

# Fluctuations of polyspectra in spherical and Euclidean random wave models\*

Francesco Grotto<sup>†</sup>    Leonardo Maini<sup>‡</sup>    Anna Paola Todino<sup>§</sup>

## Abstract

We consider polynomial transforms (polyspectra) of Berry’s model – the Euclidean Random Wave model – and of Random Hyperspherical Harmonics. We determine the asymptotic behavior of variance for polyspectra of any order in the high-frequency limit. In particular, we are able to treat polyspectra of any odd order  $q \geq 5$ , whose asymptotic behavior was left as a conjecture in the case of Random Hyperspherical Harmonics by Marinucci and Wigman (*Comm. Math. Phys.* 2014). To this end, we exploit a relation between the variance of polyspectra and the distribution of uniform random walks on Euclidean space with finitely many steps, which allows us to rely on technical results in the latter context.

**Keywords:** Gaussian eigenfunctions; Bessel functions; random walk.

**MSC2020 subject classifications:** 42C10; 60D05; 33C10; 60G50.

Submitted to ECP on March 16, 2023, final version accepted on February 16, 2024.

## 1 Introduction and main results

A *Random Wave* model (RW) can be defined as a random field on a Riemannian manifold given by a random linear combination of eigenfunctions of the Laplace-Beltrami operator at a fixed frequency, or within a small frequency bandwidth. The study of such stochastic objects dates back to the works of Berry and Zelditch [1, 30] and it is now a well-established area of research, in which properties of nodal sets of RWs and the comparison with their deterministic counterpart have garnered significant focus (*cf.* [29] for a recent survey). In homogeneous spaces such as  $\mathbb{R}^d$ , the hypersphere  $S^d$  or the hyperbolic space  $\mathbb{H}^d$  (*cf.* respectively [13, 16, 9] for specific discussions), RWs also appear naturally in the spectral decomposition of isotropic Gaussian random fields, that is Gaussian fields whose law (thus, its covariance function) is invariant under isometries of the underlying manifold, as detailed in [8].

---

\*F. G. and A. P. T. acknowledge support of INdAM through the INdAM-GNAMPA Projects CUP\_E55F22000270001 and CUP\_E53C23001670001. L.M. acknowledges support of the Luxembourg National Research Fund PRIDE17/1224660/GPS.

<sup>†</sup>Università di Pisa, Dipartimento di Matematica, 5 Largo B. Pontecorvo, 56127 Pisa, Italia.

E-mail: francesco.grotto@unipi.it

<sup>‡</sup>Université du Luxembourg, Maison du Nombre, 6 Avenue de la Fonte, 4364 Esch-sur-Alzette, Luxembourg.

E-mail: leonardo.maini@uni.lu

<sup>§</sup>Università del Piemonte Orientale, Dipartimento di Scienze e Innovazione Tecnologica, V.le Teresa Michel, 11, 15121 Alessandria, Italia.

E-mail: annapaola.todino@uniupo.it

In this note, we focus on Gaussian RWs on Euclidean space  $\mathbb{R}^d$  and on the (hyper)sphere  $S^d \subset \mathbb{R}^{d+1}$ . Our main result, Theorem 1.3, establishes asymptotics for (variances of) polynomial transforms of those RWs. In particular, we complete the high-frequency asymptotic description of the so-called *polyspectra*, that is integrals of Hermite polynomials of the RWs on fixed space domains: the case of Hermite polynomials of large odd order was left as a conjecture on hyperspheres in [19, 17] and was not discussed in works (such as [22, 13]) in the Euclidean case (see Remark 1.4 for a precise comparison with previous results). Polyspectra constitute the elementary objects in the Wiener chaos decomposition of functionals of Gaussian RWs, therefore their asymptotics play a crucial role in the Wiener chaos approach to limit theorems.

In order to prove Theorem 1.3 we will exploit a peculiar relation with uniform random walks on Euclidean space, and the well-established theory of those random processes. To some extent, this resembles a connection with short random walks considered in the case of random spherical harmonics in [14, 15].

**Notation**

We regard the  $d$ -dimensional sphere  $S^d = \{x \in \mathbb{R}^{d+1} : |x| = 1\}$  as an embedded manifold of codimension 1; we denote by  $\sigma_d$  the induced volume measure and by  $\omega_d = \sigma_d(S^d) = 2\pi^{(d+1)/2}/\Gamma(\frac{d+1}{2})$  the total volume. The geodesic distance on  $S^d$  is given by  $d(x, y) = \cos^{-1}(x \cdot y)$ ,  $x \cdot y$  denoting the canonical scalar product in  $\mathbb{R}^{d+1}$ .

We denote by  $B_d^E(x, R) \subset \mathbb{R}^d$  the ball of radius  $R$  in  $\mathbb{R}^d$  centered at the point  $x$ . When the context allows it, we lighten notation omitting dependencies, writing for instance  $B_R$  to denote a ball of radius  $R$  whose center can be chosen arbitrarily. The same applies to geodesic balls  $B_d^S(x, R) \subset S^d$ . The symbol  $\chi_A$  denotes the indicator function of a set  $A$ .

The symbol  $C$  denotes a *positive* constant, possibly differing in any of its occurrences, and depending only on eventual subscripts, as in  $C_{a,b}$ . Landau’s  $O$  and  $o$  symbols have their usual meaning, with constants involved in upper and lower bounds depending again only on eventual subscripts, as in  $O_{a,b}(1)$ .

**1.1 Random wave models**

*Berry’s model*, that is the RW model on Euclidean space  $\mathbb{R}^d$ , is the centered Gaussian random field  $U_\lambda(x)$ ,  $\lambda > 0$ ,  $x \in \mathbb{R}^d$ , with covariance function

$$\mathbb{E}[U_\lambda(x)U_\lambda(y)] = j_d(\lambda|x - y|), \quad x, y \in \mathbb{R}^d, \tag{1.1}$$

where  $j_d$  is (a multiple of) the Fourier transform of the volume measure  $\sigma_{d-1}$  of  $S^{d-1}$ , regarded as a generalized function on the ambient space  $\mathbb{R}^d$ ,

$$j_d(|x|) = \frac{1}{\omega_{d-1}} \int_{S^{d-1}} e^{ix \cdot \theta} d\sigma_{d-1}(\theta) = \frac{\nu!2^\nu}{(|x - y|)^\nu} J_\nu(|x - y|), \quad \nu = \nu_d := \frac{d}{2} - 1, \tag{1.2}$$

$J_\nu$  being the Bessel function of first kind and order  $\nu$ . In particular,  $j_2(r) = J_0(r)$  and  $j_3(r) = \text{sinc}(r) = \frac{\sin(r)}{r}$ . Notation  $\nu = d/2 - 1$  will remain valid throughout the paper.

