# Central Limit Theorems and Quadratic Variations in terms of Spectral Density[*]

Hermine BIERMÉ[†]     Aline BONAMI[‡]     José R. LEÓN[§]

### Abstract

We give a new proof and provide new bounds for the speed of convergence in the Central Limit Theorem of Breuer Major on stationary Gaussian time series, which generalizes to particular triangular arrays. Our assumptions are given in terms of the spectral density of the time series. We then consider generalized quadratic variations of Gaussian fields with stationary increments under the assumption that their spectral density is asymptotically self-similar and prove Central Limit Theorems in this context.

**Key words:** Central limit theorem, Gaussian stationary process, spectral density, periodogram, quadratic variations, fractional Brownian Motion.

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[†]Hermine BIERMÉ, MAP5 UMR 8145 Université Paris Descartes, 45 rue des Saints-Pères, 75006 Paris France, hermine.bierme@mi.parisdescartes.fr

[‡]Aline BONAMI, MAPMO UMR 6628, Université d'Orleans, 45067 Orléans Cedex 2, France, aline.bonami@univ-orleans.fr

[§]José R. LEÓN, Escuela de Matemáticas, Facultad de Ciencias, UCV. AP: 47197. Los Chaguaramos. Caracas 1041-A. Venezuela,jose.leon@ciens.ucv.ve

# 1   Introduction

In this paper we essentially develop Central Limit Theorems that are well adapted to obtain asymptotic properties of quadratic variations of Gaussian fields with stationary increments. Moreover, we give bounds for the speed of convergence, which partially improve the bounds given by Nourdin and Peccati in [22]. We rely heavily on their methods but adopt a spectral point of view, which is particularly adapted to applications in signal processing.

Before describing our theoretical results, let us describe the scope of applications that we have in mind. The finite distributional properties of a real valued Gaussian field $\{Y(t); t \in \mathbb{R}^{\nu}\}$, indexed by $\mathbb{R}^{\nu}$ ($\nu \geq 1$), with stationary increments, may be described from its variogram, that is, the function

$$v_Y(t) := \mathbb{E}(((Y(s+t) - Y(s))^2) \tag{1}$$

or from its spectral measure $\tau$, which is such that

$$v_Y(t) = 2 \int_{\mathbb{R}^{\nu}} |e^{-it \cdot x} - 1|^2 \, d\tau(x), \ \forall t \in \mathbb{R}^{\nu}. \tag{2}$$

Here $t \cdot x$ stands for the scalar product of the two vectors in $\mathbb{R}^{\nu}$ and $|x|$ denotes the Euclidean norm of the vector $x$. The spectral measure $\tau$ is a non negative even measure on $\mathbb{R}^{\nu}$. We will only consider absolutely continuous spectral measures, that is, measures that can be written as $d\tau(x) := F(x)dx$, where $dx$ is the Lebesgue measure on $\mathbb{R}^{\nu}$. The function $F$, called the *spectral density* of $Y$, is assumed to be a non-negative even function of $L^1\left(\mathbb{R}^{\nu}, \min\left(1, |x|^2\right) dx\right)$. A typical example of such random fields is given by

$$Y(t) = \int_{\mathbb{R}^{\nu}} \left(e^{-it \cdot x} - 1\right) F(x)^{1/2} d\widetilde{W}^{\nu}(x), t \in \mathbb{R}^{\nu}, \tag{3}$$

where $\widetilde{W}^{\nu}$ is a complex centered Gaussian random measure on $\mathbb{R}^{\nu}$ with Lebesgue control measure, such that $\widetilde{W}^{\nu}(-A) = \overline{\widetilde{W}^{\nu}(A)}$ a.s. for any Borel set $A$ of $\mathbb{R}^{\nu}$. In fact, if we are only interested by finite distributions of the random field $Y$, we can always assume that $Y$ is given by such a spectral representation (3).

Centered Gaussian fields with stationary increments are widely used as models for real data, for example to describe rough surfaces or porous media that possess some homogeneity properties. In particular the fractional Brownian field (fBf), first defined in dimension $\nu = 1$ through a stochastic integral of moving-average type by Mandelbrot and Van Ness [20], admits such a representation with spectral density given by

$$F_H(x) = \frac{c}{|x|^{2H+\nu}}, \text{ with } c > 0 \text{ and } H \in (0,1) \text{ called the Hurst parameter.}$$

The homogeneity of $F_H$ implies a self-similarity property of the corresponding random field $Y_H$, namely

$$\forall \lambda > 0, \ \{Y_H(\lambda t); t \in \mathbb{R}^{\nu}\} \overset{fdd}{=} \lambda^H \{Y_H(t); t \in \mathbb{R}^{\nu}\}.$$

The choice of this power of the Euclidean norm for the spectral density is equivalent to the fact that the variogram is $v_{Y_H}(t) = c_H |t|^{2H}$, for some positive constant $c_H$. When $\nu \geq 2$, it induces the

isotropy of the field $Y_H$ (its law is invariant under vectorial rotations). Such a model is not adapted when anisotropic features are observed. Anisotropic but still self-similar generalizations are simply obtained by considering a spectral density given by $F(x) = \Omega(x)F_H(x)$ with $\Omega$ an homogeneous function of degree 0 satisfying $\Omega(x) = \Omega(x/|x|)$. Then the corresponding variogram is given in a similar form

$$v(t) = \omega(t/|t|)|t|^{2H}, \text{ with } \omega(t) = c_{H,v} \int_{|x|=1} |t \cdot x|^{2H} \Omega(x)\sigma(dx),$$

where $\sigma(dx)$ denotes the Lebesgue measure on $S^{v-1} := \{x \in \mathbb{R}^v; |x| = 1\}$. When using such a model, a typical question is the identification of the Hurst parameter $H$ from real data. Many estimators for the Hurst parameter of a one-dimensional fBf (called fractional Brownian motion) have been proposed, based for example on time domain methods or spectral methods (see [10] and [3] and references therein). Quadratic variations are relevant estimators when considering $H$ as the critical index of Hölder regularity for the sample paths. Moreover in [18] the authors give precise bounds of the bias of the variance and show that minimax rates are achieved for this kind of estimators. Generalized quadratic variations also apply to more general Gaussian processes and fields with stationary increments with the same Hölder regularity (see [16, 17] or [8, 9] for instance), for which the variogram satisfies

$$v(t) = \omega(t/|t|)|t|^{2H} + \underset{|t|\to 0}{O}\left(|t|^{2H+s}\right)$$

for $H \in (0, 1)$ and $s \in (0, 2 - 2H)$ with $\omega$ a positive function on the sphere $S^{v-1}$ (and additional assumptions of regularity). This kind of assumption can be replaced by an assumption on the spectral density $F$ (which is a priori stronger but does not require any extra assumption of regularity). More precisely, we will be interested in random fields for which

$$F(x) = \frac{\Omega(x)}{|x|^{2H+v}} + \underset{|x|\to+\infty}{O}\left(\frac{1}{|x|^{2H+v+\gamma}}\right), \tag{4}$$

with $\Omega$ an even function on the sphere $S^{v-1}$ (or a constant when $v = 1$), $H > 0$ and $\gamma > 0$. Our particular interest in this situation, where the self-similar spectral density is perturbed by a rest that decreases more rapidly at infinity, may be understood from previous work [7, 5]. This arises in particular when one considers a weighted projection of a self-similar random field. We develop here methods that reveal to be stable when adding such a perturbation to the spectral density. A source of inspiration for us has also been the paper of Chan and Wood [8], which deals with stationary random Gaussian fields with asymptotic self-similar properties.

The estimation of $\Omega$ or $H$ goes through the consideration of quadratic variations of $Y$, observed on finer and finer grids. Typically, we assume to have observed values of the random field on a grid with uniform mesh, that is, $\{Y(k/n); k = (k_1, \ldots, k_v) \in \mathbb{Z}^v$ with $0 \le k_1, \ldots, k_v \le n-1\}$. We want to have Central Limit Theorems for the quadratic variation of this sequence when $n$ tends to $\infty$. For fixed $n$, this quadratic variation is related to the means of a discrete time series, whose spectral density is obtained by periodization of $F$. Moreover, a central idea used in this paper consists in a change of scale, so that we can as well consider a fixed mesh, but for a different discrete time series at each scale. Because of the fact that $F$ is asymptotically homogeneous, the rest does not appear in the limit, and acts only on the speed of convergence, which is in $n^{-\alpha}$, for some $\alpha > 0$ depending in particular on $\gamma$. Once we have Central Limit Theorems for finite distributions through this scaling argument,

we can also recover asymptotic properties for continuous time quadratic variations, which may be used when dealing with increments of non linear functionals of $Y$ instead of increments of $Y$.

Let us come back to the theoretical part of this paper, which constitutes its core. We revisit Breuer Major's Theorem, which is our main tool to obtain Central Limit Theorems, and use the powerful theory developed by Nourdin, Nualart, Ortiz-Latorre, Peccati, Tudor and others to do so. This is described in the next section and we refer to it for more details. We would like to attract attention to a remark, which has its own interest: under appropriate additional assumptions, the square of the norm of the Malliavin derivative of $\dfrac{1}{\sqrt{n}} \sum_{k=0}^{n-1} H(X(k))$, where $(X(k))_{k \in \mathbb{Z}}$ is a stationary Gaussian time series and where the Hermite expansion of $H$ starts with terms of order 2, can be written in terms of the integrated periodogram of the sequence $H'(X(k))$. Recall that the periodogram of this time series is defined as

$$\frac{1}{n} \left| \sum_{k=0}^{n-1} H'(X(k)) e^{ikx} \right|^2 .$$

Up to our knowledge, this link between two different theories had not been given before. As a consequence, the techniques that we use for having the speed of convergence in Central Limit Theorems may be used for consistency of estimators given in terms of integrated periodograms.

Section 2 is devoted to the theoretical aspects (Central Limit Theorems, integrated periodograms, speed of convergence) in dimension one. We chose to give the proofs in this context, so that the reader can easily follow them. Once this has been done, we hope that it is not difficult to see how to adapt them in higher dimension, which we do more rapidly in Section 3. We then apply this to generalized quadratic variations in Section 4.

## 2    Breuer-Major Theorem revisited

In this section we will be interested in stationary centered Gaussian time series $X = (X(k))_{k \in \mathbb{Z}}$ as well as approximate ones. We will start from Breuer-Major Theorem and give a proof of it which is based on the Malliavin Calculus, as exploited by Nourdin, Nualart, Ortiz-Latorre, Peccati, among others, to develop Central Limit Theorems in the context of Wiener Chaos (see [27, 22] for instance). This kind of proof is implicit in the work of these authors, and explicit in the last paper of Nourdin, Peccati and Podolskij [23], where speeds of convergence are given in a very general context. Our interest, here, is to see that assumptions are particularly simple and meaningful when they are given on the spectral density of the time series. Meanwhile, we improve the estimates for the speed

of convergence to the best possible through this method under the assumption that the spectral density is in some Sobolev space. Also, this study will lead us to asymptotic estimates on integrated periodograms, which have their own interest and are particularly relevant when one interests to spectral densities.

Let us first state the theorem of Breuer Major in the simplest one dimensional case. For $l \geq 1$, we consider the stationary centered time series $H_l(X) = \left( H_l(X(k)) \right)_{k \in \mathbb{Z}}$ where $H_l$ is the $l$-th Hermite polynomial defined as

$$H_l(t) = (-1)^l e^{\frac{t^2}{2}} \frac{d^l}{dt^l} e^{-\frac{t^2}{2}}, \ t \in \mathbb{R}.$$

**Theorem 2.1** (Breuer-Major)**.** *Let $(X(k))_{k \in \mathbb{Z}}$ be a centered stationary Gaussian time series. Assume that for $l \geq 1$, the sequence $r(k) = \mathbb{E}(X(j)X(j+k))$ satisfies the condition*

$$\sum_{k \in \mathbb{Z}} |r(k)|^l < \infty. \tag{5}$$

*Then we have the following asymptotic properties for n tending to infinity:*

(i)

$$\mathrm{Var}\left( \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} H_l(X(k)) \right) \longrightarrow \sigma_l^2,$$

(ii)

$$\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} H_l(X(k)) \xrightarrow{d} \mathcal{N}(0, \sigma_l^2),$$

*with*

$$\sigma_l^2 = l! \sum_{k \in \mathbb{Z}} r(k)^l. \tag{6}$$

For $p \geq 1$, we introduce the Banach space $\ell^p(\mathbb{Z})$ of $p$-summable sequences equipped with the norm

$$\|u\|_{\ell^p(\mathbb{Z})} = \left( \sum_{k \in \mathbb{Z}} |u(k)|^p \right)^{1/p} \quad \text{for } u = (u(k))_{k \in \mathbb{Z}} \in \ell^p(\mathbb{Z}). \text{ Next, we identify } 2\pi\text{-periodic functions}$$

both with functions on the torus $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$ and functions on $[-\pi, +\pi)$ and introduce the spaces $L^p(\mathbb{T})$ of measurable functions $f$ on $[-\pi, +\pi)$ such that

$$\|f\|_p^p := \|f\|_{L^p(\mathbb{T})}^p := \frac{1}{2\pi} \int_{\mathbb{T}} |f(x)|^p dx = \frac{1}{2\pi} \int_{-\pi}^{+\pi} |f(x)|^p dx. \tag{7}$$

Then, Assumption (5) can be written as $r = (r(k))_{k \in \mathbb{Z}} \in \ell^l(\mathbb{Z})$. We recall that the sequence $r(k)$ can be seen as the Fourier coefficients of a positive even periodic finite measure, called the spectral measure of the time series (see [12] or [29] for instance). Most of the time, for the sake of simplicity, we assume that the spectral measure of the stationary process is absolutely continuous with respect to the Lebesgue measure and we call $f_X$ its density with respect to the Lebesgue $\mu$. We speak of *spectral density of the time series* as it is classical. Let us recall that $f_X$ is an even function of $L^1(\mathbb{T})$ whose Fourier coefficients are given by the sequence $r = (r(k))_{k \in \mathbb{Z}}$, which means that

$$r(k) := \frac{1}{2\pi} \int_{\mathbb{T}} e^{-ikx} f_X(x) dx = \frac{1}{2\pi} \int_{-\pi}^{+\pi} e^{-ikx} f_X(x) dx. \tag{8}$$

Remark that the absolute continuity is only an additional property under (5) when $l > 2$. Actually, when $l = 2$ the sequence $r(k)$ is square summable and one can find $f_X \in L^2(\mathbb{T})$ by Plancherel's Theorem.

