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On the critical point of the Random Walk Pinning Model in dimension $d = 3$*

Quentin Berger
Laboratoire de Physique, ENS Lyon
Université de Lyon
46 Allée d'Italie, 69364 Lyon, France
e-mail: quentin.berger@ens.fr

Fabio Lucio Toninelli
CNRS and Laboratoire de Physique, ENS Lyon
Université de Lyon
46 Allée d'Italie, 69364 Lyon, France
e-mail: fabio-lucio.toninelli@ens-lyon.fr

Abstract

We consider the Random Walk Pinning Model studied in [3] and [2]: this is a random walk X on $\mathbb{Z}^d$, whose law is modified by the exponential of β times $L_N(X,Y)$, the collision local time up to time N with the (quenched) trajectory Y of another d-dimensional random walk. If β exceeds a certain critical value β_c, the two walks stick together for typical Y realizations (localized phase). A natural question is whether the disorder is relevant or not, that is whether the *quenched* and *annealed* systems have the same critical behavior. Birkner and Sun [3] proved that β_c coincides with the critical point of the *annealed* Random Walk Pinning Model if the space dimension is $d = 1$ or $d = 2$, and that it differs from it in dimension $d \geq 4$ (for $d \geq 5$, the result was proven also in [2]). Here, we consider the open case of the *marginal dimension* $d = 3$, and we prove non-coincidence of the critical points.

Key words: Pinning Models, Random Walk, Fractional Moment Method, Marginal Disorder.

AMS 2000 Subject Classification: Primary 82B44, 60K35, 82B27, 60K37.

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1 Introduction

We consider the Random Walk Pinning Model (RWPM): the starting point is a zero-drift random walk X on $\mathbb{Z}^d$ ($d \geq 1$), whose law is modified by the presence of a second random walk, Y. The trajectory of Y is fixed (quenched disorder) and can be seen as the random medium. The modification of the law of X due to the presence of Y takes the Boltzmann-Gibbs form of the exponential of a certain interaction parameter, β, times the collision local time of X and Y up to time N, $L_N(X,Y) := \sum_{1\leq n\leq N} \mathbf{1}_{\{X_n=Y_n\}}$. If β exceeds a certain threshold value β_c, then for almost every realization of Y the walk X sticks together with Y, in the thermodynamic limit $N \to \infty$. If on the other hand $\beta < \beta_c$, then $L_N(X,Y)$ is $o(N)$ for typical trajectories.

Averaging with respect to Y the partition function, one obtains the partition function of the so-called annealed model, whose critical point β_c^{ann} is easily computed; a natural question is whether $\beta_c \neq \beta_c^{ann}$ or not. In the renormalization group language, this is related to the question whether disorder is *relevant* or not. In an early version of the paper [2], Birkner *et al.* proved that $\beta_c \neq \beta_c^{ann}$ in dimension $d \geq 5$. Around the same time, Birkner and Sun [3] extended this result to $d = 4$, and also proved that the two critical points *do coincide* in dimensions $d = 1$ and $d = 2$.

The dimension $d = 3$ is the *marginal dimension* in the renormalization group sense, where not even heuristic arguments like the "Harris criterion" (at least its most naive version) can predict whether one has disorder relevance or irrelevance. Our main result here is that quenched and annealed critical points differ also in $d = 3$.

For a discussion of the connection of the RWPM with the "parabolic Anderson model with a single catalyst", and of the implications of $\beta_c \neq \beta_c^{ann}$ about the location of the weak-to-strong transition for the directed polymer in random environment, we refer to [3, Sec. 1.2 and 1.4].

Our proof is based on the idea of bounding the fractional moments of the partition function, together with a suitable change of measure argument. This technique, originally introduced in [6; 9; 10] for the proof of disorder relevance for the random pinning model with tail exponent $\alpha \geq 1/2$, has also proven to be quite powerful in other cases: in the proof of non-coincidence of critical points for the RWPM in dimension $d \geq 4$ [3], in the proof that "disorder is always strong" for the directed polymer in random environment in dimension $(1+2)$ [12] and finally in the proof that quenched and annealed large deviation functionals for random walks in random environments in two and three dimensions differ [15]. Let us mention that for the random pinning model there is another method, developed by Alexander and Zygouras [1], to prove disorder relevance: however, their method fails in the marginal situation $\alpha = 1/2$ (which corresponds to $d = 3$ for the RWPM).

To guide the reader through the paper, let us point out immediately what are the novelties and the similarities of our proof with respect to the previous applications of the fractional moment/change of measure method:

- the change of measure chosen by Birkner and Sun in [3] consists essentially in correlating positively each increment of the random walk Y with the next one. Therefore, under the modified measure, Y is more diffusive. The change of measure we use in dimension three has also the effect of correlating positively the increments of Y, but in our case the correlations have long range (the correlation between the i^{th} and the j^{th} increment decays like $|i-j|^{-1/2}$). Another ingredient which was absent in [3] and which is essential in $d = 3$ is a coarse-graining step, of the type of that employed in [14; 10];

- while the scheme of the proof of our Theorem 2.8 has many points in common with that of [10, Th. 1.7], here we need new renewal-type estimates (e.g. Lemma 4.7) and a careful application of the Local Limit Theorem to prove that the average of the partition function under the modified measure is small (Lemmas 4.2 and 4.3).

2 Model and results

2.1 The random walk pinning model

Let $X = \{X_n\}_{n\geq 0}$ and $Y = \{Y_n\}_{n\geq 0}$ be two independent discrete-time random walks on $\mathbb{Z}^d$, $d \geq 1$, starting from 0, and let $\mathbb{P}^X$ and $\mathbb{P}^Y$ denote their respective laws. We make the following assumption:

Assumption 2.1. The random walk X is aperiodic. The increments $(X_i - X_{i-1})_{i\geq 1}$ are i.i.d., symmetric and have a finite fourth moment ($\mathbb{E}^X\left[\|X_1\|^4\right] < \infty$, where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{Z}^d$). Moreover, the covariance matrix of X_1, call it Σ_X, is non-singular.

The same assumptions hold for the increments of Y (in that case, we call Σ_Y the covariance matrix of Y_1).

For $\beta \in \mathbb{R}, N \in \mathbb{N}$ and for a fixed realization of Y we define a Gibbs transformation of the path measure $\mathbb{P}^X$: this is the polymer path measure $\mathbb{P}^\beta_{N,Y}$, absolutely continuous with respect to $\mathbb{P}^X$, given by

$$\frac{\mathrm{d}\mathbb{P}^\beta_{N,Y}}{\mathrm{d}\mathbb{P}^X}(X) = \frac{e^{\beta L_N(X,Y)}\, \mathbf{1}_{\{X_N=Y_N\}}}{Z^\beta_{N,Y}}, \tag{1}$$

where $L_N(X,Y) = \sum_{n=1}^N \mathbf{1}_{\{X_n=Y_n\}}$, and where

$$Z^\beta_{N,Y} = \mathbb{E}^X[e^{\beta L_N(X,Y)}\, \mathbf{1}_{\{X_N=Y_N\}}] \tag{2}$$

is the partition function that normalizes $\mathbb{P}^\beta_{N,Y}$ to a probability.

The *quenched* free energy of the model is defined by

$$F(\beta) := \lim_{N\to\infty} \frac{1}{N} \log Z^\beta_{N,Y} = \lim_{N\to\infty} \frac{1}{N} \mathbb{E}^Y[\log Z^\beta_{N,Y}] \tag{3}$$

(the existence of the limit and the fact that it is $\mathbb{P}^Y$-almost surely constant and non-negative is proven in [3]). We define also the *annealed* partition function $\mathbb{E}^Y[Z^\beta_{N,Y}]$, and the *annealed* free energy:

$$F^{ann}(\beta) := \lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}^Y[Z^\beta_{N,Y}]. \tag{4}$$

We can compare the *quenched* and *annealed* free energies, via the Jensen inequality:

$$F(\beta) = \lim_{N\to\infty} \frac{1}{N} \mathbb{E}^Y[\log Z^\beta_{N,Y}] \leqslant \lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}^Y[Z^\beta_{N,Y}] = F^{ann}(\beta). \tag{5}$$

The properties of $F^{ann}(\cdot)$ are well known (see the Remark 2.3), and we have the existence of critical points [3], for both *quenched* and *annealed* models, thanks to the convexity and the monotonicity of the free energies with respect to β:

Definition 2.2 (Critical points)**.** *There exist* $0 \leqslant \beta_c^{ann} \leqslant \beta_c$ *depending on the laws of* X *and* Y *such that:* $F^{ann}(\beta) = 0$ *if* $\beta \leqslant \beta_c^{ann}$ *and* $F^{ann}(\beta) > 0$ *if* $\beta > \beta_c^{ann}$; $F(\beta) = 0$ *if* $\beta \leqslant \beta_c$ *and* $F(\beta) > 0$ *if* $\beta > \beta_c$.

The inequality $\beta_c^{ann} \leqslant \beta_c$ comes from the inequality (5).

Remark 2.3. As was remarked in [3], the *annealed* model is just the homogeneous pinning model [8, Chapter 2] with partition function

$$\mathbb{E}^Y[Z_{N,Y}^{\beta}] = \mathbb{E}^{X-Y}\left[\exp\left(\beta \sum_{n=1}^{N} \mathbf{1}_{\{(X-Y)_n=0\}}\right) \mathbf{1}_{\{(X-Y)_N=0\}}\right]$$

which describes the random walk $X - Y$ which receives the reward β each time it hits 0. From the well-known results on the homogeneous pinning model one sees therefore that

- If $d = 1$ or $d = 2$, the *annealed* critical point β_c^{ann} is zero because the random walk $X - Y$ is recurrent.
- If $d \geq 3$, the walk $X - Y$ is transient and as a consequence
$$\beta_c^{ann} = -\log\left[1 - \mathbb{P}^{X-Y}\left((X-Y)_n \neq 0 \ \text{ for every } \ n > 0\right)\right] > 0.$$

Remark 2.4. As in the pinning model [8], the critical point β_c marks the transition from a delocalized to a localized regime. We observe that thanks to the convexity of the free energy,

$$\partial_\beta F(\beta) = \lim_{N\to\infty} \mathbb{E}_{N,Y}^{\beta}\left[\frac{1}{N}\sum_{n=1}^{N} \mathbf{1}_{\{X_N=Y_N\}}\right], \tag{6}$$

almost surely in Y, for every β such that $F(\cdot)$ is differentiable at β. This is the contact fraction between X and Y. When $\beta < \beta_c$, we have $F(\beta) = 0$, and the limit density of contact between X and Y is equal to 0: $\mathbb{E}_{N,Y}^{\beta} \sum_{n=1}^{N} \mathbf{1}_{\{X_N=Y_N\}} = o(N)$, and we are in the delocalized regime. On the other hand, if $\beta > \beta_c$, we have $F(\beta) > 0$, and there is a positive density of contacts between X and Y: we are in the localized regime.

2.2 Review of the known results

The following is known about the question of the coincidence of quenched and annealed critical points:

Theorem 2.5. *[3] Assume that* X *and* Y *are discrete time simple random walks on* $\mathbb{Z}^d$.

If $d = 1$ *or* $d = 2$*, the quenched and annealed critical points coincide:* $\beta_c = \beta_c^{ann} = 0$.

If $d \geq 4$*, the quenched and annealed critical points differ:* $\beta_c > \beta_c^{ann} > 0$.

Actually, the result that Birkner and Sun obtained in [3] is valid for slightly more general walks than simple symmetric random walks, as pointed out in the last Remark in [3, Sec.4.1]: for instance, they allow symmetric walks X and Y with common jump kernel and finite variance, provided that $\mathbb{P}^X(X_1 = 0) \geq 1/2$.

In dimension $d \geq 5$, the result was also proven (via a very different method, and for more general random walks which include those of Assumption 2.1) in an early version of the paper [2].

Remark 2.6. The method and result of [3] in dimensions $d = 1, 2$ can be easily extended beyond the simple random walk case (keeping zero mean and finite variance). On the other hand, in the case $d \geq 4$ new ideas are needed to make the change-of-measure argument of [3] work for more general random walks.

Birkner and Sun gave also a similar result if X and Y are continuous-time symmetric simple random walks on $\mathbb{Z}^d$, with jump rates 1 and $\rho \geq 0$ respectively. With definitions of (quenched and annealed) free energy and critical points which are analogous to those of the discrete-time model, they proved:

Theorem 2.7. *[3] In dimension $d = 1$ and $d = 2$, one has $\beta_c = \beta_c^{ann} = 0$. In dimensions $d \geq 4$, one has $0 < \beta_c^{ann} < \beta_c$ for each $\rho > 0$. Moreover, for $d = 4$ and for each $\delta > 0$, there exists $a_\delta > 0$ such that $\beta_c - \beta_c^{ann} \geq a_\delta \rho^{1+\delta}$ for all $\rho \in [0, 1]$. For $d \geq 5$, there exists $a > 0$ such that $\beta_c - \beta_c^{ann} \geq a\rho$ for all $\rho \in [0, 1]$.*

Our main result completes this picture, resolving the open case of the critical dimension $d = 3$ (for simplicity, we deal only with the discrete-time model).

Theorem 2.8. *Under the Assumption 2.1, for $d = 3$, we have $\beta_c > \beta_c^{ann}$.*

We point out that the result holds also in the case where X (or Y) is a simple random walk, a case which a priori is excluded by the aperiodicity condition of Assumption 2.1; see the Remark 2.11.

Also, it is possible to modify our change-of-measure argument to prove the non-coincidence of quenched and annealed critical points in dimension $d = 4$ for the general walks of Assumption 2.1, thereby extending the result of [3]; see Section 4.4 for a hint at the necessary steps.

Note After this work was completed, M. Birkner and R. Sun informed us that in [4] they independently proved Theorem 2.8 for the continuous-time model.

2.3 A renewal-type representation for $Z_{N,Y}^{\beta}$

From now on, we will assume that $d \geq 3$.

As discussed in [3], there is a way to represent the partition function $Z_{N,Y}^{\beta}$ in terms of a renewal process τ; this rewriting makes the model look formally similar to the random pinning model [8].

In order to introduce the representation of [3], we need a few definitions.

Definition 2.9. *We let*

1. $p_n^X(x) = \mathbb{P}^X(X_n = x)$ *and* $p_n^{X-Y}(x) = \mathbb{P}^{X-Y}\big((X-Y)_n = x\big)$*;*
2. $\mathbf{P}$ *be the law of a recurrent renewal* $\tau = \{\tau_0, \tau_1, \dots\}$ *with* $\tau_0 = 0$*, i.i.d. increments and inter-arrival law given by*

$$K(n) := \mathbf{P}(\tau_1 = n) = \frac{p_n^{X-Y}(0)}{G^{X-Y}} \quad \text{where} \quad G^{X-Y} := \sum_{n=1}^{\infty} p_n^{X-Y}(0) \tag{7}$$

(note that $G^{X-Y} < \infty$ in dimension $d \geq 3$);

3. $z' = (e^\beta - 1)$ *and* $z = z'\, G^{X-Y}$*;*

4. *for $n \in \mathbb{N}$ and $x \in \mathbb{Z}^d$,*

$$w(z,n,x) = z\frac{p_n^X(x)}{p_n^{X-Y}(0)}; \tag{8}$$

5. $\check{Z}^z_{N,Y} := \frac{z'}{1+z'} Z^\beta_{N,Y}$.

