) = \left(\frac{2}{\lambda}\right)^\beta \frac{\Gamma(\beta)}{2\Gamma(1-\beta)}$, see [2, p.133].

Remark 1.7. *Let us also mention that this kind of results no longer holds for positively recurrent diffusions. Indeed, it is shown in [16] that if $(X_t, t \geq 0)$ is a recurrent diffusion reflected on an interval, then, under mild assumptions, the penalization principle is satisfied by the functional $(\Gamma_t = e^{-\alpha L_t^0}, t \geq 0)$ with $\alpha \in \mathbb{R}$, but unlike in Theorem 1.5, the penalized process so obtained remains a positively recurrent diffusion.*

Example 1.8. *Assume that $\nu \in \mathcal{R}$ and let h be a positive and decreasing function with compact support on $\mathbb{R}^+$.*

- *Let us take* $(F_t, t \geq 0) = (h(L_t^a), t \geq 0)$.
  Then $\mathbb{E}_0\left[\int_0^{+\infty} h(L_s^a)dL_s^a\right] = \int_0^{+\infty} h(\ell)d\ell < \infty$ *and, since* $L^a_{g_a^{(t)}} = L_t^a$,

$$\mathbb{E}_0\left[h(L_t^a)\right] \underset{t\to+\infty}{\sim} \nu([t,+\infty[) \int_0^{+\infty} h(\ell)d\ell,$$

  and the martingale $(M_t(L_{g_a}^a), t \geq 0)$ *is an Azéma-Yor type martingale:*

$$M_t(L_{g_a}^a) = h(L_t^a)(s(X_t) - s(a))^+ + \int_{L_t^a}^{+\infty} h(\ell)d\ell.$$

- *Let us take* $(F_t, t \geq 0) = (h(t), t \geq 0)$.
  Then $\mathbb{E}_0\left[\int_0^{+\infty} h(u)dL_u^a\right] = \int_0^{+\infty} h(u)\mathbb{E}_0[dL_u^a] = \int_0^{+\infty} h(u)q(u,0,a)du < \infty$ *and therefore:*

$$\mathbb{E}_0\left[h(g_a^{(t)})\right] \underset{t\to+\infty}{\sim} \nu([t,+\infty[) \int_0^{+\infty} h(u)q(u,0,a)du,$$

  and the martingale $(M_t(g_a), t \geq 0)$ *is given by:*

$$M_t(g_a) = h(g_a^{(t)})(s(X_t) - s(a))^+ + \int_0^{+\infty} h(v+t)q(v,X_t,a)dv.$$

- *One may also take for instance* $(F_t, t \geq 0) = (h(S_t), t \geq 0)$ *where* $S_t := \sup_{s\leq t} X_s$ *or* $(F_t, t \geq 0) = h\left(\int_0^t f(X_s)ds\right)$ *where* $f : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ *is a Borel function. These were the first kind of weights studied by Roynette, Vallois and Yor, see [19] and [20].*

1.4 Organization

The remainder of the paper is organized as follows:

- In Section 2, we introduce some notation and recall a few known results that we shall use in the sequel. They are mainly taken from [26] and [29].
- Section 3 is devoted to the proof of Theorem 1.2. The two Points 1. and 2. are dealt with separately: when $\nu \in \mathcal{R}$, the asymptotic is obtained via a Laplace transform and a Tauberien theorem, while in the case $\nu \in \mathcal{L}$, we shall use a basic result on integrated convolution products.
- Section 4 gives the proof of Point 1. of Theorem 1.5, which essentially relies on a meta-theorem, see [21].
- In Section 5, we derive a integral representation for the penalized measure $\mathbb{Q}_x^{(F)}$ which implies Point 2. of Theorem 1.5.
- Finally, we shall use several times in the paper the fact that, with our normalizations, the process $(N_t^{(a)} := (s(X_t) - s(a))^+ - L_t^a, t \geq 0)$ is a martingale. The proof of this result is postponed to Section 6.

2 Preliminaries

In this section, we essentially recall some known results that we shall need in the sequel.

• Let $T_a := \inf\{u \geq 0; X_u = a\}$ be the first passage time of X to level a. Its Laplace transform is given by

$$\mathbb{E}_x\left[e^{-\lambda T_a}\right] = \frac{u_\lambda(a,x)}{u_\lambda(a,a)}. \tag{2.1}$$

Since $(X_t, t \geq 0)$ is assumed to be null recurrent, we have for $x > a$, $\mathbb{E}_x[T_a] = +\infty$.
• We define $(\widehat{X}_t, t \geq 0)$ as the diffusion $(X_t, t \geq 0)$ killed at a:

$$\widehat{X}_t := \begin{cases} X_t & t < T_a, \\ \partial & t \geq T_a. \end{cases}$$

where ∂ is a cemetary point. We denote by $\widehat{q}(t,x,y)$ its transition density with respect to m:

$$\widehat{\mathbb{P}}_x(\widehat{X}_t \in dy) = \widehat{q}(t,x,y)m(dy) = \mathbb{P}_x\left(X_t \in dy; t < T_a\right).$$

• We also introduce $(X_t^{\uparrow a}, t \geq 0)$ the diffusion $(\widehat{X}_t, t \geq 0)$ conditionned not to touch a, following the construction in [29]. For $x > a$ and F_t a positive, bounded and $\mathcal{F}_t$-measurable r.v.:

$$\mathbb{E}_x^{\uparrow a}\left[F_t\right] = \frac{1}{s(x)-s(a)}\mathbb{E}_x\left[F_t(s(X_t)-s(a))1_{\{t<T_a\}}\right].$$

By taking $F_t = f(X_t)$, we deduce in particular that, for $x, y > a$:

$$q^{\uparrow a}(t,x,y) = \frac{\widehat{q}(t,x,y)}{(s(x)-s(a))(s(y)-s(a))} \quad \text{and} \quad m^{\uparrow a}(dy) = (s(y)-s(a))^2 m(dy).$$

Letting x tend towards a, we obtain:

$$q^{\uparrow a}(t,a,y) = \frac{n_{y,a}(t)}{s(y)-s(a)} \quad \text{where } \mathbb{P}_y(T_a \in dt) =: n_{y,a}(t)dt.$$

• We finally define $(X_u^{x,t,y}, u \leq t)$ the bridge of X of length t going from x to y. Its law may be obtained as a h-transform, for $u < t$:

$$\mathbb{E}^{x,t,y}\left[F_u\right] = \mathbb{E}_x\left[\frac{q(t-u,X_u,y)}{q(t,x,y)}F_u\right]. \tag{2.2}$$

With these notation, we may state the two following Propositions which are essentially due to Salminen.

Proposition 2.1 ([26]).

1. *The law of* $g_a^{(t)} := \sup\{u \leq t; X_u = a\}$ *is given by:*

$$\mathbb{P}_x(g_a^{(t)} \in du) = \mathbb{P}_x(T_a > t)\delta_0(du) + q(u,x,a)\nu^{(a)}([t-u,+\infty[)du. \tag{2.3}$$

2. *On the event* $\{X_t > a\}$, *the density of the couple* $(g_a^{(t)}, X_t)$ *reads :*

$$\mathbb{P}_x\left(g_a^{(t)} \in du, X_t \in dy\right) = \mathbb{P}_x(T_a > t, X_t \in dy)\delta_0(du) + \frac{q(u,x,a)}{s(y)-s(a)}\mathbb{P}_a^{\uparrow a}(X_{t-u} \in dy)du \quad (y > a) \tag{2.4}$$

Remark 2.2. *From the definitions of* $q^{\uparrow a}$ *and* $m^{\uparrow a}$, *Equation (2.4) may be rewritten:*

$$\mathbb{P}_x\left(g_a^{(t)} \in du, X_t \in dy\right) = \mathbb{P}_x(T_a > t, X_t \in dy)\delta_0(du) + q(u,x,a)n_{y,a}(t-u)m(dy)\,1_{\{0<u\leq t\}}du, \tag{2.5}$$

and this last expression is actually valid for every $y \geq 0$, see [26]. Observe now that we may deduce Point 1. of Proposition 2.1 from this relation as follow. First, integrating (2.5) with respect to dy, we obtain:

$$\mathbb{P}_x(g_a^{(t)} \in du) = \mathbb{P}_x(T_a > t)\delta_0(du) + q(u,x,a)\left(\int_0^{+\infty} n_{y,a}(t-u)m(dy)\right) 1_{\{0<u\leq t\}} du,$$

so it remains to show that, for $0 < u \leq t$:

$$\int_0^{+\infty} n_{y,a}(t-u)m(dy) = \nu^{(a)}([t-u,+\infty[).$$

To this end, let us take the Laplace transform of the left hand side:

$$\begin{aligned}
\int_u^{+\infty} e^{-\lambda t}\int_0^{+\infty} & n_{y,a}(t-u)m(dy)dt \\
&= \int_0^{+\infty} e^{-\lambda u}\int_0^{+\infty} e^{-\lambda v} n_{y,a}(v)dv\, m(dy) \\
&= e^{-\lambda u}\int_0^{+\infty} \mathbb{E}_y\left[e^{-\lambda T_a}\right] m(dy) \\
&= \frac{e^{-\lambda u}}{\lambda u_\lambda(a,a)} \qquad (\textit{from (2.1), (1.1) and Fubini's theorem}) \\
&= \int_0^{+\infty} e^{-\lambda(u+v)}\nu^{(a)}([v,+\infty[)dv \qquad (\textit{from Lemma 3.1 below}) \\
&= \int_u^{+\infty} e^{-\lambda t}\nu^{(a)}([t-u,+\infty[)dt,
\end{aligned}$$

and (2.3) follows from the injectivity of the Laplace transform. We also refer to [29, Section 2] where some similar relationships between hitting times and Lévy measures are discussed via Itô excursion measure.

