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CORRELATION MEASURES

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Abstract:
We study a class of Borel probability measures, called correlation measures. Our results are of two types: first, we give explicit constructions of non-trivial correlation measures; second, we examine some of the properties of the set of correlation measures. In particular, we show that this set of measures has a convexity property. Our work is related to the so-called Gaussian correlation conjecture.

1 Introduction

In this article, we study a class of Borel probability measures on $\mathbb{R}^d$, which we call correlation measures. Our work is related to the so-called Gaussian correlation conjecture; to place our results in context, we will review this important conjecture.
Given $x, y \in \mathbb{R}^d$, let (x, y) and $\|x\|$ denote the canonical inner product and norm on $\mathbb{R}^d$, respectively. As is customary, given $A, B \subset \mathbb{R}^d$ and $t \in \mathbb{R}$, we will write $tA = \{ta : a \in A\}$ and $A + B = \{a + b : a \in A,\, b \in B\}$; the set A is said to be *symmetric* provided that $-A = A$ and *convex* provided that $tA + (1-t)A \subset A$ for each $t \in [0,1]$. Let $\mathcal{C}_d$ denote the set of all closed,

convex, symmetric subsets of $\mathbb{R}^d$, and let γ_d be the standard Gaussian measure on $\mathbb{R}^d$, that is,

$$\gamma_d(A) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_A \exp\left(-\|x\|^2/2\right) dx.$$

The *Gaussian correlation conjecture* states that

$$\gamma_d(A \cap B) \geq \gamma_d(A)\gamma_d(B) \tag{1.1}$$

for each pair of sets $A, B \in \mathcal{C}_d$, $d \geq 1$. For $d = 1$, this conjecture is trivially true, and Pitt [5] has shown that it is true for $d = 2$. For $d \geq 3$, the conjecture remains unsettled, but a variety of partial results are known. Borell [1] establishes (1.1) for sets A and B in a certain class of (not necessarily convex) sets in $\mathbb{R}^d$, which for $d = 2$ includes all symmetric, convex sets. The conjecture can be reformulated as follows: if $(X_1, \cdots, X_n)$ is a centered, Gaussian random vector, then

$$P\left(\max_{1\leq i\leq n} |X_i| \leq 1\right) \geq P\left(\max_{1\leq i\leq k} |X_i| \leq 1\right) P\left(\max_{k+1\leq i\leq n} |X_i| \leq 1\right) \tag{1.2}$$

for each $1 \leq k < n$. Khatri [4] and Šidák [7, 8] have shown that (1.2) is true for $k = 1$. In part, the paper of Das Gupta, Eaton, Olkin, Perlman, Savage, and Sobel [2] generalizes the results of Khatri and Šidák for elliptically contoured distributions.

The recent paper of Schechtman, Schlumprecht and Zinn [6] sheds new light on the Gaussian correlation conjecture. Their results are of two types: first, they show that the conjecture is true whenever the sets satisfy additional geometric restrictions (additional symmetry, centered ellipsoids); second, they show that the conjecture is true provided that the sets are not too large.

Here is the central question of this article: to what extent is the correlation inequality (1.1) a Gaussian result? In other words, are there any non-trivial probability measures on $\mathbb{R}^d$ satisfying (1.1)? We answer the question in the affirmative.

We will call a Borel probability measure λ on $\mathbb{R}^d$ a *correlation measure* provided that

$$\lambda(A \cap B) \geq \lambda(A)\lambda(B)$$

for each pair of sets $A, B \in \mathcal{C}_d$; we will denote the set of all correlation measures on $\mathbb{R}^d$ by $\mathcal{M}_d$. In section 2 we give sufficient conditions for membership in $\mathcal{M}_d$ and show that $\mathcal{M}_d$ contains non-trivial elements for each $d \geq 2$. In section 3, we examine some properties of correlation measures. In particular, we show that non-trivial correlation measures have unbounded support, and that $\mathcal{M}_d$ has a certain convexity property. Using this convexity property, we construct an element of $\mathcal{M}_2$ based on a model introduced by Kesten and Spitzer [3]. Our results can thus be roughly summarized as:

Measures	**Correlation property**
bounded support	no (except in dimension 1)
exponential tail (including Gaussian)	unknown
heavy tail	some examples known

The correlation measures that we construct in section 2 are heavy-tailed, with the measure of the complement of the ball of radius r decaying only as a power of r. As our result of section

3 demonstrates, the measure of the complement of the ball of radius r must be positive for each $r \geq 0$. Thus it is natural to ask whether there is a minimal rate with which the measure of the complement of the ball of radius r approaches 0. Perhaps the Gaussian measures lie close to, or on, the "boundary" of $\mathcal{M}_d$, which may account for the difficulty of the Gaussian correlation conjecture.

2 The construction of correlation measures

For $d \geq 2$, let $B[0,1]$ denote the closed unit ball of $\mathbb{R}^d$; for $r \geq 0$, let $B[0,r] = rB[0,1]$. Throughout this section, μ will denote a spherically-symmetric, Borel probability measure on $\mathbb{R}^d$. For $r \geq 0$, let

$$F(r) = \mu\left(B[0,r]\right).$$

The main result of this section is Theorem 2.2, which gives sufficient conditions on F for μ to be a correlation measure; through this result, we produce explicit, nontrivial correlation measures.

The proof of Theorem 2.2 rests on a geometric fact, which we describe presently. Let S^{d-1} denote the unit sphere of $\mathbb{R}^d$. A subset S of $\mathbb{R}^d$ is called a *symmetric slab* if there exists a number $h \in [0,+\infty]$ and a $v \in S^{d-1}$ such that

$$S = \left\{x \in \mathbb{R}^d : |(v,x)| \leq h\right\}$$

The number $h = h(S)$ is called the *half-width* of S; when $h = 0$, S is a hyperplane of dimension $d-1$. Let $\mathcal{S}_d$ denote the set of all symmetric slabs in $\mathbb{R}^d$, and, for $A \in \mathcal{C}_d$, let

$$\rho(A) = \sup\{r \geq 0 : B[0,r] \subset A\}$$
$$h(A) = \inf\{h(S) : S \in \mathcal{S}_d,\ S \supset A\}$$

It is immediate that $\rho(A) \leq h(A)$; in fact, since A is convex and symmetric, $\rho(A) = h(A)$. Since A is closed, $A \supset B[0,\rho(A)]$; since S^{d-1} is compact, there exists a symmetric slab of half-width $h(A)$ containing A. We can summarize these findings as follows:

Lemma 2.1 *For each $A \in \mathcal{C}_d$, there exists a symmetric slab S of half-width $\rho(A)$ such that $B[0,\rho(A)] \subset A \subset S$.*

Let σ be uniform surface measure on S^{d-1}, normalized so that $\sigma(S^{d-1}) = 1$. Since μ is spherically symmetric, we can represent μ in polar form: for any Borel subset A of $\mathbb{R}^d$,

$$\mu(A) = \int_0^\infty \sigma(t^{-1}A \cap S^{d-1})dF(t). \tag{2.3}$$

For $0 \leq t \leq 1$, let

$$g_d(t) = \sigma\{x \in S^{d-1} : |x_1| \leq t\}.$$

This special function may be expressed as

$$g_d(t) = K_d \int_0^t (1-s^2)^{(d-3)/2} ds,$$

where

$$K_d = 2\pi^{-1/2}\left(\frac{\Gamma(d/2)}{\Gamma((d-1)/2)}\right).$$

Let S be a symmetric slab of finite half-width h, and let $p \geq h$ ($p > 0$). Then, by symmetry and scaling,

$$\sigma(p^{-1}S \cap S^{d-1}) = \sigma\{x \in S^{d-1} : |x_1| \leq h/p\} = g_d(h/p). \tag{2.4}$$

Here is the main result of this section.