Then, via the binomial expansion of $e^{\beta L_N(X,Y)} = (1+z')^{L_N(X,Y)}$ one gets [3]

$$\begin{aligned} \check{Z}^z_{N,Y} &= \sum_{m=1}^{N} \sum_{\tau_0=0<\tau_1<\ldots<\tau_m=N} \prod_{i=1}^{m} K(\tau_i-\tau_{i-1}) w(z,\tau_i-\tau_{i-1},Y_{\tau_i}-Y_{\tau_{i-1}}) \\ &= \mathbf{E}\left[W(z,\tau\cap\{0,\ldots,N\},Y)\mathbf{1}_{N\in\tau}\right], \end{aligned} \tag{9}$$

where we defined for any finite increasing sequence $s=\{s_0,s_1,\ldots,s_l\}$

$$W(z,s,Y) = \frac{\mathbb{E}^X\left[\prod_{n=1}^l z\mathbf{1}_{\{X_{s_n}=Y_{s_n}\}}\middle| X_{s_0}=Y_{s_0}\right]}{\mathbb{E}^{X-Y}\left[\prod_{n=1}^l \mathbf{1}_{\{X_{s_n}=Y_{s_n}\}}\middle| X_{s_0}=Y_{s_0}\right]} = \prod_{n=1}^{l} w(z,s_n-s_{n-1},Y_{s_n}-Y_{s_{n-1}}). \tag{10}$$

We remark that, taking the $\mathbb{E}^Y$−expectation of the weights, we get

$$\mathbb{E}^Y\left[w(z,\tau_i-\tau_{i-1},Y_{\tau_i}-Y_{\tau_{i-1}})\right] = z.$$

Again, we see that the *annealed* partition function is the partition function of a homogeneous pinning model:

$$\check{Z}^{z,ann}_{N,Y} = \mathbb{E}^Y[\check{Z}^z_{N,Y}] = \mathbf{E}\left[z^{R_N}\mathbf{1}_{\{N\in\tau\}}\right], \tag{11}$$

where we defined $R_N := |\tau\cap\{1,\ldots,N\}|$.
Since the renewal τ is recurrent, the *annealed* critical point is $z_c^{ann}=1$.

In the following, we will often use the Local Limit Theorem for random walks, that one can find for instance in [5, Theorem 3] (recall that we assumed that the increments of both X and Y have finite second moments and non-singular covariance matrix):

Proposition 2.10 (Local Limit Theorem)**.** *Under the Assumption 2.1, we get*

$$\mathbb{P}^X(X_n=x) = \frac{1}{(2\pi n)^{d/2}(\det\Sigma_X)^{1/2}}\exp\left(-\frac{1}{2n}x\cdot\left(\Sigma_X^{-1}x\right)\right) + o(n^{-d/2}), \tag{12}$$

where $o(n^{-d/2})$ is uniform for $x\in\mathbb{Z}^d$.
Moreover, there exists a constant $c>0$ such that for all $x\in\mathbb{Z}^d$ and $n\in\mathbb{N}$

$$\mathbb{P}^X(X_n=x) \leqslant cn^{-d/2}. \tag{13}$$

Similar statements hold for the walk Y.

(We use the notation $x \cdot y$ for the canonical scalar product in $\mathbb{R}^d$.)

In particular, from Proposition 2.10 and the definition of $K(\cdot)$ in (7), we get $K(n) \sim c_K n^{-d/2}$ as $n \to \infty$, for some positive c_K. As a consequence, for $d = 3$ we get from [7, Th. B] that

$$\mathbf{P}(n \in \tau) \overset{n\to\infty}{\sim} \frac{1}{2\pi c_K \sqrt{n}}. \tag{14}$$

Remark 2.11. In Proposition 2.10, we supposed that the walk X is aperiodic, which is not the case for the simple random walk. If X is the symmetric simple random walk on $\mathbb{Z}^d$, then [13, Prop. 1.2.5]

$$\mathbb{P}^X(X_n = x) = \mathbf{1}_{\{n \leftrightarrow x\}} \frac{2}{(2\pi n)^{d/2} (\det \Sigma_X)^{1/2}} \exp\left(-\frac{1}{2n} x \cdot \left(\Sigma_X^{-1} x \right) \right) + o(n^{-d/2}), \tag{15}$$

where $+o(n^{-d/2})$ is uniform for $x \in \mathbb{Z}^d$, and where $n \leftrightarrow x$ means that n and x have the same parity (so that x is a possible value for X_n). Of course, in this case Σ_X is just $1/d$ times the identity matrix. The statement (13) also holds.

Via this remark, one can adapt all the computations of the following sections, which are based on Proposition 2.10, to the case where X (or Y) is a simple random walk. For simplicity of exposition, we give the proof of Theorem 2.8 only in the aperiodic case.

3 Main result: the dimension $d = 3$

With the definition $\check{F}(z) := \lim_{N\to\infty} \frac{1}{N} \log \check{Z}^z_{N,Y}$, to prove Theorem 2.8 it is sufficient to show that $\check{F}(z) = 0$ for some $z > 1$.

3.1 The coarse-graining procedure and the fractional moment method

We consider without loss of generality a system of size proportional to $L = \frac{1}{z-1}$ (the coarse-graining length), that is $N = mL$, with $m \in \mathbb{N}$. Then, for $\mathcal{I} \subset \{1, \dots, m\}$, we define

$$Z^{\mathcal{I}}_{z,Y} := \mathbf{E}\left[W(z, \tau \cap \{0, \dots, N\}, Y) \mathbf{1}_{N \in \tau} \mathbf{1}_{E_{\mathcal{I}}}(\tau) \right], \tag{16}$$

where $E_{\mathcal{I}}$ is the event that the renewal τ intersects the blocks $(B_i)_{i\in\mathcal{I}}$ and only these blocks over $\{1, \dots, N\}$, B_i being the i^{th} block of size L:

$$B_i := \{(i-1)L + 1, \dots, iL\}. \tag{17}$$

Since the events $E_{\mathcal{I}}$ are disjoint, we can write

$$\check{Z}^z_{N,Y} := \sum_{\mathcal{I} \subset \{1,\dots,m\}} Z^{\mathcal{I}}_{z,Y}. \tag{18}$$

Note that $Z^{\mathcal{I}}_{z,Y} = 0$ if $m \notin \mathcal{I}$. We can therefore assume $m \in \mathcal{I}$. If we denote $\mathcal{I} = \{i_1, i_2, \dots, i_l\}$ $(l = |\mathcal{I}|)$, $i_1 < \dots < i_l$, $i_l = m$, we can express $Z^{\mathcal{I}}_{z,Y}$ in the following way:

$$Z^{\mathscr{I}}_{z,Y} := \sum_{\substack{a_1,b_1\in B_{i_1}\\ a_1\leqslant b_1}} \sum_{\substack{a_2,b_2\in B_{i_2}\\ a_2\leqslant b_2}} \dots \sum_{a_l\in B_{i_l}} K(a_1)w(z,a_1,Y_{a_1})Z^z_{a_1,b_1} \dots K(a_l-b_{l-1})w(z,a_l-b_{l-1},Y_{a_l}-Y_{b_{l-1}})Z^z_{a_l,N}, \tag{19}$$

where

$$Z^z_{j,k} := \mathbf{E}\left[W(z,\tau\cap\{j,\dots,k\},Y)\mathbf{1}_{k\in\tau}\,\middle|\, j\in\tau\right] \tag{20}$$

is the partition function between j and k.

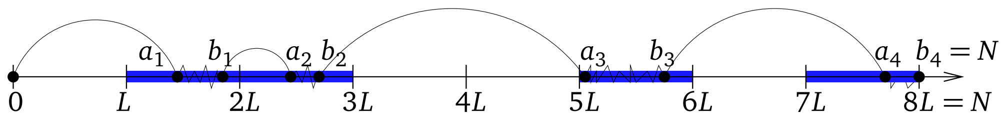

Figure 1: The coarse-graining procedure. Here $N = 8L$ (the system is cut into 8 blocks), and $\mathscr{I} = \{2,3,6,8\}$ (the gray zones) are the blocks where the contacts occur, and where the change of measure procedure of the Section 3.2 acts.

Moreover, thanks to the Local Limit Theorem (Proposition 2.10), one can note that there exists a constant $c > 0$ independent of the realization of Y such that, if one takes $z \leqslant 2$ (we will take z close to 1 anyway), one has

$$w(z,\tau_i-\tau_{i-1},Y_{\tau_i}-Y_{\tau_{i-1}}) = z\frac{p^X_{\tau_i-\tau_{i-1}}(Y_{\tau_i}-Y_{\tau_{i-1}})}{p^{X-Y}_{\tau_i-\tau_{i-1}}(0)} \leq c.$$

So, the decomposition (19) gives

$$Z^{\mathscr{I}}_{z,Y} \leqslant c^{|\mathscr{I}|} \sum_{\substack{a_1,b_1\in B_{i_1}\\ a_1\leqslant b_1}} \sum_{\substack{a_2,b_2\in B_{i_2}\\ a_2\leqslant b_2}} \dots \sum_{a_l\in B_{i_l}} K(a_1)Z^z_{a_1,b_1}K(a_2-b_1)Z^z_{a_2,b_2}\dots K(a_l-b_{l-1})Z^z_{a_l,N}. \tag{21}$$

We now eliminate the dependence on z in the inequality (21). This is possible thanks to the choice $L = \frac{1}{z-1}$. As each $Z^z_{a_i,b_i}$ is the partition function of a system of size smaller than L, we get $W(z,\tau\cap\{a_i,\dots,b_i\},Y) \leqslant z^L W(z=1,\tau\cap\{a_i,\dots,b_i\},Y)$ (recall the definition (10)). But with the choice $L = \frac{1}{z-1}$, the factor z^L is bounded by a constant c, and thanks to the equation (20), we finally get

$$Z^z_{a_i,b_i} \leqslant c Z^{z=1}_{a_i,b_i}. \tag{22}$$

Notational warning: in the following, c, c', etc. will denote positive constants, whose value may change from line to line.

We note $Z_{a_i,b_i} := Z^{z=1}_{a_i,b_i}$ and $W(\tau,Y) := W(z=1,\tau,Y)$. Plugging this in the inequality (21), we finally get

$$Z^{\mathscr{I}}_{z,Y} \leqslant c'^{|\mathscr{I}|} \sum_{\substack{a_1,b_1\in B_{i_1}\\ a_1\leqslant b_1}} \sum_{\substack{a_2,b_2\in B_{i_2}\\ a_2\leqslant b_2}} \dots \sum_{a_l\in B_{i_l}} K(a_1)Z_{a_1,b_1}K(a_2-b_1)Z_{a_2,b_2}\dots K(a_l-b_{l-1})Z_{a_l,N}, \tag{23}$$

where there is no dependence on z anymore.

The fractional moment method starts from the observation that for any $\gamma \neq 0$

$$\check{F}(z) = \lim_{N\to\infty} \frac{1}{\gamma N} \mathbb{E}^Y \left[\log \left(\check{Z}^z_{N,Y} \right)^\gamma \right] \leqslant \liminf_{N\to\infty} \frac{1}{N\gamma} \log \mathbb{E}^Y \left[\left(\check{Z}^z_{N,Y} \right)^\gamma \right]. \tag{24}$$

Let us fix a value of $\gamma \in (0,1)$ (as in [10], we will choose $\gamma = 6/7$, but we will keep writing it as γ to simplify the reading). Using the inequality $\left(\sum a_n\right)^\gamma \leqslant \sum a_n^\gamma$ (which is valid for $a_i \geq 0$), and combining with the decomposition (18), we get

$$\mathbb{E}^Y \left[\left(\check{Z}^z_{N,Y} \right)^\gamma \right] \leqslant \sum_{\mathscr{I} \subset \{1,\dots,m\}} \mathbb{E}^Y \left[\left(Z^{\mathscr{I}}_{z,Y} \right)^\gamma \right]. \tag{25}$$

Thanks to (24) we only have to prove that, for some $z > 1$, $\limsup_{N\to\infty} \mathbb{E}^Y \left[\left(\check{Z}^z_{N,Y} \right)^\gamma \right] < \infty$.

We deal with the term $\mathbb{E}^Y \left[(Z^{\mathscr{I}}_{z,Y})^\gamma \right]$ via a change of measure procedure.

3.2 The change of measure procedure

The idea is to change the measure $\mathbb{P}^Y$ on each block whose index belongs to $\mathscr{I}$, keeping each block independent of the others. We replace, for fixed $\mathscr{I}$, the measure $\mathbb{P}^Y(\,\mathrm{d}Y)$ with $g_{\mathscr{I}}(Y)\mathbb{P}^Y(\,\mathrm{d}Y)$, where the function $g_{\mathscr{I}}(Y)$ will have the effect of creating long range positive correlations between the increments of Y, inside each block separately. Then, thanks to the Hölder inequality, we can write

$$\mathbb{E}^Y \left[\left(Z^{\mathscr{I}}_{z,Y} \right)^\gamma \right] = \mathbb{E}^Y \left[\frac{g_{\mathscr{I}}(Y)^\gamma}{g_{\mathscr{I}}(Y)^\gamma} \left(Z^{\mathscr{I}}_{z,Y} \right)^\gamma \right] \quad \leqslant \quad \mathbb{E}^Y \left[g_{\mathscr{I}}(Y)^{-\frac{\gamma}{1-\gamma}} \right]^{1-\gamma} \mathbb{E}^Y \left[g_{\mathscr{I}}(Y) Z^{\mathscr{I}}_{z,Y} \right]^\gamma. \tag{26}$$

In the following, we will denote $\Delta_i = Y_i - Y_{i-1}$ the i^{th} increment of Y. Let us introduce, for $K > 0$ and ε_K to be chosen, the following "change of measure":

$$g_{\mathscr{I}}(Y) = \prod_{k\in\mathscr{I}} (\mathbf{1}_{F_k(Y) \leqslant K} + \varepsilon_K \mathbf{1}_{F_k(Y) > K}) \equiv \prod_{k\in\mathscr{I}} g_k(Y), \tag{27}$$

where

$$F_k(Y) = -\sum_{i,j\in B_k} M_{ij} \Delta_i \cdot \Delta_j, \tag{28}$$

and

$$\begin{cases} M_{ij} = \frac{1}{\sqrt{L\log L}} \frac{1}{\sqrt{|j-i|}} & \text{if } i \neq j \\ M_{ii} = 0. \end{cases} \tag{29}$$

Let us note that from the form of M, we get that $\|M\|^2 := \sum_{i,j\in B_1} M_{ij}^2 \leqslant C$, where the constant $C < \infty$ does not depend on L. We also note that F_k only depends on the increments of Y in the block labeled k.

