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The First Hitting Time of A Single Point for Random Walks

Kôhei Uchiyama
Department of Mathematics
Tokyo Institute of Technology
Oh-okayama, Meguro Tokyo 152-8551, Japan
uchiyama@math.titech.ac.jp

Abstract

This paper concerns the first hitting time τ_0 of the origin for random walks on d-dimensional integer lattice with zero mean and a finite $2+\delta$ absolute moment ($\delta \geq 0$). We derive detailed asymptotic estimates of the probabilities $P_x[\tau_0 = n]$ as $n \to \infty$ that are valid uniformly in x, the position at which the random walks start .

Key words: hitting time, asymptotic expansion, Fourier analysis, random walk.

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Introduction

Let $S_n^x = x + X_1 + \cdots + X_n$ be a random walk on the d-dimensional square lattice $\mathbf{Z}^d$ starting at x where the increments X_j are i.i.d. random variables defined on some probability space $(\Omega, \mathscr{F}, P)$ and taking values in $\mathbf{Z}^d$. Let X be a random variable having the same law as X_1 and $\psi(\theta)$ the characteristic function of X: $\psi(\theta) = Ee^{iX\cdot\theta}$, $\theta \in T^d$, where T^d stands for the d-dimensional torus $\mathbf{R}^d/(2\pi\mathbf{Z})^d \cong [-\pi, \pi)^d$ and E indicates the expectation by P. Throughout the paper we suppose unless explicitly stated otherwise that the distribution of X is *aperiodic* (strongly aperiodic in the sense of Spitzer [11]), i.e., $|\psi(\theta)| < 1$ for $\theta \in T^d \setminus \{0\}$ (which imposes no essential restriction) and that

$$EX = 0 \quad \text{and} \quad E|X|^{2+\delta} < \infty, \tag{0.1}$$

for a constant $\delta \geq 0$.

This paper concerns the asymptotic evaluation of the hitting-time distribution

$$f_x(k) = P[\tau_0^x = k] \quad (k = 1, 2, \ldots)$$

as $k \to \infty$, where $\tau_0^x = \inf\{n > 0 : S_n^x = 0\}$, the first time that S_n^x hits the origin after time 0 ($\inf\emptyset = \infty$), which plays a fundamental role in the theory of random walk and its applications. We derive asymptotic formulae of $f_x(k)$ with certain bounds for error terms valid uniformly in x for each dimension $d = 1, 2, \ldots$: under $\delta = 0$, in particular, the asymptotic form is determined in any parabolic region $x^2 \leq ak$. In general the estimates will depend on δ and we shall mainly (or essentially) consider the case $0 \leq \delta \leq 2$ and only occasionally the case $\delta > 2$. For the computation of $f_x(k)$ we use the Fourier analytic method as in [10].

When the walk is started at the origin there are several results. In Kesten [10] it is proved, among many other things, that if the walk is one-dimensional and aperiodic, and satisfies that for some $1 \leq \alpha \leq 2$, $|\theta|^{-\alpha}(1 - \psi(\theta))$ converges to a positive constant, C say, as $\theta \to 0$, then the asymptotic form of $f_0(k)$ is $C_\alpha k^{-2+1/\alpha}(1 + o(1))$ if $1 < \alpha \leq 2$ (with $C_\alpha = (\alpha - 1)\sin(\pi/\alpha)C^{1/\alpha}/\Gamma(1/\alpha)$) and $\pi C[k(\lg k)^2]^{-1}(1 + o(1))$ if $\alpha = 1$; in the particular case $\alpha = 2$ this implies in our setting that if $d = 1$, $f_0(k) = \sqrt{E|X|^2/2\pi}\, k^{-3/2}(1 + o(1))$. For the two dimensional walk satisfying (0.1) with $\delta = 0$ Jain and Pruitt show that $f_0(k) = c[k(\lg k)^2]^{-1}(1 + o(1))$ (Section 4 of [9]). (This result actually follows from Kesten's result (for $\alpha = 1$), the latter being based only on an estimate of the characteristic function of f_0 (see Remark at the end of the subsection **4.1**). The proof of [9] is rather probabilistic and quite different from Kesten's proof.). Combined with the ratio limit theorem ([10], [11]) these give the asymptotic form of the tail $P[\tau_0^x > k]$ (for each x) in the cases $d = 1$ and 2 (with $\delta = 0$) but there seems to be no results on estimation of f_x uniformly valid for x except for a few special cases. Recently Y. Hamana ([7]) has proved that for the simple random walk, $f_0(2k) = \pi[k \lg^2 k]^{-1}[1 + O((\lg\lg k)/\lg k)]$ if $d = 2$, $f_0(2k) = c_d k^{-d/2}[1 + O((\lg k)^{-N})]$ if $d \geq 3$ (c_d is a certain positive constant and $N > 0$ may be arbitrary) and applied these results to the study of the range of the pinned walk. (In [6] the error term is improved to $O(k^{-5/8})$ for $d = 3$.)

For $d \geq 2$ we have studied in [15] the random variable $Z_n = \sharp\{S_1^0, S_2^0, \ldots, S_n^0\}$, the number of sites visited by $S_\cdot^0$ until the n-th step. The expectation EZ_n is equal to $e_0 n + \sum_{k=0}^{n-1} F_k$ where $F_k = \sum_{j>k} f_0(j)$ and $e_0 = 1 - F_0$ and readily computed (up to $O(\lg\lg n)$ if $d = 2$ and $O(1)$ if $d \geq 3$) from the estimates of $f_0(k)$ obtained in this paper. In [15] we are interested in the conditional expectation $E[Z_n | S_n^0 = x]$, i.e. the expectation under the law of the random walk bridge, of which the asymptotic evaluation, being made by means of Fourier analytic method, depends on several subsidiary results from the present paper. For $d=1$ the estimate of $f_x(k)$ is effectively used to compute the transition probability of one dimensional walk killed at the origin [16]. One of the results obtained is applied in a very recent work [1], where from it is deduced a precise asymptotic estimate for the coalescing probability of a finite number of independent random walks.

1 Statements of Results

Let S_n^x, X, $\psi(\theta)$ and $f_x(k)$ be as in Introduction and suppose the condition (0.1) to hold true with some $0 \le \delta \le 2$ unless otherwise is stated explicitly. Set $p^n(x) = P[S_n^0 = x]$, $p(x) = p^1(x)$, $a(x) = \sum_{n=0}^{\infty}[p^n(0) - p^n(-x)]$ (cf. [5], [11] for convergence of the series) and

$$a^*(x) = \mathbf{1}_{\{0\}}(x) + a(x) = 1 + \sum_{n=1}^{\infty}[p^n(0) - p^n(-x)],$$

where $\mathbf{1}_{\{0\}}(x) = 1$ or 0 according as $x = 0$ or $x \neq 0$. Denote by Q the covariance matrix of X and by $Q(\theta)$ its quadratic form: $Q(\theta) = E(X \cdot \theta)^2 = \theta \cdot Q\theta$ ($\theta \in \mathbf{R}^d$). For $x \in \mathbf{Z}^d$ put

$$\tilde{x} = Q^{-1/2}x.$$

If $d = 1$, let $\sigma^2 = E|X|^2$ so that $\tilde{x} = x/\sigma$.

The following notation will be used: $\operatorname{sgn} t = t/|t|$ $(t \neq 0)$; $a \wedge b = \min\{a, b\}$, $a \vee b = \max\{a, b\}$ $(a, b \in \mathbf{R})$; $|\theta|$ denotes the Euclidean length of $\theta \in \mathbf{R}^d$, $\theta^2 = |\theta|^2$; for functions g and G of a variable ξ, $g(\xi) = O(G(\xi))$ means that there exists a constant C such that $|g(\xi)| \le C|G(\xi)|$ whenever ξ ranges over a specified set; $\lg^+ a = \lg(a \vee 1)$ $(a \ge 0)$ and

$$|x|_+ = |x| \vee 1 \ \ (x \in \mathbf{Z}^d).$$

1.1. Here we consider the one dimensional case.

Theorem 1.1. *Let $d = 1$ and $0 \le \delta \le 2$. Then, uniformly in $x \in \mathbf{Z}$, as $k \to \infty$*

$$f_x(k) = \frac{\sigma}{\sqrt{2\pi}} \frac{a^*(x)}{k^{3/2}} \left(1 + o\left(\frac{1}{k^{\delta/2}}\right) + O\left(\frac{|x|_+^2}{k}\right)\right). \tag{1.1}$$

The estimate given in Theorem 1.1 is poor in the case $x^2 > k$, when the second error term (represented by O symbol) is not smaller than the principal part. The following theorem is complementary in this respect. If $\delta \ge 1$ and $d = 1$, define constants β_3 and C^* by

$$\beta_3 = \frac{1}{6}E[X^3] \quad \text{and} \quad C^* = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\frac{\sigma^2}{1 - \psi(\theta)} - \frac{1}{1 - \cos\theta}\right]d\theta. \tag{1.2}$$

The integral above is understood to be the principal value; the imaginary part vanishes and the real part is absolutely convergent (see (3.5) in Section 3). For convenience sake we put $\beta_3 = C^* = 0$ if $\delta < 1$.

Theorem 1.2. *Let $d=1$ and $0\le\delta<3$. Then, as $k\wedge|x|\to\infty$*

$$f_x(k)=\frac{|\tilde x|e^{-\tilde x^2/2k}}{\sqrt{2\pi}\,k^{3/2}}\left[1+\frac{\mathrm{P}_1(\tilde x^2/k)+(\operatorname{sgn}x)\beta_3\mathrm{P}_2(\tilde x^2/k)}{|\tilde x|}+\frac{J(x,k)}{k}\right]+o\left(\frac{1}{|x|^{2+\delta}}\right),$$

where

$$\mathrm{P}_1(z)=\sigma^{-1}C^*(1-z),\quad \mathrm{P}_2(z)=-\sigma^{-3}(2-5z+z^2),$$

and $J(x,k)=\mathrm{P}_3(\tilde x^2/k)+(\operatorname{sgn}x)\beta_3\mathrm{P}_4(\tilde x^2/k)$ with P_3 and P_4 being certain polynomials of at most degree 3. Moreover, in the formula above, the error term can be replaced by $o(k^{-1-\delta/2})$ if $0\le\delta<1$ (but not if $\delta\ge1$), by $o(|x|^{-1}k^{-1-(\delta-1)/2})$ if $1\le\delta<2$ and by $k^{-3/2}|x|^{-2}o(k^{(1-\delta)/2}\wedge|x|^{(1-\delta)}\lg|x|)$ if $2\le\delta<3$.

Theorems 1.1 and 1.2 together entails the following result.

Corollary 1.1. *Uniformly in x as $k\to\infty$*

$$f_x(k)=\frac{\sigma a^*(x)}{\sqrt{2\pi}k^{3/2}}\,e^{-\tilde x^2/2k}+o\left(\frac{|x|_+}{k^{3/2}}\wedge\frac{1}{|x|_+^2}\right).$$

REMARK 1. Theorem 1.2 can be extended to the case $\delta\ge3$ with a higher order expansion.

REMARK 2. (i) Suppose that $1\le\delta<2$. According to [16] (Appendix) $\sigma a(x)=|\tilde x|+\sigma^{-1}C^*-2\sigma^{-3}\beta_3\operatorname{sgn}x+o(|x|^{1-\delta})$ if $d=1$. (ii) If $p(x)=0$ either for all $x\le-2$ (LC) or for all $x\ge2$ (RC), then $C^*=2\sigma^{-2}|\beta_3|$ (the converse is also true [16]). This is a consequence of the asymptotic form of $a(x)$ just mentioned together with the fact ([11]:P30.3) that $a(x)=|x|/\sigma^2$ for all $x>0$ in the case (LC) and for all $x<0$ in the case (RC). If both (LC) and (RC) are the case, we have $C^*=\beta_3=0$ and both $\mathrm{P}_1(z)$ and $\mathrm{P}_2(z)$ disappear from the formula of Theorem 1.2 (otherwise $C^*>0$, provided that $E|X|^3<\infty$).

REMARK 3. The random walk on $\mathbf{Z}^d$ with $P[X=\omega]=1/2d$ for all $\omega\in\mathbf{Z}^d$ with $|\omega|=1$ is said simple. The simple random walk is not aperiodic; it is of period $\nu=2$. (The period is the smallest integer r such that $p^{rn}(0)>0$ for all sufficiently large n). In general, if X is irreducible but not aperiodic, we obtain the correct formula for $f_x(k)$ by simply multiplying by the factor

$$\nu\,\mathbf{1}(p^k(-x)\ne0)\tag{1.3}$$

the leading term of the formulae obtained under aperiodicity assumption, where $\mathbf{1}(\mathscr{S})$ is 1 or 0 according as the statement $\mathscr{S}$ is true or false. (For $d=1$ this is, in effect, ascertained in [16]: page 692; for the case $d\ge2$ see Appendix (D) of this paper.)

For the one dimensional simple random walk we have a simple explicit expression of $f_x(k)$ (cf. [3]: III.4), from which, with the help of Stirling's formula, one deduces that uniformly for $x\ne0$ with $k+x$ even, as $x^4/k^3\to0$

$$f_x(k)=\frac{2|x|}{\sqrt{2\pi}\,k^{3/2}}e^{-x^2/2k}\left[1+\frac{1}{k}\mathrm{P}\left(\frac{x^2}{k}\right)+O\left(\frac{x^2}{k^3}+\frac{x^6}{k^5}\right)\right],$$

where $\mathrm{P}(z)=-\frac14+\frac12z-\frac1{12}z^2$. (The factor 2 is due to (1.3). It is also remarked that $f_0(2k)=f_1(2k-1)$.)

1.2. Next consider the case $d = 2$. In order to have a formula more or less parallel to that of Theorem 1.1 we introduce the function

$$W(\lambda) = \int_0^\infty \frac{e^{-\lambda u} du}{[\lg u]^2 + \pi^2} \qquad (\lambda > 0).$$

We also bring in the constants

$$c_1 = \frac{1}{(2\pi)^2} \int_{T^2} \Re\left(\frac{1}{1-\psi(\theta)} - \frac{2}{Q(\theta)}\right) d\theta \ \ (\le \infty),$$

$$c_2 = \frac{\lg(s^2/2)}{2\pi |Q|^{1/2}} + \frac{1}{(2\pi)^2} \int_{\{Q > s^2\} \cap T^2} \frac{2}{Q(\theta)} d\theta,$$

where T^2 is the two dimensional torus (as in Introduction), $|Q| = \det Q$ and s is a positive number chosen so small that $\{\theta : Q(\theta) < s^2\} \subset T^2$, and define

$$c_\circ = 2\pi |Q|^{1/2} (c_1 + c_2) \quad \text{if } E[X^2 \lg^+ |X|] < \infty.$$

The negative part of the integrand of the integral defining c_1 is integrable so that c_1 is well defined, whereas c_1 itself (possibly $+\infty$) is finite if and only if $E[X^2 \lg^+ |X|] < \infty$ (see [12]: proof of Theorem 1, p. 227); and c_2 does not depend on the choice of s. If $Q(\theta)$ is of the form $\sigma^2\theta^2$, then, on examining the proof of Proposition 12.3 of [11],

$$\pi\sigma^2 c_2 = \lg(\pi\sqrt{2}) - 2\varepsilon(2)/\pi,$$

where $\varepsilon(2) = \sum_{n=0}^\infty (-1)^n (2n+1)^2 = 0.9159\ldots$ (Cataran's constant) (see also REMARK 6 below). In the case $E[X^2 \lg^+ |X|] = \infty$ $c_\circ$ may be any number and in what follows we put $c_\circ = 0$ for definiteness.

Theorem 1.3. *Let $d = 2$ and $0 \le \delta \le 2$. Then, uniformly in $x \in \mathbf{Z}^2$, as $k \to \infty$*

$$f_x(k) = 2\pi |Q|^{1/2} a^*(x) e^{c_\circ} W(e^{c_\circ} k) \left[1 + o\left(k^{-\delta/2}\right)\right] + O\left(\frac{|x|_+^2}{k^2 \lg k}\right).$$

REMARK 4. $\lambda W(\lambda)$ admits the following asymptotic expansion in powers of $1/\lg\lambda$:

$$\lambda W(\lambda) = \frac{1}{(\lg\lambda)^2} - \frac{2\gamma}{(\lg\lambda)^3} - \frac{\frac{1}{2}\pi^2 - 3\gamma^2}{(\lg\lambda)^4} + \cdots$$

valid in the both limits as $\lambda \to \infty$ and as $\lambda \downarrow 0$, where $\gamma = -\int_0^\infty (\lg u) e^{-u} du$ (Euler's constant). The Fourier representation of $W(\lambda)$ takes the form

$$\frac{1}{2\pi} \int_{-\infty}^\infty \frac{e^{-i\lambda u}}{\lg(-iu)} du = \begin{cases} W(\lambda) & (\lambda > 0), \\ -e^\lambda & (\lambda < 0), \end{cases} \tag{1.4}$$

as is readily derived by Cauchy's theorem (cf. Appendix of [17]). A class of integrals containing that defining $W(\lambda)$ is studied by Ramanujan (cf. [8], sections 11.4 through 11.10 of Chapter XI, where it is in effect shown that $W(\lambda) = \int_0^\infty [\Gamma(x)]^{-1} \lambda^{x-1} dx - e^\lambda$).

According to the rule (1.3), a remedy for periodic walks, Theorem 1.3 has the following corollary (note that $c_\circ$ is well defined under irreducibility of the walk).

Corollary 1.2. *For simple random walk on $\mathbf{Z}^2$, it holds that $c_\circ = \lg 8$ (see* Remark 6 *below) and uniformly for $x = (x_1, x_2)$ with $x_1 + x_2 - k$ even,*

$$f_x(k) = 16\pi a^*(x) W(8k) + O\left(\frac{|x|_+^2}{k^2 \lg k}\right).$$

The asymptotic form of $f_x(k)$ as $|x|$ becomes large comparably to $\sqrt{k}$ is provided not by Theorem 1.3 but by Theorems 1.4 and 1.5 given below. (But it should be kept in mind that for $|x|$ much larger than $\sqrt{k}$, the trivial upper bound $f_x(k) \le p^k(x)$ may provide fairly nice estimates (see Remark 10 below).)

Theorem 1.4. *Let $d = 2$. Then as $k \wedge |x| \to \infty$, in general*

$$f_x(k) = \frac{2 \lg |\tilde{x}|}{k(\lg k)^2} e^{-\tilde{x}^2/2k} + \frac{1}{k \lg k} \cdot o\left(1 \wedge \frac{\sqrt{k}}{|x|}\right);$$

and if $\delta > 0$,

$$f_x(k) = \frac{\lg(\frac{1}{2} e^{c_\circ} \tilde{x}^2)}{k(\lg(e^{c_\circ} k))^2} e^{-\tilde{x}^2/2k} + \begin{cases} \dfrac{2\gamma \lg(k/\tilde{x}^2)}{k(\lg k)^3} + O\left(\dfrac{1}{k(\lg k)^3}\right) & \text{for } \tilde{x}^2 < k, \\ O\left(\dfrac{|\lg(\tilde{x}^2/k)|_+^2}{x^2(\lg k)^3}\right) & \text{for } \tilde{x}^2 \ge k. \end{cases} \tag{1.5}$$

Remark 5. If $d = 2$, $a(x)$ has the asymptotic form $(\pi |Q|^{1/2})^{-1} (\lg |\tilde{x}|)(1 + o(1))$ as $|x| \to \infty$, ensuring the consistency between Theorems 1.3 and 1.4. The second term on the right side of (1.5) in its first case is significant for properly evaluating the probability $\sum_{j \le k} f_x(j) = P[\tau_0^x \le k]$ so as to have its correct asymptotic form that turns out to be $(\lg k)^{-1} \int_0^k u^{-1} e^{-x^2/2u} du (1 + o(1))$ as $k \to \infty$ valid uniformly at least for $|x| \le \sqrt{3k \lg \lg k}$ (see [17] for more details).