The Gaussian field  $U_\lambda$  is named after M. Berry, who introduced it in order to describe local behavior of high frequency eigenstates in quantum billiards [1, 2]. It arises as a central limit of random linear combinations of planar waves on  $\mathbb{R}^d$  of the form

$$\frac{1}{\sqrt{N}} \sum_{i=1}^N \cos(\lambda x \cdot y_i + \phi_i), \quad x \in \mathbb{R}^d, \quad N \rightarrow \infty,$$

with the  $y_i$ ’s and  $\phi_i$ ’s being i.i.d. uniform random variables respectively in  $S^{d-1}$  and  $[0, 2\pi]$  (cf. [2, 22]). Samples of  $U_\lambda$  are themselves smooth  $\lambda^2$ -eigenfunctions of the Laplace operator.

Random (Hyper)Spherical Harmonics, that is the RW model on the sphere  $S^d$ , consist in the centered Gaussian random field  $T_\ell(x)$ ,  $\ell \in \mathbb{N}$ ,  $x \in S^d$ , with covariance

$$\mathbb{E} [T_\ell(x)T_\ell(y)] = \mathcal{G}_{d,\ell}(\cos d(x, y)) = \binom{\ell + \nu}{\ell}^{-1} P_\ell^{\nu,\nu}(\cos d(x, y)) \quad x, y \in S^d, \quad (1.3)$$

where  $\mathcal{G}_{d,\ell}$  is the normalized Gegenbauer polynomial of degree  $\ell$  (see [25, 4.7]), the right-hand side providing an alternative representation in terms of Jacobi polynomials. This random field can be regarded as the analog of Berry's model on  $\mathbb{R}^d$ . Indeed, the role of Euclidean planar waves is played on the sphere  $S^d$  by hyperspherical harmonics, that is Laplace-Beltrami eigenfunctions

$$\{Y_{\ell,m,d}\}_{m=1,\dots,\eta_{\ell,d}} \subset C^\infty(S^d), \ell \in \mathbb{N}, \quad \Delta_{S^d} Y_{\ell,m,d} + \ell(\ell + d - 1)Y_{\ell,m,d} = 0,$$

forming a complete orthonormal basis of  $L^2(S^d, \sigma_d)$ , spanning at each fixed  $\ell \in \mathbb{N}$  a distinct eigenspace<sup>1</sup> of dimension  $\eta_{\ell,d} = \frac{2\ell+d-1}{\ell} \binom{\ell+d-2}{\ell-1}$ . The Gaussian field  $T_\ell$  has the law of a random superposition of waves at a fixed wavenumber:

$$T_\ell(x) \sim \sum_{m=1}^{\eta_{\ell,d}} a_{\ell,m,d} Y_{\ell,m,d}(x), \quad x \in S^d,$$

with  $\{a_{\ell,m,d}\}_{m=1}^{\eta_{\ell,d}}$  being i.i.d. Gaussian variables  $N(0, \omega_d/\eta_{\ell,d})$ . The index  $\ell$  parametrizes the spectrum of Laplace-Beltrami operator on  $S^d$  through  $\ell(\ell + d - 1)$ , thus in the limit  $\ell \rightarrow \infty$  it plays the same asymptotic role of  $\lambda \rightarrow \infty$  in the Euclidean setting, where  $\lambda^2$  is the eigenvalue of the wave  $x \mapsto e^{i\lambda x \cdot \theta}$ .

### 1.2 Polyspectra

Relevant geometric functionals of  $U_\lambda$ , such as the  $(d - 1)$ -dimensional volume of the nodal set  $\{U_\lambda = 0\}$ , depend in general also on derivatives (i.e. the gradient) of the random field. However, already in the case of integral functionals of the form

$$\mathcal{F}_\lambda = \int_D F(U_\lambda(x))dx, \quad F : \mathbb{R} \rightarrow \mathbb{R}, \lambda > 0, D \subset \mathbb{R}^d, \quad (1.4)$$

(and the analog for spherical RWs on measurable subsets  $D \subseteq S^d$ ) the description of the asymptotic behavior as  $\lambda \rightarrow \infty$  is not trivial, essentially because covariance functions do not satisfy integrability conditions allowing direct applications of basic results (e.g. [7]), and they oscillate between positive and negative values.

**Remark 1.1.** The covariance function (1.1) depends on  $\lambda|x - y|$ , therefore the high-frequency limit for functionals of the form (1.4) is equivalent to a (fixed-frequency) large domain limit. Indeed, the random fields  $B_\lambda(\cdot) \sim B_1(\lambda \cdot)$  have the same law, and thus integral functionals

$$\int_D F(U_\lambda(x))dx \sim \frac{1}{\lambda^d} \int_{\lambda D} F(U_1(x))dx,$$

are also equidistributed. There is no equivalent infinite-domain limit on the compact  $S^d$ .

Since the RWs we consider are Gaussian random fields, functionals of the form (1.4) can be studied considering their Wiener chaos expansion. Hermite polynomials,  $H_n(t) = (-1)^n \phi^{(n)}(t)/\phi(t)$ ,  $n \in \mathbb{N}$ , form an orthogonal basis of  $L^2(\mathbb{R}, \phi(t)dt)$ ,  $\phi$  being the p.d.f. of

<sup>1</sup>In fact, the arbitrary choice of an orthogonal basis of the eigenspace relative to  $\ell(\ell + d - 1)$  is irrelevant in our scope.

the standard Gaussian variable. If  $F \in L^2(\mathbb{R}; \phi(x)dx)$ , the Wiener chaos decomposition of  $\mathcal{F}_\lambda$  is given by (cf. [22])

$$\mathcal{F}_\lambda = \sum_{q=0}^{\infty} \frac{1}{q!} \left( \int_{\mathbb{R}} F(t)H_q(t)\phi(t)dt \right) \left( \int_D H_q(U_\lambda(x))dx \right).$$

The stochastic terms  $\int_D H_q(U_\lambda(x))dx$  in the decomposition are called *polyspectra*: Fourth Moment Theorems – by now a standard tool in Gaussian analysis, cf. [21] – allow to deduce Central Limit Theorems for single polyspectra and functionals of the above form as  $\lambda \rightarrow \infty$ , provided that asymptotics of variance and fourth cumulants are available.