Let us deal with the first factor of (26):

$$\mathbb{E}^Y\left[g_{\mathscr{I}}(Y)^{-\frac{\gamma}{1-\gamma}}\right]=\prod_{k\in\mathscr{I}}\mathbb{E}^Y\left[g_k(Y)^{-\frac{\gamma}{1-\gamma}}\right]=\left(\mathbb{P}^Y(F_1(Y)\leqslant K)+\varepsilon_K^{-\frac{\gamma}{1-\gamma}}\mathbb{P}^Y(F_1(Y)>K)\right)^{|\mathscr{I}|}. \tag{30}$$

We now choose

$$\varepsilon_K:=\mathbb{P}^Y(F_1(Y)>K)^{\frac{1-\gamma}{\gamma}} \tag{31}$$

such that the first factor in (26) is bounded by $2^{(1-\gamma)|\mathscr{I}|}\leqslant 2^{|\mathscr{I}|}$. The inequality (26) finally gives

$$\mathbb{E}^Y\left[\left(Z_{z,Y}^{\mathscr{I}}\right)^{\gamma}\right]\leqslant 2^{|\mathscr{I}|}\mathbb{E}^Y\left[g_{\mathscr{I}}(Y)Z_{z,Y}^{\mathscr{I}}\right]^{\gamma}. \tag{32}$$

The idea is that when $F_1(Y)$ is large, the weight $g_1(Y)$ in the change of measure is small. That is why the following lemma is useful:

Lemma 3.1. *We have*

$$\lim_{K\to\infty}\limsup_{L\to\infty}\varepsilon_K=\lim_{K\to\infty}\limsup_{L\to\infty}\mathbb{P}^Y(F_1(Y)>K)=0 \tag{33}$$

Proof. We already know that $\mathbb{E}^Y[F_1(Y)]=0$, so thanks to the standard Chebyshev inequality, we only have to prove that $\mathbb{E}^Y[F_1(Y)^2]$ is bounded uniformly in L. We get

$$\begin{aligned}\mathbb{E}^Y[F_1(Y)^2] &= \sum_{\substack{i,j\in B_1\\ k,l\in B_1}}M_{ij}M_{kl}\mathbb{E}^Y\left[(\Delta_i\cdot\Delta_j)(\Delta_k\cdot\Delta_l)\right]\\ &= \sum_{\{i,j\}}M_{ij}^2\mathbb{E}^Y\left[(\Delta_i\cdot\Delta_j)^2\right]\end{aligned} \tag{34}$$

where we used that $\mathbb{E}^Y\left[(\Delta_i\cdot\Delta_j)(\Delta_k\cdot\Delta_l)\right]=0$ if $\{i,j\}\neq\{k,l\}$. Then, we can use the Cauchy-Schwarz inequality to get

$$\mathbb{E}^Y[F_1(Y)^2]\leqslant\sum_{\{i,j\}}M_{ij}^2\mathbb{E}^Y\left[\left\|\Delta_i\right\|^2\left\|\Delta_j\right\|^2\right]\leqslant\|M\|^2\sigma_Y^4:=\|M\|^2\left[\mathbb{E}^Y(\|Y_1\|^2)\right]^2. \tag{35}$$

□

We are left with the estimation of $\mathbb{E}^Y\left[g_{\mathscr{I}}(Y)Z_{z,Y}^{\mathscr{I}}\right]$. We set $P_{\mathscr{I}}:=\mathbf{P}\left(E_{\mathscr{I}},N\in\tau\right)$, that is the probability for τ to visit the blocks $(B_i)_{i\in\mathscr{I}}$ and only these ones, and to visit also N. We now use the following two statements.

Proposition 3.2. *For any $\eta>0$, there exists $z>1$ sufficiently close to 1 (or L sufficiently big, since $L=(z-1)^{-1}$) such that for every $\mathscr{I}\subset\{1,\dots,m\}$ with $m\in\mathscr{I}$, we have*

$$\mathbb{E}^Y\left[g_{\mathscr{I}}(Y)Z_{z,Y}^{\mathscr{I}}\right]\leqslant\eta^{|\mathscr{I}|}P_{\mathscr{I}}. \tag{36}$$

Proposition 3.2 is the core of the paper and is proven in the next section.

Lemma 3.3. *[10, Lemma 2.4] There exist three constants $C_1 = C_1(L)$, C_2 and L_0 such that (with $i_0 := 0$)*

$$P_{\mathscr{I}} \leqslant C_1 C_2^{|\mathscr{I}|} \prod_{j=1}^{|\mathscr{I}|} \frac{1}{(i_j - i_{j-1})^{7/5}} \tag{37}$$

for $L \geq L_0$ and for every $\mathscr{I} \in \{1, \ldots, m\}$.

Thanks to these two statements and combining with the inequalities (25) and (32), we get

$$\mathbb{E}^Y \left[\left(\check{Z}^z_{N,Y} \right)^\gamma \right] \leqslant \sum_{\mathscr{I} \subset \{1,\ldots,m\}} \mathbb{E}^Y \left[\left(Z^{\mathscr{I}}_{z,Y} \right)^\gamma \right] \leqslant C_1^\gamma \sum_{\mathscr{I} \subset \{1,\ldots,m\}} \prod_{j=1}^{|\mathscr{I}|} \frac{(3C_2\eta)^\gamma}{(i_j - i_{j-1})^{7\gamma/5}}. \tag{38}$$

Since $7\gamma/5 = 6/5 > 1$, we can set

$$\widetilde{K}(n) = \frac{1}{\widetilde{c} n^{6/5}}, \text{where } \widetilde{c} = \sum_{i=1}^{+\infty} i^{-6/5} < +\infty, \tag{39}$$

and $\widetilde{K}(\cdot)$ is the inter-arrival probability of some recurrent renewal $\widetilde{\tau}$. We can therefore interpret the right-hand side of (38) as a partition function of a homogeneous pinning model of size m (see Figure 2), with the underlying renewal $\widetilde{\tau}$, and with pinning parameter $\log[\widetilde{c}(3C_2\eta)^\gamma]$:

$$\mathbb{E}^Y \left[\left(\check{Z}^z_{N,Y} \right)^\gamma \right] \leqslant C_1^\gamma \mathbf{E}_{\widetilde{\tau}} \left[\left(\widetilde{c}(3C_2\eta)^\gamma \right)^{|\widetilde{\tau} \cap \{1,\ldots,m\}|} \right]. \tag{40}$$

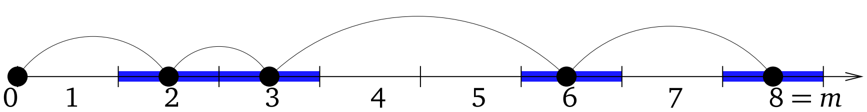

Figure 2: The underlying renewal $\widetilde{\tau}$ is a subset of the set of blocks $(B_i)_{1 \leqslant i \leqslant m}$ (i.e the blocks are reinterpreted as points) and the inter-arrival distribution is $\widetilde{K}(n) = 1/\left(\widetilde{c} n^{6/5}\right)$.

Thanks to Proposition 3.2, we can take η arbitrary small. Let us fix $\eta := 1/((4C_2)\widetilde{c}^{1/\gamma})$. Then,

$$\mathbb{E}^Y \left[\left(\check{Z}^z_{N,Y} \right)^\gamma \right] \leqslant C_1^\gamma \tag{41}$$

for every N. This implies, thanks to (24), that $\check{F}(z) = 0$, and we are done. □

Remark 3.4. The coarse-graining procedure reduced the proof of delocalization to the proof of Proposition 3.2. Thanks to the inequality (23), one has to estimate the expectation, with respect to the $g_{\mathscr{I}}(Y)$−modified measure, of the partition functions Z_{a_i,b_i} in each visited block. We will show (this is Lemma 4.1) that the expectation with respect to this modified measure of $Z_{a_i,b_i}/\mathbf{P}(b_i - a_i \in \tau)$ can be arbitrarily small if L is large, and if $b_i - a_i$ is of the order of L. If $b_i - a_i$ is much smaller, we can deal with this term via elementary bounds.

4 Proof of the Proposition 3.2

As pointed out in Remark 3.4, Proposition 3.2 relies on the following key lemma:

Lemma 4.1. *For every ε and $\delta > 0$, there exists $L > 0$ such that*

$$\mathbb{E}^Y \left[g_1(Y) Z_{a,b} \right] \leqslant \delta \mathbf{P}(b-a \in \tau) \tag{42}$$

for every $a \leqslant b$ in B_1 such that $b - a \geq \varepsilon L$.

Given this lemma, the proof of Proposition 3.2 is very similar to the proof of [10, Proposition 2.3], so we will sketch only a few steps. The inequality (23) gives us

$$\begin{aligned}
&\mathbb{E}^Y \left[g_{\mathscr{I}}(Y) Z^{\mathscr{I}}_{z,Y} \right] \\
&\leqslant \quad c^{|\mathscr{I}|} \sum_{\substack{a_1,b_1 \in B_{i_1} \\ a_1 \leqslant b_1}} \sum_{\substack{a_2,b_2 \in B_{i_2} \\ a_2 \leqslant b_2}} \cdots \sum_{a_l \in B_{i_l}} K(a_1) \mathbb{E}^Y \left[g_{i_1}(Y) Z_{a_1,b_1} \right] K(a_2 - b_1) \mathbb{E}^Y \left[g_{i_2}(Y) Z_{a_2,b_2} \right] \ldots \\
&\qquad \ldots K(a_l - b_{l-1}) \mathbb{E}^Y \left[g_{i_l}(Y) Z_{a_l,N} \right] \\
&= \quad c^{|\mathscr{I}|} \sum_{\substack{a_1,b_1 \in B_{i_1} \\ a_1 \leqslant b_1}} \sum_{\substack{a_2,b_2 \in B_{i_2} \\ a_2 \leqslant b_2}} \cdots \sum_{a_l \in B_{i_l}} K(a_1) \mathbb{E}^Y \left[g_1(Y) Z_{a_1 - L(i_1-1), b_1 - L(i_1-1)} \right] K(a_2 - b_1) \ldots \\
&\qquad \ldots K(a_l - b_{l-1}) \mathbb{E}^Y \left[g_1(Y) Z_{a_l - L(m-1), N - L(m-1)} \right].
\end{aligned} \tag{43}$$

The terms with $b_i - a_i \geq \varepsilon L$ are dealt with via Lemma 4.1, while for the remaining ones we just observe that $\mathbb{E}^Y[g_1(Y) Z_{a,b}] \leq \mathbf{P}(b-a \in \tau)$ since $g_1(Y) \leq 1$. One has then

$$\begin{aligned}
\mathbb{E}^Y \left[g_{\mathscr{I}}(Y) Z^{\mathscr{I}}_{z,Y} \right] \quad &\leqslant \quad c^{|\mathscr{I}|} \sum_{\substack{a_1,b_1 \in B_{i_1} \\ a_1 \leqslant b_1}} \sum_{\substack{a_2,b_2 \in B_{i_2} \\ a_2 \leqslant b_2}} \cdots \sum_{a_l \in B_{i_l}} K(a_1) \left(\delta + \mathbf{1}_{\{b_1 - a_1 \leqslant \varepsilon L\}} \right) \mathbf{P}(b_1 - a_1 \in \tau) \\
&\qquad \ldots K(a_l - b_{l-1}) \left(\delta + \mathbf{1}_{\{N - a_l \leqslant \varepsilon L\}} \right) \mathbf{P}(N - a_l \in \tau).
\end{aligned} \tag{44}$$

From this point on, the proof of Theorem 3.2 is identical to the proof of Proposition 2.3 in [10] (one needs of course to choose $\varepsilon = \varepsilon(\eta)$ and $\delta = \delta(\eta)$ sufficiently small). □

4.1 Proof of Lemma 4.1

Let us fix a, b in B_1, such that $b - a \geq \varepsilon L$. The small constants δ and ε are also fixed. We recall that for a fixed configuration of τ such that $a, b \in \tau$, we have $\mathbb{E}^Y \left[W(\tau \cap \{a, \ldots, b\}, Y) \right] = 1$ because $z = 1$. We can therefore introduce the probability measure (always for fixed τ)

$$\mathrm{d}\mathbb{P}_\tau(Y) = W(\tau \cap \{a, \ldots, b\}, Y) \, \mathrm{d}\mathbb{P}^Y(Y) \tag{45}$$

where we do not indicate the dependence on a and b. Let us note for later convenience that, in the particular case $a = 0$, the definition (10) of W implies that for any function $f(Y)$

$$\mathbb{E}_\tau[f(Y)] = \mathbb{E}^X \mathbb{E}^Y \left[f(Y) | X_i = Y_i \ \forall i \in \tau \cap \{1, \ldots, b\} \right]. \tag{46}$$

With the definition (20) of $Z_{a,b} := Z_{a,b}^{z=1}$, we get

$$\mathbb{E}^Y\left[g_1(Y)Z_{a,b}\right] = \mathbb{E}^Y\mathbf{E}\left[g_1(Y)W(\tau\cap\{a,\dots,b\},Y)\mathbf{1}_{b\in\tau}\,|a\in\tau\,\right] = \widehat{\mathbf{E}}\mathbb{E}_\tau[g_1(Y)]\mathbf{P}(b-a\in\tau), \quad (47)$$

where $\widehat{\mathbf{P}}(\cdot) := \mathbf{P}(\cdot|a,b\in\tau)$, and therefore we have to show that $\widehat{\mathbf{E}}\mathbb{E}_\tau[g_1(Y)] \leqslant \delta$.
With the definition (27) of $g_1(Y)$, we get that for any K

$$\widehat{\mathbf{E}}\mathbb{E}_\tau[g_1(Y)] \leqslant \varepsilon_K + \widehat{\mathbf{E}}\mathbb{P}_\tau\left(F_1 < K\right). \quad (48)$$

If we choose K big enough, ε_K is smaller than $\delta/3$ thanks to the Lemma 3.1. We now use two lemmas to deal with the second term. The idea is to first prove that $\mathbb{E}_\tau[F_1]$ is big with a $\widehat{\mathbf{P}}-$probability close to 1, and then that its variance is not too large.

Lemma 4.2. *For every $\zeta > 0$ and $\varepsilon > 0$, one can find two constants $u = u(\varepsilon,\zeta) > 0$ and $L_0 = L_0(\varepsilon,\zeta) > 0$, such that for every $a, b \in B_1$ such that $b - a \geq \varepsilon L$,*

$$\widehat{\mathbf{P}}\left(\mathbb{E}_\tau[F_1] \leq u\sqrt{\log L}\right) \leq \zeta, \quad (49)$$

for every $L \geq L_0$.