We now study the pre- and post- $g_a^{(t)}$-process:

Proposition 2.3. *Under $\mathbb{P}_x$:*

$i)$ *Conditionnally to $g_a^{(t)}$, the process $(X_s, s \leq g_a^{(t)})$ and $(X_{g_a^{(t)}+s}, s \leq t - g_a^{(t)})$ are independent.*

$ii)$ *Conditionnally to $g_a^{(t)} = u$,*

$$(X_s, s \leq u) \overset{\text{(law)}}{=} (X_s^{x,u,a}, s \leq u).$$

$iii)$ *Conditionnally to $g_a^{(t)} = u$ and $X_t = y > a$,*

$$(X_{u+s}, s \leq t-u) \overset{\text{(law)}}{=} \left(X_s^{\uparrow a\; a,t-u,y}, s \leq t-u\right).$$

Proof. $i)$ Point (i) follows from Proposition 5.5 of [14] applied to the diffusion

$$X_s^{(t)} := \begin{cases} X_s & s < t \\ \partial & s \geq t \end{cases}$$

so that $\xi := \inf\{s \geq 0;\ X_s^{(t)} \notin \mathbb{R}^+\} = t$.

$ii)$ Point (ii) is taken from [26].
$iii)$ As for Point (iii), still from [26], conditionnally to $g_a^{(t)} = u$ and $X_t = y > a$, we have:

$$(X_{u+s}, s \leq t-u) \overset{\text{(law)}}{=} \left(\widehat{X}_s^{a,t-u,y}, s \leq t-u\right).$$

But the bridges of $\widehat{X}$ et $X^{\uparrow}$ have the same law. Indeed, for $y, x > a$:

$$\begin{aligned}
&\widehat{\mathbb{P}}^{\,x,t,y}\left(X_{t_1} \in dx_1, \ldots, X_{t_n} \in dx_n\right) \\
&\qquad = \widehat{\mathbb{E}}_x\left[\frac{\widehat{q}(t-t_n, X_{t_n}, y)}{\widehat{q}(t,x,y)} 1_{\{X_{t_1}\in dx_1,\ldots,X_{t_n}\in dx_n\}}\right] \quad \text{(from (2.2))} \\
&\qquad = \mathbb{E}_x\left[\frac{(s(X_{t_n}) - s(a))q^{\uparrow a}(t-t_n, X_{t_n}, y)}{(s(x)-s(a))q^{\uparrow a}(t,x,y)} 1_{\{X_{t_1}\in dx_1,\ldots,X_{t_n}\in dx_n\}} 1_{\{t_n < T_a\}}\right] \\
&\qquad = \mathbb{E}_x^{\uparrow a}\left[\frac{q^{\uparrow a}(t-t_n, X_{t_n}, y)}{q^{\uparrow a}(t,x,y)} 1_{\{X_{t_1}\in dx_1,\ldots,X_{t_n}\in dx_n\}}\right] \quad \text{(by definition of } \mathbb{P}_x^{\uparrow a}) \\
&\qquad = \mathbb{P}^{\uparrow a\ x,t,y}\left(X_{t_1} \in dx_1, \ldots, X_{t_n} \in dx_n\right).
\end{aligned}$$

and the result follows by letting x tend toward a.

□

3 Study of asymptotics

The aim of this section is to prove Theorem 1.2. We start with the case $\nu \in \mathcal{R}$.

3.1 Proof of Theorem 1.2 when $\nu \in \mathcal{R}$

Let $(F_t, t \geq 0)$ be a decreasing, positive and predictable process such that

$$0 < \mathbb{E}_x\left[\int_0^{+\infty} F_u dL_u^a\right] < \infty.$$

Our approach in this section is based on the study of the Laplace transform of $t \longmapsto \mathbb{E}_x\left[F_{g_a^{(t)}}\right]$. Indeed, from Propositions 2.1 and 2.3, we may write, applying Fubini's Theorem:

$$\begin{aligned}
&\int_0^{+\infty} e^{-\lambda t}\mathbb{E}_x\left[F_{g_a^{(t)}}\right] dt \\
&= \int_0^{+\infty} e^{-\lambda t} \int_0^t \mathbb{E}_x\left[F_u | g_a^{(t)} = u\right] \mathbb{P}(g_a^{(t)} \in du) dt \\
&= \mathbb{E}_x[F_0] \int_0^{+\infty} e^{-\lambda t}\mathbb{P}_x(T_a > t)dt + \int_0^{+\infty} e^{-\lambda t} \int_0^t \mathbb{E}_x\left[F_u | X_u = a\right] q(u,x,a)\nu^{(a)}([t-u, +\infty[)du\, dt \\
&= \mathbb{E}_x\left[F_0\right] \frac{1-\mathbb{E}_x\left[e^{-\lambda T_a}\right]}{\lambda} + \int_0^{+\infty} e^{-\lambda t}\mathbb{P}^{x,t,a}(F_t)q(t,x,a)dt \times \int_0^{+\infty} e^{-\lambda t}\nu^{(a)}([t,+\infty[)dt
\end{aligned} \tag{3.1}$$

We shall now study the asymptotic (when $\lambda \to 0$) of each term separately. To this end, we state and prove two Lemmas.

3.1.1 The Laplace transform of $t \to \nu^{(a)}([t, +\infty[)$

Lemma 3.1. *The following formula holds:*

$$\frac{1}{\lambda u_\lambda(a,a)} = \int_0^{+\infty} e^{-\lambda t}\nu^{(a)}([t,+\infty[)dt$$

Proof. Since τ is a subordinator and m has no atoms, from the Lévy-Khintchine formula:

$$\mathbb{E}_a\left[e^{-\lambda\tau_l^{(a)}}\right] = \exp\left(-l\int_0^{+\infty}(1-e^{-\lambda t})\nu^{(a)}(dt)\right).$$

Then, from the classic relation (see [18] for instance):

$$\mathbb{E}_a\left[e^{-\lambda\tau_l^{(a)}}\right] = e^{-l/u_\lambda(a,a)},$$

we deduce that

$$\frac{1}{u_\lambda(a,a)} = \int_0^{+\infty}(1-e^{-\lambda t})\nu^{(a)}(dt).$$

Now, let $\varepsilon > 0$:

$$\begin{aligned}\int_\varepsilon^\infty (1-e^{-\lambda t})\nu^{(a)}(dt) &= \left[(e^{-\lambda t}-1)\nu^{(a)}([t,+\infty[)\right]_\varepsilon^{+\infty} + \int_\varepsilon^\infty \lambda e^{-\lambda t}\nu^{(a)}([t,+\infty[)dt \\ &= (1-e^{-\lambda\varepsilon})\nu^{(a)}([\varepsilon,+\infty[) + \int_\varepsilon^\infty \lambda e^{-\lambda t}\nu^{(a)}([t,+\infty[)dt\end{aligned}$$

Since both terms are positive, we may let $\varepsilon \to 0$ to obtain:

$$\frac{1}{\lambda u_\lambda(a,a)} = \int_0^\infty e^{-\lambda t}\nu^{(a)}([t,+\infty[)dt + \ell,$$

where $\ell := \lim\limits_{\varepsilon\to 0}\varepsilon\nu([\varepsilon,+\infty[)$, and it remains to prove that $\ell = 0$. Assume that $\ell > 0$. Then: $\nu^{(a)}([\varepsilon,+\infty[) \underset{\varepsilon\to 0}{\sim} \dfrac{\ell}{\varepsilon}$ and :

$$\begin{aligned}\int_\varepsilon^1 t\nu^{(a)}(dt) &= \left[-t\nu^{(a)}([t,1])\right]_\varepsilon^1 + \int_\varepsilon^1 \nu^{(a)}([t,1])dt \\ &= \varepsilon\nu^{(a)}([\varepsilon,1]) + \int_\varepsilon^1 \nu^{(a)}([t,1])dt \\ &\xrightarrow[\varepsilon\to 0]{} +\infty,\end{aligned}$$

since, from our hypothesis, $\nu^{(a)}([t,1]) \underset{t\to 0}{\sim} \dfrac{\ell}{t}$, i.e. $t \mapsto \nu^{(a)}([t,1])$ is not integrable at 0. This contradicts the fact that $\nu^{(a)}$ is the Lévy measure of a subordinator, hence $\ell = 0$ and the proof is completed.