**Remark 2.2.** *Recall that, for U and V two centered Gaussian variables such that $\mathbb{E}(UV) = \rho$, we have $\mathbb{E}(H_l(U)H_l(V)) = l!\rho^l$. So, whenever the time series X has the spectral density $f_X$, the time series $H_l(X)$ has the density $f_{H_l(X)} = l!f_X^{*l}$, where the notation $f_X^{*l}$ stands for l times the convolution of $f_X$ by itself on the torus. Assumption (5) implies the absolute convergence of the Fourier series of the spectral density of $H_l(X)$. It also means that $f_X^{*l}$ is continuous on $\mathbb{T}$ and that $\sigma_l^2 = l!f_X^{*l}(0)$ according to (6). For all $l \geq 2$, Assumption (5) is in particular implied by the stronger assumption*

$$f_X \in L^{\frac{l}{l-1}}(\mathbb{T}),\tag{9}$$

*since $\|r\|_{\ell^l(\mathbb{Z})} \leq \|f_X\|_{\frac{l}{l-1}}$ by Hausdorff-Young Inequality (see [19] for instance).*

**Remark 2.3.** *Note also that the assumption that the Gaussian time series X has a spectral density $f_X$ implies in particular that the time series $H_l(X)$ is a strictly stationary ergodic one, for any $l \geq 1$, (see [11] for instance).*

We will give a new proof of the theorem of Breuer Major. We will use for instance Theorem 4 in [27], which asserts that the Central Limit Theorem is a consequence of the convergence of the variance given in (i) on one side, then of a quantity related to the Malliavin derivative on another side, so that one does not need to consider all moments as in the original proof of Breuer and Major. In a first subsection we recall the main tools in our framework in the classical context of stochastic integrals of a Brownian Motion, which is adapted here to a spectral point view when using harmonizable representation (see [12] for instance). This could be generalized to isonormal Gaussian processes, as it is developed in the first chapter of [26] and used in [23], but we preferred to restrict to the classical case for simplification, even if the general context is necessary in the vectorial case. The price to pay is the fact that we only prove Breuer Major under Assumption (8) but the interested reader can easily generalize our proof (see Remark 2.4).

## 2.1 Complex Wiener chaos and Malliavin calculus

Let $W$ be a complex centered Gaussian random measure on $[-\pi, +\pi)$ with Lebesgue control measure $\frac{1}{2\pi}dx$ such that, for any Borel set $A$ of $[-\pi, \pi)$ we have $W(-A) = \overline{W(A)}$ almost surely. We consider complex-valued functions $\psi$ defined on $[-\pi, +\pi)$, considered as periodic functions of the torus that satisfy, for almost every $x \in \mathbb{T}$,

$$\overline{\psi(x)} = \psi(-x).$$

We write $L_e^2(\mathbb{T})$ for the real vector space of such functions that are square integrable with respect to the Lebesgue measure on $\mathbb{T}$. Endowed with the scalar product of $L^2(\mathbb{T})$, which we also note

$$\langle \psi, \varphi \rangle_{L^2(\mathbb{T})} = \frac{1}{2\pi} \int_{\mathbb{T}} \psi(x)\overline{\varphi(x)}dx,$$

$L_e^2(\mathbb{T})$ is a real separable Hilbert space. Moreover, for any $\psi \in L_e^2(\mathbb{T})$, one can define its stochastic integral with respect to $W$ as

$$I_1(\psi) = \int_{-\pi}^{+\pi} \psi(x)dW(x).$$

Then $I_1(\psi)$ is a real centered Gaussian variable with variance given by $\|\psi\|_2^2$, where $\|\cdot\|_2$ is the norm induced by the scalar product $\langle \cdot, \cdot \rangle_{L^2(\mathbb{T})}$. To introduce the $k$-th Itô-Wiener integral, with $k \geq 1$, we consider the complex functions belonging to

$$L_e^2(\mathbb{T}^k) = \{\psi \in L^2(\mathbb{T}^k) : \psi(-x) = \overline{\psi(x)}\}.$$

The inner product in the real Hilbert space of complex functions of $L_e^2(\mathbb{T}^k)$ is given by

$$\langle \psi, \varphi \rangle_{L^2(\mathbb{T}^k)} = \frac{1}{(2\pi)^k} \int_{\mathbb{T}^k} \psi(x)\overline{\varphi(x)}dx.$$

The space $L_s^2(\mathbb{T}^k)$ denotes the subspace of functions of $L_e^2(\mathbb{T}^k)$ a.e. invariant under permutations of their arguments. By convention $L_s^2(\mathbb{T}^k) = \mathbb{R}$ for $k = 0$. Let us define $H(W)$ the subspace of random variables in $L^2(\Omega, \mathbb{P})$ measurable with respect to $W$. The $k$-Itô-Wiener integral $I_k$ is defined in such a way that $(k!)^{-1/2}I_k$ is an isometry between $L_s^2(\mathbb{T}^k)$ and its range $\mathcal{H}_k \subset H(W)$, so that we have the orthogonal decomposition

$$H(W) = \bigoplus_{k=0}^{\infty} \mathcal{H}_k,$$

where $\mathcal{H}_0$ is the space of real constants. Each $Y \in H(W)$ has an $L^2(\Omega, \mathbb{P})$ convergent decomposition

$$Y = \sum_{k=0}^{\infty} I_k(\psi_k), \quad \psi_k \in L_s^2(\mathbb{T}^k).$$

When moreover $\sum_{k=1}^{+\infty}(k+1)!\|\psi_k\|_2^2 < +\infty$, with $\|\psi_k\|_2^2 = \langle \psi_k, \psi_k \rangle_{L^2(\mathbb{T}^k)}$, the Malliavin derivative of $Y$, denoted by $DY$, is defined as the complex valued random process given on $\mathbb{T}$ by

$$D_tY = \sum_{k=1}^{+\infty} kI_{k-1}\left(\psi_k(\cdot, t)\right), t \in \mathbb{T}.$$

Furthermore if $H_k$ is the $k$-th Hermite polynomial for the standard Gaussian measure and denoting by $\psi^{\odot_k}$ the $k$-tensor product of the function $\psi \in L_e^2(\mathbb{T})$ we have

$$H_k(I_1(\psi)) = I_k(\psi^{\odot_k}) := \int_{[-\pi, +\pi)^k} \psi(x_1)\ldots\psi(x_k)dW(x_1)\ldots dW(x_k), \tag{10}$$

so that its Malliavin derivative is given by $DH_k(I_1(\psi)) = kH_{k-1}(I_1(\psi))\psi$.

## 2.2 New proof of Breuer Major Theorem

For sake of simplicity we provide a new proof under Assumption (8). As far as finite distributions are concerned, when the time series $X$ admits a covariance function given by (8) for some even function $f_X \in L^1(\mathbb{T})$, we can assume, without loss of generality, that

$$X(k) := \int_{-\pi}^{+\pi} e^{-ikx} g(x) dW(x), \tag{11}$$

with $g = f_X^{1/2} \in L_e^2(\mathbb{T})$. Up to normalization, we can also assume that $r(0) = 1$, or equivalently that $\|g\|_2^2 = \frac{1}{2\pi} \int_{\mathbb{T}} |g(x)|^2 dx = \frac{1}{2\pi} \int_{\mathbb{T}} f_X(x) dx = 1$. For any $k \in \mathbb{Z}$, we write $g_k(x) = e^{-ikx} g(x) \in L_e^2(\mathbb{T})$ so that $X(k)$ may be written as the Itô-Wiener integral $I_1(g_k)$. Moreover $H_l(X)$ is in the Wiener chaos of order $l$ with

$$H_l(X(k)) = I_l\left(g_k^{\odot l}\right), k \in \mathbb{Z}.$$

Let us now proceed to the proof. The computation of the variance in (i) is direct. Let us write

$$Y_n = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} H_l(X(k)).$$

Then,

$$\begin{aligned}
\mathrm{Var}(Y_n) &= \frac{1}{n} \sum_{k=0}^{n-1} \sum_{k'=0}^{n-1} \mathrm{Cov}(H_l(X(k)), H_l(X(k'))) \\
&= \frac{l!}{n} \sum_{k=0}^{n-1} \sum_{k'=0}^{n-1} r(k-k')^l \\
&= l! \sum_{k=-(n-1)}^{n-1} \left(1 - \frac{|k|}{n}\right) r(k)^l,
\end{aligned} \tag{12}$$

which tends to $l! \sum_{k \in \mathbb{Z}} r(k)^l = l!\sigma_l^2$. Recall that this last sum is absolutely convergent because of the assumption on $r$. This concludes the proof when $l = 1$ since $Y_n$ is a Gaussian variable in this case.

When $l \geq 2$ we write $Y_n = I_l(F_n)$, with $F_n = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} g_k^{\odot l}$. Then, by Theorem 4 of [27], to prove Part (ii) it is necessary and sufficient to prove that

$$\|DY_n\|_2^2 \underset{n\to+\infty}{\longrightarrow} l\sigma_l^2 \text{ in } L^2(\Omega, \mathbb{P}),$$

with $DY_n$ the Malliavin's Derivative of $Y_n$, which is given by

$$DY_n = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} lH_{l-1}(I_1(g_k))g_k.$$

We first remark that

$$
\begin{aligned}
\left\| DY_n \right\|_2^2 &= \frac{l^2}{n} \sum_{k,k'=0}^{n-1} H_{l-1}(I_1(g_k)) H_{l-1}(I_1(g_{k'})) \langle g_k, g_{k'} \rangle_{L^2(\mathbb{T})} \\
&= \frac{l^2}{n} \sum_{k,k'=0}^{n-1} H_{l-1}(X(k)) H_{l-1}(X(k')) r(k-k') \\
&= \frac{l^2}{2\pi} \int_{-\pi}^{\pi} \Pi_n^{(l-1)}(x) f_X(x) dx,
\end{aligned}
$$

where

$$
\Pi_n^{(l-1)}(x) = \frac{1}{n} \sum_{k,k'=0}^{n-1} H_{l-1}(X(k)) H_{l-1}(X(k')) e^{i(k'-k)x} = \frac{1}{n} \left| \sum_{k=0}^{n-1} H_{l-1}(X(k)) e^{ikx} \right|^2,
$$

is the periodogram of order $n$ of the stationary sequence $\left( H_{l-1}(X(k)) \right)_{k \in \mathbb{Z}}$ (see [14] for instance). The end of the proof is a direct consequence of the next subsection, which is devoted to the limit of integrated periodograms. The fact that the square of the norm of the Malliavin derivative may be written in terms of the periodogram is an unexpected phenomenon.

**Remark 2.4.** *The additional assumption (8) is not necessary to use the method above. Indeed, there is always an isonormal Gaussian process $\{W(u) : u \in \mathfrak{H}\}$, where $\mathfrak{H}$ is a separable Hilbert space, such that $X(k)$ may be seen as $W(u_k)$, with $u_k$ a sequence in $\mathfrak{H}$ such that $\langle u_k, u_{k'} \rangle_{\mathfrak{H}} = r(k-k')$. Then our proof of Breuer Major is the same replacing $L^2(\mathbb{T})$ by $\mathfrak{H}$. This is in particular used in [23] to get the speed of convergence under stronger assumptions than ours.*

## 2.3 Integrated periodograms

We keep the notations of the last subsection, so that for $l \geq 1$,

$$
\Pi_n^{(l)}(x) := \frac{1}{n} \left| \sum_{k=0}^{n-1} H_l(X(k)) e^{ikx} \right|^2. \tag{13}
$$

The periodogram $\Pi_n^{(l)}$ is used as an estimator of the spectral density of the stationary sequence $\left( H_l(X(k)) \right)_{k \in \mathbb{Z}}$, that is $l! f_X^{*l}$ since its Fourier coefficients are equal to $l! r(k)^l$. It is well known that $\Pi_n^{(l)}(x)$ is not a consistent estimate of $l! f_X^{*l}(x)$, even when well defined because of continuity (see [14] for instance). However we can hope consistency results for

$$
\Pi_{\phi,n}^{(l)} := \frac{1}{2\pi} \int_{-\pi}^{+\pi} \Pi_n^{(l)}(x) \phi(x) dx. \tag{14}
$$

Here $\phi$ is a test function, which is real, even, integrable and has some smoothness properties to be stated later on. Such quantities $\Pi_{\phi,n}^{(l)}$ are called *integrated periodograms*.

We have the following proposition, which gives in particular the asymptotic properties that are required for the proof of Breuer Major Theorem. We introduce $c_k(\phi) = \frac{1}{2\pi} \int_{\mathbb{T}} \phi(x) e^{-ikx} dx$ the $k$-th Fourier coefficient of $\phi$.

**Proposition 2.5.** *Assume that $(r(k))_{k\in\mathbb{Z}} \in \ell^{l+1}(\mathbb{Z})$ and $\sum_{k\in\mathbb{Z}} |c_k(\phi)|^{l+1} < \infty$. Then, as $n$ tends to $\infty$,*

*(i)* $\mathbb{E}(\Pi^{(l)}_{\phi,n}) \longrightarrow \frac{l!}{2\pi} \int_{\mathbb{T}} f_X^{*l}(x)\phi(x)dx$;

*(ii)* $Var\left(\Pi^{(l)}_{\phi,n}\right) \longrightarrow 0$.

Remark that conditions on $r$ and $\phi$ imply that we may give meaning to the integral $\frac{1}{2\pi}\int f_X^{*l}(x)\phi(x)dx$ as $\sum_{k\in\mathbb{Z}} r(k)^l c_k(\phi)$, using the absolute convergence of the series. When $\phi = f_X$, as in the proof of Breuer Major Theorem, the limit of the expectation is $l! \sum_{k\in\mathbb{Z}} r(k)^{l+1} = \sigma^2_{l+1}/(l+1)$ in view of (6).