Choose $\zeta = \delta/3$ and fix $u > 0$ such that (49) holds for every L sufficiently large. If $2K = u\sqrt{\log L}$ (and therefore we can make ε_K small enough by choosing L large), we get that

$$\begin{aligned} \widehat{\mathbf{E}}\mathbb{P}_\tau\left(F_1 < K\right) &\leqslant \widehat{\mathbf{E}}\mathbb{P}_\tau\left[F_1 - \mathbb{E}_\tau[F_1] \leqslant -K\right] + \widehat{\mathbf{P}}\left(\mathbb{E}_\tau[F_1] \leqslant 2K\right) & (50)\\ &\leqslant \frac{1}{K^2}\widehat{\mathbf{E}}\mathbb{E}_\tau\left[\left(F_1 - \mathbb{E}_\tau[F_1]\right)^2\right] + \delta/3. & (51)\end{aligned}$$

Putting this together with (48) and with our choice of K, we have

$$\widehat{\mathbf{E}}\mathbb{E}_\tau[g_1(Y)] \leqslant 2\delta/3 + \frac{4}{u^2\log L}\widehat{\mathbf{E}}\mathbb{E}_\tau\left[\left(F_1 - \mathbb{E}_\tau[F_1]\right)^2\right] \quad (52)$$

for $L \geq L_0$. Then we just have to prove that $\widehat{\mathbf{E}}\mathbb{E}_\tau\left[\left(F_1 - \mathbb{E}_\tau[F_1]\right)^2\right] = o(\log L)$. Indeed,

Lemma 4.3. *For every $\varepsilon > 0$ there exists some constant $c = c(\varepsilon) > 0$ such that*

$$\widehat{\mathbf{E}}\mathbb{E}_\tau\left[\left(F_1 - \mathbb{E}_\tau[F_1]\right)^2\right] \leqslant c\left(\log L\right)^{3/4} \quad (53)$$

for every $L > 1$ and $a, b \in B_1$ such that $b - a \geq \varepsilon L$.

We finally get that

$$\widehat{\mathbf{E}}\mathbb{E}_\tau[g_1(Y)] \leqslant 2\delta/3 + c(\log L)^{-1/4}, \quad (54)$$

and there exists a constant $L_1 > 0$ such that for $L > L_1$

$$\widehat{\mathbf{E}}\mathbb{E}_\tau[g_1(Y)] \leqslant \delta. \quad (55)$$

□

4.2 Proof of Lemma 4.2

Up to now, the proof of Theorem 2.8 is quite similar to the proof of the main result in [10]. Starting from the present section, instead, new ideas and technical results are needed.

Let us fix a realization of τ such that $a, b \in \tau$ (so that it has a non-zero probability under $\widehat{\mathbf{P}}$) and let us note $\tau \cap \{a, \dots b\} = \{\tau_{R_a} = a, \tau_{R_a+1}, \dots, \tau_{R_b} = b\}$ (recall that $R_n = |\tau \cap \{1, \dots, n\}|$). We observe (just go back to the definition of $\mathbb{P}_\tau$) that, if f is a function of the increments of Y in $\{\tau_{n-1} + 1, \dots, \tau_n\}$, g of the increments in $\{\tau_{m-1} + 1, \dots, \tau_m\}$ with $R_a < n \neq m \leq R_b$, and if h is a function of the increments of Y not in $\{a + 1, \dots, b\}$ then

$$\begin{aligned}
&\mathbb{E}_\tau\left[f\left(\{\Delta_i\}_{i\in\{\tau_{n-1}+1,\dots,\tau_n\}}\right)g\left(\{\Delta_i\}_{i\in\{\tau_{m-1}+1,\dots,\tau_m\}}\right)h\left(\{\Delta_i\}_{i\notin\{a+1,\dots,b\}}\right)\right] \\
&\quad= \mathbb{E}_\tau\left[f\left(\{\Delta_i\}_{i\in\{\tau_{n-1}+1,\dots,\tau_n\}}\right)\right]\mathbb{E}_\tau\left[g\left(\{\Delta_i\}_{i\in\{\tau_{m-1}+1,\dots,\tau_m\}}\right)\right]\mathbb{E}^Y\left[h\left(\{\Delta_i\}_{i\notin\{a+1,\dots,b\}}\right)\right],
\end{aligned} \tag{56}$$

and that

$$\begin{aligned}
\mathbb{E}_\tau\left[f\left(\{\Delta_i\}_{i\in\{\tau_{n-1}+1,\dots,\tau_n\}}\right)\right] &= \mathbb{E}^X\mathbb{E}^Y\left[f\left(\{\Delta_i\}_{i\in\{\tau_{n-1}+1,\dots,\tau_n\}}\right)|X_{\tau_{n-1}} = Y_{\tau_{n-1}}, X_{\tau_n} = Y_{\tau_n}\right] \\
&= \mathbb{E}^X\mathbb{E}^Y\left[f\left(\{\Delta_{i-\tau_{n-1}}\}_{i\in\{\tau_{n-1}+1,\dots,\tau_n\}}\right)|X_{\tau_n-\tau_{n-1}} = Y_{\tau_n-\tau_{n-1}}\right].
\end{aligned} \tag{57}$$

We want to estimate $\mathbb{E}_\tau[F_1]$: since the increments Δ_i for $i \in B_1 \setminus \{a+1, \dots, b\}$ are i.i.d. and centered (like under $\mathbb{P}^Y$), we have

$$\mathbb{E}_\tau[F_1] := \sum_{i,j=a+1}^{b} M_{ij}\mathbb{E}_\tau[-\Delta_i \cdot \Delta_j]. \tag{58}$$

Via a time translation, one can always assume that $a = 0$ and we do so from now on.

The key point is the following

Lemma 4.4. *1. If there exists $1 \leq n \leq R_b$ such that $i, j \in \{\tau_{n-1} + 1, \dots, \tau_n\}$, then*

$$\mathbb{E}_\tau[-\Delta_i \cdot \Delta_j] = A(r) \overset{r\to\infty}{\sim} \frac{C_{X,Y}}{r} \tag{59}$$

where $r = \tau_n - \tau_{n-1}$ (in particular, note that the expectation depends only on r) and $C_{X,Y}$ is a positive constant which depends on $\mathbb{P}^X, \mathbb{P}^Y$;

2. otherwise, $\mathbb{E}_\tau[-\Delta_i \cdot \Delta_j] = 0$.

Proof of Lemma 4.4 **Case (2).** Assume that $\tau_{n-1} < i \leq \tau_n$ and $\tau_{m-1} < j \leq \tau_m$ with $n \neq m$. Thanks to (56)-(57) we have that

$$\mathbb{E}_\tau[\Delta_i \cdot \Delta_j] = \mathbb{E}^X\mathbb{E}^Y[\Delta_i|X_{\tau_{n-1}} = Y_{\tau_{n-1}}, X_{\tau_n} = Y_{\tau_n}] \cdot \mathbb{E}^X\mathbb{E}^Y[\Delta_j|X_{\tau_{m-1}} = Y_{\tau_{m-1}}, X_{\tau_m} = Y_{\tau_m}] \tag{60}$$

and both factors are immediately seen to be zero, since the laws of X and Y are assumed to be symmetric.

Case (1). Without loss of generality, assume that $n = 1$, so we only have to compute

$$\mathbb{E}^Y\mathbb{E}^X\left[\Delta_i \cdot \Delta_j \,\middle|\, X_r = Y_r\right]. \tag{61}$$

where $r = \tau_1$. Let us fix $x \in \mathbb{Z}^3$, and denote $\mathbb{E}^Y_{r,x}[\cdot] = \mathbb{E}^Y[\cdot \,|Y_r = x]$.

$$\begin{aligned}
\mathbb{E}^Y[\Delta_i \cdot \Delta_j \,|Y_r = x] &= \mathbb{E}^Y_{r,x}\left[\Delta_i \cdot \mathbb{E}^Y_{r,x}\left[\Delta_j \,|\Delta_i\right]\right] \\
&= \mathbb{E}^Y_{r,x}\left[\Delta_i \cdot \frac{x - \Delta_i}{r-1}\right] = \frac{x}{r-1} \cdot \mathbb{E}^Y_{r,x}\left[\Delta_i\right] - \frac{1}{r-1}\mathbb{E}^Y_{r,x}\left[\left\|\Delta_i\right\|^2\right] \\
&= \frac{1}{r-1}\left(\frac{\|x\|^2}{r} - \mathbb{E}^Y_{r,x}\left[\left\|\Delta_1\right\|^2\right]\right),
\end{aligned}$$

where we used the fact that under $\mathbb{P}^Y_{r,x}$ the law of the increments $\{\Delta_i\}_{i \le r}$ is exchangeable. Then, we get

$$\begin{aligned}
&\mathbb{E}_\tau[\Delta_i \cdot \Delta_j] = \mathbb{E}^X\mathbb{E}^Y\left[\Delta_i \cdot \Delta_j \mathbf{1}_{\{Y_r = X_r\}}\right]\mathbb{P}^{X-Y}(Y_r = X_r)^{-1} \\
&= \mathbb{E}^X\left[\mathbb{E}^Y\left[\Delta_i \cdot \Delta_j \,|Y_r = X_r\right]\mathbb{P}^Y(Y_r = X_r)\right]\mathbb{P}^{X-Y}(Y_r = X_r)^{-1} \\
&= \frac{1}{r-1}\left(\mathbb{E}^X\left[\frac{\left\|X_r\right\|^2}{r}\mathbb{P}^Y(Y_r = X_r)\right]\mathbb{P}^{X-Y}(Y_r = X_r)^{-1}\right. \\
&\qquad\left. - \mathbb{E}^X\mathbb{E}^Y\left[\left\|\Delta_1\right\|^2 \mathbf{1}_{\{Y_r = X_r\}}\right]\mathbb{P}^{X-Y}(Y_r = X_r)^{-1}\right) \\
&= \frac{1}{r-1}\left(\mathbb{E}^X\left[\frac{\left\|X_r\right\|^2}{r}\mathbb{P}^Y(Y_r = X_r)\right]\mathbb{P}^{X-Y}(Y_r = X_r)^{-1} - \mathbb{E}^X\mathbb{E}^Y\left[\left\|\Delta_1\right\|^2 \,|Y_r = X_r\right]\right).
\end{aligned}$$

Next, we study the asymptotic behavior of $A(r)$ and we prove (59) with $C_{X,Y} = tr(\Sigma_Y) - tr\left((\Sigma_X^{-1} + \Sigma_Y^{-1})^{-1}\right)$. Note that $tr(\Sigma_Y) = \mathbb{E}^Y(||Y_1||^2) = \sigma_Y^2$. The fact that $C_{X,Y} > 0$ is just a consequence of the fact that, if A and B are two positive-definite matrices, one has that $A - B$ is positive definite if and only if $B^{-1} - A^{-1}$ is [11, Cor. 7.7.4(a)].

To prove (59), it is enough to show that

$$\mathbb{E}^X\mathbb{E}^Y\left[\left\|\Delta_1\right\|^2 \,|Y_r = X_r\right] \overset{r\to\infty}{\to} \mathbb{E}^X\mathbb{E}^Y\left[\left\|\Delta_1\right\|^2\right] = \sigma_Y^2, \tag{62}$$

and that

$$B(r) := \frac{\mathbb{E}^X\left[\frac{\|X_r\|^2}{r}\mathbb{P}^Y(Y_r = X_r)\right]}{\mathbb{P}^{X-Y}(X_r = Y_r)} \overset{r\to\infty}{\to} tr\left((\Sigma_X^{-1} + \Sigma_Y^{-1})^{-1}\right). \tag{63}$$

To prove (62), write

$$\begin{aligned}
\mathbb{E}^X\mathbb{E}^Y\left[\left\|\Delta_1\right\|^2 \,|Y_r = X_r\right] &= \mathbb{E}^Y\left[\left\|\Delta_1\right\|^2\mathbb{P}^X(X_r = Y_r)\right]\mathbb{P}^{X-Y}(X_r = Y_r)^{-1} \\
&= \sum_{y,z \in \mathbb{Z}^d} \|y\|^2\mathbb{P}^Y(Y_1 = y)\frac{\mathbb{P}^Y(Y_{r-1} = z)\mathbb{P}^X(X_r = y + z)}{\mathbb{P}^{X-Y}(X_r - Y_r = 0)}.
\end{aligned} \tag{64}$$

We know from the Local Limit Theorem (Proposition 2.10) that the term $\frac{\mathbb{P}^X(X_r = y+z)}{\mathbb{P}^{X-Y}(X_r - Y_r = 0)}$ is uniformly bounded from above, and so there exists a constant $c > 0$ such that for all $y \in \mathbb{Z}^d$

$$\sum_{z \in \mathbb{Z}^d} \frac{\mathbb{P}^Y(Y_{r-1} = z)\mathbb{P}^X(X_r = y + z)}{\mathbb{P}^{X-Y}(X_r - Y_r = 0)} \leqslant c. \tag{65}$$

If we can show that for every y fixed $\mathbb{Z}^d$ the left-hand side of (65) goes to 1 as r goes to infinity, then from (64) and a dominated convergence argument we get that

$$\mathbb{E}^X\mathbb{E}^Y\left[\left\|\Delta_1\right\|^2 \middle| Y_r = X_r\right] \stackrel{r\to\infty}{\longrightarrow} \sum_{y\in\mathbb{Z}^d} \|y\|^2\mathbb{P}^Y(Y_1 = y) = \sigma_Y^2. \tag{66}$$

We use the Local Limit Theorem to get

$$\begin{aligned}\sum_{z\in\mathbb{Z}^d} \mathbb{P}^Y(Y_{r-1} = z)\mathbb{P}^X(X_r = y+z) &= \sum_{z\in\mathbb{Z}^d} \frac{c_X c_Y}{r^d} e^{-\frac{1}{2(r-1)}z\cdot\left(\Sigma_Y^{-1}z\right)} e^{-\frac{1}{2r}(y+z)\cdot\left(\Sigma_X^{-1}(y+z)\right)} + o(r^{-d/2})\\ &= (1+o(1))\sum_{z\in\mathbb{Z}^d} \frac{c_X c_Y}{r^d} e^{-\frac{1}{2r}z\cdot\left(\Sigma_Y^{-1}z\right)} e^{-\frac{1}{2r}z\cdot\left(\Sigma_X^{-1}z\right)} + o(r^{-d/2}) \end{aligned} \tag{67}$$

where $c_X = (2\pi)^{-d/2}(\det\Sigma_X)^{-1/2}$ and similarly for c_Y (the constants are different in the case of simple random walks: see Remark 2.11), and where we used that y is fixed to neglect $y/\sqrt{r}$.

