DOI: 10.1214/ECP.v7-1042

Elect. Comm. in Probab. **7** (2002) 1–12

ELECTRONIC COMMUNICATIONS in PROBABILITY

A REPRESENTATION FOR NON-COLLIDING RANDOM WALKS

NEIL O'CONNELL
BRIMS, HP Labs, Bristol BS34 8QZ, UK
email: Neil.O.connell@ens.fr

MARC YOR
Laboratoire de Probabilités, Université Pierre et Marie Curie,
4 Place Jussieu F-75252 Paris Cedex 05, France

submitted March 7, 2001 *Final version accepted* July 28, 2001

AMS 2000 Subject classification: 15A52, 60J27, 60J65, 60J45, 60K25
GUE, eigenvalues of random matrices, Hermitian Brownian motion, non-colliding Brownian motions, Weyl chamber, queues in series, Burke's theorem, reversibility, Pitman's representation theorem, Charlier ensemble.

Abstract
Let $D_0(\mathbb{R}_+)$ denote the space of cadlag paths $f : \mathbb{R}_+ \to \mathbb{R}$ with $f(0) = 0$. For $f, g \in D_0(\mathbb{R}_+)$, define $f \otimes g \in D_0(\mathbb{R}_+)$ and $f \odot g \in D_0(\mathbb{R}_+)$ by

$$(f \otimes g)(t) = \inf_{0\le s\le t}[f(s) + g(t) - g(s)],$$

and

$$(f \odot g)(t) = \sup_{0\le s\le t}[f(s) + g(t) - g(s)].$$

Unless otherwise deleniated by parentheses, the default order of operations is from left to right; for example, when we write $f \otimes g \otimes h$, we mean $(f \otimes g) \otimes h$. Define a sequence of mappings $\Gamma_k : D_0(\mathbb{R}_+)^k \to D_0(\mathbb{R}_+)^k$ by

$$\Gamma_2(f, g) = (f \otimes g, g \odot f),$$

and, for $k > 2$,

$$\Gamma_k(f_1, \dots, f_k) = (f_1 \otimes f_2 \otimes \cdots \otimes f_k, \Gamma_{k-1}(f_2 \odot f_1, f_3 \odot (f_1 \otimes f_2), \dots, f_k \odot (f_1 \otimes \cdots \otimes f_{k-1}))).$$

Let $N_1, \dots, N_n$ be the counting functions of independent Poisson processes on $\mathbb{R}_+$ with respective intensities $\mu_1 < \mu_2 < \cdots < \mu_n$. Our main result is that the conditional law of $N_1, \dots, N_n$, given

$$N_1(t) \le \cdots \le N_n(t), \quad \text{for all } t \ge 0,$$

is the same as the unconditional law of $\Gamma_n(N)$. From this, we deduce the corresponding results for independent Poisson processes of equal rates and for independent Brownian motions (in *both of these cases the conditioning is in the sense of Doob). This extends a recent observation, independently due to Baryshnikov (2001) and Gravner, Tracy and Widom (2001), that if B is a standard Brownian motion in $\mathbb{R}^n$, then $(B_1 \otimes \cdots \otimes B_n)(1)$ has the same law as the smallest eigenvalue of a $n \times n$ GUE random matrix.*

1 Introduction and Summary

Let $B = (B_1, \ldots, B_n)$ be a standard n-dimensional Brownian motion and set

$$R_n(t) = \inf_{0=t_0<t_1<\cdots<t_{n-1}<t_n=t} \sum_{k=1}^{n} [B_k(t_k) - B_k(t_{k-1})]. \tag{1}$$

The process R_n was introduced in [14]. It has recently been observed [3, 16] that:

Theorem 1 *The random variable $R_n(1)$ has the same law as the smallest eigenvalue of a $n \times n$ GUE random matrix.*

A $n \times n$ GUE random matrix $A \in \mathbb{C}^{n\times n}$ is constructed as follows: it is Hermitian, that is, $A = A^*(= (\bar{A})^t)$; the entries $\{A_{ij},\ i \le j\}$ are independent; on the diagonal A_{ii} are standard real normal random variables; below the diagonal, $\{A_{ij},\ i < j\}$ are standard complex normal random variables, that is, the real and imaginary parts of A_{ij} are independent centered real normal random variables, each with variance 1/2; above the diagonal we set $A_{ji} = \bar{A}_{ij}$. Here, $\bar{z} = x - iy$ denotes the complex conjugate of $z = x + iy$. *Hermitian Brownian motion* is constructed in the same way as a GUE random matrix, but with Brownian motions instead of normal random variables. It is well-known (see, for example, [11, 15, 26]) that the eigenvalues of Hermitian Brownian motion evolve like independent Brownian motions started from the origin and conditioned (in the sense of Doob) never to collide. To make this more precise, the function

$$h(x) = \prod_{i<j} (x_j - x_i) \tag{2}$$

is harmonic on $\mathbb{R}^n$, and moreover, is a strictly positive harmonic function for Brownian motion killed when it exits the Weyl chamber

$$W = \{x \in \mathbb{R}^n :\ x_1 < x_2 < \cdots < x_n\}; \tag{3}$$

the conditioned process we refer to is the corresponding Doob h-transform, started at the entrance point $(0, 0, \ldots, 0)$. For related work on non-colliding diffusions and random matrices, see [4, 9, 18, 22], and references therein. Thus, if $\hat{B}$ is a realisation of this conditioned process, then the smallest eigenvalue of a $n \times n$ GUE random matrix has the same law as $\hat{B}_1(1)$, and Theorem 1 states that $R_n(1)$ and $\hat{B}_1(1)$ have the same law.