Theorem 2.2 *If $F(a) > 0$ for $a > 0$ and*

$$F(b) + \int_b^\infty \left[g_d\left(\frac{b}{t}\right) + \frac{1}{F(a)}g_d\left(\frac{a}{t}\right)\right] dF(t) \leq 1 \tag{2.5}$$

for each pair of real numbers a and b with $0 < a \leq b < +\infty$, then $\mu \in \mathcal{M}_d$.

Proof Let $A, B \in \mathcal{C}_d$ and let $a = \rho(A)$ and $b = \rho(B)$. We will assume, without loss of generality, that $a \leq b$.
We need to treat the cases $a = 0$ and $b = +\infty$ separately. If $a = 0$, then, by Lemma 2.1, A is contained within a symmetric slab S of half-width 0. By (2.3) and (2.4), $\mu(A) \leq \mu(S) = 0$; thus, $\mu(A \cap B) \geq \mu(A)\mu(B)$. If $b = +\infty$, then $B = \mathbb{R}^d$ and, once again, $\mu(A \cap B) \geq \mu(A)\mu(B)$. Hereafter let $0 < a \leq b < +\infty$. By Lemma 2.1, let S_1 be a symmetric slab of half-width b, satisfying $B[0, b] \subset B \subset S_1$. Then, by (2.3) and (2.4),

$$\mu(B) \leq \mu(B[0, b]) + \mu(S_1 \cap B[0, b]^c) \leq F(b) + \int_b^\infty g_d\left(\frac{b}{t}\right) dF(t). \tag{2.6}$$

By Lemma 2.1, let S_2 be a symmetric slab of half-width a, satisfying $B[0, a] \subset A \subset S_2$. Then, by (2.3) and (2.4),

$$\begin{aligned}\mu(A) &= \mu(A \cap B[0, b]) + \mu(A \cap B[0, b]^c)\\ &\leq \mu(A \cap B) + \mu(S_2 \cap B[0, b]^c)\\ &= \mu(A \cap B) + \int_b^\infty g_d\left(\frac{a}{t}\right) dF(t).\end{aligned}$$

Since $0 < F(a) \leq \mu(A)$,

$$\frac{\mu(A \cap B)}{\mu(A)} \geq 1 - \frac{1}{F(a)} \int_b^\infty g_d\left(\frac{a}{t}\right) dF(t). \tag{2.7}$$

Combining (2.6) and (2.7),

$$\begin{aligned}&\frac{\mu(A \cap B)}{\mu(A)} - \mu(B)\\ &\qquad \geq 1 - F(b) - \int_b^\infty \left[g_d\left(\frac{b}{t}\right) + \frac{1}{F(a)}g_d\left(\frac{a}{t}\right)\right] dF(t),\end{aligned}$$

which, according to (2.5), is nonnegative. As such, $\mu(A \cap B) \geq \mu(A)\mu(B)$, as was to be shown. $\square$

A simpler form of this result can be obtained by strengthening the conditions on F. Let $L_2 = 1$ and, for $d \geq 3$, let $L_d = K_d$. With this convention,

$$g_d(t) \leq L_d t \tag{2.8}$$

for $d \geq 2$ and $t \in [0, 1]$.

Corollary 2.3 *If F is concave and*

$$F(b) + L_d b \left(1 + \frac{1}{F(b)}\right) \int_b^\infty t^{-1} dF(t) \leq 1 \tag{2.9}$$

for each $b \in (0, \infty)$, then $\mu \in \mathcal{M}_d$.