Using the same reasoning, we also have (with the same constants c_X and c_Y)

$$\begin{aligned}\mathbb{P}^{X-Y}(X_r = Y_r) &= \sum_{z\in\mathbb{Z}^d} \mathbb{P}^Y(Y_r = z)\mathbb{P}^X(X_r = z)\\ &= \sum_{z\in\mathbb{Z}^d} \frac{c_X c_Y}{r^d} e^{-\frac{1}{2r}z\cdot\left(\Sigma_Y^{-1}z\right)} e^{-\frac{1}{2r}z\cdot\left(\Sigma_X^{-1}z\right)} + o(r^{-d/2}). \end{aligned} \tag{68}$$

Putting this together with (67) (and considering that $\mathbb{P}^{X-Y}(X_r = Y_r) \sim c_{X,Y} r^{-d/2}$), we have, for every $y\in\mathbb{Z}^d$

$$\sum_{z\in\mathbb{Z}^d} \frac{\mathbb{P}^Y(Y_{r-1} = z)\mathbb{P}^X(X_r = y+z)}{\mathbb{P}^{X-Y}(X_r - Y_r = 0)} \stackrel{r\to\infty}{\longrightarrow} 1. \tag{69}$$

To deal with the term $B(r)$ in (63), we apply the Local Limit Theorem as in (68) to get

$$\mathbb{E}^X\left[\frac{\left\|X_r\right\|^2}{r}\mathbb{P}^Y(Y_r = X_r)\right] = \frac{c_Y c_X}{r^d}\sum_{z\in\mathbb{Z}^d} \frac{\|z\|^2}{r} e^{-\frac{1}{2r}z\cdot\left(\Sigma_Y^{-1}z\right)} e^{-\frac{1}{2r}z\cdot\left(\Sigma_X^{-1}z\right)} + o(r^{-d/2}). \tag{70}$$

Together with (68), we finally get

$$B(r) = \frac{\frac{c_Y c_X}{r^d}\sum_{z\in\mathbb{Z}^d} \frac{\|z\|^2}{r} e^{-\frac{1}{2r}z\cdot\left((\Sigma_Y^{-1}+\Sigma_X^{-1})z\right)} + o(r^{-d/2})}{\frac{c_Y c_X}{r^d}\sum_{z\in\mathbb{Z}^d} e^{-\frac{1}{2r}z\cdot\left((\Sigma_Y^{-1}+\Sigma_X^{-1})z\right)} + o(r^{-d/2})} = (1+o(1))E\left[\|\mathcal{N}\|^2\right], \tag{71}$$

where $\mathcal{N} \sim \mathcal{N}\left(0, (\Sigma_Y^{-1}+\Sigma_X^{-1})^{-1}\right)$ is a centered Gaussian vector of covariance matrix $(\Sigma_Y^{-1}+\Sigma_X^{-1})^{-1}$. Therefore, $E\left[\|\mathcal{N}\|^2\right] = tr\left((\Sigma_Y^{-1}+\Sigma_X^{-1})^{-1}\right)$ and (63) is proven.

□

Remark 4.5. For later purposes, we remark that with the same method one can prove that, for any given $k_0 \geq 0$ and polynomials U and V of order four (so that $\mathbb{E}^Y[|U\left(\{\left\|\Delta_k\right\|\}_{k\leqslant k_0}\right)|] < \infty$ and $\mathbb{E}^X[V(\|X_r\|/\sqrt{r})] < \infty$), we have

$$\mathbb{E}^X\mathbb{E}^Y\left[U\left(\{\left\|\Delta_k\right\|\}_{k\leqslant k_0}\right)V\left(\frac{\left\|X_r\right\|}{\sqrt{r}}\right)\middle| Y_r = X_r\right] \stackrel{r\to\infty}{\to} \mathbb{E}^Y\left[U\left(\{\|\Delta_k\|\}_{k\leqslant k_0}\right)\right]E\left[V(\|\mathcal{N}\|)\right], \tag{72}$$

where $\mathcal{N}$ is as in (71).

Let us quickly sketch the proof: as in (64), we can write

$$\mathbb{E}^X\mathbb{E}^Y\left[U\left(\{\|\Delta_k\|\}_{k\leqslant k_0}\right)V\left(\frac{\|X_r\|}{\sqrt{r}}\right)\middle| Y_r=X_r\right]= \tag{73}$$
$$\sum_{y_1,\dots,y_{k_0}\in\mathbb{Z}^d}U\left(\{\|y_k\|\}_{k\leqslant k_0}\right)\sum_{z\in\mathbb{Z}^d}V\left(\frac{\|z\|}{\sqrt{r}}\right)\mathbb{P}^X(X_r=z)\frac{\mathbb{P}^Y(Y_{r-k_0}=z-y_1-\ldots-y_{k_0})}{\mathbb{P}^{X-Y}(X_r-Y_r=0)}$$
$$\times\mathbb{P}^Y(\Delta_i=y_i, i\le k_0).$$

Using the Local Limit Theorem the same way as in (68) and (71), one can show that for any $y_1,\dots,y_{k_0}$

$$\sum_{z\in\mathbb{Z}^d}V\left(\frac{\|z\|}{\sqrt{r}}\right)\mathbb{P}^X(X_r=z)\frac{\mathbb{P}^Y(Y_{r-k_0}=z-y_1-\ldots-y_{k_0})}{\mathbb{P}^{X-Y}(X_r-Y_r=0)}\stackrel{r\to\infty}{\to}E\left[V(\|\mathcal{N}\|)\right]. \tag{74}$$

The proof of (72) is concluded via a domination argument (as for (62)), which is provided by uniform bounds on $\mathbb{P}^Y(Y_{r-k_0}=z-y_1-\ldots-y_{k_0})$ and $\mathbb{P}^{X-Y}(X_r-Y_r=0)$ and by the fact that the increments of X and Y have finite fourth moments.

Given Lemma 4.4, we can resume the proof of Lemma 4.2, and lower bound the average $\mathbb{E}_\tau[F_1]$. Recalling (58) and the fact that we reduced to the case $a=0$, we get

$$\mathbb{E}_\tau[F_1] = \sum_{n=1}^{R_b}\left(\sum_{\tau_{n-1}<i,j\le\tau_n}M_{ij}\right)A(\Delta\tau_n), \tag{75}$$

where $\Delta\tau_n:=\tau_n-\tau_{n-1}$. Using the definition (29) of M, we see that there exists a constant $c>0$ such that for $1<m\le L$

$$\sum_{i,j=1}^{m}M_{ij}\ge\frac{c}{\sqrt{L\log L}}m^{3/2}. \tag{76}$$

On the other hand, thanks to Lemma 4.4, there exists some $r_0>0$ and two constants c and c' such that $A(r)\ge\frac{c}{r}$ for $r\ge r_0$, and $A(r)\ge -c'$ for every r. Plugging this into (75), one gets

$$\sqrt{L\log L}\,\mathbb{E}_\tau[F_1]\ge c\sum_{n=1}^{R_b}\sqrt{\Delta\tau_n}\mathbf{1}_{\{\Delta\tau_n\geq r_0\}}-c'\sum_{n=1}^{R_b}(\Delta\tau_n)^{3/2}\mathbf{1}_{\{\Delta\tau_n\leqslant r_0\}}\ge c\sum_{n=1}^{R_b}\sqrt{\Delta\tau_n}-c'R_b. \tag{77}$$

Therefore, we get for any positive $B>0$ (independent of L)

$$\begin{aligned}\widehat{\mathbf{P}}\left(\mathbb{E}_\tau[F_1]\leqslant g\sqrt{\log L}\right)&\leqslant\widehat{\mathbf{P}}\left(\frac{1}{\sqrt{L\log L}}\left(c\sum_{n=1}^{R_b}\sqrt{\Delta\tau_n}-c'R_b\right)\leqslant u\sqrt{\log L}\right)\\ &\leqslant\widehat{\mathbf{P}}\left(\frac{1}{\sqrt{L\log L}}\left(c\sum_{n=1}^{R_b}\sqrt{\Delta\tau_n}-c'\sqrt{L}B\right)\leqslant u\sqrt{\log L}\right)+\widehat{\mathbf{P}}\left(R_b>B\sqrt{L}\right)\\ &\leqslant\widehat{\mathbf{P}}\left(\sum_{n=1}^{R_{b/2}}\sqrt{\Delta\tau_n}\le(1+o(1))\frac{u}{c}\sqrt{L}\log L\right)+\widehat{\mathbf{P}}(R_b>B\sqrt{L}).\end{aligned} \tag{78}$$

Now we show that for B large enough, and $L \geq L_0(B)$,

$$\widehat{\mathbf{P}}(R_b > B\sqrt{L}) \leqslant \zeta/2, \tag{79}$$

where ζ is the constant which appears in the statement of Lemma 4.2. We start with getting rid of the conditioning in $\widehat{\mathbf{P}}$ (recall $\widehat{\mathbf{P}}(\cdot) = \mathbf{P}(\cdot|b \in \tau)$ since we reduced to the case $a = 0$). If $R_b > B\sqrt{L}$, then either $|\tau \cap \{1, \ldots, b/2\}|$ or $|\tau \cap \{b/2+1, \ldots, b\}|$ exceeds $\frac{B}{2}\sqrt{L}$. Since both random variables have the same law under $\widehat{\mathbf{P}}$, we have

$$\widehat{\mathbf{P}}(R_b > B\sqrt{L}) \leqslant 2\widehat{\mathbf{P}}\left(R_{b/2} > \frac{B}{2}\sqrt{L}\right) \leq 2c\mathbf{P}\left(R_{b/2} > \frac{B}{2}\sqrt{L}\right), \tag{80}$$

where in the second inequality we applied Lemma A.1. Now, we can use the Lemma A.3 in the Appendix, to get that (recall $b \leq L$)

$$\mathbf{P}\left(R_{b/2} > \frac{B}{2}\sqrt{L}\right) \leq \mathbf{P}\left(R_{L/2} > \frac{B}{2}\sqrt{L}\right) \stackrel{L\to\infty}{\to} \mathbf{P}\left(\frac{|\mathscr{Z}|}{\sqrt{2\pi}} \geq B\frac{c_K}{\sqrt{2}}\right), \tag{81}$$

with $\mathscr{Z}$ a standard Gaussian random variable and c_K the constant such that $K(n) \sim c_K n^{-3/2}$. The inequality (79) then follows for B sufficiently large, and $L \geq L_0(B)$.

We are left to prove that for L large enough and u small enough

$$\widehat{\mathbf{P}}\left(\sum_{n=1}^{R_{b/2}} \sqrt{\Delta\tau_n} \leqslant \frac{u}{c}\sqrt{L}\log L\right) \leqslant \zeta/2. \tag{82}$$

The conditioning in $\widehat{\mathbf{P}}$ can be eliminated again via Lemma A.1. Next, one notes that for any given $A > 0$ (independent of L)

$$\mathbf{P}\left(\sum_{n=1}^{R_{b/2}} \sqrt{\Delta\tau_n} \leqslant \frac{u}{c}\sqrt{L}\log L\right) \leqslant \mathbf{P}\left(\sum_{n=1}^{A\sqrt{L}} \sqrt{\Delta\tau_n} \leqslant \frac{u}{c}\sqrt{L}\log L\right) + \mathbf{P}\left(R_{b/2} < A\sqrt{L}\right). \tag{83}$$

Thanks to the Lemma A.3 in Appendix and to $b \geq \varepsilon L$, we have

$$\limsup_{L\to\infty} \mathbf{P}\left(\frac{R_{b/2}}{\sqrt{L}} < A\right) \leq \mathbf{P}\left(\frac{|\mathscr{Z}|}{\sqrt{2\pi}} < Ac_K\sqrt{\frac{2}{\varepsilon}}\right),$$

which can be arbitrarily small if $A = A(\varepsilon)$ is small enough, for L large. We now deal with the other term in (83), using the exponential Bienaymé-Chebyshev inequality (and the fact that the $\Delta\tau_n$ are i.i.d.):

$$\mathbf{P}\left(\frac{1}{\sqrt{L\log L}}\sum_{n=1}^{A\sqrt{L}} \sqrt{\Delta\tau_n} < \frac{u}{c}\sqrt{\log L}\right) \leqslant e^{(u/c)\sqrt{\log L}}\mathbf{E}\left[\exp\left(-\sqrt{\frac{\tau_1}{L\log L}}\right)\right]^{A\sqrt{L}}. \tag{84}$$

To estimate this expression, we remark that, for L large enough,

$$\begin{aligned}\mathbf{E}\left[1 - \exp\left(-\sqrt{\frac{\tau_1}{L\log L}}\right)\right] &= \sum_{n=1}^{\infty} K(n)\left(1 - e^{-\sqrt{\frac{n}{L\log L}}}\right) \\ &\geq c'\sum_{n=1}^{\infty} \frac{1 - e^{-\sqrt{\frac{n}{L\log L}}}}{n^{3/2}} \geq c''\sqrt{\frac{\log L}{L}},\end{aligned} \tag{85}$$

where the last inequality follows from keeping only the terms with $n \le L$ in the sum, and noting that in this range $1 - e^{-\sqrt{\frac{n}{L\log L}}} \ge c\sqrt{n/(L\log L)}$. Therefore,

$$\mathbf{E}\left[\exp\left(-\sqrt{\frac{\tau_1}{L\log L}}\right)\right]^{A\sqrt{L}} \leqslant \left(1 - c''\sqrt{\frac{\log L}{L}}\right)^{A\sqrt{L}} \le e^{-c''A\sqrt{\log L}}, \tag{86}$$

and, plugging this bound in the inequality (84), we get

$$\mathbf{P}\left(\frac{1}{\sqrt{L\log L}}\sum_{n=1}^{A\sqrt{L}}\sqrt{\Delta\tau_n} \leqslant \frac{u}{c}\sqrt{\log L}\right) \leqslant e^{[(u/c)-c''A]\sqrt{\log L}}, \tag{87}$$

that goes to 0 if $L \to \infty$, provided that u is small enough. This concludes the proof of Lemma 4.2. □

4.3 Proof of Lemma 4.3

We can write

$$-F_1 + \mathbb{E}_\tau[F_1] = S_1 + S_2 := \sum_{i\neq j=a+1}^{b} M_{ij}D_{ij} + \sum_{i\neq j}{}' M_{ij}D_{ij} \tag{88}$$

where we denoted

$$D_{ij} = \Delta_i\cdot\Delta_j - \mathbb{E}_\tau[\Delta_i\cdot\Delta_j] \tag{89}$$

and $\sum'$ stands for the sum over all $1 \le i \neq j \le L$ such that either i or j (or both) do not fall into $\{a+1,\dots,b\}$. This way, we have to estimate

$$\begin{aligned}\mathbb{E}_\tau[(F_1 - \mathbb{E}_\tau[F_1])^2] &\le 2\mathbb{E}_\tau[S_1^2] + 2\mathbb{E}_\tau[S_2^2] \\ &= 2\sum_{i\neq j=a+1}^{b}\sum_{k\neq l=a+1}^{b} M_{ij}M_{kl}\mathbb{E}_\tau[D_{ij}D_{kl}] + 2\sum_{i\neq j}{}'\sum_{k\neq l}{}' M_{ij}M_{kl}\mathbb{E}_\tau[D_{ij}D_{kl}].\end{aligned} \tag{90}$$

Remark 4.6. We easily deal with the part of the sum where $\{i,j\} = \{k,l\}$. In fact, we trivially bound $\mathbb{E}_\tau\left[(\Delta_i\cdot\Delta_j)^2\right] \le \mathbb{E}_\tau\left[\left\|\Delta_i\right\|^2\left\|\Delta_j\right\|^2\right]$. Suppose for instance that $\tau_{n-1} < i \le \tau_n$ for some $R_a < n \le R_b$: in this case, Remark 4.5 tells that $\mathbb{E}_\tau\left[\left\|\Delta_i\right\|^2\left\|\Delta_j\right\|^2\right]$ converges to $\mathbb{E}^Y[\left\|\Delta_1\right\|^2\left\|\Delta_2\right\|^2] = \sigma_Y^4$ as $\tau_n - \tau_{n-1} \to \infty$. If, on the other hand, $i \notin \{a+1,\dots,b\}$, we know that $\mathbb{E}_\tau\left[\left\|\Delta_i\right\|^2\left\|\Delta_j\right\|^2\right]$ equals exactly $\mathbb{E}^Y\left[\left\|\Delta_1\right\|^2\right]\mathbb{E}_\tau\left[\left\|\Delta_j\right\|^2\right]$ which is also bounded. As a consequence, we have the following inequality, valid for every $1 \le i,j \le L$:

$$\mathbb{E}_\tau\left[(\Delta_i\cdot\Delta_j)^2\right] \le c \tag{91}$$

and then

$$\sum_{i\neq j=1}^{L}\sum_{\{k,l\}=\{i,j\}} M_{ij}M_{kl}\mathbb{E}_\tau[D_{ij}D_{kl}] \leqslant c\sum_{i\neq j=1}^{L} M_{ij}^2 \leqslant c' \tag{92}$$

since the Hilbert-Schmidt norm of M was choosen to be finite.