Similar connections between directed percolation random variables, such as $R_n(1)$, and random matrix or discrete orthogonal polynomial ensembles have also been observed in [19, 20]. See also [1, 13]. These are all related to the amazing fact, recently discovered and proved by Baik, Deift and Johansson [2], that the asymptotic distribution of the longest increasing subsequence in a random permutation is the same as the asymptotic distribution of the largest eigenvalue in a GUE random matrix, which had earlier been identified by Tracy and Widom [33].

Before stating our main result we will introduce some notation. Let $D_0(\mathbb{R}_+)$ denote the space of cadlag paths $f : \mathbb{R}_+ \to \mathbb{R}$ with $f(0) = 0$. For $f, g \in D_0(\mathbb{R}_+)$, define $f \otimes g \in D_0(\mathbb{R}_+)$ and

$f \odot g \in D_0(\mathbb{R}_+)$ by

$$(f \otimes g)(t) = \inf_{0 \le s \le t} [f(s) + g(t) - g(s)], \tag{4}$$

and

$$(f \odot g)(t) = \sup_{0 \le s \le t} [f(s) + g(t) - g(s)]. \tag{5}$$

Unless otherwise delineated by parentheses, the default order of operations is from left to right; for example, when we write $f \otimes g \otimes h$, we mean $(f \otimes g) \otimes h$. Define a mapping $\Gamma : D_0(\mathbb{R}_+)^2 \to D_0(\mathbb{R}_+)^2$ by

$$\Gamma(f, g) = (f \otimes g, g \odot f). \tag{6}$$

We now define a sequence of mappings $\Gamma_k : D_0(\mathbb{R}_+)^k \to D_0(\mathbb{R}_+)^k$ recursively, as follows. Set $\Gamma_2 = \Gamma$. For $k > 2$ and $f = (f_1, \ldots, f_k) \in D_0(\mathbb{R}_+)^k$, set

$$\Gamma_k(f_1, \ldots, f_k) = (f_1 \otimes f_2 \otimes \cdots \otimes f_k, \tag{7}$$
$$\Gamma_{k-1}(f_2 \odot f_1, f_3 \odot (f_1 \otimes f_2), \ldots, f_k \odot (f_1 \otimes \cdots \otimes f_{k-1}))). \tag{8}$$

Let $N^{(\mu)} = (N_1^{(\mu_1)}, \ldots, N_n^{(\mu_n)})$ be the counting functions of n independent Poisson processes on $\mathbb{R}_+$ with respective intensities $\mu_1 < \mu_2 < \cdots < \mu_n$. That is, $N_k^{(\mu_k)}(t)$ is the measure induced by the k^{th} Poisson process on the interval $(0, t]$, with the convention that $N_k^{(\mu_k)}(0) = 0$.

Theorem 2 *The conditional law of $N^{(\mu)}$, given that*

$$N_1^{(\mu_1)}(t) \le \cdots \le N_n^{(\mu_n)}(t), \quad \textit{for all } t \ge 0,$$

is the same as the unconditional law of $\Gamma_n(N^{(\mu)})$.

The proof of Theorem 2, presented in the next section, is based on some natural independence and reversibility properties of M/M/1 queues in series. At the heart of the proof is a generalisation of the celebrated theorem, due to Burke, which states that (in equilibrium) the output of a stable M/M/1 queue is Poisson.

In Section 3, we recover the analogue of Theorem 2 for independent Poisson processes of equal rates. (This is interesting in its own right: in [23] the conditioned process is shown to be closely connected with the Charlier ensemble.) In Section 4, by carefully applying Dönsker's theorem, we deduce that the n-dimensional process $\Gamma_n(B)$ has the same law as $\hat{B}$. To see that Theorem 1 follows, note that there is equality between the one-dimensional processes:

$$R_n = B_1 \otimes \cdots \otimes B_n = \Gamma_n(B)_1, \tag{9}$$

where $\Gamma_n(B)_1$ denotes the first component of the n-dimensional process $\Gamma_n(B)$.

In the case $n = 2$, the fact that $\Gamma_n(B)$ has the same law as $\hat{B}$ is essentially equivalent to Pitman's representation [29, 30] for the three-dimensional Bessel process; this connection is discussed in [28]. Note that, for $n = 2$ in the Poisson case, Theorem 2 (see also Theorem 5) yields the following discrete analogue of Pitman's theorem if X_t is a simple random walk with non-negative drift (in continuous or discrete time) and $M_t = \max_{0 \le s \le t} X_s$, then $2M - X$ has the same law as that of X conditioned to stay positive (in the case of a symmetric random walk, this conditioning is in the sense of Doob). This result was obtained in [29] for the symmetric random walk; Pitman's original proof for Brownian motion used Dönsker's theorem and this simple random walk result.