In the formula (1.5) the estimate does not depend on $\delta > 0$, although it is best possible for $x^2 < k$. This is because the bottle neck for the estimate comes from a term that does not depend on $\psi(\theta)$ except via Q and $c_\circ$ (see Lemma 4.5). The situation becomes different if $f_x(k)$ is compared with the corresponding Brownian object

$$q_r(t, x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{K_0(|x| \sqrt{-i2u})}{K_0(r \sqrt{-i2u})} e^{-itu} du, \tag{1.6}$$

where $q_r(t, x)$ is the density for Brownian hitting time of $D(r)$ the disc of radius $r > 0$ centered at the origin and K_0 is the modified Bessel function of order 0 (see (6.7) of Appendix B) . Define the constant b_3 to be 1 if $\delta \ge 1$ and at least one of the third moments of X does not vanish, and to be 0 otherwise. The result is stated only in the case $\delta = 2$ (see Subsection **4.3** for more details).

Theorem 1.5. *Let $d = 2$, $\delta = 2$ and $r_\circ = \sqrt{2} e^{-(\gamma + c_\circ/2)}$. Then as $k \wedge |x| \to \infty$*

$$f_x(k) = q_{r_\circ}(k, \tilde{x}) + O\left(\frac{1}{k \lg k}\left(\frac{b_3}{|x|} + \frac{1}{|x|^2}\right)\right).$$

REMARK 6. Of the radius $r_\circ$ defined in Theorem 1.5 we have another formula

$$\pi\sqrt{|Q|}\,a(x) = \lg(|\tilde{x}|/r_\circ) + o(1) \quad \text{as} \quad |x| \to \infty, \tag{1.7}$$

provided $E[|X|^2 \lg^+ |X|] < \infty$. (This relation is essentially proved in [11] under the condition $E[|X|^{2+\delta}] < \infty$, $\delta > 0$; see also [12] and [4]; it also is a result of consistency between Theorems 1.3 and 1.4.) The function $\lg(|\tilde{x}|/r_\circ)$ equals apart from a constant factor the corresponding potential (Green's function with pole at infinity) of the process $Q^{1/2}B_t$ killed on the ellipse $Q^{1/2}D(r_\circ)$ where B_t is the standard two-dimensional Brownian motion. For the two dimensional simple random walk we know that $a(x) = 2\pi^{-1}\lg(\sqrt{8}e^\gamma |x|_+) + (8(x_1x_2)^2 - |x|_+^4)/6\pi|x|^6 + O(1/|x|_+^4)$ (cf. [4]). Comparing this with (1.7) and, noting $Q = \frac{1}{2}\mathrm{id}$ and $|\tilde{x}| = \sqrt{2}|x|$, we find that $r_\circ = e^{-\gamma}/2$ and $c_\circ = \lg 8$.

Asymptotic form of the distribution function $P[\tau_0^x \le n] = \sum_{k=1}^n f_x(k)$ is easily computed from Theorem 1.3 if x^2 is much smaller than n (it is sharper than one given in the next theorem if $n^{-1}x^2\lg|x|_+ \to 0$), while the corresponding computation based on Theorem 1.4 is somewhat complicated. The result becomes as follows.

Theorem 1.6. *Let $d = 2$ and $\delta > 0$ and write $\xi = |\tilde{x}|/\sqrt{n}$. Then as $n \wedge |x| \to \infty$,*

$$P[\tau_0^x \le n] = D(e^{c_\circ}n, \xi^2/2) + \frac{1}{(\lg n)^3} \times \begin{cases} O(|\lg \frac{1}{2}\xi|) & \text{for } \tilde{x}^2 < n, \\ O(|\lg 2\xi|^2/\xi^2) & \text{for } \tilde{x}^2 \ge n, \end{cases} \tag{1.8}$$

where

$$D(t,\alpha) = \frac{1}{\lg t}\left[1 - \frac{\gamma}{\lg t}\right]\int_\alpha^\infty \frac{e^{-u}}{u}du - \frac{1}{[\lg t]^2}\int_1^\infty \frac{e^{-\alpha u}}{u}\lg\left(1 - \frac{1}{u}\right)du, \quad \alpha > 0.$$

The following upper bounds are obtained as a corollary of Theorems 1.1 through 1.4.

Corollary 1.3. *For some constant C,*

$$f_x(k) \le \begin{cases} C\left(\dfrac{|x|_+}{k^{3/2}} \wedge \dfrac{1}{|x|_+^2}\right) & (d=1), \\ \dfrac{C}{k\lg k}\left(\dfrac{\lg(|x| \vee 2)}{\lg k} \wedge \dfrac{\sqrt{k}}{|x|_+}\right) & (d=2). \end{cases}$$

1.3. Suppose $d \ge 3$ and let e_x be the probability that the random walk starting at x escapes the origin after time 0:

$$e_x = P[S_k^x \ne 0 \text{ for all } k \ge 1],$$

and $G(x) = \sum_{n=0}^\infty P[S_n^0 = x]$. It holds that

$$1 - e_x = \frac{G(-x)}{G(0)}\ (x \ne 0) \ \text{ and } \ e_0 = \frac{1}{G(0)} \tag{1.9}$$

(cf. [11]) and $G(x) = O(1/|x|_+)$ for $d = 3$ (see [12] for $d \ge 4$).

Theorem 1.7. *Let $d \ge 3$. Then uniformly in $x \in \mathbf{Z}^d$, as $k \to \infty$*

$$f_x(k) \;=\; e_0\Big[\, e_x p^k(0) - p^k(0) + p^k(-x)\,\Big] \tag{1.10}$$
$$+\frac{1}{k^{1+d/2}} \times \begin{cases} O\Big(|x|_+ \wedge \dfrac{k}{|x|_+}\Big) + o\Big(k^{(1-\delta)/2}\Big) & \text{if } d=3, \delta \le 2, \\ (\lg k) \times O\Big(1 \wedge \dfrac{k^{1/2}}{|x|_+}\Big) & \text{if } d=4, \\ O\Big(1 \wedge \Big(\dfrac{k^{1/2}}{|x|_+}\Big)^{d-3}\Big) & \text{if } d \ge 5. \end{cases}$$

REMARK 7. If $x^2 = o(k)$, the leading term in (1.10) may be written as $e_0 e_x p^k(0)[1+O(x^2/k)]$; in view of (1.9) it is also written as $e_0 p^k(-x)[1+O(|x|^{-d+2})]$ for $0<|x|<M\sqrt{k}$ (for each $M>1$), provided that $|S_1^0|$ satisfies a sufficient moment condition so that $G(x) = O(|x|_+^{-d+2})$. It is noted that $a^*(x) = \mathbf{1}_{\{0\}}(x) + G(0) - G(-x) = e_x/e_0$, so the factor $a^*(x)$ persists to appear in the leading term (for the case $x^2 = o(k)$) in all dimensions.

For $d=3$ our actual estimation yields a better error estimate than that in (1.10): the first one (i.e. O term) may be explicit and the second one improved, and moreover it leads to the following

Theorem 1.8. *Let $d=3$, $0 \le \delta \le 2$ and $r_\circ = 1/2\pi|Q|^{1/2}G(0)$. Then uniformly in $x \in \mathbf{Z}^3$, as $k \to \infty$*

$$f_x(k) \;=\; \frac{r_\circ e^{-\tilde{x}^2/2k}}{k\sqrt{2\pi k}}\left[e_x + \frac{r_\circ|\tilde{x}|}{k} + \frac{b_3}{\sqrt{k}}\mathrm{P}_3\Big(\frac{-\tilde{x}}{\sqrt{k}}\Big) + \frac{O(1+x^4/k^2)}{k}\right]$$
$$+\frac{1}{k^{3/2}}\left[b_3 O\Big(\frac{1}{k} \wedge \frac{1}{|x|_+^2}\Big) + o\Big(\frac{1}{k^{\delta/2}} \wedge \frac{k}{|x|_+^{2+\delta}}\Big)\right], \tag{1.11}$$

where $\mathrm{P}_3(z)$ is the odd polynomial of degree 3 (involving the thirds moments of X) that appears in the Edgeworth expansion of $p^k(x)$; b_3 is defined as in Theorem 1.5 (i.e., $b_3 = 0$ if either $\delta < 1$ or all the third moments of X vanish and $b_3 = 1$ otherwise).

REMARK 8. The asymptotic form of $f_x(k)$ given in Corollary 1.1 is in good accordance with $(2\pi)^{-1/2}|x|t^{-3/2}e^{-x^2/2t}$, the density of corresponding distribution of the standard one dimensional Brownian motion started at $x \in \mathbf{R}$. In the higher dimensions $d \ge 2$ let $\mathbf{t}_r^{(d)}$ denote the Brownian hitting time of the ball of radius $r>0$ centered at the origin. The probability $f_x(k)$ is to be compared with the density of the distribution of $\mathbf{t}_r^{(d)}$. For $d=3$ it holds (see Appendix B) that for $|x|>r$,

$$P_x[\mathbf{t}_r^{(3)} \in dt]/dt = \frac{re^{-(|x|-r)^2/2t}}{t\sqrt{2\pi t}}\Big(1 - \frac{r}{|x|}\Big). \tag{1.12}$$

Taking $r = r_\circ$ exhibits a close similarity between this formula and that of Theorem 1.8. Indeed, if $\delta = 2$, the latter implies

$$f_x(k) = \frac{r_\circ e^{-(|\tilde{x}|-r_\circ)^2/2k}}{k\sqrt{2\pi k}}\left[e_x + \frac{b_3}{\sqrt{k}}\mathrm{P}_3\Big(\frac{-\tilde{x}}{\sqrt{k}}\Big)\right] + O\Big(\frac{1 \wedge (k/|x|_+^2)^2}{k^{5/2}}\Big);$$

we also know $e_x = 1-|\tilde{x}|^{-1}r_\circ + O(1/|x|_+^2)$ (cf. [12]). In the case $b_3 = 0$ this is because $e_x + r_\circ|\tilde{x}|/k = e^{r_\circ|\tilde{x}|/k}e_x + O(1/k)$ uniformly for $|\tilde{x}| \le k$; the case $b_3 = 1$ is dealt with in a similar way. Also note that $1 - r/|x|$ is the Brownian escape probability from the sphere.

REMARK 9. The proofs of Theorems stated above largely depend on the expansion of the characteristic function: $e^{it}\psi(\theta) = 1 + it - \frac{1}{2}Q(\theta) + o(|\theta|^{2+\delta}) + O(t^2 + |t|\theta^2)$ (if $0 \le \delta < 1$), and can be adapted for finding the asymptotic form of the hitting distribution to the first coordinate axis $x_1 = |x|e_1$ for the random walks that is biased to the direction e_1, the present problem of hitting time being the extreme case where the first coordinate of the walk deterministically increases by one at each step. (Cf. [13] for the unbiased case.)

REMARK 10. When x^2/k is large the trivial bound $f_x(k) \le p^k(x)$ may be useful as noted previously. For example, from the theorems above together with the estimate

$$p^k(x) = \frac{e^{-\tilde{x}^2/2k}}{|Q|^{1/2}(2\pi k)^{d/2}}\left[1 + O\left(\sum_{1\le j\le\delta}\frac{1+(x^2/k)^{1+j/2}}{k^{j/2}}\right)\right] + o\left(\frac{1}{k^{d/2}}\left[\frac{1}{k^{\delta/2}} \wedge \frac{k}{|x|^{2+\delta}}\right]\right) \qquad (1.13)$$

valid under (0.1) with any $\delta \ge 0$ (cf. [14]) one can readily deduce that as $|x| \to \infty$

$$\max_{k\ge1} f_x(k) \sim \begin{cases} [(3/e)^{3/2}/\sqrt{2\pi}]|\tilde{x}|^{-2} & \text{if } d = 1, \\ 8e^{-2}(|\tilde{x}|^2 \lg|x|)^{-1} & \text{if } d = 2,\ \delta > 0, \\ \left[(d/2\pi e)^{d/2}/|Q|^{1/2}G(0)\right]|\tilde{x}|^{-d} & \text{if } d \ge 3,\ \delta = d-2 \end{cases}$$

(the bound (1.13) is applied for $k < \tilde{x}^2/3\lg\lg|x|$ if $d = 2$ and for $k < \tilde{x}^2/6\lg|x|$ if $d \ge 3$).

The rest of the paper is organized as follows. In Section 2 we shall provide some preliminary formulae and lemmas which will be applied throughout Sections 3, 4 and 5, where we shall give proofs of Theorems above for the cases $d = 1$, $d = 2$ and $d \ge 3$, respectively. The final section consists of four appendices: we shall prove a lemma of Fourier analytic nature in the first one and give a simple comment on the formula (1.12) and a Brownian counterpart of Theorem 1.6 in the second and third ones, respectively.

2 Preliminary formulae and lemmas

Set

$$\pi_x(t) = \frac{1}{(2\pi)^d}\int_{T^d}\frac{e^{-ix\cdot\theta}}{1 - e^{it}\psi(\theta)}d\theta \qquad (t \ne 0, x \in \mathbf{Z}^d)$$

and

$$\hat{f}_x(t) = \sum_{k=1}^{\infty} f_x(k)e^{ikt}.$$

Since $p^n(x) = (2\pi)^{-d}\int_{T^d}[\psi(\theta)]^n e^{-ix\cdot\theta}d\theta$, we have

$$\pi_x(t) = \lim_{r\uparrow1}\sum_{n=0}^{\infty} p^n(x)e^{itn}r^n.$$

Substituting from the identity $p^n(-x) = \sum_{k=1}^{n} f_x(k)p^{n-k}(0)$ and making usual manipulation for the convolution sum, we infer that for $t \neq 0$, $\pi_{-x}(t) = \delta_{0x} + \hat{f}_x(t)\pi_0(t)$, or on solving for $\hat{f}_x(t)$,

$$\hat{f}_x(t) = -\frac{\delta_{0x}}{\pi_0(t)} + \frac{\pi_{-x}(t)}{\pi_0(t)}; \tag{2.1}$$

in particular

$$\hat{f}_0(t) = 1 - \frac{1}{\pi_0(t)}.$$

Note that $\pi_0(t)$ is smooth in $t \in T^1 \setminus \{0\}$ owing to the aperiodicity; also that since f_x is a probability supported on the positive integers we have three expressions of $f_x(k)$:

$$f_x(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{f}_x(t)e^{-ikt}dt = \frac{1}{\pi}\int_{-\pi}^{\pi} \hat{f}_x(t)\left\{\begin{array}{c} \cos kt \\ -i\sin kt \end{array}\right\} dt \tag{2.2}$$

among which we may choose one that is suitable to each occasion.

Bring in the functions $R_1(t,\theta)$ and $R_2(t,\theta)$ by

$$R_1 = \frac{1}{1-e^{it}\psi(\theta)} - \frac{1}{-it+1-\psi(\theta)} \quad \text{and} \quad R_2 = \frac{1}{-it+1-\psi(\theta)} - \frac{1}{-it+\frac{1}{2}Q(\theta)}$$

so that

$$\frac{1}{1-e^{it}\psi(\theta)} = \frac{1}{-it+\frac{1}{2}Q(\theta)} + R_1 + R_2. \tag{2.3}$$

Observe that $(-it+1-\psi) - (1-e^{it}\psi) = (e^{-it}-1)(1-e^{it}\psi) - (e^{-it}-1+it)$ to have

$$R_1 = \frac{\frac{1}{2}t^2 - r_3(t)}{[1-e^{it}\psi(\theta))][-it+1-\psi(\theta)]} + \frac{-it+r_2(t)}{-it+1-\psi(\theta)}, \tag{2.4}$$

where $r_2(t) = e^{-it} - 1 + it$ and $r_3(t) = r_2(t) + \frac{1}{2}t^2 = O(t^3)$ (the contributions of r_2, r_3 will be negligible in our analysis); also

$$R_2 = \frac{\psi(\theta)-1+\frac{1}{2}Q(\theta)}{(-it+\frac{1}{2}Q(\theta))^2} + \frac{\psi(\theta)-1+\frac{1}{2}Q(\theta)}{-it+\frac{1}{2}Q(\theta)}R_2. \tag{2.5}$$

The second fraction in (2.5) tends to zero as $|\theta| \to 0$ uniformly in t. Hence the first term on the right side is the principal term, i.e., R_n divided by it approaches 1 as $|t|+|\theta| \to 0$. Substituting into (2.4) from the defining expression of R_2 (to eliminate $\psi - 1 + it$) as well as from (2.3) yields

$$\begin{aligned} R_1 &= [2^{-1}t^2 - r_3(t)]\left(\frac{1}{-it+\frac{1}{2}Q(\theta)} + R_1 + R_2\right)\left(\frac{1}{-it+\frac{1}{2}Q(\theta)} - R_2\right) \\ &\quad + [-it+r_2(t)]\left(\frac{1}{-it+\frac{1}{2}Q(\theta)} - R_1\right) \\ &= \frac{\frac{1}{2}t^2}{[-it+\frac{1}{2}Q(\theta)]^2} + \frac{-it}{-it+\frac{1}{2}Q(\theta)} + R_3 \qquad (say). \end{aligned} \tag{2.6}$$

The next lemma (or its variants), stating elementary results, will be repeatedly used throughout the proofs of Theorems 1.1 to 1.8.

Lemma 2.1. *Let j and k be real constants such that $j > 0$ and $2k > -d$ and put $\alpha = j - k - d/2$. Then as $t \to 0\pm$*

$$\int_{T^d} \frac{[\frac{1}{2}Q(\theta)]^k d\theta}{(-it + \frac{1}{2}Q(\theta))^j} = \begin{cases} C_{\pm}^I |t|^{-\alpha} + O(1) & \text{if} \quad \alpha > 0 \\ \frac{1}{2}A \lg |t|^{-1} & \text{if} \quad \alpha = 0 \\ C^{II} + \eta(t) & \text{if} \quad \alpha < 0 \end{cases} \tag{2.7}$$

with

$$A = \frac{2^{d/2}|S^{d-1}|}{|Q|^{1/2}}, \quad C_{\pm}^I = A\int_0^\infty \frac{r^{2k+d-1}dr}{(\mp i + r^2)^j}, \quad C^{II} = \int_{T^d} \left[\frac{1}{2}Q(\theta)\right]^{k-j} d\theta$$

($|S^{d-1}|$ *is the hyper-area of $d-1$ dimensional unit sphere if $d \geq 2$ and* $|S^0| = 2$) *and for $\alpha < 0$,*

$$\eta(t) = \begin{cases} O(|t|^{|\alpha|}) & \text{if} \quad \alpha > -1 \\ O(t \lg |t|) & \text{if} \quad \alpha = -1 \\ O(t) & \text{if} \quad \alpha < -1. \end{cases}$$

Proof. Denote by $V(t)$ the integral to be estimated. In the case $\alpha > 0$ the change of variable (scaling θ by $\sqrt{|t|}$) then shows that

$$V(t) = \frac{1}{|t|^\alpha} \int_{T^d/\sqrt{|t|}} \frac{[\frac{1}{2}Q]^k d\theta}{(-i \operatorname{sgn} t + \frac{1}{2}Q)^j} = \frac{1}{|t|^\alpha} \int_{\mathbf{R}^d} \frac{[\frac{1}{2}Q]^k d\theta}{(-i \operatorname{sgn} t + \frac{1}{2}Q)^j} + O(1).$$

In the case $\alpha = 0$ we have only to replace the right most member above by $A \lg |t|^{-1/2}$.