*Proof.* The first assertion follows from the fact that

$$\mathbb{E}(\Pi^{(l)}_{\phi,n}) = l! \sum_{k=-n+1}^{n-1} \left(1 - \frac{|k|}{n}\right) r(k)^l c_k(\phi).$$

Next, in view of the second assertion, we consider the components of $\left\|DY_n\right\|_2^2$ in the Wiener chaos and use for this the multiplication formula (see [15] for instance), which we recall now:

$$H_l(X(k))H_l(X(k')) = \sum_{p=0}^{l} p!(2(l-p))!\binom{l}{p}^2 I_{2l-2p}(g_k^{\odot l} \tilde{\otimes}_p g_{k'}^{\odot l}),$$

with

$$g_k^{\odot l} \tilde{\otimes}_p g_{k'}^{\odot l} = \langle g_k, g_{k'}\rangle^p_{L^2(\mathbb{T})} \left(g_k^{\odot l-p} \otimes g_{k'}^{\odot l-p}\right)_s.$$

Here, when $\psi \in L_e^2(\mathbb{T}^k)$, we write $(\psi)_s$ its symmetrization in $L_s^2(\mathbb{T}^k)$, for $k \geq 2$. For simplification, we note $s(k) := c_k(\phi)$. Then we have

$$\Pi^{(l)}_{\phi,n} - \mathbb{E}(\Pi^{(l)}_{\phi,n}) = \sum_{p=0}^{l-1} p!(2(l-p))!\binom{l}{p}^2 U_{p,n},$$

with

$$
\begin{aligned}
U_{p,n} &= \frac{1}{n}\sum_{k,k'=0}^{n-1} s(k-k')I_{2l-2p}(g_k^{\odot l} \tilde{\otimes}_p g_{k'}^{\odot l}) \\
&= \frac{1}{n}\sum_{k,k'=0}^{n-1} s(k-k')r(k-k')^p I_{2l-2p}((g_k^{\odot l-p} \otimes g_{k'}^{\odot l-p})_s).
\end{aligned}
$$

Using orthogonality between components, it is sufficient to consider each of them separately. The next lemma gives the convergence in $L^2(\Omega, \mathbb{P})$ of each term.

**Lemma 2.6.** *Assume that $\sum_{k\in\mathbb{Z}} |r(k)|^{l+1} < \infty$ and $\sum_{k\in\mathbb{Z}} |s(k)|^{l+1} < \infty$. Let $p < l$. Then $\mathbb{E}(|U_{p,n}|^2)$ tends to $0$ for $n$ tending to $\infty$.*

*Proof.* We can write $U_{p,n}$ as $I_{2(l-p)}\left(\left(F_{p,n}\right)_s\right)$, with

$$F_{p,n} := \frac{1}{n}\sum_{k,k'=0}^{n-1} s(k-k')r(k-k')^p g_k^{\odot l-p} \otimes g_{k'}^{\odot l-p}.$$

By isometry, the $L^2(\Omega,\mathbb{P})$ norm of $U_{p,n}$ is equal, up to the constant $(2(l-p))!^{1/2}$, to the $L_s^2\left(\mathbb{T}^{2(l-p)},\right)$ norm of $\left(F_{p,n}\right)_s$. Now $\left(F_{p,n}\right)_s$ may be written as the mean of $(2(l-p))!$ terms, corresponding to permutations of the $2(l-p)$ variables. It is easy to see that all terms have the same norm, so that the $L_s^2\left(\mathbb{T}^{2(l-p)}\right)$ norm of $\left(F_{p,n}\right)_s$ is bounded by the $L_e^2\left(\mathbb{T}^{2(l-p)}\right)$ norm of one of the terms, that is

$$\left\|F_{p,n}\right\|_{L^2\left(\mathbb{T}^{2(l-p)}\right)} = \left\|\frac{1}{n}\sum_{k,k'=0}^{n-1} s(k-k')r(k-k')^p g_k^{\odot l-p} \otimes g_{k'}^{\odot l-p}\right\|_{L^2\left(\mathbb{T}^{2(l-p)}\right)}.$$

Finally, $\left\|F_{p,n}\right\|_{L^2\left(\mathbb{T}^{2(l-p)}\right)}^2$ is equal to

$$\frac{(2(l-p))!}{(2\pi)^{2l-2p}}\int_{(-\pi,+\pi)^{l-p}\times(-\pi,+\pi)^{l-p}} K_{p,n}(x,y)f_X(x_1)\cdots f_X(x_{l-p})f_X(y_1)\cdots f_X(y_{l-p})dxdy,$$

with

$$K_{p,n}(x,y) := \left|\frac{1}{n}\sum_{k,k'=0}^{n-1} r(k-k')^p s(k-k')e^{-ik(x_1+\cdots+x_{l-p})}e^{ik'(y_1+\cdots+y_{l-p})}\right|^2$$

$$= \frac{1}{n^2}\sum_{j,j',k,k'=0}^{n-1} r(k-k')^p s(k-k')r(j-j')^p s(j-j')e^{-i(k-j)(x_1+\cdots+x_{l-p})}e^{i(k'-j')(y_1+\cdots+y_{l-p})}.$$

Then

$$\left\|F_{p,n}\right\|_{L^2\left(\mathbb{T}^{2(l-p)}\right)}^2 = \frac{1}{n^2}\sum_{j,j',k,k'=0}^{n-1} r(k-k')^p s(k-k')r(j-j')^p s(j-j')r(k-j)^{l-p}r(k'-j')^{l-p}. \quad (15)$$

We pose $\rho_1(k) := |r(k)|^p|s(k)|$ and $\rho_2(k) = |r(k)|^{l-p}$, for $k \in \mathbb{Z}$ and denote by $\rho_{1,n}(k)$, (resp. $\rho_{2,n}(k)$) the truncated sequence $\rho_{1,n}(k) = \rho_1(k)$ (resp. $\rho_{2,n}(k) = \rho_2(k)$) when $|k| \leq n-1$ and $0$ otherwise. Then,

$$\left\|F_{p,n}\right\|_{L^2\left(\mathbb{T}^{2(l-p)}\right)}^2 \leq \frac{1}{n^2}\sum_{j,j',k,k'=0}^{n-1} \rho_{1,n}(k-k')\rho_{1,n}(j-j')\rho_{2,n}(k-j)\rho_{2,n}(j'-k')$$

$$\leq \frac{1}{n^2}\sum_{k,j'=0}^{n-1} \left(\rho_{1,n}*\rho_{2,n}(k-j')\right)^2.$$

It follows that

$$\mathbb{E}\left(|U_{p,n}|^2\right) \leq \frac{(2(l-p))!}{n}\sum_{|j|\leq n-1}\left(1-\frac{|j|}{n}\right)\left(\rho_{1,n}*\rho_{2,n}(j)\right)^2, \quad (16)$$

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The convolution product of $\rho_{1,n}$ and $\rho_{2,n}$ is bounded by

$$\rho_{1,n} * \rho_{2,n}(k) = \sum_{k' \in \mathbb{Z}} \rho_{1,n}(k - k')\rho_{2,n}(k') \le \|\rho_{1,n}\|_{\ell^{\frac{l+1}{p+1}}(\mathbb{Z})} \|\rho_{2,n}\|_{\ell^{\frac{l+1}{l-p}}(\mathbb{Z})}.$$

Assumption on $r$ implies that $\rho_2 \in \ell^{\frac{l+1}{l-p}}(\mathbb{Z})$ with $\|\rho_{2,n}\|_{\ell^{\frac{l+1}{l-p}}(\mathbb{Z})} \le \|\rho_2\|_{\ell^{\frac{l+1}{l-p}}(\mathbb{Z})} \le \|r\|_{\ell^{l+1}(\mathbb{Z})}^{l-p}$. Together with Assumption on $s$ and Hölder Inequality, $\rho_1 \in \ell^{\frac{l+1}{p+1}}(\mathbb{Z})$, with $\|\rho_{1,n}\|_{\ell^{\frac{l+1}{p+1}}(\mathbb{Z})} \le \|r\|_{\ell^{l+1}(\mathbb{Z})}^{p}\|s\|_{\ell^{l+1}(\mathbb{Z})}$. Therefore $\mathbb{E}\left(|U_{p,n}|^2\right)$ is uniformly bounded with

$$\mathbb{E}(|U_{p,n}|^2) \le (2(l-p))! \|r\|_{\ell^{l+1}(\mathbb{Z})}^{2l} \|s\|_{\ell^{l+1}(\mathbb{Z})}^{2}.$$

Let us now prove that $\mathbb{E}(|U_{p,n}|^2)$ tends to 0. We will use a density argument. For $\widetilde{s}$ a sequence with finite support, the quantity

$$\frac{1}{n^2}\mathbb{E}\left| \sum_{k,k'=0}^{n-1} \widetilde{s}(k - k') I_{2l-2p}(g_k^{\odot l-1} \otimes_p g_{k'}^{\odot l-1})(k') \right|^2$$

tends to 0. To prove the same with $s$ in place of $\widetilde{s}$, for a given $\varepsilon > 0$ we write $s$ as the sum of some $\widetilde{s}$ with finite support such that $\sum_{k \in \mathbb{Z}} |\widetilde{s}(k) - s(k)|^l < \varepsilon$. We conclude by a standard argument. $\qquad\square$

We have completed the proof of Proposition 2.5, and in the same time the proof of Breuer Major Theorem under the assumption that the spectral measure has a density, see also Remark 2.4 for the general case. $\qquad\square$

In the present context, this proposition on periodograms seems new. Actually Part (i) proves the asymptotic unbiasedness of the estimator $\Pi_{\phi,n}^{(l)}$, while Part (ii) implies its consistency.

**Remark 2.7.** *If we are only interested in asymptotic unbiasedness, continuity of the function $f^{*l} * \phi$ at 0 is sufficient, see [13]. Remark that the assumptions given here imply that its Fourier series is absolutely convergent. Note also that consistency is proved through asymptotic normality under stronger assumptions of integrability in [13]. This proposition may also be compared with [2], where Central Limit Theorems are developed for integrated periodograms when the test functions are in the Sobolev space $\mathscr{H}^\alpha$ for $\alpha > 1/2$.*

Recall that $\mathscr{H}^\alpha := \mathscr{H}^\alpha(\mathbb{T})$ is the space of functions $\psi \in L^2(\mathbb{T})$ such that

$$\sum_{k \in \mathbb{Z}} |c_k(\psi)|^2 (1 + |k|)^{2\alpha} < \infty.$$

We now give a bound for the speed of convergence in Proposition 2.5 when $\phi$ is a test function that satisfies a condition of Sobolev type. More precisely, we have the following proposition.

**Proposition 2.8.** *Assume that $r \in \ell^{l+1}(\mathbb{Z})$ and $\sum_{k \in \mathbb{Z}} |c_k(\phi)|^{l+1}(1 + |k|)^{\alpha(l+1)} < \infty$, for some $\alpha > 0$. Then, for some constant $C_\alpha$ and for all $n \ge 1$, we have*

$$\mathrm{Var}\left(\Pi_{\phi,n}^{(l)}\right) \le C_\alpha \begin{cases} \max(n^{-1}, n^{-2\alpha}) & \text{if } \alpha \ne \frac{1}{2} \\ n^{-1}\log(n) & \text{if } \alpha = \frac{1}{2} \end{cases}$$

*Proof.* Going back to the last proof and its notations, in view of (16), it is sufficient to prove that

$$\sum_{|j|\leq n-1} (\rho_{1,n} * \rho_{2,n}(j))^2 \leq C_\alpha \begin{cases} \max(1, n^{1-2\alpha}) & \text{if } \alpha \neq \frac{1}{2} \\ \log(n) & \text{if } \alpha = \frac{1}{2} \end{cases}$$

We will only consider the case $p = 0$ and let the reader see that the proof is the same for the other terms. In this case, $\|\rho_{2,n}\|_{\ell^{\frac{l+1}{l}}(\mathbb{Z})} \leq \|r\|_{\ell^{l+1}(\mathbb{Z})}^l$. By Young Inequality, one has the inclusion $\ell^{\frac{l+1}{l}}(\mathbb{Z}) * \ell^q(\mathbb{Z}) \subset \ell^2(\mathbb{Z})$ when $\frac{1}{q} = \frac{1}{2} + \frac{1}{l+1}$, with the corresponding norm inequality. So it is sufficient to compute the norm of $\rho_{1,n}$ in $\ell^q(\mathbb{Z})$, which is elementary by Hölder Inequality. For this, we use the fact that

$$\sum_{|k|\leq n}(1 + |k|)^{-2\alpha} \leq C_\alpha \max(n^{1-2\alpha}, 1) \text{ when } \alpha \neq \frac{1}{2} \text{ and } \sum_{|k|\leq n}(1 + |k|)^{-1} \leq C_{\frac{1}{2}} \log(n).$$

$\square$

This kind of proof can be generalized to other assumptions on data. We give now one computation that leads to a bound for the speed of convergence in Breuer Major Theorem.

**Proposition 2.9.** *Assume that* $\sum_{k\in\mathbb{Z}} |r(k)|^{l+1}(1 + |k|)^{\alpha(l+1)} < \infty$ *for some* $\alpha > 0$. *Then, when* $l = 1$, *for some* $C_\alpha > 0$ *and for all* $n \geq 1$ *we have the uniform estimate*

$$\text{Var}\left(\Pi_{f_X,n}^{(1)}\right) \leq C_\alpha \max(n^{-1}, n^{-4\alpha}).$$

*For* $l \geq 2$, *for some* $C_\alpha > 0$ *and for all* $n \geq 1$ *we have*

$$\text{Var}\left(\Pi_{f_X,n}^{(l)}\right) \leq C_\alpha \begin{cases} n^{-2\alpha(l+1)} & : & \alpha < \frac{1}{l(l+1)} \\ n^{-2\alpha-\frac{2}{l+1}} & : & \frac{1}{l(l+1)} < \alpha < \frac{1}{2} - \frac{1}{l+1} \\ n^{-1} & : & \alpha > \frac{1}{2} - \frac{1}{l+1} \end{cases}.$$

*Proof.* Again, we go back to the previous notations and estimate $\sum_{|j|\leq n-1} (\rho_{1,n} * \rho_{2,n}(j))^2$ in view of (16). Let us first consider $l = 1$. The only case to consider is $p = 0$, and we want to prove the estimate

$$\sum_{|j|\leq n-1} (\rho_{1,n} * \rho_{2,n}(j))^2 \leq C \max(1, n^{1-4\alpha}).$$

Here $\rho_{1,n} = \rho_{2,n}$ coincides with $|r|$ for $|k| \leq n - 1$. Assume first that $\alpha < 1/4$. It follows from Hölder inequality that $\|\rho_{1,n}\|_{\ell^{4/3}(\mathbb{Z})} \leq C_\alpha n^{1/4-\alpha}$. Now the convolution of two sequences in $\ell^{4/3}(\mathbb{Z})$ is in $\ell^2(\mathbb{Z})$, which allows to conclude in this case. For $\alpha > 1/4$, the sequence $\rho_1$ is in $\ell^{4/3}(\mathbb{Z})$ and we conclude in the same way.