### 1.3 Variance asymptotics

For  $d, q \geq 2, R > 0$ , we will write:

$$V_{d,R}^E(q, \lambda) = \text{Var} \left( \int_{B^E(x_0, R)} H_q(U_\lambda(x))dx \right), \quad \lambda > 0,$$

$$V_{d,R}^S(q, \ell) = \text{Var} \left( \int_{B^S(x_0, R)} H_q(T_\ell(x))d\sigma_d(x) \right), \quad \ell \in \mathbb{N}.$$

**Remark 1.2.** The case  $q = 0$  needs no discussion. As for  $q = 1$ , it turns out that both  $V_{d,R}^E(1, \lambda)$  and  $V_{d,R}^S(1, \ell)$  have an oscillatory behavior as  $\lambda, \ell \rightarrow \infty$  and they can vanish (see respectively [13, 26] in the two geometrical settings). Anyways, when  $q = 1$  the polyspectrum is a Gaussian variable, and the study of its variance is simpler, so we will omit that case in our discussion.

Our main result is the following:

**Theorem 1.3.** Let  $d, q \geq 2, R > 0$ . There exist finite **positive** constants  $c_q^d$  such that:

- (Euclidean) as  $\lambda \rightarrow \infty$ ,

$$V_{d,R}^E(q, \lambda) = c_q^d \omega_d \omega_{d-1} R^d (1 + o_{q,d}(1)) \cdot \begin{cases} \lambda^{1-d} & q = 2, \\ \lambda^{-2} \log(\lambda) & q = 4, d = 2, \\ \lambda^{-d} & \text{all other } d \geq 2, q \geq 3. \end{cases}$$

- (Hyperspherical) as  $\ell \rightarrow \infty$ , if  $R \in (0, \pi)$ ,

$$V_{d,R}^S(q, \ell) = c_q^d \omega_{d-1} \sigma_d(B_R^S) (1 + o_{q,d}(1)) \cdot \begin{cases} \ell^{1-d} & q = 2, \\ \ell^{-2} \log(\ell) & q = 4, d = 2, \\ \ell^{-d} & \text{all other } d \geq 2, q \geq 3, \end{cases}$$

and when  $R = \pi$  (that is in the case of polyspectra obtained integrating over the whole  $S^d$ ),

$$V_{d,\pi}^S(q, \ell) = 2c_q^d \omega_{d-1} \omega_d (1 + o_{q,d}(1)) \cdot \begin{cases} \ell^{1-d} & q = 2, \\ \ell^{-2} \log(\ell) & q = 4, d = 2, \\ 0 & q, \ell \text{ both odd}, \\ \ell^{-d} & \text{all other } d \geq 2, q \geq 3. \end{cases}$$

**Remark 1.4.** Let us clarify the relation of Theorem 1.3 with the previous literature.

- (Euclidean case) [22] proved the result in dimension  $d = 2$  for even  $q \geq 2$  and upper bounds for odd  $q \geq 3$ , for integration domains  $D \subset \mathbb{R}^2$  with boundary of class  $C^1$ ; [20] generalizes the discussion to arbitrary  $d \geq 2$  and even  $q \geq 2$ ; for  $d \geq 2$  and even  $q \geq 2$  [12] considers compact domains  $D \subset \mathbb{R}^d$  with finite perimeter, treating the larger class of weakly stationary random fields and establishing asymptotics for covariances of functionals;
- (case of  $S^2$ ) [18, 19] proved the above asymptotics on the whole  $S^2$  (case  $R = \pi$ ) leaving as an open problem the positivity of  $c_q^2$  for odd  $q \geq 5$ ; [26] discussed the case  $R < \pi$  for  $q = 2$  and provided upper bounds for larger  $q \geq 3$ ;
- (Hyperspherical case) concerning polyspectra over the whole  $S^d$  ( $R = \pi$ ), in larger dimension the above asymptotic was established for all even  $q \geq 2$  in [17] and for  $q = 3$  [24], leaving positivity of  $c_q^d$  for odd  $q \geq 5$  as a conjecture (see [17, p. 2386]).

The remainder of this note is organized as follows. Section 2 details the relation between variance of polyspectra and the density of uniform random walks, and introduces results on the latter that we can exploit to control the former. We prove Theorem 1.3 in Section 3.

## 2 Pearson’s random walk

Let  $X_n^d, n = 1, 2, \dots$  denote the uniform random walk on  $\mathbb{R}^d$ , that is

$$X_n^d = U_1^d + \dots + U_n^d,$$

with  $(U_k^d)_{k \in \mathbb{N}}$  a sequence of i.i.d. uniform random variables on  $S^{d-1} \subset \mathbb{R}^d$ . The stochastic process  $X_n^d$  is known as *Pearson’s Random Walk*, because of a very early mention in [23].

Much is known on Pearson’s random walk: in particular, its distribution and moments for a small number of steps  $n$  are of interest in Number Theory because of their relation with Mahler measures. We can thus rely on the series of works [3, 6, 5, 4], to which we refer for a comprehensive background on the matter and the related literature.

The basic result on the distribution of  $X_n$  for small  $n$  is a representation formula for the p.d.f. dating back to the seminal work [10].

**Lemma 2.1** (Kluyver’s Formula). *For  $n \geq 2$  the law  $\mu_n^d$  of  $X_n^d$  is absolutely continuous with respect to Lebesgue’s measure on  $\mathbb{R}^d$ , and its density is a radial function, i.e.  $\mu_q^d(dx) = f_q^d(\|x\|)dx$  with  $f_q^d(\|\cdot\|) \in L^1(\mathbb{R}^d)$ .*

*The (Euclidean) distance from the origin  $\|X_n^d\|$  is absolutely continuous with respect to Lebesgue’s measure on  $(0, \infty)$ , and its density is given by*

$$\rho_n^d(r) = \frac{1}{(\nu!)^2 4^\nu} \int_0^\infty (tr)^{2\nu+1} j_d(tr) j_d(t)^n dt, \quad r > 0, \quad \nu = \frac{d}{2} - 1.$$