Now consider the case $\alpha < 0$. It follows that $V(0) = C^{II} < \infty$ and $\eta(t) = V(t) - V(0) = \int_0^t V'(u)du$. The required estimates of η are then obtained by the result just proved (with j replaced by $j+1$) since it yields that $V'(t) = O(|t|^{-\alpha-1})$ if $0 < -\alpha < 1$, $V'(t) = O(\lg |t|)$ if $-\alpha = 1$ and $V'(t) = O(1)$ if $-\alpha > 1$. The proof of the lemma is complete.

In the first case of Lemma 2.1 the integral is unbounded and the order of growth as $t \to 0$ is found out by scaling the variable of integration by $\sqrt{|t|}$, while in the third case the integral is bounded and the convergence is trivial. In Lemma 2.1 the results are exhibited only on typical integrals of which there are many variants we shall encounter in the proofs of the main Theorems. In dealing with such variants, we shall refer to Lemma 2.1 even if it is not directly applicable but the adaptation is easy.

When $d = 2$ we shall need to evaluate integrals such that

$$I_j^s(k) := \int_0^a \frac{g(t)}{t^j |\lg t|^p} \sin kt \, dt \quad \text{and} \quad I_j^c(k) := \int_0^a \frac{g(t)}{t^j |\lg t|^p} \cos kt \, dt \quad (j = 0, 1)$$

where p is a positive constant, a is a constant from the unit open interval $(0, 1)$ and g is twice differentiable in $t > 0$. The way of computation involved in the proof of the following lemma will also be employed throughout the paper. The moral underlying therein will roughly be this: separate the integral near the origin and for the rest, perform integration by parts repeatedly until the integral becomes divergent if extended to the origin.

Lemma 2.2. *Let α be a constant such that $0 \le \alpha < 1$ and suppose that $g(a) = 0$, $g = O(t^\alpha)$ and $g'(t) = O(t^{\alpha-1})$ as $t \downarrow 0$. Then $I_1^s(k) = O(1/k^\alpha(\lg k)^p)$ (for $k \ge 2$); and if $\alpha > 0$, then $I_1^c = O(1/k^\alpha(\lg k)^p)$ (as $k \to \infty$) and if $p > 1$ and $\alpha = 0$, then $I_1^c = O(1/(\lg k)^{p-1})$. If $g''(t) = O(t^{\alpha-2})$ in addition, then $I_0^s(k) = O(1/k^{1+\alpha}(\lg k)^p)$ and $I_0^c(k) = O(1/k^{1+\alpha}(\lg k)^p)$.*

Proof. Splitting the range of integration and integrating by parts one obtains that

$$|I_0^s(k)| \le \int_0^{1/k} \frac{|g(t)|}{|\lg t|^p} \sin kt \, dt + \frac{|g(1/k)|}{k(\lg k)^p} + \left| \int_{1/k}^a \left(\frac{d}{dt} \frac{g(t)}{(-\lg t)^p} \right) \frac{\cos kt}{k} \, dt \right|.$$

On using $\sin kt \le 1$ the first integral on the right side is evaluated to be $O(1/k^{1+\alpha}(\lg k)^p)$. Integrating the second one by parts once more shows that the last integral equals

$$\left(\frac{d}{dt} \frac{g(t)}{(-\lg t)^p} \right) \frac{\sin kt}{k^2} \Bigg|_{t=1/k}^{a} - \int_{1/k}^a \left(\frac{d^2}{dt^2} \frac{g(t)}{(-\lg t)^p} \right) \frac{\sin kt}{k^2} \, dt,$$

which we evaluate (by using $|\sin kt| \le 1$) to be $O(k^{-1-\alpha}(\lg k)^{-p})$, the boundary contribution from a being $O(1/k^2)$, hence negligible since $\alpha < 1$. Thus the required estimate of $I_0^s(k)$ obtains. Estimations of $I_1^s(k)$ and $I_0^c(k)$ are made in the same way (except that for $I_1^s(k)$ we use the bound $\sin kt \le kt$ instead of $\sin kt \le 1$ in the case $\alpha = 0$). The evaluation of I_1^c is also made in the same way if $\alpha > 0$. If $\alpha = 0$ and $p > 1$, one has only to note that $\int_0^{1/k} |\lg t|^{-p} t^{-1} dt = (p-1)^{-1}(\lg k)^{-p+1}$, the integral on the interval $[\varepsilon/k, a)$ being evaluated, by integrating by parts as above, to be $O(1/|\lg k|^p)$, hence negligible.

The following argument or its modifications will also be made throughout the paper. For simplicity we consider $f_0(k)$ of the case $d = 2$. Let $w = w(t)$ be a function on $\mathbf{R}$ that is even, smooth, equal to 1 in a neighborhood of the origin and identically zero for $|t| \ge 1$. Employing (2.1) we write the first equality in (2.2) in the form

$$2\pi f_0(k) = -\int_{-\pi}^{\pi} w(t) \frac{e^{-ikt}}{\pi_0(t)} dt - \int_{-\pi}^{\pi} (1 - w(t)) \frac{e^{-ikt}}{\pi_0(t)} dt \qquad (k \ne 0).$$

Since $(1 - w(t))/\pi_0(t)$ may be regarded as a smooth (differentiable arbitrary times) function on the torus $\mathbf{R}/2\pi\mathbf{Z}$, the second integral gives a rapidly decreasing function of k. On using (2.3) the principal part of $1/\pi_0(t)$ takes on the form $-C/[\lg(-it) - c_\circ]$ ($C = 2\pi|Q|^{1/2}$) as we shall see in Section 4. Writing $h(t)$ for the remainder term and further decomposing the first integral above we find that

$$\begin{aligned} 2\pi f_0(k) &= \int_{-\infty}^{\infty} \frac{Ce^{-ikt}}{\lg(-it) - c_\circ} dt - \int_{-\infty}^{\infty} (1 - w(t)) \frac{Ce^{-ikt}}{\lg(-it) - c_\circ} dt \\ &\quad - \int_{-\pi}^{\pi} w(t) h(t) e^{-ikt} dt - \int_{-\pi}^{\pi} (1 - w(t)) \frac{e^{-ikt}}{\pi_0(t)} dt. \end{aligned} \tag{2.8}$$

The first integral represents the principal part. The second one as well as the last one is rapidly decreasing. Thus our task is to evaluate the third integral to reasonable accuracy.

3 The case d=1

3.1. Let $d=1$. We use the letter l ($|l|\le\pi$) instead of θ for Fourier parameter. Let $R_1(t,l)$ and $R_2(t,l)$ be the functions introduced in the preceding section and define $\lambda(t)$ by

$$\lambda(t)=-\frac{1}{2\pi}\int_{\mathbf{R}\setminus[-\pi,\pi)}\frac{dl}{-it+\frac{1}{2}Q(l)}+\frac{1}{2\pi}\int_{-\pi}^{\pi}(R_1(t,l)+R_2(t,l))dl$$

so that

$$\pi_0(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{dl}{-it+\frac{1}{2}Q(l)}+\lambda(t).$$

From the formula $\int_0^\infty(x^2+a^2)^{-1}dx=\pi/2a$ valid if $\Re a>0$ ($\Re$ designates the real part) we derive for $n=1,2,\dots$

$$\int_{-\infty}^{\infty}\frac{du}{(-it+u^2)^n}=\frac{\pi A_n}{(\sqrt{-it}\,)^{2n-1}},\tag{3.1}$$

where $A_1=1, A_2=\frac{1}{2}$ and in general $A_n=2^{-n+1}(2n-3)(2n-5)\cdots 1/(n-1)!$. This with $n=1$ and a simple change of variable of integration give

$$\pi_0(t)=\frac{1}{\sigma\sqrt{-2it}}+\lambda(t)\qquad(t\ne 0).\tag{3.2}$$

Moreover

$$\lambda(t)=\sigma^{-2}C^*+o\Big(|t|^{(\delta-1)/2}\Big)+O\Big(|t|^{1/2}\Big)\quad(t\to 0)\quad\text{if}\quad 0\le\delta\le 2,\tag{3.3}$$

as will be verified shortly. Here C^* is a (real) constant which may be arbitrary if $\delta<1$ (since then it may be absorbed in the first error term); in the case $\delta\ge 1$ it is given by

$$\frac{C^*}{\sigma^2}=-\frac{2}{\sigma^2\pi^2}+\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\psi(l)-1+\frac{1}{2}Q(l)}{\frac{1}{2}(1-\psi(l))Q(l)}dl,\tag{3.4}$$

where the last integral (understood to be the principal value at 0: see (3.5) below) arises as the limit as $t\to 0$ of $\int_{-\pi}^{\pi}R_2dl$ (it comes out as a constant in the third case of Lemma 2.1 if $\delta>1$). This constant agrees with that defined just before Theorem 1.2 owing to the identity $\int_0^\pi\Big[(1-\cos l)^{-1}-2l^{-2}\Big]dl=[-\cot y+y^{-1}]_{y=0+}^{\pi/2}=2/\pi$.

Proof of (3.3). In the expression defining $\lambda(t)$ the first integral defines a smooth function of t which is of the form $a_0+a_1t+\cdots$ with $a_0=-2/(\sigma\pi)^2$, and we have only to examine the second integral, of which one observes, using (2.4), that the contribution of R_1 to $\lambda(t)$ is $O(|t|^{1/2})$ (actually of the form $3\pi(2\sqrt{2|Q|})^{-1}(1-i\,\mathrm{sgn}\,t)|t|^{1/2}[1++O(|t|^{\delta\wedge 1})]$).

Let $\delta<1$. Then $\psi-1+\frac{1}{2}Q=o(|l|^{2+\delta})$ and an application of Lemma 2.1 (the first case) deduces that $\int_{-\pi}^{\pi}R_2dl=o(|t|^{(\delta-1)/2})$, which implies (3.3).

In the case $\delta=1$, we need to verify the convergence of $\int_{-\pi}^{\pi}R_2dl$ to $\int_{-\pi}^{\pi}R_2(0,l)dl$ as $t\to 0$. To this end we have only to deal with the first term of the expression (2.5) of R_2, for if $\delta=1$, $R_2=O(1/l)$

so that the second one is bounded. By symmetry $E[\sin Xl]$ involved in ψ then can be deleted from the integrand. Now the dominated convergence theorem concludes that the integral thus modified converges to the constant

$$\int_{-\pi}^{\pi} \frac{\psi(l)-1+\frac{1}{2}Q(l)}{(\frac{1}{2}Q(l))^2} dl = E\left[|X|^3 \int_{-|X|\pi}^{|X|\pi} \frac{\cos u - 1 + \frac{1}{2}u^2}{(\frac{1}{2}Q(u))^2} du\right] < \infty. \tag{3.5}$$

Combined with the bounds $|E[\sin Xl]| = O(l^3)$ and $\psi(l)-\psi(-l) = O(l^3)$ this in particular verifies the existence of the integral in (3.4).

If $1<\delta<2$, then $\psi - 1 + \frac{1}{2}Q = -i\beta_3 l^3 + o(|l|^{2+\delta})$, and (3.3) follows from (2.5) and Lemma 2.1 (the third case). It is readily seen that $\lambda(t) = \sigma^{-2}C^* + O(\sqrt{|t|})$ if $\delta = 2$. Thus (3.3) has been proved.

We write down the estimate (3.3) in the following form

$$\frac{1}{\pi_0(t)} = \sigma\sqrt{-2it} + i2C^* t + o\left(|t|^{(1+\delta)/2}\right) + O(|t|^{3/2}) \qquad (t \to 0). \tag{3.6}$$

3.2. Here the asymptotic estimate of $f_0(k)$ is obtained. Let $0 \le \delta \le 2$. It follows that as $t \to 0$,

$$(d/dt)^j \lambda(t) = o(|t|^{(\delta-1)/2}|t|^{-j}) + O(|t|^{-j+1/2}) \quad \text{for} \quad j = 1,2,3. \tag{3.7}$$

For the proof we have only to consider the second integral of the defining expression of λ for the same reason as noted in the proof of (3.3). By applying Lemma 2.1 (the first case only) it is readily obtained that $\int \partial_t^j R_2 dl = O(|t|^{-j+1/2})$. Also we evaluate $\int \partial_t^j R_1 dl$ to be $o(|t|^{(\delta-1)/2}|t|^{-j})$ if $0 \le \delta < 2$ and $O(|t|^{-j+1/2})$ if $\delta = 2$; here one needs to note that if $\delta = 1$, the even part of $\psi(l) - 1 + \frac{1}{2}Q(l)$ is $o(l^3)$ and the odd part makes no contribution.

From (3.2) and (3.7) we especially obtain that

$$\pi_0'(t) = \frac{i}{\sigma(\sqrt{-2it}\,)^3}\left(1 + o(|t|^{\delta/2}) + O(t)\right). \tag{3.8}$$

By (2.1) $\hat{f}_0(t) = 1 - 1/\pi_0(t)$ and if $\eta(t)$ is defined by

$$\hat{f}_0'(t) = \frac{\pi_0'(t)}{[\pi_0(t)]^2} = \frac{i\sigma}{\sqrt{-2it}} + i2C^* + \eta(t),$$

we infer from (3.6), (3.7) and (3.8) that for $j = 0,1,2.$,

$$(d/dt)^j \eta(t) = o(|t|^{(\delta-1)/2})|t|^{-j} + O(|t|^{-j+1/2}).$$

Let $w(t)$ be a smooth cutoff function introduced at the end of Section 2. Then by Fourier inversion and integration by parts

$$\begin{aligned} f_0(k) &= \frac{1}{\pi}\int_{-\pi}^{\pi} \hat{f}_0(t)\cos kt\,dt \\ &= -\frac{1}{\pi}\int_{-\pi}^{\pi} \hat{f}_0'(t)\frac{\sin kt}{k}dt = -\frac{1}{\pi}\int_{-\pi}^{\pi} w(t)\hat{f}_0'(t)\frac{\sin kt}{k}dt + \varepsilon(k) \\ &= -\frac{\sigma}{\pi k}\int_{-\pi}^{\pi} \frac{iw(t)}{\sqrt{-2it}}\sin kt\,dt - \frac{1}{\pi k}\int_{-\pi}^{\pi} w(t)\eta(t)\sin kt\,dt + \varepsilon(k) \\ &= K_1 + K_2 + \varepsilon(k) \quad \text{(say)}. \end{aligned}$$

Here, as well as in what follows, $\varepsilon(k)$ denotes any function approaching zero faster than k^{-N} as $k \to \infty$ for all N that needs not be the same at each occurrence. On using $\int_0^\infty t^{-1/2} \sin t\, dt = \sqrt{\pi/2}$ and $1/\sqrt{-2it} = (1 + i \operatorname{sgn} t)/2\sqrt{|t|}$,

$$K_1 = \frac{\sigma}{\pi k} \int_{-\pi}^{\pi} \frac{-i \sin kt}{\sqrt{-2it}} w(t) dt = \frac{\sigma}{\sqrt{2\pi}} \cdot \frac{1}{k\sqrt{k}} + \varepsilon(k). \tag{3.9}$$

For $0 \le \delta < 1$, the estimation of K_2 is carried out as in the proof of Lemma 2.2 and it is found that $K_2 = o((1/\sqrt{k})^{3+\delta})$. In the case $1 \le \delta < 2$ we perform integration by parts once more and use $\eta''(t) = o(|t|^{(-5+\delta)/2})$ to have the desired estimate. Similarly, K_2 is easily evaluated to be $O(k^{-5/2})$ if $\delta = 2$. Thus we have shown

Proposition 3.1. *For $0 \le \delta \le 2$, as $k \to \infty$*

$$f_0(k) = \frac{\sigma}{\sqrt{2\pi}} \cdot \frac{1}{k\sqrt{k}} \left(1 + o\left(\frac{1}{k^{\delta/2}}\right) + O\left(\frac{1}{k}\right)\right).$$

3.3. In this subsection Theorem 1.1 is proved when $0 \le \delta < 1$. (The case $1 \le \delta \le 2$ will be treated in the next subsection.) Recalling $\hat{f}_x(t) = \pi_{-x}(t)/\pi_0(t)$ $(x \neq 0)$ we introduce

$$\mathrm{e}_x(t) := \pi_{-x}(t) - \pi_0(t) + a(x)$$

so that

$$\hat{f}_x(t) = \frac{\mathrm{e}_x(t)}{\pi_0(t)} + 1 - \frac{a^*(x)}{\pi_0(t)}.$$

The integral representation $a(x) = (2\pi)^{-1} \int_{-\pi}^{\pi} (1-\psi)^{-1}(1 - e^{ixl}) dl$ yields

$$\mathrm{e}_x(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{1}{1 - e^{it}\psi(l)} - \frac{1}{1 - \psi(l)} \right) (e^{ixl} - 1)\, dl.$$

We make the decomposition $(2\pi)\mathrm{e}_x(t) = \mathrm{c}_x(t) + i\, \mathrm{s}_x(t)$, where

$$\mathrm{c}_x(t) = \int_{-\pi}^{\pi} \left(\frac{1}{1 - e^{it}\psi(l)} - \frac{1}{1 - \psi(l)} \right) (\cos xl - 1) dl$$

$$\mathrm{s}_x(t) = \int_{-\pi}^{\pi} \left(\frac{1}{1 - e^{it}\psi(l)} - \frac{1}{1 - \psi(l)} \right) \sin xl\, dl.$$

Lemma 3.1. *There exists a constant C such that $|\mathrm{c}_x^{(j)}(t)| \le C x^2 |t|^{-j+1/2}$ for $j = 0, 1, 2$. (Here $\mathrm{c}^{(j)}$ denotes the j-th derivative w.r.t. t.)*

Proof. Writing

$$\mathrm{c}_x(t) = \int_{-\pi}^{\pi} \frac{(e^{it} - 1)\psi(l)}{1 - e^{it}\psi(l)} \cdot \frac{\cos xl - 1}{(1 - \psi(l))} dl \tag{3.10}$$

one applies Lemma 2.1 together with $|1-e^{it}\psi| \ge C^{-1}(|t|+l^2)$ to see that $|\mathrm{c}_x(t)|$ is dominated by $Cx^2|t|\int_{-\pi}^{\pi}(|t|+l^2)^{-1}\,dl \le C'x^2\sqrt{|t|}$, hence the asserted bound of c_x. Differentiating the defining expression of c_x we have

$$\mathrm{c}_x'(t) = ie^{it}\int_{-\pi}^{\pi}\frac{\psi(l)(\cos xl-1)}{[1-e^{it}\psi(l)]^2}\,dl,$$

which yields the second bound of the lemma similarly to the above. The third one is similar.