Finally, we mention that Bougerol and Jeulin [5] have recently found a proof of Theorem 1 by considering Brownian motion on symmetric spaces and applying a kind of Laplace method.

2 Proof of Theorem 2

We will first state and prove a generalisation of Burke's theorem [8], which states that the output of a stationary M/M/1 queue is Poisson. As was observed by Reich [31], there is an elementary proof of Burke's theorem using reversibility. We will state a slightly stronger result and prove it using essentially the same reversibility argument.

The stationary M/M/1 queue can be constructed as follows. Let A and S be independent Poisson processes on $\mathbb{R}$ with respective intensities $0 < \lambda < \mu$. For intervals I, open, half-open or closed, we will denote by $A(I)$ the measure of I with respect to dA; for $I = (0, t]$ we will simply write $A(t)$, with the convention that $A(0) = 0$. Similarly for S and any other point process we introduce. For $t \in \mathbb{R}$, set

$$Q(t) = \sup_{s \le t} [A(s, t] - S(s, t]]^+, \tag{10}$$

and for $s < t$,

$$D(s, t] = A(s, t] + Q(s) - Q(t). \tag{11}$$

In the language of queueing theory, A is the *arrivals* process, S is the *service* process, Q is the *queue-length* process, and D is the *departure* process. With this construction it is also natural (and indeed very important for what follows) to define the *unused service* process by

$$U(s, t] = S(s, t] - D(s, t]. \tag{12}$$

We will use the following notation for reversed processes. For a point process X, the reversed process $\bar{X}$ is defined by $\bar{X}(s, t) = X(-t, -s)$. The reversed queue-length process $\bar{Q}$ is defined to be the right-continuous modification of $\{Q(-t),\ t \in \mathbb{R}\}$.

Burke's theorem states that D is a homogeneous Poisson process with intensity λ. On a historical note, this fact was anticipated by O'Brien [27] and Morse [24], and proved in 1956 by Burke [8]. In 1957, Reich [31] gave the following very elegant proof which uses reversibility. The process Q is reversible (in fact, all birth and death processes are reversible). It follows that the joint law of A and D is the same as the joint law of $\bar{D}$ and $\bar{A}$. In particular, $\bar{D}$, and hence D, is a Poisson process with intensity λ.

Burke also proved that, for each t, $\{D(s, t],\ s \le t\}$ is independent of $Q(t)$. This property is now called *quasi-reversibility*. Note that it also follows from Reich's reversibility argument. Discussions on Burke's theorem and related material can be found in the books of Brémaud [6, 7], Kelly [21] and Robert [32].

For $s < t$, set

$$T(s, t] = A(s, t] + U(s, t] = S(s, t] - Q(s) + Q(t). \tag{13}$$

Theorem 3 *The processes D and T are independent Poisson processes with respective intensities λ and μ.*

Proof. First note that, given Q, U is a homogeneous Poisson process with intensity μ on the set $I = \{s \in \mathbb{R} :\ Q(s) = 0\}$, and if we let V be another Poisson process with intensity μ on the complement of I, which is conditionally independent of U given Q, then (unconditionally) $N = U + V$ is a homogeneous Poisson process with intensity μ on $\mathbb{R}$ which is independent of Q. Now, (A, S) can be written as a simple function of (Q, N), $(A, S) = \varphi(Q, N)$ say. By construction, we have $(\bar{D}, \bar{T}) = \varphi(\bar{Q}, \bar{N})$. Now we use the reversibility of Q and N to deduce that $(\bar{D}, \bar{T})$, and hence (D, T), has the same law as (A, S), as required. □

Some Remarks. The analogue of Theorem 3 holds for Brownian motions with drift—several proofs of this fact are given in [28]. (In fact, there is also a version given there which holds for exponential functionals of Brownian motion.) It is closely related to Pitman's representation for the three-dimensional Bessel process [29] and Williams' path-decomposition [35]. See also [17, 25], for related work.

Note that

$$Q(t) = \sup_{u>t}[D(t,u) - T(t,u)]. \tag{14}$$

We also have, on $\{Q(0)=0\}$,

$$\{(D(t), T(t)),\ t \geq 0\} = \{\Gamma(A,S)(t),\ t \geq 0\}. \tag{15}$$

Theorem 3 has the following multi-dimensional extension, which relates to a sequence of M/M/1 queues in tandem. Let $A, S_1, \ldots, S_n$ be independent Poisson processes with respective intensities $\lambda, \mu_1, \ldots, \mu_n$, and assume that $\lambda < \min_{i\leq n} \mu_i$. Set $D_0 = A$ and, for $k \geq 1$, $t \in \mathbb{R}$, set

$$Q_k(t) = \sup_{s\leq t}[D_{k-1}(s,t] - S_k(s,t]]^+, \tag{16}$$

and for $s < t$,

$$D_k(s,t] = D_{k-1}(s,t] + Q_k(s) - Q_k(t), \tag{17}$$

$$T_k(s,t] = S_k(s,t] - Q_k(s) + Q_k(t). \tag{18}$$

Theorem 4 *The processes $D_n, T_1, \ldots, T_n$ are independent Poisson processes with respective intensities $\lambda, \mu_1, \ldots, \mu_n$.*