□

Remark 3.2. *Since we assume that $(X_t, t \geq 0)$ is a null recurrent diffusion, we have $m(\mathbb{R}^+) = +\infty$ and from Salminen [24]:*

$$\lim_{\lambda\to 0}\lambda u_\lambda(a,a) = \frac{1}{m(\mathbb{R}^+)} = 0. \tag{3.2}$$

Thus, from the monotone convergence theorem, the function $t \to \nu^{(a)}([t,+\infty[)$ is not integrable at $+\infty$. On the other hand, if $(X_t, t \geq 0)$ is positively recurrent, we obtain:

$$\int_0^{+\infty}\nu^{(a)}([t,+\infty[)dt = m(\mathbb{R}^+) < +\infty.$$

We now study the asymptotic of the first hitting time of X to level a.

Lemma 3.3. *Let $x > a$ and assume that ν belongs to class $\mathcal{R}$. Then:*

$i)$ The tails of ν and $\nu^{(a)}$ are equivalent:

$$\nu^{(a)}([t,+\infty[) \underset{t\to+\infty}{\sim} \nu([t,+\infty[).$$

$ii)$ The survival function of T_a satisfies the following property:

$$\mathbb{P}_x(T_a \geq t) \underset{t\to+\infty}{\sim} (s(x)-s(a))\nu([t,+\infty[). \tag{3.3}$$

Proof. We shall use the following Tauberian theorem (see Feller [7, Chap. XIII.5, p.446] or [1, Section 1.7]):

Let f be a positive and decreasing function, $\beta \in]0,1[$ and η a slowly varying function. Then,

$$f(t) \underset{t\to+\infty}{\sim} \frac{\eta(t)}{t^\beta} \iff \int_0^\infty e^{-\lambda t} f(t)dt \underset{\lambda\to 0}{\sim} \frac{\Gamma(1-\beta)}{\lambda^{1-\beta}}\eta\left(\frac{1}{\lambda}\right). \tag{3.4}$$

In particular, with $f(t)=\nu([t,+\infty[)$ (since $\nu \in \mathcal{R}$), we obtain:

$$\int_0^\infty e^{-\lambda t}\nu([t,+\infty[)dt = \frac{1}{\lambda u_\lambda(0,0)} \underset{\lambda\to 0}{\sim} \frac{\Gamma(1-\beta)}{\lambda^{1-\beta}}\eta\left(\frac{1}{\lambda}\right).$$

Now, from Krein's Spectral Theory (see for instance [5, Chap.5], [10], [12] or [9]), $u_\lambda(x,y)$ admits the representation, for $x \leq y$:

$$u_\lambda(x,y) = \Phi(x,\lambda)\left(u_\lambda(0,0)\Phi(y,\lambda) - \Psi(y,\lambda)\right) \tag{3.5}$$

where the eigenfunctions Φ and Ψ are solutions of:

$$\begin{cases} \Phi(x,\lambda) = 1 + \lambda \displaystyle\int_0^x s'(dy)\int_0^y \Phi(z,\lambda)m(dz), \\ \Psi(x,\lambda) = s(x) + \lambda \displaystyle\int_0^x s'(dy)\int_0^y \Psi(z,\lambda)m(dz), \end{cases}$$

We deduce then, since $\lim\limits_{\lambda\to 0}\Phi(x,\lambda)=1$, $\lim\limits_{\lambda\to 0}\Psi(x,\lambda)=s(x)$ and $\lim\limits_{\lambda\to 0}u_\lambda(0,0)=+\infty$ that:

$$\frac{u_\lambda(a,a)}{u_\lambda(0,0)} = \Phi(a,\lambda)^2 - \frac{\Phi(a,\lambda)\Psi(a,\lambda)}{u_\lambda(0,0)} \xrightarrow[\lambda\to 0]{} 1.$$

Therefore, from the Tauberien theorem (3.4) with $f(t)=\nu^{(a)}([t,+\infty[)$, we obtain:

$$\nu^{(a)}([t,+\infty[) \underset{t\to+\infty}{\sim} \frac{\eta(t)}{t^\beta}$$

i.e. Point (i) of Lemma 3.3.
To prove Point (ii), let us compute the Laplace transform of $\mathbb{P}_x(T_a \geq t)$, using (2.1):

$$\int_0^{+\infty} e^{-\lambda t}\mathbb{P}_x(T_a\geq t)dt = \frac{1-\mathbb{E}_x\left[e^{-\lambda T_a}\right]}{\lambda} = \frac{1}{\lambda} - \frac{u_\lambda(x,a)}{\lambda u_\lambda(a,a)} = \frac{u_\lambda(a,a)-u_\lambda(x,a)}{\lambda u_\lambda(a,a)}. \tag{3.6}$$

Now, for $x > a$, we get from (3.5):

$$\begin{aligned} u_\lambda(a,a) - u_\lambda(a,x) &= \Phi(a,\lambda)(u_\lambda(0,0)\Phi(a,\lambda)-\Psi(a,\lambda)) - \Phi(a,\lambda)(u_\lambda(0,0)\Phi(x,\lambda)-\Psi(x,\lambda)) \\ &= \Phi(a,\lambda)u_\lambda(0,0)\left(\Phi(a,\lambda)-\Phi(x,\lambda)\right) + \Phi(a,\lambda)\left(\Psi(x,\lambda)-\Psi(a,\lambda)\right) \\ &= \Phi(a,\lambda)u_\lambda(0,0)\left(\lambda\int_a^x s'(y)dy\int_0^y \Phi(z,\lambda)m(dz)\right) + \Phi(a,\lambda)\left(\Psi(x,\lambda)-\Psi(a,\lambda)\right), \end{aligned}$$

and, letting λ tend toward 0 and using (3.2):

$$\lim_{\lambda\to 0}(u_\lambda(a,a)-u_\lambda(a,x)) = s(x)-s(a).$$

Therefore,

$$\int_0^{+\infty} e^{-\lambda t}\mathbb{P}_x(T_a\geq t)dt \underset{\lambda\to 0}{\sim} \frac{s(x)-s(a)}{\lambda u_\lambda(a,a)} \underset{\lambda\to 0}{\sim} (s(x)-s(a))\frac{\Gamma(1-\beta)}{\lambda^{1-\beta}}\eta\left(\frac{1}{\lambda}\right)$$

and Point (ii) follows once again from the Tauberian theorem (3.4).

□

3.1.2 Proof of Point 2. of Theorem 1.2

We now let λ tend toward 0 in (3.1). Observe first that, from our hypothesis on $(F_u, u\geq 0)$:

$$\int_0^{+\infty}\mathbb{P}^{x,u,a}(F_u)q(u,x,a)du = \int_0^{+\infty}\mathbb{E}_x\left[F_u|X_u=a\right]\mathbb{E}_x[dL_u^a] = \mathbb{E}_x\left[\int_0^{+\infty}F_u dL_u^a\right] < +\infty.$$

Then, from Lemmas 3.1 and 3.3, we obtain

- if $x\leq a$,
$$\int_0^{+\infty} e^{-\lambda t}\mathbb{E}_x\left[F_{g_a^{(t)}}\right]dt \underset{\lambda\to 0}{\sim} \frac{1}{\lambda u_\lambda(a,a)}\mathbb{E}_x\left[\int_0^{+\infty}F_u dL_u^a\right]$$
since $\lim\limits_{\lambda\to 0}\int_0^{+\infty} e^{-\lambda t}\mathbb{P}_x(T_a\geq t)dt = \mathbb{E}_x\left[T_a\right]<+\infty$,
- if $x>a$,
$$\int_0^{+\infty} e^{-\lambda t}\mathbb{E}_x\left[F_{g_a^{(t)}}\right]dt \underset{\lambda\to 0}{\sim} \frac{1}{\lambda u_\lambda(a,a)}\left(\mathbb{E}_x[F_0](s(x)-s(a))+\mathbb{E}_x\left[\int_0^{+\infty}F_u dL_u^a\right]\right).$$

Therefore, for every $x\geq 0$:

$$\int_0^{+\infty} e^{-\lambda t}\mathbb{E}_x\left[F_{g_a^{(t)}}\right]dt \underset{\lambda\to 0}{\sim} \left(\mathbb{E}_x[F_0](s(x)-s(a))^+ +\mathbb{E}_x\left[\int_0^{+\infty}F_u dL_u^a\right]\right)\frac{\Gamma(\beta)}{\lambda^{1-\beta}}\eta\left(\frac{1}{\lambda}\right)$$

and Point 2. follows from the Tauberian theorem (3.4) since $t\longmapsto \mathbb{E}_x\left[F_{g_a^{(t)}}\right]$ is decreasing.