Lemma 3.2. *Let $0 \le \delta < 1$. Then $\mathrm{s}_x^{(j)}(t)/x = o(|t|^{-j+\delta/2})$ as $t \to 0$ uniformly in $x \ne 0$ for $j = 0, 1, 2$.*

Proof. Write

$$\begin{aligned} I_x(t) := \frac{1}{x}\int_{-\pi}^{\pi}\frac{\sin xl\,dl}{1-e^{it}\psi(l)} &= \frac{1}{2}\int_{-\pi}^{\pi}\left[\frac{1}{1-e^{it}\psi(l)} - \frac{1}{1-e^{it}\psi(-l)}\right]\frac{\sin xl}{x}\,dl \\ &= ie^{it}\int_{-\pi}^{\pi}\frac{lE[\sin Xl]}{(1-e^{it}\psi(l))(1-e^{it}\psi(-l))}\cdot\frac{\sin xl}{xl}\,dl. \end{aligned} \tag{3.11}$$

Putting $\Lambda(l) = E[\sin Xl]/l^3$ so that $lE[\sin Xl] = l^4\Lambda(l)$ one observes that $\Lambda(l)$ is integrable and then applies the dominated convergence theorem to see that $\mathrm{s}_x(t)/x = I_x(t) - I_x(0) \to 0$ as $t \to 0$ uniformly in x. We also have $\Lambda(l) = o(|l|^{\delta-1})$ and $|1-e^{it}\psi(l)| \ge C^{-1}(|t|+l^2)$ and, employing Lemma 2.1 (the first case), obtain $\mathrm{s}_x(t)/x = o(|t|^{\delta/2})$. Thus the first estimate of the lemma has been verified. The other two are readily shown by differentiating the last expression of $I_x(t)$ and applying the estimates just obtained together with the inequality $|1-e^{it}\psi(l)| \ge C^{-1}|t|$.

In the case $0 \le \delta < 1$ Theorem 1.1 is proved by the same argument as made in the proof of Proposition 3.1 with the help of it as well as of Lemmas 3.1 and 3.2. The details are omitted.

3.4. We prove Theorem 1.1 in the case $1 \le \delta \le 2$. We need to make more detailed estimation of c_x and s_x than we have made above. We continue to suppose $x \ne 0$.

Lemma 3.3. *Let $\delta = 1$. Then uniformly in $x \ne 0$, as $k \to \infty$*

$$\int_{-\pi}^{\pi}\frac{c_x(t)}{\pi_0(t)}e^{-ikt}\,dt = O\left(\frac{x^3}{k^2\sqrt{k}}\right).$$

Proof. Rewrite the expression of c_x in (3.10) in the form

$$c_x(t)/x^2 = -i2\pi t\pi_0(t)/\sigma^2 + r_1(t) + r_2(t),$$

where

$$r_1 = \int_{-\pi}^{\pi}\frac{(e^{it}-1)\psi(l)-it}{1-e^{it}\psi(l)}\cdot\frac{\cos xl-1}{x^2(1-\psi(l))}\,dl$$

and

$$r_2 = \int_{-\pi}^{\pi}\frac{it}{1-e^{it}\psi(l)}\left[\frac{\cos xl-1}{x^2(1-\psi(l))} + \frac{1}{\sigma^2}\right]dl.$$

Write $(e^{it}-1)\psi - it = (e^{it}-1-it)\psi - it(1-\psi)$. Then, using $1-e^{it}\psi = (-it+\frac{1}{2}Q)(1+o(1))$, we deduce that $|r_1| \le C(|t|/|x|)\int_0^\infty (1-\cos u)u^{-2}du$ for some constant C; similarly, by using

$$|1-e^{it}\psi(l)|^{-1-j} = O(|l|^{-2}|t|^{-j} \wedge |t|^{-j-1}), \tag{3.12}$$

we obtain bounds for the j-th derivatives, yielding

$$r_1 = O(t/x), \quad r_1' = O(1/x), \quad r_1'' = O(1/xt), \quad r_1''' = O(1/xt^2).$$

Also observe that

$$\begin{aligned} r_2 &= it\int_{-\pi}^{\pi} \frac{\cos xl - 1 + \frac{1}{2}(xl)^2}{x^2(1-e^{it}\psi)(1-\psi)}dl - it\int_{-\pi}^{\pi} \frac{\psi - 1 + \frac{1}{2}Q}{(1-e^{it}\psi)(1-\psi)\sigma^2}dl \\ &= O(xt) + O(t), \end{aligned} \tag{3.13}$$

where the first and second terms in the last line represent the corresponding ones in the preceding line and we have used the integrability $\int_0^\infty |\cos l - 1 + \frac{1}{2}l^2|l^{-4}dl < \infty$ for the first integral and (3.5) for the second (due to the condition $\delta = 1$), and similarly that

$$|r_2'| \le C|x|, \quad |r_2''| \le C|x/t|, \quad |r_2'''| \le C|x/t^2|.$$

(Use (3.12) for the first integral; apply Lemma 2.2 along with $\delta = 1$ for the second .) Now it is easy to see

$$\begin{aligned} \int \frac{c_x}{\pi_0}e^{-ikt}dt = \frac{1}{ik}\int \left(\frac{c_x}{\pi_0}\right)' e^{-ikt}dt &= \frac{x^2}{ik}\int \left(\frac{r_1+r_2}{\pi_0}\right)' e^{-ikt}dt \\ &= -\frac{x^2}{k^2}\int \left(\frac{r_1+r_2}{\pi_0}\right)'' e^{-ikt}dt. \end{aligned}$$

The integrand of the last integral is $O(x/\sqrt{|t|})$ and in the same way as in the proof of Lemma 2.2 the integral itself is shown to be $O(x/\sqrt{k})$. Thus we conclude the assertion of the lemma.

Lemma 3.4. *Let $1 \le \delta \le 2$. Then uniformly in $x \ne 0$, as $k \to \infty$*

$$\int_{-\pi}^{\pi} \frac{s_x(t)}{\pi_0(t)}e^{-ikt}dt = o\left(\frac{x}{k^{(3+\delta)/2}}\right) + O\left(\frac{x^2}{k^2\sqrt{k}}\right).$$

Proof. Recalling (3.11) and $s_x/x = I_x(t) - I_x(0)$ we write s_x/x in the form

$$\frac{s_x(t)}{x} = ie^{it}\int_{-\pi}^{\pi} \frac{F(t,l)l^4\Lambda(l)}{(1-e^{it}\psi(l))(1-e^{it}\psi(-l))(1-\psi(l))(1-\psi(-l))} \cdot \frac{\sin xl}{xl}\,dl, \tag{3.14}$$

where $\Lambda(l) = E[\sin Xl]/l^3$ (as before) and

$$F(t,l) = (1-\psi(l))(1-\psi(-l)) - e^{-it}(1-e^{it}\psi(l))(1-e^{it}\psi(-l)).$$

Observing $F = 1 - e^{-it} + (1-e^{it})\psi(l)\psi(-l)$ as well as $\psi(l)\psi(-l) = 1 - 2E[1-\cos Xl] + O(l^6)$, one obtains the expansion

$$F = t^2 + itQ(l) + O(t^2l^2 + |t||l|^{2+\delta}) \tag{3.15}$$

and those of the derivatives $\partial_t^j F$; the contribution of the error terms of F in (3.15) to the integral above is readily seen to be $O(|t|^{(1+\delta)/2})$. In the denominator of the integrand in (3.14) the first two factors $1 - e^{it}\psi(\pm l)$ and the remaining two $1 - \psi(\pm l)$ may then be replaced by $-it + Q(l)/2$ and $Q(l)/2$, respectively, the error caused by the replacement being negligible. We wish further replace $\sin xl/xl$ by 1. The error by this replacement is shown to be $O(xt)$ in the same way as r_2 is estimated in the preceding proof but this time using $\int_0^\infty |\sin u - u|u^{-3}du < \infty$. Finally note that $\Lambda(l) = \beta_3 + o(|l|^{\delta-1})$. These considerations then lead to

$$\frac{\mathrm{s}_x(t)}{x} = \frac{i4\beta_3}{\sigma^4}\int_{-\pi}^{\pi} \frac{t^2 + itQ(l)}{(-it + \frac{1}{2}Q(l))^2}\,dl + O(|xt|) + o(|t|^{\delta/2}), \tag{3.16}$$

provided that $1 \le \delta < 2$. Here the last error term comes from the replacement of $\Lambda(l)$ by β_3. If $\delta = 2$, it may be replaced by $O(t)$ (hence superfluous), as assured by the inequality $\int E|\sin Xl - Xl + \frac{1}{6}(Xl)^3||l|^{-5}dl \le CE|X|^4$. Also the factor e^{it} that exists in (3.14) is replaced by 1 in (3.16), causing only the error of the magnitude $O(|t|^{3/2})$.

Differentiating the last expression of I_x in (3.11), we derive in the same way as above that for $t \ne 0$,

$$\frac{\mathrm{s}_x'(t)}{x} = -2\beta_3\int_{-\pi}^{\pi} \frac{l^4}{(-it + \frac{1}{2}Q(l))^3}\,dl + O(|x|) + o(|t|^{(\delta-2)/2}). \tag{3.17}$$

From the formula (3.1) (with $n = 1, 2, 3$) it follows that for any complex numbers α, β,

$$\int_{-\infty}^{\infty} \frac{\alpha t^2 + \beta t u^2}{(-it + u^2)^2}du = A_{\alpha,\beta}\sqrt{-i2t} \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{\alpha t u^2 + \beta u^4}{(-it + u^2)^3}du = B_{\alpha,\beta}\frac{1}{\sqrt{-i2t}}$$

where $A_{\alpha,\beta}$ and $B_{\alpha,\beta}$ are certain complex numbers whose values are not important for our present purpose. A simple computation then deduces from (3.16), (3.17), (3.2) and (3.8) that

$$\left(\frac{\mathrm{s}_x}{x\pi_0}\right)'(t) = \alpha_\circ + r(t)$$

with some complex number $\alpha_\circ$ and the remainder term $r(t) = O\left(x\sqrt{|t|}\right) + o(|t|^{(\delta-1)/2})$, so that

$$\int_{-\pi}^{\pi} \frac{\mathrm{s}_x(t)}{\pi_0(t)}e^{-ikt}dt = \frac{x}{ik}\int_{-\pi}^{\pi} r(t)e^{-ikt}dt = O\left(\frac{x^2}{k^{5/2}}\right) + o\left(\frac{x}{k^{(\delta+3)/2}}\right),$$

where the estimation of the last integral is carried out by estimating the derivatives r' and r'' as those of the corresponding ones in the preceding proof. The proof of the lemma is complete.

Theorem 1.1 is now immediate from Lemmas 3.3 and 3.4 and Proposition 3.1.

3.5. Here we give a proof of Theorem 1.2. We apply the Fourier inversion formula (2.2) as before, but, unlike the proof of Theorem 1.1, here we make no use of the decomposition of $\hat{f}_x$ given in **3.3** and rather directly evaluate the Fourier integral in (2.2). We suppose $0 \le \delta < 3$.

We truncate the Fourier integral by a smooth cutoff function $w(t)$ as in **3.2**, with the remainder term (the contribution of $1 - w$) being plainly negligible. Here we also truncate the l-integral (i.e., the integral w.r.t. the variable l) by $w(l)$ and define

$$\lambda_x(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{(1-w)e^{ixl}dl}{1 - e^{it}\psi(l)} - \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{(1-w)e^{ixl}dl}{-it + \frac{1}{2}Q(l)} + \frac{1}{2\pi}\int_{-\pi}^{\pi} (R_1 + R_2)we^{ixl}dl, \tag{3.18}$$

so that

$$\pi_x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\cos xl\,dl}{-it+\frac{1}{2}Q(l)} + \lambda_x(t). \tag{3.19}$$

The last integral can be explicitly computed (see (3.28)). The first and second terms on the right side of (3.18) make only a negligible contribution to $f_x(k)$ (for the first one we use Lemma 6.1 in Appendix A; see the discussion around (3.26)), so that

$$f_x(k) = \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi} \frac{w(t)e^{-ikt}}{\pi_0(t)}dt\left[\int_{-\infty}^{\infty} \frac{\cos xl\,dl}{-it+\frac{1}{2}Q(l)} + \int_{-\pi}^{\pi}(R_1+R_2)w(l)e^{ixl}dl\right] + \mathrm{ngl}, \tag{3.20}$$

where the ngl designates the remainder term that is smaller or the same order of magnitude compared to the required error estimate. From (2.6) one sees that the integral $\int_{-\pi}^{\pi} R_1we^{ixl}dl$ would be much easier to evaluate than $\int_{-\pi}^{\pi} R_2we^{ixl}dl$ (see the remark at the end of the next paragraph), and we shall concentrate on the latter in what follows.

Let $b_4 = \frac{1}{24}E[X^4]$ if $\delta = 2$ and $b_4 = 0$ otherwise. Decompose

$$R_2 = \frac{\psi - 1 + \frac{1}{2}Q}{(-it+\frac{1}{2}Q(l))^2} + \frac{(\psi - 1 + \frac{1}{2}Q)^2}{(-it+\frac{1}{2}Q(l))^3} + \left[\frac{\psi - 1 + \frac{1}{2}Q}{-it+\frac{1}{2}Q(l)}\right]^2 R_2,$$

and write $R_2 = T + U + V$, where $T = T(t,l)$, $U = U(t,l)$ are defined by

$$T = \frac{(-i\beta_3 l^3 + b_4 l^4)w(l)}{(-it+\frac{1}{2}Q(l))^2} + \frac{-\beta_3^2 l^6 w(l)}{(-it+\frac{1}{2}Q(l))^3}, \tag{3.21}$$

$$U = \frac{\psi(l) - 1 + \frac{1}{2}Q(l) + i\beta_3 l^3 - b_4 l^4}{(-it+\frac{1}{2}Q(l))^2}w(l)$$

and V is the rest. Put $T_x^{\wedge}(t) = \int_{-\pi}^{\pi} T(t,l)e^{ixl}dl$ etc., so that

$$\int_{-\pi}^{\pi} R_2we^{ixl}dl = T_x^{\wedge}(t) + U_x^{\wedge}(t) + V_x^{\wedge}(t).$$

Changing the variable of integration, writing $\tilde{w}(l) = w(l/\sigma)$ and appropriately arranging the terms we have

$$T_x^{\wedge}(t) = \int_{-\sigma\pi}^{\sigma\pi} \frac{-4i\beta_3 l^3\tilde{w}(l)e^{i\tilde{x}l}\,dl}{(-2it+l^2)^2\sigma^4} + \int_{-\sigma\pi}^{\sigma\pi} \frac{4\sigma^2 b_4 l^4(-2it+l^2) - 8\beta_3^2 l^6}{(-2it+l^2)^3\sigma^6}\tilde{w}(l)e^{i\tilde{x}l}\,\frac{dl}{\sigma}. \tag{3.22}$$

The evaluation of the contribution to $f_x(k)$ of the two integrals on the right side will be made by rather explicit computations as given shortly. For the evaluation of the error term we need some estimates of $U_x^{\wedge}(t)$ and $V_x^{\wedge}(t)$. To this end we shall consider only $U_x^{\wedge}(t)$, $V_x^{\wedge}(t)$ being much easier to deal with. It is incidentally remarked that the contribution of the leading two terms appearing in the expression (2.6) of R_1 is comparable to the second integral in (3.22); that of R_3 defined there is evaluated to be negligible similarly to $V^{\wedge}(t)$.

Computation of the error term corresponding to U. We begin with easy estimates of $U_x^\wedge(t)$ that lead to the second assertion of Theorem 1.2. Put

$$r(l)=\psi(l)-1+2^{-1}Q(l)+i\beta_3 l^3-b_4 l^4.$$

Then the j-th derivative $r^{(j)}(l)$ is $o(|l|^{2+\delta-j})$ if $j\le 2+\delta$ and it follows from it that

$$U_x^\wedge(t)=\int_{-\pi}^{\pi}\frac{r(l)}{(-it+\frac12 Q(l))^2}w(l)e^{ixl}dl=\frac{o(|t|^{(\delta-1-j)/2})}{|x|_+^j}\quad\text{if}\quad j\le\delta<j+1. \tag{3.23}$$

(Perform integration by parts j times and apply Lemma 2.1.) We can differentiate w.r.t. t without any cost, so that

$$(d/dt)^m U_x^\wedge(t)=\frac{o(|t|^{-m+(\delta-1-j)/2})}{|x|_+^j}\quad\text{if}\quad j\le\delta<j+1+2m. \tag{3.24}$$

Recalling the estimates of the derivatives of $1/\pi_0=1-\hat f_0$ obtained in **3.2** as well as $1/\pi_0(t)=\sigma\sqrt{-2it}+\cdots$, we apply the method used for Lemma 2.1 to conclude that for $j=0,1,2$,

$$\int\frac{U_x^\wedge(t)}{\pi_0(t)}w(t)e^{ikt}dt=\frac{o(k^{-1-(\delta-j)/2})}{|x|_+^j}\qquad(j\le\delta<j+1). \tag{3.25}$$

Note that the formula (3.24) may become false if $\delta\ge j+1$; in particular if $1\le\delta<2$, we must not to take $j=0$ on the right side of (3.25).

In the case $2\le\delta<3$, the estimate (3.25) is not satisfactory, being not sharp when $x^2=o(k)$. It may be natural that the integration by parts is made just once w.r,t. each variables l and t, which yields

$$\int\frac{U_x^\wedge(t)}{\pi_0(t)}w(t)e^{ikt}dt=\frac{1}{kx}\int\frac{i\sigma w(t)}{\sqrt{-2it}}e^{-ikt}dt\int_{-\pi}^{\pi}\left(\frac{d}{dl}\frac{r(l)w(l)}{(-it+\frac12 Q(l))^2}\right)e^{ixl}dl+o\left(\frac{1}{k^{3/2}|x|^{\delta-1}}\right).$$

Here we have repeated the same argument made right above by using (3.24) with $j=1,m=1$ for obtaining the error term. The double integral on the right side is evaluated by using Lemma 6.2 (see Remark in Appendix A)) to yield the error term asserted at the end of Theorem 1.2.

As for the error estimate $o(|x|^{-2-\delta})$ of Theorem 1.2 we are to employ Lemma 6.1 in Appendix A. Remember that we have the three expressions of $f_x(k)$ given in (2.2). Here we use the last one of them because of the better estimate of the second formula in Lemma 6.1. The estimate that we need to verify may accordingly be written as

$$I(x,k):=\int_{-\pi}^{\pi}\frac{U_x^\wedge(t)}{\pi_0(t)}w(t)\sin kt\,dt=o(|x|^{-2-\delta}). \tag{3.26}$$

(Synchronously we must replace e^{-ikt} by $-2i\sin kt$ in (3.20), which causes no problem: see a remark after (3.29).) For the proof of (3.26) suppose $\delta>0$ and let m be the non-negative integer such that $m<\delta\le m+1$ so that $r(l)$ is differentiable $m+2$ times but may not be $m+3$ times. Then performing integration by parts $m+2$ times for the integral that defines $U_x^\wedge$ (see (3.23)) results in

$$I(x,k)=\frac{1}{x^{m+2}}\sum_{j=0}^{m+2}\int_{-\pi}^{\pi}\frac{1}{\pi_0(t)}\sin kt\,dt\int_{-\pi}^{\pi}\frac{v_j(l)}{(-it+\frac12 Q(l))^{2+j}}e^{ixl}dl+\text{ngl},$$

where $v_j = c_j l^j r^{(m+2-j)}(l)$ ($j = 0, \ldots, m+2, c_j$ are certain real constants) and the ngl arises by differentiation of $w(l)$. If v_j's satisfy the condition of Lemma 6.1 with $\nu = \delta - m + 2$ (and $\beta = 2j$), then applications of Lemma 6.1 then will lead to the desired estimate. That this premise holds true follows from $r^{(m+2)}(l+h) - r^{(m+2)}(l) = E[(iX)^{m+2}e^{ilX}(e^{ihX}-1)] = o(|h|^\nu)$ ($h \to 0$) (see Remark in Appendix A). The case $\delta = 0$ remains to be considered, but this case is directly treated by integrating by parts twice for the l-integral and then proceeding as in the proof of Lemma 2.1 for the t-integral (here also we need to work not with $\cos kt$ but with $\sin kt$).