Proof We will show that the conditions of Theorem 2.2 are satisfied. Since F is concave,

$$\frac{F(a)}{a} \geq \frac{F(b)}{b} \tag{2.10}$$

for $0 < a \leq b$. Since F is ultimately positive, this shows that $F(a) > 0$ for $a > 0$.
Let $0 < a \leq b < \infty$. Then

$$\begin{aligned} F(b) &+ \int_b^\infty \left[g_d\left(\frac{b}{t}\right) + \frac{1}{F(a)} g_d\left(\frac{a}{t}\right)\right] dF(t) \\ &\leq F(b) + L_d \left(b + \frac{a}{F(a)}\right) \int_b^\infty t^{-1} dF(t) && \text{(by (2.8))} \\ &\leq F(b) + L_d b \left(1 + \frac{1}{F(b)}\right) \int_b^\infty t^{-1} dF(t), && \text{(by (2.10))} \end{aligned}$$

which shows that (2.9) implies (2.5). □

Our next result uses Corollary 2.3 to demonstrate the existence of non-trivial correlation measures in each dimension $d \geq 2$.

Theorem 2.4 *For each $L \geq 1$, there exists a differentiable, concave, increasing function $F : [0, \infty) \to [0, 1]$ satisfying*

$$F(r) + Lr \left(1 + \frac{1}{F(r)}\right) \int_r^\infty \frac{F'(t)}{t} \, dt \leq 1 \tag{2.11}$$

for each $r \in (0, \infty)$.

Proof Let

$$F(r) = \begin{cases} \frac{1}{2} r^{1/4L}, & \text{for } r \leq 1; \\ 1 - \frac{1}{2} r^{-1/4L}, & \text{for } r \geq 1. \end{cases}$$

This makes F differentiable, concave, and increasing on $[0,\infty)$. For $r \geq 1$, the left-hand side of (2.11) is

$$\begin{aligned}
1 - \frac{1}{2} r^{-1/4L} + Lr\left(\frac{4 - r^{-1/4L}}{2 - r^{-1/4L}}\right)\frac{1}{8L}\int_r^\infty t^{-2-1/4L}\,dt \\
\leq 1 - \frac{1}{2} r^{-1/4L} + 4r\frac{1}{8}\int_r^\infty t^{-2-1/4L}\,dt \\
= 1 - \frac{1}{2}\left(\frac{1}{4L+1}\right) r^{-1/4L} \\
\leq 1.
\end{aligned}$$

For $r \leq 1$, the left-hand side of (2.11) is

$$\begin{aligned}
&\frac{1}{2} r^{1/4L} + Lr\left(1 + 2r^{-1/4L}\right)\left\{\int_r^1 \frac{1}{8L} t^{-2+1/4L}\,dt + \int_1^\infty \frac{1}{8L} t^{-2-1/4L}\,dt\right\} \\
&= \frac{1}{2} r^{1/4L} + Lr\left(1 + 2r^{-1/4L}\right)\left\{\frac{1}{2(4L-1)}\left(r^{-1+1/4L} - 1\right) + \frac{1}{2(4L+1)}\right\} \\
&\leq \frac{1}{2} r^{1/4L} + Lr\left(1 + 2r^{-1/4L}\right)\frac{1}{2(4L-1)} r^{-1+1/4L} \\
&= \frac{1}{2} r^{1/4L} + \left(\frac{L}{4L-1}\right)\left(\frac{1}{2} r^{1/4L} + 1\right) \\
&\leq \frac{1}{2} + \frac{1}{3}\left(\frac{1}{2} + 1\right) = 1,
\end{aligned}$$

as was to be shown. □

When $L = 1$, another solution to (2.11) is given by $F(r) = (r/(1+r))^{1/2}$, for which the inequality (2.11) becomes an equality. This function F is thus the best possible solution to (2.11) in that sense.

3 Some properties of correlation measures

Let μ denote a Borel probability measure on $\mathbb{R}^d$. As is customary, let the *support of* μ (denoted by $\operatorname{supp}(\mu)$) be the intersection of the closed subsets of $\mathbb{R}^d$ having full measure.

Theorem 3.1 *If μ has compact support and* $\dim(\operatorname{supp}(\mu)) > 1$, *then* $\mu \notin \mathcal{M}_d$.