It remains to conclude for $\alpha = 1/4$. We want to prove that $\sum_{|j|\leq n-1} (\rho_{1,n} * \rho_{2,n}(j))^2$ is uniformly bounded under the assumption that $\sum_{k\in\mathbb{Z}} |r(k)|^2(1 + |k|)^{1/2} < \infty$. Let $h_n$ be the trigonometric polynomial with $\rho_{1,n}$ as Fourier coefficients. Then $\rho_{1,n} * \rho_{2,n} = \rho_{1,n} * \rho_{1,n}$ are the Fourier coefficients of the

function $h_n^2$. The function $h_n$ is uniformly in the Sobolev space $\mathcal{H}^{1/4}$. Now it follows from Sobolev Theorem (see [19] for instance) that such functions are uniformly in $L^4(\mathbb{T})$. By Plancherel Identity,

$$\sum_{j\in\mathbb{Z}}(\rho_{1,n}*\rho_{1,n}(j))^2 = \frac{1}{2\pi}\int_{\mathbb{T}}|h_n(x)|^4dx \leq C. \tag{17}$$

Let us now consider $l \geq 2$ and estimate again the norm of $\rho_{1,n}*\rho_{2,n}$ in $\ell^2(\mathbb{Z})$. The worst case is obtained for $p = 0$. Then $\rho_{1,n}$ coincides with $|r|$ while $\rho_2$ is equal to $|r|^l$. Then

$$\|\rho_{1,n}*\rho_{2,n}\|_{\ell^2(\mathbb{Z})} \leq \|\rho_{1,n}\|_{\ell^2(\mathbb{Z})}\|\rho_{2,n}\|_{\ell^1(\mathbb{Z})}.$$

The first estimate is obtained by taking the norm of $\rho_{1,n}$ in $\ell^2(\mathbb{Z})$ and the norm of $\rho_{2,n}$ in $\ell^1(\mathbb{Z})$, as long as this last one is not uniformly bounded. For larger values of $\alpha$, $\rho_2$ is in $\ell^1(\mathbb{Z})$ and the bound is given by the the norm of $\rho_{1,n}$ in $\ell^2(\mathbb{Z})$, as long as this last one is not uniformly bounded. $\qquad\square$

## 2.4 Variable spectral densities

In practice, the spectral density may change at each step of computation of the mean. This is what happens for instance when we look at increments of a Gaussian process at different scales. It is important to have still Central Limit Theorems in this context, as well as methods to compute the speed of convergence. Let us first state a CLT in this framework, which may also be seen as a CLT for particular triangular arrays.

**Theorem 2.10.** *Let $X_n = (X_n(k))_{k\in\mathbb{Z}}$ be centered stationary Gaussian time series with spectral densities $f_{X_n}$. Let $l \geq 2$. We assume that the functions $f_{X_n}$ belong uniformly to the space $L^{\frac{l}{l-1}}(\mathbb{T})$ and converge in this space to a function $f_X$. We call $r(k) := \frac{1}{2\pi}\int_{\mathbb{T}}e^{-ikx}f_X(x)dx$ and assume, without loss of generality, that $r(0) = 1$. Then we have the following asymptotic properties for $n$ tending to infinity:*

(i)

$$\mathrm{Var}\left(\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1}H_l(X_n(k))\right) \longrightarrow \sigma_l^2,$$

(ii)

$$\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1}H_l(X_n(k)) \xrightarrow{d} \mathcal{N}(0,\sigma_l^2),$$

*with*

$$\sigma_l^2 = l!\sum_{k\in\mathbb{Z}}r(k)^l.$$

*Proof.* Let $X$ be a centered stationary Gaussian time series with spectral density $f_X$. We define $Y_n$ as before. Let us define

$$Z_n := \frac{1}{\sqrt{n}}\sum_{k=0}^{n-1}H_l(X_n(k)).$$

Similar computations as for $Y_n$ imply that

$$\mathrm{Var}(Z_n) = l! \sum_{k=-(n-1)}^{n-1} \left(1 - \frac{|k|}{n}\right) r_n(k)^l.$$

It follows from the assumption and Hausdorff-Young Inequality that the sequence $r_n$ tends to $r$ in $\ell^l(\mathbb{Z})$. Then, one has the inequalities

$$
\begin{aligned}
\left|\mathrm{Var}(Y_n) - \mathrm{Var}(Z_n)\right| &= l! \left| \sum_{k=-(n-1)}^{n-1} \left(1 - \frac{|k|}{n}\right) \left(r(k)^l - r_n(k)^l\right) \right| \\
&\leq C_\ell \|r - r_n\|_{\ell^l(\mathbb{Z})} \\
&\leq C_\ell \|f_X - f_{X_n}\|_{\frac{l}{l-1}},
\end{aligned}
\tag{18}
$$

according to Hausdorff-Young Inequality. This implies that $\mathrm{Var}(Y_n)$ and $\mathrm{Var}(Z_n)$ have the same limit.

We then have to prove that $\mathrm{Var}(\|DZ_n\|_2^2)$ tends to 0 in place of $\mathrm{Var}(\|DY_n\|_2^2)$, with a variable spectral density in place of a fixed one. For this, it is sufficient to revisit the proof of Lemma 2.6, where we consider $r_n$ in place of $r$ and $s$, and so, later on, $\widetilde{\rho}_{1,n}$ and $\widetilde{\rho}_{2,n}$ instead of $\rho_{1,n}$ and $\rho_{2,n}$, which are obtained when replacing $r$ by $r_n$. We write $r_n = r + r - r_n$ and develop the corresponding formulas by multi-linearity. When all $r_n$'s are replaced by $r$, we have the limit 0 by Lemma 2.6. Now, when one $r_n$ is replaced by $r - r_n$ somewhere, the proof goes the same way, except for the fact that $r - r_n$ has an arbitrarily small norm in $\ell^l(\mathbb{Z})$. So the limit is 0.

$\square$

**Remark 2.11.** *The conclusion of Theorem 2.10 holds true under the weaker assumption that $r_n$ tends to $r$ in $\ell^l(\mathbb{Z})$. Moreover, one does not need to have spectral densities, according to Remark 2.4.*

## 2.5 Speed of convergence in Breuer Major Central Limit Theorem

We are now able to bound the speed of convergence in Theorem 2.1 under the assumption that $\sum |r(k)|^l (1 + |k|)^{l\alpha}$ is finite for some $\alpha > 0$, as well as in Theorem 2.10. We recall that the Kolmogorov distance between the random variables $Y$ and $Z$ is defined as

$$d_{Kol}(Y, Z) = \sup_{z \in \mathbb{R}} |P(Y \leq z) - P(Z \leq z)|. \tag{19}$$

We will be particularly interested by the distance of Kolomogorov to some normal random variable $\sigma N$, where $N \sim \mathcal{N}(0, 1)$. We recall that (see [22] for instance), for $Z$ a centered random variable with variance 1 in the $l$-th Wiener chaos,

$$d_{Kol}(Z, N) \leq \sqrt{\left(\mathrm{Var}\left(\frac{1}{l}\|DZ\|_2^2\right)\right)}. \tag{20}$$

The following lemma will be used to compute the required Kolmogorov distances.

**Lemma 2.12.** *For Y a centered random variable in the l-th Wiener chaos we have the inequality*

$$d_{Kol}(Y, \sigma N) \leq \frac{2}{\sigma^2} |\text{Var}(Y)) - \sigma^2| + \sqrt{\left( \text{Var}\left( \frac{1}{l\sigma^2} \|DY\|_2^2 \right) \right)}.$$

*Proof.* When $\frac{|\text{Var}(Y) - \sigma^2|}{\sigma^2} > \frac{1}{2}$ there is nothing to prove. Otherwise we write

$$d_{Kol}(Y, \sigma N) \leq d_{Kol}(Y, \sqrt{\text{Var}(Y)}N) + d_{Kol}(\sqrt{\text{Var}(Y)}N, \sigma N).$$

We use the Malliavin derivative for the first distance, then a direct computation of the distance between $N$ and a multiple of $N$. More precisely, for example for $z > 0$ and $\sigma > 1$, one has the inequality $P(z < N \leq \sigma z) \leq (\sigma - 1)ze^{-z^2/2} \leq \sigma - 1$. $\square$

We can now state the first theorem of this subsection, which gives the speed of convergence in Breuer Major Theorem.

**Theorem 2.13.** . *Let $(X(k))_{k \in \mathbb{Z}}$ be a centered stationary Gaussian time series with an absolutely continuous spectral measure. Assume that $r$ satisfies $r(0) = 1$ and the assumption*

$$\sum_{k \in \mathbb{Z}} |r(k)|^l (1 + |k|)^{l\alpha} < \infty \tag{21}$$

*for some $\alpha > 0$. Then, for $l = 2$, for some constant $C_\alpha > 0$ and for all $n \geq 1$,*

$$d_{Kol}\left( \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} H_2(X(k)), \sigma_2 N \right) \leq C_\alpha \max(n^{-2\alpha}, n^{-1/2}), \tag{22}$$

*while, for $l \geq 3$, for some constant $C_\alpha > 0$ and for all $n \geq 1$,*

$$d_{Kol}\left( \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} H_l(X(k)), \sigma_l N \right) \leq C_\alpha \begin{cases} n^{-\alpha l} & : & \alpha < \frac{1}{l(l-1)} \\ n^{-\alpha - \frac{1}{l}} & : & \frac{1}{l(l-1)} < \alpha < \frac{1}{2} - \frac{1}{l} \\ n^{-\frac{1}{2}} & : & \alpha > \frac{1}{2} - \frac{1}{l} \end{cases} . \tag{23}$$

*with $\sigma_l^2 = l! \sum_{k \in \mathbb{Z}} r(k)^l$.*

*Proof.* Let us first prove that

$$|\sigma_l^2 - \text{Var}(Y_n)| \leq C \max(n^{-\alpha l}, n^{-1}). \tag{24}$$

From the expression of $\text{Var}(Y_n)$ given in (12) we deduce that

$$l!^{-1}\left| \sigma_l^2 - \text{Var}(Y_n) \right| \leq \frac{1}{n} \sum_{k=-(n-1)}^{n-1} |k||r(k)|^l + \sum_{|k| \geq n} |r(k)|^l.$$

We conclude directly using the fact that $\sum_{k \in \mathbb{Z}} |r(k)|^l (1 + |k|)^{l\alpha} < \infty$. Now the required estimate for the Malliavin derivative is given by Proposition 2.9. $\square$

Next we give the speed of convergence in Theorem 2.10, which also depends on the speed of convergence of $f_{X_n}$ to $f_X$.

**Theorem 2.14.** *Let $X_n = (X_n(k))_{k \in \mathbb{Z}}$ be centered stationary Gaussian time series with spectral densities $f_{X_n}$ which satisfy the assumptions of Theorem 2.10 with $r(k) := \frac{1}{2\pi} \int_{\mathbb{T}} e^{-ikx} f_X(x) dx$. We assume moreover that the two following properties are satisfied.*

$$\sum_{k \in \mathbb{Z}} |r(k)|^l (1 + |k|)^{\alpha l} < \infty. \tag{25}$$

$$\|f_{X_n} - f_X\|_{\frac{l}{l-1}} \leq C n^{-\beta}. \tag{26}$$

*Then, for $l = 2$, for some constant $C_{\alpha,\beta}$ and for all $n \geq 1$, we have*

$$d_{Kol}\left(\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} H_2(X_n(k)), \sigma_2 N\right) \leq C_{\alpha,\beta} \max(n^{-\beta}, n^{-2\alpha}, n^{-1/2}), \tag{27}$$

*while, for $l \geq 3$, for some constant $C_{\alpha,\beta}$ and for all $n \geq 1$, we have*

$$d_{Kol}\left(\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} H_l(X_n(k)), \sigma_l N\right) \leq C_{\alpha,\beta} \begin{cases} \max(n^{-\beta}, n^{-\alpha l}) & : & \alpha < \frac{1}{l(l-1)} \\ \max(n^{-\beta}, n^{-\alpha - \frac{1}{l}}) & : & \frac{1}{l(l-1)} < \alpha < \frac{1}{2} - \frac{1}{l} \\ \max(n^{-\beta}, n^{-\frac{1}{2}}) & : & \alpha > \frac{1}{2} - \frac{1}{l} \end{cases}. \tag{28}$$

*Proof.* We go back to the notations used in the proof of Theorem 2.10. We will use Lemma 2.12 with $Z_n$ in place of $Y$. We first want to have the speed of convergence of $\text{Var}(Z_n) - \sigma_l^2$ to 0. This is given by (18) and (24). We then have to bound $\text{Var}(\|DZ_n\|_2^2)$ in place of $\text{Var}(\|DY_n\|_2^2)$, with a variable spectral density in place of a fixed one. So we have to consider $\widetilde{\rho}_{1,n}$ and $\widetilde{\rho}_{2,n}$ instead of $\rho_{1,n}$ and $\rho_{2,n}$, with $r$ replaced by $r_n$.