Proof. By Theorem 3, D_1, T_1 and S_2 are independent Poisson processes with respective intensities λ, μ_1 and μ_2. Applying Theorem 3 again we see that D_2 and T_2 are independent Poisson processes with respective intensities λ and μ_2, and since D_2 and T_2 are determined by D_1 and S_2 they are independent of T_1. Thus D_2, T_1, T_2 and S_3 are independent Poisson processes with respective intensities λ, μ_1, μ_2 and μ_3. And so on. The condition $\lambda < \min_{i\leq n} \mu_i$ ensures that this procedure is well-defined. □

Remark. Again, the analogue of Theorem 4 can be shown to hold for Brownian motions with drifts, by exactly the same argument.

By repeated iteration of (16) and (17), we obtain (almost surely)

$$Q_1(0) + \cdots + Q_n(0) = \sup_{s\geq 0}[\bar{A}(s) - (\bar{S}_n \otimes \cdots \otimes \bar{S}_1)(s)]. \tag{19}$$

To see this, first recall that $\bar{A}(s) = A(-s, 0)$, and

$$Q_1(t-) = \sup_{s\leq t}[A(s,t) - S_1(s,t)]. \tag{20}$$

This yields (19) for $n = 1$. For $n = 2$, almost surely,

$$\begin{aligned} Q_1(0) + Q_2(0) &= Q_1(0-) + Q_2(0-) \\ &= Q_1(0) + \sup_{s\le 0}[D_1(s,0] - S_2(s,0]] \\ &= \sup_{s\le 0}[A(s,0] + Q_1(s) - S_2(s,0]] \\ &= \sup_{s\le 0}[A[s,0] + Q_1(s-) - S_2[s,0]] \\ &= \sup_{s'\le s\le 0}[A(s',0] - S_1(s',s) - S_2[s,0]] \\ &= \sup_{s'\ge 0}[\bar{A}(s') - (\bar{S}_2 \otimes \bar{S}_1)(s')]. \end{aligned}$$

And so on. In particular, $Q_1(0) + \cdots + Q_n(0)$ depends only on the restriction of $A, S_1, \ldots, S_n$ to $(-\infty, 0]$.
Iterating (17) we obtain, for each $k \le n$,

$$D_k(t) + Q_1(t) + \cdots + Q_k(t) = A(t) + Q_1(0) + \cdots + Q_k(0). \tag{21}$$

We also have, by (14),

$$Q_k(t) = \sup_{u>t}[D_k(t,u) - T_k(t,u)]. \tag{22}$$

Applying this repeatedly (as in the derivation of (19) above) we obtain

$$Q_1(0) + \cdots + Q_n(0) = \sup_{t>0}[D_n(t) - (T_1 \otimes \cdots \otimes T_n)(t)]. \tag{23}$$

Note that, on $\{Q_1(0) + \cdots + Q_n(0) = 0\}$,

$$D_n(t) = (A \otimes S_1 \otimes \cdots \otimes S_n)(t), \tag{24}$$

and

$$T_k(t) = (S_k \odot (A \otimes S_1 \otimes \cdots \otimes S_{k-1}))(t), \tag{25}$$

for $t \ge 0$, $k \le n$.
We will prove Theorem 2 by induction on n.
We first prove it for $n = 2$: By Theorem 3 and the formula (14), the conditional law of $\{(A(t), S(t)),\ t \ge 0\}$ given that $A(t) \le S(t)$ for all $t \ge 0$ is the same as the conditional law of $\{(D(t), T(t)),\ t \ge 0\}$ given that $Q(0) = 0$. But when $Q(0) = 0$, $(D(t), T(t)) = \Gamma(A, S)(t)$ for $t \ge 0$. Moreover, by (10) and the independence of increments of A and S, $\{\Gamma(A, S)(t),\ t \ge 0\}$ is independent of $Q(0)$. Therefore, the conditional law of $\{(A(t), S(t)),\ t \ge 0\}$ given that $A(t) \le S(t)$ for all $t \ge 0$ is the same as the unconditional law of $\{\Gamma(A, S)(t),\ t \ge 0\}$, as required.
Now we will assume that Theorem 3 is true as stated for a particular value of n, and moreover holds for any choice of $\mu_1 < \ldots < \mu_n$. In the above setting we have, by Theorem 4, that $D_n, T_1, \ldots, T_n$ are independent Poisson processes with respective intensities $\lambda, \mu_1, \ldots, \mu_n$. Assume that $\lambda < \mu_1 < \ldots < \mu_n$. By the induction hypothesis, the conditional law of

$$\{(D_n(t), T_1(t), \ldots, T_n(t)),\ t \ge 0\}, \tag{26}$$

given that $T_1(t) \le \cdots \le T_n(t)$ for all $t \ge 0$, is the same as the (unconditional) law of