To be precise, the integral in Kluyver’s formula has to be regarded as an improper integral that either converges to a non-negative number (for large enough  $d, n$  even absolutely) or diverges to  $+\infty$  (see Remark 2.8 below). We refer to [4, Theorem 2.1] for a proof of Lemma 2.1 and a thorough general discussion. Let us note that  $f_q^d$  and  $\rho_q^d$  are related by

$$\omega_{d-1} \int_0^s r^{d-1} f_q^d(r) dr = \int_{B_s} \mu_q^d(dx) = \mathbb{P}(X_q^d \in B_s) = \mathbb{P}(\|X_q^d\| \leq s) = \int_0^s \rho_q^d(r) dr. \quad (2.1)$$

### 2.1 Random walks and polyspectra

Before we outline further properties of Pearson’s random walk to be used in the last Section, let us explain heuristically how one is led to consider this apparently unrelated

object in studying RW's polyspectra. The following computation concerns the Euclidean case only, and has no direct analog in the case of hyperspherical RWs. Nevertheless, in the high-frequency limiting regime the Euclidean and hyperspherical RW models turn out to exhibit the same behavior. It holds

$$V_{d,R}^E(q, \lambda) = q! \int_{B_R} \int_{B_R} j_d(\lambda|x - y|)^q dx dy,$$

since  $j_d(\lambda|x - y|)$  is the covariance function of  $U_\lambda$ , and if  $X, Y$  are two centered jointly Gaussian variables  $\mathbb{E}[H_q(X)H_q(Y)] = q! \mathbb{E}[XY]^q$ . By (1.2), the function

$$j_d(|x|) = \mathcal{F}[\mu_1^d](x) := \int_{\mathbb{R}^d} e^{ix \cdot y} \mu_1^d(dy) \tag{2.2}$$

can be represented as the Fourier transform of  $\mu_1^d$ . Hence we can write

$$\begin{aligned} V_{d,R}^E(q, \lambda) &= q! \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\chi_{B_{\lambda R}}(x)\chi_{B_{\lambda R}}(y)}{\lambda^{2d}} \mathcal{F}[\mu_1^d](x - y)^q dx dy \\ &= q! \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\chi_{B_{\lambda R}}(x)\chi_{x+B_{\lambda R}}(z)}{\lambda^{2d}} \underbrace{\mathcal{F}[\mu_1^d * \dots * \mu_1^d]}_{q \text{ times}}(z) dx dz, \end{aligned}$$

where we recognize the  $q$ -fold convolution of  $\mu_1^d$  with itself as the law  $\mu_q^d$  of  $X_q^d$ . As  $\lambda \rightarrow \infty$ , indicator functions in the last displayed equation converge pointwise to 1 on  $\mathbb{R}^d$ , therefore we expect – at least intuitively – that the limiting integral

$$\int_{\mathbb{R}^d} \mathcal{F}[\mu_q^d](z) dz = f_q^d(0)$$

plays a relevant role in determining the asymptotic behavior of  $V_{d,R}^E(q, \lambda)$ . By (2.1) and Kluyver's formula we can also write

$$\omega_{d-1} f_q^d(0) = \lim_{r \rightarrow 0} \frac{\rho_q^d(r)}{r^{d-1}} = \frac{1}{(\nu!)^2 4^\nu} \int_0^\infty j_d(t)^q t^{d-1} dt = \rho_{q-1}^d(1),$$

therefore the crucial objects appear to be the integrals

$$I_q^d := \int_0^\infty j_d(t)^q t^{d-1} dt = \frac{\rho_{q-1}^d(1)}{(\nu!)^2 4^\nu}, \tag{2.3}$$

and these numbers can *a priori* take any value in  $[0, \infty]$ .

In fact,  $I_q^d$  coincides with the coefficient  $c_q^d$  appearing in Theorem 1.3 in the cases  $d \geq 2, q \geq 3$  except  $d = 2, q = 4$ . Before we prove it in the last Section, we need to discuss finiteness and positivity of the  $I_q^d$ 's, that is the task for which we need to resort to previous results on Pearson's random walk.

### 2.2 Density of short random walks

We report an explicit expression for the densities  $\rho_2^d$  and a recursion formula allowing to treat the density at higher  $n$ . Representations for  $\rho_n^d, n = 3, 4$ , in terms of hypergeometric functions are available in [4, Equations (74),(79)]. We recall that  $\nu = d/2 - 1$  in all its occurrences.

**Lemma 2.2** ([4, Lemma 4.1]). *For  $d \geq 2$  and  $0 < r < 2$  it holds*

$$\rho_2^d(r) = \frac{2}{\pi \binom{2\nu}{\nu}} r^{2\nu} (4 - r^2)^{\nu-1/2} \chi_{(0,2)}(r).$$

**Lemma 2.3** ([4, Theorem 2.9]). *Let  $d, n \geq 2$ ,  $r > 0$  and define  $\psi_n^d(r) = \rho_n^d(r)/r^{d-1}$ . It holds, for  $n \geq 3$ ,*

$$\psi_n^d(r) = \frac{(\nu!)^2 4^\nu}{\pi(2\nu)!} \int_{-1}^1 \psi_{n-1}^d \left( \sqrt{1 + 2sr + r^2} \right) (1 - s^2)^{\nu-1/2} ds, \quad r \in (0, n). \quad (2.4)$$

**Corollary 2.4.** *For any  $d \geq 2$  and  $n \geq 3$ ,  $\psi_n^d$  is strictly positive (possibly infinite) on  $(0, n)$ .*

*Proof.* By induction on  $n$ , the case  $n = 2$  follows by definition of  $\psi_2^d$  and Lemma 2.2. Assume now that  $\psi_{n-1}^d(r) > 0$  for  $r \in (0, n - 1)$  (by definition it vanishes for  $r > n - 1$ ); by (2.4) if the open set

$$\{s \in (-1, 1) : \psi_{n-1}^d \left( \sqrt{1 + 2sr + r^2} \right) > 0\} = \{s \in (-1, 1) : 1 + 2sr + r^2 < (n - 1)^2\}$$

is not empty, then  $\psi_n^d(r) > 0$ . This is in fact the case, because  $p(s) = 1 + 2sr + r^2$  is increasing and continuous and  $p(-1) = 1 - 2r + r^2 = (r - 1)^2 < (n - 1)^2$  for  $r < n$ .  $\square$

Moving back to the specific integrals  $I_q^d$  we are interested in, we combine the latter Corollary and the following approximation result on Bessel functions to show that they are both positive and finite.