□

3.2 Proof of Theorem 1.2 when $\nu\in\mathcal{L}$

Let $(F_t, t\geq 0)$ be a positive and predictable process such that

$$0<\mathbb{E}_x\left[\int_0^{+\infty}F_u dL_u^a\right]<\infty.$$

From Propositions 2.1 and 2.3 we have the decomposition:

$$\begin{aligned}\int_0^t \mathbb{E}_x\left[F_{g_a^{(s)}}\right]ds &= \int_0^t\int_0^s \mathbb{E}_x\left[F_u|g_a^{(s)}=u\right]\mathbb{P}(g_a^{(s)}\in du)\,ds\\ &= \mathbb{E}_x\left[F_0\right]\int_0^t \mathbb{P}_x(T_a>s)ds + \int_0^t\int_0^s \mathbb{E}_x\left[F_u|X_u=a\right]q(u,a,x)\nu^{(a)}([s-u,+\infty[)du\,ds.\end{aligned} \tag{3.7}$$

But, inverting the Laplace transform (3.6), we deduce that:

$$\mathbb{P}_x(T_a > s) = \int_0^s (q(u,a,a) - q(u,a,x))\nu^{(a)}([s-u, +\infty[)du,$$

hence, we may rewrite:

$$\int_0^t \mathbb{E}_x\left[F_{g_a^{(s)}}\right] ds = \int_0^t f * \overline{\nu}^{(a)}(s) ds$$

with $f(u) = \mathbb{E}_x[F_0](q(u,a,a) - q(u,a,x)) + \mathbb{P}^{x,u,a}(F_u)q(u,x,a)$ and $\overline{\nu}^{(a)}(u) = \nu^{(a)}([u,+\infty[)$. As in the previous section, the study of the asymptotic (when $t \to +\infty$) will rely on a few Lemmas.

3.2.1 Asymptotic of an integrated convolution product

Lemma 3.4. *Let μ be a measure whose tail $\overline{\mu}(t) = \mu([t, +\infty[)$ satisfies the following property:*

$$\text{for every } u \geq 0, \qquad \int_0^{t-u} \overline{\mu}(s)ds \underset{t\to+\infty}{\sim} \int_0^t \overline{\mu}(s)ds,$$

and let $f : \mathbb{R}^+ \to \mathbb{R}$ be a continuous function such that $\int_0^{+\infty} f(u)du < +\infty$. Then,

$$\int_0^t f * \overline{\mu}(s)\, ds \underset{t\to+\infty}{\sim} \int_0^{+\infty} f(u)du \int_0^t \overline{\mu}(s)ds.$$

Proof. Let $\varepsilon > 0$. There exists $A > 0$ such that, for every $t \geq A$, $\left|\int_t^{+\infty} f(u)du\right| < \varepsilon$. From Fubini's Theorem, we may write:

$$\begin{aligned}\int_0^t f * \overline{\mu}(s)ds &= \int_0^t f(u)du \int_u^t \overline{\mu}(s-u)ds \\ &= \int_0^t f(u)du \int_0^{t-u} \overline{\mu}(s)ds \\ &= \int_0^A f(u)du \int_0^{t-u} \overline{\mu}(s)ds + \int_A^t f(u)du \int_0^{t-u} \overline{\mu}(s)ds\end{aligned}$$

Using this decomposition, we obtain

$$\begin{aligned}&\left|\int_0^{+\infty} f(u)du - \frac{\int_0^t f * \overline{\mu}(s)ds}{\int_0^t \overline{\mu}(s)ds}\right| \\ &\leq \left|\int_0^A f(u)\left(1 - \frac{\int_0^{t-u} \overline{\mu}(s)ds}{\int_0^t \overline{\mu}(s)ds}\right)du\right| + \left|\int_A^t f(u)\frac{\int_0^{t-u} \overline{\mu}(s)ds}{\int_0^t \overline{\mu}(s)ds}du\right| + \left|\int_A^{+\infty} f(u)du\right| \\ &\leq \int_0^A |f(u)|\left(1 - \frac{\int_0^{t-A} \overline{\mu}(s)ds}{\int_0^t \overline{\mu}(s)ds}\right)du + \left|\int_A^t f(u)\frac{\int_0^{t-u} \overline{\mu}(s)ds}{\int_0^t \overline{\mu}(s)ds}du\right| + \varepsilon. \qquad (3.8)\end{aligned}$$

Then, applying the second mean value theorem, there exists $c \in]A, t[$ such that

$$\int_A^t f(u)\frac{\int_0^{t-u} \overline{\mu}(s)ds}{\int_0^t \overline{\mu}(s)ds}du = \frac{\int_0^{t-A} \overline{\mu}(s)ds}{\int_0^t \overline{\mu}(s)ds}\int_A^c f(u)du$$

hence,

$$\left|\int_A^t f(u)\frac{\int_0^{t-u} \overline{\mu}(s)ds}{\int_0^t \overline{\mu}(s)ds}du\right| = \frac{\int_0^{t-A} \overline{\mu}(s)ds}{\int_0^t \overline{\mu}(s)ds}\left|\int_A^{+\infty} f(u)du - \int_c^{+\infty} f(u)du\right| \leq 2\varepsilon\,\frac{\int_0^{t-A} \overline{\mu}(s)ds}{\int_0^t \overline{\mu}(s)ds}$$

and, letting t tend to $+\infty$ in (3.8), we finally obtain:

$$\limsup_{t\to+\infty}\left|\int_0^{+\infty} f(u)du - \frac{\int_0^t f*\overline{\mu}(s)ds}{\int_0^t \overline{\mu}(s)ds}\right| \leq 3\varepsilon.$$

□

Remark 3.5. *Assume that* $\nu \in \mathcal{L}$*. Then* ν *satisfies the hypothesis of Lemma 3.4. Indeed for* $u \geq 0$*, since* $\overline{\nu}(s-u) \underset{s\to+\infty}{\sim} \overline{\nu}(s)$ *and* $\overline{\nu}$ *is not integrable at* $+\infty$ *(see Remark 3.2), we have:*

$$\int_0^t \overline{\nu}(s)ds \underset{t\to+\infty}{\sim} \int_u^t \overline{\nu}(s)ds \underset{t\to+\infty}{\sim} \int_u^t \overline{\nu}(s-u)ds = \int_0^{t-u} \overline{\nu}(s)ds.$$

Lemma 3.6. *The following formula holds, for* $x > a$*:*

$$\int_0^{+\infty}(q(u,a,a) - q(u,a,x))du = s(x) - s(a).$$

Proof. We set $f(t) = \int_0^t (q(u,a,a) - q(u,a,x))du$. From Borodin-Salminen [2, p.21], we have:

$$f(t) = \mathbb{E}_a\left[L_t^a\right] - \mathbb{E}_a\left[L_t^x\right].$$

Since $(N_t^{(a)} = (s(X_t) - s(a))^+ - L_t^a, t \geq 0)$ is a martingale (see Lemma 6.1), this relation may be rewritten:

$$\begin{aligned} f(t) = \mathbb{E}_a\left[(s(X_t)-s(a))^+\right] - \mathbb{E}_a\left[(s(X_t)-s(x))^+\right] \\ = (s(x)-s(a))\mathbb{P}_a(X_t \geq x) + \mathbb{E}_a\left[(s(X_t)-s(a))1_{\{a\leq X_t\leq x\}}\right]. \end{aligned}$$