Upper bound on $\mathbb{E}_\tau[S_2^2]$. This is the easy part, and this term will be shown to be bounded even without taking the average over $\widehat{\mathbf{P}}$.

We have to compute $\sum'_{i\neq j}\sum'_{k\neq l} M_{ij}M_{kl}\mathbb{E}_\tau[D_{ij}D_{kl}]$. Again, thanks to (56)-(57), we have $\mathbb{E}_\tau[D_{ij}D_{kl}] \neq 0$ only in the following case (recall that thanks to Remark 4.6 we can disregard the case $\{i,j\}=\{k,l\}$):

$$i = k \notin \{a+1,\dots,b\} \text{ and } \tau_{n-1} < j \neq l \leq \tau_n \text{ for some } R_a < n \leq R_b. \tag{93}$$

One should also consider the cases where i is interchanged with j and/or k with l. Since we are not following constants, we do not keep track of the associated combinatorial factors. Under the assumption (93), $\mathbb{E}_\tau[\Delta_i\cdot\Delta_j] = \mathbb{E}_\tau[\Delta_i\cdot\Delta_l] = 0$ (cf. (56)) and we will show that

$$\mathbb{E}_\tau[D_{ij}D_{il}] = \mathbb{E}_\tau[(\Delta_i\cdot\Delta_j)(\Delta_i\cdot\Delta_l)] \leq \frac{c}{r} \tag{94}$$

where $r = \tau_n - \tau_{n-1} =: \Delta\tau_n$. Indeed, using (56)-(57), we get

$$\begin{aligned}\mathbb{E}_\tau[(\Delta_i\cdot\Delta_j)(\Delta_i\cdot\Delta_l)] &= \sum_{\nu,\mu=1}^{3}\mathbb{E}^Y[\Delta_i^{(\nu)}\Delta_i^{(\mu)}]\mathbb{E}^X\mathbb{E}^Y[\Delta^{(\nu)}_{j-\tau_{n-1}}\Delta^{(\mu)}_{l-\tau_{n-1}}|X_{\tau_n-\tau_{n-1}} = Y_{\tau_n-\tau_{n-1}}] \\ &= \sum_{\nu,\mu=1}^{3}\Sigma_Y^{\nu\mu}\,\mathbb{E}^X\mathbb{E}^Y\left[\Delta^{(\nu)}_{j-\tau_{n-1}}\Delta^{(\mu)}_{l-\tau_{n-1}}|X_r = Y_r\right].\end{aligned} \tag{95}$$

In the remaining expectation, we can assume without loss of generality that $\tau_{n-1}=0, \tau_n = r$. Like for instance in the proof of (59), one writes

$$\mathbb{E}^X\mathbb{E}^Y\left[\Delta_j^{(\nu)}\Delta_l^{(\mu)}|X_r = Y_r\right] = \frac{\mathbb{E}^X\left[\mathbb{E}^Y[\Delta_j^{(\nu)}\Delta_l^{(\mu)}|Y_r = X_r]\mathbb{P}^Y(Y_r = X_r)\right]}{\mathbb{P}^{X-Y}(X_r = Y_r)} \tag{96}$$

and

$$\mathbb{E}^Y\left[\Delta_j^{(\nu)}\Delta_l^{(\mu)}\Big| Y_r = X_r\right] = \frac{1}{r(r-1)}X_r^{(\nu)}X_r^{(\mu)} - \frac{1}{r-1}\mathbb{E}^Y[\Delta_j^{(\nu)}\Delta_j^{(\mu)}|Y_r = X_r]. \tag{97}$$

An application of the Local Limit Theorem like in (62), (63) then leads to (94).

We are now able to bound

$$\begin{aligned}\mathbb{E}_\tau\left[S_2^2\right] &= c\sum_{i\notin\{a+1,\dots,b\}}\sum_{n=R_a+1}^{R_b}\sum_{\tau_{n-1}<j\neq l\leqslant\tau_n} M_{ij}M_{il}\mathbb{E}_\tau[D_{ij}D_{il}] \\ &\leqslant \frac{c}{L\log L}\sum_{i\notin\{a+1,\dots,b\}}\sum_{n=R_a+1}^{R_b}\sum_{\tau_{n-1}<j,l\leqslant\tau_n}\frac{1}{\sqrt{|i-j|}}\frac{1}{\sqrt{|i-l|}}\frac{1}{\Delta\tau_n}.\end{aligned} \tag{98}$$

Assume for instance that $i > b$ (the case $i\leq a$ can be treated similarly):

$$\begin{aligned}&\frac{c}{L\log L}\sum_{i>b}\sum_{n=R_a+1}^{R_b}\sum_{\tau_{n-1}<j,l\leqslant\tau_n}\frac{1}{\sqrt{i-j}}\frac{1}{\sqrt{i-l}}\frac{1}{\Delta\tau_n} \\ &\leq \frac{c}{L\log L}\sum_{i>b}\sum_{n=R_a+1}^{R_b}\sum_{\tau_{n-1}<j,l\leqslant\tau_n}\frac{1}{(i-\tau_n)\Delta\tau_n} \leq \frac{c}{L\log L}(b-a)\sum_{i=1}^{L}\frac{1}{i} \leq c'.\end{aligned}$$

Upper bound on $\mathbb{E}_\tau[S_1^2]$. Thanks to time translation invariance, one can reduce to the case $a = 0$. We have to distinguish various cases (recall Remark 4.6: we assume that $\{i, j\} \neq \{k, l\}$).

1. Assume that $\tau_{n-1} < i, j \leq \tau_n$, $\tau_{m-1} < k, l \leq \tau_m$, with $1 \leq n \neq m \leq R_b$. Then, thanks to (56), we get $\mathbb{E}_\tau[D_{ij}D_{kl}] = \mathbb{E}_\tau[D_{ij}]\mathbb{E}_\tau[D_{kl}] = 0$, because $\mathbb{E}_\tau[D_{ij}] = 0$. For similar reasons, one has that $\mathbb{E}_\tau[D_{ij}D_{kl}] = 0$ if one of the indexes, say i, belongs to one of the intervals $\{\tau_{n-1} + 1, \dots, \tau_n\}$, and the other three do not.

2. Assume that $\tau_{n-1} < i, j, k, l \leq \tau_n$ for some $n \leq R_b$. Using (57), we have

$$\mathbb{E}_\tau[D_{ij}D_{kl}] = \mathbb{E}^Y \mathbb{E}^X \left[D_{ij}D_{kl} \middle| X_{\tau_{n-1}} = Y_{\tau_{n-1}}, X_{\tau_n} = Y_{\tau_n} \right],$$

   and with a time translation we can reduce to the case $n = 1$ (we call $\tau_1 = r$). Thanks to the computation of $\mathbb{E}_\tau[\Delta_i \cdot \Delta_j]$ in Section 4.2, we see that $\mathbb{E}_\tau[\Delta_i \cdot \Delta_j] = \mathbb{E}_\tau[\Delta_k \cdot \Delta_l] = -A(r)$ so that

$$\mathbb{E}_\tau[D_{ij}D_{kl}] = \mathbb{E}_\tau[(\Delta_i \cdot \Delta_j)(\Delta_k \cdot \Delta_l)] - A(r)^2 \leqslant \mathbb{E}_\tau[(\Delta_i \cdot \Delta_j)(\Delta_k \cdot \Delta_l)]. \tag{99}$$

   (a) If $i = k$, $j \neq l$ (and $\tau_{n-1} < i, j, l \leq \tau_n$ for some $n \leq R_b$), then

$$\mathbb{E}_\tau[(\Delta_i \cdot \Delta_j)(\Delta_i \cdot \Delta_l)] \leq \frac{c}{\Delta\tau_n}. \tag{100}$$

   The computations are similar to those we did in Section 4.2 for the computation of $\mathbb{E}_\tau[\Delta_i \cdot \Delta_j]$. See Appendix A.1 for details.

   (b) If $\{i, j\} \cap \{k, l\} = \emptyset$ (and $\tau_{n-1} < i, j, k, l \leq \tau_n$ for some $n \leq R_b$), one gets

$$\mathbb{E}_\tau[(\Delta_i \cdot \Delta_j)(\Delta_k \cdot \Delta_l)] \leq \frac{c}{(\Delta\tau_n)^2}. \tag{101}$$

   See Appendix A.2 for a (sketch of) the proof, which is analogous to that of (100).

3. The only remaining case is that where $i \in \{\tau_{n-1} + 1, \dots, \tau_n\}$, $j \in \{\tau_{m-1} + 1, \dots, \tau_m\}$ with $m \neq n \leq R_b$, and each of these two intervals contains two indexes in i, j, k, l. Let us suppose for definiteness $n < m$ and $k \in \{\tau_{n-1} + 1, \dots, \tau_n\}$. Then $\mathbb{E}_\tau[\Delta_i \cdot \Delta_j] = \mathbb{E}_\tau[\Delta_k \cdot \Delta_l] = 0$ (cf. Lemma 4.4), and $\mathbb{E}_\tau[D_{ij}D_{kl}] = \mathbb{E}_\tau[(\Delta_i \cdot \Delta_j)(\Delta_k \cdot \Delta_l)]$. We will prove in Appendix A.3 that

$$\mathbb{E}_\tau[(\Delta_i \cdot \Delta_j)(\Delta_k \cdot \Delta_l)] \leqslant \frac{c}{\Delta\tau_n \Delta\tau_m} \tag{102}$$

   and that

$$\mathbb{E}_\tau[(\Delta_i \cdot \Delta_j)(\Delta_i \cdot \Delta_l)] \leqslant \frac{c}{\Delta\tau_m}. \tag{103}$$

We are now able to compute $\mathbb{E}_\tau[S_1^2]$. We consider first the contribution of the terms whose indexes i, j, k, l are all in the same interval $\{\tau_{n-1} + 1, \dots, \tau_n\}$, i.e. case (2) above. Recall that we drop the terms $\{i,j\}=\{k,l\}$ (see Remark 4.6):

$$\begin{aligned}\sum_{\substack{\tau_{n-1}<i,j,k,l\leqslant\tau_n\\ \{i,j\}\neq\{k,l\}}} M_{ij}M_{kl}\mathbb{E}_\tau[D_{ij}D_{kl}] &\leqslant \frac{c}{\Delta\tau_n}\sum_{\substack{l\in\{i,j\}\ or\ k\in\{i,j\}\\ \tau_{n-1}<i,j,k,l\leqslant\tau_n}} M_{ij}M_{kl}+\frac{c}{\Delta\tau_n^2}\sum_{\substack{\{i,j\}\cap\{k,l\}=\emptyset\\ \tau_{n-1}<i,j,k,l\leq\tau_n}} M_{ij}M_{kl}\\ &\leqslant \frac{c'}{L\log L}\left[\frac{1}{\Delta\tau_n}\sum_{1\leq i<j<k\leq\Delta\tau_n}\frac{1}{\sqrt{j-i}}\frac{1}{\sqrt{k-j}}+\frac{1}{\Delta\tau_n^2}\left(\sum_{1\leq i<j\leq\Delta\tau_n}\frac{1}{\sqrt{j-i}}\right)^2\right]\\ &\leqslant \frac{c''}{L\log L}\Delta\tau_n. \end{aligned} \tag{104}$$

Altogether, we see that

$$\begin{aligned}&\sum_{i\neq j=1}^{b}\sum_{\substack{k\neq l=1\\ \{i,j\}\neq\{k,l\}}}^{b} M_{ij}M_{kl}\mathbb{E}_\tau[D_{ij}D_{kl}]\mathbf{1}_{\{\exists n\leq R_b: i,j\in\{\tau_{n-1}+1,\dots,\tau_n\}\}}\\ &\qquad = \sum_{n=1}^{R_b}\sum_{\substack{\tau_{n-1}<i,j,k,l\leq\tau_n\\ \{i,j\}\neq\{k,l\}}} M_{ij}M_{kl}\mathbb{E}_\tau[D_{ij}D_{kl}]\leqslant\frac{c}{L\log L}\sum_{n=1}^{R_b}\Delta\tau_n\leq\frac{c}{\log L}. \end{aligned}\tag{105}$$

Finally, we consider the contribution to $\mathbb{E}_\tau[S_1^2]$ coming from the terms of point (3). We have (recall that $n<m$)

$$\begin{aligned}\sum_{\substack{\tau_{n-1}<i,k\leq\tau_n\\ \tau_{m-1}<j,l\leq\tau_m\\ \{i,j\}\neq\{k,l\}}} M_{ij}M_{kl}\mathbb{E}_\tau[D_{ij}D_{kl}] \quad\leqslant\quad & \frac{c}{L\log L}\frac{1}{\Delta\tau_n\Delta\tau_m}\sum_{\substack{\tau_{n-1}<i\neq k\leq\tau_n\\ \tau_{m-1}<j\neq l\leq\tau_m}}\frac{1}{\sqrt{j-i}}\frac{1}{\sqrt{l-k}}\\ &+\frac{c}{L\log L}\frac{1}{\Delta\tau_n}\sum_{\substack{\tau_{n-1}<i\neq k\leq\tau_n\\ \tau_{m-1}<j\leq\tau_m}}\frac{1}{\sqrt{j-i}}\frac{1}{\sqrt{j-k}}\\ &+\frac{c}{L\log L}\frac{1}{\Delta\tau_m}\sum_{\substack{\tau_{n-1}<i\leq\tau_n\\ \tau_{m-1}<j\neq l\leq\tau_m}}\frac{1}{\sqrt{j-i}}\frac{1}{\sqrt{l-i}}. \end{aligned}\tag{106}$$