Derivation of the principal part. In order to include the case $\delta \geq 2$ we need to know a precise function form corresponding to $O(|t|^{3/2})$ in the expansion of $1/\pi_0$ given in (3.6). This turns out to be of the form $B_4(-i2t)^{3/2}$ (if $\delta \geq 2$), where B_4 is some constant. We shall present this fact as Lemma 3.5 at the end of this section. For simplicity we suppose $\delta \geq 2$ and in the expression (3.20) we substitute the expansion of $1/\pi_0$ given in Lemma 3.5. That the contribution of the error term in (3.6) is negligible is shown in the same way as before. Also we may replace R_1 by T defined in (3.21) and R_2 by the sum of the two leading terms on the right side of (2.6) as discussed above. Now in the double integral that then comes out we replace two w's by 1 and extend the range of integration to the whole real line for both the inner and outer integrals, which results in

$$\begin{aligned} f_x(k) &= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\left[\sigma\sqrt{-2it} + i2C^*t + B_4(-2it)^{3/2}\right]e^{-ikt}dt \\ &\quad \times \int_{-\infty}^{\infty}\left[\frac{2\cos\tilde{x}l}{-i2t+l^2} + \frac{-4i\beta_3 l^3}{(-i2t+l^2)^2\sigma^3} + K + L\right]e^{i\tilde{x}l}\frac{dl}{\sigma} + \text{ngl}, \end{aligned} \tag{3.27}$$

where K denotes the second fraction appearing in (3.22) and L the sum of the two terms from (2.6) but with $Q(l)$ replaced by l^2.

The evaluation of the double integral above is performed by elementary calculus based on the following formulae: for $\alpha > 0$ and $y > 0$,

$$\int_{-\infty}^{\infty}\frac{\cos yl}{-i2\alpha t + l^2}dl = \frac{\pi}{\sqrt{-i2\alpha t}}e^{-y\sqrt{-i2\alpha t}} \tag{3.28}$$

and

$$\int_{-\infty}^{\infty}\frac{1}{\sqrt{-i2\alpha t}}e^{-y\sqrt{-i2\alpha t}}e^{-ikt}dt = \begin{cases} \dfrac{\sqrt{2\pi}}{\sqrt{\alpha k}}e^{-\alpha y^2/2k} & (k>0), \\ 0 & (k \leq 0). \end{cases} \tag{3.29}$$

The latter formula is the Laplace inversion of the well known formula for the resolvent kernel of the one-dimensional Brownian motion ([2], p.146 (27)). Since the real and the imaginary parts of the function in (3.28) are even and odd, respectively, we can replace e^{-ikt} by $-2i\sin kt$ in all the formulae given above (and also in below as is easily checked), so that the choice of *sin* transform made at (3.26) causes no problem.

Now, applying (3.28) and (3.29) successively, we find that for $\alpha > 0$, $k > 0$ and $y \in \mathbf{R} \setminus \{0\}$,

$$\begin{aligned} S = S_k(y,\alpha) := \int_{-\infty}^{\infty}e^{-ikt}dt\int_{-\infty}^{\infty}\frac{e^{iyl}dl}{-i2\alpha t + l^2} &= \int_{-\infty}^{\infty}\frac{\pi}{\sqrt{-i2\alpha t}}e^{-|y|\sqrt{-i2\alpha t}}e^{-ikt}dt \\ &= \frac{\pi\sqrt{2\pi}}{\sqrt{\alpha k}}e^{-\alpha y^2/2k}, \end{aligned}$$

and then on differentiating the two sides of the last equality

$$\int_{-\infty}^{\infty}\sqrt{-i2\alpha t}\,e^{-ikt}dt\int_{-\infty}^{\infty}\frac{e^{iyl}dl}{-i2\alpha t+l^2}=-\mathrm{sgn}\,y\,\partial_y S=\frac{\pi\sqrt{2\alpha\pi}\,|y|}{k\sqrt{k}}e^{-\alpha y^2/2k}, \tag{3.30}$$

$$\int_{-\infty}^{\infty}ite^{-ikt}dt\int_{-\infty}^{\infty}\frac{e^{iyl}dl}{-i2\alpha t+l^2}=-\frac{1}{2\alpha}\partial_y^2 S=\left(1-\frac{\alpha y^2}{k}\right)\frac{\pi\sqrt{2\pi}}{2k\sqrt{\alpha k}}e^{-\alpha y^2/2k},$$

of which the first and second formulae give the principal term and the polynomial P_1, respectively, in the expansion of $f_x(k)$ in Theorem 1.2, in view of (3.27).

Keeping (3.28) in mind we derive from (3.30) first

$$\begin{aligned}\int_{-\infty}^{\infty}\sqrt{-i2t}\,e^{-ikt}dt\int_{-\infty}^{\infty}\frac{ile^{iyl}dl}{-i2\alpha t+l^2} &= -\alpha^{-1/2}\partial_y\left(\mathrm{sgn}\,y\,\partial_y S\right)=-\alpha^{-1/2}\mathrm{sgn}\,y\,\partial_y^2 S\\ &= \mathrm{sgn}\,y\left(1-\frac{y^2}{k}\alpha\right)\frac{\pi\sqrt{2\pi}}{k\sqrt{k}}e^{-\alpha y^2/2k},\end{aligned}$$

and then

$$\begin{aligned}\int_{-\infty}^{\infty}\sqrt{-i2t}\,e^{-ikt}dt\int_{-\infty}^{\infty}\frac{il^3e^{iyl}dl}{(-i2\alpha t+l^2)^2} &= \mathrm{sgn}\,y\left[-\alpha\partial_\alpha(\alpha^{-1/2}\partial_y^2 S)-\alpha^{-1/2}\partial_y^2 S\right]\\ &= \mathrm{sgn}\,y\left(1-\frac{5y^2}{2k}\alpha+\frac{y^4}{2k^2}\alpha^2\right)\frac{\pi\sqrt{2\pi}}{k\sqrt{k}}e^{-\alpha y^2/2k},\end{aligned}$$

and you see that in (3.27) this last formula evaluates the contribution of $\sqrt{-2it}$ multiplied by the second fraction in the square brackets, giving the polynomial P_2 in the expansion of Theorem 1.2. Those of the remaining terms together yield the term involving $J(x,k)$ apart from some higher order terms. Computation is made as above by differentiation of the formulae obtained above w.r.t. α, y (in the last formula the double integral does not allow differentiation by y under the (inner) integral symbol, so we truncate the integrand by $w(l)$; note that the remainder is a nice smooth function of α and y that together with derivatives rapidly approaches zero as $k,|y|\to\infty$). The further details are omitted.

The proof of Theorem 1.2 is finished by proving

Lemma 3.5. *For some constant B_4, $1/\pi_0(t)=\sigma\sqrt{-2it}+i2C^*t+B_4(-2it)^{3/2}+o(|t|^{\delta+1)/2})$.*

Proof. We can suppose $2\le\delta<3$. Remembering the procedure by which (3.6) is derived we have only to show

$$\int_{-\pi}^{\pi}R_1(t,l)dl=\frac{3\sqrt{-2it}}{8\sigma}+o(|t|^{3/2})$$

and

$$\int_{-\pi}^{\pi}\left[R_2(t,l)-R_2(0,l)\right]dl=b_4C\sqrt{-2it}+o(|t|^{\delta-1)/2}),$$

where C is a constant. For the first formula use (3.1) and note that $-it/\sqrt{-2it} = \frac{1}{2}\sqrt{-2it}$ and $\frac{1}{2}t^2/(-2it)^{3/2} = -\frac{1}{8}\sqrt{-2it}$; estimation of the error term is made as before. By observing that

$$\begin{aligned} R_2(t,l) - R_2(0,l) &= \frac{16b_4 l^4(itQ - t^2)}{(-2it+Q)^2 Q^2}(1 + o(|l|^{\delta-2})) \\ &= \frac{16b_4}{\sigma^4}\frac{it((-2it+Q) - 3t^2}{(-2it+Q)^2}(1 + o(|l|^{\delta-2})), \end{aligned}$$

the second one is obtained by the same argument as for the first formula.

4 The case $d = 2$

This section consists of three subsections. In the first subsection we evaluate $\pi_0(t)$ and derive an asymptotic estimate of $f_0(k)$. The second one is devoted to the proof of Theorem 1.3. The proofs of Theorems 1.4, 1.5 and 1.6 are given in the third one.

4.1. Since $(-it + \frac{1}{2}Q(\theta))^{-1}$ is not integrable on $\{\theta \in \mathbf{R}^2\}$ we proceed somewhat differently from the case $d = 1$.
Suppose $E[|X|^2 \lg^+ |X|] < \infty$. From (2.3) we deduce as in the case $d = 1$ that

$$\pi_0(t) = \frac{1}{(2\pi)^2}\int_{T^2} \frac{1}{-it + \frac{1}{2}Q(\theta)} d\theta + c_1 + \lambda(t), \tag{4.1}$$

where

$$c_1 = \frac{1}{(2\pi)^2}\int_{T^2} R_2(0,\theta)d\theta = \frac{1}{(2\pi)^2}\int_{T^2}\frac{\psi(\theta) - 1 + \frac{1}{2}Q(\theta)}{(1-\psi(\theta))\frac{1}{2}Q(\theta)}d\theta > -\infty$$

and $\lambda(t) = (2\pi)^{-2}\int_{T^2}[(R_1 + R_2)(t,\theta) - R_2(0,\theta)]d\theta$. We write

$$\lambda(t) = \frac{1}{(2\pi)^2}\int_{T^2}\left(\frac{[t^2 + it(1-\psi+\frac{1}{2}Q)](\psi - 1 + \frac{1}{2}Q)}{\frac{1}{2}(-it+1-\psi)(-it+\frac{1}{2}Q)(1-\psi)Q} + R_1\right)d\theta. \tag{4.2}$$

The present moment condition guarantees that $c_1 < \infty$ as is verified in the same way as in (3.5). It follows that

$$\lambda(t) = o(|t|^{\delta/2}) + O(t\lg|t|). \tag{4.3}$$

Here the first (second) error term is superfluous if $\delta = 2$ (respectively if $\delta < 2$); if $\delta = 1$ there appear the third order monomials of θ as leading terms in the numerator, but they do not cause the magnitude of $O(|t|^{1/2})$ because they are odd; the contribution of R_1 is $O(t\lg|t|)$, which the first integrand in (4.2) also contributes if $\delta = 2$. For the derivatives we have

$$\lambda'(t) = o(|t|^{\delta/2-1}) + O(\lg|t|);$$

$$(d/dt)^j\lambda(t) = o(|t|^{\delta/2-j}) + O(|t|^{-(j-1)}) \quad (j = 2,3) \tag{4.4}$$

as being shown below. The situation that if $\delta \geq 2$, the contribution from R_1 is dominant (which are mostly estimated independently of δ) remains true for the derivatives. The contributions from the other term or its derivatives are evaluated by the first case of Lemma 2.1, giving the $o(\cdot)$ terms in (4.4). As for R_1 the first fraction in (2.4) is evaluated in the same way. The other fraction causes the terms involving logarithm but only for the first derivative; indeed its second derivative is of the form

$$\frac{r_2''(t)}{-it+1-\psi}+\frac{1+ir_2'(t)}{(-it+1-\psi)^2}+\frac{2(-it+r_2(t))}{(-it+1-\psi)^3},$$

of which the first term is plainly negligible and the other two terms only contribute the estimate $O(1/t)$, and similarly for the higher order derivatives.

Splitting T^2, the range of integration, into two parts by the curve $\{Q(\cdot)=a\}$ with a constant $a>0$ chosen arbitrarily so far as $\{Q(\cdot)\le a\}\subset T^2$, we obtain

$$\int_{T^2}\frac{1}{-it+\frac{1}{2}Q(\theta)}d\theta=\frac{2\pi}{|Q|^{1/2}}\int_0^{a/2}\frac{du}{-it+u}+\int_{\{Q>a\}\cap T^2}\frac{1}{-it+\frac{1}{2}Q(\theta)}d\theta,$$

of which the first integral on the right side equals $\lg(-it+a/2)-\lg(-it)=-\lg(-it)+\lg(a/2)+O(t)$ so that $(2\pi)^{-2}$ times the integral on the left side above may be written as $-\lg(-it)/2\pi|Q|^{1/2}+c_2+\eta(t)$ with the constant c_2 introduced in Section 1 and a smooth function $\eta(t)$ which vanishes at $t=0$. Thus, with $c_\circ=2\pi\sqrt{|Q|}\,(c_1+c_2)$ (also introduced in Section 1) and $\tilde{\lambda}(t)=\lambda(t)+\eta(t)$,

$$\pi_0(t)=\frac{-\lg(-it)+c_\circ}{2\pi|Q|^{1/2}}+\tilde{\lambda}(t). \tag{4.5}$$

Define $h(t)$ via

$$\frac{1}{\pi_0(t)}=\frac{-2\pi|Q|^{1/2}}{\lg(-it)-c_\circ}\left(1-\frac{\tilde{\lambda}(t)}{\pi_0(t)}\right)=\frac{-2\pi|Q|^{1/2}}{\lg(-it)-c_\circ}+h(t). \tag{4.6}$$

Employing (4.3) and (4.4), which are satisfied by $\tilde{\lambda}$ in place of λ, we then see that for $j=0,1,2,$

$$\frac{d^j}{dt^j}h(t)=o\left(\frac{|t|^{\delta/2-j}}{(\lg|t|)^2}\right)+O\left(\frac{t^{1-j}}{\lg|t|}\right) \tag{4.7}$$

and, proceeding as in the subsection **3.2** (or rather by (2.8)), that

$$f_0(k)\;=\;\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{2\pi|Q|^{1/2}}{\lg(-it)-c_\circ}\cos kt\,dt-\frac{1}{\pi}\int_{-\pi}^{\pi}h(t)w(t)\cos kt\,dt+\varepsilon(k).$$

On changing the variable of integration the first term on the right side may be written as

$$|Q|^{1/2}e^{c_\circ}\int_{-\infty}^{\infty}\frac{1}{\lg(-it)}\cos(e^{c_\circ}kt)\,dt,$$

which equals $2\pi|Q|^{1/2}e^{c_\circ}\left[W(e^{c_\circ}k)-e^{-e^{c_\circ}k}\right]$ as is easily deduced from the identity (1.4) (cf. [17] Appendix). The second term is easily evaluated by integrating by parts (cf. Lemma 2.2) and we can conclude that if $E[X^2\lg^+|X|]<\infty$,

$$f_0(k)=2\pi|Q|^{1/2}e^{c_\circ}W(e^{c_\circ}k)+\frac{o(k^{-\delta/2})}{k(\lg k)^2}+O\left(\frac{1}{k^2\lg k}\right). \tag{4.8}$$

Without assuming the condition $E[X^2 \lg^+ |X|] < \infty$ it holds that if

$$g(t) = \int_{T^2} R_2(t,\theta)d\theta = \int_{T^2} \left[\frac{1}{-it+1-\psi(\theta)} - \frac{1}{-it+\frac{1}{2}Q(\theta)} \right] d\theta,$$

then

$$g_e(t) = o(\lg|t|) \quad \text{and} \quad g_o(t) = o(1), \tag{4.9}$$

as $t \to 0$, where g_e and g_o denote the even and odd parts of g, respectively. In fact the odd part of the integrand takes on the form

$$\frac{it\Big([\frac{1}{2}Q(\theta)]^2 - [1-\psi(\theta)]^2\Big)}{|-it+1-\psi(\theta)|^2|-it+\frac{1}{2}Q(\theta)|^2}$$

and an application of Lemma 2.1 shows the second relation of (4.9); the first one is shown in the same way. Similarly we obtain $g'(t) = o(1/t)$ and $g''(t) = o(1/t^2)$. The integral $\int_{T^2} R_1 d\theta$ is negligible in comparison with g. With the term $c_1 + \lambda(t)$ in (4.1) replaced by $(2\pi)^{-1}\int_{T^2}(R_1+R_2)d\theta$ and with $c_\circ$ by $2\pi|Q|^{1/2}c_2$ the functions $h(t)$ and $\tilde{\lambda}(t)$ defined via (4.6) and (4.5), respectively, satisfy (4.7) (with $\delta = 0$) for $j = 1,2$ and $\tilde{\lambda}^{(j)}(t) = o(1/t^j)$ $(j > 0)$; on the other hand, for the even and odd parts of $h(t)$ we have

$$h_e(t) = o\left(\frac{1}{\lg|t|}\right) \quad \text{and} \quad h_o(t) = O\left(\frac{1}{(\lg|t|)^2}\right). \tag{4.10}$$

On using Lemma 2.2 the same argument as above shows (4.8) with the new $c_\circ$.

Thus the asymptotic formula of Theorem 1.3 have been verified for $x = 0$.

REMARK. The proof of (4.8) for the case $\delta = 0$ given above is essentially the same as that in [10] given to the one dimensional result mentioned in Introduction (the case $\alpha = 1$). The imbedded walk that consists of traces on the horizontal axis of our walk on $\mathbf{Z}^2$ is a one dimensional walk whose characteristic function is $|Q|^{1/2}|t|(1+o(1))$ as $t \to 0$ ([13]), so that for its hitting time distribution, $f_0(k) = \pi|Q|^{1/2}[k(\lg k)^2]^{-1}(1+o(1))$ according to Kesten's result. It may be worth noticing that this asymptotic form differs from the one for the two dimensional walk itself only by the factor 1/2 and this factor is the same as we might compute as if the successive time intervals spent outside the horizontal axis were independent not only one another but also of the imbedded walk.

4.2. Define $\mathrm{e}_x(t)$ as in the case $d = 1$, namely

$$\begin{aligned} \mathrm{e}_x(t) &= \pi_{-x}(t) - \pi_0(t) + a(x) \\ &= \frac{1}{(2\pi)^2}\int_{T^2}\left(\frac{1}{1-e^{it}\psi(\theta)} - \frac{1}{1-\psi(\theta)}\right)(e^{ix\cdot\theta}-1)\,d\theta, \end{aligned}$$

so that $\hat{f}_x(t) = \mathrm{e}_x(t)/\pi_0(t) + 1 - a^*(x)/\pi_0(t)$, and $\mathrm{c}_x(t)$ and $\mathrm{s}_x(t)$ analogously to those given in the subsection **3.3** so that $\mathrm{e}_x = (2\pi)^{-2}[\mathrm{c}_x + i\mathrm{s}_x]$.