Then

$$\begin{aligned} |f(t) - (s(x)-s(a))| &\leq (s(x)-s(a))\mathbb{P}_a(X_t \leq x) + \mathbb{E}_a\left[(s(X_t)-s(a))1_{\{a\leq X_t\leq x\}}\right] \\ &\leq (s(x)-s(a))\left(\mathbb{P}_a(X_t\leq x) + \mathbb{P}_a(a \leq X_t \leq x)\right) \\ &\leq 2(s(x)-s(a))\mathbb{P}_a(X_t \leq x) \\ &\leq 2(s(x)-s(a))\mathbb{P}_0(X_t \leq x) \xrightarrow[t\to+\infty]{} 0 \end{aligned}$$

from [17, Chap.8, p.226], since $(X_t, t \geq 0)$ is null recurrent. □

Lemma 3.7. *Assume that* ν *belongs to class* $\mathcal{L}$*. Then:*

$$\forall a \geq 0, \qquad \int_0^t \nu^{(a)}([s,+\infty[)ds \underset{t\to+\infty}{\sim} \int_0^t \nu([s,+\infty[)ds$$

Proof. Let us define the function:

$$f_a(t) = \int_0^t q(u,0,0)\nu^{(a)}([t-u,+\infty[)du.$$

We claim that $\lim_{t\to+\infty} f_a(t) = 1$. Indeed, let us decompose f_a as follows, with $\varepsilon > 0$:

$$\begin{aligned} f_a(t) &= \int_0^t (q(u,0,0) - q(u,0,a))\nu^{(a)}([t-u,+\infty[)du + \mathbb{P}_0(T_a \leq t) \\ &= \int_0^{t-\varepsilon} (q(u,0,0) - q(u,0,a))\nu^{(a)}([t-u,+\infty[)du \\ &\qquad + \int_{t-\varepsilon}^t (q(u,0,0) - q(u,0,a))\nu^{(a)}([t-u,+\infty[)du + \mathbb{P}_0(T_a \leq t). \\ &= \int_0^{+\infty} (q(u,0,0) - q(u,0,a))1_{\{u\leq t-\varepsilon\}}\nu^{(a)}([t-u,+\infty[)du \\ &\qquad + \int_0^{\varepsilon} (q(t-u,0,0) - q(t-u,0,a))\nu^{(a)}([u,+\infty[)du + \mathbb{P}_0(T_a \leq t). \end{aligned}$$

From [17, Chap.8, p.224], we know that for every $u \geq 0$ the function $z \longmapsto q(u,0,z)$ is decreasing, hence the function

$$u \longmapsto q(u,0,0) - q(u,0,a)$$

is a positive and integrable function from Lemma 3.6. Therefore, from the dominated convergence theorem, the first integral tends toward 0 as $t \to +\infty$. Moreover, it is known from Salminen [25] that for every $x, y \geq 0$,

$$\lim_{t\to+\infty} q(t,x,y) = \frac{1}{m(\mathbb{R}^+)} = 0,$$

which proves, still from the dominated convergence theorem, that the second integral also tends toward 0 as $t \to +\infty$. Finally, we deduce that $\lim_{t\to+\infty} f_a(t) = \mathbb{P}_0(T_a < +\infty) = 1$. Observe now that, since $\overline{\nu} * q(t) = \int_0^t \nu([u,+\infty[)q(t-u,0,0)du = 1$, we have from Fubini-Tonelli:

$$\int_0^t \nu^{(a)}([s,+\infty[)ds = 1 * \overline{\nu}^{(a)}(t) = (\overline{\nu} * q) * \overline{\nu}^{(a)}(t) = \overline{\nu} * f_a(t) = \int_0^t f_a(s)\nu([t-s,+\infty[)ds.$$

Let $\varepsilon > 0$. There exists $A > 0$ such that, for every $s \geq A$:

$$1 - \varepsilon \leq f_a(s) \leq 1 + \varepsilon.$$

Integrating this relation, we deduce that, for $t > A$:

$$(1-\varepsilon)\int_A^t \overline{\nu}(t-s)ds \leq \int_A^t f_a(s)\overline{\nu}(t-s)ds \leq (1+\varepsilon)\int_A^t \overline{\nu}(t-s)ds.$$

Therefore:

$$\left|\int_0^t f_a(s)\overline{\nu}(t-s)ds - \int_A^t \overline{\nu}(t-s)ds - \int_0^A f_a(s)\overline{\nu}(t-s)ds\right| \leq \varepsilon \int_A^t \overline{\nu}(t-s)ds = \varepsilon \int_0^{t-A} \overline{\nu}(s)ds,$$

and it only remains to divide both terms by $\int_0^t \overline{\nu}(s)ds$ and let t tend toward $+\infty$ to conclude, thanks to Remark 3.5, that:

$$\limsup_{t\to+\infty} \left|\frac{\int_0^t \overline{\nu}^{(a)}(s)ds}{\int_0^t \overline{\nu}(s)ds} - 1\right| \leq \varepsilon.$$

□

3.2.2 Proof of Point 1. of Theorem 1.2

Going back to (3.7), we have, with $f(u) = \mathbb{P}^{x,u,a}(F_u)q(u,x,a)$ and $\overline{\nu}^{(a)}(u) = \nu^{(a)}([u,+\infty[)$:

$$\int_0^t \mathbb{E}_x\left[F_{g_a^{(s)}}\right]ds = \left(\mathbb{E}_x\left[F_0\right]\int_0^t \mathbb{P}_x(T_a > s)ds + \int_0^t f * \overline{\nu}^{(a)}(s)ds\right).$$

From Lemmas 3.4 and 3.6, we deduce that:

$$\lim_{t\to+\infty} \frac{1}{\int_0^t \overline{\nu}(s)ds}\int_0^t \mathbb{P}_x(T_a > s)ds = (s(x)-s(a))^+$$

since, for $x \leq a$, $\int_0^{+\infty} \mathbb{P}_x(T_a > s)ds = \mathbb{E}_x\left[T_a\right] < +\infty$. Then, Point 1. of Theorem 1.2 follows from Lemmas 3.4 and 3.7 and the fact that:

$$\int_0^{+\infty} f(u)du = \int_0^{+\infty} \mathbb{P}^{x,u,a}(F_u)q(u,x,a)du = \mathbb{E}_x\left[\int_0^{+\infty} F_u dL_u^a\right] < +\infty.$$

□

4 The penalization principle

4.1 Preliminaries: a meta-theorem and some notations

To prove Theorem 1.5, we shall apply a meta-theorem, whose proof relies mainly on Scheffé's Lemma (see Meyer [13, p.37]):

Theorem 4.1 ([21]). *Let $(\Gamma_t, t \geq 0)$ be a positive stochastic process satisfying for every $t > 0$, $0 < \mathbb{E}[\Gamma_t] < +\infty$. Assume that, for every $s \geq 0$:*

$$\lim_{t\to+\infty} \frac{\mathbb{E}[\Gamma_t|\mathcal{F}_s]}{\mathbb{E}[\Gamma_t]} =: M_s$$

exists a.s., and that,

$$\mathbb{E}[M_s] = 1.$$

Then,

i) for every $s \geq 0$ and $\Lambda_s \in \mathcal{F}_s$:

$$\lim_{t\to+\infty} \frac{\mathbb{E}[1_{\Lambda_s}\Gamma_t]}{\mathbb{E}[\Gamma_t]} = \mathbb{E}[M_s 1_{\Lambda_s}].$$

ii) there exists a probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F}_\infty)$ such that for every $s \geq 0$:

$$\mathbb{Q}(\Lambda_s) = \mathbb{E}[M_s 1_{\Lambda_s}].$$

In the following, we shall use Biane-Yor's notations [3]. We denote by Ω_{loc} the set of continuous functions ω taking values in $\mathbb{R}^+$ and defined on an interval $[0,\xi(\omega)] \subset [0,+\infty]$. Let $\mathbb{P}$ and $\mathbb{Q}$ be two probability measures, such that $\mathbb{P}(\xi = +\infty) = 0$. We denote by $\mathbb{P}\circ\mathbb{Q}$ the image measure $\mathbb{P}\otimes\mathbb{Q}$ by the concatenation application :

$$\begin{array}{rccc} \circ: & \Omega_{\text{loc}}\times\Omega_{\text{loc}} & \longrightarrow & \Omega_{\text{loc}} \\ & (\omega_1,\omega_2) & \longmapsto & \omega_1\circ\omega_2 \end{array}$$

defined by $\xi(\omega_1\circ\omega_2) = \xi(\omega_1)+\xi(\omega_2)$, and

$$(\omega_1\circ\omega_2)(t) = \begin{cases} \omega_1(t) & \text{if } 0 \leq t \leq \xi(\omega_1) \\ \omega_1(\xi(\omega_1)) + \omega_2(t-\xi(\omega_1)) - \omega_2(0) & \text{if } \xi(\omega_1) \leq t \leq \xi(\omega_1)+\xi(\omega_2). \end{cases}$$

To simplify the notations, we define the following measure, which was first introduced by Najnudel, Roynette and Yor [15]:

Definition 4.2. *Let $\mathcal{W}_x^{(a)}$ be the measure defined by:*

$$\mathcal{W}_x^{(a)} = \int_0^{+\infty} du\, q(u,x,a)\mathbb{P}^{x,u,a} \circ \mathbb{P}_a^{\uparrow a} + (s(x)-s(a))^+ \mathbb{P}_x^{\uparrow a}$$

$\mathcal{W}_x^{(a)}$ is a sigma-finite measure with infinite mass.