**Lemma 2.5** ([11, Theorem 4, Equation 6]). *Let  $d \geq 2$ . There exist constants  $\phi_d > 0$  such that*

$$0 < \sup_{r \geq 0} r^{3/2} \left| J_\nu(r) - \sqrt{\frac{2}{\pi r}} \cos(r - \phi_d) \right| < \infty. \quad (2.5)$$

*In particular by (1.2),*

$$j_d(r) = C_d r^{-(d-1)/2} \cos(r - \phi_d) + O_d(r^{-(d+1)/2}), \quad r \rightarrow \infty. \quad (2.6)$$

**Lemma 2.6.** *For any  $d \geq 2, q \geq 3$  except the single case  $d = 2, q = 4$ ,  $I_q^d \in (0, \infty)$  is positive and finite.*

*Proof.* By Corollary 2.4, for all  $d \geq 2, q \geq 3$  it holds

$$I_n^d = \frac{\rho_{n-1}^d(1)}{(\nu!)^2 4^\nu} = \frac{\psi_{n-1}^d(1)}{(\nu!)^2 4^\nu} > 0.$$

Concerning finiteness, by (2.6) and (2.3) we have

$$I_q^d \leq C_{d,q} \int_1^\infty r^{(d-1)(1-q/2)} dr;$$

in particular, the integral in (2.3) is absolutely convergent if  $q > \frac{d}{d-1}$ . We are left to discuss the cases  $d = 2, 3, q = 3$ , since we are excluding  $d = 2, q = 4$  (see Remark 2.8 below). A classical result on Bessel functions (cf. [27, Section 13.46]) implies  $I_3^2 = \frac{2}{\pi\sqrt{3}}$ , and  $I_3^3 = \int_0^\infty t^{-1} \sin(t)^3 dt = \pi/4$  is easily computed reducing it to Dirichlet's integral.  $\square$

**Remark 2.7.** The following exact expressions

$$I_3^d = \frac{2}{\pi\sqrt{3}} \cdot \frac{12^\nu (\nu!)^4}{(2\nu)!}, \quad d \geq 2, \quad I_5^2 = \frac{\sqrt{5} \Gamma\left(\frac{1}{15}\right) \Gamma\left(\frac{2}{15}\right) \Gamma\left(\frac{4}{15}\right) \Gamma\left(\frac{8}{15}\right)}{40\pi^4},$$

can be deduced respectively from [4, Proposition 4.3] and [6, Theorem 5.1] (concerning the asymptotics of  $\rho_n^d(r)$  as  $r \rightarrow 0$ ). To the best of our knowledge, no representation is available in the general case.

**Remark 2.8.** In the case  $d = 2, q = 4$  excluded from Lemma 2.6, one can prove by means of Lemma 2.5 that  $I_4^2 = \int_0^\infty J_0(t)^4 t dt = +\infty$ , coherently with the fact that the density  $\rho_2^3(1) = I_2^4 = +\infty$  is not finite. This is actually the only case of a density  $\rho_n^d$ ,  $d, n \geq 2$ , taking value  $+\infty$  at some point, as for larger  $n$  the regularizing effect of the multiple convolution defining the law of Pearson's random walk prevails (cf. again [4]).

### 3 Fluctuation asymptotics and integration domains

#### 3.1 Variance of Euclidean polyspectra

We first rewrite the variances we are interested in as integrals of a single variable.

**Lemma 3.1.** *Let  $R > 0$ ,  $B_R = B^E(x_0, R) \subseteq \mathbb{R}^d$  a ball of radius  $R$  centered at a (fixed, arbitrary) point  $x_0 \in \mathbb{R}^d$ , and  $f \in L^1_{loc}(\mathbb{R})$ ; it holds*

$$\int_{B_R} \int_{B_R} f(|x - y|) dx dy = \int_0^{2R} f(r) W_{d,R}(r) r^{d-1} dr, \tag{3.1}$$

where, denoting by  $m_d$  Lebesgue's measure on  $\mathbb{R}^d$ ,

$$W_{d,R}(r) = \omega_{d-1} m_d(B(x, R) \cap B(y, R)), \quad |x - y| = r, \tag{3.2}$$

whose value does not depend on the choice of points  $x, y \in \mathbb{R}^d$ .

The proof is a consequence of the fact that translations on  $\mathbb{R}^d$  are isometries of the whole space: it holds

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \chi_{B_R}(x) \chi_{B_R}(y) f(|x - y|) dx dy = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \chi_{B_R}(x) \chi_{B_R+z}(x) f(|z|) dx dz,$$

from which (3.1) follows moving to polar coordinates for  $z$ . It is easy to observe that  $W_{d,R} : [0, \infty) \rightarrow [0, \infty)$  is supported by  $[0, 2R]$ , it is differentiable on  $(0, \infty) \setminus \{2R\}$  and  $W_{d,R}(0) = \omega_{d-1} m_d(B_R)$ ,  $W_{d,R}(2R) = 0$ .

*Proof of Theorem 1.3, Euclidean part.* In sight of Lemma 3.1 it holds,

$$V_{d,R}^E(q, \lambda) = q! \int_{B_R} \int_{B_R} j_d(\lambda|x - y|)^q dx dy = q! \int_0^{2R} j_d(\lambda r)^q W_{d,R}(r) r^{d-1} dr. \tag{3.3}$$

For  $q = 2$ , (2.6) implies that, as  $\lambda \rightarrow \infty$ ,

$$V_{d,R}^E(2, \lambda) = C_d \int_0^{2R} \frac{\cos(\lambda r - \phi_d)^2}{\lambda^{d-1}} W_{d,R}(r) dr + O_{d,R} \left( \frac{1}{\lambda^d} \right) = C_{d,R} \frac{1 + o_{d,R}(1)}{\lambda^{d-1}},$$

the second step following by expanding  $2 \cos(\theta)^2 = 1 + \cos(2\theta)$  and applying Riemann-Lebesgue lemma to oscillatory terms.

We also treat separately the case  $d = 2, q = 4$ , since it involves a logarithmic discrepancy with the general case  $q \geq 3$ . (2.6) implies that, as  $\lambda \rightarrow \infty$ ,

$$V_{2,R}^E(4, \lambda) = C_d \int_0^{2R} \frac{\cos(\lambda r - \phi_d)^4}{\lambda^2 r} W_{d,R}(r) dr + O_R \left( \frac{1}{\lambda^3} \right) = C_R \frac{\log(\lambda) + o_R(1)}{\lambda^2},$$

where, as in the case  $q = 2$ , the last equality follows by  $8 \cos(\theta)^4 = 3 + 4 \cos(2\theta) + \cos(4\theta)$  and Riemann-Lebesgue lemma.