Lemma 4.1. *There exists a constant C such that for $0 < |t| < 1/2$,*

(i) $\quad |\mathrm{c}_x(t)| \le Cx^2|t|\lg|t|^{-1}$ *and* $\quad |\mathrm{c}'_x(t)| \le Cx^2\lg|t|^{-1}$,

(ii) $|c''_x(t)| \le C|x|^2/|t|$ *and* $|c'''_x(t)| \le C|x|^2/|t|^2$,

(iii) $c'_x(t)/i = a(t,x)\lg|t|^{-1} + ib(t,x)\,\mathrm{sgn}\,t$

with the functions a and b both even in t and dominated by Cx^2 (in absolute value).

Proof. From the expression of c_x corresponding to (3.10) we have

$$|c_x(t)| \le C_1 \int_{T^2} \frac{|t|(1-\cos x\cdot\theta)d\theta}{|-it+\frac{1}{2}Q(\theta)|Q(\theta)} \le C_2 x^2 |t| \lg |t|^{-1}. \tag{4.11}$$

where for the last inequality we have dominated $1-\cos x\cdot\theta$ by $x^2\theta^2$ and applied Lemma 2.1 (the second case). Thus the first bound of (i) is verified.

Differentiate the defining expression of c_x we see that

$$c'_x(t) = \int_{T^2} \frac{i(\cos x\cdot\theta - 1)d\theta}{[-it+1-\psi(\theta)]^2} + \int_{T^2} \partial_t R_1(t,\theta)(\cos x\cdot\theta - 1)d\theta$$

On employing (2.4) and the inequality $1-\cos x\cdot\theta \le |x||\theta|$ the second integral is evaluated to be $O(|x|)$. The first one being evaluated as above, this verifies not only the second bound of (i) but also (iii). For the proof of (ii) we have only to observe the bound

$$|c''_x(t)| \le C_1 \int_{T^2} \frac{1-\cos x\cdot\theta}{(|t|+\theta^2)^3} d\theta \le \frac{C_2 x^2}{|t|},$$

and a similar one for $c'''_x(t)$. The proof of Lemma 4.1 is complete.

Lemma 4.2. *Let $0 \le \delta < 1$. Then, as $t \to 0$, uniformly in $x \ne 0$*

$$|s_x(t)| = o(|t|^{\delta/2}), \quad |s'_x(t)| = o(|x||t|^{(\delta-1)/2}), \quad |s''_x(t)| = o(|x||t|^{(\delta-3)/2}).$$

Proof. The proof of the first bound is the same as that of Lemma 3.2 except that we have $|\sin x\cdot\theta|$ dominated by 1 (instead of $|x\cdot\theta|$). For estimation of s'_x we differentiate the analogue for s_x of the expression of I_x given in (3.11) to see that for any $\varepsilon > 0$,

$$\begin{aligned} |s'_x(t)| &\le C_1 \int_{T^2} \frac{|E[\sin X\cdot\theta]\sin x\cdot\theta|}{(|t|+\theta^2)^3} d\theta + C_2 \\ &\le \varepsilon |x||t|^{(\delta-1)/2} \int_{R^2} \frac{|\theta|^{3+\delta} d\theta}{(1+|\theta|)^6} + C(\varepsilon) \end{aligned}$$

for some positive constant $C(\varepsilon)$ depending on ε but not on x nor on t, showing the second bound. The third one is proved in the same way. The proof of the lemma is complete.

In the second half of the subsection **4.1** it is noticed that the bounds for the derivatives of $h^{(j)}$ and $\tilde{\lambda}^{(j)}(t)$ $(j > 0)$ derived in its first half are valid without assuming $E[X^2 \lg|X|] < \infty$. Taking this as well as (4.10) into account we infer from Lemmas 4.1 and 4.2 the following

Corollary 4.1. *Uniformly in $x \in \mathbf{Z}$, as $t \to 0$*

$$\pi_x'(t) = -(2\pi|Q|^{1/2}\,t)^{-1} + o\Big(|t|^{\delta/2-1}(1+|x||t|^{1/2})\Big) + O\Big(|x|_+^2 \lg|t|\Big).$$

In what follows of this section any estimates are insignificant unless $k \to \infty$, so k is understood large unless the contrary is explicitly stated.

Lemma 4.3.

$$\int_{-\pi}^{\pi} \frac{\mathrm{c}_x(t)}{\pi_0(t)} e^{-ikt}\,dt = O\Big(\frac{x^2}{k^2 \lg k}\Big).$$

Proof. Write $g(t)$ for $\mathrm{c}_x(t)/\pi_0(t)$. First we verify that

$$g'(t) = \tilde{a}(t,x) + \tilde{b}(t,x)(\operatorname{sgn} t)/\lg|t|^{-1} \qquad (0<|t|<1/2), \tag{4.12}$$

where both $\tilde{a}$ and $\tilde{b}$ are even in t and bounded by Cx^2. To this end we employ the estimate of $h(t)$ in (4.10) together with Lemma 4.1 (iii) to see that $c_x'(t)/\pi_0(t)$ may be written in the same form as the right side of (4.12). On the other hand, using the estimates $\pi_0(t) = C\lg|t| + O(1)$ and $\pi_0' = O(1/t)$ as well as the bound of $c_x(t)$ in Lemma 4.1 (i), one infers that $|c_x(t)\pi_0'(t)/\pi_0^2(t)| \le Cx^2/\lg|t|^{-1}$. Thus (4.12) holds true.

Integrating by parts (once / twice), splitting the range of integration at $t = \pm 1/k, \pm\varepsilon$ and letting $\varepsilon \downarrow 0$ with the help of $\lim_{\varepsilon\downarrow 0}[g'(\varepsilon) - g'(-\varepsilon)] = 0$, which follows from (4.12), one obtains

$$\int_{-\pi}^{\pi} g(t)e^{-ikt}\,dt = \frac{1}{(ik)^2}\Big[\lim_{\varepsilon\downarrow 0}\int_{\varepsilon<|t|\le 1/k} e^{-ikt}\,dg'(t) + \int_{1/k<|t|\le\pi} g''(t)e^{-ikt}\,dt\Big]. \tag{4.13}$$

The last integral is easily evaluated to be $O(x^2/\lg k)$ by applying the bounds

$$|g''(t)| \le Cx^2/|t|\lg|t|^{-1}, \;\; |g'''(t)| \le Cx^2/t^2\lg|t|^{-1} \quad (0<|t|<1/2),$$

which follow from Lemma 4.1 and the bounds $\pi_0^{(j)}(t) = O(t^{-j})$, $(j \ge 1)$. The limit on the right side of (4.13) is bounded by

$$|g'(1/k) - g'(-1/k)| + \int_{|t|<1/k} |1-e^{-ikt}||g''(t)|\,dt \le \frac{2C\|\tilde{b}\|_\infty x^2}{\lg k} + Cx^2 k\int_0^{1/k}\frac{2dt}{\lg|t|^{-1}}.$$

The integral in the right-most member being $O(1/k\lg k)$, this concludes the assertion of the lemma.

Lemma 4.4. *If $1 \le \delta \le 2$,*

$$\int_{-\pi}^{\pi} \frac{\mathrm{s}_x(t)}{\pi_0(t)} e^{-ikt}\,dt = O\Big(\frac{|x|}{k^2\lg k}\Big).$$

Proof. We proceed as in the proof of Lemma 4.1 starting with a two dimensional analogue of (3.14) (instead of (3.10)) or with (3.11) (for derivatives) to see that

$$|\mathrm{s}_x(t)| \le C_1|t|\int_{T^2}\frac{|\sin x\cdot\theta|}{(|t|+\theta^2)|\theta|}\,d\theta$$

and similar bounds for the derivatives, which reduce to

$$\mathrm{s}_x(t) = O(|x|t\lg|t|^{-1}),\quad \mathrm{s}''_x(t) = O(|x|/t) \quad\text{and}\quad \mathrm{s}'''_x(t) = O(|x|/t^2)$$

(for $0 < |t| < 1/2$). Further employing (3.16) (of which only the term involving itQ is relevant here) we also deduce (as in the proof of Lemma 4.1 (iii)) that

$$\begin{aligned}\mathrm{s}'_x(t) &= \int_{T^2} \frac{-i2E[\sin X\cdot\theta]}{(1-e^{it}\psi(\theta))(1-e^{it}\psi(-\theta))Q(\theta)}\sin x\cdot\theta\,d\theta + O(|x|)\\ &= ia(t,x)\lg|t|^{-1} + b(t,x)\operatorname{sgn} t,\end{aligned}$$

where a and b are even in t and bounded by $C|x|$ (see the proof of (iii) of Lemma 4.1). By these bounds we derive that of the lemma as in the proof of Lemma 4.3.

Proof of Theorem 1.3. The case $1 \le \delta < 2$ is immediate from the last two lemmas (together with the result on $f_0(k)$ in **4.1**). For $0 \le \delta < 1$, the same argument as made in the proof of Lemma 4.3 deduces from Lemma 4.2 that

$$\int_{-\pi}^{\pi} \frac{\mathrm{s}_x(t)}{\pi_0(t)} e^{-ikt}dt = o\left(\frac{|x|}{k^{(3+\delta)/2}\lg k}\right), \tag{4.14}$$

which in turn shows the asserted estimate of Theorem 1.3 in view of Lemma 4.3 and the inequality $|x|/\sqrt{k} \le \lg|x|/\lg k$ $(3 \le x^2 \le k)$. The case $\delta = 2$ is similarly dealt with. The proof of Theorem 1.3 is complete.

4.3. Here we prove Theorems 1.4, 1.5 and 1.6. Recalling (2.1) we have

$$f_x(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{-2\pi|Q|^{1/2}}{\lg(-it)-c_\circ} + h(t)\right)\pi_{-x}(t)e^{-ikt}dt,$$

where $h = h(t)$ is defined via (4.6) (see the second half of **4.1** in the case $E[X^2\lg^+|X|] = \infty$).We truncate this integral by $w(t)$ (as in (3.18) but with t in place of θ). The $(1-w)$ part is plainly negligible, so that we may multiply the integrand by $w(t)$. We further truncate the integral defining $\pi_x(\theta)$ by means $w(|\theta|)$. The $(1-w(|\theta|)$ part that accordingly arises equals

$$\frac{1}{(2\pi)^3}\int_{-\pi}^{\pi}\frac{w(t)}{\pi_0(t)}e^{-ikt}dt\int_{T^2}\frac{1-w(|\theta|)}{1-e^{it}\psi(\theta)}e^{ix\cdot\theta}d\theta = o\left(\frac{1}{x^{2+\delta}k(\lg k)^2}\right). \tag{4.15}$$

For the proof of this estimate we may replace $1-e^{it}\psi$ by $1-\psi$ in the second integrand, the error being of smaller order. This results in the product of two independent integral, of which the first is already evaluated in **4.1** and the second is $o(|x|^{-2-\delta})$ (use a two dimensional analogue of Lemma 6.1 (cf.[14]:Appendix) if δ is not integral, otherwise Riemann-Lebesgue lemma disposes).

Let $x \ne 0$ and define

$$q_x(k) = -\frac{2\pi|Q|^{1/2}}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{e^{-ikt}dt}{\lg(-it)-c_\circ}\int_{\mathbf{R}^2}\frac{e^{ix\cdot\theta}d\theta}{-it+\frac{1}{2}Q(\theta)}$$

and

$$r_x(t) = \frac{1}{(2\pi)^2}\int_{T^2}[R_1(t,\theta)+R_2(t,\theta)]w(|\theta|)e^{ix\cdot\theta}\,d\theta.$$

Then, employing (2.3) and (4.15) together with what is discussed preceding the latter, one deduces that as $|x| \wedge k \to \infty$,

$$\begin{aligned} f_x(k) &= q_x(k) + \frac{1}{2\pi}\int_{-\pi}^{\pi} h(t)\pi_{-x}(t)w(t)e^{-ikt}dt \\ &\quad - |Q|^{1/2}\int_{-\pi}^{\pi} \frac{r_x(t)w(t)e^{-ikt}dt}{\lg(-it) - c_\circ} + o\left(\frac{1}{x^{2+\delta}k(\lg k)^2}\right). \end{aligned} \tag{4.16}$$

One can write $q_x(k)$ in the form

$$q_x(k) = -\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} \frac{e^{-ikt}dt}{\lg(-it) - c_\circ}\int_{\mathbf{R}^2} \frac{e^{i\tilde{x}\cdot\theta}d\theta}{-it + \frac{1}{2}\theta^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{K_0(|\tilde{x}|\sqrt{-i2t})}{-\lg\sqrt{-ie^{-c_\circ}t}}e^{-ikt}dt$$

(K_0 is the usual modified Bessel function of order 0). The following lemma is proved in [17].

Lemma 4.5. *As $k \wedge |x| \to \infty$*

$$q_x(k) = \begin{cases} \dfrac{\lg(\frac{1}{2}e^{c_\circ}\tilde{x}^2)}{k(\lg(e^{c_\circ}k))^2}e^{-\tilde{x}^2/2k} + \dfrac{2\gamma\lg(k/x^2)}{k(\lg k)^3} + O\left(\dfrac{1}{k(\lg k)^3}\right) & \text{for } \tilde{x}^2 < k, \\ \dfrac{\lg(\frac{1}{2}e^{c_\circ}\tilde{x}^2)}{k(\lg(e^{c_\circ}k))^2}e^{-\tilde{x}^2/2k} + O\left(\dfrac{1 + [\lg(\tilde{x}^2/k)]^2}{x^2(\lg k)^3}\right) & \text{for } \tilde{x}^2 \geq k. \end{cases} \tag{4.17}$$

For the proof of Theorems 1.4 and 1.5 the two integrals in (4.16) need to be evaluated and we prove the following estimates (i) through (iii) valid whenever $k \wedge |x| \to \infty$.

(i) If $E[\,|X|^2\lg^+|X|\,] < \infty$, then

$$H := \int_{-\pi}^{\pi} h(t)\pi_{-x}(t)w(t)e^{-ikt}dt = o\left(\frac{1}{|x|k^{(\delta+1)/2}(\lg k)^2}\right) \qquad \text{for} \quad 0 \leq \delta < 2.$$

In general, $\quad H = o\left(\dfrac{1}{|x|k^{1/2}\lg k}\right).$

(ii)
$$R := \int_{-\pi}^{\pi} \frac{r_x(t)w(t)e^{-ikt}dt}{\lg(-it) - c_\circ} = \frac{1}{|x|k\lg k}\left[o(k^{(1-\delta)/2}) + b_3 O(1)\right] \qquad \text{for} \quad 0 \leq \delta < 2.$$

(iii)
$$H = O\left(\frac{1}{|x|k^{3/2}\lg k}\right) \quad \text{and} \quad R = O\left(\frac{1}{k\lg k}\left(\frac{b_3}{|x|} + \frac{1}{|x|^2}\right)\right) \qquad \text{if} \quad \delta = 2.$$

Proof of (i) *through* (iii). Regarding $(1 - e^{it}\psi)^{-1}e^{ix\cdot\theta}$ as the inner product of the vector function $(i|x|(1 - e^{it}\psi))^{-1}|x|^{-1}x$ and the gradient of $e^{ix\cdot\theta}$ and noting that ψ is periodic we apply the divergence theorem to find

$$\pi_{-x}(t) = \frac{-e^{it}}{i|x|(2\pi)^2}\int_{T^2} \frac{|x|^{-1}x\cdot\nabla\psi}{(1 - e^{it}\psi)^2}e^{ix\cdot\theta}d\theta \tag{4.18}$$

and using this we deduce that $\pi_{-x}(t) = O(1/|x||t|^{1/2})$ and $\pi'_{-x}(t) = O(1/|x||t|^{3/2})$. Combined with the estimate of h given in (4.7) and (4.10) these yield the bounds of H in (i), in view of Lemma 2.2.

For the proof of (ii) first we see, by using Lemma 2.1, that for $\delta < 1$,

$$r_x(t) = o(|t|^{(\delta-1)/2}/|x|) \quad \text{and} \quad r'_x(t) = o(|t|^{(\delta-3)/2}/|x|). \tag{4.19}$$

Next let $\delta \geq 1$. Then

$$\int \frac{\psi(\theta) - 1 + \frac{1}{2}Q(\theta)}{(-i2t + Q(\theta))^2} w(|\theta|) e^{ix\cdot\theta}\, d\theta = \frac{b_3 O(1) + o(1)}{|x|}, \tag{4.20}$$

giving the estimate of the essential part of $r_x(t)$, so that

$$r_x(t) = b_3 O(1/|x|) + o(1/|x|). \tag{4.21}$$

The proof of (4.20) may proceed analogously to that of Lemma 2.2: split the range T^2 by means of the circle $|\theta| = 1/|x|$ and apply the divergence theorem twice for the integral on $|\theta| > 1/|x|$, in which the quantity arising in the last step is dominated by a positive multiple of

$$\frac{1}{x^2}\int_{1/|x|<|\theta|<\pi} \frac{b_3 + o(1)}{|-i2t + Q(\theta)|^{3/2}}\, d\theta \leq \frac{C}{x^2}\int_{1/|x|<|\theta|<\pi} \frac{b_3 + o(1)}{|\theta|^3} d\theta = C\frac{b_3 + o(1)}{|x|}$$

plus the two boundary integrals that admit the same bound as above. The first formula of (4.19) does not hold for $\delta > 1$ (we have the third case of Lemma 2.1), but we still have

$$r'_x(t) = o(|t|^{(\delta-3)/2}/|x|) + b_3 O(1/|x|\sqrt{|t|}) \tag{4.22}$$

as is readily seen. Now, (ii) follows from (4.19), (4.21) and (4.22) on using Lemma 2.2.

For (iii), i.e. in case $\delta = 2$, first integrate by parts relative to θ, and then proceed as above.

Proof of Theorem 1.4. In view of (4.16) the assertion is readily deduced from (i), (ii) and Lemma 4.5 if one also employs Theorem 1.3 and the trivial bound $f_x(k) \leq p^k(x)$ (in disposing of the case $x^2 < k/\lg k$ and of the case $x^2 > k(\lg k)$, respectively).

Proof of Theorem 1.5. This follows from (iii) given above and the following lemma.

Lemma 4.6. *If $r_\circ = \sqrt{2}\, e^{-\gamma - c_\circ/2}$, then uniformly for $|x| > r_\circ$, as $k \to \infty$*

$$q_{r_\circ}(k, \tilde{x}) - q_x(k) = O\left(\frac{1}{k^2(\lg k)} \wedge \frac{1}{|x|^4 \lg(|x|+1)}\right).$$

Proof. This is Lemma 4 of [17].

Proof of Theorem 1.6. Let $\xi^2 = x^2/n$. We derive the formula (1.8) from Theorem 1.3 if $\xi^2 < 1/(\lg n)^2$ and from Theorem 1.4 if $\xi^2 \geq 1/(\lg n)^2$. First let $\xi^2 < 1/(\lg n)^2$. Then an elementary computation shows that $1 - D(e^{c_\circ} n, \xi^2/2)$ agrees with $2\lg(|x|/r_\circ)\int_{e^{c_\circ}}^\infty W(u)du$ within the error of magnitude $O((\lg \xi)/(\lg n)^3)$, where $r_\circ$ is given in Lemma 4.6 (see Remark 4 of [17]). Now (1.8) readily follows from Theorem 1.3. Next let $\xi^2 \geq 1$ and integrate the error term in (1.5):

$$\int_2^n \frac{|\lg(x^2/t)|^2+1}{x^2(\lg t)^3}dt = O\left(\frac{|\lg \xi|_+^2}{(\lg n)^3\xi^2}\right).$$

In view of the results of [17] (as presented in Appendix (C) of this paper) this combined with Theorem 1.5 shows (1.8). A similar argument applies in the case $1/(\lg n)^2 \leq \xi^2 < 1$. The proof of Theorem 1.6 is finished.