When $l = 2$ we use the fact that $\|\widetilde{\rho}_{1,n}\|_{\ell^{4/3}(\mathbb{Z})} \leq \|\widetilde{\rho}_{1,n} - \rho_{1,n}\|_{\ell^{4/3}(\mathbb{Z})} + \|\rho_{1,n}\|_{\ell^{4/3}(\mathbb{Z})}$, with

$$\|\widetilde{\rho}_{1,n} - \rho_{1,n}\|_{\ell^{4/3}(\mathbb{Z})}^4 \leq n \|r_n - r\|_{\ell^2(\mathbb{Z})}^4 \leq C n^{1-4\beta}.$$

So $\|\widetilde{\rho}_{1,n}\|_{\ell^{4/3}(\mathbb{Z})} \leq C n^{1/4 - \alpha \wedge \beta}$. When $l \geq 3$, similarly we use the fact that

$$\|\widetilde{\rho}_{1,n}\|_{\ell^2(\mathbb{Z})}^2 \leq C \left(n^{\frac{l-2}{l}} \|r_n - r\|_{\ell^l(\mathbb{Z})}^2 + \|\rho_{1,n}\|_{\ell^2(\mathbb{Z})}^2\right), \text{ and } \|\widetilde{\rho}_{2,n}\|_{\ell^1(\mathbb{Z})}^2 \leq C \left(n^{\frac{2}{l}} \|r_n - r\|_{\ell^l(\mathbb{Z})}^2 + \|\rho_{2,n}\|_{\ell^1(\mathbb{Z})}^2\right).$$

$\square$

## 2.6 Comparisons with the results of [23]

The aim of this sub-section is to compare our results with the results of [23]. It should be emphasized that these two papers have been written independently, with different purposes, even if both papers address the use of the Malliavin derivative (as developed by Nourdin, Peccati, Tudor, Nualart, etc.) in view of the theorem of Breuer Major and its quantitative generalizations. While we only consider Hermite polynomials, the authors of [23] consider a general functional $H$, which they expand in the basis of Hermite polynomials. This means that their results are certainly more general, but estimates

for Hermite polynomials play a crucial role in this theory, so that their results can be improved by our techniques. In fact a synthesis of the two papers has to be done in the future.

The link given here between the Malliavin derivative and the integrated periodogram is new. It induces different notations, which may be helpful for readers coming from signal processing. But the method that we develop is inspired by [22, 21] and is essentially the same up to the identity (15). Then the way we estimate this sum is more accurate. The key point is the fact that it can be bounded by an expression that involves convolution in sequence spaces. Then we can use Hausdorff-Young inequalities in this context.

Let us first compare the results in an emblematic case that is used when dealing with the increments (or generalized increments) of the Fractional Brownian Motion, see Section 4 below. Let us quote that the sequence of the normalized increments of the Fractional Brownian motion with Hurst parameter $H \in (0,1)$ is a stationary Gaussian sequence with spectral density given by (36), whose Fourier coefficients satisfy $r(k) = O(|k|^{-2+2H})$. Because of the same application, this is also considered in Example 2.7 in [23], and had been considered before in Theorem 4.1 of [21] and Theorem 3.10 of [22]. We give our results as a proposition.

**Proposition 2.15.** *Assume that $r(k) = O(|k|^{-a})$. Then, for $a > \frac{1}{2}$, $l = 2$ and $a \neq \frac{3}{4}$,*

$$d_{Kol}\left(\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} H_2(X(k)), \sigma_2 N\right) \leq C \max(n^{1-2a}, n^{-1/2}), \tag{29}$$

*while, for $l \geq 3$ and $a > \frac{1}{l}$*

$$d_{Kol}\left(\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} H_l(X(k)), \sigma_l N\right) \leq C \begin{cases} n^{-la+1} & : & a < \frac{1}{l-1} \\ n^{-a} & : & \frac{1}{l-1} < a < \frac{1}{2} \\ n^{-\frac{1}{2}} & : & a > \frac{1}{2} \end{cases}. \tag{30}$$

*Proof.* The proof is a small variation of the proof of Theorem 2.13. See the proposition below, which is the main step of the proof. $\square$

In comparison, the estimates given in [23] are the following: for $l = 2$,

$$d_{Kol}\left(\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} H_2(X(k)), \sigma_2 N\right) \leq C \max(n^{\frac{1-2a}{2}}, n^{-1/2}), \tag{31}$$

while, for $l \geq 3$ and $a > \frac{1}{l}$

$$d_{Kol}\left(\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} H_l(X(k)), \sigma_l N\right) \leq C \begin{cases} n^{\frac{-la+1}{2}} & : & a < \frac{1}{l-1} \\ n^{-a/2} & : & \frac{1}{l-1} < a < 1 \\ n^{-\frac{1}{2}} & : & a > 1 \end{cases}. \tag{32}$$

Except for $a > 1$, these estimates of [23] are weaker than ours. This allows to answer negatively to their question on the optimality of their bounds. Let us in particular mention that, for $l = 2$, one has the speed $n^{-1/2}$ from the value 3/4 of the parameter $a$. This means that for the increments of

the Fractional Brownian Motion, one has this speed for $H < 5/8$ (and not only for $H \leq 1/2$), which had not been observed before.

Again, one may ask the question of the optimality of our estimates. We cannot answer this question, since we do not know whether there is a loss or not when using the Malliavin derivative for the computation of the Kolmogorov distance. But we can ask the same question for the variance of the integrated periodogram, which we give now.

**Proposition 2.16.** *Assume that $r(k) = O(|k|^{-a})$ and $|c_k(\phi)| = O(|k|^{-a})$. Then, for $a > \frac{1}{2}$, $l = 1$ and $a \neq \frac{3}{4}$, for some constant $C$ and for all $n \geq 1$, we have*

$$\mathrm{Var}\left(\Pi_{\phi,n}^{(l)}\right) \leq C \max(n^{2(1-2a)}, n^{-1}).$$

*For all $l \geq 2$, for some constant $C$ and for all $n \geq 1$, we have*

$$\mathrm{Var}\left(\Pi_{\phi,n}^{(l)}\right) \leq C \begin{cases} n^{-2(l+1)a+2} & : & a < \frac{1}{l} \\ n^{-2a} & : & \frac{1}{l} < a < \frac{1}{2} \\ n^{-1} & : & a > \frac{1}{2} \end{cases}.$$

So let us discuss the optimality of this proposition. We will prove it for $l = 1$. Again, for $l \geq 2$, there is a step of symmetrization and we only get an upper bound in (16). However, when $l = 1$, then

$$\mathrm{Var}\left(\Pi_{\phi,n}^{(1)}\right) = 4\|F_{0,n}\|_{L^2(\mathbb{T}^2)}^2,$$

where $\|F_{0,n}\|_{L^2(\mathbb{T}^2)}^2$ is given by (15). We will use a particular spectral density $f_X$. Namely, there exists $f_X$ with positive Fourier coefficients given by $r(k) = c|k|^{-a}$ for $k \geq 1$ (see [31] p.70). We also choose $\phi = f_X$. Let us note $r_n(k) = r(k)$ or $0$, depending whether $|k| < n$ or not. Remark that, because of the positivity of the Fourier coefficients, we not only have an estimate above as in (16), but also an estimate below:

$$\mathrm{Var}\left(\Pi_{\phi,2n}^{(1)}\right) \geq \frac{1}{n} \sum_{|j| \leq n-1} \left(1 - \frac{|j|}{n}\right) (r_n * r_n(j))^2. \tag{33}$$

Then $r_n * r_n(j) > c' \max(|j|^{-a}, |j|^{1-2a})$ for $a \neq 1$ and $\sum_{|j| \leq n-1} \left(1 - \frac{|j|}{n}\right) (r_n * r_n(j))^2 \geq c'' \max(1, n^{3-4a})$, which proves the optimality.

Let us also consider the exceptional values. The same kind of computations gives that, for $a = 3/4$,

$$\mathrm{Var}\left(\Pi_{\phi,n}^{(1)}\right) \leq \frac{C \log n}{n}$$

for all $r$ such that $r(k) = O(|k|^{-3/4})$, while such a converse inequality holds for our particular example of sequence $r$. So again, this convergence is optimal.

For the value $a = 1/2$ (which corresponds to the case $H = 3/4$), the same kind of computations shows that

$$\mathrm{Var}\left(\Pi_{\phi,n}^{(1)}\right) \leq \frac{C}{n}.$$

But then we may have $\sum |r(k)|^2 = \infty$. If we come back to the notations of the sub-section 2.2, and the choice of the random sequence $Y_n$, it was important that its variance be equivalent to a constant for $n$ tending to $\infty$. To have convergence in this case, one has to divide $Y_n$ by $\log n$. We then obtain a speed of convergence in $C/\log n$, which improves the rate $C/(\log n)^{1/2}$ given in [22], Theorem 3.1. Moreover our estimate is optimal in the same way as the other ones.

Let us finally remark that the use of the linearity in $\phi$ of the integrated periodogram as well as a standard density argument has allowed us to give a new proof of Breuer Major's Theorem through the consideration of Malliavin's derivative. This had not been observed before, up to our knowledge.

We have deliberately restricted to the dimension one for this comparison, but this could be pursued in higher dimension.

# 3 Vector-valued central limit theorem and generalizations

## 3.1 Vector-valued central limit theorem

We now describe a very useful extension of Theorem 2.10 to the vectorial case. Our main tool is [28] where it is proved that vectorial Central Limit Theorems follow from Central Limit Theorems for marginals and convergence of covariance matrix. For $d \geq 2$ we consider a vector-valued centered stationary Gaussian time series defined by $\overrightarrow{X}(k) = (X_1(k), X_2(k), \cdots, X_d(k))$. We assume that the covariance matrix of $\overrightarrow{X}$ is given by

$$r_{i,j}(k) = \text{Cov}(X_i(k'+k), X_j(k')) := \frac{1}{2\pi} \int_{-\pi}^{+\pi} e^{-ikx} \left( f_{\overrightarrow{X}} \right)_{i,j}(x) dx.$$

With a little abuse we say that the Hermitian $d$ times $d$ matrix $f_{\overrightarrow{X}}$ is the spectral density of $\overrightarrow{X}$. Then we can consider the stationary vector-valued processes

$$H_l(\overrightarrow{X}(k)) = (H_l(X_1(k)), \ldots, H_l(X_d(k))), \; k \in \mathbb{Z}, \; l \geq 1.$$

In fact we are interested in the more general case of variable spectral densities.

**Theorem 3.1.** *Let $\left( \overrightarrow{X_n}(k) \right)_{k \in \mathbb{Z}}$ be centered stationary time series with values in $\mathbb{R}^d$. We call $f_{\overrightarrow{X_n}}$ the spectral density matrix of $\overrightarrow{X_n}$. Let $l \geq 2$. We assume that $f_{\overrightarrow{X_n}}$ belongs uniformly to the space $L^{\frac{l}{l-1}}(\mathbb{T})$ and converges in this space to a function $f_{\overrightarrow{X}}$ (in the sense that $\| \left( f_{\overrightarrow{X_n}} \right)_{i,j} - \left( f_{\overrightarrow{X}} \right)_{i,j} \|_{\frac{l}{l-1}}$ tends to 0 as n tends to infinity for all $1 \leq i, j \leq d$). We call $r_{i,j}(k) := \frac{1}{2\pi} \int_{\mathbb{T}} e^{-ikx} \left( f_{\overrightarrow{X}} \right)_{i,j}(x) dx$ and assume, without loss of generality, that $r_{i,i}(0) = 1$. Then we have the following vectorial CLT for n tending to infinity:*

$$\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} H_l(\overrightarrow{X_n}(k)) \xrightarrow{d} \mathcal{N}(0, \Sigma_l),$$

*with $(\Sigma_l)_{i,j} = l! \sum_{k \in \mathbb{Z}} r_{i,j}(k)^l$.*

*Proof.* It is not very natural to write $\overrightarrow{X_n}$ as a Brownian stochastic integral, but we can still use an isonormal Gaussian process (see Remark 2.4 and [23]), which allows us to use the results of [28]. Let us introduce as before

$$\overrightarrow{Z_n} = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} H_l(\overrightarrow{X_n}(k)).$$

Once we have a one dimensional CLT for each coordinate of $\overrightarrow{Z_n}$, then it is sufficient to prove that the covariance matrix of $\overrightarrow{Z_n}$ tends to the matrix $\Sigma_l$ (that is, Assumption 6 of Proposition 2 of [28]). First, let us remark that for any $1 \leq i \leq d$, $X_{n,i}$ admits $\left(f_{\overrightarrow{X_n}}\right)_{i,i}$ for spectral density. Since $\|\left(f_{\overrightarrow{X_n}}\right)_{i,i} - \left(f_{\overrightarrow{X}}\right)_{i,i}\|_{\frac{l}{l-1}}$ tends to 0 as $n$ tends to infinity, we already know by Theorem 2.10 that the random variable $Z_{n,i}$ converges in distribution to $\mathscr{N}\left(0, (\Sigma_l)_{i,i}\right)$. Now, for $1 \leq i, j \leq d$, we have

$$
\begin{aligned}
\mathrm{Cov}(Z_{n,i}, Z_{n,j}) &= \frac{1}{n} \sum_{k=0}^{n-1} \sum_{k'=0}^{n-1} \mathbb{E}(H_l(X_{n,i}(k)) H_l(X_{n,j}(k'))) \\
&= \frac{l!}{n} \sum_{k=0}^{n-1} \sum_{k'=0}^{n-1} r_{n,i,j}(k-k')^l \\
&= l! \sum_{k=-(n-1)}^{n-1} \left(1 - \frac{|k|}{n}\right) r_{n,i,j}(k)^l.
\end{aligned}
$$

From this point, the proof that this quantity tends to $(\Sigma_l)_{i,j}$ is the same as for a scalar valued time series. $\qquad \square$

**Remark 3.1.** *One can also have a bound for the speed of convergence as in Section 2, based on results of [25, 24]. One considers now the distance of Wasserstein*

$$d_W(\overrightarrow{Y}, \overrightarrow{Z}) = \sup |\mathbb{E}(\Phi(\overrightarrow{Y})) - \mathbb{E}(\Phi(\overrightarrow{Z}))|,$$

*where the supremum is taken on all Lipschitz functions with Lipschitz constant bounded by 1, under the assumption that the matrix $\Sigma_l$ is positive definite. When it is not the case, the function $\Phi$ is taken of class $\mathscr{C}^2$, with bounded second derivatives. Mutatis mutandis, the bounds obtained for the speed of convergence are the same as in the last section, see Theorem 2.14.*

### 3.2 Extension to stationary centered Gaussian fields

Until now, we have chosen to restrict our study to stationary centered Gaussian processes, essentially for notational sake of simplicity. However, all previous results have their counterpart in the framework of Gaussian random fields that are indexed by $\mathbb{Z}^\nu$ for some integer $\nu \geq 2$ instead of $\mathbb{Z}$. Then Theorems 2.10 and 3.1 are generalized in the following setting.