This measure enjoys many remarkable properties, and was the main ingredient in the proof of the penalization results they obtained for Brownian motion. A similar construction was made by Yano, Yano and Yor for symmetric stable Lévy processes, see [30].
With this new notation, we shall now write:

$$\begin{aligned}\mathcal{W}_x^{(a)}(F_{g_a}) &= \mathbb{E}_x\left[\int_0^{+\infty} F_u dL_u^a\right] + \mathbb{E}_x^{\uparrow a}[F_0](s(x)-s(a))^+ \\ &= \mathbb{E}_x\left[\int_0^{+\infty} F_u dL_u^a\right] + \mathbb{E}_x[F_0](s(x)-s(a))^+.\end{aligned}$$

4.2 Proof of Point $i)$ of Theorem 1.5

Let $0 \le u \le t$. Using Biane-Yor's notation, we write:

$$(X_s, s\le t) = (X_s, s\le u) \circ (X_{s+u}, 0 \le s \le t-u)$$

hence, from the Markov property, denoting $F_{g_a^{(t)}} = F(X_s, s\le t)$:

$$\mathbb{E}_x[F(X_s, s\le t)1_{\{u\le t\}}|\mathcal{F}_u] = \widehat{\mathbb{E}}_{X_u}\left[F((X_s, s\le u)\circ(\widehat{X}_s, 0 < s \le t-u))1_{\{u\le t\}}\right].$$

Let us assume first that $\nu \in \mathcal{R}$ and that $(F_t, t\ge 0)$ is decreasing. Then, from Theorem 1.2 with $\Gamma_t = F_{g_a^{(t)}}$:

$$\begin{aligned}&\lim_{t\to+\infty} \frac{\widehat{\mathbb{E}}_{X_u}\left[F((X_s, s\le u)\circ(\widehat{X}_s, 0\le s\le t-u))1_{\{u\le t\}}\right]}{\nu([t,+\infty[)} \\ &= \widehat{\mathbb{E}}_{X_u}\left[F((X_s, s\le u)\circ \widehat{X}_0)\right](s(X_u)-s(a))^+ + \widehat{\mathbb{E}}_{X_u}\left[\int_u^{+\infty} F((X_s, s\le u)\circ(\widehat{X}_s, 0\le s\le v-u))d\widehat{L}_v^a\right] \\ &= F(X_s, s\le u)(s(X_u)-s(a))^+ + \mathbb{E}_x\left[\int_u^{+\infty} F((X_s, s\le u)\circ(X_s, 0\le s\le v-u))dL_v^a|\mathcal{F}_u\right] \\ &= F_{g_a^{(u)}}(s(X_u)-s(a))^+ + \mathbb{E}_x\left[\int_u^{+\infty} F_{g_a^{(v)}} dL_v^a|\mathcal{F}_u\right] \\ &= F_{g_a^{(u)}}(s(X_u)-s(a))^+ + \mathbb{E}_x\left[\int_u^{+\infty} F_v dL_v^a|\mathcal{F}_u\right],\end{aligned}$$

hence,

$$\lim_{t\to+\infty} \frac{\mathbb{E}_x\left[F_{g_a^{(t)}}|\mathcal{F}_u\right]}{\mathbb{E}_x\left[F_{g_a^{(t)}}\right]} = \frac{M_u(F_{g_a})}{\mathcal{W}_x^{(a)}(F_{g_a})}.$$

On the other hand, if $\nu \in \mathcal{L}$ and $\Gamma_t = \int_0^t F_{g_a^{(s)}} ds$, a similar computation gives:

$$\begin{aligned}&\lim_{t\to+\infty} \frac{\int_0^t \widehat{\mathbb{E}}_{X_u}\left[F((X_s, s\le u)\circ(\widehat{X}_s, 0\le s\le v-u))1_{\{u\le t\}}\right]dv}{\int_0^t \nu([s,+\infty[)ds} \\ &\qquad = F_{g_a^{(u)}}(s(X_u)-s(a))^+ + \mathbb{E}_x\left[\int_u^{+\infty} F_v dL_v^a|\mathcal{F}_u\right],\end{aligned}$$

and

$$\lim_{t\to+\infty} \frac{\mathbb{E}_x\left[\int_0^t F_{g_a^{(s)}} ds|\mathcal{F}_u\right]}{\mathbb{E}_x\left[\int_0^t F_{g_a^{(s)}} ds\right]} = \frac{M_u(F_{g_a})}{\mathcal{W}_x^{(a)}(F_{g_a})}.$$

Therefore, to apply Theorem 4.1, it remains to prove that:

$$\forall t \geq 0, \qquad \mathbb{E}_x\left[M_t(F_{g_a})\right] = \mathcal{W}_x^{(a)}(F_{g_a}).$$

We shall make a direct computation, applying Proposition 2.1:

• if $x > a$,

$$\begin{aligned}
\mathbb{E}_x\left[M_t(F_{g_a})\right] &= \mathbb{E}_x\left[F_{g_a^{(t)}}(s(X_t)-s(a))^+ + \mathbb{E}_x\left[\int_t^{+\infty} F_u dL_u^a|\mathcal{F}_t\right]\right] \\
&= \int_a^{+\infty} \mathbb{E}_x[F_0|X_t=y, T_a>t](s(y)-s(a))\mathbb{P}_x(T_a>t, X_t\in dy) \\
&\quad + \int_0^t \int_a^{+\infty} \mathbb{P}^{x,u,a}(F_u)q(u,a,x)\mathbb{P}_a^{\uparrow}(X_{t-u}\in dy)du + \int_t^{+\infty} \mathbb{P}^{x,u,a}(F_u)q(u,a,x)du \\
&= \mathbb{E}_x[F_0(s(X_t)-s(a))1_{\{t<T_a\}}] + \int_0^{+\infty} \mathbb{P}^{x,u,a}(F_u)q(u,a,x)du \\
&= \mathbb{E}_x^{\uparrow a}[F_0](s(x)-s(a)) + \int_0^{+\infty} \mathbb{P}^{x,u,a}(F_u)q(u,a,x)du = \mathcal{W}_x^{(a)}(F_{g_a}),
\end{aligned}$$

• if $x \leq a$, then, for $y > a$, $\mathbb{P}_x\left(T_a > t, X_t \in dy\right) = 0$ since X has continuous paths, and the same computation leads to:

$$\mathbb{E}_x\left[M_t(F_{g_a})\right] = \int_0^{+\infty} \mathbb{P}^{x,u,a}(F_u)q(u,a,x)du = \mathcal{W}_x^{(a)}(F_{g_a}).$$

Therefore, for every $x \geq 0$, $\mathbb{E}_x\left[\dfrac{M_t(F_{g_a})}{\mathcal{W}_x^{(a)}(F_{g_a})}\right] = 1$, and the proof is completed.