$$\{(D_n(t), \Gamma_n(T_1, \ldots, T_n)(t)),\ t \ge 0\}; \tag{27}$$

therefore, the conditional law of

$$\{(D_n(t), T_1(t), \ldots, T_n(t)),\ t \geq 0\}, \tag{28}$$

given that $D_n(t) \leq T_1(t) \leq \cdots \leq T_n(t)$ for all $t \geq 0$, is the same as the conditional law of

$$\{(D_n(t), \Gamma_n(T_1, \ldots, T_n)(t)),\ t \geq 0\}, \tag{29}$$

given that $D_n(t) \leq (T_1 \otimes \cdots \otimes T_n)(t)$ for all $t \geq 0$. But, by (23), this is precisely the condition that $Q_1(0) + \cdots + Q_n(0) = 0$ or, equivalently, $Q_1(0) = \cdots = Q_n(0) = 0$, and in this case we have, by (24) and (25),

$$(D_n(t), \Gamma_n(T_1, \ldots, T_n)(t)) = \Gamma_{n+1}(A, S_1, \ldots, S_n)(t) \tag{30}$$

for $t \geq 0$; since this latter expression, by independence of increments, is independent of $Q_1(0) + \cdots + Q_n(0)$, we are done.

□

3 The case of equal rates

Let $N = (N_1, \ldots, N_n)$ be a collection of independent unit-rate Poisson processes, with $N(0) = (0, \ldots, 0)$. The function h given by (2) is a strictly positive harmonic function for the restriction of the transition kernel of N to the discrete Weyl chamber $E = W \cap \mathbb{Z}^n$ (this follows from a more general result presented in [23]). Let $\hat{N}$ be a realisation of the corresponding Doob h-transform of N, started at $x^* = (0, 1, \ldots, n-1) \in E$.
Apart from providing a convenient framework in which we can apply Dönsker's theorem and deduce the Brownian analogue of Theorem 2—this will be presented in the next section—the process $\hat{N}$ is interesting in its own right. In [23] it is shown (see the identity (40) below) that the random vector $\hat{N}(1)$ is distributed according to the *Charlier* ensemble, a discrete orthogonal polynomial ensemble. Thus, the next result, which follows from Theorem 2, yields a representation for the Charlier ensemble. For more on discrete orthogonal polynomial ensembles, see [20].

Theorem 5 *The processes $\hat{N} - x^*$ and $\Gamma_n(N)$ have the same law.*

Proof. Let $D(\mathbb{R}_+)$ denote the space of cadlag paths $f : \mathbb{R}_+ \to \mathbb{R}$, equipped with the Skorohod topology. Let $D(\mathbb{R}_+)^n$ be equipped with the corresponding product topology, and $M_1(D(\mathbb{R}_+)^n)$, the space of probability measures on $D(\mathbb{R}_+)^n$, with the corresponding weak topology. In this section, all weak convergence statements for processes will be with respect to this topology.
Note that we can restate Theorem 2 as follows. Let $x^* = (0, 1, \ldots, n-1)$. Theorem 2 states that the conditional law of $x^* + N^{(\mu)}$, given that $x^* + N^{(\mu)}(t) \in E$, for all $t \geq 0$, is the same as the unconditional law of $x^* + \Gamma_n(N^{(\mu)})$. It is easy to see that the operations $\otimes$ and $\odot$ are continuous (with respect to the Skorohod topology); it follows that Γ_n is continuous. The statement of Theorem 5 therefore follows from Lemma 6 below. □

Lemma 6 *As $W \ni \mu \to (1, \ldots, 1)$,*

$N^{(\mu)}$ converges in distribution to N

the conditional law of $x^* + N^{(\mu)}$, given that $x^* + N^{(\mu)}(t) \in E$, for all $t \geq 0$, converges to that of $\hat{N}$.

Proof. The first claim is easy to check (see, for example, [12, Exercise 7.6.1]). To prove the second claim, we need to introduce some notation. Denote by $\Pi^{(\mu)}$ the restriction of the transition kernel associated with $N^{(\mu)}$ to E, and let $T_x^{(\mu)}$ denote the first time the process $x + N^{(\mu)}$ exits E. The conditional law of $x^* + N^{(\mu)}$, given that $x^* + N^{(\mu)}(t) \in E$, for all $t \geq 0$, has transition kernel $\hat{\Pi}^{(\mu)}$ given by the Doob transform of $\Pi^{(\mu)}$ with the strictly positive harmonic function $h_\mu(x) = P(T_x^{(\mu)} = +\infty)$. In other words, for $x, y \in E$,

$$\hat{\Pi}_t^{(\mu)}(x,y) = \frac{h_\mu(y)}{h_\mu(x)} \Pi_t^{(\mu)}(x,y). \tag{31}$$