Finally, we consider the general case  $q \geq 3$ , excluding  $d = 2, q = 4$ . By (3.3),

$$\begin{aligned} V_{d,R}^E(q, \lambda) &= q! \int_0^{2R} j_d(\lambda r)^q r^{d-1} \left( \int_r^{2R} -W'_{d,R}(s) ds \right) dr \\ &= -q! \int_0^{2R} W'_{d,R}(s) \left( \int_0^s j_d(\lambda r)^q r^{d-1} dr \right) ds \\ &= \frac{q!}{\lambda^d} \int_0^{2R} -W'_{d,R}(s) \left( \int_0^{s\lambda} j_d(r)^q r^{d-1} dr \right) ds, \end{aligned}$$

integrating by parts in the second step. Since  $|W'_{d,R}(r)|$  is bounded and supported by  $[0, 2R]$ , the thesis follows by dominated convergence and Lemma 2.6 (implying convergence of the inner integral on the right-hand side as  $\lambda \rightarrow \infty$ ).  $\square$



**3.2 Variance of hyperspherical polyspectra**

The forthcoming computations are in fact close analogs of the ones in the previous paragraph, only in a perhaps less familiar geometrical setting. The change of variables of Lemma 3.1 takes on  $S^d$  the following form, with flat translations of  $\mathbb{R}^d$  being replaced by isometries of  $S^d$ .

**Lemma 3.2.** *Let  $R \in [0, \pi]$ ,  $B_R = B^S(x_0, R) \subseteq S^d$  a ball of radius  $R$  centered at a (fixed, arbitrary) point  $x_0 \in S^d$ , and  $f \in L^1(\mathbb{R})$ ; it holds*

$$\int_{B_R} \int_{B_R} f(d(x, y)) d\sigma_d(x) d\sigma_d(y) = \int_0^\pi f(r) \tilde{W}_{d,R}(r) \sin(r)^{d-1} dr, \tag{3.4}$$

where

$$\tilde{W}_{d,R}(r) = \omega_{d-1} \sigma_d(B(x, R) \cap B(y, R)), \quad d(x, y) = r, \tag{3.5}$$

whose value does not depend on the choice of points  $x, y \in S^d$ .

The proof is conceptually analogous to the one of Lemma 3.1. First, let us recall that the isometry group of  $S^d$  can be identified with that of linear orthogonal transformations of the ambient  $\mathbb{R}^{d+1}$ , and it acts transitively on  $S^d$ . The volume  $\sigma_d$  and the spherical geodesic distance are invariant under isometries, that is  $I_\# \sigma_d = \sigma_d \circ I = \sigma_d$  and  $d(a, b) = d(I(a), I(b))$  for any isometry  $I : S^d \rightarrow S^d$  and points  $a, b \in S^d$ .

*Proof.* We first observe that for any isometry  $I : S^d \rightarrow S^d$  it holds

$$\begin{aligned} \sigma_d(B(Ix, R) \cap B(Iy, R)) &= \sigma_d \{z : d(Ix, z), d(Iy, z) \leq R\} \\ &= \sigma_d \{z : d(x, I^{-1}z), d(y, I^{-1}z) \leq R\} = \sigma_d \{Iw : d(x, w), d(y, w) \leq R\} \\ &= I_\#^{-1} \sigma_d \{w : d(x, w), d(y, w) \leq R\} = \sigma_d(B(x, R) \cap B(y, R)), \end{aligned}$$

therefore the right-hand side only depends on  $d(x, y)$ , as it is a function of  $x, y$  invariant under isometries. In particular,  $\tilde{W}_{d,R}(r)$  in (3.5) is well-defined.

For any two points  $x, y \in S^d$  we can choose isometries  $I_x, I_y : S^d \rightarrow S^d$  such that  $I_x x_0 = x$  and  $I_y x_0 = y$ . Letting  $z = I_y I_x^{-1} x_0$ , by Fubini-Tonelli Theorem it holds

$$\begin{aligned} \int_{S^d} \int_{S^d} \chi \{x \in B_R\} \chi \{y \in B_R\} f(d(x, y)) d\sigma_d(x) d\sigma_d(y) \\ = \int_{S^d} d\sigma_d(z) f(d(x_0, z)) \int_{S^d} d\sigma_d(x) \chi \{x \in B_R\} \chi \{I_x z \in B_R\}, \end{aligned}$$

where the inner integral can be rewritten as follows: since  $I_x^{-1} x_0 = I_x^{-2} x$ ,

$$\begin{aligned} \int_{S^d} \chi \{x \in B_R\} \chi \{I_x z \in B_R\} d\sigma_d(x) &= \sigma_d \{x : d(x, x_0), d(I_x z, x_0) \leq R\} \\ &= I_\#^2 \sigma_d \{x : d(x_0, I_x^{-1} x_0), d(z, I_x^{-1} x_0) \leq R\} \\ &= \sigma_d \{I_x^{-2} x : d(x_0, I_x^{-1} x_0), d(z, I_x^{-1} x_0) \leq R\} \\ &= \sigma_d \{w : d(x_0, w), d(z, w) \leq R\}, \end{aligned}$$

thus in particular it only depends on  $d(x_0, z)$ , thanks to the argument at the beginning of the proof. Since the volume element of  $S^d$  reduces to

$$\int_{S^d} h(d(x_0, x)) d\sigma_d(x) = \omega_{d-1} \int_0^\pi h(r) \sin(r)^{d-1} dr, \quad h \in L^1([0, \pi]),$$

for radial functions, combining the above equations concludes the proof. □

The function  $\tilde{W}_{d,R} : [0, \pi] \rightarrow [0, \infty)$  is differentiable, and attains its maximum value in  $\tilde{W}_{d,R}(0) = \omega_{d-1}\sigma_d(B(x_0, R))$  and minimum in  $\tilde{W}_{d,R}(\pi)$  (which, unlike the Euclidean  $W_{d,R}(2R)$ , does not vanish in general). We can now proceed to complete the proof of Theorem 1.3 by reducing ourselves to the computations performed in the Euclidean case thanks to the following:

**Lemma 3.3** (Hilb's Asymptotic Formula [25, Theorem 8.21.12]). *Let  $d \geq 2, \ell \in \mathbb{N}$ . Given  $a \in (0, \pi)$ , for all  $\theta \in (0, a)$  it holds:*

$$(\sin \theta)^\nu \mathcal{G}_{d,\ell}(\cos \theta) = \frac{2^\nu}{(\ell + \nu)} \left( \frac{\Gamma(\ell + d/2)}{L^\nu \ell!} \left( \frac{\theta}{\sin \theta} \right)^{1/2} J_\nu(L\theta) + \delta(\theta) \right), \tag{3.6}$$

where  $L = \ell + \frac{d-1}{2}$  and  $\delta(\theta) \leq C_{d,a}(\chi_{(0,1/\ell)}(\theta)\theta^{\nu+2}\ell^\nu + \chi_{(1/\ell,a)}(\theta)\sqrt{\theta}\ell^{-3/2})$ .