□

Remark 4.3. *Consider the martingale* $(N_t^{(a)} = (s(X_t)-s(a))^+ - L_t^a, t \geq 0)$. *We apply the balayage formula to the semimartingale* $((s(X_t)-s(a))^+, t\geq 0)$*:*

$$\begin{aligned}
F_{g_a^{(t)}}(s(X_t)-s(a))^+ &= F_0(s(x)-s(a))^+ + \int_0^t F_{g_a^{(u)}} d(s(X_u)-s(a))^+ \\
&= F_0(s(x)-s(a))^+ + \int_0^t F_{g_a^{(u)}} dN_u^{(a)} + \int_0^t F_{g_a^{(u)}} dL_u^a \\
&= F_0(s(x)-s(a))^+ + \int_0^t F_{g_a^{(u)}} dN_u^{(a)} + \int_0^t F_u dL_u^a.
\end{aligned}$$

Therefore, the martingale $(M_t(F_{g_a}), t \geq 0)$ *may be rewritten:*

$$M_t(F_{g_a}) = F_0(s(x)-s(a))^+ + \int_0^t F_{g_a^{(u)}} dN_u^{(a)} + \mathbb{E}_x\left[\int_0^{+\infty} F_s dL_s^a|\mathcal{F}_t\right].$$

5 An integral representation of $\mathbb{Q}_x^{(F_{g_a})}$

Finally, Point 2. of Theorem 1.5 is a direct consequence of the following result:

Theorem 5.1. $\mathbb{Q}_x^{(F_{g_a})}$ *admits the following integral representation:*

$$\mathbb{Q}_x^{(F_{g_a})} = \frac{1}{\mathcal{W}_x^{(a)}(F_{g_a})} \left(\int_0^{+\infty} q(u,x,a) F_u \, \mathbb{P}^{x,u,a} \circ \mathbb{P}_a^{\uparrow a} + (s(x)-s(a))^+ F_0 \, \mathbb{P}_x^{\uparrow a} \right)$$

Proof. Let G, H and φ be three bounded Borel functionals, with H depending only on the trajectory up to a finite time. We write:

$$\begin{aligned}
&\mathcal{W}_x^{(a)}(F_{g_a}) \mathbb{Q}_x^{(F_{g_a})} \left(G(X_s, s \le g_a^{(t)}) \varphi(g_a^{(t)}) H(X_{g_a^{(t)}+s}, s \le t - g_a^{(t)}) \right) \\
&= \mathbb{E}_x \left[G(X_s, s \le g_a^{(t)}) \varphi(g_a^{(t)}) H(X_{g_a^{(t)}+s}, s \le t - g_a^{(t)}) M_t(F_{g_a}) \right] \\
&= \mathbb{E}_x \left[G(X_s, s \le g_a^{(t)}) \varphi(g_a^{(t)}) H(X_{g_a^{(t)}+s}, s \le t - g_a^{(t)}) \left(F_{g_a^{(t)}} (s(X_t) - s(a))^+ + \mathbb{E}_x \left[\int_t^{+\infty} F_u dL_u^a | \mathcal{F}_t \right] \right) \right] \\
&= I_1(t) + I_2(t).
\end{aligned}$$

On the one hand,

$$I_2(t) = \mathbb{E}_x \left[G(X_s, s \le g_a^{(t)}) \varphi(g_a^{(t)}) H(X_{g_a^{(t)}+s}, s \le t - g_a^{(t)}) \int_t^{+\infty} F_u dL_u^a \right] \xrightarrow[t \to +\infty]{} 0$$

from the dominated convergence theorem.
On the other hand, from Propositions 2.1 and 2.3:

$$\begin{aligned}
I_1(t) = &\int_a^{+\infty} \int_0^t \mathbb{P}_x \left(g_a^{(t)} \in du, X_t \in dy \right) \times \\
&\qquad \mathbb{E}_x \left[G(X_s, s \le u) \varphi(u) H(X_{u+s}, s \le t-u) F_u (s(y) - s(a)) | g_a^{(t)} = u, X_t = y \right] \\
= &\int_a^{+\infty} \int_0^t \mathbb{P}_x \left(g_a^{(t)} \in du, X_t \in dy \right) \times \\
&\qquad \mathbb{P}^{x,u,a} \left(G(X_s, s \le u) F_u \right) \varphi(u) (s(y) - s(a)) \mathbb{E}_x \left[H(X_{u+s}, s \le t-u) | g_a^{(t)} = u, X_t = y \right].
\end{aligned}$$

We now separate the two cases $g_a^{(t)} = 0$ and $g_a^{(t)} > 0$ as in relation (2.4).

• First, when $g_a^{(t)} = 0$ and $x \le a$, this term is null. Indeed, for $x \le a < y$, $\mathbb{P}_x (T_a > t, X_t \in dy) = 0$ since X has continuous paths. Next, for $x > a$:

$$\begin{aligned}
&\int_a^{+\infty} \mathbb{P}_x (T_a > t, X_t \in dy) \, G(x) \mathbb{E}_x[F_0] \varphi(0) (s(y) - s(a)) \mathbb{E}_x \left[H(X_s, s \le t) | T_a > t, X_t = y \right] \\
&= G(x) \mathbb{E}_x[F_0] \varphi(0) \mathbb{E}_x \left[(s(X_t) - s(a))^+ H(X_s, s \le t) 1_{\{T_a > t\}} \right] \\
&= G(x) \mathbb{E}_x[F_0] \varphi(0) (s(x) - s(a)) \mathbb{E}_x^{\uparrow a} \left[H(X_s, s \le t) \right] \\
&\xrightarrow[t \to +\infty]{} G(x) \mathbb{E}_x[F_0] \varphi(0) (s(x) - s(a))^+ \mathbb{E}_x^{\uparrow a} \left[H(X_s, s \ge 0) \right].
\end{aligned}$$

• Second, when $g_a^{(t)} > 0$:

$$\int_a^{+\infty} \int_0^t \frac{q(u,x,a)}{s(y)-s(a)} \mathbb{P}_a^{\uparrow a}(X_{t-u} \in dy) du \times$$
$$\mathbb{P}^{x,u,a}\left(G(X_s, s \le u) F_u\right) \varphi(u)(s(y)-s(a)) \mathbb{E}_x\left[H(X_{u+s}, s \le t-u) | g_a^{(t)} = u, X_t = y\right]$$
$$= \int_a^{+\infty} \int_0^t q(u,x,a) \mathbb{P}_a^{\uparrow a}(X_{t-u} \in dy) du \times$$
$$\mathbb{P}^{x,u,a}\left(G(X_s, s \le u) F_u\right) \varphi(u) \mathbb{E}_a^{\uparrow a}\left[H(X_s, s \le t-u) | X_{t-u} = y\right]$$
$$= \int_0^t du\, q(u,x,a) \mathbb{P}^{x,u,a}\left(G(X_s, s \le u) F_u\right) \varphi(u) \mathbb{E}_a^{\uparrow a}\left[H(X_s, s \le t-u)\right]$$
$$\xrightarrow[t \to +\infty]{} \int_0^{+\infty} du\, q(u,x,a) \mathbb{P}^{x,u,a}\left(G(X_s, s \le u) F_u\right) \varphi(u) \mathbb{E}_a^{\uparrow a}\left[H(X_s, s \ge 0)\right].$$

□

Remark 5.2. *From Theorem 5.1,* $\mathbb{Q}_x^{(F_{g_a})}(g_a < +\infty) = 1$ *and we deduce that, conditionally to* g_a*,*

1. *on the event* $g_a > 0$*, the law of the process* $(X_{g_a+u}, u \ge 0)$ *under* $\mathbb{Q}_x^{(F_{g_a})}$ *is the same as the law of* $(X_u, u \ge 0)$ *under* $\mathbb{P}_a^{\uparrow a}$*,*
2. *on the event* $g_a = 0$*, the law of the process* $(X_u, u \ge 0)$ *under* $\mathbb{Q}_x^{(F_{g_a})}$ *is the same as the law of* $(X_u, u \ge 0)$ *under* $\mathbb{P}_x^{\uparrow a}$*.*

Observe that the process $(F_u, u \ge 0)$ *plays no role in these results.*

Example 5.3. *Let* h *be a positive and decreasing function on* $\mathbb{R}^+$*.*

• *Let us take* $(F_t, t \ge 0) = (h(L_t^a), t \ge 0)$ *and assume that* $\int_0^{+\infty} h(\ell) d\ell = 1$*:*

$$\mathbb{Q}_0^{(h(L_{g_a}^a))} = \int_0^{+\infty} du\, q(u,0,a) h(L_u^a) \mathbb{P}^{0,u,a} \circ \mathbb{P}_a^{\uparrow a}.$$

Now, if G *and* φ *are two bounded Borel functionals, we may write*

$$\begin{aligned} \mathbb{Q}_0^{(h(L_{g_a}^a))}\big(G(X_t, t \le g_a)\varphi(L_\infty^a)\big) &= \int_0^{+\infty} du\, q(u,0,a) \mathbb{P}^{0,u,a}\big(G(X_t, t \le u)\varphi(L_u^a) h(L_u^a)\big) \\ &= \mathbb{E}_0\left[\int_0^{+\infty} G(X_t, t \le u)\varphi(L_u^a) h(L_u^a) dL_u^a\right] \\ &= \mathbb{E}_0\left[\int_0^{+\infty} G\big(X_t, t \le \tau_\ell^{(a)}\big)\varphi(\ell) h(\ell) d\ell\right], \end{aligned}$$

which leads to:

$$\int_0^{+\infty} \mathbb{Q}_0^{(h(L_{g_a}^a))}\big(G(X_t, t \le g_a) | L_\infty^a = \ell\big)\varphi(\ell) \mathbb{Q}_0^{(h(L_{g_a}^a))}(L_\infty^a \in d\ell)$$
$$= \int_0^{+\infty} \mathbb{E}_0\left[G(X_t, t \le \tau_\ell^{(a)})\right] \varphi(\ell) h(\ell) d\ell.$$