Now, if $\mu_n < \delta$ and $\mu_1 > 1/\delta$, for some fixed $\delta > 1$, there exist strictly positive functions k_δ and l_δ on E (independent of μ) such that

$$k_\delta(x) \leq \frac{h_\mu(x)}{h_\mu(x^*)} \leq l_\delta(x), \tag{32}$$

for all $x \in E$. To see this, note that the probability $h_\mu(x)$ is at least the probability of (the embedded discrete chain in) $N^{(\mu)}$ following a direct path in E to the point $(x_n - n+1, \ldots, x_n)$, times the probability $h_\mu(x_n - n+1, \ldots, x_n)$. But this latter probability (by translation invariance) equals $h_\mu(x^*)$. Thus,

$$\begin{aligned} h_\mu(x) &\geq \left(\frac{\mu_{n-1}}{\sum_i \mu_i}\right)^{x_n - x_{n-1} - 1} \cdots \left(\frac{\mu_1}{\sum_i \mu_i}\right)^{x_n - x_1 - n+1} h_\mu(x^*) \\ &\geq \left(\frac{1}{n\delta^2}\right)^{(n-1)x_n - \sum_{i=1}^{n-1} x_i - n(n-1)/2} h_\mu(x^*) \\ &=: k_\delta(x) h_\mu(x^*). \end{aligned}$$

This yields the lower bound in (32); the upper bound is obtained similarly.
Denote by Π the restriction of the transition kernel associated with N to E, and let T_x denote the first time the process $x+N$ exits E. It follows from (32) that, for any sequence $\mu(m)$ in W converging to $(1,1,\ldots,1)$, there exists a further subsequence $\mu(m)'$ such that $h_{\mu(m)'}/h_{\mu(m)'}(x^*)$ converges pointwise to some strictly positive function g on E, with $g(x^*) = 1$. Since $h_{\mu(m)'}$ is harmonic for $\Pi^{(\mu(m)')}$ for all m, $\Pi^{(\mu(m)')} \to \Pi$ pointwise and the processes have bounded jumps, we deduce that g is harmonic for Π.
Now, since $N^{(\mu(m)')}$ converges in law to N there exists an almost-sure realisation of this convergence, and since any fixed t is almost surely not a point of discontinuity of N, we will also simultaneously realise the convergence of $N^{(\mu(m)')}(t)$ and $\{T_{x^*}^{(\mu(m)')} > t\}$. Using this, the bound (32), and the easy fact that $El_\delta(N^{(\mu(m)')}(t))^2$ is uniformly bounded in m, we see that, for any bounded continuous function ϕ on $D(\mathbb{R}_+)^n$ such that $\phi(f)$ only depends on $\{f(s),\ s \leq t\}$, we have, as $m \to \infty$,

$$E\left[\frac{h_{\mu(m)'}(x^* + N^{(\mu(m)')}(t))}{h_{\mu(m)'}(x^*)} 1_{\{T_{x^*}^{(\mu(m)')} > t\}} \phi(N^{(\mu(m)')})\right] \to E\left[\frac{g(x^* + N(t))}{g(x^*)} 1_{\{T_{x^*} > t\}} \phi(N)\right]. \tag{33}$$

It follows that the conditional law of $x^* + N^{(\mu(m)')}$, given that $x^* + N^{(\mu(m)')}(t) \in E$, for all $t \geq 0$, converges to the Doob transform of N by the strictly positive harmonic function g on E, started at x^*; let N^g be a realisation of this Doob transform. Now we apply Theorem 2, from which it follows that $\Gamma_n(N)$ has the same law as N^g. In particular, the limiting function g must be the same for any choice of sequence $\mu(m)$. It remains to show that $g = h/h(x^*)$. The Martin boundary associated with Π is analysed in [23]. If $k(x, y)$ is the Martin kernel associated with Π, and $y \to \infty$ in such a way that $y/\sum_i y_i \to (1/n, \ldots, 1/n)$, then $k(x, y) \to h(x)/h(x^*)$. Thus, by standard Doob-Hunt theory (see, for example, [10, 34]) if we can show that, with probability one, $N^g(t)/t \to (1, \ldots, 1)$ as $t \to \infty$, we are done. Since $\Gamma_n(N)$ has the same law as N^g, we need only check that

$$\Gamma_n(N)(t)/t = \Gamma_n(N(\cdot t)/t)(1) \to (1, \ldots, 1).$$

But this follows from the continuity of Γ_n and the fact that, as $t \to \infty$, the function $s \mapsto N(st)/t$ converges almost surely in $D(\mathbb{R}_+)^n$ to the function $g(s) = (s, \ldots, s)$, for which $\Gamma_n(g)(1) = (1, \ldots, 1)$.

□

4 The corresponding result for Brownian motion

In this section we recover the analogous result for Brownian motion. For $x \in \mathbb{R}^n$, let $\mathbb{P}_x$ denote the law of B started at x, and, for $x \in W$, let $\hat{\mathbb{P}}_x$ denote the law of the h-transform of B started at x, where h is given by (2). The laws $\hat{\mathbb{P}}_x$ and $\mathbb{P}_x$ are related as follows. If T denotes the first exit time of B from W, and $\mathcal{F}_t$ the natural filtration of B, then for $A \in \mathcal{F}_t$,

$$\hat{\mathbb{P}}_x(A) = \mathbb{P}_x\left(\frac{h(B_t)}{h(x)} 1_{T>t} A\right). \tag{34}$$