In other words, unless a correlation measure is supported on a one-dimensional subspace, it must have unbounded support.

Proof Let $x_0 \in \operatorname{supp}(\mu)$ have maximal distance from 0. Without loss of generality we may assume that $x_0 = e_1 = (1, 0, \ldots, 0)$. For $\epsilon \in (0, 1)$, let

$$\begin{aligned}
A_\epsilon &= \{x \in \mathbb{R}^d : x_2^2 + \cdots + x_d^2 \leq \epsilon^2\} \\
B_\epsilon &= \{x \in \mathbb{R}^d : |x_1| \leq \sqrt{1-\epsilon^2}\}
\end{aligned}$$

Observe that $A_\epsilon \cup B_\epsilon \supset B[0,1] \supset \operatorname{supp}(\mu)$; thus, $\mu(A_\epsilon^c \cap B_\epsilon^c) = 0$. Since $\dim(\operatorname{supp}(\mu)) > 1$, we can choose $\epsilon > 0$ such that $\mu(A_\epsilon^c \cap B_\epsilon) = \mu(A_\epsilon^c) > 0$. Since $e_1 \in B_\epsilon^c$, $\mu(A_\epsilon \cap B_\epsilon^c) = \mu(B_\epsilon^c) > 0$. Finally,

$$\begin{aligned}&\mu(A_\epsilon \cap B_\epsilon) - \mu(A_\epsilon)\mu(B_\epsilon)\\ &\qquad = \mu(A_\epsilon \cap B_\epsilon)\mu(A_\epsilon^c \cap B_\epsilon^c) - \mu(A_\epsilon \cap B_\epsilon^c)\mu(A_\epsilon^c \cap B_\epsilon) < 0,\end{aligned}$$

which shows that $\mu \notin \mathcal{M}_d$. □

Our next result shows that $\mathcal{M}_d$ remains closed under certain convex combinations. Let μ and λ be Borel probability measures on $\mathbb{R}^d$. We will say that μ *dominates* λ (written $\mu \succ \lambda$) provided that $\mu(A) \geq \lambda(A)$ for each $A \in \mathcal{C}_d$.

Theorem 3.2 *Let $\mu, \lambda \in M_d$ with $\mu \succ \lambda$, and let a, b be nonnegative real numbers with $a + b = 1$. Then $a\mu + b\lambda \in \mathcal{M}_d$.*

Proof Let $m = a\mu + b\lambda$, and let $A, B \in \mathcal{C}_d$. Then

$$m(A)m(B) = a^2\mu(A)\mu(B) + ab\mu(A)\lambda(B) + ab\mu(B)\lambda(A) + b^2\lambda(A)\lambda(B).$$

Since $a + b = 1$ and μ and λ are correlation measures,

$$\begin{aligned}m(A \cap B) &= (a+b)m(A \cap B)\\ &= a^2\mu(A \cap B) + ab\mu(A \cap B) + ab\lambda(A \cap B) + b^2\lambda(A \cap B)\\ &\geq a^2\mu(A)\mu(B) + ab\mu(A)\mu(B) + ab\lambda(A)\lambda(B) + b^2\lambda(A)\lambda(B).\end{aligned}$$

Recalling that $\mu \succ \lambda$, we have

$$\begin{aligned}&m(A \cap B) - m(A)m(B)\\ &\qquad \geq ab\left(\mu(A)\mu(B) + \lambda(A)\lambda(B) - \mu(A)\lambda(B) - \mu(B)\lambda(A)\right)\\ &\qquad = ab\left(\mu(A) - \lambda(A)\right)\left(\mu(B) - \lambda(B)\right) \geq 0,\end{aligned}$$

which shows that $m \in \mathcal{M}_d$, completing our proof. □

In general, a linear combination of correlation measures need not be a correlation measure. For example, let μ and λ be the centered Gaussian measures on $\mathbb{R}^2$ with covariance matrices

$$Q_\mu = \begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix} \qquad \text{and} \qquad Q_\lambda = \begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix},$$

respectively. By the theorem of Pitt [5], μ and λ are correlation measures; however, the measure $m = (\mu + \lambda)/2$ is not a correlation measure. To see this, let

$$\begin{aligned}A &= \{(x_1, x_2) \in \mathbb{R}^2 : |x_1| \leq 1\}\\ B &= \{(x_1, x_2) \in \mathbb{R}^2 : |x_2| \leq 1\}.\end{aligned}$$