**Theorem 3.2.** *Let $\nu, d \geq 1$ integers. Let $\overrightarrow{X_n} = \left(\overrightarrow{X_n}(k)\right)_{k \in \mathbb{Z}^\nu}$ be centered stationary Gaussian fields with values in $\mathbb{R}^d$. Let $l \geq 2$. We call $f_{\overrightarrow{X_n}}$ the spectral density matrix of $\overrightarrow{X_n}$ and assume that $f_{\overrightarrow{X_n}}$ belongs to the space $L^{\frac{l}{l-1}}(\mathbb{T}^\nu)$ and converges in this space to a function $f_{\overrightarrow{X}}$. We call $r_{i,j}(k) := \frac{1}{(2\pi)^\nu} \int_{\mathbb{T}^\nu} e^{-ik \cdot x} \left(f_{\overrightarrow{X}}\right)_{i,j}(x) dx$ and assume that $r_{i,i}(0) = 1$ for $1 \leq i \leq d$. Then, for $n$ tending to infinity,*

(i)

$$\text{Var}\left( \frac{1}{n^{\nu/2}} \sum_{k_1,\dots,k_d=0}^{n-1} H_l(X_{n,i}(k)) \right) \longrightarrow (\Sigma_l)_{i,i},$$

(ii)

$$\frac{1}{n^{\nu/2}} \sum_{k_1,\dots,k_d=0}^{n-1} H_l(\overrightarrow{X_n}(k)) \overset{d}{\longrightarrow} \mathcal{N}(0,\Sigma_l),$$

with

$$(\Sigma_l)_{i,j} = l! \sum_{k \in \mathbb{Z}^\nu} r_{i,j}(k)^l. \tag{34}$$

In the case where $\overrightarrow{X_n} = \overrightarrow{X}$, this result is a particular consequence of Theorem 4 of Arcones [1].

**Remark 3.3.** *One can also have a bound for the speed of convergence (written in terms of the distance of Wasserstein if $d > 1$) as in the last subsection, but with different exponents coming from the generalization of Proposition 2.9. In this proposition, the new bound is $\max(n^{-\nu}, n^{-4\alpha})$ when $l = 1$, due to the fact that the Sobolev space $\mathcal{H}^\alpha$ is contained in $L^4(\mathbb{T}^\nu)$ for $\alpha = \nu/4$. When $l \geq 2$, the new bound is, up to a constant,*

$$\max(n^{-2(l+1)\alpha}, n^{-2\alpha - \frac{2\nu}{l+1}}, n^{-\nu}).$$

*In Theorem 3.2, whenever $\|f_{\overrightarrow{X_n}} - f_{\overrightarrow{X_n}}\|_{\frac{l}{l-1}} \leq C n^{-\beta}$, the speed of convergence is $O(\max(n^{-\beta}, n^{-2\alpha}, n^{-1/2}))$ for $l = 2$. Whenever $l > 2$, it is bounded, up to a constant, by*

$$\begin{cases} \max(n^{-\beta}, n^{-\alpha l}) & : & \frac{\alpha}{\nu} < \frac{1}{l(l-1)} \\ \max(n^{-\beta}, n^{-\alpha - \frac{\nu}{l}}) & : & \frac{1}{l(l-1)} < \frac{\alpha}{\nu} < \frac{1}{2} - \frac{1}{l} \\ \max(n^{-\beta}, n^{-\frac{\nu}{2}}) & : & \frac{\alpha}{\nu} > \frac{1}{2} - \frac{1}{l} \end{cases}.$$

# 4   Application to generalized quadratic variations

In this section we consider a continuous time real-valued centered Gaussian field with stationary increments, defined through a spectral representation

$$Y(t) = \int_{\mathbb{R}^\nu} \left( e^{-it \cdot x} - 1 \right) F(x)^{1/2} d\widetilde{W}^\nu(x), t \in \mathbb{R}^\nu, \tag{35}$$

where $\widetilde{W}^\nu$ is a complex centered Gaussian random measure on $\mathbb{R}^\nu$ with Lebesgue control measure, such that $\widetilde{W}^\nu(-A) = \overline{\widetilde{W}^\nu(A)}$ a.s. for any Borel set $A$ of $\mathbb{R}^\nu$. The function $F$ satisfies the integrability condition

$$\int_{\mathbb{R}^\nu} \min(1, |x|^2) dx < \infty.$$

Remark that this condition could be relaxed when only higher order increments are stationary (see [29]). We refer to the introduction for more notations and comments.

We assume that only $\{Y(k/n); k = (k_1, \ldots, k_\nu) \in \mathbb{Z}^\nu \text{ with } 0 \le k_1, \ldots, k_\nu \le n - 1\}$ are known for some large $n$.

Let us first describe our method in the simplest possible example, that is, the Fractional Brownian Motion in one dimension. So, for a moment we assume that $\nu = 1$ and $F(x) := |x|^{-2H-1}$, with $0 < H < 1$. We are interested in first increments $X_n(k) := Y((k+1)/n) - Y(k/n)$ and want to have a CLT for the means

$$\frac{1}{n} \sum_{k=0}^{n-1} |X_n(k)|^2$$

which are also the quadratic variations of the sequence $Y(k/n)$. We remark that

$$\mathrm{Cov}(X_n(k), X_n(k')) = \int_{\mathbb{R}} e^{-i\frac{(k-k')x}{n}} \left| e^{-i\frac{x}{n}} - 1 \right|^2 F(x)dx.$$

We use the homogeneity of $F$ to write this covariance as

$$\mathrm{Cov}(X_n(k), X_n(k')) = n^{-2H} \int_{\mathbb{R}} e^{-i(k-k')x} \left| e^{-ix} - 1 \right|^2 F(x)dx.$$

Then, by a standard argument (which may be seen as an elementary version of Poisson's Formula), the spectral density of this time series is given by a periodization of $F$, that is,

$$\mathrm{Cov}(X_n(k), X_n(k')) = n^{-2H} \int_{-\pi}^{+\pi} e^{-i(k-k')\cdot x} \left| e^{-ix} - 1 \right|^2 \sum_{j \in \mathbb{Z}} \frac{1}{|x + 2\pi j|^{2H+1}} dx.$$

So, if we consider asymptotic properties of normalized increments $n^H X_n(k)$, they have the same law as the ones of a unique time series, whose spectral density is the periodic function

$$f_X(x) := \left| e^{-ix} - 1 \right|^2 \sum_{j \in \mathbb{Z}} \frac{1}{|x + 2\pi j|^{2H+1}}. \tag{36}$$

In particular, one observes a Central Limit Theorem for the quadratic variations $\frac{1}{n} \sum_{k=0}^{n-1} |X(k)|^2$ (once centralized and reduced) if one has the same for the means of $H_2(X)$. This can be deduced from Section 2 as soon as the function $f_X$ is in $L^2(\mathbb{T})$, which is the case for $H < 3/4$ (we give the proof of this fact in the general case).

If $F$ is only asymptotically homogeneous, we will still be able to use the same argument, but with a variable spectral density.

We now consider the general case $\nu \ge 1$ and define generalized quadratic variations (recall that one has to deal with higher increments for $H \ge 3/4$ in dimension one). We first define generalized increments. More precisely, our first step is to consider a stationary field induced by these observations. This is obtained through a filtering of this sequence. In particular we consider the discrete time stationary field

$$Z_{n,\mathbf{a}}(k) = \sum_{m_1, \ldots, m_\nu = 0}^{p} a_1(m_1) \ldots a_\nu(m_\nu) Y\left( \frac{k_1 + m_1}{n}, \ldots, \frac{k_\nu + m_\nu}{n} \right), \quad \text{for } k = (k_1, \ldots, k_\nu) \in \mathbb{Z}^\nu$$

and $\mathbf{a} = (\mathbf{a_1}, \ldots, \mathbf{a_\nu})$ with $\mathbf{a_j} = (a_j(0), \ldots, a_j(p)) \in \mathbb{R}^{p+1}$ a discrete filter of length $p+1$ and of order $K_j$ ($p, K_j \in \mathbb{N}$ with $p \geq K_j$), which means that $\sum_{m_j=0}^{p} a_j(m_j) \neq 0$ when $K_j = 0$ and otherwise

$$\sum_{m_j=0}^{p} a_j(m_j) m_j^r = 0 \quad \text{for } 0 \leq r \leq K_j - 1 \quad \text{and} \quad \sum_{m_j=0}^{p} a_j(m_j) m_j^{K_j} \neq 0.$$

For $\nu = 1$, examples are given

- the increments of $Y$: $Z_{n,\mathbf{a}}(k) = Y\left(\frac{k+1}{n}\right) - Y\left(\frac{k}{n}\right)$ for $\mathbf{a} = (-1, 1)$, which is a filter of order 1.

- the second order increments of $Y$: $Z_{n,\mathbf{a}}(k) = Y\left(\frac{k+2}{n}\right) - 2Y\left(\frac{k+1}{n}\right) + Y\left(\frac{k}{n}\right)$ for $\mathbf{a} = (1, -2, 1)$, which is a filter of order 2.

In dimension $\nu = 2$, following the works of Chan & Wood [8] and Zu & Stein [30] we can also consider the following types of increments:

- Vertical $Z_{n,\mathbf{a}}(k) = Y\left(\frac{k_1}{n}, \frac{k_2+2}{n}\right) - 2Y\left(\frac{k_1}{n}, \frac{k_2+1}{n}\right) + Y\left(\frac{k_1}{n}, \frac{k_2}{n}\right)$ for $\mathbf{a_1} = (1)$ filter of order 0 and $\mathbf{a_2} = (1, -2, 1)$ filter of order 2.

- Horizontal $Z_{n,\mathbf{a}}(k) = Y\left(\frac{k_1+2}{n}, \frac{k_2}{n}\right) - 2Y\left(\frac{k_1+1}{n}, \frac{k_2}{n}\right) + Y\left(\frac{k_1}{n}, \frac{k_2}{n}\right)$ for $\mathbf{a_1} = (1, -2, 1)$ filter of order 2 and $\mathbf{a_2} = (1)$ filter of order 0.

- Superficial $Z_{n,\mathbf{a}}(k) := \square_{k_1, k_2}^n(Y) = Y\left(\frac{k_1+1}{n}, \frac{k_2+1}{n}\right) - Y\left(\frac{k_1+1}{n}, \frac{k_2}{n}\right) - Y\left(\frac{k_1}{n}, \frac{k_2+1}{n}\right) + Y\left(\frac{k_1}{n}, \frac{k_2}{n}\right)$ for $\mathbf{a_1} = \mathbf{a_2} = (-1, 1)$ filter of order 1.

Coming back to the general case, let us associate to the filter $\mathbf{a_j}$ the real polynomial

$$P_{\mathbf{a_j}}(x_j) = \sum_{m_j=0}^{p} a_j(m_j) x_j^{m_j}, \text{ for } x_j \in \mathbb{R}.$$

Then $\mathbf{a_j}$ is a filter of order $K_j \geq 1$ if and only if $P_{\mathbf{a_j}}^{(r)}(1) = 0$, for $0 \leq r \leq K_j - 1$ and $P_{\mathbf{a_j}}^{(K_j)}(1) \neq 0$. By Taylor formula, this implies that there exists $c_j > 0$ such that

$$\left| P_{\mathbf{a_j}}\left(e^{-ix_j}\right) \right| \leq c_j \min\left(|x_j|^{K_j}, 1\right), x_j \in \mathbb{R}. \tag{37}$$

Moreover using the spectral representation of $Y$, one has

$$Z_{n,\mathbf{a}}(k) = \int_{\mathbb{R}^\nu} e^{-i\frac{k \cdot x}{n}} \prod_{j=1}^{\nu} P_{\mathbf{a_j}}\left(e^{-i\frac{x_j}{n}}\right) F(x)^{1/2} d\widetilde{W^\nu}(x).$$

We will note $P_{\mathbf{a}}(x) := \prod_{j=1}^{\nu} P_{\mathbf{a_j}}(x_j)$. The only assumption that we will use is the fact that $P_{\mathbf{a}}$ has a zero of order $K := K_1 + K_2 + \cdots K_\nu$ at $(1, \cdots, 1)$. We say that the filter $\mathbf{a}$ has order $K$. Then we have

$$\begin{aligned}
\text{Cov}(Z_{n,\mathbf{a}}(k), Z_{n,\mathbf{a}}(k')) &= \int_{\mathbb{R}^\nu} e^{-i\frac{(k-k') \cdot x}{n}} \left| P_{\mathbf{a}}\left(e^{-i\frac{x_1}{n}}, \cdots, e^{-i\frac{x_\nu}{n}}\right) \right|^2 F(x) dx \\
&= \frac{1}{(2\pi)^\nu} \int_{[-\pi, \pi)^\nu} e^{-i(k-k') \cdot x} \left| P_{\mathbf{a}}\left(e^{-ix_1}, \cdots, e^{-ix_\nu}\right) \right|^2 \sum_{k \in \mathbb{Z}^\nu} (2\pi n)^\nu F(nx + 2n\pi k) dx.
\end{aligned}$$

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So the spectral density of $Z_{n,\mathbf{a}}$ is given by

$$f_{n,\mathbf{a}}(x) = \left| P_{\mathbf{a}}\left(e^{-ix_1}, \cdots, e^{-ix_v}\right) \right|^2 \sum_{k \in \mathbb{Z}^v} (2\pi n)^v F(nx + 2n\pi k), \quad x \in [-\pi, \pi)^v. \tag{38}$$

Because of the assumption on $\mathbf{a}$, one can find a positive constant $c > 0$ such that

$$\left| P_{\mathbf{a}}\left(e^{-ix_1}, \cdots, e^{-ix_v}\right) \right| \leq c|x|^K, \quad x = (x_1, \ldots, x_v) \in [-\pi, \pi)^v, \tag{39}$$

Then, the generalized quadratic variations of $Y$ are defined as

$$V_{n,\mathbf{a}} = \frac{1}{(n-p+1)^v} \sum_{k_1,\ldots,k_v=0}^{n-p} \left( Z_{n,\mathbf{a}}(k) \right)^2. \tag{40}$$

Such quantities are very helpful to estimate the $H$ parameter as explained below.