**Corollary 3.4.** *Let  $d \geq 2, \ell \in \mathbb{N}$ . For all  $a \in (0, \pi)$  it holds, as  $\ell \rightarrow \infty$ ,*

$$\int_0^a \mathcal{G}_{d,\ell}(\cos r)^q (\sin r)^{d-1} dr = (1 + o(1)) \begin{cases} C_d \ell^{1-d} & q = 2 \\ C \log(\ell)/\ell^2 & q = 4, d = 2 \\ I_q^d / \ell^d & \text{all other } d \geq 2, q \geq 3 \end{cases} \tag{3.7}$$

with  $I_q^d$  being as in Lemma 2.6 and  $C, C_d > 0$  positive constants.

**Remark 3.5.** The limit case  $a = \pi$  can be reduced to the one with  $a = \pi/2$  splitting the integration domain in half and exploiting the following symmetry property: for all  $d \geq 2$ ,

$$\mathcal{G}_{d,\ell}(t) = (-1)^\ell \mathcal{G}_{d,\ell}(-t), \quad t \in \mathbb{R}, \ell \in \mathbb{N}.$$

In particular, the integral in Corollary 3.4 identically vanishes if  $a = \pi$  and both  $q, \ell$  are odd. If this is not the case, then the thesis of Corollary 3.4 holds with an additional factor 2 on the right-hand side.

Corollary 3.4 actually collects computations already appeared in [28, 18, 17]. Because of this and the similarity with computations in the previous paragraph, we only present a sketch of the proof.

*Proof.* We first observe that as  $\ell \rightarrow \infty$ ,

$$\frac{\Gamma(\ell + d/2)}{L^\nu \ell!} = 1 + o(1), \quad \binom{\ell + \nu}{\ell} = \frac{\ell^\nu}{\nu!} (1 + o(1)). \tag{3.8}$$

When  $q = 2$ , Lemma 3.3 and (3.8) imply

$$\begin{aligned} \int_0^a \mathcal{G}_{d,\ell}(\cos r)^2 (\sin r)^{d-1} dr &= \frac{4^\nu (\nu!)^2}{\ell^{2\nu}} (1 + o(1)) \int_0^a J_\nu(Lr)^2 r dr \\ &\quad + O\left(\frac{1}{\ell^{d-2}} \int_0^a |J_\nu(Lr)| \delta(r) \sin(r) dr\right). \end{aligned}$$

The thesis now follows by applying Lemma 2.5 in the same fashion of the Euclidean case (the remainder term on the right-hand side is proved to be  $O(\ell^{-d})$ ). The analysis of the case  $d = 2, q = 4$  is similar, we refer again to [28] for full details.

As for the remaining cases, by Lemma 3.3, (3.8) and the definition of  $j_d$  in (1.2),

$$\begin{aligned} \int_0^a \mathcal{G}_{d,\ell}(\cos r)^q (\sin r)^{d-1} dr &= (1 + o(1)) \int_0^a \left( \frac{r}{\sin r} \right)^{q\nu + q/2 - (d-1)} j_d(Lr)^q r^{d-1} dr \\ &\quad + O\left(\frac{1}{\ell^{q\nu}} \int_0^a (\sin r)^{-q\nu} |J_\nu(Lr)|^{q-1} \delta(r) (\sin r)^{d-1} dr\right). \end{aligned}$$

The remainder term can be shown to be  $O(\ell^{-(d+2)\wedge(q(d/2-1/2)-1)}) = o(\ell^{-d})$  by means of Lemma 2.5 (changing variables and splitting the integration domain similarly to the previous case). The thesis now follows if

$$\int_0^a \left(\frac{r}{\sin r}\right)^{q\nu+q/2-(d-1)} j_d(Lr)^q r^{d-1} dr = (1 + o(1)) \int_0^\infty j_d(r)^q r^{d-1} dr, \quad \ell \rightarrow \infty,$$

which can be proved with the same computation exploiting integration by parts we used at the end of the proof of Theorem 1.3, Euclidean case, that is replacing  $W_{d,R}$  of that computation with  $\chi_{[0,a]}(r)(r/\sin r)^{q\nu+q/2-(d-1)}$ .  $\square$

*Proof of Theorem 1.3, Hyperspherical part.* Let  $R \in (0, \pi)$  (the case  $R = \pi$  is dealt with similarly, by Remark 3.5). By Lemma 3.2,

$$\begin{aligned} V_{d,R}^S(q, \ell) &= \text{Var} \left( \int_{B_R} H_q(T_\ell(x)) d\sigma_d(x) \right) \\ &= q! \int_{B_R} \int_{B_R} \mathcal{G}_{d,\ell}(x, y)^q d\sigma_d(x) d\sigma_d(y) = q! \int_0^\pi \mathcal{G}_{d,\ell}(\cos r)^q \sin(r)^{d-1} \tilde{W}_{d,R}(r) dr. \end{aligned}$$

Integrating by parts as in Section 3.1 we rewrite:

$$\begin{aligned} \int_0^\pi \mathcal{G}_{d,\ell}(\cos r)^q (\sin r)^{d-1} \tilde{W}_{d,R}(r) dr &= \tilde{W}_{d,R}(\pi) \int_0^\pi \mathcal{G}_{d,\ell}(\cos r)^q (\sin r)^{d-1} dr \\ &\quad - \int_0^\pi \left( \int_0^r \mathcal{G}_{d,\ell}(\cos s)^q (\sin s)^{d-1} ds \right) \tilde{W}'_{d,R}(r) dr. \end{aligned}$$