Then, by a calculation as in the proof of Theorem 3.2, $m(A \cap B) - m(A)m(B) < 0$, which shows that $m \notin \mathcal{M}_2$.
Theorem 3.2 be extended by induction:

Corollary 3.3 *Let* $\{\mu_i : 1 \leq i \leq n\} \subset \mathcal{M}_d$ *with* $\mu_1 \succ \mu_2 \succ \cdots \succ \mu_{n-1} \succ \mu_n$*, and let* $\{a_i : 1 \leq i \leq n\}$ *be a set of nonnegative real numbers with* $\sum_{i=1}^n a_i = 1$*. Then* $\sum_{i=1}^n a_i \mu_i \in \mathcal{M}_d$*.*

Dominating measures can be constructed through scaling. Given $\mu \in \mathcal{M}_d$ and $s > 0$, let $\mu_s(A) = \mu(sA)$ for each Borel subset of $\mathbb{R}^d$. If $r \geq s$, then $rA \supset sA$ for each $A \in \mathcal{C}_d$; thus, $\mu_r \succ \mu_s$. We will use this notion of domination through scaling in conjunction with Corollary 3.3 to construct elements of $\mathcal{M}_2$.
Let $\{S_n : n \geq 0\}$ ($S_0 = 0$) be simple random walk on $\mathbb{Z}$, and let $\{Y(k) : k \in \mathbb{Z}\}$ be a sequence of independent and identically distributed, two-dimensional, standard Gaussian random vectors. We will assume that the random walk and the Gaussian vectors are defined on a common probability space and generate independent independent σ-algebras. For $n \geq 0$, let

$$Z_n = \sum_{k=0}^{n} Y(S_k).$$

The process $\{Z_n : n \geq 0\}$, called *random walk in random scenery*, was introduced by Kesten and Spitzer [3], who investigated its weak limits.

Theorem 3.4 *For each* $n \geq 0$*, the law of* Z_n *is an element of* $\mathcal{M}_2$*.*

Proof For $n \geq 0$, let ζ_n denote the law of Z_n. For $j \in \mathbb{Z}$ and $n \geq 0$, let

$$\ell_n^j = \sum_{k=0}^{n} I(S_k = j)$$

and observe that $Z_n = \sum_{j \in \mathbb{Z}} \ell_n^j Y(j)$. For $n \geq 0$, let

$$V_n = \sum_{j \in \mathbb{Z}} \left(\ell_n^j\right)^2.$$

The process $\{V_n : n \geq 0\}$ is called the *self-intersection local time* of the random walk. Conditional on the σ-field generated by the random walk, Z_n is a Gaussian random vector with covariance matrix V_n times the identity matrix. Thus, for each Borel set $A \in \mathbb{R}^2$,

$$\begin{aligned}\zeta_n(A) &= \sum_{k=0}^{\infty} P(Z_n \in A \,|\, V_n = k) P(V_n = k) \\ &= \sum_{k=0}^{\infty} \gamma_2(k^{-1/2} A) P(V_n = k).\end{aligned}$$

By the theorem of Pitt [5], the measures $\{\gamma_2(k^{-1/2}\,\cdot\,) : k \geq 1\}$ are in $\mathcal{M}_2$, and, by scaling, the measures can be ordered by domination; thus, by Corollary 3.3, ζ_n is in $\mathcal{M}_2$, as was to be shown. □

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