5 The case $d \geq 3$

This section is divided into three subsections. In the first one we provide some preliminary formulae. Theorems 1.7 and 1.8 will be proved in the second and third, respectively. Details of the proofs are quite similar to that for the case $d = 1$ and only main steps of the proof will be indicated. Here, however, we use the fact that

$$(2\pi)^{-1}\int_{-\pi}^{\pi} \pi_x(t)e^{-ikt}dt = \begin{cases} p^k(x) & (k \geq 0), \\ 0 & (k < 0). \end{cases} \tag{5.1}$$

(This holds true in all dimensions $d \geq 1$.)

5.1. Let $d \geq 3$. Since $(1-\psi)^{-1}$ is integrable over T^d, it is appropriate to subtract the term $(1-\psi)^{-1}$ from $(1-e^{it}\psi)^{-1}$ and is accordingly convenient to bring in

$$R_4 = R_4(t,\theta) := R_2(t,\theta) - \frac{1}{1-\psi(\theta)} + \frac{1}{\frac{1}{2}Q(\theta)}$$

so that

$$\frac{1}{1-e^{it}\psi(\theta)} - \frac{1}{1-\psi(\theta)} = \frac{it}{(-it+\frac{1}{2}Q(\theta))\frac{1}{2}Q(\theta)} + R_1 + R_4; \tag{5.2}$$

also

$$R_4 = \left[\frac{1}{\frac{1}{2}Q} + \frac{1}{-it+1-\psi}\right]\frac{it(\psi-1+\frac{1}{2}Q)}{(1-\psi)(-it+\frac{1}{2}Q)}. \tag{5.3}$$

For computation of $f_x(k)$ we decompose

$$\frac{\pi_{-x}(t)}{\pi_0(t)} = \frac{G(-x)}{\pi_0(t)} + \frac{\pi_{-x}(t)-\pi_{-x}(0)}{G(0)} + \left(\frac{1}{\pi_0(t)} - \frac{1}{\pi_0(0)}\right)(\pi_{-x}(t)-\pi_{-x}(0)),$$

where the identity $G(-x) = \pi_{-x}(0)$ is used. The contribution of the first term on the right side to $f_x(k)$ with $x \neq 0$ is $-G(-x)f_0(k)$ and that of the second term equals $p^k(-x)/G(0)$ ($k > 0$) owing to (5.1). Hence putting

$$m_x(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{1}{\pi_0(t)} - \frac{1}{\pi_0(0)}\right)(\pi_{-x}(t)-\pi_{-x}(0))w(t)e^{-ikt}dt,$$

we have

$$f_x(k) = e_0 p^k(-x) - G(-x) f_0(k) + m_x(k) + |x|_+^{-1}\varepsilon(k) \qquad (x \neq 0), \tag{5.4}$$

where the error term is caused by truncation by means of $w(t)$ ($\varepsilon(k)$ denotes a rapidly decreasing term as in **3.2**). Decomposing $-\pi_0(0)/\pi_0(t)$ in a similar way we also have

$$f_0(k) = e_0^2 p^k(0) + e_0 m_0(k) + \varepsilon(k). \tag{5.5}$$

5.2. Here we prove Theorem 1.7. First consider the case $d = 3$ and suppose $0 \le \delta \le 2$. Put $C^\circ = 0$ if $\delta < 1$ and

$$C^\circ = \frac{1}{(2\pi)^3} \int_{T^3} \frac{(\psi - 1 + \frac{1}{2}Q)(1 - \psi + \frac{1}{2}Q)}{[\frac{1}{2}(1-\psi)Q]^2} d\theta \tag{5.6}$$

if $\delta \ge 1$. Then $(2\pi)^{-3}\int_{T^3} R_4 d\theta = iC^\circ t + o(|t|^{(1+\delta)/2}) + O(|t|^{3/2})$. For verification we apply the third case of Lemmas 2.1 (with $\alpha = (1-\delta)/2$) if $\delta \neq 1$ and an obvious analogue of (3.5) if $\delta = 1$; the last O-term needs to be given if $\delta = 2$ and is superfluous if $\delta < 2$. Likewise, $(2\pi)^{-3}\int_{T^3} R_1 d\theta = -itG(0) + O(|t|^{3/2})$, as required.

For simplicity let $0 \le \delta < 2$; if $\delta = 2$ we have only to replace o-terms by the corresponding O-terms. Then, taking what is obtained right above into account, we make the same manipulation with a cutoff function $w(\theta)$ as before and then apply the formula (3.1) with $n = 1$ to find

$$\pi_0(t) = G(0) - \frac{\sqrt{-i2t}}{2\pi|Q|^{1/2}} - iC_0 t + o(|t|^{(1+\delta)/2}) \qquad (C_0 := -C^\circ + G(0)). \tag{5.7}$$

A little inspection assures that the derivative of the error term is $o(|t|^{(\delta-1)/2})$, hence

$$\pi_0'(t) = \frac{1}{2\pi|Q|^{1/2}} \cdot \frac{i}{\sqrt{-i2t}} - iC_0 + o(|t|^{(\delta-1)/2}); \tag{5.8}$$

and similarly for $\pi_0''(t)$, $\pi_0'''(t)$. Using $e_0 = 1/G(0)$ as well as (5.7) one infers that

$$\frac{1}{\pi_0(t)} - \frac{1}{\pi_0(0)} = -\hat{f}_0(t) + 1 - e_0 = \frac{e_0^2}{2\pi|Q|^{1/2}}\sqrt{-i2t} - i2C_1 t + o(|t|^{(1+\delta)/2}) \tag{5.9}$$

with $C_1 = -\frac{1}{2}C_0 e_0^2 + [(2\pi)^2|Q|]^{-1} e_0^3$.

Using (5.7) together with estimates of the derivatives of $\pi_0(t)$ one can obtain (also using Lemma 5.1 in **5.3**) $m_x(k) = O(k^{-5/2}(|x|_+ \wedge (k/|x|_+)) + o(k^{-2-\delta/2})$. (This will be refined in Lemma 5.2 below, so details are omitted.) From (5.5) it in particular follows that

$$f_0(k) = e_0^2 p^k(0) + o(k^{-2-\delta/2}) + O(k^{-5/2}). \tag{5.10}$$

Substitution of these estimates of $m_x(k)$ and $f_0(k)$ into (5.4) yields the formula (1.10) of Theorem 1.7 for $d = 3$ since in view of (1.9) the leading term of (1.10) may be written as

$$e_0^2\, p^k(0)\mathbf{1}_{\{0\}}(x) + e_0\, p^k(-x) - e_0^2 G(-x) p^k(0).$$

In the same way the formula (1.10) for $d \ge 4$ follows if we prove that for some constant C,

$$|m_x(k)| \le \begin{cases} \dfrac{C \lg k}{k^3}\left(1 \wedge \dfrac{\sqrt{k}}{|x|_+}\right) & \text{if } d = 4 \\ \dfrac{C}{k^3}\left[1 \wedge \left(\dfrac{\sqrt{k}}{|x|_+}\right)^{d-3}\right] & \text{if } d \ge 5. \end{cases} \tag{5.11}$$

For the proof one has only to look at the main part of $\pi_x(t) - \pi_0(0)$ which is a constant times

$$\int_{Q^{1/2}T^d} \frac{t e^{i\tilde{x}\cdot\theta}\tilde{w}}{(-i2t+\theta^2)\theta^2} d\theta = c_d t \int_0^\pi \sin\alpha\, d\alpha \int_0^1 \frac{r^{d-3}\cos(|\tilde{x}| r\cos\alpha)}{-2it + r^2} w(r) dr + t\eta(t),$$

where $\eta(t)$ is smooth and the cut-off is made with $w(Q^{1/2}\theta)$. It is easy to see that if $d = 4$,

$$\left|\partial_t^j[\pi_x(t) - \pi_0(0)]\right| \le \begin{cases} [(|t|\lg|t|) \wedge (\sqrt{|t|}/|x|_+)]|t|^{-j} & \text{for } j = 0,1 \\ [|t| \wedge (\sqrt{|t|}/|x|_+)]|t|^{-j} & \text{for } j = 2,3, \end{cases}$$

from which one evaluates the integrand of the integral defining $m_x(k)$ and its derivatives to obtain the estimate (5.11) for $d = 4$; that for $d \ge 5$ is obtained similarly.

5.3. For the proof of Theorem 1.8 we need to find a finer evaluation of $m_x(k)$. To this end we make an exact computation based on the formula

$$\int_{-\infty}^{\infty} e^{-y\sqrt{-2it}} e^{-ikt} dt = \sqrt{2\pi} \cdot \frac{y}{k\sqrt{k}} e^{-y^2/2k} \quad (y > 0), \tag{5.12}$$

which follows from (3.29). The result is formulated in the next lemma. Set

$$H(t,x) = \frac{1}{(2\pi)^3} \int_{\mathbf{R}^3} \frac{ite^{ix\cdot\theta} d\theta}{(-it + \frac{1}{2}Q(\theta))\frac{1}{2}Q(\theta)}. \tag{5.13}$$

Lemma 5.1. *For $x \in \mathbf{R}^3$ and $k > 0$,*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \sqrt{-i2t}\, H(t,x) w(t) e^{-ikt} dt = \frac{1}{|Q|^{1/2}|x|_+} \cdot \frac{1}{(2\pi k)^{3/2}}\left[1 - e^{-\tilde{x}^2/2k}\left(1 - \frac{\tilde{x}^2}{k}\right)\right] + \frac{\varepsilon(k)}{|x|_+},$$

where the first term on the right side is understood to be zero if $x = 0$.

Proof. First we compute $(2\pi)^3 H(t,x)$, which may be written as

$$\frac{2\pi}{|Q|^{1/2}} \int_0^\pi \sin\alpha\, d\alpha \int_0^\infty \frac{i4t}{-i2t + r^2} \cos[|\tilde{x}| r\cos\alpha]\, dr.$$

Applying (3.28) to the inner integral above and then performing the outer integration we find

$$H(t,x) = \frac{1}{2\pi |Q|^{1/2} |\tilde{x}|}\left(e^{-\sqrt{-2it}\,|\tilde{x}|} - 1\right) \qquad (x \ne 0) \tag{5.14}$$

and by continuity $H(t,0) = -(2\pi|Q|^{1/2})^{-1}\sqrt{-2it}$. The formula (5.12) as well as (3.9) (and its cosine companion) is now used to verify that for $x \neq 0$,

$$\int_{-\pi}^{\pi}(e^{-\sqrt{-i2t}|\tilde{x}|}-1)\sqrt{-i2t}\,w(t)e^{-ikt}dt = \frac{\sqrt{2\pi}}{k\sqrt{k}}\left[1-e^{-\tilde{x}^2/2k}\left(1-\frac{\tilde{x}^2}{k}\right)\right]+\varepsilon(k),$$

showing the formula of the lemma. For the verification it suffices to see that for $y > 0$, the integral $\int_{-\infty}^{\infty}(1-w(t))\sqrt{-i2t}\,e^{-y\sqrt{-2it}}e^{-ikt}dt$ is $O(k^{-N})$ for any N, but its absolute value is indeed at most $C_N k^{-N}\left[1+y^N\int_0^{\infty}e^{-y\sqrt{t}}(1-w(t/2))dt\right]$, which is $O(k^{-N})$. The proof of Lemma 5.1 is complete.

The next lemma provides an asymptotic form of $m_x(k)$. It follows from (5.2) and (5.13) that

$$\pi_{-x}(t)-\pi_{-x}(0) = H(t,x)+\frac{1}{(2\pi)^3}\int_{T^3}(R_1+R_4)\tilde{w}e^{ix\cdot\theta}d\theta+\eta_x(t) \tag{5.15}$$

where $\tilde{w} = \tilde{w}(\theta) := w(|\theta|)$ and

$$\eta_x(t) = \frac{1}{(2\pi)^3}\int_{T^3}\frac{(1-\tilde{w})(e^{it}-1)\psi}{(1-e^{it}\psi)(1-\psi)}e^{ix\cdot\theta}d\theta+\frac{it}{(2\pi)^3}\int_{\mathbf{R}^3}\frac{(1-\tilde{w})e^{ix\cdot\theta}d\theta}{(-it+\frac{1}{2}Q)\frac{1}{2}Q}.$$

It is readily seen that the contribution of $\eta_x(t)$ to m_x is $O(k^{-5/2}|x|_+^{-2-\delta})$.

Lemma 5.2. *Uniformly in $x \in \mathbf{R}^3$, as $k \to \infty$*

$$\begin{aligned} m_x(k) \;&=\; \frac{e_0^2}{2\pi|Q|(2\pi k)^{3/2}|\tilde{x}|_+}\left[1-e^{-\tilde{x}^2/2k}\left(1-\frac{\tilde{x}^2}{k}\right)\right]+\frac{C_1(3-k^{-1}\tilde{x}^2)}{|Q|^{1/2}(2\pi)^{3/2}k^{5/2}}e^{-\tilde{x}^2/2k}+\frac{\varepsilon(k)}{|x|_+} \\ &+\begin{cases} o\left(\dfrac{\sqrt{k}\wedge|x|_+}{k^{2+\delta/2}|x|_+}\right) & \text{if } \delta<1; \\ \dfrac{\sqrt{k}\wedge|x|_+}{k^{5/2}}\times o\left(|x|_+^{-\delta}\lg(|x|^{\delta-1}\vee e)\right)+b_3 O\left(\dfrac{k\wedge x^2}{k^{5/2}|x|_+^2}\right) & \text{if } 1\le\delta<2, \end{cases}\end{aligned}$$

where b_3 is the same as in Theorem 1.8; $o\left(|x|_+^{-\delta}\lg(|x|^{\delta-1}\vee e)\right)$ is bounded and approaches zero faster than $|x|^{-\delta}\lg|x|^{\delta-1}$ as $|x|\to\infty$ (uniformly in k).

Proof. Recall (5.9) as well as (5.15) and observe that the preceding lemma gives the leading term. The contribution to $m_x(k)$ of $-i2C_1t$ involved in (5.9) equals

$$\frac{C_1}{2\pi}\int_{-\pi}^{\pi}(-i2t)H(t,x)w(t)e^{-ikt}dt = \frac{C_1}{|Q|^{1/2}(2\pi k)^{3/2}}\left(\frac{3}{k}-\frac{|\tilde{x}|^2}{k^2}\right)e^{-\tilde{x}^2/2k}+\frac{\varepsilon(k)}{|x|_+} \tag{5.16}$$

as is readily proved in a similar way to Lemma 5.1. That of the error term in (5.9) is small enough to be absorbed into the estimate of the one coming from R_4.

It remains to appraise the contribution of the integral in (5.15) that involves R_1+R_4. The contribution of R_1 turns out to be negligible. This is easily seen if $\delta<1$. We verify that if $\delta\ge 1$,

$$\int_{-\pi}^{\pi}\left(\frac{1}{\pi_0(t)}-\frac{1}{\pi_0(0)}\right)w(t)e^{-ikt}dt\int_{T^3}R_1\tilde{w}e^{ix\cdot\theta}d\theta = O\left(\frac{k\wedge|x|_+^2}{k^{5/2}|x|_+^3}\right),$$

which is also negligible. Performing the same computation as before with the help of (5.9) and (2.4) we observe that the case $x^2 > k$ is plain and the verification is reduced to verifying

$$\frac{1}{k^2}\int_{-\pi}^{\pi}\frac{w(t)}{\sqrt{-i2t}}e^{-ikt}dt\int_{T^3}\frac{\tilde{w}(\theta)}{-it+1-\psi(\theta)}e^{ix\cdot\theta}d\theta = O\biggl(\frac{1}{k^{5/2}|x|_+}\biggr). \tag{5.17}$$

(This one is distinct from other similar integrals: the manner having been practiced above gives rise to a term involving logarithm.) We can replace the denominator in the inner integral by $1-e^{it}\psi(\theta)$ and express the resulting double integral in the form

$$(2\pi)^3\sum_{n=0}^{\infty}p^n(x)\int_{-\pi}^{\pi}\frac{1}{\sqrt{-i2t}}w(t)e^{i(n-k)t}dt.$$

Write the integral above as $2\sqrt{2\pi}/\sqrt{k-n}+\varepsilon(k-n)$ if $k-n>0$ and $\varepsilon(n-k)$ if $k-n<0$. Then the bound (5.17) is deduced by using the estimate of $p^n(x)$ as given in (1.13) (with $\delta=0$) (cf. [11]: Proof of P26.1).

We have to prove that the same double integral as above but with R_4 replacing R_1 is appraised with the error term given in the formula of the lemma. Denote by $I_x(k)$ this double integral. Then on integrating by parts

$$I_x(k) = \frac{1}{ik}\int_{-\pi}^{\pi}w(t)e^{-ikt}dt\int_{T^3}\partial_t\biggl[\biggl(\frac{1}{\pi_0(t)}-\frac{1}{\pi_0(0)}\biggr)R_4\biggr]\tilde{w}e^{ix\cdot\theta}d\theta. \tag{5.18}$$

Note that R_4-R_2 is independent of t and integrable on T^3. At first suppose that $b_3=0$ if $\delta\geq 1$. Then with the help of $\partial_t^j R_2 = (\psi-1+\frac{1}{2}Q)\times O(|t|+|\theta|^2)^{-2-j}$ for $j=0,1,2,\dots$ and $\psi-1+\frac{1}{2}Q = o(|\theta|^{2+\delta})$ we apply Lemma 2.1 (the first case) to deduce that

$$\int_{T^3}\partial_t^j R_4\tilde{w}e^{ix\cdot\theta}d\theta = o(t^{(1+\delta)/2-j})\quad\left(\begin{array}{ll}\text{for } j=1,2,3 & \text{if } \ \delta<1\\ \text{for } j=2,3 & \text{if } 1\leq\delta<2 \text{ and } b_3=0\end{array}\right)$$

and

$$\int_{T^3}R_4\tilde{w}e^{ix\cdot\theta}d\theta = \left\{\begin{array}{ll} o(|t|^{(1+\delta)/2}),\ |x|_+^{-1}\times o(|t|^{\delta/2}) & \text{if } \ \delta<1\\ t\times o(|x|_+^{1-\delta}\lg(|x|^{\delta-1}\vee e)),\ |x|_+^{-2}\times o(|t|^{(\delta-1)/2}) & \text{if } \ 1\leq\delta<2\end{array}\right.$$

(for the latter (with $x\neq 0$) the integration by parts in θ has been applied once if $\delta<1$ and twice if $\delta\geq 1$ but further application is not allowed in each case; in the cases $\delta=0,1$ split the range of the integral with the spherical surface $|\theta|=1/|x|$ for integrating by parts as in the proof of Lemma 2.2; also, in the case $\delta\geq 1$, we have used an analogue of Lemma 6.1 of Appendix A (cf. [14]: Appendix) as well as the fact that the integral defining C° in (5.6) is absolutely convergent). From these it is inferred that for $\delta<1$, $I_x(k) = o\Bigl((\sqrt{k}\wedge|x|_+)\Big/k^{2+\delta/2}|x|_+\Bigr)$ and that for $1\leq\delta<2$ with $b_3=0$,

$$I_x(k) = \frac{1\vee\lg|x|_+^{\delta-1}}{k^{5/2}}\times o(|x|_+^{1-\delta})\ \ (x^2\leq k)\quad\text{and}\quad = o\biggl(\frac{1}{k^{(2+\delta)/2}x^2}\biggr)\ \ (x^2\geq k),$$

which together imply the required estimates.