But as $j>\tau_{m-1}$

$$\sum_{\tau_{n-1}<i\leqslant\tau_n}\frac{1}{\sqrt{j-i}}\leqslant\sum_{\tau_{n-1}<i\leqslant\tau_n}\frac{1}{\sqrt{\tau_{m-1}-i+1}}\leqslant c\left(\sqrt{\tau_{m-1}-\tau_{n-1}}-\sqrt{\tau_{m-1}-\tau_n}\right), \tag{107}$$

and as $k\leqslant\tau_n$

$$\sum_{\tau_{m-1}<l\leqslant\tau_m}\frac{1}{\sqrt{l-k}}\leqslant\sum_{\tau_{m-1}<l\leqslant\tau_m}\frac{1}{\sqrt{l-\tau_n}}\leqslant c\left(\sqrt{\tau_m-\tau_n}-\sqrt{\tau_{m-1}-\tau_n}\right), \tag{108}$$

so that

$$\sum_{\substack{\tau_{n-1}<i,k\leq\tau_n\\ \tau_{m-1}<j,l\leq\tau_m\\ \{i,j\}\neq\{k,l\}}} M_{ij}M_{kl}\mathbb{E}_\tau[D_{ij}D_{kl}]\leqslant\frac{c}{L\log L}\left(\sqrt{T_{nm}+\Delta\tau_n}-\sqrt{T_{nm}}\right)\left(\sqrt{T_{nm}+\Delta\tau_m}-\sqrt{T_{nm}}\right), \tag{109}$$

where we noted $T_{nm} = \tau_{m-1} - \tau_n$. Recalling (105) and the definition (90) of S_1, we can finally write

$$\widehat{\mathbf{E}}\left[\mathbb{E}_\tau[S_1^2]\right] \leq c\left(1 + \widehat{\mathbf{E}}\left[\sum_{n=1}^{R_b-1} \sum_{n<m\leq R_b} \sum_{\substack{\tau_{n-1}<i,k\leq\tau_n \\ \tau_{m-1}<j,l\leq\tau_m}} M_{ij}M_{kl}\mathbb{E}_\tau[D_{ij}D_{kl}]\right]\right)$$

$$\leqslant \quad c + \frac{c}{L\log L}\widehat{\mathbf{E}}\left[\sum_{1\leq n<m\leq R_b}\left(\sqrt{T_{nm}+\Delta\tau_n} - \sqrt{T_{nm}}\right)\left(\sqrt{T_{nm}+\Delta\tau_m} - \sqrt{T_{nm}}\right)\right].$$

The remaining average can be estimated via the following Lemma.

Lemma 4.7. *There exists a constant $c > 0$ depending only on $K(\cdot)$, such that*

$$\widehat{\mathbf{E}}\left[\sum_{1\leq n<m\leq R_b}\left(\sqrt{T_{nm}+\Delta\tau_n} - \sqrt{T_{nm}}\right)\left(\sqrt{T_{nm}+\Delta\tau_m} - \sqrt{T_{nm}}\right)\right] \leqslant cL(\log L)^{7/4}. \tag{110}$$

Of course this implies that $\widehat{\mathbf{E}}\mathbb{E}_\tau[S_1^2] \leq c(\log L)^{3/4}$, which together with (98) implies the claim of Lemma 4.3. □

Proof of Lemma 4.7. One has the inequality

$$\left(\sqrt{T_{nm}+\Delta\tau_n} - \sqrt{T_{nm}}\right)\left(\sqrt{T_{nm}+\Delta\tau_m} - \sqrt{T_{nm}}\right) \leqslant \sqrt{\Delta\tau_n}\sqrt{\Delta\tau_m}, \tag{111}$$

which is a good approximation when T_{nm} is not that large compared with $\Delta\tau_n$ and $\Delta\tau_m$, and

$$\left(\sqrt{T_{nm}+\Delta\tau_n} - \sqrt{T_{nm}}\right)\left(\sqrt{T_{nm}+\Delta\tau_m} - \sqrt{T_{nm}}\right) \leqslant c\frac{\Delta\tau_n\Delta\tau_m}{T_{nm}}, \tag{112}$$

which is accurate when T_{nm} is large. We use these bounds to cut the expectation (110) into two parts, a term where $m - n \leqslant H_L$ and one where $m - n > H_L$, with H_L to be chosen later:

$$\widehat{\mathbf{E}}\left[\sum_{n=1}^{R_b}\sum_{m=n+1}^{R_b}\left(\sqrt{T_{nm}+\Delta\tau_n} - \sqrt{T_{nm}}\right)\left(\sqrt{T_{nm}+\Delta\tau_m} - \sqrt{T_{nm}}\right)\right]$$

$$\leqslant \quad \widehat{\mathbf{E}}\left[\sum_{n=1}^{R_b}\sum_{m=n+1}^{(n+H_L)\wedge R_b}\sqrt{\Delta\tau_n}\sqrt{\Delta\tau_m}\right] + c\,\widehat{\mathbf{E}}\left[\sum_{n=1}^{R_b}\sum_{m=n+H_L+1}^{R_b}\frac{\Delta\tau_n\Delta\tau_m}{T_{nm}}\right]. \tag{113}$$

We claim that there exists a constant c such that for every $l \geq 1$,

$$\widehat{\mathbf{E}}\left[\sum_{n=1}^{R_b-l}\sqrt{\Delta\tau_n}\sqrt{\Delta\tau_{n+l}}\right] \leqslant c\sqrt{L}(\log L)^{2+\frac{1}{12}} \tag{114}$$

(the proof is given later). Then the first term in the right-hand side of (113) is

$$\widehat{\mathbf{E}}\left[\sum_{n=1}^{R_b}\sum_{m=n+1}^{(n+H_L)\wedge R_b}\sqrt{\Delta\tau_n}\sqrt{\Delta\tau_m}\right] = \sum_{l=1}^{H_L}\widehat{\mathbf{E}}\left[\sum_{n=1}^{R_b-l}\sqrt{\Delta\tau_n}\sqrt{\Delta\tau_{n+l}}\right] \leqslant cH_L\sqrt{L}(\log L)^{2+1/12}.$$

If we choose $H_L = \sqrt{L}(\log L)^{-1/3}$, we get from (113)

$$\widehat{\mathbf{E}}\left[\sum_{n=1}^{R_b}\sum_{m=n+1}^{R_b}\left(\sqrt{T_{nm}+\Delta\tau_n}-\sqrt{T_{nm}}\right)\left(\sqrt{T_{nm}+\Delta\tau_m}-\sqrt{T_{nm}}\right)\right] \tag{115}$$
$$\leqslant \quad cL(\log L)^{7/4} + c\,\widehat{\mathbf{E}}\left[\sum_{n=1}^{R_b}\sum_{m=n+H_L+1}^{R_b}\frac{\Delta\tau_n\Delta\tau_m}{T_{nm}}\right].$$

As for the second term in (113), recall that $T_{nm} = \tau_{m-1} - \tau_n$ and decompose the sum in two parts, according to whether T_{nm} is larger or smaller than a certain $K_L > 1$ to be fixed:

$$\begin{aligned}
&\widehat{\mathbf{E}}\left[\sum_{n=1}^{R_b}\sum_{m=n+H_L+1}^{R_b}\frac{\Delta\tau_n\Delta\tau_m}{T_{nm}}\right]\\
&\quad\leq \widehat{\mathbf{E}}\left[\sum_{n=1}^{R_b}\sum_{m=n+H_L+1}^{R_b}\frac{\Delta\tau_n\Delta\tau_m}{T_{nm}}\mathbf{1}_{\{T_{nm}>K_L\}}\right] + \widehat{\mathbf{E}}\left[\sum_{n=1}^{R_b}\sum_{m=n+H_L+1}^{R_b}\Delta\tau_n\Delta\tau_m\mathbf{1}_{\{T_{nm}\leqslant K_L\}}\right]\\
&\quad\leqslant \frac{1}{K_L}\widehat{\mathbf{E}}\left[\left(\sum_{n=1}^{R_b}\Delta\tau_n\right)^2\right] + L^2\widehat{\mathbf{E}}\left[\sum_{n=1}^{R_b}\sum_{m=n+H_L+1}^{R_b}\mathbf{1}_{\{\tau_{n+H_L}-\tau_n\leqslant K_L\}}\right]\\
&\quad\leqslant \frac{L^2}{K_L} + L^4\widehat{\mathbf{P}}\left(\tau_{H_L}\leqslant K_L\right).
\end{aligned} \tag{116}$$

We now set $K_L = L(\log L)^{-7/4}$, so that we get in the previous inequality

$$\widehat{\mathbf{E}}\left[\sum_{n=1}^{R_b}\sum_{m=n+H_L+1}^{R_b}\frac{\Delta\tau_n\Delta\tau_m}{T_{nm}}\right] \leqslant L(\log L)^{7/4} + L^4\widehat{\mathbf{P}}\left(\tau_{H_L}\leqslant K_L\right), \tag{117}$$

and we are done if we prove for instance that $\widehat{\mathbf{P}}\left(\tau_{H_L}\leqslant K_L\right) = o(L^{-4})$. Indeed,

$$\widehat{\mathbf{P}}\left(\tau_{H_L}\leqslant K_L\right) \;=\; \widehat{\mathbf{P}}\left(R_{K_L}\geq H_L\right) \leqslant c\mathbf{P}\left(R_{K_L}\geq H_L\right) \tag{118}$$

where we used Lemma A.1 to take the conditioning off from $\widehat{\mathbf{P}} := \mathbf{P}(\cdot|b\in\tau)$ (in fact, $K_L \leqslant b/2$ since $b \geq \varepsilon L$). Recalling the choices of H_L and K_L, we get that $H_L/\sqrt{K_L} = (\log L)^{13/24}$ and, combining (118) with Lemma A.2, we get

$$\widehat{\mathbf{P}}\left(\tau_{H_L}\leqslant K_L\right) \leqslant c'\,e^{-c(\log L)^{13/12}} = o(L^{-4}) \tag{119}$$

which is what we needed.

To conclude the proof of Lemma 4.7, we still have to prove (114). Note that

$$\begin{aligned}
\widehat{\mathbf{E}}\left[\sum_{n=1}^{R_b-l}\sqrt{\Delta\tau_n}\sqrt{\Delta\tau_{n+l}}\mathbf{1}_{\{R_b>l\}}\right] &= \widehat{\mathbf{E}}\left[\mathbf{1}_{\{R_b>l\}}\sum_{n=1}^{R_b-l}\widehat{\mathbf{E}}\left[\sqrt{\Delta\tau_n}\sqrt{\Delta\tau_{n+l}}\,\middle|R_b\right]\right]\\
&= \widehat{\mathbf{E}}\left[\mathbf{1}_{\{R_b>l\}}(R_b-l)\widehat{\mathbf{E}}\left[\sqrt{\tau_1}\sqrt{\tau_2-\tau_1}\,\middle|R_b\right]\right]\\
&\leq \widehat{\mathbf{E}}\left[R_b\sqrt{\tau_1}\sqrt{\tau_2-\tau_1}\mathbf{1}_{\{R_b\geq 2\}}\right]
\end{aligned} \tag{120}$$

where we used the fact that, under $\widehat{\mathbf{P}}(\cdot|R_b = p)$ for a fixed p, the law of the jumps $\{\Delta\tau_n\}_{n\le p}$ is exchangeable. We first bound (120) when R_b is large:

$$\begin{aligned}\widehat{\mathbf{E}}\left[R_b\sqrt{\tau_1}\sqrt{\tau_2-\tau_1}\mathbf{1}_{\{R_b\ge\kappa\sqrt{L\log L}\}}\right] &\leqslant L^2\widehat{\mathbf{P}}\left(R_b\ge\kappa\sqrt{L\log L}\right)\\ &\leqslant L^2\mathbf{P}(b\in\tau)^{-1}\mathbf{P}\left(R_b\ge\kappa\sqrt{L\log L}\right). \qquad (121)\end{aligned}$$

In view of (14), we have $\mathbf{P}(b\in\tau)^{-1} = O(\sqrt{L})$. Thanks to Lemma A.2 in the Appendix, and choosing κ large enough, we get

$$\mathbf{P}\left(R_b\ge\kappa\sqrt{L\log L}\right)\leqslant e^{-c\kappa^2\log L+o(\log L)} = o(L^{-5/2}), \qquad (122)$$

and therefore

$$\widehat{\mathbf{E}}\left[R_b\sqrt{\tau_1}\sqrt{\tau_2-\tau_1}\mathbf{1}_{\{R_b\ge\kappa\sqrt{L\log L}\}}\right] = o(1). \qquad (123)$$

As a consequence,

$$\begin{aligned}\widehat{\mathbf{E}}\left[R_b\sqrt{\tau_1}\sqrt{\tau_2-\tau_1}\mathbf{1}_{\{R_b\ge 2\}}\right] &= \widehat{\mathbf{E}}\left[R_b\sqrt{\tau_1}\sqrt{\tau_2-\tau_1}\mathbf{1}_{\{2\le R_b<\kappa\sqrt{L\log L}\}}\right]+o(1)\\ &\leqslant \sqrt{L}(\log L)^{1/12}\widehat{\mathbf{E}}\left[\sqrt{\tau_1}\sqrt{\tau_2-\tau_1}\mathbf{1}_{\{R_b\ge 2\}}\right]\\ &\quad +\kappa\sqrt{L\log L}\widehat{\mathbf{E}}\left[\sqrt{\tau_1}\sqrt{\tau_2-\tau_1}\mathbf{1}_{\{R_b>\sqrt{L}(\log L)^{1/12}\}}\right]+o(1). \qquad (124)\end{aligned}$$

Let us deal with the second term:

$$\begin{aligned}&\widehat{\mathbf{E}}\left[\mathbf{1}_{\{R_b>\sqrt{L}(\log L)^{1/12}\}}\sqrt{\tau_1}\sqrt{\tau_2-\tau_1}\right]\\ &= \frac{1}{\mathbf{P}(b\in\tau)}\sum_{i=1}^{b}\sum_{j=1}^{b-i}\sqrt{i}\sqrt{j}\mathbf{P}\left(\tau_1=i,\tau_2-\tau_1=j,b\in\tau,R_b>\sqrt{L}(\log L)^{1/12}\right)\\ &= \frac{1}{\mathbf{P}(b\in\tau)}\sum_{i=1}^{b}\sum_{j=1}^{b-i}\sqrt{i}\sqrt{j}K(i)K(j)\mathbf{P}\left(b-i-j\in\tau,R_{b-i-j}>\sqrt{L}(\log L)^{1/12}-2\right). \qquad (125)\end{aligned}$$