Thus, taking $G = 1$*, we deduce that, under* $\mathbb{Q}_0^{(h(L_{g_a}^a))}$*, the r.v.* L_∞^a *is a.s. finite and admits* $\ell \longmapsto h(\ell)$ *as its density function. Furthermore, conditionally to* $L_\infty^a = \ell$ *the process* $(X_t, t \le g_a)$ *has the same law as* $(X_t, t \le \tau_\ell^{(a)})$ *under* $\mathbb{P}_0$*.*

- *Let us take* $(F_t, t \geq 0) = (h(t), t \geq 0)$ *and assume that* $\int_0^{+\infty} h(u)q(u, 0, a)du = 1$*:*

$$\mathbb{Q}_0^{(h(g_a))} = \int_0^{+\infty} du\, q(u, 0, a)h(u)\mathbb{P}^{0,u,a} \circ \mathbb{P}_a^{\uparrow a}.$$

Then, under $\mathbb{P}_0^{(h(g_a))}$*, the r.v.* g_a *admits as density function* $u \longmapsto h(u)q(u, 0, a)$ *and, conditionally to* $g_a = u$ *the process* $(X_t, t \leq g_a)$ *has the same law as* $(X_t, t \leq u)$ *under* $\mathbb{P}^{0,u,a}$*.*

6 Appendix

Let $a \geq 0$ and define $(N_t^{(a)} := (s(X_t) - s(a))^+ - L_t^a, t \geq 0)$. The aim of this section is to prove the following lemma:

Lemma 6.1. *The process* $(N_t^{(a)}, t \geq 0)$ *is a martingale in the filtration* $(\mathcal{F}_t, t \geq 0)$*.*

Proof. Applying the Markov property to the diffusion $(X_t, t \geq 0)$ we deduce that:

$$\mathbb{E}_0\left[N_{t+s}^{(a)} | \mathcal{F}_s\right] = \widehat{\mathbb{E}}_{X_s}\left[(s(\widehat{X}_t) - s(a))^+\right] - L_s^a - \widehat{\mathbb{E}}_{X_s}\left[\widehat{L}_t^a\right].$$

We set $x = X_s$, so we need to prove that for every $x \geq 0$:

$$(s(x) - s(a))^+ = \mathbb{E}_x\left[(s(X_t) - s(a))^+\right] - \mathbb{E}_x\left[L_t^a\right],$$

or rather:

$$\int_0^{+\infty} (s(y) - s(a))^+ q(t, x, y)m(dy) = \int_0^t q(u, x, a)du + (s(x) - s(a))^+.$$

Let us take the Laplace transform of this last relation (applying Fubini-Tonelli):

$$\int_0^{+\infty} (s(y) - s(a))^+ u_\lambda(x, y)m(dy) = \frac{u_\lambda(x, a)}{\lambda} + \frac{(s(x) - s(a))^+}{\lambda}. \tag{6.1}$$

Our aim now is to prove (6.1). To this end, we shall use the following representation of the resolvent kernel $u_\lambda(x, y)$ (see [2, p.19]):

$$u_\lambda(x, y) = \omega_\lambda^{-1}\psi_\lambda(x)\varphi_\lambda(y) \qquad x \leq y$$

where ψ_λ and φ_λ are the fundamental solutions of the generalized differential equation

$$\frac{d^2}{dm\, ds}u = \lambda u \tag{6.2}$$

such that ψ_λ is increasing (resp. φ_λ is decreasing) and the Wronskian ω_λ is given, for all $z \geq 0$ by:

$$\omega_\lambda = \varphi_\lambda(z)\frac{d\psi_\lambda}{ds}(z) - \psi_\lambda(z)\frac{d\varphi_\lambda}{ds}(z).$$

Note that since m has no atoms, the meaning of (6.2) is as follows:

$$\forall y \geq x, \quad \lambda\int_x^y u(z)m(dz) = \frac{d\,u}{ds}(y) - \frac{d\,u}{ds}(x) \qquad \text{where} \quad \frac{d\,u}{ds}(x) := \lim_{h\to 0}\frac{u(x+h) - u(x)}{s(x+h) - s(x)}.$$

• Assume first that $x \leq a$.

$$\begin{aligned}
&\int_a^{+\infty} (s(y) - s(a)) u_\lambda(x,y) m(dy) \\
&= \frac{1}{\omega_\lambda} \int_a^{+\infty} \left(\int_a^y ds(z) \right) \psi_\lambda(x) \varphi_\lambda(y) m(dy) \\
&= \frac{\psi_\lambda(x)}{\omega_\lambda} \int_a^{+\infty} ds(z) \int_z^{+\infty} \varphi_\lambda(y) m(dy) \quad \text{(applying Fubini-Tonelli's theorem since } \varphi_\lambda \geq 0) \\
&= -\frac{\psi_\lambda(x)}{\lambda \omega_\lambda} \int_a^{+\infty} ds(z) \frac{d\varphi_\lambda}{ds}(z) \quad \left(\text{since } \lim_{y \to +\infty} \frac{d\varphi_\lambda}{ds}(y) = 0 \quad \text{as } +\infty \text{ is a natural boundary} \right) \\
&= \frac{\psi_\lambda(x)}{\lambda \omega_\lambda} \varphi_\lambda(a) \quad \left(\text{since } \lim_{z \to +\infty} \varphi_\lambda(z) = 0 \quad \text{as } +\infty \text{ is a natural boundary} \right) \\
&= \frac{u_\lambda(x,a)}{\lambda}
\end{aligned}$$

which gives (6.1) for $x \leq a$.

• Now, let us suppose that $x > a$. We have, with the same computation:

$$\begin{aligned}
&\int_a^{+\infty} (s(y) - s(a)) u_\lambda(x,y) m(dy) \\
&= \int_a^x (s(y) - s(a)) u_\lambda(x,y) m(dy) + \int_x^{+\infty} (s(y) - s(a)) u_\lambda(x,y) m(dy) \\
&= I_1 + I_2.
\end{aligned}$$

On the one hand:

$$\begin{aligned}
I_1 &= \frac{\varphi_\lambda(x)}{\omega_\lambda} \int_a^x ds(z) \int_z^x \psi_\lambda(y) m(dy) \\
&= \frac{\varphi_\lambda(x)}{\lambda \omega_\lambda} \int_a^x ds(z) \left(\frac{d\psi_\lambda}{ds}(x) - \frac{d\psi_\lambda}{ds}(z) \right) \\
&= \frac{\varphi_\lambda(x)}{\lambda \omega_\lambda} \left((s(x) - s(a)) \frac{d\psi_\lambda}{ds}(x) - (\psi_\lambda(x) - \psi_\lambda(a)) \right) \\
&= \frac{s(x) - s(a)}{\lambda \omega_\lambda} \varphi_\lambda(x) \frac{d\psi_\lambda}{ds}(x) - \frac{u_\lambda(x,x)}{\lambda} + \frac{u_\lambda(x,a)}{\lambda}.
\end{aligned}$$

On the other hand:

$$\begin{aligned}
I_2 &= \int_x^{+\infty} (s(y) - s(x)) u_\lambda(x,y) m(dy) + (s(x) - s(a)) \int_x^{+\infty} u_\lambda(x,y) m(dy) \\
&= \frac{u_\lambda(x,x)}{\lambda} + \frac{s(x) - s(a)}{\omega_\lambda} \psi_\lambda(x) \int_x^{+\infty} \varphi_\lambda(y) m(dy) \quad \text{(from the previous computations)} \\
&= \frac{u_\lambda(x,x)}{\lambda} - \frac{s(x) - s(a)}{\lambda \omega_\lambda} \psi_\lambda(x) \frac{d\varphi_\lambda}{ds}(x).
\end{aligned}$$

Finally, gathering both terms, we obtain for $x > a$:

$$\begin{aligned}
\int_a^{+\infty} (s(y) - s(a)) u_\lambda(x,y) m(dy) &= \frac{s(x) - s(a)}{\lambda \omega_\lambda} \left(\varphi_\lambda(x) \frac{d\psi_\lambda}{ds}(x) - \psi_\lambda(x) \frac{d\varphi_\lambda}{ds}(x) \right) + \frac{u_\lambda(x,a)}{\lambda}, \\
&= \frac{s(x) - s(a)}{\lambda} + \frac{u_\lambda(x,a)}{\lambda},
\end{aligned}$$

which is the desired result (6.1) from the definition of the Wronskian.

□

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