The thesis now follows from Corollary 3.4. We observe in particular that when we are not in the cases  $q = 2$  nor in the one  $d = 2, q = 4$ , by Corollary 3.4 and dominated convergence we have

$$\begin{aligned} \ell^d \int_0^\pi \mathcal{G}_{d,\ell}(\cos r)^q (\sin r)^{d-1} \tilde{W}_{d,R}(r) dr \\ \xrightarrow{\ell \rightarrow \infty} I_q^d \tilde{W}_{d,R}(\pi) - I_q^d \int_0^\pi \tilde{W}'_{d,R}(r) dr = I_q^d \tilde{W}_{d,R}(0). \quad \square \end{aligned}$$

## References

- [1] M. V. Berry, *Regular and irregular semiclassical wavefunctions*, J. Phys. A Math. Gen. **10** (1977), no. 12, 2083–2091. MR489542
- [2] M. V. Berry, *Statistics of nodal lines and points in chaotic quantum billiards: perimeter corrections, fluctuations, curvature*, J. Phys. A Math. Gen. **35** (2002), no. 13, 3025–3038. MR1913853
- [3] J. M. Borwein, D. Nuyens, A. Straub, and J. Wan, *Some arithmetic properties of short random walk integrals*, Ramanujan J. **26** (2011), no. 1, 109–132. MR2837721
- [4] J. M. Borwein, A. Straub, and C. Vignat, *Densities of short uniform random walks in higher dimensions*, J. Math. Anal. Appl. **437** (2016), no. 1, 668–707. MR3451989
- [5] J. M. Borwein, A. Straub, and J. Wan, *Three-step and four-step random walk integrals*, Exp. Math. **22** (2013), no. 1, 1–14. MR3038778
- [6] J. M. Borwein, A. Straub, J. Wan, and W. Zudilin, *Densities of short uniform random walks*, Canad. J. Math. **64** (2012), no. 5, 961–990, With an appendix by Don Zagier. MR2979573
- [7] P. Breuer and P. Major, *Central limit theorems for nonlinear functionals of Gaussian fields*, J. Multivariate Anal. **13** (1983), no. 3, 425–441. MR716933
- [8] S. Cohen and M. A. Lifshits, *Stationary Gaussian random fields on hyperbolic spaces and on Euclidean spheres*, ESAIM Probab. Stat. **16** (2012), 165–221. MR2946126

- [9] F. Grotto and G. Peccati, *Nonlinear Functionals of Hyperbolic Random Waves: the Wiener Chaos Approach*, arXiv e-prints (2023), arXiv:2301.08315.
- [10] J. C. Kluyver, *Some formulae concerning the integers less than  $n$  and prime to  $n$* , Koninklijke Nederlandse Akademie van Wetenschappen Proceedings Series B Physical Sciences **9** (1906), 408–414.
- [11] I. Krasikov, *Approximations for the Bessel and Airy functions with an explicit error term*, LMS J. Comput. Math. **17** (2014), no. 1, 209–225. MR3230865
- [12] L. Maini, *Asymptotic covariances for functionals of weakly stationary random fields*, Stochastic Process. Appl. **170** (2024), Paper no. 104297. MR4689941
- [13] L. Maini and I. Nourdin, *Spectral central limit theorem for additive functionals of isotropic and stationary Gaussian fields*, Ann. Probab., in press (2024+), arXiv:2206.14458.
- [14] D. Marinucci and G. Peccati, *High-frequency asymptotics for subordinated stationary fields on an abelian compact group*, Stochastic Process. Appl. **118** (2008), no. 4, 585–613. MR2394764
- [15] D. Marinucci and G. Peccati, *Group representations and high-resolution central limit theorems for subordinated spherical random fields*, Bernoulli **16** (2010), no. 3, 798–824. MR2730649
- [16] D. Marinucci and G. Peccati, *Random fields on the sphere*, London Mathematical Society Lecture Note Series, vol. 389, Cambridge University Press, Cambridge, 2011, Representation, limit theorems and cosmological applications. MR2840154
- [17] D. Marinucci and M. Rossi, *Stein-Malliavin approximations for nonlinear functionals of random eigenfunctions on  $S^d$* , J. Funct. Anal. **268** (2015), no. 8, 2379–2420. MR3318653
- [18] D. Marinucci and I. Wigman, *The defect variance of random spherical harmonics*, J. Phys. A Math. Theor. **44** (2011), no. 35, 355206.
- [19] D. Marinucci and I. Wigman, *On nonlinear functionals of random spherical eigenfunctions*, Commun. Math. Phys. **327** (2014), no. 3, 849–872. MR3192051
- [20] M. Notarnicola, *Probabilistic limit theorems and the geometry of random fields*, Ph.D. thesis, Luxembourg University, 2021. MR4678940
- [21] I. Nourdin and G. Peccati, *Normal approximations with Malliavin calculus*, Cambridge Tracts in Mathematics, vol. 192, Cambridge University Press, Cambridge, 2012, From Stein’s method to universality. MR2962301
- [22] I. Nourdin, G. Peccati, and M. Rossi, *Nodal statistics of planar random waves*, Comm. Math. Phys. **369** (2019), no. 1, 99–151. MR3959555
- [23] K. Pearson, *The problem of the random walk*, Nature **72** (1905), no. 1865, 294–294.
- [24] M. Rossi, *The defect of random hyperspherical harmonics*, J. Theoret. Probab. **32** (2019), no. 4, 2135–2165. MR4020703
- [25] G. Szego, *Orthogonal polynomials*, American Math. Soc: Colloquium publ, American Mathematical Society, 1939. MR0000077
- [26] A. P. Todino, *A quantitative central limit theorem for the excursion area of random spherical harmonics over subdomains of  $S^2$* , J. Math. Phys. **60** (2019), no. 2, 023505, 33. MR3916834
- [27] G. N. Watson, *A treatise on the theory of Bessel functions*, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1995, Reprint of the second (1944) edition. MR1349110
- [28] I. Wigman, *Fluctuations of the nodal length of random spherical harmonics*, Comm. Math. Phys. **298** (2010), no. 3, 787–831. MR2670928
- [29] Igor Wigman, *On the nodal structures of random fields: a decade of results*, Journal of Applied and Computational Topology (2023), 1–43.
- [30] S. Zelditch, *Real and complex zeros of Riemannian random waves*, Spectral analysis in geometry and number theory, Contemp. Math., vol. 484, Amer. Math. Soc., Providence, RI, 2009, pp. 321–342. MR1500155

**Acknowledgments.** F. G. wishes to thank the University of Luxembourg for the warm hospitality during part of the preparation of the present work.