## 4.1 Central Limit Theorem for quadratic variations

We now consider random fields $Y$ for which Assumption (4) is valid. More precisely, let $\Omega$ be a strictly positive homogeneous function of degree 0 that is continuous on the sphere $S^{v-1}$. We assume that

$$F(x) = \frac{\Omega(x)}{|x|^{2H+v}} + R(x), \tag{41}$$

where the rest $R$ satisfies the estimate

$$|R(x)| \leq \frac{\kappa}{|x|^{2H+v+\gamma}} \qquad \text{for} \quad |x| > A. \tag{42}$$

We will prove a Central Limit Theorem for the generalized quadratic variations related to $H$. We will see that the limit does not depend on the rest. We use the notations given above.

**Theorem 4.1.** *Let us assume that $F$, the spectral density of $Y$ satisfies (41) and (42) for some $H > 0$ and $\gamma > 0$. Let $\mathbf{a}$ be a filter of order $K$. Moreover, we assume that $|x|^{4K} F(x)^2$ is integrable on compact sets. If $K > H + \frac{v}{4}$, then for n tending to infinity,*

(i) $(n-p+1)^v \mathrm{Var}\left( \frac{V_{n,\mathbf{a}}}{\mathbb{E}(V_{n,\mathbf{a}})} \right) \longrightarrow \sigma_{\mathbf{a}}^2(H)$

(ii) $(n-p+1)^{v/2} \left( \frac{V_{n,\mathbf{a}}}{\mathbb{E}(V_{n,\mathbf{a}})} - 1 \right) \xrightarrow{d} \mathcal{N}(0, \sigma_{\mathbf{a}}^2(H)),$

*with*

$$\sigma_{\mathbf{a}}^2(H) = \frac{2(2\pi)^v}{C_{\mathbf{a}}(H)^2} \int_{[-\pi,\pi)^v} \left| P_{\mathbf{a}}\left(e^{-ix_1}, \cdots, e^{-ix_v}\right) \right|^4 \left| \sum_{k \in \mathbb{Z}^v} \frac{\Omega(x + 2\pi k)}{|x + 2\pi k|^{2H+v}} \right|^2 dx, \tag{43}$$

*where*

$$C_{\mathbf{a}}(H) = \int_{\mathbb{R}^v} \left| P_{\mathbf{a}}\left(e^{-ix_1}, \cdots, e^{-ix_v}\right) \right|^2 \frac{\Omega(x)}{|x|^{2H+v}} dx. \tag{44}$$

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*Proof.* This will be a direct consequence of the previous sections. We can write

$$(n-p+1)^{v/2}\left(\frac{V_{n,\mathbf{a}}}{\mathbb{E}\left(V_{n,\mathbf{a}}\right)}-1\right)=\frac{1}{(n-p+1)^{v/2}}\sum_{k_1,\dots,k_v=0}^{n-p}H_2(X_{n,\mathbf{a}}(k)),$$

with $\{X_{n,\mathbf{a}}(k),\,k\in\mathbb{Z}^v\}$ a stationary Gaussian time series, given by

$$X_{n,\mathbf{a}}(k):=\frac{Z_{n,\mathbf{a}}(k)}{\sqrt{\operatorname{Var}\left(Z_{n,\mathbf{a}}(k)\right)}}.$$

The spectral density of $X_{n,\mathbf{a}}$ is easily deduced from the one of $Z_{n,\mathbf{a}}$ given in (38), using the fact that

$$C_{n,\mathbf{a}}:=\operatorname{Var}\left(Z_{n,\mathbf{a}}(k)\right)=\int_{\mathbb{R}^v}\left|P_{\mathbf{a}}\left(e^{-ix_1},\cdots,e^{-ix_v}\right)\right|^2 n^v F(nx)dx.$$

This means that $f_{X_{n,\mathbf{a}}}$ is given by

$$f_{X_{n,\mathbf{a}}}(x):=\frac{(2\pi n)^v}{C_{n,\mathbf{a}}}|P_{\mathbf{a}}\left(e^{-ix_1},\cdots,e^{-ix_v}\right)|^2\sum_{k\in\mathbb{Z}^v}F(nx+2n\pi k). \tag{45}$$

We are in position to apply Theorem 2.10 or Theorem 3.2 depending on the dimension. It is sufficient to prove that $f_{X_{n,\mathbf{a}}}$ is uniformly in $L^2(\mathbb{T})$ and converges to $f_{X_{\mathbf{a}}}$, with

$$f_{X_{\mathbf{a}}}(x)=\frac{(2\pi n)^v}{C_{\mathbf{a}}(H)}|P_{\mathbf{a}}\left(e^{-ix_1},\cdots,e^{-ix_v}\right)|^2\sum_{k\in\mathbb{Z}^v}\frac{\Omega(x+2\pi k)}{|x+2\pi k|^{2H+v}} \tag{46}$$

and $C_{\mathbf{a}}(H)$ given by (44). The required convergence properties are contained in the following lemma.

**Lemma 4.2.** *We have the following.*

$$n^{2H}\mathbb{E}\left(V_{n,\mathbf{a}}\right)-C_{\mathbf{a}}(H)=\begin{cases}\displaystyle\operatorname*{O}_{n\to+\infty}\left(n^{-2(K-H)}\right)&\text{if }K-H<\gamma/2\\[2mm]\displaystyle\operatorname*{O}_{n\to+\infty}\left(n^{-\gamma}\log n\right)&\text{if }K-H=\gamma/2\\[2mm]\displaystyle\operatorname*{O}_{n\to+\infty}\left(n^{-\gamma}\right)&\text{if }K-H>\gamma/2\end{cases}.$$

*Moreover $f_{X_{n,\mathbf{a}}}$ and $f_{X_{\mathbf{a}}}$ are in $L^2(\mathbb{T})$ and*

$$\|f_{X_{n,\mathbf{a}}}-f_{X_{\mathbf{a}}}\|_2=\begin{cases}\displaystyle\operatorname*{O}_{n\to+\infty}\left(n^{-(2K-2H-v/2)}\right)&\text{if }K-H<\gamma+v/4\\[2mm]\displaystyle\operatorname*{O}_{n\to+\infty}\left(n^{-2\gamma}\log n\right)&\text{if }K-H=\gamma+v/4\\[2mm]\displaystyle\operatorname*{O}_{n\to+\infty}\left(n^{-2\gamma}\right)&\text{if }K-H>\gamma+v/4\end{cases}.$$

*Proof.* For the first estimates, we have to bound

$$\int_{\mathbb{R}^v}\min(|x|^{2K},1)n^{2H+v}|R(nx)|dx=\int_{|x|<A/n}+\int_{A/n<|x|<1}+\int_{|x|>1}.$$

For the first term, since we have no assumption on $R$ except for the fact that it is the difference between $F$ and $\frac{\Omega(x)}{|x|^{2H+v}}$, we consider each quantity separately. For the integral in $F$, we change of variable and use the assumption of integrability on $F$ to conclude that it is a term in $n^{-2(K-H)}$. The other estimates are straightforward.

Next, let us prove that $f_{X_a}$ is in $L^2(\mathbb{T})$. We write that

$$
\begin{aligned}
\int_{\mathbb{T}^v} (f_{X_a}(x))^2 dx &= c \int_{[-\pi,\pi)^v} \left| P_a\left(e^{-ix_1}, \cdots, e^{-ix_v}\right)\right|^4 \left(\sum_{k\in\mathbb{Z}^v} \frac{\Omega(x+2\pi k)}{|x+2\pi k|^{2H+v}}\right)^2 dx \\
&\leq c \int_{[-\pi,\pi)^v} |x|^{4K} \frac{\Omega(x)^2}{|x|^{4H+2v}} dx + C \\
&< +\infty.
\end{aligned}
$$

We have used the fact that, for $|x| \leq \pi$, the sum $\displaystyle\sum_{k\neq 0} \frac{1}{|x+2\pi k|^{2H+v}}$ is uniformly bounded since $K > H + \frac{v}{4}$. Next, in order to bound the norm of $f_{X_{n,a}} - f_{X_a}$, we have to consider the quantity

$$
\Delta_n(x) := n^{v+2H} \sum_{k\in\mathbb{Z}^v} R(n(x+2k\pi)).
$$

From Assumptions (41) and (42) and from the fact that $\Omega$ is bounded below on the unit sphere, we deduce that, for $|x| > A/n$ (with some constant $C$ that varies from line to line)

$$
|R(x)| \leq \frac{C}{|x|^\gamma} \frac{\Omega(x)}{|x|^{2H+v}}.
$$

It follows that, for $|x| > \frac{2C}{n}$, we have the inequality

$$
\Delta_n(x) \leq \frac{C}{n^\gamma |x|^{2H+v+\gamma}}.
$$

Using this inequality, we estimate easily

$$
\int_{|x|>\frac{2C}{n}} (f_{X_{n,a}}(x) - f_{X_a}(x))^2 dx.
$$

To conclude, it is sufficient, after a change of variables, to bound

$$
n^{4H+v-4K} \int_{|x|\leq 2C} |x|^{4K} F(x)^2 dx,
$$

which is direct upon the local integrability assumption on $F$. $\qquad\square$

This finishes the proof of the theorem. $\qquad\square$

## 4.2 Remark on the speed of convergence

We keep the notations of the last subsection and interest ourselves to the speed of convergence towards a Gaussian law. We want to give a bound for the Kolmogorov distance in one variable, or the distance of Wasserstein in general, between $\frac{V_{n,\mathbf{a}}}{\mathbb{E}(V_{n,\mathbf{a}})}$ and a Gaussian random variable of law $\mathcal{N}(0, \sigma_{\mathbf{a}}^2(H))$. By Lemma 4.2, we have a bound for the speed of convergence of $f_{X_{n,\mathbf{a}}}$ towards $f_{X_{\mathbf{a}}}$. So we can use Theorem 2.14 or Remark 3.3 as soon as $f_{X_{\mathbf{a}}}$ belongs to some Sobolev space $\mathcal{H}^\alpha$. Or, if the Fourier coefficients of $f_{X_{\mathbf{a}}}$ behave like a power $|k|^{-a}$, then we can also use Remark 2.15 to have a bound for the speed of convergence. This is what we discuss now.

Let us start with dimension one, where $\Omega$ is a constant. Then the Fourier coefficients of $f_{X_{\mathbf{a}}}$, which we note $r_{\mathbf{a}}$, are also, up to a constant, the values of the Fourier transform of the function

$$\frac{|P_{\mathbf{a}}(e^{-ix})|^2}{|x|^{2H+v}}.$$

It is classical that these last ones may be written as

$$r_{\mathbf{a}}(k) := \sum_{m=-p}^{p} b_m |k+m|^{2H},$$

where $b_m$ are coefficients of the polynomial $Q := |P|^2$. Using the fact that $Q$ vanishes at order $2K$ and Taylor's Formula, one sees that

$$r_{\mathbf{a}}(k) = O(|k|^{2H-2K}), \tag{47}$$

In higher dimension, we will show that we can conclude with some regularity assumption on $\Omega$. Specifically, if we assume that $\Omega$ is of class $\mathscr{C}^1$ on the unit sphere, then the first partial derivatives of

$$h_{\mathbf{a}}(x) := \frac{|P_{\mathbf{a}}(e^{-ix})|^2 \Omega(x)}{|x|^{2H+v}}$$

satisfy the same kind of estimates as the function itself, apart from the loss of 1 in the power of $|x|$ in one term, and the fact that $P$ or $\overline{P}$ has been replaced by its derivative in another one. If again $r_{\mathbf{a}}(k) := \int_{\mathbb{R}^v} e^{-ik \cdot x} h_{\mathbf{a}}(x) dx$, then $k_j r_{\mathbf{a}}(k)$ appears as the Fourier coefficients of the $j$th partial derivative of $h_{\mathbf{a}}$. Remark first that we have proved in Lemma 4.2 that the sequence $r_{\mathbf{a}}(k)$ is in $\ell^2(\mathbb{Z}^v)$ by proving that the periodization of $h_{\mathbf{a}}$ is in $L^2(\mathbb{T}^v)$, which is equivalent by Plancherel Theorem. For the same reason, to prove that $k_j r_{\mathbf{a}}(k)$ is a sequence in $\ell^2(\mathbb{Z}^v)$, it is equivalent to prove that the periodization of the $j$th derivative of $h_{\mathbf{a}}$ is in $L^2(\mathbb{T}^v)$. Under the assumption that $\Omega$ is of class $\mathscr{C}^1$ on the unit sphere, we do this by the same method, but on the stronger assumption that $K-1-H > v/4$. The necessity of a stronger assumption is linked to the loss of 1 in the power of $|x|$. Finally, under these two assumptions, we conclude that $\sum_{k \in \mathbb{Z}^v} |k|^2 |r_{\mathbf{a}}(k)|^2 < \infty$ and one can apply Theorem 3.3.

One can weaken or strengthen these assumptions on $\Omega$ to obtain the full range of Sobolev spaces. In all cases we have a speed of convergence towards a Gaussian law in $|n|^{-\delta}$ with $\delta$ depending on $\gamma$, $K$, $H$, and the regularity of $\Omega$.