The point $(0, \ldots, 0)$ is an entrance point for $\hat{\mathbb{P}}$; we denote the corresponding law by $\hat{\mathbb{P}}_{0+}$. The law $\hat{\mathbb{P}}_{0+}$ is defined, for $A \in \mathcal{T}_t = \sigma(B_u,\ u \geq t)$, $t > 0$, by

$$\hat{\mathbb{P}}_{0+}(A) = \mathbb{P}_0\left[C_t h(B_t)^2 \hat{\mathbb{P}}_{B_t}(\theta_t A)\right], \tag{35}$$

where θ is the shift operator (so that $\theta_t A \in \mathcal{T}_0$) and

$$C_t = \left[t^{n(n-1)/2} \prod_{j=1}^{n-1} j!\right]^{-1} \tag{36}$$

is a normalisation constant. To see that this makes sense, we recall the following well-known connection between $\hat{\mathbb{P}}$ and the GUE ensemble, as remarked upon in the introduction:

$$\lim_{W \ni x \to 0} \hat{\mathbb{P}}_x(X_t \in dy) = C_t h(y)^2 \mathbb{P}_0(X_t \in dy). \tag{37}$$

(See, for example, [22].) Let $\hat{B}$ be a realisation of $\hat{\mathbb{P}}_{0+}$.

Theorem 7 *The processes $\hat{B}$ and $\Gamma_n(B)$ have the same law.*

Proof. We will use Dönsker's theorem. It is convenient to switch topologies: we now equip $D(\mathbb{R}_+)$ with the topology of uniform convergence on compacts, $D(\mathbb{R}_+)^n$ with the corresponding product topology, and $M_1(D(\mathbb{R}_+)^n)$ with the corresponding weak topology. In this section, all weak convergence statements for processes will be with respect to this topology. Note that the mapping Γ_n is still continuous in this setting.
In the the context of the previous section, for $m \in \mathbb{Z}$, set $X_m(t) = [N(mt) - mt]/\sqrt{m}$ and $\hat{X}_m(t) = [\hat{N}(mt) - mt]/\sqrt{m}$. The theorem will be proved if we can show that X_m converges in law to $\hat{B}$. It is convenient to introduce general initial positions for the Markov processes X_m and $\hat{X}_m$. Denote by $\mathbb{P}_x^{(m)}$ (respectively $\hat{\mathbb{P}}_x^{(m)}$) the law of X_m (respectively $\hat{X}_m$) started at $x \in E/\sqrt{m}$. Note that, by scaling properties of h, $\hat{\mathbb{P}}_x^{(m)}$ is the Doob h-transform of $\mathbb{P}_x^{(m)}$. In this notation, all we need to show is that

$$\hat{\mathbb{P}}^{(m)}_{x^*/\sqrt{m}} \to \hat{\mathbb{P}}_{0+}. \tag{38}$$

By an appropriate version of Dönsker's theorem (see, for example, [12, Section 7.5]), if $x_m \to x$ in W, then $\mathbb{P}^{(m)}_{x_m} \to \mathbb{P}_x$. Using the easy fact that $\mathbb{P}_{x^*/\sqrt{m}} h(X_m(t))^2$ is uniformly bounded in m, we deduce that for $x_m \to x$ in W, we have

$$\hat{\mathbb{P}}^{(m)}_{x_m} \to \hat{\mathbb{P}}_x. \tag{39}$$

To deduce (38), we use the formula (see [23]):

$$P(\hat{N}(t) = y) = C_t h(y)^2 P(x^* + N(t) = y), \tag{40}$$

where C_t is the same normalisation constant as in the Brownian case, given by (36). Since $m^{n(n-1)/2} C_{mt} = C_t$, this translates as: for $y \in E/\sqrt{m}$,

$$\hat{\mathbb{P}}^{(m)}_{x^*/\sqrt{m}}(X_m(t) = y) = C_t h(y)^2 \mathbb{P}^{(m)}_{x^*/\sqrt{m}}(X_m(t) = y), \tag{41}$$

Thus, we need to show that, for any bounded continuous function $\phi : D(\mathbb{R}_+)^n \to \mathbb{R}$,

$$\mathbb{P}^{(m)}_{x^*/\sqrt{m}} \left(C_t h(X_m(t))^2 \hat{\mathbb{P}}^{(m)}_{X_m(t)}(\phi(X_m)) \right) \to \mathbb{P}_0 \left(C_t h(B(t))^2 \hat{\mathbb{P}}_{B(t)}(\phi(B)) \right). \tag{42}$$

To do this, we simply take an almost-sure realisation of the convergence $\mathbb{P}^{(m)}_{x^*/\sqrt{m}} \to \mathbb{P}_0$, appeal to (39), and use the easy fact that $\mathbb{P}_{x^*/\sqrt{m}} h(X_m(t))^4$ is uniformly bounded in m. □
Acknowledgements. We would like to thank the anonymous referee for careful reading and helpful comments.

References

[1] J. Baik. Random vicious walks and random matrices. *Comm. Pure Appl. Math.* 53 (2000) 1385–1410.