But we have

$$\begin{aligned}\mathbf{P}\left(R_{b-i-j}>\sqrt{L}(\log L)^{1/12}-2\,\middle|\,b-i-j\in\tau\right) &\leqslant 2\mathbf{P}\left(R_{(b-i-j)/2}>\frac{1}{2}\sqrt{L}(\log L)^{1/12}-1\,\middle|\,b-i-j\in\tau\right)\\ &\leqslant c\mathbf{P}\left(R_{(b-i-j)/2}>\frac{1}{2}\sqrt{L}(\log L)^{1/12}-1\right)\\ &\leqslant c\mathbf{P}\left(R_L>\frac{1}{2}\sqrt{L}(\log L)^{1/12}-1\right)\leqslant c'e^{-c(\log L)^{1/6}} \qquad (126)\end{aligned}$$

where we first used Lemma A.1 to take the conditioning off, and then Lemma A.2. Putting (125) and (126) together, we get

$$\begin{aligned}&\widehat{\mathbf{E}}\left[\mathbf{1}_{\{R_b>\sqrt{L}(\log L)^{1/12}\}}\sqrt{\tau_1}\sqrt{\tau_2-\tau_1}\right]\\ &\leqslant c'e^{-c(\log L)^{1/6}}\frac{1}{\mathbf{P}(b\in\tau)}\sum_{i=1}^{b}\sum_{j=1}^{b-i}\sqrt{i}\sqrt{j}K(i)K(j)\mathbf{P}\left(b-i-j\in\tau\right)\\ &= c'e^{-c(\log L)^{1/6}}\widehat{\mathbf{E}}\left[\sqrt{\tau_1}\sqrt{\tau_2-\tau_1}\mathbf{1}_{\{R_b\ge 2\}}\right]. \qquad (127)\end{aligned}$$

So, recalling (124), we have

$$\widehat{\mathbf{E}}\left[R_b\sqrt{\tau_1}\sqrt{\tau_2-\tau_1}\mathbf{1}_{\{R_b\geq 2\}}\right] \leqslant 2\sqrt{L}(\log L)^{1/12}\widehat{\mathbf{E}}\left[\sqrt{\tau_1}\sqrt{\tau_2-\tau_1}\mathbf{1}_{\{R_b\geq 2\}}\right]+o(1) \tag{128}$$

and we only have to estimate (recall (14))

$$\begin{aligned}\widehat{\mathbf{E}}\left[\sqrt{\tau_1}\sqrt{\tau_2-\tau_1}\mathbf{1}_{\{R_b\geq 2\}}\right] &= \sum_{p=1}^{b-1}\sum_{q=1}^{b-p}\sqrt{p}\sqrt{q}K(p)K(q)\frac{\mathbf{P}(b-p-q\in\tau)}{\mathbf{P}(b\in\tau)}\\ &\leqslant c\sqrt{b}\sum_{p=1}^{b-1}\sum_{q=1}^{b-p}\frac{1}{p\,q}\frac{1}{\sqrt{b+1-p-q}}.\end{aligned} \tag{129}$$

Using twice the elementary estimate

$$\sum_{k=1}^{M-1}\frac{1}{k}\frac{1}{\sqrt{M-k}}\leq c\frac{1}{\sqrt{M}}\log M,$$

we get

$$\widehat{\mathbf{E}}\left[\sqrt{\tau_1}\sqrt{\tau_2-\tau_1}\mathbf{1}_{\{R_b\geq 2\}}\right] \leqslant c\sqrt{b}\sum_{p=1}^{b-1}\frac{1}{p}\frac{1}{\sqrt{b-p+1}}\log(b-p+1)\leqslant c\sqrt{b}\frac{1}{\sqrt{b}}(\log L)^2. \tag{130}$$

Together with (128), this proves the desired estimate (114).

□

4.4 Dimension $d=4$ (a sketch)

As we mentioned just after Theorem 2.8, it is possible to adapt the change-of-measure argument to prove non-coincidence of quenched and annealed critical points in dimension $d\geq 4$ for the general walks of Assumption 2.1, while the method of Birkner and Sun [3] does not seem to adapt easily much beyond the simple random walk case. In this section, we only deal with the case $d=4$, since the Theorem 2.8 is obtained for $d\geq 5$ in [2], with more general condition than Assumption 2.1. We will not give details, but for the interested reader we hint at the "right" change of measure which works in this case.

The "change of measure function" $g_{\mathscr{I}}(Y)$ is still of the form (27), factorized over the blocks which belong to $\mathscr{I}$, but this time M is a matrix with a finite bandwidth:

$$F_k(Y)=-\frac{1}{\sqrt{L}}\sum_{i=L(k-1)+1}^{kL-p_0}\Delta_i\cdot\Delta_{i+p_0}, \tag{131}$$

where p_0 is an integer. The role of the normalization $L^{-1/2}$ is to guarantee that $\|M\|<\infty$. The integer p_0 is to be chosen such that $A(p_0)>0$, where $A(\cdot)$ is the function defined in Lemma 4.4. The existence of such p_0 is guaranteed by the asymptotics (59), whose proof for $d=4$ is the same as for $d=3$.

For the rest, the scheme of the proof of $\beta_c\neq\beta_c^{ann}$ (in particular, the coarse-graining procedure) is analogous to that we presented for $d=3$, and the computations involved are considerably simpler.

A Some technical estimates

Lemma A.1. *(Lemma A.2 in [9]) Let* $\mathbf{P}$ *be the law of a recurrent renewal whose inter-arrival law satisfies* $K(n) \overset{n\to\infty}{\sim} c_K n^{-3/2}$ *for some* $c_K > 0$. *There exists a constant* $c > 0$, *that depends only on* $K(\cdot)$, *such that for any non-negative function* $f_N(\tau)$ *which depends only on* $\tau \cap \{1,\dots,N\}$, *one has*

$$\sup_{N>0} \frac{\mathbf{E}[f_N(\tau)|2N \in \tau]}{\mathbf{E}[f_N(\tau)]} \leqslant c. \tag{132}$$

Lemma A.2. *Under the same assumptions as in Lemma A.1, and with* $R_N := |\tau \cap \{1,\dots,N\}|$, *there exists a constant* $c > 0$, *such that for any positive function* $\alpha(N)$ *which diverges at infinity and such that* $\alpha(N) = o(\sqrt{N})$, *we have*

$$\mathbf{P}\left(R_N \geq \sqrt{N}\alpha(N)\right) \leqslant e^{-c\alpha(N)^2 + o\left(\alpha(N)^2\right)}. \tag{133}$$

Proof. For every $\lambda > 0$

$$\begin{aligned}
\mathbf{P}\left(R_N \geq \sqrt{N}\alpha(N)\right) &= \mathbf{P}\left(\tau_{\sqrt{N}\alpha(N)} \leqslant N\right) = \mathbf{P}\left(\lambda\alpha(N)^2 \frac{\tau_{\sqrt{N}\alpha(N)}}{N} \leqslant \lambda\alpha(N)^2\right) \\
&\leqslant e^{\lambda\alpha(N)^2}\mathbf{E}\left[e^{-\lambda\frac{\alpha(N)^2}{N}\tau_{\sqrt{N}\alpha(N)}}\right] = e^{\lambda\alpha(N)^2}\mathbf{E}\left[e^{-\lambda\alpha(N)^2\frac{\tau_1}{N}}\right]^{\sqrt{N}\alpha(N)}.
\end{aligned} \tag{134}$$

The asymptotic behavior of $\mathbf{E}\left[e^{-\lambda\alpha(N)^2\frac{\tau_1}{N}}\right]$ is easily obtained:

$$\begin{aligned}
1 - \mathbf{E}\left[e^{-\lambda\alpha(N)^2\frac{\tau_1}{N}}\right] &= \sum_{n\in\mathbb{N}} K(n)\left(1 - e^{-n\lambda\alpha(N)^2/N}\right) \\
&\overset{N\to\infty}{\sim} c\frac{\sqrt{\lambda}\alpha(N)}{\sqrt{N}},\ c = c_K\int_0^\infty \frac{1-e^{-x}}{x^{3/2}}\,\mathrm{d}x,
\end{aligned} \tag{135}$$

where the condition $\alpha(N)^2/N \to 0$ was used to transform the sum into an integral. Therefore, we get

$$\begin{aligned}
\mathbf{E}\left[e^{-\lambda\alpha(N)^2\frac{\tau_1}{N}}\right]^{\sqrt{N}\alpha(N)} &= \left(1 - c\frac{\sqrt{\lambda}\alpha(N)}{\sqrt{N}} + o\left(\frac{\alpha(N)}{\sqrt{N}}\right)\right)^{\sqrt{N}\alpha(N)} \\
&= e^{-c\sqrt{\lambda}\alpha(N)^2 + o\left(\alpha(N)^2\right)}.
\end{aligned} \tag{136}$$

Then, for any $\lambda > 0$,

$$\mathbf{P}\left(R_N \geq \sqrt{N}\alpha(N)\right) \leqslant e^{(\lambda - c\sqrt{\lambda})\alpha(N)^2 + o\left(\alpha(N)^2\right)} \tag{137}$$

and taking $\lambda = c^2/4$ we get the desired bound. □

We need also the following standard result (cf. for instance [10, Section 5]):

Lemma A.3. *Under the same hypothesis as in Lemma A.1, we have the following convergence in law:*

$$\frac{c_K}{\sqrt{N}}R_N \overset{N\to\infty}{\Rightarrow} \frac{1}{\sqrt{2\pi}}|\mathscr{Z}| \qquad (\mathscr{Z} \sim \mathscr{N}(0,1)). \tag{138}$$

A.1 Proof of (100)

We wish to show that for distinct i, j, l smaller than r,

$$\mathbb{E}^X \mathbb{E}^Y [(\Delta_i \cdot \Delta_j)(\Delta_i \cdot \Delta_l) | X_r = Y_r] \leq \frac{c}{r}. \tag{139}$$

We use the same method as in Section 4.2: we fix $x \in \mathbb{Z}^d$, and we use the notation $\mathbb{E}^Y_{r,x}[\cdot] = \mathbb{E}^Y[\cdot \,\big|\, Y_r = x\,]$. Then,

$$\begin{aligned}
\mathbb{E}^Y_{r,x}\left[(\Delta_i \cdot \Delta_j)(\Delta_i \cdot \Delta_l)\right] &= \mathbb{E}^Y_{r,x}\left[(\Delta_i \cdot \Delta_j)\left(\Delta_i \cdot \mathbb{E}^Y_{r,x}\left[\Delta_l \,\middle|\, \Delta_i, \Delta_j\,\right]\right)\right] \\
&= \frac{1}{r-2}\mathbb{E}^Y_{r,x}\left[(\Delta_i \cdot \Delta_j)\left(\Delta_i \cdot (x - \Delta_i - \Delta_j)\right)\right] \\
&= \frac{1}{r-2}\mathbb{E}^Y_{r,x}\left[(\Delta_i \cdot \Delta_j)\left((x \cdot \Delta_i) - \left\|\Delta_i\right\|^2\right) - (\Delta_i \cdot \Delta_j)^2\right] \\
&\leqslant \frac{1}{r-2}\mathbb{E}^Y_{r,x}\left[\left((x \cdot \Delta_i) - \left\|\Delta_i\right\|^2\right)\left(\Delta_i \cdot \mathbb{E}^Y_{r,x}\left[\Delta_j \,\middle|\, \Delta_i\,\right]\right)\right] \\
&= \frac{1}{(r-1)(r-2)}\mathbb{E}^Y_{r,x}\left[\left((x \cdot \Delta_i) - \left\|\Delta_i\right\|^2\right)^2\right] \\
&\leqslant \frac{2}{(r-1)(r-2)}\mathbb{E}^Y_{r,x}\left[\|x\|^2 \left\|\Delta_i\right\|^2 + \left\|\Delta_i\right\|^4\right]
\end{aligned}$$

and we can take by symmetry $i = 1$. Therefore,

$$\begin{aligned}
\mathbb{E}^X \mathbb{E}^Y \left[\, (\Delta_i \cdot \Delta_j)(\Delta_i \cdot \Delta_l) \middle| X_r = Y_r\right] &= \frac{\mathbb{E}^X\left[\mathbb{E}^Y\left[(\Delta_i \cdot \Delta_j)(\Delta_i \cdot \Delta_l) \middle| Y_r = X_r\,\right] \mathbb{P}^Y(Y_r = X_r)\right]}{\mathbb{P}^{X-Y}(Y_r = X_r)} \qquad (140) \\
&\leq \frac{c}{r}\frac{\mathbb{E}^X\left[\left(\frac{\|X_r\|^2}{r}\mathbb{E}^Y\left[\|\Delta_1\|^2 | Y_r = X_r\right] + \frac{1}{r}\mathbb{E}^Y(\|\Delta_1\|^4 | Y_r = X_r)\right)\mathbb{P}^Y(Y_r = X_r)\right]}{\mathbb{P}^{X-Y}(Y_r = X_r)} \\
&= \frac{c}{r}\,\mathbb{E}^X\mathbb{E}^Y\left[Q\left(\frac{\|X_r\|}{\sqrt{r}}, \|\Delta_1\|\right)\middle| Y_r = X_r\right],
\end{aligned}$$

where

$$Q\left(\frac{\|X_r\|}{\sqrt{r}}, \|\Delta_1\|\right) = \frac{\|X_r\|^2}{r}\|\Delta_1\|^2 + \frac{\|\Delta_1\|^4}{r}. \tag{141}$$

At this point, one can apply directly the result of Remark 4.5. □

A.2 Proof of (101)

We wish to prove that, for distinct $i, j, k, l \leq r$,

$$\mathbb{E}_\tau[(\Delta_i \cdot \Delta_j)(\Delta_k \cdot \Delta_l)] \leqslant \frac{c}{r^2}. \tag{142}$$

The proof is very similar to that of (139), so we skip details. What one gets is that

$$\mathbb{E}_\tau\left[(\Delta_i \cdot \Delta_j)(\Delta_k \cdot \Delta_l)\right] \leqslant \frac{c}{r^2}\frac{\mathbb{E}^X\left[\mathbb{E}^Y\left[Q'\left(\frac{\|X_r\|}{r^{1/2}}, \{\|\Delta_i\|\}_{i=1,2,3}\right)\middle| Y_r = X_r\right]\mathbb{P}^Y(Y_r = X_r)\right]}{\mathbb{P}^{X-Y}(Y_r = X_r)}, \tag{143}$$

where Q' is a polynomial of degree four. Again, like after (140), one uses the Remark 4.5 to get the desired result.

A.3 Proof of (102)-(103)

In view of (56), in order to prove (102) it suffices to prove that for $0 < i \neq k \leq r, 0 < j \neq l \leq s$

$$\sum_{\nu,\mu=1}^{3} \mathbb{E}^X \mathbb{E}^Y [\Delta_i^{(\nu)} \Delta_k^{(\mu)} | X_r = Y_r] \mathbb{E}^X \mathbb{E}^Y [\Delta_j^{(\nu)} \Delta_l^{(\mu)} | X_s = Y_s] \leq \frac{c}{rs}. \tag{144}$$

Both factors in the left-hand side have already been computed in (96)-(97). Using these two expressions and once more the Local Limit Theorem, one arrives easily to (144). The proof of (103) is essentially identical.

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