In order to complete the proof we must deal with the part of the right side of (5.18) that involves $E[(X\cdot\theta)^3]$ in the case $\delta \geq 1$ with $b_3 = 1$. Its essential term is

$$J_x(k) := \frac{1}{k}\int_{-\pi}^{\pi}\sqrt{-i2t}\,w(t)e^{-ikt}dt\int_{T^3}\frac{E[(X\cdot\theta)^3]\,\tilde{w}(\theta)e^{ix\cdot\theta}}{(-it+\frac{1}{2}Q(\theta))Q(\theta)(1-\psi(\theta))}d\theta;$$

the variants of this integral that we must actually compute are treated similarly to it. On further integrating by parts in θ (twice) as well as in t this term is evaluated to be $O(1/k^{3/2}|x|^2)$; on observing that the inner integral is bounded uniformly for x and t it is also evaluated to be $O(1/k^{5/2})$. Hence

$$J_x(k) = b_3 O\left(\frac{k\wedge|x|_+^2}{k^{5/2}|x|_+^2}\right),$$

and this completes the proof of Lemma 5.2.

Proof of Theorem 1.8. From (5.10) and the expansion of $p^k(0)$ it follows that

$$f_0(k) = e_0^2|Q|^{-1/2}(2\pi k)^{-3/2}\Big[1+o(k^{-\delta/2})+O(1/k)\Big]. \tag{5.19}$$

First consider the case $\delta < 2$. We have

$$p^k(-x) = \frac{e^{-\tilde{x}^2/2k}}{|Q|^{1/2}(2\pi k)^{3/2}}\left(1+\frac{b_3}{\sqrt{k}}\mathrm{P}_3\left(\frac{-\tilde{x}}{\sqrt{k}}\right)\right)+o\left(\frac{1}{k^{(3+\delta)/2}}\wedge\frac{1}{\sqrt{k}\,|x|_+^{2+\delta}}\right), \tag{5.20}$$

$$(\pi|Q|^{1/2}\,|\tilde{x}|)^{-1} - G(-x) = O(1/|x|^{1+\delta}) + b_3 O(1/|x|^2) \tag{5.21}$$

(cf. [14]). Substitute from (5.19) into the right side of the decomposition (5.4), Lemma 5.2 and (5.20), and you find that for $x \neq 0$,

$$\begin{aligned} f_x(k) \;\; = \;\; & \frac{e_0}{|Q|^{1/2}(2\pi k)^{3/2}}\left[\left(1-\frac{G(-x)}{G(0)}+\frac{e_0|\tilde{x}|}{2\pi|Q|^{1/2}\,k}+\frac{b_3}{\sqrt{k}}\mathrm{P}_3\left(\frac{-\tilde{x}}{\sqrt{k}}\right)\right)e^{-\tilde{x}^2/2k}\right. \\ & \left. \qquad + \left(\frac{e_0}{2\pi|Q|^{1/2}\,|\tilde{x}|}-\frac{G(-x)}{G(0)}\right)(1-e^{-\tilde{x}^2/2k})\right] \\ & + o\left(\frac{1}{k^{(3+\delta)/2}}\wedge\frac{1}{\sqrt{k}\,|x|^{2+\delta}}\right)+o\left(\frac{\sqrt{k}\wedge|x|_+}{k^{2+\delta/2}|x|_+}\right) \end{aligned} \tag{5.22}$$

(the bottle neck here is the error term in (5.20) for $x^2 < k$ and that involved in $G(-x)f_0(k)$ for $x^2 \geq k$). In view of (5.21) the second term inside the big square brackets is at most

$$[1\wedge(x^2/k)]\times o\Big(|x|^{-1-\delta}\Big) + b_3 O\Big((k\wedge x^2)/k|x|^2\Big). \tag{5.23}$$

In the region $|\tilde{x}| \leq 4\sqrt{k\lg k}$ the error terms to $f_x(k)$ resulting from (5.23) as well as the one exhibited as the last term in (5.22) are all dominated by the first error term in (5.22) (i.e. the one in (5.20)), hence superfluous and may be deleted: note that for $|\tilde{x}| > 4\sqrt{k\lg k}$, the latter error term is dominant on the right side of (5.20), hence in (5.22) since $f_x(k) < p^k(-x)$, so that every other term is superfluous. Consequently we have the formula of Theorem 1.8 if $\delta < 2$.

Finally consider the case $\delta = 2$. Then both (5.19) and (5.21) hold true. The expansion of $p^k(-x)$ is also true if we add the third term of the Edgeworth expansion, which together with the quantities evaluated in (5.16) and (5.21) constitute the term involving $O(1+x^4/k^2)$ in the formula of Theorem 1.8. Terms of order at most $O(k^{-5/2}|x|^{-1})$ are absorbed in this one for the same reason as mentioned at the end of the last paragraph.

The proof of Theorem 1.8 is complete.

6 Appendices

(A) Let α, β, j and ν be real constants and $v(l)$ a continuous function of $l \geq 0$.

Lemma 6.1. *Suppose that $-\beta < 1+\nu < 2j-\beta \leq 2\alpha+3 < 1+2j$, or what is the same thing,*

$$2\alpha - 2j + \beta \geq -3,\ 2j-\beta-\nu > 1,\ \nu+\beta > -1,\ \alpha - j < -1 \tag{6.1}$$

and that $0 < \nu \leq 1$, $v(l) = O(l^{\beta+\nu})$ and for some constant C

$$|v(l+h) - v(l)| \leq C l^{\beta} h^{\nu} \qquad \text{whenever} \quad l \geq h > 0. \tag{6.2}$$

Let $a(t)$ be a differentiable function of $t \geq 0$ such that $a(t) = O(t^{\alpha})$ and $a'(t) = O(t^{\alpha-1})$. Then there exists a constant C' such that for $k > 0$, $x \in \mathbf{R}$

$$\left| \int_0^1 a(t) \left\{ \begin{array}{c} \cos kt \\ \sin kt \end{array} \right\} dt \int_0^1 \frac{v(l)}{(-it+l^2)^j} e^{ixl} dl \right| \leq \frac{C'}{|x|^{\nu}+1} \times \left\{ \begin{array}{l} \lg[(x^2/k) \vee e] \\ 1; \end{array} \right.$$

if $2\alpha - 2j + \beta > -3$, then the logarithmic term above may be replaced by 1.

Remark. (i) In our application we take $\alpha = 1/2$, $2j - \beta = 4$, $j = 2, 3, 4, 5, 6$.

(ii) If β is a positive integer, v has continuous derivatives of order up to and including β whose values at 0 all vanish, and the last derivative $v^{(\beta)}$ satisfies $v^{(\beta)}(l) = O(|l|^{\nu})$, then $|v(l+h) - v(l)| \leq C l^{\beta+\nu-1} h$ $(l \geq h > 0)$, which is stronger than (6.2). It is warned that if $\beta = 0$, the condition $v(l) = O(|l|^{\nu})$ is not sufficient for (6.2).

Proof. Let $g(t,l) = v(l)/(-it+l^2)^j$. Suppose that $x \geq 1$, which gives rise to no loss of generality. We consider the critical case $2\alpha - 2j + \beta = -3$ only; the other case is easy. Then $\int_0^{1/x^2} |a(t)| dt \int_0^1 |g(t,l)| dl \leq C \int_0^{1/x^2} t^{\nu/2-1} dt = O(x^{-\nu})$ (since $\nu > 0$) and

$$\int_{1/x^2}^1 |a(t)| dt \int_0^{\pi/x} |g(t,l)| dl \leq C \int_{1/x^2}^1 t^{\alpha} t^{-j} dt \int_0^{\pi/x} l^{\nu+\beta} dl = O(x^{-\nu}). \tag{6.3}$$

From the first two inequalities it follows that $\alpha > -1$, so that $\sup_{1/2<l<1} |g(t,l)|$ is integrable on $(0,1)$. Since $\int_{\pi/x}^1 g(t,l) e^{ixl} dl = -\int_0^{1-\pi/x} g(t, l+\pi/x) e^{ixl} dl$ and since the upper limit of the inner integrals in (6.3) may be $2\pi/x$ instead of π/x, we have

$$\begin{aligned} 2\int_{1/x^2}^1 a(t) e^{-ikt} dt \int_{\pi/x}^1 g(t,l) e^{ixl} dl \quad &= \quad \int_{1/x^2}^1 a(t) e^{-ikt} dt \int_{\pi/x}^{1-\pi/x} [g(t,l) - g(t, l+\pi/x)] e^{ixl} dl \\ &\quad + O(x^{-\nu}). \end{aligned}$$

By the hypothesis of the lemma we also have

$$g(t,l)-g(t,l+\pi/x)=\frac{l^{\beta}A(x,l)}{(-it+l^2)^j}\times x^{-\nu}\quad\text{ for }\quad l>\frac{\pi}{x} \tag{6.4}$$

where A is uniformly bounded. Up to now e^{-ikt} may be replaced by either of $\cos kt$ or $\sin kt$. We now evaluate the last double integral with e^{ikt} replaced by $\cos kt$ and $\sin kt$. To this end suppose $x^2>k$ in below; the case $x^2<k$ is easy to deal with. We make decomposition

$$\left(\int_{1/x^2}^{1/k}+\int_{1/k}^{1}\right)\left\{\begin{array}{l}\cos kt\\ \sin kt\end{array}\right\}dt\int_{\pi/x}^{1}\frac{a(t)l^{\beta}A(x,l)}{(-it+l^2)^j}e^{ixl}dl=I+II\quad\text{(say)}.$$

From the hypothesis $2\alpha-2j+\beta=-3$ it follows that the inner integral and its derivative are $O(1/t)$ and $O(1/t^2)$, respectively. Then an integration by parts shows that II is bounded, while I is dominated by a constant multiple of

$$\left|\int_{1/x^2}^{1/k}\frac{1}{t}\left\{\begin{array}{l}\cos kt\\ \sin kt\end{array}\right\}dt\right|\le\left\{\begin{array}{l}\lg[(x^2/k)\vee e]\\ 1.\end{array}\right.$$

This completes the proof of the lemma.

Lemma 6.1 concerns the situation that the inner integral in its formula diverges for $t=0$. The next lemma deals with the case when it converges.

Lemma 6.2. *Let a be as in Lemma 6.1 and ν satisfy $\nu(l)=O(l^{\beta+\nu})$ as well as the condition (6.1) as in Lemma 6.1 but with $0\le\nu\le1$. Suppose that $-1<\alpha<0$ and $\kappa:=\beta-2j+1\ge0$. Then there exists a constant C' such that*

$$\left|\int_0^1 a(t)e^{ikt}dt\int_0^1\frac{\nu(l)}{(-it+l^2)^j}e^{ixl}dl\right|\le\frac{C'}{k^{1+\alpha}(|x|^{\nu+\kappa}+1)}\times\left\{\begin{array}{ll}\lg[|x|\vee e] & if\ \ \kappa=0,\\ 1 & if\ \ \kappa>0.\end{array}\right.$$

Proof. Let $g(t,l)=\nu(l)/(-it+l^2)^j$ and $k\ge1,x\ge1$ as in the preceding proof. Then

$$\int_0^{\pi/x}|g|dl\le C\int_0^{1/x}l^{\nu+\beta-2j}\le C_1x^{-\nu-\kappa}\times\left\{\begin{array}{ll}\lg x & (\kappa=\nu=0),\\ 1 & (\nu+\kappa>0).\end{array}\right. \tag{6.5}$$

and from (6.4), which is available here also,

$$\left|\int_{\pi/x}^{1}g(t,l)e^{ixl}dl\right|\le C_2x^{-\nu-\kappa}\times\left\{\begin{array}{ll}\lg x & (\kappa=0),\\ 1 & (\kappa>0).\end{array}\right. \tag{6.6}$$

These together gives the bound of the lemma for $\int_0^{1/k}\left|a(t)\int_0^1 g(t,l)e^{ixl}dl\right|dt$. We must still examine the integral $\int_{1/k}^1 ae^{ikt}dt\int_0^1 ge^{ixl}dl$. To this end we integrate by parts w.r.t t. The boundary term is easily disposed of by (6.5) and (6.6) . It suffices to show that each of

$$I=\frac{1}{k}\int_{1/k}^{1}|a'(t)|\left|\int_0^1 g(t,l)e^{ixl}dl\right|dt,\quad II=\frac{1}{k}\int_{1/k}^{1}|a(t)|\left|\int_0^1\frac{\nu(l)}{(-it+l^2)^{j+1}}e^{ixl}dl\right|dt$$

admits the required bound. The first one is immediate from (6.5) and (6.6) . For the second one we see that its inner integral multiplied by $|t|$ admits the bounds in (6.5) and (6.6), which gives the required bound of II.

(B) Let $d \geq 2$ and $\mathbf{t}_r^{(d)}$ be as in REMARK 8. Then for $|x| > r > 0$,

$$E_x[\exp\{-\lambda \mathbf{t}_r^{(d)}\}] = \frac{G_\lambda(|x|, r)}{G_\lambda(r,r)} = \frac{K_{d/2-1}(|x|\sqrt{2\lambda})|x|^{1-d/2}}{K_{d/2-1}(r\sqrt{2\lambda})r^{1-d/2}} \quad (\lambda > 0), \tag{6.7}$$

where G_λ denotes the resolvent kernel for the d-dimensional Bessel process and K_ν is the usual modified Bessel function. For $d = 3$ the Laplace transform is easily inverted to yield the formula (1.12) (see (5.12)), which also follows from the one dimensional result since the three dimensional Bessel process conditioned on its eventually arriving at r is a one-dimensional Brownian motion.

(C) Here we give an asymptotic estimate of $P_x[\mathbf{t}_{r_\circ}^{(2)} \leq t] = \int_0^t q_{r_\circ}(s,x)ds$ for large t. Put

$$\varphi(\alpha) = -\int_1^\infty \frac{e^{-\alpha y}}{y} \lg\left(1 - \frac{1}{y}\right) dy \quad (\alpha > 0),$$

and

$$A_x(t) = \frac{1}{\lg(e^{c_\circ}t)}\left[1 - \frac{\gamma}{\lg(e^{c_\circ}t)}\right]\int_{x^2/2t}^\infty \frac{e^{-u}}{u} du + \frac{\varphi(x^2/2t)}{[\lg(e^{c_\circ}t)]^2}$$

so that $D(e^{c_\circ}n, x^2/2n) = A_x(t)$. (The function $D(t,\alpha)$ is defined in Theorem 1.6.) The following result belongs to [17], but not explicitly stated there.

Theorem 6.1. *Let $\xi = |x|/\sqrt{t}$. Then, uniformly for $|x| > r_\circ$, as $t \to \infty$*

$$P_x[\mathbf{t}_{r_\circ}^{(2)} \leq t] = A_x(t) + \frac{1}{(\lg t)^3} \times \begin{cases} O(\lg \frac{1}{2}\xi) & \textit{for } \; x^2 < t \\ O((\lg 2\xi)^2/\xi^2) & \textit{for } \; x^2 \geq t. \end{cases} \tag{6.8}$$

Proof. Immediate from Lemma 6 and Eq (26) of [17].

REMARK. (i) It holds that $\varphi(\alpha) = O(\alpha^{-1}e^{-\alpha}\log\alpha)$ as $\alpha \to \infty$ and $\varphi(\alpha) = \frac{1}{6}\pi^2 + \alpha \lg \alpha + O(\alpha)$ as $\alpha \downarrow 0$.

(ii) On using the identities $\int_1^\infty e^{-u}u^{-1}du + \int_0^1 (e^{-u} - 1)u^{-1}du = -\gamma$ and $2\gamma = \lg[2/e^{c_\circ}r_\circ^2]$

$$\int_{\xi^2/2}^\infty \frac{e^{-u}}{u} du = -\gamma - \lg(\xi^2/2) - \int_0^{\xi^2/2} \frac{e^{-u} - 1}{u} du = \gamma - \lg\frac{x^2}{r_\circ^2} + \lg(e^{c_\circ}t) + \frac{\xi^2}{2} + O(\xi^4).$$

With the help of this we deduce that for $x^2/t \leq 1$,

$$A_x(t) = 1 - \frac{2\lg(|x|/r_\circ)}{\lg(e^{c_\circ}t)}\left[1 - \frac{\gamma}{\lg(e^{c_\circ}t)}\right] + \frac{\frac{1}{6}\pi^2 - \gamma^2}{[\lg(e^{c_\circ}t)]^2} + \frac{\xi^2 \lg|x| + O(\xi^2)}{(\lg(e^{c_\circ}t))^2}.$$

(iii) Integrating the formula of Theorem 1 of [17] leads to

$$P_x[\mathbf{t}_{r_\circ}^{(2)} > t] = \frac{2\lg(|x|/r_\circ)}{\lg(e^{c_\circ}t)}\left[1 - \frac{\gamma}{\lg(e^{c_\circ}t)} - \frac{\frac{1}{6}\pi^2 - \gamma^2}{[\lg(e^{c_\circ}t)]^2} + \cdots\right] + O\left(\frac{\xi^2\lg|x|}{(\lg t)^2}\right)$$

($r_\circ < |x| < \sqrt{t}$), which can be squared with the expression of $1 - A_x(t)$ obtained from (ii). On equating the error terms $O(|\lg \frac{1}{2}\xi|/(\lg t)^3)$ and $O(\xi^2 \lg|x|)/(\lg t)^2)$ the latter formula is sharper than the former if $t^{-1}x^2\lg|x| \to 0$.

(D) Suppose that the period ν of the walk S_n is greater than 1. Let $d = 2$ for simplicity. Then, because of the irreducibility of the walk, there exists a proper subgroup $H \subset \mathbf{Z}^2$ and $\xi \in \mathbf{Z}^2$ such that $H + j\xi = H$ if and only if $j = 0 \pmod{\nu}$ and that $P[S_j^0 \in H + j\xi] = 1$. Let H be spanned by $h_1, h_2 \in \mathbf{Z}^2$ and determine $\lambda_1, \lambda_2 \in \mathbf{R}^2$ by the condition $\lambda_j \cdot h_j = 2\pi\delta_{lj}$. Then one may write $\xi = (\alpha_1 h_1 + \alpha_2 h_2)/\nu$ with some integers α_1, α_2, so that $\xi \cdot \lambda_l = 2\pi\alpha_l/\nu$ ($l = 1, 2$) and it holds that for each $j \in \{1, \dots, \nu - 1\}$, either $j\alpha_1 \neq 0 \pmod{\nu}$ or $j\alpha_2 \neq 0 \pmod{\nu}$. This condition implies that three integers α_1, α_2, ν have no common devisor except 1, so that there exists two integers k_1, k_2 such that $k_1\alpha_1 + k_2\alpha_2 = 1 \pmod{\nu}$. Putting $\lambda = k_1\lambda_1 + k_2\lambda_2$, we have $\xi \cdot \lambda = 2\pi/\nu \pmod{2\pi}$, hence $x \cdot \lambda = j2\pi/\nu \pmod{2\pi}$ if $x \in H + j\xi$ and it follows that $\psi(\theta + \lambda) = \psi(\theta)e^{i2\pi/\nu}$. The rest is the same as in [16].

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