[2] J. Baik, P. Deift and K. Johansson. On the distribution of the length of the longest increasing subsequence of random permutations. *J. Amer. Math. Soc.* 12 (1999), no. 4, 1119–1178.

[3] Yu. Baryshnikov. GUES and QUEUES. *Probab. Theor. Rel. Fields* 119 (2001) 256–274.

[4] Ph. Biane. Quelques propriétés du mouvement brownien dans un cone. *Stoch. Proc. Appl.* 53 (1994), no. 2, 233–240.

[5] Ph. Bougerol and Th. Jeulin. Paths in Weyl chambers and random matrices. Prépublication No. 703 du Laboratoire de Probabilités et Modèles Aléatoires, Paris 6.

[6] P. Brémaud. *Point Processes and Queues: Martingale Dynamics.* Springer-Verlag, Berlin, 1981.

[7] P. Brémaud. *Markov Chains, Gibbs Fields, Monte-Carlo Simulation, and Queues.* Texts in App. Maths., vol. 31. Springer, 1999.

[8] P.J. Burke. The output of a queueing system. *Operations Research* 4 (1956), no. 6, 699–704.

[9] E. Cépa and D. Lépingle. Diffusing particles with electrostatic repulsion. *Probab. Th. Rel. Fields* 107 (1997), no. 4, 429–449.

[10] J.L. Doob. *Classical Potential Theory and its Probabilistic Counterpart.* Springer, 1984.

[11] F.J. Dyson. A Brownian-motion model for the eigenvalues of a random matrix. *J. Math. Phys.* 3 (1962) 1191–1198.

[12] S.N. Ethier and T.G. Kurtz. *Markov Processes: Characterization and Convergence.* Wiley, New York, 1986.

[13] P.J. Forrester. Random walks and random permutations. Preprint, 1999. (XXX: math.CO/9907037)

[14] P.W. Glynn and W. Whitt. Departures from many queues in series. *Ann. Appl. Prob.* 1 (1991), no. 4, 546–572.

[15] D. Grabiner. Brownian motion in a Weyl chamber, non-colliding particles, and random matrices. *Ann. IHP* 35 (1999), no. 2, 177–204.

[16] J. Gravner, C.A. Tracy and H. Widom. Limit theorems for height fluctuations in a class of discrete space and time growth models. *J. Stat. Phys.* 102 (2001), nos. 5-6, 1085–1132.

[17] J. M. Harrison and R.J. Williams. On the quasireversibility of a multiclass Brownian service station. *Ann. Probab.* 18 (1990) 1249–1268.

[18] D. Hobson and W. Werner. Non-colliding Brownian motion on the circle, *Bull. Math. Soc.* 28 (1996) 643–650.

[19] K. Johansson. Shape fluctuations and random matrices. *Commun. Math. Phys.* 209 (2000) 437–476.

[20] K. Johansson. Discrete orthogonal polynomial ensembles and the Plancherel measure, Preprint 1999, to appear in *Ann. Math.*. (XXX: math.CO/9906120)

[21] F.P. Kelly. *Reversibility and Stochastic Networks.* Wiley, 1979.

[22] Wolfgang König and Neil O'Connell. Eigenvalues of the Laguerre process as non-colliding squared Bessel processes. Elect. Comm. in Probab. 6 (2001) 107–114.

[23] Wolfgang König, Neil O'Connell and Sebastien Roch. Non-colliding random walks, tandem queues and the discrete ensembles. Elect. J. Probab., to appear.

[24] P.M. Morse. Stochastic properties of waiting lines. *Operations Research* 3 (1955) 256.

[25] H. Matsumoto and M. Yor. A version of Pitman's $2M-X$ theorem for geometric Brownian motions. *C.R. Acad. Sci. Paris* 328 (1999), Série I, 1067–1074.

[26] M.L. Mehta. *Random Matrices: Second Edition.* Academic Press, 1991.

[27] G.G. O'Brien. Some queueing problems. *J. Soc. Indust. Appl. Math.* 2 (1954) 134.

[28] Neil O'Connell and Marc Yor. Brownian analogues of Burke's theorem. *Stoch. Proc. Appl.*, 96 (2) (2001) 285–304.

[29] J. W. Pitman. One-dimensional Brownian motion and the three-dimensional Bessel process. *Adv. Appl. Probab.* 7 (1975) 511–526.

[30] J. W. Pitman and L.C.G. Rogers. Markov functions. *Ann. Probab.* 9 (1981) 573–582.

[31] E. Reich. Waiting times when queues are in tandem. *Ann. Math. Statist.* 28 (1957) 768–773.

[32] Ph. Robert. *Réseaux et files d'attente: méthodes probabilistes.* Math. et Applications, vol. 35. Springer, 2000.

[33] C.A. Tracy and H. Widom. Fredholm determinants, differential equations and matrix models. Comm. Math. Phys. 163 (1994), no. 1, 33–72.

[34] David Williams. *Diffusions, Markov Processes and Martingales. Volume 1: Foundations.* Wiley, 1979.

[35] David Williams. Path decomposition and continuity of local time for one-dimensional diffusions I. *Proc. London Math. Soc.* 28 (1974), no. 3, 738–768.