## 4.3 Application to the identification of $H$

As we said before, a central question is the estimation of $H$ from real data. As an application of generalized quadratic variations, we obtain Proposition 1.3 of [5] without additional assumption of regularity on the spectral density. Actually, let us fix the dimension $\nu = 1$ and, following [17], let us consider the filtered process of $Y$ with a dilated filter. More precisely, let $U \geq 2$ an integer. For an integer $u \in \{1, \ldots, U\}$, the dilation $\mathbf{a}^u$ of $\mathbf{a}$ is defined by, for $0 \leq m \leq pu$,

$$a_m^u = \begin{cases} a_{m'} & \text{if } m' = mu \\ 0 & \text{otherwise.} \end{cases}$$

Since $\sum_{m=0}^{pu} m^r a_m^u = u^r \sum_{m=0}^{p} m^r a_m$, the filter $\mathbf{a}^u$ has the same order than $\mathbf{a}$ but length $pu$. Then,

$$\left\{ X_{n,\mathbf{a}^u}(k) ; k \in \mathbb{Z}^\nu, u \in \{1, \ldots, U\} \right\} \stackrel{f\,dd}{=} \left\{ \frac{Z_{n,\mathbf{a}^u}(k)}{\sqrt{\mathrm{Var}\left( Z_{n,\mathbf{a}^u}(k) \right)}} ; k \in \mathbb{Z}^\nu, u \in \{1, \ldots, U\} \right\},$$

so that $\left( \frac{V_{n,\mathbf{a}^u}}{\mathbb{E}\left( V_{n,\mathbf{a}^u} \right)}, \frac{V_{n,\mathbf{a}^\nu}}{\mathbb{E}\left( V_{n,\mathbf{a}^\nu} \right)} \right) \xrightarrow[n \to +\infty]{} (1,1)$ almost surely with asymptotic normality for $K > H + \frac{1}{4}$, according to Theorems 3.2 and 4.1, for any $u, \nu \in \{1, \ldots, U\}$. According to Proposition 1.1 of [5] (see also Lemma 4.2 above), Assumption (4) implies that

$$n^{2H} \mathbb{E} \left( V_{n,\mathbf{a}^u} \right) = u^{2H} C_{\mathbf{a}}(H) + \begin{cases} \underset{n \to +\infty}{O} \left( n^{-2(K-H)} \right) & \text{if } K - H < \gamma/2 \\ \underset{n \to +\infty}{O} \left( n^{-\gamma} \log n \right) & \text{if } K - H = \gamma/2 \\ \underset{n \to +\infty}{O} \left( n^{-\gamma} \right) & \text{if } K - H > \gamma/2 \end{cases},$$

so that for $u, \nu \in \{1, \ldots, U\}$,

$$\widehat{H_{n,\mathbf{a}}}(u, \nu) := \frac{1}{2 \log(u/\nu)} \log \left( \frac{V_{n,\mathbf{a}^u}}{V_{n,\mathbf{a}^\nu}} \right) \xrightarrow[n \to +\infty]{} H \text{ a.s.}$$

with asymptotic normality when $K > H + 1/4$ and $\gamma > 1/2$. We refer to [5] for details. The main difference here is the fact that we only need an assumption on the behavior of the rest, not on its derivatives, due to the use of Theorem 3.2.

## 4.4 Functional Central Limit Theorem for Quadratic variations of a stationary Gaussian random process

Up to now, we have only considered finite distributions. In this last subsection we want to prove that one can have convergence for continuous time processes as well. We will restrict our study to the case $\nu = 1$. So let us consider the random process $Y$ given by (3) in dimension one. Then Assumption (4) on the spectral density $F$ of $Y$ can be written as

$$F(x) = \frac{c}{|x|^{2H+1}} + \underset{|x| \to +\infty}{O} \left( \frac{1}{|x|^{2H+1+\gamma}} \right). \tag{48}$$

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We keep the notations of the previous section and consider for a discrete filter **a** of length $p+1$ and order $K \geq 1$ the filtered process of discrete observations of $Y$ defined for $n \geq p$ by

$$Z_{n,\mathbf{a}}(k) = \sum_{m=0}^{p} a_m Y \left( \frac{k+m}{n} \right), \quad \text{for } k \in \mathbb{Z}.$$

Following Donsker's Theorem we consider a functional version of the Central Limit Theorem obtained in Theorem 4.1. For this purpose let us introduce the continuous time random process defined for $t \in [\frac{p}{n}, 1]$ by

$$S_{n,\mathbf{a}}(t) = \frac{1}{\sqrt{n-p+1}} \sum_{k=0}^{[nt]-p} \left( \frac{Z_{n,\mathbf{a}}(k)^2}{C_{n,\mathbf{a}}} - 1 \right),$$

and $S_{n,\mathbf{a}}(t) = 0$ for $0 \leq t < \frac{p}{n}$, where $C_{n,\mathbf{a}} = \mathbb{E}\left( Z_{n,\mathbf{a}}(k)^2 \right)$. Then $S_{n,\mathbf{a}}(1) = \sqrt{n-p+1} \left( \frac{V_{n,\mathbf{a}}}{\mathbb{E}(V_{n,\mathbf{a}})} - 1 \right)$. Moreover, $S_{n,\mathbf{a}}$ is a.s. a càdlàg process on $[0,1]$ and we denote as usual $\mathscr{D}([0,1])$ the set of such processes. We also introduce the càdlàg random process defined on $[0,1]$ by $Y_n(t) = Y \left( \frac{[nt]}{n} \right)$. Let us recall that according to (48) and Proposition 1 of [7] we can assume that $Y$ is a.s. continuous on $[0,1]$. It follows that $Y_n$ converges in law to $Y$ in the space $\mathscr{D}([0,1])$ equipped with the Skorohod topology. Then, Theorem 6 of [4] has the following counterpart in our setting.

**Theorem 4.3.** *We keep notations introduced in the previous section. Let us assume that $F$, the spectral density of $Y$ satisfies (48) for some $H > 0$ and $\gamma > 0$. Let $\mathbf{a}$ be a filter of order $K$. Moreover, we assume that $|x|^{4K} F(x)^2$ is integrable on compact sets. If $K > H + \frac{1}{4}$, then for $n$ tending to infinity, we obtain the weak convergence (in the space $\mathscr{D}([0,1])^2$ equipped with the Skorohod topology)*

$$\left( Y_n(t), S_{n,\mathbf{a}}(t) \right) \longrightarrow \left( Y(t), \sigma_{\mathbf{a}}(H) B(t) \right),$$

*where $B$ is a standard Brownian motion on $[0,1]$ that is defined on the same probability space than $Y$, independent of $Y$ and*

$$\sigma_{\mathbf{a}}^2(H) = \frac{4\pi}{C_{\mathbf{a}}(H)^2} \int_{-\pi}^{+\pi} \left| P_{\mathbf{a}} \left( e^{-ix} \right) \right|^4 \left| \sum_{k \in \mathbb{Z}} \frac{c}{|x + 2\pi k|^{2H+1}} \right|^2 dx, \tag{49}$$

*where*

$$C_{\mathbf{a}}(H) = \int_{\mathbb{R}} \left| P_{\mathbf{a}} \left( e^{-ix} \right) \right|^2 \frac{c}{|x|^{2H+1}} dx. \tag{50}$$

*Proof.* Let us first consider the convergence of $S_{n,\mathbf{a}}(t)$. For fixed $t$, this is a small modification of the data of the previous section, and we immediately have

$$S_{n,\mathbf{a}}(t) \xrightarrow[n \to +\infty]{d} \mathcal{N}\left( 0, \sigma_{\mathbf{a}}^2(H) t \right) \overset{d}{=} \sigma_{\mathbf{a}}(H) B(t).$$

Next, we want to deal with a finite vector $(S_{n,\mathbf{a}}(t_1), \cdots, S_{n,\mathbf{a}}(t_d))$. We are not exactly in the same setting as in Theorem 3.2 since for the computation of each coordinate we use the same discrete time series, but modify the mean that we are taking depending on the coordinate. But it is easy to see that the same strategy is available, that is, it is sufficient to have the convergence of the

covariance matrix according to Proposition 2 of [28]. Therefore, we are linked to prove, for any fixed $0 < t < s$, that

$$\mathrm{Cov}\left(S_{n,\mathbf{a}}(t), S_{n,\mathbf{a}}(s)\right) \xrightarrow[n \to +\infty]{} \sigma_{\mathbf{a}}^2(H)t,$$

or, which is equivalent, to prove the convergence to 0 of $\mathrm{Cov}\left(S_{n,\mathbf{a}}(t), S_{n,\mathbf{a}}(s) - S_{n,\mathbf{a}}(t)\right)$. As previously we introduce the function

$$f_{\mathbf{a}}(x) = \frac{1}{C_{\mathbf{a}}(H)^2} |P_{\mathbf{a}}\left(e^{-ix}\right)|^2 \sum_{k \in \mathbb{Z}} \frac{2\pi c}{|x + 2\pi k|^{2H+1}}$$

and denote by $r_{\mathbf{a}}$ the Fourier coefficients of $f_{\mathbf{a}}$. We consider the centered stationary discrete Gaussian time series $X_{\mathbf{a}}$ which admits $f_{\mathbf{a}}$ for spectral density, and therefore $r_{\mathbf{a}}$ as covariance sequence. Then, let us define the random process

$$\widetilde{S}_{n,\mathbf{a}}(t) = \frac{1}{\sqrt{n-p+1}} \sum_{k=0}^{[nt]-p} H_2(X_{\mathbf{a}}(k)),$$

for $t \in [\frac{p}{n}, 1]$ and $\widetilde{S}_{n,\mathbf{a}}(t) = 0$ otherwise. It is easily seen, as in the proof of Theorem 2.10, that limits are the same for $\widetilde{S}_{n,\mathbf{a}}$ or $S_{n,\mathbf{a}}$. So, let us compute

$$
\begin{aligned}
\mathrm{Cov}\left(\widetilde{S}_{n,\mathbf{a}}(t), \widetilde{S}_{n,\mathbf{a}}(s) - \widetilde{S}_{n,\mathbf{a}}(t)\right) &= \frac{2}{n-p+1} \sum_{k=0}^{[nt]-p} \sum_{l=[nt]-p+1}^{[ns]-p} r_{\mathbf{a}}^2(l-k) \\
&\leq \frac{2}{n-p+1} \sum_{j=1}^{[ns]} j r_{\mathbf{a}}^2(j) + \sum_{j=\min([nt]-p,[ns]-[nt]-1)}^{[ns]} r_{\mathbf{a}}^2(j).
\end{aligned}
$$

The second term tends to zero as a rest of a convergent series, since $t < s$. For the first term, recall that, by (47),

$$\left|r_{\mathbf{a}}(k)\right| \leq C(1 + |k|)^{-2(K-H)},$$

so that for $\alpha \in (0, \min(2(K-H-1/4), 1/2))$

$$\frac{2}{n-p+1} \sum_{j=1}^{[ns]} j r_{\mathbf{a}}^2(j) \leq C n^{-2\alpha} s^{1-2\alpha},$$

which tends to zero as $n$ tends to infinity. This ends the proof of the convergence in finite dimensional distributions of $\widetilde{S}_{n,\mathbf{a}}$ and thus $S_{n,\mathbf{a}}$ to $\sigma_{\mathbf{a}}(H)B$.

Let us prove the tightness. We clearly have for $0 < t \leq s$

$$
\begin{aligned}
\mathbb{E}\left(\left(S_{n,\mathbf{a}}(t) - S_{n,\mathbf{a}}(s)\right)^2\right) &\leq C \|f_{X_{n,\mathbf{a}}}\|_2^2 \left(\frac{[ns] - [nt]}{n}\right) \\
&\leq C' \left(\frac{[ns] - [nt]}{n}\right).
\end{aligned}
$$

Finally for $t \le s \le r$, by Hölder inequality and using the equivalence of $L^p(\Omega, \mathbb{P})$ norms in the second chaos,

$$
\begin{aligned}
\mathbb{E}\left(\left(S_{n,\mathbf{a}}(t) - S_{n,\mathbf{a}}(s)\right)^2 \left(S_{n,\mathbf{a}}(r) - S_{n,\mathbf{a}}(s)\right)^2\right) &\le \mathbb{E}\left(\left(S_{n,\mathbf{a}}(t) - S_{n,\mathbf{a}}(s)\right)^4\right)^{1/2} \mathbb{E}\left(\left(S_{n,\mathbf{a}}(r) - S_{n,\mathbf{a}}(s)\right)^4\right)^{1/2} \\
&\le C \mathbb{E}\left(\left(S_{n,\mathbf{a}}(t) - S_{n,\mathbf{a}}(s)\right)^2\right) \mathbb{E}\left(\left(S_{n,\mathbf{a}}(r) - S_{n,\mathbf{a}}(s)\right)^2\right) \\
&\le C \left(\frac{[ns] - [nt]}{n}\right)\left(\frac{[nr] - [ns]}{n}\right).
\end{aligned}
$$

This quantity is bounded by $(r - t)^2$. Indeed, it is clearly the case when $r - t \ge 1/n$. When $r - t < 1/n$, either $[ns] = [nt]$ or $[nt] = [nr]$, so that it vanishes. The tightness of $S_{n,\mathbf{a}}$ follows from Theorem 13.5 of [6].

Now, let us consider the sequence of vectorial processes $\left(Y_n, S_{n,\mathbf{a}}\right)$, which belong to $\mathscr{D}([0,1])^2$. Each coordinate is tight, thus $\left(Y_n, S_{n,\mathbf{a}}\right)$ is also tight. It remains to study the finite dimensional convergence. Any linear combination of the coordinates of the above vector belongs to the order one and order two Chaos respectively. Moreover they have both a Gaussian limit. Thus Theorem 1 (item $(iv)$) of [28] allows to conclude of the vector itself and that the two Gaussian limits are independent. Summarizing we have

$$
\left(Y_n, S_{n,\mathbf{a}}\right) \xrightarrow{d} (Y, \sigma_{\mathbf{a}}(H)B),
$$

where the convergence is in distribution in the space $\mathscr{D}^2([0,1])$ and the two processes coordinates are Gaussian and independent. This also implies that the convergence is stable in law. $\qquad\square$

Such a result is a fundamental tool when one deals with a non linear function of a Gaussian field, see [9] for instance. In applications to porous media for instance, it is natural to consider that the observed field is $U(t) = g(Y(t))$, $t \in \mathbb{R}$, where $g$ is a non-linear function $g$ with extra assumptions of smoothness and integrability. In this context we are interested in the asymptotic behavior of the quadratic variations of $U$ instead of $Y$. A Central Limit Theorem can still be obtained, using a Taylor expansion of $g$ and similar methods to the ones that have been developed in the proof of Theorem 7 in [4]. In this case the limit variable is no more Gaussian. It is given by the stochastic integral $\sigma_{\mathbf{a}}(H) \int_0^1 \left(g'(Y(t))\right)^2 dB(t)$. We intend to develop this, in connection with applications, in another work.

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